

THE PERIODIC POINTS OF RENORMALIZATION

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Abstract.

It will be shown that the renormalization operator, acting on the space of smooth unimodal maps with critical exponent $\alpha > 1$, has periodic points of any combinatorial type.

1. Introduction

A central question in the theory of dynamical systems is whether small scale geometrical properties of dynamical systems are universal or not, whether they are imposed by the combinatorial properties of the systems. The empirical discovery of universality of geometry in dynamical systems was made by Couillet-Tresser and Feigenbaum. They studied infinitely renormalizable period doubling unimodal maps and observed that the geometry of the invariant Cantor set of such maps converges when looking at smaller and smaller scales. Furthermore they observed that the limiting geometry was universal, in the sense that the small scale geometry of these Cantor sets depends only on the local behavior of the map around the critical point. This local behavior is specified by the critical exponent. To explain the universality of geometry they defined the period doubling renormalization operator on a suitable space of unimodal maps. This operator acts like a microscope: the image under the renormalization operator is a unimodal map describing the geometry and dynamics on a smaller scale. The universality of geometry could be understood by conjecturing that the renormalization operator has a hyperbolic fixed point. In particular, the infinitely renormalizable unimodal maps form the stable manifold of the fixed point of

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the renormalization operator. The first step in proving these Conjectures is showing the existence of a renormalization fixed point.

In 1981 Lanford showed the existence of a period doubling renormalization fixed point with critical exponent $\alpha = 2$. This fixed point was obtained by rigorous numerical analysis of the renormalization operator. In 1986,88 H.Epstein obtained fixed points for the period doubling renormalization operator with critical exponents $\alpha > 1$.

In 1992 D.Sullivan gave his conceptual explanation for universality. This beautiful combination of real and complex analysis opened new directions in the Theory of dynamical systems. It explains the existence of periodic points of the renormalization operator in the class of unimodal maps with an even critical exponent. Moreover it showed the convergence of renormalization. Left was to show that these periodic points are hyperbolic. The recent work of Lyubich shows that there is a 1-dimensional unstable manifold. Furthermore McMullen showed that the infinitely renormalizable maps form the codimension 1 stable manifolds.

The Theory of Sullivan considers holomorphic extensions of the system on the interval. These extensions only exist if the critical exponent is even. However the experiments indicate also universality for any other critical exponent. What has been lacking to the present is a proof of the universality Conjectures which does not leave the interval. The specific knowledge of the critical exponent should be irrelevant for the understanding of universality. The reason for wishing more general universality theorems is that universality is observed in many different fields but not explained. For example universality in statistical mechanics, is observed but not explained. Maybe an explanation of universality in interval dynamics which uses less structure can spread some light on the other universality phenomena.

In this work an approach to interval dynamics is developed based on real methods. The main result is the Existence Theorem of Periodic Points for the renormalization operator.

Theorem. *The renormalization operator, acting on the space of smooth unimodal maps with critical exponent $\alpha > 1$, has periodic points of any combinatorial type.*

The Theorem is formulated precisely in section 2. This introduction will be used to outline the method. It will discuss the reason for the definitions given in the sections 3,4,...,8. The technical Lemmas are proved in section 9 and the appendix.

Outline of the Argument

Let us discuss in some detail the period doubling renormalization operator and outline the construction of a fixed point. There is no essential difference between the construction of this fixed point and the construction of a general periodic orbit.

A *unimodal map* is an endomorphism of the interval $[-1, 1]$. It is of the form $f = \phi \circ q_t$ where $\phi \in \text{Diff}_+^2([-1, 1])$ is an orientation preserving C^2 diffeomorphism of $[-1, 1]$ and $q_t : [-1, 1] \rightarrow [-1, 1]$, $t \in [0, 1]$ defined by

$$q_t(x) = -2t|x|^\alpha + 2t - 1.$$

The exponent $\alpha > 1$ is called the *critical exponent* of f . We will fix it once and for all throughout the text. The map q_t is called the *canonical folding map* with *peak-value* $t \in [0, 1]$. The peak-value determines the maximum $q_t(0) = 2t - 1$. The above form for the canonical folding map is not just a choice for convenience. The canonical folding maps are naturally presented to us in section 4. They have a property intrinsically related to universality. The diffeomorphism ϕ is called the *diffeomorphic part* of f . Observe that $f(1) = f(-1) = -1$. The collection of unimodal maps with the chosen critical exponent $\alpha > 1$ is denoted by \mathcal{U} .

Let \mathcal{U}^+ be the collection of unimodal maps whose peak-value is high enough such that the unimodal map has a fixed point $p \in (0, 1)$. For every $f \in \mathcal{U}^+$ we can consider the first return map to the interval $[-p, p]$. If the peak-value is not too high this first return map will be just $f^2|_{[-p, p]}$, the unimodal map f is called *renormalizable*. The unimodal map obtained by rescaling this first return-map to $[-p, p]$ is called the *renormalization* of f . The period doubling renormalization operator is illustrated in Figure 1.

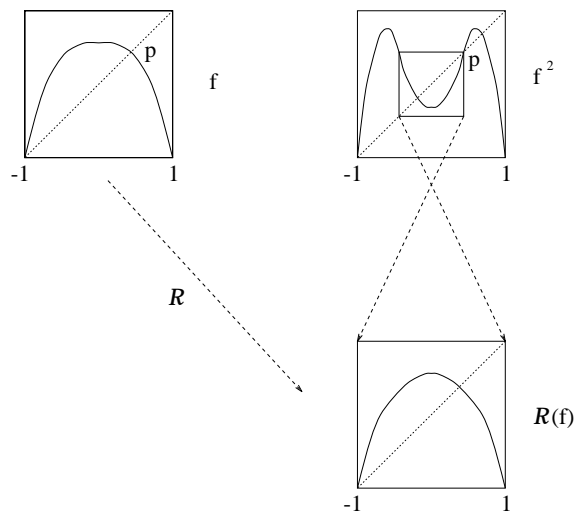


Figure 1. The Renormalization Operator

The Theorem states that the renormalization operator has a fixed point. The naive reason for the existence is the following. The space of unimodal maps can be represented as $\mathcal{U} = \text{Diff}_+^2([-1, 1]) \times [0, 1]$. The renormalization operator is only defined on the subset of renormalizable unimodal maps. For every diffeomorphism ϕ there will be a range of peak-values t such that $f = \phi \circ q_t$ is renormalizable. It seems that the subset of renormalizable maps forms a strip, see Figure 2. This is naive but let us assume this situation. Maps close to the lower boundary will have renormalizations with peak-value close to 0. Moving towards the upper boundary we will see renormalizations whose peak-value tend to 1. The global action of the renormalization operator is illustrated in Figure 2.

If we furthermore allow us to think about $\text{Diff}_+^2([-1, 1])$ as being a compact Euclidean ball then the renormalization operator can be modeled by a map

$$R : D \times [0, 1] \rightarrow D \times (-\infty, \infty)$$

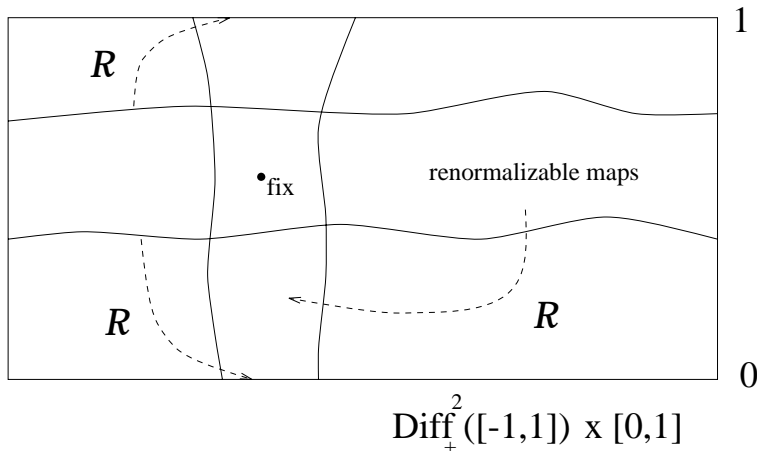


Figure 2. Bottom Moves Down, Top Moves Up

where D is an Euclidean ball and $R(D \times \{0\}) \subset D \times (-\infty, 0)$ and $R(D \times \{1\}) \subset D \times (1, \infty)$: bottom goes down and top goes up. A slight variation on Brouwer's Fixed Point Theorem gives a fixed point for any map of the above form.

There are two difficulties. The renormalizable maps do, maybe, not form a strip and secondly the space of diffeomorphisms is far from an Euclidean ball. There will be a very simple way around the first problem. The way to deal with the second problem is more elaborate and we will concentrate on this. The idea is to construct a (thin) subspace in the space of diffeomorphism which is parametrized by the Hilbert-cube, a countable product of closed intervals. The Hilbert-cube is close enough to an Euclidean ball to apply the above idea. This Hilbert-cube in the space of diffeomorphism is constructed in a natural way: it gives rise to an invariant set of the renormalization operator. Moreover it attracts exponentially every orbit of the renormalization operator, the fixed point of renormalization has to be in this subspace. Moreover, the universality phenomena are caused by the very special nature of these diffeomorphisms in this Hilbert-cube.

The construction of this attracting set of unimodal maps is based on a generalization of the renormalization operation. The usual renormalization operator will, from now on, be called the *classical* renormalization operator. A careful study of the classical renormalization operator will lead us to this generalization.

Fix $f = \phi \circ q_t \in \mathcal{U}^+$ with fixed point $p \in (0, 1)$. We will need the following objects to construct the renormalization of f : the *central interval* $S_2 = [p', p]$ where $p' = -p$ and the *side interval* $S_1 = [p, b]$ where $b \in (p, 1)$ is such that $f(b) = p'$. Furthermore let $S_2(1.) = \phi^{-1}(S_2)$ and $S_1(1.) = \phi^{-1}(S_1)$ be the corresponding pull-backs. The notation looks too complicated. However, it corresponds to the notation used in the general case.

Usually, unimodal maps are represented by their graph. The *dynamical picture* in Figure 3 is more convenient for the renormalization discussion. It contains all the objects needed to construct the renormalization.

