HAUSDORFF DIMENSION OF BOUNDARIES OF SELF-AFFINE TILES IN $\mathbb{R}^N$

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Abstract

We present a new method to calculate the Hausdorff dimension of a certain class of fractals: boundaries of self-affine tiles. Among the interesting aspects are that even if the affine contraction underlying the iterated function system is not conjugated to a similarity we obtain an upper- and and lower-bound for its Hausdorff dimension. In fact, we obtain the exact value for the dimension if the moduli of the eigenvalues of the underlying affine contraction are all equal (this includes Jordan blocks). The tiles we discuss play an important role in the theory of wavelets. We calculate the dimension for a number of examples.

1 Introduction

The object of this study is a class of self-affine (or self-similar) sets generated by pairs.

Definition 1.1 *A pair (M, R) is a linear isomorphism $M : \mathbb{R}^n \to \mathbb{R}^n$ with all eigenvalues outside the unit circle together with a finite subset R of $\mathbb{R}^n$.*

The space of closed and subsets of a fixed closed ball $\mathcal{B} \subset \mathbb{R}^n$ will be denoted by $H(\mathcal{B})$. Endow this space with the usual Hausdorff distance between two compact sets (the infimum of ϵ such that an ϵ-neighborhood of each one of the two sets contains the other). This distance induces a topology on $H(\mathcal{B})$ with respect to which $H(\mathcal{B})$ is a complete compact metric space. In $H(\mathcal{B})$, we define

$$\tau : H(\mathcal{B}) \to H(\mathcal{B})$$

by

$$\tau(A) \stackrel{\text{def}}{=} \cup_{r \in R} M^{-1}(A + r) \quad .$$

Such systems are affine examples of what are known as iterated function systems (see [**?**]). It is easy to prove that τ is a contraction (see [**?**]) and its unique fixed point is a compact set which we

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denote by $\Lambda(M, R)$. Hence we obtain that Λ is 'self-affine' (or 'self-similar'):

$$\Lambda = \cup_{r \in R} M^{-1}(\Lambda + r) \quad .$$

$$\text{or} \quad M\Lambda = \cup_{r \in R}(\Lambda + r) \quad . \tag{1.1}$$

Equivalently (see [**?**]), we may consider the set Λ of expansions on the base M using the set of digits R, or

$$\Lambda(M, R) = \{x \in \mathbb{R}^n | x = \sum_{i=1}^{\infty} M^{-i} r_i \text{ with } r_i \in R\} \quad .$$

Now we restrict our attention to a particularly interesting subclass of pairs, the class of standard pairs (in accordance with the nomenclature of [**?**]).

Definition 1.2 *A standard pair (M, R) is a pair such that the isomorphism M preserves $\mathbb{Z}^n$ (that is: its matrix has integer entries) and the set R is contained in $\mathbb{Z}^n$ and is a complete set of coset representatives of $\mathbb{Z}^n / M\mathbb{Z}^n$ (R contains one representative in $\mathbb{Z}^n$ of each the classes $\mathbb{Z}^n / M\mathbb{Z}^n$).*

By performing a translation we may assume that R contains the origin. Note that R contains $m = |\det M|$ elements. This subclass of pairs is a boundary case in the following sense. In the above equation (**??**), the measure on both sides can be positive, but only if the sets on the right hand side intersect in sets of (Lebesgue) measure zero.

For later reference, we include the following definition. Denote by $\mathbb{Z}[M, R]$ the smallest M-invariant sublattice of $\mathbb{Z}^n$ that contains all differences in R (smallest in the sense that it contains no subset satisfying the same requirements).

Definition 1.3 *A standard primitive pair is a standard pair with $\mathbb{Z}[M, R] = \mathbb{Z}^n$.*

Definition 1.4 *Let $N : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism preserving $\mathbb{Z}^n$ and $\pi_N : \mathbb{R}^n \to \mathbb{R}^n / N\mathbb{Z}^n$ the canonical projection. A compact set A of positive Lebesgue measure is called a tile by $N\mathbb{Z}^n$ if $\pi_N : A \to \mathbb{R}^n / N\mathbb{Z}^n$ is a bijection for Lebesgue almost every point of A.*

When the matrix N is not specified (as in most of this paper), we assume it to be the identity. In this case, we see that a tile is a compact set such that the union of its translates by $\mathbb{Z}^n$ covers $\mathbb{R}^n$, but any two translates by distinct elements of $\mathbb{Z}^n$ may intersect in sets of measure zero only.

The study of these tiles has important applications in various areas of mathematics. In fact, interest in these objects originally arose (see [**?**] and references therein), because they are intimately related to certain types of wavelets (Haar bases) (see [**?**] for a discussion of wavelets). Connections with number theory (see [**?**] and dynamical systems (see [**?**]) have been made by other authors.

The main result here was proved by Lagarias and Wang. To state it, we need the following definition first.

Definition 1.5 *A standard pair (M, R) is called exceptional if there exists an integer matrix $P \in GL(n, \mathbb{Z})$ such that*

- *PMP^{-1} is a 'block-triangular' matrix $\begin{pmatrix} A & B \\ \emptyset & C \end{pmatrix}$ and*

- $P(R)$ *is of so-called quasi-product form (the definition is rather complicated, see [?]).*

A regular standard pair is one that is not exceptional.

Here is the result of Lagarias and Wang [?].

Theorem 1.6 *If (M, R) is a regular standard pair then $\Lambda(M, R)$ is a tile by $\mathbb{Z}[M, R]$. If (M, R) is an exceptional standard pair, then $\Lambda(M, R)$ is also a tile, though possibly by some other lattice.*

Lagarias and Wang based their proof upon an earlier work by Gröchenig and Haas [?], who proved a one dimensional version of this result. The proof by Gröchenig and Haas was given a much more geometric flavor and was substantially simplified in [?].

The object of the current work is to study the Hausdorff dimension (for the definition, we refer to section 2 of this work) of the boundary $\delta\Lambda(M, R)$ of $\Lambda(M, R)$. In [?], it was proved that this boundary has Lebesgue measure zero. In the present work, we turn once again to the work by Gröchenig and Haas. The calculation of dimension usually involves a counting argument. The counting procedure used in lemma **??** is related to results of Gröchenig and Haas (see [?], lemma's 4.4 and 4.6).