The unimodal map f is renormalizable if $q_t(0) \in S_1(1.)$. To describe precisely the classical renormalization operator we need to define the *Zoom*-operators. Let $T \subset \mathbb{R}$ be a closed

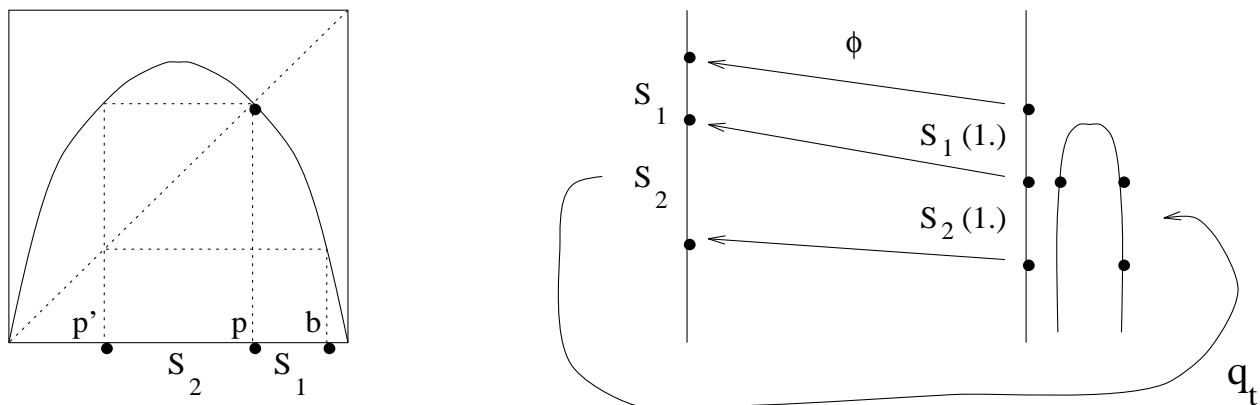


Figure 3. A Renormalizable Unimodal Map

interval and $\psi : T \rightarrow \mathbb{R}$ be a diffeomorphism onto its image. Then $[\psi|_T] \in \text{Diff}_+^2([-1, 1])$ is the orientation preserving diffeomorphism of $[-1, 1]$ obtained by rescaling the map $\psi : T \rightarrow \psi(T)$. If $T \subset (-1, 1)$ then the operator $Z_T : \text{Diff}_+^2([-1, 1]) \rightarrow \text{Diff}_+^2([-1, 1])$ defined by

$$Z_T : \phi \rightarrow [\phi|_T]$$

is called the *zoom-operator* to T .

The classical renormalization of f , the rescaled first return map to S_2 , is given by

$$\begin{aligned} \mathcal{R}_{\text{class}}(f) &= ([\phi|_{S_2(1.)} \circ q_t|_{S_1} \circ \phi|_{S_1(1.)}]) \circ q_\rho \\ &= ([\phi|_{S_2(1.)}] \circ [q_t|_{S_1}] \circ [\phi|_{S_1(1.)}]) \circ q_\rho \end{aligned}$$

where $q_\rho : [-1, 1] \rightarrow [-1, 1]$ is the canonical folding part of the renormalization, obtained by rescaling domain and range of $q_t : S_2 \rightarrow S_1(1.)$. Indeed this rescaled map is again a canonical folding map. The peak-value $\rho \in [0, 1]$ of the renormalization is, in general, not the original peak-value t . Using the zoom-operators we get the following expression for the classical renormalization operator.

$$\mathcal{R}_{\text{class}}(f) = Z_{S_2(1.)}(\phi) \circ Z_{S_1}(q_t) \circ Z_{S_1(1.)}(\phi) \circ q_\rho.$$

The classical renormalization of a unimodal map involves three operations:

- 1) Finding the dynamical intervals $S_1(1.)$, $S_2(1.)$, S_2 and S_1 ,
- 2) Zooming in onto these dynamical intervals to obtain the diffeomorphisms $Z_{S_1(1.)}(\phi)$, $Z_{S_1}(q_t)$, $Z_{S_2(1.)}(\phi)$ and the canonical folding map q_ρ ,
- 3) Compositions of these diffeomorphisms and the canonical folding map q_ρ .

The first aspect of renormalization deals with the mystery of universality. Universality says that these intervals have very special positions. In the construction of the fixed point

the Brouwer Theorem will take care of this mystery, it will allow us to avoid a careful discussion. We will concentrate on zooming and composition.

The zoom operators are expected to behave well. After all, if $T \subset (-1, 1)$ is a very small interval then $Z_T(\phi)$ is very close to the identity map of $[-1, 1]$. In section 2. we will define a vector space structure and the *non-linearity*-norm on $\text{Diff}_+^2([-1, 1])$ which make it into the Banach space \mathcal{D} . The zoom-operators on \mathcal{D} become linear contractions. The second aspect of renormalization, the zooming part, is under control.

The composition operator is known to behave badly. Here we propose a simple way around this problem: do not compose! Consider a unimodal map which can be renormalized repeatedly. After taking a few renormalizations we will obtain a unimodal map whose diffeomorphic part is a long composition of interval diffeomorphisms, restrictions of the original unimodal map. The actual composition of these diffeomorphisms will eliminate the information contained in the factorization. We do not want to lose this information. This leads us to the idea of a *decomposition*. Instead of considering the diffeomorphic part of a renormalization as an interval diffeomorphism we will consider it to be a non-composed chain of diffeomorphisms, called a decomposition.

These decompositions are not arbitrary chains of diffeomorphisms. There is essentially one way to label the diffeomorphisms in these chains. Let us discuss this labeling in some detail.

A finite ordered set $(T^{(n)}, \succ)$ is called a set of *decomposition times* if it has the following properties

- 1) $T^{(n)} = \bigcup_{i=0}^n L_i$ a pairwise disjoint union of levels L_i ,
- 2) $L_0 = \{1.\}$,
- 3) there exist order preserving bijections

$$A_1 : T_1 \rightarrow \bigcup_{i=0}^{n-1} L_i \text{ and } A_2 : T_2 \rightarrow \bigcup_{i=0}^{n-1} L_i,$$

where $T_1 = \{\tau \in T^{(n)} | \tau \prec 1.\}$ and $T_2 = \{\tau \in T^{(n)} | \tau \succ 1.\}$.

Clearly such sets exists and can be modeled by the vertices of a finite binary tree, see Figure 4. A decomposition $\underline{\phi}$ is a chain of diffeomorphisms labeled by a set of decomposition times $T^{(n)}$, with $n \geq 0$,

$$\underline{\phi} : T^{(n)} \rightarrow \mathcal{D}.$$

To understand the reason for this labeling we will analyze how this labeling behaves under renormalization. We can compose the diffeomorphisms in the decomposition according to the order of times. Doing so we obtain a diffeomorphism $O(\underline{\phi}) \in \mathcal{D}$. Now choose a peak-value $t \in [0, 1]$ such that the unimodal map $f = O(\underline{\phi}) \circ q_t$ is renormalizable, see Figure 4. We will use the notation $f = (\underline{\phi}, t)$ to indicate that the diffeomorphic part of f comes from a decomposition.

The fixed point of f is $p \in (0, 1)$, the central interval $S_2 = [-p, p]$ and the side interval $S_1 = [p, b]$ are defined as before. We have to pull back these intervals through the decomposition

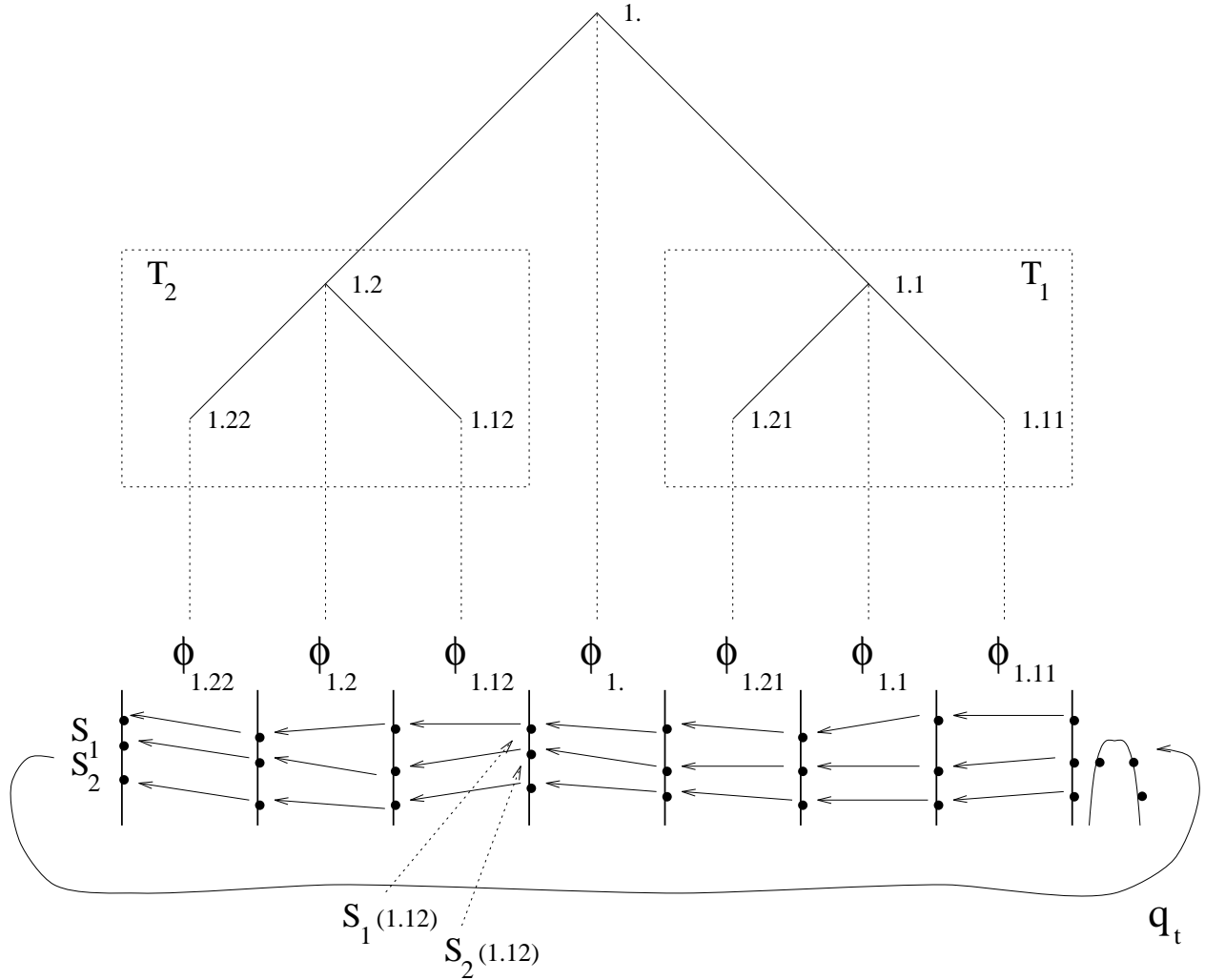


Figure 4. The Decomposed Unimodal Map f

to construct the renormalization of f . These preimages of S_2 and S_1 are the dynamical intervals

$$S_1(\tau) = (O^\tau(\underline{\phi}))^{-1}(S_1) \text{ and } S_2(\tau) = (O^\tau(\underline{\phi}))^{-1}(S_2)$$

where $\tau \in T^{(n)}$ and $O^\tau(\underline{\phi}) \in \mathcal{D}$ the diffeomorphism obtained by composition of the diffeomorphisms ϕ_w with $w \succ \tau$, see Figure 4.

The next step in the renormalization process, is to zoom in onto these dynamical intervals. Let $\underline{S}_1 = (S_1(\tau))_{\tau \in T^{(n)}}$ and $\underline{S}_2 = (S_2(\tau))_{\tau \in T^{(n)}}$. This defines the decompositions $Z_{\underline{S}_1}(\underline{\phi}), Z_{\underline{S}_2}(\underline{\phi}) : T^{(n)} \rightarrow \mathcal{D}$ with

$$Z_{\underline{S}_1}(\underline{\phi})(\tau) = Z_{S_1(\tau)}(\phi_\tau) \text{ and } Z_{\underline{S}_2}(\underline{\phi})(\tau) = Z_{S_2(\tau)}(\phi_\tau)$$

for $\tau \in T^{(n)}$. Then form the decomposition

$$(Z_{\underline{S}_2}(\underline{\phi}), Z_{S_1}(q_t), Z_{\underline{S}_1}(\underline{\phi})) : T^{(n+1)} \rightarrow \mathcal{D},$$

as illustrated in Figure 5. The renormalization of $f = (\underline{\phi}, t)$ becomes

$$\mathcal{R}(f) = ((Z_{\underline{s}_2}(\underline{\phi}), Z_{s_1}(q_t), Z_{\underline{s}_1}(\underline{\phi})), \rho)$$

where ρ is, as before, the peak-value of the renormalization. Compare this with the expression for the classical renormalization. The renormalization operation in terms of decompositions is illustrated in Figure 4. and 5.

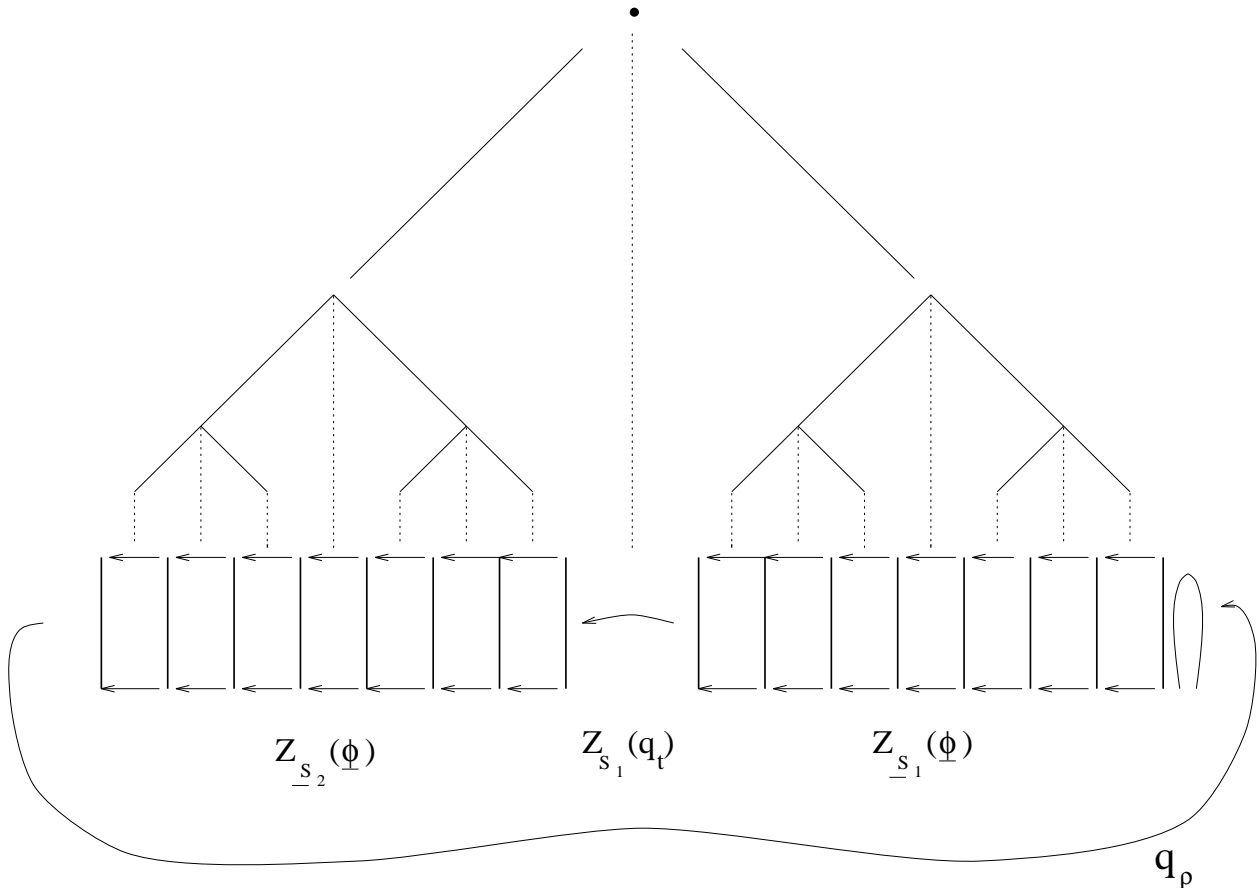


Figure 5. The Renormalization of the Decomposed Map f

Observe that the diffeomorphic part of the renormalization $\mathcal{R}(f)$ is a decomposition labeled by a tree with one level more. Each renormalization step will give rise to decompositions labeled by a tree with one more level. We would like to have the renormalization operator to act on a space. The obvious definition of a decomposition will be a chain of diffeomorphisms labeled by the infinite binary tree T or more precisely by an infinite set of decomposition times.

The Banach space of decompositions X , the corresponding space of *decomposed unimodal maps* $U = X \times [0, 1]$ and the action of the corresponding renormalization operator will be

discussed in the sections 4,5 and 6. This renormalization operator is called the *dynamical* renormalization operator \mathcal{R}_{dyn} . The notion of decompositions eliminated the difficulty of the third aspect of renormalization, composition.

Instead of discussing the classical renormalization operator $\mathcal{R}_{\text{class}}$ we will discuss the dynamical renormalization operator \mathcal{R}_{dyn} . This dynamical renormalization operator is a lift of the classical renormalization operator to the space of decomposed unimodal maps. In particular, the construction of a fixed point for the dynamical renormalization operator will give a fixed point for the classical renormalization operator. The action of the dynamical renormalization operator was informally discussed above: zoom in to dynamically defined intervals and move the obtained diffeomorphisms to the right place in the tree. The zoom-operators are contractions. If the dynamically defined intervals were always at the same place then the renormalization operator would essentially be a zoom-operator. In particular renormalization would be a contraction and the fixed point would be found immediately.

This discussion leads to the definition of *geometrical renormalization operators*. Eliminate the difficulty that the intervals \underline{S}_2 , S_1 and \underline{S}_1 have dynamical definitions. Make an arbitrary choice for these intervals. Such a choice $g = (\underline{S}_2, S_1, \underline{S}_1)$ is called a *geometry*. The space of all a priori possible geometries is denoted by G . These geometries are abstract objects. They have no dynamical meaning. For every geometry $g \in G$ define the geometrical renormalization operator

$$\mathcal{R}_g : X \rightarrow X.$$

These operators act on the space of decompositions like the dynamical renormalization operator. The difference is that they do not zoom in onto dynamically defined intervals but they just zoom in onto the chosen intervals of the geometry g . They are essentially zoom-operators. Putting all the definitions together we see that these geometrical renormalization operators are affine contractions on the Banach space of decompositions. Therefore each geometrical renormalization operator has a unique fixed point. The decompositions which are fixed point of some geometrical renormalization operator are called *pure decompositions*. For each geometry there is a unique pure decomposition. These pure decompositions form a thin subspace in the space X of decompositions, parametrized by the Hilbert-cube G of geometries.

The geometrical renormalization operators are generalizations of the dynamical renormalization. Now we will explain the relation between geometrical and dynamical renormalization and show that fixed points of the dynamical renormalization operator have to be in the set of unimodal maps formed by these pure decompositions.

A decomposed unimodal map is a pair $f = (\underline{\phi}, t)$, where $\underline{\phi} \in X$ is a decomposition and $t \in [0, 1]$ a peak-value. In section 4 it will be shown that decompositions can be composed to an actual diffeomorphism, $O(\underline{\phi})$. This diffeomorphism can be composed with the canonical folding map q_t and we obtain a classical unimodal map. It could be called the *observed* unimodal map. If the peak-value is high enough we can define, as before, the

central interval S_2 and the side interval S_1 depending on the decomposed unimodal map $f = (\underline{\phi}, t)$. The collection $U^+ \subset X \times [0, 1]$ consists of those decomposed unimodal maps for which the central and side interval is defined.

For each unimodal maps in $f = (\underline{\phi}, t) \in U^+$ we can pull back the central and side interval through the decomposition to obtain the dynamical intervals

$$\underline{S}_1(\underline{\phi}, t) \text{ and } \underline{S}_2(\underline{\phi}, t).$$

This defines the *dynamical* geometry of $f = (\underline{\phi}, t)$:

$$d : U^+ \rightarrow G$$

with

$$d(\underline{\phi}, t) = (\underline{S}_2(\underline{\phi}, t), S_1(\underline{\phi}, t), \underline{S}_1(\underline{\phi}, t)).$$

The dynamical renormalization operator $\mathcal{R}_{\text{dyn}} : U^+ \rightarrow X \times \mathbb{R}$ becomes

$$\mathcal{R}_{\text{dyn}}(\underline{\phi}, t) = (\mathcal{R}_{d(\underline{\phi}, t)}(\underline{\phi}), \rho(\underline{\phi}, t)),$$

where, as before,

$$\rho : U^+ \rightarrow \mathbb{R}$$

is the peak-value of the renormalization and $\mathcal{R}_{d(\underline{\phi}, t)}$ the geometrical renormalization operator using the dynamical geometry $d(\underline{\phi}, t)$.

This indicates the candidates for a renormalization fixed point. Namely if $(\underline{\phi}, t) \in U^+$ is a renormalization fixed point then its diffeomorphic part $\underline{\phi}$ is the unique fixed point of the geometrical renormalization operator corresponding to the dynamical geometry $d(\underline{\phi}, t)$. Decompositions which are the fixed point of some geometrical renormalization operator are called *pure* decompositions. Let P be the space of pure decompositions. Observe that P is homeomorphic to G .