To state our main result formally, we need some notation. First, define the (direct) sum of two sets A and B as follows:

$$Z = A + B \stackrel{\text{def}}{\equiv} \{z|z = a + b, a \in A, b \in B\} \quad .$$

Now define the difference set D:

$$D \stackrel{\text{def}}{\equiv} R - R \stackrel{\text{def}}{\equiv} \{d \in \mathbb{Z}^n | \exists r_1, r_2 \in R \text{ such that } d = r_1 - r_2\} \quad ,$$

and the multiplicity μ:

$$\mu(d) = \text{ card} \{r_1, r_2 \in R | r_1 - r_2 = d\} \quad .$$

(For instance, the element 0 occurs exactly $|\det M|$ times in D, namely $0 = r - r$ for all elements r in R.) Denote

$$S \stackrel{\text{def}}{\equiv} \{\Lambda - \Lambda\} \cap \mathbb{Z}^n \quad , \tag{1.2}$$

that is, the set of integer differences contained in Λ. Note that S and D are not equal! Let $\mathbb{R}^S$ be the space obtained by associating a fibre $\mathbb{R}$ to each point of S. Define a linear map $T : \mathbb{R}^S \to \mathbb{R}^S$, the transition operator, whose matrix elements are given by:

$$T_{ij} \stackrel{\text{def}}{\equiv} \text{ card} \{(Mj + D) \cap i\} \quad , \tag{1.3}$$

where i and j are in S. Note that this matrix is the transpose of the one defined in [?] as contact-matrix and in [?] as transition operator. The choice we make here is more natural in view of the calculations done in the next sections.

Let us consider this transition operator in some more detail. First of all, $\frac{1}{m}T$ is a non-negative stochastic matrix:

$$\sum_{j \in S} T_{ij} = \sum_{j \in S} \text{ card} \{(Mj + D) \cap i\} = \text{ card} \{(M\mathbb{Z}^n + D) \cap i\} = m \quad . \tag{1.4}$$

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The second equality follows from the definition of S: i being in S implies that if $j \notin S$, then $\operatorname{card}\{(Mj + D) \cap i\} = 0$. The last equality is implied by the fact that $M\mathbb{Z}^n + D$ covers all elements of $\mathbb{Z}^n$ exactly m times. Thus T has leading eigenvalue m with associated eigenspace $E_1 = \lambda(1, 1, \cdots 1)$. (In fact, if Λ is a tile, then all other eigenvalues have modulus smaller than m see for example [**?**]).

There is also a useful symmetry in the problem, namely, by its definition, D is symmetric, that is $D = -D$. The same holds for S. It follows immediately from equation (**??**) that $T_{-i-j} = T_{ij}$. Thus T preserves the subspace E_+ of symmetric vectors ($v_i = v_{-i}$) as well as the subspace of anti-symmetric vectors ($v_i = -v_{-i}$) in $\mathbb{R}^S$. (Vectors in $\mathbb{R}^S$ are written as components v_i where $i \in S$.)

We decrease the dimensionality of the system by 'quotienting out' this symmetry. Let $S^+ \subset S$ be such that of each pair $x \in S$ and $-x \in S$, precisely one is contained in S^+. Let v^+ denote the restriction of v to $\mathbb{R}^{S^+}$. It is now easy to calculate the affine map $T^+ : \mathbb{R}^{S^+} \to \mathbb{R}^{S^+}$ induced by T acting on E_+. Indeed,

$$
\begin{aligned}
(T^+ v^+)_i &= \Big(\sum_{j \in S} T_{ij} v_j\Big)^+ = \sum_{j \in S^+ - \{0\}} (T_{ij} v_j + T_{i,-j} v_{-j}) + T_{i,\{0\}} v_{\{0\}} \\
&= \sum_{j \in S^+ - \{0\}} (T_{ij} + T_{i,-j}) v_j + T_{i,\{0\}} v_{\{0\}}
\end{aligned}
$$

Notice that T^+ is a square matrix of dimension $\dfrac{|S| + 1}{2}$ and one of its eigenvalues is m. The reader having trouble with these definitions can see them illustrated in an easy case (example **??**).

The modulus of the eigenvalue of M that is closest to the unit circle will denoted by m_- (its reciprocal is the spectral radius of M^{-1}). Recall that $m = |\det M|$. Note that $1 \leq m_-^n \leq m$.

Definition 1.7 λ is a special eigenvalue of T if it is real, λ is contained in $[m_-^{n-1}, m)$, and λ is an eigenvalue of T^+.

Here is our main result.

Theorem 1.8 Let (M, R) be a regular standard pair.
1) The Hausdorff dimension of $\delta\Lambda$ satisfies:

$$
n + \frac{\ln \lambda - \ln m}{\ln m_-} \leq \operatorname{Hdim}(\delta\Lambda) \leq \frac{\ln \lambda}{\ln m_-} \quad,
$$

where λ is the leading special eigenvalue of T (which is the next-to-leading eigenvalue of T).
2) Let V be an open ball intersecting the boundary of Λ. The Hausdorff dimension of $\delta\Lambda \cap V$ is:

$$
n + \frac{\ln \lambda_p - \ln m}{\ln m_-} \leq \operatorname{Hdim}(\delta\Lambda \cap V) \leq \frac{\ln \lambda_p}{\ln m_-} \quad,
$$

where λ_p is a special eigenvalue of T.

Remark: In fact, the result also appears to hold for exceptional standard pairs, but we do not pursue this here.

Note that our result depends on the choice of V, that is: if T has more than one positive real eigenvalue in the appropiate range, the dimension may not be constant. We will give an example of this in section **??** (example **??**).

The most interesting case arises when the eigenvalues of M are equal in modulus. We then have $m_-^n = m$, and the above inequalities become equalities.

Corollary 1.9 *Let* (M, R) *be a regular standard pair and suppose that all eigenvalues of M are equal in modulus.*
1) The Hausdorff dimension of $\delta\Lambda$ satisfies:

$$\mathrm{Hdim}\,(\delta\Lambda) = \frac{\ln\lambda}{\ln m_-} \quad,$$

where λ is the leading special eigenvalue of T (which is the next-to-leading eigenvalue of T).
2) For an open ball V intersecting $\delta\Lambda$, we have

$$n - 1 \leq \mathrm{Hdim}\,(\delta\Lambda \cap V) = \frac{\ln\lambda_p}{\ln m_-} < n \quad,$$

where λ_p is a special eigenvalue of T.

In [**?**], Kenyon obtained an equality for the Hausdorff dimension similar to the one in the first part of the corollary. The difference is that in his case λ is the principal eigenvalue of the transition operator of a Markov partition (in the usual sense) for $\delta\Lambda$. Our matrix T is certainly not a transition matrix of a Markov partition, since it contains integers greater than one. Moreover, Kenyon's result does not include an algorithm to calculate the Markov transition operator from the initial data (M, R). On the other hand, our transition operator is easily calculated (see section **??**, where we calculate the dimension of $\delta\Lambda$ in various cases). In addition, Kenyon's result is much more restricted than ours for various reasons. First of all, he assumes that M is conformal, which in our corollary is not necessary (think of Jordan matrices). A more serious restriction is that he assumes that Λ is homeomorphic to a ball in $\mathbb{R}^n$. This is generally not the case. A non-trivial set in one dimension, for example, cannot be connected. Thus his result excludes the one-dimensional tiles. In higher dimension, Λ may have a complicated topology: in [**?**] an example of a connected set with infinitely many holes is given, or even one with infinitely many components, each of which has infinitely many holes (see also example **??**).