Our search for a renormalization fixed point can be limited to the subspace

$$U_P^+ \subset U^+$$

of unimodal maps whose diffeomorphic part is a pure decomposition. This space U_P^+ is homeomorphic to $P \times [0, 1]$. A renormalization fixed point corresponds to a fixed point of

$$\mathcal{R}_{\text{dyn}} : U_P^+ \approx P \times [0, 1] \rightarrow P \times \mathbb{R}.$$

It is the above map $\mathcal{R}_{\text{dyn}}|_{U_P^+}$, to which we are going to apply the Brouwer Theorem mentioned in the naive discussion in the beginning: any map from $P \times [0, 1]$ into $P \times \mathbb{R}$ whose bottom goes down and top goes up has a fixed point. Unfortunately the space P of pure decompositions is homeomorphic to the space of geometries G , which is not exactly a Hilbert cube, it is a countable product of open intervals.

The last step in the construction is to find a priori bounds on the geometry: If we know that

$$d(U_P^+) \subset G$$

is compact we could apply Brouwer to this compact piece and the fixed point would be constructed. It turned out to be difficult to find such an invariant piece of P . It is not known whether there is such a piece. We have to proceed differently. We will work in two steps.

First, construct finite dimensional approximations of this map $\mathcal{R}_{\text{dyn}}|_{U_P^+}$. These finite dimensional approximations, better truncations, allow a priori bounds. The Brouwer Theorem gives for every truncation a fixed point, called truncation fixed points.

These truncation fixed points are almost fixed under the renormalization operator. This extra information enables us to show uniform bounds on the geometries of the truncation fixed points. These uniform bounds allow us to take a limit, the renormalization fixed point is found.

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2. The Classical Renormalization Operator

In this section we will introduce combinatorial notions needed to describe the classical renormalization operator and formulate the Existence Theorem for Periodic Points of Renormalization. The combinatorial statements made are folklore, their proof will be omitted. The critical exponent $\alpha > 1$ will be fixed throughout the text.

A unimodal map $f \in \mathcal{U}$ is called *renormalizable* iff there exists an expanding periodic point $p \in (-1, 1)$ such that the first return map to the *central* interval $C = [-p, p]$ is a of the form $f^q : C \rightarrow C$ with $f^q(p) = p$ and $q \geq 2$. The first return map to C will be, up to rescaling, a unimodal map. This unimodal map is a *renormalization* of f . Observe that a renormalization is completely determined by the periodic point p .

The combinatorial aspects of a renormalization are described by unimodal permutations. A permutation on a finite ordered set is a *unimodal* permutation if the following holds. Embed the set monotonically into the real line. Draw the graph of the permutation. If this graph can be extended to the graph of a unimodal map then the permutation is called unimodal.

Definition 2.1. A collection $\mathcal{S} = \{S_1, S_2, \dots, S_{q-1}, S_q\}$ of oriented closed intervals in $[-1, 1]$ is called a *cycle* for the unimodal map f if it has following properties

- 1) there is an expanding periodic point $p \in (-1, 1)$ with $S_q = [-|p|, |p|]$,
- 2) $f : S_i \rightarrow S_{i+1}$, $i = 1, 2, \dots, q-1$, is monotone onto,
- 3) $f(S_q) \subset S_1$ with $f(p) \in \partial S_1$, the boundary of S_1 ,
- 4) the interiors of S_1, S_2, \dots, S_q are pairwise disjoint, \mathcal{S} inherits an order from $[-1, 1]$,

5) the map

$$\sigma(\mathcal{S}) : S_i \rightarrow S_{i+1 \pmod q}$$

on \mathcal{S} is a unimodal permutation,

6) the orientation

$$o_{\mathcal{S}} : \mathcal{S} \rightarrow \{-1, 1\}$$

is such that $o_{\mathcal{S}}(S_i) = 1$ when $f^i(p)$ is the left boundary point of S_i and $o_{\mathcal{S}}(S_i) = -1$ otherwise.

Observe that

- 1) a unimodal map is renormalizable iff it has a cycle,
- 2) properties 4), 5) and 6) follow automatically once a unimodal map has a periodic point with the first three properties.
- 3) the orientation $o_{\mathcal{S}}$ depends only on σ . We will use the notation o_{σ} .

Let σ be a unimodal permutation and

$$\mathcal{U}_{\sigma} = \{f \in \mathcal{U} \mid f \text{ has a cycle } \mathcal{S} \text{ with } \sigma(\mathcal{S}) = \sigma\}.$$

The unimodal maps in \mathcal{U}_{σ} are sometimes called σ -renormalizable to emphasize the type of renormalization under consideration. The renormalization operator

$$\mathcal{R}_{\text{class},\sigma} : \mathcal{U}_{\sigma} \rightarrow \mathcal{U}$$

is defined to be the rescaled first return map to the smallest central interval giving rise to a cycle \mathcal{S} with $\sigma(\mathcal{S}) = \sigma$.

These sets of renormalizable maps \mathcal{U}_{σ} are not empty, every family $t \rightarrow \phi \circ q_t \in \mathcal{U}$ contains points in each \mathcal{U}_{σ} . Often a unimodal has different cycles, the sets \mathcal{U}_{σ} are not disjoint. However they are nested. For each σ there exists a unique maximal factorization $\sigma = \langle \sigma_n, \dots, \sigma_2, \sigma_1 \rangle$ such that

$$\mathcal{R}_{\text{class},\sigma} = \mathcal{R}_{\text{class},\sigma_n} \circ \dots \circ \mathcal{R}_{\text{class},\sigma_2} \circ \mathcal{R}_{\text{class},\sigma_1}.$$

A unimodal permutation σ is called *prime* iff $\sigma = \langle \sigma \rangle$. Clearly each permutation in the maximal factorization is prime. Using the prime unimodal permutations we obtain a partition of the set of renormalizable unimodal maps and the *classical renormalization* operator becomes

$$\mathcal{R}_{\text{class}} : \{\text{renormalizable maps}\} = \bigcup_{\text{prime } \sigma} \mathcal{U}_{\sigma} \rightarrow \mathcal{U},$$

with $\mathcal{R}_{\text{class}}|_{\mathcal{U}_{\sigma}} = \mathcal{R}_{\text{class},\sigma}$.

Theorem. For each choice $\sigma_n, \dots, \sigma_2, \sigma_1$ of prime unimodal permutations there exists $f \in \mathcal{U}$ with

$$\mathcal{R}_{\text{class}}^n(f) = f$$

and

$$\mathcal{R}_{\text{class}}^i(f) \in \mathcal{U}_{\sigma_{i+1 \bmod n}}$$

for $i \geq 0$.

The Existence Theorem for Periodic Points is a direct consequence of

Theorem 2.2. For every unimodal permutation σ there exists $f \in \mathcal{U}_\sigma$ with

$$\mathcal{R}_{\text{class}, \sigma}(f) = f.$$

This Theorem 2.2 will be proved in the next sections. Fix the critical exponent $\alpha > 1$ and the unimodal permutation σ .

3. Zoom Operators

Let $\mathcal{D} = \text{Diff}_+^2([-1, 1])$ be the C^2 orientation preserving diffeomorphisms of the interval $[-1, 1]$. Consider the *non-linearity* $N : \mathcal{D} \rightarrow C^0([-1, 1])$ with

$$N(\phi)(x) = \frac{D^2\phi(x)}{D\phi(x)} = D \ln D\phi(x).$$

This map is a bijection with inverse

$$N^{-1}(\eta)(x) = 2 \frac{\int_{-1}^x e^{\int_{-1}^s \eta ds}}{\int_{-1}^1 e^{\int_{-1}^s \eta ds}} - 1.$$

We will identify \mathcal{D} with $C^0([-1, 1])$ and use the supremum norm of $C^0([-1, 1])$. In this context we will speak about the *non-linearity norm* on \mathcal{D} . Observe that these linear and metric structures on \mathcal{D} are not the usual structures on $\text{Diff}_+^2([-1, 1])$. An appendix is added in which some properties of this norm are discussed. The Sandwich Lemma 10.5 is the most important property. Usually we will denote $N(\phi)$ by η_ϕ .

Let $I \subset [-1, 1]$ be an oriented closed interval. Let $o(I) = \pm 1$, according to whether the orientation of I and the natural orientation of $[-1, 1]$ matches or not. Furthermore define

$$i_I : [-1, 1] \rightarrow I$$

to be the affine orientation preserving map onto I . Now we can define the zoom operator

$$Z_I : \mathcal{D} \rightarrow \mathcal{D}$$

by

$$Z_I(\phi) = (i_{\phi(I)})^{-1} \circ \phi \circ i_I,$$

where $\phi(I)$ and I are oriented in the same direction, $o(\phi(I)) = o(I)$.

Lemma 3.1. *The zoom operator Z_I is a linear contraction. In particular*

$$|Z_I(\phi) - Z_I(\psi)| \leq \frac{|I|}{2} |\phi - \psi|.$$

Proof. If $\psi = \phi_2 \circ \phi_1$ then we have the following chain rule for non-linearities:

$$\eta_\psi = (\eta_{\phi_2} \circ \phi_1) \times D\phi_1 + \eta_{\phi_1}.$$

So

$$\eta_{Z_I(\phi)} = (\eta_\phi \circ i_I) \times Di_I = o(I) \frac{|I|}{2} \eta_\phi(i_I).$$

And the statement follows. \square (Lemma 3.1)

In the next section we will use the Sandwich Lemma 10.5. It deals with C^3 diffeomorphisms. We will need

$$\mathcal{D}_C = \{\phi \in \mathcal{D} | \eta_\phi \in C^1([-1, 1]) \text{ and } \forall x \in [-1, 1] |\eta'_{\phi_\tau}(x)| \leq C |\eta_{\phi_\tau}(x)|\}$$

where $C > 0$ is a big constant which will be defined in section 7, the proof of Theorem 2.2.

4. The Space of Decompositions

The objects we are going to define will depend on the unimodal permutation σ which is fixed throughout the text. We will suppress subscripts σ . Let $q = |\sigma|$.

The set T of *decomposition times* is a countable set with the properties

- 1) T carries an order \prec .
- 2) $\{1, 2, \dots, q-1\} \subset T$ and is naturally ordered by \prec .
- 3) the intervals

$$T_q = \{\tau \in T | \tau \succ q-1\}$$

$$T_i = \{\tau \in T | i \succ \tau \succ i-1\}, i = 2, 3, \dots, q-1$$

$$T_1 = \{\tau \in T | 1 \succ \tau\}$$

admit order preserving bijections $A_i : T_i \rightarrow T$, $i = 1, 2, \dots, q$.