There is yet another approach to the calculation of the Hausdorff dimension for fractals generated by iterated function systems. This goes by means of a Mauldin-Williams graph (see [**?**]). This technique is also based on partitioning the boundary in a finite number of pieces and determining which ones are mapped where by the contractions of the iterated functions system. This sort of knowledge is not a priori present for the fractals under discussion here. In addition, the technique also requires the contractions to be conjugate to similarities.

Techniques by which one can calculate the Hausdorff dimension of a set that is invariant under a system of transformation that are not similarities are rare. We know of only the socalled Sierpinski Carpets (see [**?**], also explained in [**?**]). Here the transformations are *diagonal* matrices

with integer entries. There is also an expression for the Hausdorff dimension of more general sets, sometimes called Falconer's Formula. One can find this formula in [?]. This formula holds 'almost always', but its proof does not indicate what the exceptional cases are (but see [?]).

Let d_{λ_p} denote the size of the largest Jordan block associated with λ_p. Denote the dimension of $\delta\Lambda \cap V$ by

$$\beta = \frac{\ln \lambda_p}{\ln m_-} \tag{1.5}$$

Denote the size of the largest Jordan block associated with m_- by d_M. Suppose X is a set of Hausdorff dimension β. Denote by $\mathcal{H}^\beta(X)$ the Hausdorff outer measure of the set X (for the definition, see section 2).

Our calculations needed for the above result also give the following result.

Theorem 1.10 *Let (M, R) be a regular standard primitive pair and suppose all eigenvalues of M have equal modulus.*
1) If $d_M = d_{\lambda_p} = 1$, then:
$$\mathcal{H}^\beta(\delta\Lambda \cap V) < \infty \quad .$$

2) If $d_{\lambda_p} - 1 \geq (n - \beta)(d_M - 1)$, then

$$\mathcal{H}^\beta(\delta\Lambda \cap V) > 0 \quad .$$

3) If $d_{\lambda_p} - 1 > (n - \beta)(d_M - 1)$, then

$$\mathcal{H}^\beta(\delta\Lambda \cap V) = \infty \quad .$$

The set-up of this article is as follows. In section 2 we discuss elementary notions concerning the set $\delta\Lambda$. Essentially, we decribe how to construct it. Then in the next section, we describe the operator T and its properties. This serves to facilitate the counting argument already mentioned. These counting arguments will then be spelled out in the next two sections. The first of these, section 4, gives the upper bound for our dimension estimate, and in section 5 we derive the lower bound. In section 6, we prove the result concerning the regularity of the boundary (if M has eigenvalues of equal modulus). Finally, in section 7, we calculate the dimension of the boundary of several tiles.

In all subsequent sections, we will assume, without loss of generality, that (M, R) is a standard primitive pair. We give the reduction to that case here.

Lemma 1.11 *Let (M, R) be a standard pair. Then $\Lambda(M, R)$ is conjugate to a tile $\Lambda' = \Lambda(M', R')$, where (M', R') is a standard primitive pair.*

Proof: Let B be a matrix such that $B\mathbb{Z}^n = \mathbb{Z}[M, R]$. Clearly, M preserves $B\mathbb{Z}^n$. Thus $M' = B^{-1}MB$ has integer entries. Further, $R' = B^{-1}R \subset \mathbb{Z}^n$ is a complete set of coset representatives [?]. It is easy to see that $\Lambda' = B^{-1}\Lambda$. ∎

The lattice $B\mathbb{Z}^n$ is a generalization of the notion of greatest common divisor (see also [?]).

2 Construction of the Boundary

In this section, we describe the construction of the boundary. It also serves to collect most of the definitions and notation (in so far not already discussed in the introduction) in addition to some elementary results.

Throughout this work, we will denote Lebesgue measure by μ and the ϵ-neighborhood of a set A by $N_\epsilon(A)$. The diameter of a set U is denoted by $|U|$.

Of the many definitions of dimension [**?**], Hausdorff dimension has been one of the mathematically most fruitful ones (see [**?**]). We give the definition and refer to [**?**] and [**?**] and references therein for further reading.

For a set $F \subset \mathbb{R}^n$, define

$$\mathcal{H}_\delta^s(F) = \inf\{\sum |U_i|^s\} \quad ,$$

where the infimum is over all countable covers of F whose individual sets have diameters less than δ. The s-dimensional Hausdorff outer-measure is given by:

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F) \quad .$$

There is a unique number $\beta \geq 0$ such that

$$\text{if } s < \beta \quad \text{then} \quad \mathcal{H}^s(F) = \infty$$
$$\text{if } s > \beta \quad \text{then} \quad \mathcal{H}^s(F) = 0$$

.

This number is the Hausdorff dimension of F. Part of the importance derives from the fact that it is an invariant under bi-Lipschitz homeomorphisms. A set is called an s-set if its s-dimensional Hausdorff outer-measure is positive and finite. This property is also invariant under bi-Lipschitz homeomorphisms.

Lemma 2.1 *For all $x \in \Lambda$, we have that for all $\epsilon > 0$*

$$\mu(N_\epsilon(x) \cap \Lambda) > 0 \quad .$$

Proof: By definition of Λ, for any given ϵ, there is a k such that $x_k + M^{-k}\Lambda \subset N_\epsilon(x)$ for some $x_k \in \sum_{i=1}^k M^{-i}R$. ∎

Corollary 2.2 Λ *is the closure of its interior.*

Proof: Recall that $\delta\Lambda$ has measure zero and use the previous lemma. ∎

Now we define

$$\Lambda^{(2)} = \{x \in \Lambda | x + \{\mathbb{Z}^n - \{0\}\} \cap \Lambda \neq \emptyset\} \quad . \tag{2.1}$$

Recall that we assume that (M, R) is a regular standard primitive pair. Thus by theorem **??**, $\Lambda(M, R)$ is a tile by $\mathbb{Z}^n$.

Lemma 2.3 *We have*
$$\Lambda^{(2)} = \delta\Lambda \quad .$$

Proof: First, let $x \in \delta\Lambda$. Then, for all $\epsilon > 0$

$$\pi[\{N_\epsilon(x) + \mathbb{Z}^n\} \cap \Lambda] = \pi N_\epsilon(x)$$

$$\pi[N_\epsilon(x) \cap \Lambda] \neq \pi N_\epsilon(x) \quad .$$

Thus for all $\epsilon > 0$, $[N_\epsilon(x) + \{\mathbb{Z}^n - \{0\}\}] \cap \Lambda$ is non-empty.

Now let $x \in \Lambda^{(2)}$. There is $y \in x + \{\mathbb{Z}^n - \{0\}\}$ in Λ. Pick some small ϵ_0 such that

$$N_{\epsilon_0}(x) \cap N_{\epsilon_0}(y) = \emptyset$$

For all $\epsilon < \epsilon_0$ we then have

$$\pi[(N_\epsilon(x) \cap \Lambda) \cup (N_\epsilon(y) \cap \Lambda)]$$

covers $\pi[N_\epsilon(x)]$ at most once (modulo sets of measure zero). By the previous lemma, $N_\epsilon(x) \cap \Lambda$ and $N_\epsilon(y) \cap \Lambda$ have positive measure. Thus, for all $\epsilon > 0$, neither has full measure. ∎

We define:

$$\Gamma_k \stackrel{\text{def}}{=} \sum_{i=1}^{k} M^{-i} R = \tau^k(\{0\}) \quad . \tag{2.2}$$

Γ_k is the k-th generation approximation to Λ and as an element of $H(\mathcal{B})$ it is the k-th iterate of $\{0\}$. Notice that since $0 \in R$, $\Gamma_k \subseteq \Lambda$. Similarly, recall the definition of the set S (equation (**??**)) and define

$$\Delta_k \stackrel{\text{def}}{=} \{x \in \Gamma_k | \exists i \in S - \{0\}, \exists j \in S - \{0\} \text{ such that } x + i + M^{-k}j \in \Gamma_k\} \quad , \tag{2.3}$$

the k-th generation approximation to the boundary of Λ. E_k is the k-th generation 'filled-in' approximation to the boundary of Λ:

$$E_k \stackrel{\text{def}}{=} \Delta_k + M^{-k}\Lambda \quad . \tag{2.4}$$

Lemma 2.4 *We have*
$$\Delta_k \subseteq N_{3\epsilon_k}(\delta\Lambda)$$

where $\epsilon_k = |M^{-k}\Lambda|$

Proof: If $x \in \Delta_k$, then $y = x + i + M^{-k}j \in \Delta_k$. So, both x and y are in Λ. Then $N_{\epsilon_k}(y)$ contains an open set $A \subset \Lambda$ lemma **??**. The set $A - i \subset N_{3\epsilon_k}(x)$ is not contained in Λ. Since $x \in \Lambda$, $N_{3\epsilon_k}(x)$ must also contain points of the boundary. ∎

We now construct a sequence of maps $\tau_k : H(\mathcal{B}) \to H(\mathcal{B})$ whose fixed points F_k will also approach the boundary of Λ:

$$\tau_k(X) \stackrel{\text{def}}{=} M^{-k}X + \Delta_k \quad , \tag{2.5}$$

and

$$F_k \stackrel{\text{def}}{\equiv} \Lambda(M^k, M^k \Delta_k) \quad . \tag{2.6}$$

Note that $\tau_k F_k = F_k$.

Lemma 2.5 *We have*

$$\delta \Lambda \subseteq F_k \subseteq E_k \quad .$$

Proof: The second inclusion follows immediately from the definitions of E_k and F_k.

Suppose $x_1 \in \delta \Lambda$, then by lemma **??**, we may suppose that $x_1 \in \Lambda^{(2)}$. We will show that x_1 can be written as

$$x_1 = \sum M^{-ki} v_i, \quad \text{where} \ \ v_i \in M^k \Delta_k \quad .$$

By iterating equation (**??**), we see that there is a $v_1 \in M^k \Gamma_k$ with

$$x_1 \in M^{-k}(\Lambda + v_1) \quad .$$

Thus there is an $x_2 \in \Lambda$ such that

$$x_1 = M^{-k}(x_2 + v_1) \quad .$$

Since $x_1 \in \Lambda^{(2)}$, there are $\overline{x}_1 \in \Lambda$ and $\overline{v}_1 \in M^k \Gamma_k$ such that for some $\overline{x}_2 \in \Lambda$

$$\overline{x}_1 = M^{-k}(\overline{x}_2 + \overline{v}_1) \quad , \quad x_1 - \overline{x}_1 = j_1 \in S - \{0\} \quad .$$

Now observe that

$$v_1 - \overline{v}_1 = M^k(x_1 - \overline{x}_1) + \overline{x}_2 - x_2 \quad .$$

Since $v_1 - \overline{v}_1 \in \mathbb{Z}^n$, we also have

$$v_1 - \overline{v}_1 = M^k j_1 + j_2 \quad ,$$

Note that $v_1 - \overline{v}_1 \in M^k(\Gamma_k - \Gamma_k)$. This set contains no elements of $M^k(\mathbb{Z}^n - \{0\})$ (see [**?**]). Recall that $j_1 \in S - \{0\}$. Thus $j_2 \in S - \{0\}$. Thus $v_1 \in M^k \Delta_k$. Moreover, we see that

$$\overline{x}_2 - x_2 = j_2 \quad .$$

Thus x_2 belongs to $\delta \Lambda = \Lambda^{(2)}$. Continue by induction. ∎

The following proposition establishes that the sets we have defined so far converge to $\delta \Lambda$ in the Hausdorff topology.

Proposition 2.6 *One has*

$$\text{Hlim}_{k \to \infty} F_k = \ \text{Hlim}_{k \to \infty} \Delta_k = \ \text{Hlim}_{k \to \infty} E_k = \delta \Lambda \quad .$$

Proof: Combining lemmas **??** and **??**, we have

$$\Delta_k \subseteq N_{3\epsilon_k}(\delta \Lambda) \subseteq N_{3\epsilon_k}(F_k) \subseteq N_{3\epsilon_k}(E_k) \quad .$$

By definition of the Hausdorff distance, the Hausdorff distance between the two sets Δ_k and E_k is exactly $|M^{-k} \Lambda|$, which implies the proposition. ∎

3 The Transition Matrix

Here we derive how the transition operator counts the number of points in the k-th level approximation of the boundary. Again, we assume that Λ is a tile (for example, when (M, R) is a regular standard primitive pair, according to theorem **??**). This implies that the leading eigenvalue of T, discussed in equation (**??**) is simple (see [**?**] or [**?**]).

Consider the transition matrix $T : \mathbb{R}^S \to \mathbb{R}^S$ as defined in the introduction. For an open set V with diameter smaller than one and a non-negative integer k define the contact-matrix (we borrowed the name from [**?**]) as follows.

$$T(k, V)_{ij} \stackrel{\text{def}}{\equiv} \left\{ \begin{array}{ll} \text{card}\,\{x \in \Delta_k | x \in V \ \text{ or } \ x + i + M^{-k}j \in V\} & \text{if } \ i, j \in S - \{0\} \\ 0 & \text{else} \end{array} \right. \tag{3.1}$$

In this definition, we say that x is a basepoint for the difference $i + M^{-k}j$.

Note that we have approximately

$$T(k + \ell, V) \approx T^\ell T(k, V) \quad .$$

The fact that this is not exact, is due to boundary effects whose relative error decreases exponentially (this is worked out precisely in proposition **??**). By applying T to $T(k, V)$, one sees that

$$(T^\ell T(k, V))_{\{0\},j} = (T^\ell T(k, V))_{i,\{0\}} = 0 \quad .$$

Thus for all ℓ, the angle between the span of the columns of $T^\ell T(k, V)$ and $E_1 = \lambda(1, 1 \cdots, 1)$, the eigenspace associated with the leading eigenvalue m, is bounded from below. Since, as remarked just after equation (**??**), the leading eigenvalue is simple, the growth-rate of $T^\ell T(k, V)$ is less than m.