The set of decomposition times exists and is unique up to isomorphism. In the period doubling case ($q = 2$) the vertices of the binary tree can be used to model T . Observe that for each q we get a different set of decomposition times.

For each time $\tau \in T$ there is a unique number $n(\tau) \geq 0$, called the *depth* of τ and a unique sequence $i_j \in \{1, 2, \dots, q\}$ with $j = 1, 2, \dots, n(\tau)$ such that

$$A_{i_{n(\tau)}} \circ \dots \circ A_{i_2} \circ A_{i_1}(\tau) \in \{1, 2, \dots, q-1\}.$$

The times in T are organized in pairwise disjoint levels,

$$T = \bigcup_{n \geq 0} L_n,$$

where $L_n = \{\tau \in T \mid n(\tau) = n\}$. The set L_n is called the level of depth n .

The *space of decompositions* is

$$X = \{(\phi_\tau)_{\tau \in T} \mid \phi_\tau \in \mathcal{D} \text{ and } \sum_{\tau \in T} |\phi_\tau| < \infty\}.$$

The elements in X are called *decompositions*. This set inherits the vector space structure of \mathcal{D} . The norm will be

$$|\underline{\phi} - \underline{\psi}| = \sum_{\tau \in T} |\phi_\tau - \psi_\tau|.$$

This norm makes $(X, |\cdot|)$ into a Banach space.

The second half of this section will define a natural notion of composition of decompositions.

On X there are projections $\pi_k, \pi^\tau : X \rightarrow X$, with $k \geq 0$ and $\tau \in T$, defined by

$$\pi_k(\underline{\phi})_\tau = \phi_\tau \text{ if } \tau \in \cup_{j \leq k} L_j$$

$$\pi_k(\underline{\phi})_\tau = id \text{ if } \tau \notin \cup_{j \leq k} L_j$$

and

$$\pi^\tau(\underline{\phi})_t = \phi_t \text{ if } t \succ \tau$$

$$\pi^\tau(\underline{\phi})_t = id \text{ if } t \prec \tau.$$

Let $X_k = \pi_k(X)$. On these subsets there are natural composition maps

$$O_n, O_n^\tau : X_n \rightarrow \mathcal{D},$$

where both maps compose the diffeomorphisms of a decomposition according to the order of the decomposition times. The map O_n composes all maps of a given decomposition $\underline{\phi}$ and O_n^τ composes all the maps ϕ_t with $t \succ \tau$. Observe

$$O_n^\tau = O_n \circ \pi^\tau.$$

We will not be able to extend these composition maps to the whole X . To extend we need some condition on the derivative of the non-linearities. Choose $C > 0$ sufficiently big. In section 7, in the proof of Theorem 2.2, we will make the right choice for $C > 0$. Let

$$X_C = \{\underline{\phi} \in X \mid \forall \tau \in T \ \eta_{\phi_\tau} \in \mathcal{D}_C\}.$$

Proposition 4.1. *There exist continuous composition maps*

$$O, O^\tau : X_C \rightarrow \mathcal{D},$$

such that $O|_{X_C \cap X_n} = O_n$ and $O^\tau|_{X_C \cap X_n} = O_n^\tau$. In particular restricted to any bounded set in X_C , all composition maps O, O^τ are Lipschitz with the same constant.

Proof. The proof relies heavily on the Sandwich Lemma 10.5, given in the appendix. Let us first show that the pointwise limit of $O_n \circ \pi_n$ is defined. Let $\underline{\phi} \in X_C$. We are going to apply the Sandwich Lemma with $b = |\underline{\phi}|$ and C the defining constant for X_C . Observe that for $k > n$, $O_k(\pi_k(\underline{\phi}))$ is obtained from $O_n(\pi_n(\underline{\phi}))$ by applying the Sandwich Lemma to all ϕ_τ with $\tau \in \bigcup_{j=n+1}^k L_j$: the sequence $O_n(\pi_n(\underline{\phi}))$ is a Cauchy sequence. It converges to $O(\underline{\phi})$. This defines

$$O : X_C \rightarrow \mathcal{D}.$$

Left is to prove that the function O is Lipschitz on bounded sets. This will be again a Sandwich argument. Fix a bounded set $B \subset X_C$. Take $\underline{\phi}, \underline{\psi} \in B$. The chain-rule for non-linearities gives for every $\tau \in T$

$$\begin{aligned} |\psi_\tau \circ (\phi_\tau)^{-1}| &\leq \sup_x |\eta_{\psi_\tau}(\phi_\tau^{-1}(x)) - \eta_{\phi_\tau}(\phi_\tau^{-1}(x))| \cdot (\phi_\tau^{-1})'(x) \\ &\leq K \cdot |\psi_\tau - \phi_\tau|, \end{aligned}$$

where the second estimate was obtained by applying Lemma 10.3. We are going to apply the Sandwich Lemma 10.5 with $b = K \cdot \text{diam}(B)$ and C the defining constant for X_C . Consider again $\underline{\phi}, \underline{\psi} \in B$. Because $O(\pi_k(\underline{\phi})) = O_k(\pi_k(\underline{\phi}))$ we can find for each $\epsilon > 0$ some $k \geq 0$ such that

$$|O(\underline{\phi}) - O(\pi_k(\underline{\phi}))| \leq \epsilon$$

and

$$|O(\underline{\psi}) - O(\pi_k(\underline{\psi}))| \leq \epsilon.$$

Then

$$|O(\underline{\psi}) - O(\underline{\phi})| \leq |O(\underline{\psi}) - O(\pi_k(\underline{\psi}))| + |O(\pi_k(\underline{\psi})) - O(\pi_k(\underline{\phi}))| + |O(\pi_k(\underline{\phi})) - O(\underline{\phi})|.$$

But $O(\pi_k(\underline{\psi}))$ is obtained from $O(\pi_k(\underline{\phi}))$ by Sandwiching the maps $\psi_\tau \circ (\phi_\tau)^{-1}$. We get

$$\begin{aligned} |O(\underline{\psi}) - O(\underline{\phi})| &\leq \epsilon + \text{const} \cdot \sum_\tau |\psi_\tau \circ (\phi_\tau)^{-1}| + \epsilon \\ &\leq 2\epsilon + \text{const} \cdot \sum_\tau |\psi_\tau - \phi_\tau| \\ &\leq 2\epsilon + \text{const} \cdot |\underline{\psi} - \underline{\phi}|. \end{aligned}$$

Because ϵ was taken arbitrarily we get the Lipschitz estimate for O . Let

$$O^\tau = O \circ \pi^\tau.$$

Because the projection do not increase distance we get the same Lipschitz constant for all O^τ , $\tau \in T$. □ (Proposition 4.1)

5. The Space of Geometries

The orientation of a cycle with $\sigma(\mathcal{S}) = \sigma$ is denoted by o_σ .

Definition 5.1. *Let $\epsilon \geq 0$. A collection $\mathcal{S} = \{S_1, S_2, \dots, S_q\}$ of oriented closed intervals in $[-1, 1]$ is called an ϵ -elementary geometry if it has the following properties.*

- 1) *the interiors of the intervals are pairwise disjoint, \mathcal{S} inherits an order from $[-1, 1]$,*
- 2) *the permutation $S_i \rightarrow S_{i+1 \pmod q}$ on the ordered set \mathcal{S} is isomorphic to the unimodal permutation σ ,*
- 3) *the orientation $o(S_i) = o_\sigma(i)$, $i = 1, 2, \dots, q$,*
- 4) $\bigcup S_i \subset [-1 + \epsilon, 1 - \epsilon]$.

The space of ϵ -elementary geometries is denoted by E_ϵ . Moreover

$$Q_\epsilon = \{\mathcal{S} \in E_\epsilon \mid S_q = [-|p|, |p|] \text{ for some } p \in (-1, 1) \text{ with } |p| \geq \epsilon\}.$$

The spaces E_ϵ and Q_ϵ can be considered to be convex subsets of some Euclidean space. The Euclidean topology makes them into compact Euclidean balls. The space of ϵ -geometries is

$$G_\epsilon = Q_\epsilon \times E_\epsilon^T.$$

We will use the product topology on G_ϵ . In particular it is compact. We will use the following notation: $G = G_0$ and if $g = (\mathcal{S}, (\mathcal{S}(\tau))_{\tau \in T}) \in G$ then

$$|g| = \frac{1}{2} \sup \left\{ \left\{ \sum_{i=1}^q |S_i(\tau)| \mid \tau \in T \right\} \cup \left\{ \sum_{i=1}^q |S_i| \right\} \right\}.$$

The geometries do not have any dynamical meaning. They are merely abstract generalizations of cycles. To explain this we go back to unimodal maps.

The non-linearity of $x \rightarrow |x|^\alpha$ is called the *critical non-linearity*

$$\gamma(x) = (\alpha - 1) \frac{1}{x}.$$

The *canonical folding family* is the family $q_t : [-1, 1] \rightarrow [-1, 1]$, $t \in [0, 1]$, with the properties

- 1) $\eta_{q_t} = \gamma$.
- 2) $q_t(-1) = q_t(1) = -1$.
- 3) $t = \frac{|[-1, q_t(0)]|}{|[-1, 1]|}$.

A computation gives

$$q_t(x) = -2t \cdot |x|^\alpha + 2t - 1.$$

There are two observations to be made. First, every q_t has negative Schwarzian derivative. Furthermore consider the interval $I = [-p, p] \subset [-1, 1]$ and consider $Z_I(q_t)$. It has non-linearity

$$\gamma(i_I(x)) \cdot p = (\alpha - 1) \cdot \frac{1}{px} \cdot p = \gamma(x)$$

The canonical folding family have the fundamental property that it is fixed under Zoom-operators to intervals centered around 0.

Usually a unimodal map is defined as a map $f = \phi \circ q_t$ with $t \in [0, 1]$. Here ϕ is some orientation preserving C^2 diffeomorphism of $[-1, 1]$. For renormalization purposes it is more convenient to use the following

Definition 5.2. *A decomposed unimodal map is a pair $(\underline{\phi}, t)$, where $\underline{\phi}$ is a decomposition in X_C and $t \in [0, 1]$. The decomposition $\underline{\phi}$ is called the diffeomorphic part of the decomposed unimodal map $(\underline{\phi}, t)$. The interpretation of $(\underline{\phi}, t)$ is classical unimodal map*

$$f = O(\underline{\phi}) \circ q_t \in \mathcal{U}.$$

The space of decomposed unimodal maps is denoted by $U = X_C \times [0, 1]$, with the product topology.