Since we will be doing a lot of counting, define the following counter for a non-negative integer matrix C:

$$\|C\| = \sum_{ij} c_{ij} \quad .$$

For a given ball B, we can now express the growth-rate of $\text{card}\,(\Delta_{k+\ell} \cap B)$ in terms of the growth-rate associated with the matrix T. This is done in the following two results.

Lemma 3.1 *We have*

$$\frac{\|T(k, B)\|}{2 \,\text{card}\, S} \leq \ \text{card}\,(\Delta_k \cap B) \leq \frac{\|T(k, B)\|}{2} \quad .$$

Proof: The first inequality follows from the fact $x \in \Delta_k \cap B$ can be the basepoint of at most $2\,\text{card}\, S$ differences.

The second inequality follows from the definition of Δ_k: each $x \in \Delta_k \cap B$ is the basepoint of some difference and its negative (by the definition of Δ_k). ∎

Proposition 3.2 *Let B_r be a ball of radius r intersecting $\delta\Lambda$. Fix a constant $\epsilon > 0$ and choose k such that $|M^{-k}\Lambda| < \epsilon r$. Then*

$$(T^\ell T(k, B_{r(1-\epsilon)}))_{i,j} \leq (T(k+\ell, B_r))_{i,j} \leq (T^\ell T(k, B_{r(1+\epsilon)}))_{i,j} \quad .$$

Proof: Suppose $\Delta_k \cap B$ contains $N = T(k, B)_{ba}$ basepoints of the difference $a + M^{-k}b$. Then $\Delta_k \cap B + \sum_{i=k+1}^{k+\ell} M^{-i}R$ contains

$$\sum_{b \in S} \text{card}\left(\{a + M^{-k}b + \sum_{i=k+1}^{k+\ell} M^{-i}R\} \cap \{a + M^{-k-\ell}c + \sum_{i=k+1}^{k+\ell} M^{-i}R\}\right) \cdot T(k, B)_{ba} =$$

$$= \sum_{b \in S} \text{card}\left(\{b + \sum_{i=1}^{\ell} M^{-i}D\} \cap \{M^{-\ell}c\}\right) \cdot T(k, B)_{ba} = \sum_{b \in S} (T^\ell)_{cb} T(k, B_r)_{ba}$$

basepoints of the difference $a + M^{-k-\ell}c$. There is a discrepancy due to the fact that $\Delta_k \cap B + \sum_{i=k+1}^{k+\ell} M^{-i}R \neq \Delta_{k+\ell} \cap B$. Points may 'seep' across the boundary of B. By assumption, points that do so, lie within a distance ϵr of the boundary. ∎

Lemma 3.3 *Let V be a ball.*
1) the growth-rate (in ℓ) of $\|T^\ell T(k, V)\|$ is determined by a special eigenvalue.
2) If V is sufficiently large, then this eigenvalue is the next-to-leading eigenvalue of T.

Proof: To prove the first statement, recall the definition of the T-invariant (symmetric and anti-symmetric) splitting of $\mathbb{R}^n$ in the introduction: $\mathbb{R}^n = E_+ \oplus E_-$, and the operators T_+ and T_- which are just the linear map T restricted to these respective spaces. Let $v = v^+ + v^-$ be the j-th column of $T(k, V)$. The j-th column of $T^\ell T(k, V)$ is

$$(T^+)^\ell v^+ + (T^-)^\ell v^- \quad .$$

By the previous proposition, one sees that the components of this vector are non-negative for all ℓ. Thus the growth-rate must be determined by an eigenvalue of T^+.

A similar argument shows that this eigenvalue is real positive. Let E_i denote the eigenspaces of T^+, ordered in such a way that the associated eigenvalues λ_i satisfy: $|\lambda_i| \geq |\lambda_{i+1}|$. Denote $T|_{E_i}$ by T_i. Now, consider the smallest integer j for which the span of the columns of $T^\ell T(k, V)$ intersects E_j in a linear subspace of positive dimension. The growth of some column $v = v_j + \sum_{\alpha > j} v_\alpha$ where $v_\alpha \in E_\alpha$, is dominated by $(T_j)^\ell v_j$. Suppose that the eigenvalues $\lambda_{\alpha > j}$ are less in modulus than λ_j. Then for ℓ big enough the entries of $(T_j)^\ell v_j$ have to be positive, since its contribution dominates the count of the number of differences.

Finally, the bounds on the eigenvalue follow from theorem **??** (and will not be used in the proof of that theorem), together with the observation that the boundary of an n-dimensional volume has dimension at least $n - 1$.

To prove the second statement, consider the following splitting:

$$\mathbb{R}^n = E_1 \oplus E_\perp \quad .$$

If V sufficiently big, it will contain Λ. In this case, $T(k, V)$ counts all differences in Γ_k except the ones equal to zero. Thus for any vector v, we have that

$$T(k,V)v = T(k,V)(v_1 \oplus v_\perp) = T^k v_\perp \quad .$$

■

Finally, we state a lemma that we will often use

Lemma 3.4 *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ a linear map and E_λ the invariant space associated with an eigenvalue of modulus λ.*
1) There is a polynomial p such that for all $x \in E_\lambda$

$$C_1 \lambda^k |x| \leq |A^k x| \leq C_2 \lambda^k p(k)|x| \quad .$$

The degree of p is one less than the size of the biggest Jordan block associated with A.
2) For a given $x \in E_\lambda$, we have that there is a polynomial p of degree less than the size of the biggest Jordan block associated with A such that

$$C_1 \lambda^k p(k) \leq |A^k x| \leq C_2 \lambda^k p(k) \quad .$$

Proof: Bring A into Jordan form. ■

4   The Upper Bound for the Dimension

In this section, we calculate the upper bound of the dimension of $\delta\Lambda \cap V$ where V is an arbitrary open ball intersecting the boundary of a tile Λ. We do this by showing there is a sequence of sets E_k with $\delta\Lambda \cap V \subseteq E_k$. The upper bound we calculate equals $\lim_{k\to\infty} \text{Hdim} \, E_k$.

The main tool we use to give an upper estimate for the dimension of a set is one that follows almost directly from proposition 9.6 in [**?**]. For completeness we include the proof.

Notice that by lemma **??**, there is a polynomial p of degree d_{m_-} such that $|M^{-k}\Lambda| \leq Cp(k)m_-^{-k}$.

Proposition 4.1 *Let (A, Q) be a pair as described in the introduction (not necessarily standard) and denote the spectral radius of A^{-1} by a_-^{-1}. Then*

$$\text{Hdim} \, \Lambda(A, Q) \leq \frac{\ln \, \text{card} \, (Q)}{\ln a_-} \quad .$$

Proof: Cover $\Lambda(A, Q)$ by hypercubes whose sides have length $\epsilon_k = |A^{-k}\Lambda| \leq Cp(k)a_-^{-k}$. To do so, we need at most $(\, \text{card} \, Q)^k$ hypercubes. Let $\beta = \dfrac{\ln \, \text{card} \, (Q)}{\ln a_-}$ and d any positive number. Then the $\beta + d$ dimensional Hausdorff outer measure of Λ satisfies:

$$\begin{aligned} \mathcal{H}_{\epsilon_k}^{\beta+d}(\Lambda) &\leq Cp(k)^{\beta+d}a_-^{-k(\beta+d)}(\, \text{card} \, Q)^k \\ &= Cp(k)^{\beta+d}a_-^{-kd} \quad , \end{aligned}$$

which tends to zero as k tends to infinity (and ϵ_k to zero). Thus for all positive d, the $\beta + d$-dimensional Hausdorff outer measure of Λ is zero. The proposition follows immediately from the definition of the Hausdorff dimension. ∎

Corollary 4.2 *Let (M, R) be a regular standard primitive pair and suppose that Q is a subset of $\cup_{i=0}^{k-1}M^i R$. Then*

$$\text{Hdim} \, \Lambda(M^k, Q) \leq \frac{\ln \, \text{card} \, (Q)}{k \ln m_-} \quad .$$

Remark: This estimate becomes exact if M is conformal and the pair (M, R) satisfies the open set condition (see [**?**]). As remarked in the introduction, $\Lambda(M, R)$ is a tile whose boundary has measure zero. Thus its interior has positive measure. The open set condition now holds for the pair (M^k, Q) with the interior of Λ as the open set.

Notice that by lemma **??** there is a polynomial q of degree $d_{\lambda_p} - 1$ such that

$$\|T^\ell T(k, V)\| \leq Cq(\ell)\lambda_p^\ell \quad .$$

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Lemma 4.3 *There is a polynomial q such that*

$$C_1 q(k) \lambda_p^k < \ \mathrm{card}\,(\Delta_k \cap V) \le C_2 q(k) \lambda_p^k \quad,$$

where λ_p is a special eigenvalue (next-to-leading if V big enough).

Proof: Let V_+ denote the ϵr-neighborhood of V and V_- the ball whose ϵr-neighborhood is V. Then by results **??**, **??** and **??**

$$\|T^\ell \cdot T(k, V_-)\| \le \ \mathrm{card}\,(\Delta_{k+\ell} \cap V) \le \|T^\ell \cdot T(k, V_+)\| \quad.$$

The result follows from applying lemmas **??** and **??** to this formula. ∎

Theorem 4.4 $\mathrm{Hdim}\,(\delta\Lambda \cap V) \le \dfrac{\ln \lambda_p}{\ln m_-} \quad.$

Proof: We have by lemma **??**

$$\delta\Lambda \cap V \subseteq F_k \cap V \quad.$$

Thus,

$$\mathrm{Hdim}\,(\delta\Lambda \cap V) \le \liminf_{k \to \infty}\ \mathrm{Hdim}\,(F_k \cap V) \quad.$$

Now apply corollary **??** and lemma **??** to the definition of F_k (equation (**??**)) to see that

$$\liminf_{k \to \infty}\ \mathrm{Hdim}\, F_k \le \liminf_{k \to \infty} \frac{\mathrm{card}\,(\Delta_k \cap V)}{k \ln m_-} = \frac{\ln \lambda_p}{\ln m_-} \quad.$$

∎

5 The Lower Bound for the Dimension

In this section, we calculate the lower bound for the Hausdorff dimension of $\delta\Lambda \cap V$, where V is an open ball intersecting the boundary.

The technique we use consists putting a probability measure ν on $\delta\Lambda \cap V$ where V is an open ball. If for r small enough the measure contained in a ball of radius r does not exceed Cr^s, then s is a lower bound for the Hausdorff dimension [?]. The following result is a minor extension of this.

Proposition 5.1 *Let ν be a probability measure on a set X. If for all $d > 0$ there exists a $C_d > 0$ such that for all $x \in X$*

$$\lim_{r \to 0} \frac{\nu(B_r(x))}{r^{\beta - d}} \leq C_d \quad .$$

Then

$$\mathcal{H}^{\beta - d}(X) \geq \frac{\nu(X)}{C_d} \quad .$$

And thus $\mathrm{Hdim}(X) \geq \beta$.

Proof: Let $\{U_i\}$ be a cover of X and suppose that $x_i \in U_i \cap X$. Then

$$\nu(U_i) \leq \nu(B_{|U_i|}(x_i)) < (C_d + \epsilon)|U_i|^{\beta - d} \quad ,$$

provided $|U_i| < \delta$ and δ small enough. By summing, one obtains:

$$\sum_i |U_i|^{\beta - d} > \frac{\nu(X)}{C_d + \epsilon} \quad .$$

The definition of Hausdorff dimension implies that for all $d > 0$

$$\mathrm{Hdim}(X) \geq \beta - d \quad .$$

$\blacksquare$

We now define a probability measure ν_k on $\Delta_k \cap V$. For any open set $U \subset V$:

$$\nu_k(U) = \frac{\mathrm{card}\,(\Delta_k \cap U)}{\mathrm{card}\,(\Delta_k \cap V)} \quad . \tag{5.1}$$

Presumably, the measures ν_k converge exponentially fast to a limiting measure ν. However, the local structure of Δ_k is difficult to control. Instead, we simply use the Banach-Alaoglu theorem (see [?]).

Proposition 5.2 *There is a subsequence $\{\nu_{k_i}\}$ that converges to a probability measure ν on $\delta\Lambda \cap V$.*

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Proof: Let $X = \cup_k (\Delta_k \cap V)$. Then X is compact and $\nu_k(X) = 1$. By the Banach-Alaoglu theorem, there must be a subsequence of the ν_k converging to a measure ν. By using proposition **??**, we see that ν has support in $\delta\Lambda \cap V$. ∎

We will now use proposition **??** to devise a method to count the growth rate with k of the number of elements of Δ_k contained in a ball B. This will enable us to calculate estimates for the measures ν_k and thus to determine the ν-measure contained in a ball of radius r. Without loss of generality, we may also take V to be a ball.

Proposition 5.3 *Let ν be the probability measure just constructed. Fix a small constant $\epsilon > 0$. Then there is a $K > 0$ such that if k satisfies*

$$|M^{-k}\Lambda| < \epsilon r \quad ,$$

then for a ball of radius r intersecting $\delta\Lambda \cap V$

$$\nu(B_r) < Kr^n \frac{m^k}{\lambda_p^k} \frac{1}{q(k)} \quad ,$$

where λ_p is a special eigenvalue (next-to-leading if V big enough).