Definition 5.3. *A decomposed unimodal map $f = (\underline{\phi}, t)$ is called quasi renormalizable if the classical unimodal map $O(\underline{\phi}) \circ q_t$ has the following properties*

- 1) *there exists an expanding periodic point $p \in (-1, 1)$, $f^q(p) = p$,*
 - 2) *there exist $\epsilon > 0$ and an ϵ -elementary geometry $\mathcal{S} = \{S_1, S_2, \dots, S_{q-1}, S_q = [-|p|, |p|]\} \in Q_\epsilon$ such that*
 - 3) *$O(\underline{\phi}) \circ q_t(S_i) = S_{i+1}$, $i = 1, 2, \dots, q-1$.*
 - 4) *$|p| > 0$ is minimal with the above properties.*
- A decomposed unimodal map is called renormalizable if it is quasi renormalizable and*
- 5) *$O(\underline{\phi}) \circ q_t(S_q) \subset S_1$.*

Observe that the elementary geometry \mathcal{S} is uniquely defined. It is constructed by pulling back the central interval $S_q = [-|p|, |p|]$ according to the unimodal permutation σ .

Let $U^+ \subset U$ be the set of quasi-renormalizable decomposed unimodal maps. The *dynamical geometry*

$$d : U^+ \rightarrow G$$

is defined as follows. Let \mathcal{S} be the elementary geometry of $f = (\underline{\phi}, t)$. Then

$$d(\underline{\phi}, t) = (\mathcal{S}, (\mathcal{S}_\tau)_{\tau \in T}),$$

where

$$O^\tau(\mathcal{S}_\tau) = \mathcal{S}$$

for all $\tau \in T$. The *peak-value* of the renormalization

$$\rho : U^+ \rightarrow [0, \infty)$$

is defined as follows. Let $\hat{S}_1 = (O(\underline{\phi}))^{-1}(S_1)$. Then

$$\rho(\underline{\phi}, t) = \frac{|q_t(S_q)|}{|\hat{S}_1|}.$$

A decomposed unimodal map $f = (\underline{\phi}, t) \in U^+$ is renormalizable if

$$q_t(0) \in \hat{S}_1,$$

or equivalently

$$\rho(\underline{\phi}, t) \in [0, 1].$$

Lemma 5.4. *The dynamical geometry and the function ρ are continuous.*

Proof. Let $(\underline{\phi}_n, t_n) \rightarrow (\underline{\phi}, t)$. This implies that $\mathcal{S}(n)$ will tend to \mathcal{S} of the limit because of the continuity of the composition operator O . Now we have to pull back those intervals by the partial compositions $O^\tau(\underline{\phi}_n)$, $\tau \in T$. Observe that we see a uniform convergence of $O^\tau(\underline{\phi}_n) \rightarrow O^\tau(\underline{\phi})$. Lemma 10.2 and Lemma 10.4 can be used to show that the uniform convergence implies that the elementary geometries $\mathcal{S}_\tau(n)$ tend uniformly to the elementary geometries of the limit. This means that the functions d and ρ are continuous.

□ (Lemma 5.4)

6. Geometrical Renormalization

Let $g \in G_\epsilon$, say

$$g = (\mathcal{S}, (\mathcal{S}_\tau)_{\tau \in T})$$

with $\mathcal{S} = \{S_1, S_2, \dots, S_q\}$ and $\mathcal{S}_\tau = \{S_1(\tau), S_2(\tau), \dots, S_q(\tau)\}$ and

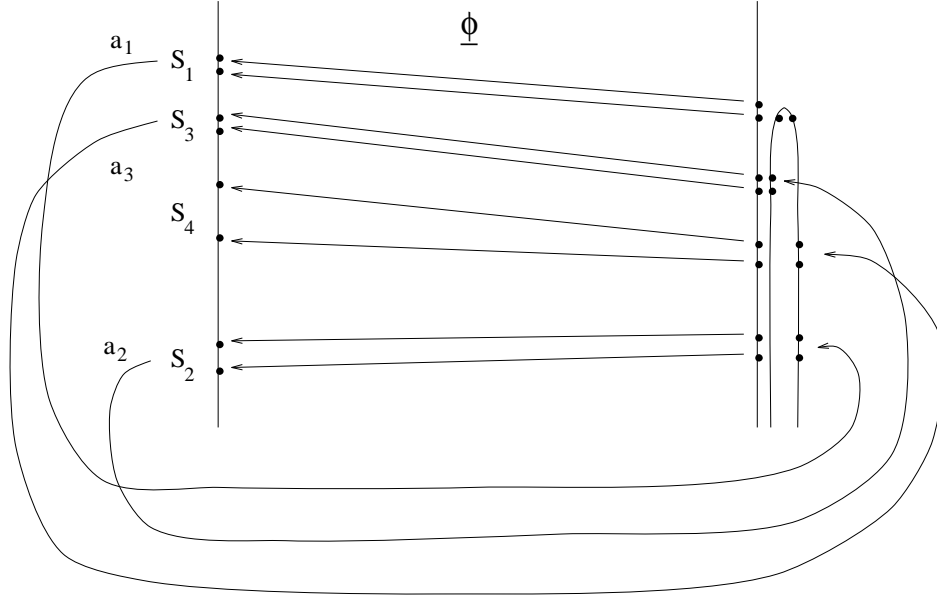
$$a_i = Z_{S_i}(q_t) \in \mathcal{D}, i = 1, 2, \dots, q - 1.$$

Observe that these diffeomorphisms a_i depend only on the intervals S_i , not on t . The *geometrical renormalization* operator for geometry $g \in G_\epsilon$ with $\epsilon > 0$

$$\mathcal{R}_g : X \rightarrow X$$

is defined by

$$\begin{aligned} \mathcal{R}_g(\underline{\phi})(\tau) &= Z_{S_1(A_1(\tau))}(\phi_{A_1(\tau)}) \text{ for } \tau \in T_1 \\ \mathcal{R}_g(\underline{\phi})(1) &= a_1 \\ \mathcal{R}_g(\underline{\phi})(\tau) &= Z_{S_2(A_2(\tau))}(\phi_{A_2(\tau)}) \text{ for } \tau \in T_2 \\ \mathcal{R}_g(\underline{\phi})(2) &= a_2 \\ &\dots = \dots \\ \mathcal{R}_g(\underline{\phi})(q-1) &= a_{q-1} \\ \mathcal{R}_g(\underline{\phi})(\tau) &= Z_{S_q(A_q(\tau))}(\phi_{A_q(\tau)}) \text{ for } \tau \in T_q, \end{aligned}$$

Figure 6. A General Dynamical Picture with $q = 4$

This definition corresponds to the informal discussion in the introduction applied to a general dynamical picture like the one shown in Figure 6.

The family of geometrical renormalization operators is denoted by

$$\mathcal{R} : X \times G_\epsilon \rightarrow X,$$

with $\mathcal{R}(\underline{\phi}, g) = \mathcal{R}_g(\underline{\phi})$. The *dynamical renormalization* operator

$$\mathcal{R}_{\text{dyn}} : U^+ \rightarrow X \times [0, \infty),$$

reflects the classical renormalization operation in the space of decomposed unimodal maps. It is defined as follows. Let $f = (\underline{\phi}, t)$ be a decomposed unimodal map in U^+ then

$$\mathcal{R}_{\text{dyn}}((\underline{\phi}, t)) = (\mathcal{R}_g(\underline{\phi}), \rho(\underline{\phi}, t)),$$

where \mathcal{R}_g is the geometrical renormalization operator with geometry $g = d(\underline{\phi}, t)$. The three renormalization operators $\mathcal{R}_{\text{class}}$, \mathcal{R}_{dyn} and \mathcal{R} are related by

$$\begin{aligned} \mathcal{R}_{\text{class}, \sigma}(O(\underline{\phi}) \circ q_t) &= O(\mathcal{R}(\underline{\phi}, d(\underline{\phi}, t))) \circ q_{\rho(\underline{\phi}, t)}, \\ \mathcal{R}_{\text{dyn}}(\underline{\phi}, t) &= (\mathcal{R}(\underline{\phi}, d(\underline{\phi}, t)), \rho(\underline{\phi}, t)), \end{aligned}$$

where $(\underline{\phi}, t) \in U^+$ is a quasi renormalizable decomposed unimodal map. The difference between the dynamical and geometrical renormalization operators is that the first uses dynamically defined geometries in the renormalization process and the others use given geometries.

The following Proposition is a central ingredient for the understanding of the classical renormalization operator.

Proposition 6.1. *For every $g \in G_\epsilon$ the geometrical renormalization operator \mathcal{R}_g is an affine contraction with contraction constant $|g| \leq 1 - \epsilon$. Its fixed point $\Phi(g)$ depends continuously on g . In particular $\Phi(G_\epsilon)$ is compact. The fixed points have the following special properties*

- 1) *Every map $O(\Phi(g))$ expands hyperbolic distance.*
- 2) *There exists $C > 0$, depending on ϵ , such that $\Phi(G_\epsilon) \in X_C$.*

The use of hyperbolic distance in interval dynamics is thoroughly discussed in [MS].

Proof. The fact that \mathcal{R}_g is an affine contraction is a direct consequence of Lemma 3.1. Observe that the affine term of \mathcal{R}_g is formed by the diffeomorphisms $a_i, i = 1, 2, \dots, q-1$. The second special property can be shown as follows. Observe that the diffeomorphisms $a_i, i = 1, 2, \dots, q-1$ are obtained by applying zoom-operators to canonical folding maps. Their non-linearities have explicit formulas and there is some $C > 0$ such that for all $g = (\mathcal{S}, (\mathcal{S}_\tau)_{\tau \in T}) \in G_\epsilon$ the diffeomorphisms $a_i, i = 1, 2, \dots, q-1$ will satisfy

$$|\eta'_{a_i}(x)| \leq C|\eta_{a_i}(x)|$$

for all $x \in [-1, 1]$. Now observe that for any oriented interval $I \subset [-1, 1]$ the diffeomorphisms $Z_I(\eta_{a_i})$ will satisfy the same property, because

$$|Z_I(\eta_{a_i})(x)| = |\eta_{a_i}(i_I(x))| \cdot \frac{|I|}{2}$$

and

$$|(Z_I(\eta_{a_i}))'(x)| = |\eta'_{a_i}(i_I(x))| \cdot \frac{|I|^2}{4}.$$

It is left is to show that Φ is continuous. Observe that for any $k \geq 0$

$$\pi_k(\Phi(g)) = \mathcal{R}_g^{k+1}(id)$$

and that this projection depends only on the elementary geometries in the levels L_0, \dots, L_k and on \mathcal{S} . The map

$$g \mapsto \pi_k(\Phi(g))$$

is continuous. The last things to observe is that for $\underline{\phi} = \Phi(g), g \in G_\epsilon$

$$\sum_{\tau \in L_k} |\phi_\tau| \leq (1 - \epsilon)^k \cdot \sum_{i=1}^{q-1} |a_i|,$$

which is true because \mathcal{R}_g is an $(1 - \epsilon)$ -contraction. So for $\underline{\phi} = \Phi(g)$ we get

$$\sum_{\tau \in \cup_{j \geq k} L_j} |\phi_\tau| \leq \frac{(1 - \epsilon)^k}{\epsilon} \cdot \sum_{i=1}^{q-1} |a_i| \leq \frac{(1 - \epsilon)^k}{\epsilon^2} \cdot (\alpha - 1) \cdot (q - 1).$$

To get two fixed points close, we only have to get them close on finite levels, which can be done because of the continuity the map $g \mapsto \pi_k(\Phi(g))$.