Proof: Let V_- be the ball whose ϵr-neighborhood equals V, denote the ϵr-neighborhood of $B_r(x)$ by B_+, and assume that $B_+ \subset V_-$. Combining the hypotheses and the results **??**, **??**, and **??**, we see

$$\mathrm{card}\,(\Delta_{k+\ell} \cap V) \geq \frac{\|T^\ell \cdot T(k, V_-)\|}{\mathrm{card}\,S} \quad .$$

Using the opposite inequalities, we arrive at

$$\mathrm{card}\,(\Delta_{k+\ell} \cap B_r) \leq \|T^\ell \cdot T(k, B_+)\| \quad .$$

Now, since $(\Delta_{k+\ell} \cap B_+) \subset (\Delta_{k+\ell} \cap V_-)$, the growth rate (in ℓ) of $T^\ell \cdot T(k, B_+)$ is dominated by the growth rate of $T^\ell \cdot T(k, V_-)$. Thus, recalling the definition of the measure ν in proposition **??**, and lemma **??**:

$$\nu(B_r) \leq \lim_i \frac{\mathrm{card}\,(\Delta_{k+\ell_i} \cap B_r)}{\mathrm{card}\,(\Delta_{k+\ell_i} \cap V)} \leq \mathrm{card}\,S \frac{\|T(k, B_+)\|}{\|T(k, V_-)\|}$$

By lemma **??**, we know that

$$\|T(k, V_-)\| \geq C\lambda_p^k q(k) \quad .$$

Furthermore,

$$\|T(k, B_+)\| \leq \frac{\mathrm{vol}(B_+)}{\mathrm{vol}(M^{-k}\Lambda)} = \frac{Cr^n}{m^{-k}} \quad .$$

Putting the estimates together yields the result. ∎

To simplify notation, put

$$\beta = n + \frac{\ln \lambda_p - \ln m}{\ln m_-} \quad . \tag{5.2}$$

Theorem 5.4 Hdim $(\delta\Lambda \cap V) \geq \beta$.

Proof: In accordance with the previous proposition, we can choose k such that

$$|M^{-k}\Lambda| < \epsilon r \leq |M^{-(k-1)}\Lambda| \quad . \tag{5.3}$$

By bringing M^{-1} in Jordan normal form applying lemma **??**, we conclude that there are a constant C and a polynomial p of degree $d_M - 1$ such that $C_1 p(k) m_-^{-k} < |M^{-(k-1)}\Lambda| \leq C_2 p(k) m_-^{-k}$. By using this and equation **??**, we calculate the dependency of k on r:

$$C_1 \epsilon^{-1} p(k) m_-^{-k} < r \leq C_2 \epsilon^{-1} p(k) m_-^{-k} \quad . \tag{5.4}$$

By proposition **??**, we will be done if for all positive d small enough

$$\lim_{r \to 0} \frac{\nu(B_r(x))}{r^{\beta-d}} \leq C_d \quad .$$

We calculate, using the previous proposition and equation (**??**):

$$\frac{\nu(B_r(x))}{r^{\beta-d}} < K r^n m^k \lambda_p^{-k} \frac{1}{q(k)} r^{-n} r^{-\frac{\ln \lambda_p}{\ln m_-}} r^{\frac{\ln m}{\ln m_-}} r^d =$$

$$K m^k \lambda_p^{-k} m^{\frac{\ln r}{\ln m_-}} \lambda_p^{\frac{-\ln r}{\ln m_-}} r^d \frac{1}{q(k)} \quad .$$

Now using equation (**??**), one easily checks that

$$m^{k+\frac{\ln r}{\ln m_-}} \leq m^{k-k+\frac{\ln p - \ln \epsilon + \ln C_2}{\ln m_-}} \quad = \quad C_3 p^{\frac{\ln m}{\ln m_-}} \quad .$$

$$\lambda_p^{k+\frac{\ln r}{\ln m_-}} \geq \lambda_p^{k-k+\frac{\ln p - \ln \epsilon + \ln C_1}{\ln m_-}} \quad = \quad C_4 p^{\frac{\ln \lambda_p}{\ln m_-}} \quad .$$

So

$$\frac{\nu(B_r(x))}{r^{\beta-d}} \leq K_2 p^{\frac{\ln m - \ln \lambda_p}{\ln m_-}} q^{-1} r^d \quad .$$

Thus the r^d-term dominates. $\blacksquare$

6   The Hausdorff Outer Measure

In this section we give some results about the β-dimensional Hausdorff outer measure of $\delta\Lambda \cap V$. These results are essentially corollaries of the calculations done before. We will deal only with the case where the eigenvalues of M have equal modulus. Thus

$$m_-^n = m \tag{6.1}$$

Proof of theorem ??: From the proof of proposition **??** and lemma **??**, we have that

$$
\begin{aligned}
\mathcal{H}_{\epsilon_k}^\beta(\delta\Lambda \cap V) &\leq Cp^\beta m_-^{-k\beta} q\lambda_p^k \\
&= Cp^\beta q \quad .
\end{aligned}
$$

Thus

$$\mathcal{H}^\beta(\delta\Lambda \cap V) \leq C \quad .$$

For the second and third statements, we see that the proof of theorem **??** together with equalities (**??**) and (**??**) imply that

$$\mathcal{H}^\beta(\delta\Lambda \cap V) \geq \frac{\nu(\delta\Lambda \cap V)}{p^{n-\beta}q^{-1}} \quad ,$$

and recall that p has degree $d_M - 1$ and q has degree $d_{\lambda_p} - 1$. ■

7 Some Examples

We illustrate the ideas in this work by calculating the dimension of the boundary of a self-affine tile in a number of examples.

We begin with an trivial example that can be understood without calculation.

Example 7.1 *The interval $[0, 1]$ is the invariant set for $(M, R) = (2, \{0, 1\})$. It is easy to check that $D=\{-1,0,1\}$ with 0 appearing with multiplicity 2. The set S is given by $\{-1, 0, 1\}$ and $S^+ = \{0, 1\}$. Thus the transition matrix and the reduced transition matrix are given by*

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad ; \quad T^+ = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad .$$

*We follow the convention that the ordering of coordinates on which T^+ (T) acts is the same as the ordering in S^+ (S). So the upper right entry of the T^+ multiplies $v_{\{0\},\{1\}}$, and so forth. Since the only special eigenvalue (definition **??**) equals 1, the dimension of the boundary equals $\dfrac{\ln 1}{\ln 2} = 0$.*

Now let us look at some non-trivial examples.

Example 7.2 *Let $(M, R) = (3, \{0, 4, 11\})$. Then $\mathrm{Hdim}\,\delta\Lambda \cap V \approx \dfrac{\ln 2.84 \cdots}{\ln 3} \approx 0.87 \cdots$.*

Proof: Check that
$$S^+ = \{0, 1, 2, 3, 4, 5\} \quad .$$

With the same convention as before (noting that D consists of the numbers -11, -7, -4, 0, 4, 7, and 11, and 0 appearing with multiplicity 3),

$$T^+ = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad .$$

Using Maple, we find that the only special eigenvalue is approximately 2.84. $\blacksquare$

We now exhibit examples in one dimension whose boundaries have dimension approximating (but not equal to) 0.