The first special property is a consequence of the fact that the diffeomorphisms a_i , $i = 1, 2, \dots, q-1$, have negative Schwarzian derivative. Observe that $\Phi(g) = \lim_{k \rightarrow \infty} \mathcal{R}^k(id)$, all the diffeomorphisms of the decomposition $\Phi(g)$ are obtained by applying zoom-operators to the diffeomorphisms a_i , they all have negative Schwarzian derivative. Then observe that all finite compositions expands definitely hyperbolic distance, which is preserved by taking the limit. \square (Proposition 6.1)

The diffeomorphisms $O(\Phi(g))$, with $g \in G_\epsilon$, are C^2 by construction. It can be shown that these diffeomorphisms are in fact analytic.

7. The fixed Point

In this section we are going to prove Theorem 2.2. The aim is to construct a fixed point for $\mathcal{R}_{\text{class},\sigma}$. The strategy is to construct a sequence of approximate fixed points for the corresponding renormalization operator \mathcal{R}_{dyn} on the space of decomposed unimodal maps. These approximate fixed points will be called truncation fixed points. The limit of these approximations is going to be a fixed point for \mathcal{R}_{dyn} . After composition we obtain a fixed point for $\mathcal{R}_{\text{class},\sigma}$.

The *critical value* of a decomposed unimodal map is given by the continuous function

$$v : \bigcup_{C < \infty} X_C \times \mathbb{R} \rightarrow \mathbb{R},$$

where

$$\begin{aligned} v(\underline{\phi}, t) &= -1 + t \text{ for } t \leq 0 \\ v(\underline{\phi}, t) &= O(\underline{\phi}) \circ q_t(0) \text{ for } t \in [0, 1] \\ v(\underline{\phi}, t) &= t \text{ for } t \geq 1. \end{aligned}$$

Note that v is strictly monotone in t .

The proofs of the following Propositions are somewhat involved and we postpone them to respectively section 8. and 9. The first states the existence of truncation fixed points and the second the a priori bounds on their dynamical geometry.

Proposition 7.1. *For every $k \geq 0$ there exists a decomposed unimodal map $(\underline{\phi}, t)$ with the following properties*

- 1) $(\underline{\phi}, t) \in U^+$,
- 2) $v(\underline{\phi}, t) = v(\mathcal{R}_{\text{dyn}}(\underline{\phi}, t))$,
- 3) $d(\underline{\phi}, t) = g \in G_\epsilon$ with $\epsilon > 0$,
- 4) $\underline{\phi} = \mathcal{R}_g^{k+1}(id)$,
- 5) *In particular*

$$\mathcal{R}_{\text{dyn}}(\underline{\phi}, t) = (\mathcal{R}_g(\underline{\phi}), \rho(\underline{\phi}, t)) = (\mathcal{R}_g^{k+2}(id), \rho(\underline{\phi}, t)).$$

A decomposed unimodal map with these properties is called a *truncation fixed point* of depth $k \geq 0$.

Proposition 7.2. *The dynamical geometry of any truncation fixed point is contained in a universal G_ϵ , $\epsilon > 0$.*

We will use the following notation to describe truncation fixed points. For $k \geq 0$ define the function

$$\Phi_k : \bigcup_{\epsilon > 0} G_\epsilon \rightarrow \bigcup_{C < \infty} X_C$$

by

$$\Phi_k(g) = \mathcal{R}_g^{k+1}(id) = \pi_k \circ \Phi(g).$$

These functions describe truncations of the fixed points $\Phi(g)$. Observe that $\Phi_k(g)$ and $\Phi_{k+1}(g)$ differ only in the level of depth $k + 1$.

Theorem 2.2. *The renormalization operator $\mathcal{R}_{\text{class},\sigma}$ (and \mathcal{R}_{dyn}) has a fixed point.*

Proof. From Proposition 7.1 we get a sequence of truncation fixed points with increasing depth $k \geq 0$, $(\Phi_k(g_k), t_k) \in U^+$. According to Proposition 7.2 the sequence of geometries $g_k \in G_\epsilon$ can be assumed to be convergent

$$(g_k, t_k) \rightarrow (g, t) \in G_\epsilon \times [0, 1]$$

for $k \rightarrow \infty$. This convergence implies

- 1) $\Phi_k(g_k), \Phi_{k+1}(g_k) \rightarrow \Phi(g)$,
- 2) $|\Phi_k(g_k) - \Phi_{k+1}(g_k)| \leq K(1 - \epsilon)^k$, where $\epsilon > 0$ and K are universal,
- 3) $\rho_k = \rho(\Phi_k(g_k), t_k) \rightarrow t$.

The first two statements follow from the fact that for all $\underline{\phi} = \Phi(g)$ with $g \in G_\epsilon$

$$\sum_{\tau \in L_j} |\phi_\tau| \leq K(1 - \epsilon)^j,$$

with $j \geq 0$. The third statement follows from continuity of v : observe

$$v(\Phi_{k+1}(g_k), \rho_k) = v(\Phi_k(g_k), t_k) \rightarrow v(\Phi(g), t).$$

So

$$(\Phi_{k+1}(g_k), \rho_k) \rightarrow v^{-1}(v(\Phi(g), t)) \cap (\Phi(g) \times [0, 1]) = \{(\Phi(g), t)\}.$$

The candidate fixed point for \mathcal{R}_{dyn} is $(\Phi(g), t)$. First we have to show that $(\Phi(g), t) \in U^+$. Let $g = (\mathcal{S}, (\mathcal{S})_{\tau \in T})$ and $S_q = [-|p|, |p|]$. By continuity we see that p is a periodic point of $O(\Phi(g)) \circ q_t$. If this periodic point is expanding then the elementary geometry \mathcal{S} makes $(\Phi(g), t)$ renormalizable. We have to show that p is an expanding periodic point. Let

$$B_0 = v^{-1}([-1, 0]) \cap (\Phi(G_\epsilon) \times [0, 1]).$$

This set is compact. Furthermore let

$$\overline{U_\epsilon^+} = \overline{U^+ \cap (\Phi(G_\epsilon) \times [0, 1])}$$

This set is also compact and $B_0 \cap U_\epsilon^+ = \emptyset$. The distance of these sets is $\delta > 0$. Assume $(\Phi(g), t) \notin U^+$. Then the periodic point p is a neutral periodic point for the map $f = O(\Phi(g)) \circ q_t$. This map expands hyperbolic distance. So the periodic orbit attracts 0. The orbit of 0 is contained in $\cup S_i$. This implies that $f^q([p, 0]) \subset [p, 0]$. Then observe that $\mathcal{R}_{\text{class}, \sigma}(O(\Phi_k(g_k)) \circ q_{t_k})$ tends to the rescaling of $f^q : S_q \rightarrow S_q$. So for k big enough we see that $\mathcal{R}_{\text{dyn}}(\Phi_k(g_k), t_k)$ is in B_0 . In particular

$$|\mathcal{R}_{\text{dyn}}(\Phi_k(g_k), t_k) - (\Phi_k(g_k), t_k)| \geq \delta,$$

which contradicts property 2 and 3 above.

We proved that $(\Phi(g), t) \in U^+$ and we can apply the dynamical renormalization operator. The continuity of the functions \mathcal{R}, d and ρ implies

$$\begin{aligned} \mathcal{R}_{\text{dyn}}(\Phi(g), t) &= (\mathcal{R}(\Phi(g), d(\Phi(g), t)), \rho(\Phi(g), t)) \\ &= \lim_{k \rightarrow \infty} (\mathcal{R}(\Phi_k(g_k), d(\Phi_k(g_k), t_k)), \rho(\Phi_k(g_k), t_k)) \\ &= \lim_{k \rightarrow \infty} (\mathcal{R}(\Phi_k(g_k), g_k), \rho_k) \\ &= \lim_{k \rightarrow \infty} (\Phi_{k+1}(g_k), \rho_k) = (\Phi(g), t). \end{aligned}$$

In particular for $\underline{\phi} = \Phi(g)$ we get

$$\mathcal{R}_{\text{class}, \sigma}(O(\underline{\phi}) \circ q_t) = O(\mathcal{R}(\underline{\phi}, d(\underline{\phi}, t))) \circ q_{\rho(\underline{\phi}, t)} = O(\underline{\phi}) \circ q_t.$$

□ (Theorem 2.2)

8. The Existence of Truncation Fixed Points

In this section we are going to prove Proposition 7.1, the existence of truncation fixed points.

Assume that $(\underline{\phi}, t)$ is a fixed point of \mathcal{R}_{dyn} . Then

$$\mathcal{R}_{\text{dyn}}(\underline{\phi}, t) = (\mathcal{R}(\underline{\phi}, d(\underline{\phi}, t)), \rho(\underline{\phi}, t)) = (\underline{\phi}, t).$$

Hence $\underline{\phi} = \Phi(d(\underline{\phi}, t))$. We have to construct a fixed using the collection

$$P = \bigcup_{\epsilon > 0} \Phi(G_\epsilon) \subset \bigcup_{C < \infty} X_C.$$

consisting of so-called *pure decompositions*. Let

$$U_P^+ = \{(\underline{\phi}, t) \in U^+ | \underline{\phi} \in P\},$$

be the collection of quasi renormalizable decomposed unimodal maps whose diffeomorphic part is pure. Observe that

$$\mathcal{R}_{\text{dyn}} : U_P^+ \rightarrow P \times (0, \infty).$$

It is this map to which we want to apply the bottom-down-top-up-Principle discussed in the introduction. Unfortunately P is homeomorphic to a countable product of open intervals. It is not possible to take a compactification of P and extend the renormalization operator to this compactification. The problem that arises is that the composition operator

$$O : \bigcup_{C < \infty} X_C \rightarrow \mathcal{D}$$

can not be extended. This extension is necessary to define the dynamical geometry. We have to proceed differently.