Example 7.3 *For $m \geq 4$, let $(M, R) = (M, \{0, 2, 3, \cdots m-1, m+1\})$. Then $\mathrm{Hdim}\,(\delta\Lambda \cap V) = \dfrac{\ln 3}{\ln m}$.*

Proof: We easily see that $S^+ = \{0, 1\}$. For the set D the following holds

$$
\begin{array}{cccc}
0 & \in D \text{ with} & \text{multiplicity} & m \\[4pt]
1 & & " & m - 3 \\[4pt]
m - 1 & & " & 1 \\[4pt]
m + 1 & & " & 1
\end{array}
$$

Thus

$$
T^+ = \begin{pmatrix} m & 0 \\ m - 3 & 3 \end{pmatrix} \quad .
$$

$\blacksquare$

Here is a family of examples such that their boundaries have dimension converging to (without being equal to) 1.

Example 7.4 *Let $m \geq 4$ be even. Let*

$$
(M, R) = (m, \{0, 2, 4, \cdots (m - 4), (m - 2), (m + 1), (m + 3), \cdots (2m - 1)\}) \quad .
$$

Then

$$
\text{Hdim} \, (\delta \Lambda \cap V) = \frac{\ln[(m - 1) + ((m - 1)^2 + 8)^{1/2}] - \ln 2}{\ln m} \quad .
$$

Proof: Again, it is easy to see that $S^+ = \{0, 1, 2\}$. Concerning D, we only need the following information:

$$
\begin{array}{cccc}
0 & \in D \text{ with} & \text{multiplicity} & m \\[4pt]
2 & & " & m - 2 \\[4pt]
m - 2 & & " & 2 \\[4pt]
m - 1 & & " & m/2 - 1 \\[4pt]
m + 1 & & " & m/2 \\[4pt]
m + 2 & & " & 0
\end{array}
$$

Thus

$$
\mathbb{T}^+ = \begin{pmatrix} m & 0 & 0 \\ 0 & m - 1 & 1 \\ m - 2 & 2 & 0 \end{pmatrix} \quad .
$$

The only special eigenvalue is $\dfrac{(m - 1) + ((m - 1)^2 + 8)^{1/2}}{2}$.

$\blacksquare$

We have not been able to find any one-dimensional examples in which the leading special eigenvalue is associated with a Jordan block.

In studying the dimension of boundaries of tiles in two (or more) dimensions, three new aspects arise. In the first place, not all expanding map are (conjugate to) similarities. So we may obtain estimates rather than equalities. That this is inevitable is clear from the following example slightly modified from [?].

Example 7.5 *Let $N > 2$ and*

$$(M, R) = \left(\begin{pmatrix} 2 & 0 \\ 0 & N \end{pmatrix}, \{(0,0), (-2\lambda, N-1)\} \right) \quad .$$

We calculate the dimension of Λ (not its boundary). When $\lambda = 0$, this gives $\text{Hdim}(\Lambda) = \dfrac{\ln 2}{\ln N}$. When $\lambda \neq 0$, we have $\text{Hdim}(\Lambda) = 1$.

Proof: See example 9.10 in [?]. ∎

A second problem is that, although the algorithm that determines the set S^+ terminates after a finite number (but not a priori bounded) number of steps, this calculation isn't nearly as straightforward as in the one-dimensional case. In fact, to check that a given set is indeed S^+ is an elementary but very longwinded calculation, which we leave to reader in the examples below. In the following examples, all eigenvalues of M have equal modulus.

Example 7.6 *The case where $m = 2$ in two dimensions. As explained in [?], there are (modulo affine coordinate transformations) only six cases. The following three are representative.*

$$i) \quad M = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}, \quad ii) \quad M = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}, \quad iii) \quad M = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix} \quad ,$$

and in all three cases $R = \{(0,0), (1,0)\}$.

We have that in the three respective cases, the Hausdorff dimension assumes the values $1, \dfrac{ln 1.5 \cdots}{\ln 2}$, and $\dfrac{ln 1.7 \cdots}{\ln 2}$.

Proof: In the three cases $D = \{(0,0), (-1,0), (1,0)\}$, the point $(0,0$ having multiplicity 2. Note that (M, R) generates the same set Λ as $(M^2, MR + R)$. In the first case, we obtain the system

$$\left(\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \{((0,0) \cup (1,0)) + ((0,0) \cup (0,1))\} \right) \quad .$$

It is easy to see that the associated set Λ is, in fact, the unit square (by explicit substitution, for example).

In the second case:
$$S^+ = \{(0,0), (1,0), (0,1), (1,-1)\} \quad ,$$

and

$$T^+ = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad .$$

The characteristic polynomial is $-\lambda^3 + \lambda + 2$, whose only zero is $\lambda \approx 1.5 \cdots$.

In the third case:

$$S^+ = \{(0,0), (1,0), (1,-1), (1,-2)\} \quad ,$$

and

$$T^+ = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \quad .$$

The characteristic polynomial is $-\lambda^3 + \lambda^2 + 2$, whose only zero is $\lambda \approx 1.7 \cdots$. ∎

Finally, we conclude by calculating the dimension of the boundaries of the two tiles depicted in figures 2 and 3 of [?]. Here, the aspect arises that M may also have Jordan blocks.

Example 7.7 *Let*

$$(M, R) = \left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \{(0,0), (1,0), (0,1), (1,1)\} \right) \quad .$$

Then $\mathrm{Hdim}\,(\delta\Lambda \cap V) = 1$.

Proof: One checks that

$$S^+ = \{(0,0), (1,0), (0,1), (1,-1)\} \quad ,$$

and

$$T^+ = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix} \quad .$$

∎

Remark: Notice that both M and T^+ have a Jordan block of size 2. So it is unclear whether $\delta\Lambda$ is an s-set.

Example 7.8 *Let*

$$\left(\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \{(-1,-1), (0,-1), (1,-1), (-2,0), (0,0), (2,0), (-1,1), (0,1), (1,1)\} \right) \quad .$$

Then $\mathrm{Hdim}\,(\delta\Lambda \cap V) = 1$ *or* $\mathrm{Hdim}\,(\delta\Lambda \cap V) = \dfrac{\ln 5}{\ln 3}$.

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Proof: We have
$$S^+ = \{(0,0), (1,0), (2,0), (0,1), (1,1), (1,-1)\} \quad,$$
and
$$T^+ = \begin{pmatrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 & 0 & 0 \\ 4 & 4 & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 3 & 0 & 0 \\ 4 & 2 & 0 & 2 & 1 & 0 \\ 4 & 2 & 0 & 2 & 0 & 1 \end{pmatrix} \quad.$$
There are now two special eigenvalues, namely 3 and 5. ∎

Remark: This last result is in fact easy to verify by inspection of the figure.

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