Let

$$\mathcal{Q} = \{Z_I(q_{\frac{1}{2}}) | I = [1 - a, 1] \text{ an oriented interval with } a \in [0, 1]\}.$$

This set can be parametrized by $[-1, 1]$: for $a \in [0, 1]$ we take the interval $[1 - a, 1]$ with $o(I) = 1$ and for $a \in [-1, 0]$ we use $o(I) = -1$. There is no ambiguity when $a = 0$, $Z_I(q_{\frac{1}{2}}) = id$.

This compact space \mathcal{Q} will be used to compactify the pure decompositions. Observe that when $g \in G_\epsilon$, $\epsilon > 0$,

$$\Phi(g)(\tau) \in \mathcal{Q},$$

for all $\tau \in T$.

Instead of working in the infinite dimensional space of pure decompositions we will work in finite dimensional truncations. Fix $k \geq 0$. Let

$$P(k) = \mathcal{Q}^{\cup_{j \leq k} L_j}$$

Observe that $P(k)$ is homeomorphic to a finite dimensional closed Euclidean ball. There is an embedding $j : P(k) \rightarrow P(k + 1)$ defined by

$$\begin{aligned} j(\underline{\phi})(\tau) &= \phi_\tau, \tau \in \cup_{j \leq k} L_j \\ j(\underline{\phi})(\tau) &= id, \tau \in L_{k+1} \end{aligned}$$

Identify $P(k)$ with this embedding, $P(k) \subset P(k + 1)$ and let $\pi_k : P(k + 1) \rightarrow P(k)$ be the projection.

Let \mathcal{H} be the collection of orientation preserving interval homeomorphisms $h : [-1, 1] \rightarrow [-1, 1]$ with the property

- 1) $h|_{(-1, 1)}$ is C^2 ,
- 2) $Dh(\pm 1) \geq 0$.

The decompositions in $P(k)$ contain only finitely many endomorphisms in \mathcal{H} . There is no difficulty in defining the composition operator

$$O : P(k) \rightarrow \mathcal{H}.$$

The definitions used before carry automatically over to the finite dimensional truncations. The objects thus obtained can also be considered to be continuous extensions to the compact space $P(k) \times [0, 1]$.

The set of quasi renormalizable decomposed unimodal maps is denoted by

$$U^+(k) \subset P(k) \times [0, 1].$$

For $\epsilon \geq 0$ let

$$G_\epsilon(k) = Q_\epsilon \times E_\epsilon^{\cup_{j \leq k} L_j}.$$

The dynamical geometry is again a continuous function

$$d : U^+(k) \rightarrow \bigcup_{\epsilon > 0} G_\epsilon(k)$$

and the peak value of the renormalization is the continuous function

$$\rho : U^+(k) \rightarrow [0, \infty).$$

The geometrical renormalization operators

$$\mathcal{R}_g : P(k) \rightarrow P(k+1),$$

$g \in G_\epsilon$, are defined as before by using the truncated set $\cup_{j \leq k} L_j$ as set of decomposition times. The dynamical renormalization operator becomes

$$\mathcal{R}_{\text{dyn}} : U^+(k) \rightarrow P(k+1) \times [0, 1]$$

with

$$\mathcal{R}_{\text{dyn}}((\underline{\phi}, t)) = (\mathcal{R}_{d((\underline{\phi}, t))}(\underline{\phi}), \rho((\underline{\phi}, t))).$$

The proof of the following Lemma is left to the reader.

Lemma 8.1. *For every $k \geq 0$ there exists a unique continuous function $\beta_k : P(k) \rightarrow (0, 1]$ such that*

$$O(\underline{\phi}) \circ q_{\beta_k(\underline{\phi})}(0) = 0.$$

In particular

$$U^+(k) \subset \{(\underline{\phi}, t) \in P(k) \times [0, 1] \mid t > \beta_k(\underline{\phi})\}$$

and $\beta_{k+1}|_{P(k)} = \beta_k$.

Lemma 8.2. For $k \geq 0$

- 1) $U^+(k)$ is open,
- 2) $P(k) \times \{1\} \subset U^+(k)$ and $\rho(P(k) \times \{1\}) \subset (1, \infty)$,
- 3) there is a continuous extension

$$\mathcal{R}_{\text{dyn}} : \overline{U^+(k)} \rightarrow P(k+1) \times [0, \infty).$$

- 4) the image of the boundary of $U^+(k)$ satisfies

$$\mathcal{R}_{\text{dyn}}(\partial U^+(k)) \subset \{(\underline{\phi}, t) \in P(k+1) \times [0, 1] \mid t < \beta_{k+1}(\underline{\phi})\}$$

The boundary of a set A is $\partial A = \overline{A} \cap \overline{A^c}$.

Proof. Observe that a map is quasi renormalizable iff it has the expanding periodic orbit of the combinatorial type determined by σ . The unimodal maps corresponding to points in $P(k) \times \{1\}$, so-called full maps, have periodic orbits of all possible combinatorial types. All of which are expanding. This explains Property 2).

The stability of expanding periodic orbits together with the above observation implies that $U^+(k)$ is open, property 1).

The boundary of $U^+(k)$ consists of maps $(\underline{\phi}, t)$ for which the unimodal map $O(\underline{\phi}) \circ q_t$ has a neutral periodic point p with combinatorics still the same as determined by σ . We can still construct an elementary geometry by pulling back the central interval $C = [-|p|, |p|]$. This elementary geometry determines the extension of d and ρ . Property 3.

Observe that this neutral point attracts the orbit of the critical point. The classical renormalization of $O(\underline{\phi}) \circ q_t$ is a unimodal map such that the image of the critical point 0 is below 0. Property 4. \square (Proposition 8.2)

The author would like to thank D.Sullivan for indicating the following

Bottum-Down-Top-Up-Proposition 8.3. Let $F : D_n \times [0, 1] \rightarrow D_n \times \mathbb{R}$ be a continuous map, D_n is the closed n -dimensional ball. If

$$F(D_n \times \{0\}) \subset D_n \times (-\infty, 0)$$

and

$$F(D_n \times \{1\}) \subset D_n \times (1, \infty)$$

then F has a fixed point in $D_n \times [0, 1]$.

Proof. Suppose not. Then let S be the boundary of $D_n \times [0, 1]$, this is an n -dimensional sphere. And consider the displacement function $f : S \rightarrow S^n$, where S^n is the unit sphere in \mathbb{R}^{n+1} and

$$f(x) = \frac{x - F(x)}{|x - F(x)|}.$$

Claim. $\deg(f) = \pm 1$.

By radial projection onto the axis $\{0\} \times \mathbb{R}$ we can find a homotopy of F , say F_t , $t \in [0, 1]$, such that

- 1) $F_0 = F$.
- 2) F_t has no fixed points in S .
- 3) the image of F_1 is an interval $\{\underline{0}\} \times [-a, b]$, $a > 0$, $b > 1$.

Then the images $F_1(D_n \times \{0\})$ and $F_1(D_n \times \{1\})$ are intervals in $\{\underline{0}\} \times [-a, b]$ outside $D_n \times [0, 1]$. Now we can perform a second homotopy to collapse these two intervals to points. Without loss of generality we may assume that

4)

$$F_1(D_n \times \{1\}) = (\underline{0}, b)$$

and

$$F_1(D_n \times \{0\}) = (\underline{0}, -a).$$

Let $f_t : S \rightarrow S^n$ be the displacement function for F_t . Because $f = f_0$ and f_1 are homotopic we get

$$\deg(f) = \deg(f_1).$$

To compute this degree observe that only $(\underline{0}, 1)$ and $(\underline{0}, 0)$ will have pure vertical displacements under the map F_1 . The first point goes up, the second goes down. So if $N \in S^n$ be the north pole of S^n then

$$f_1^{-1}(N) = \{(\underline{0}, 1)\}.$$

So the degree of f_1 equals the local degree at $(\underline{0}, 1)$. In the natural local coordinates at $(\underline{0}, 1)$ f_1 is exactly the anti-podal map in dimension n . So

$$\deg(f) = \pm 1.$$

The Claim is proved.

But f has an extension to $f : D_n \times [0, 1] \rightarrow S^n$ in the obvious way, so $\deg(f) = 0$. Contradiction. The map F has a fixed point. \square (Proposition 8.3)

The Construction of a Truncation Fixed Point, Proposition 7.1. Fix $k \geq 0$. Let $H_1 : P(k+1) \times \mathbb{R} \rightarrow P(k+1) \times \mathbb{R}$ be defined by

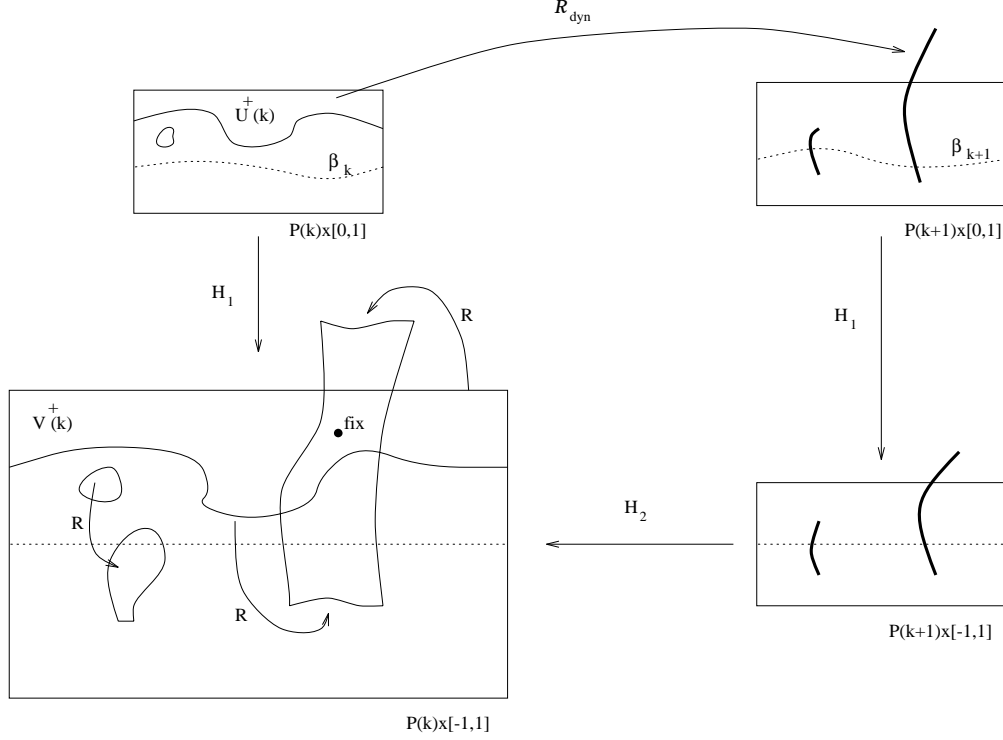
$$H_1(\underline{\phi}, t) = (\underline{\phi}, v(\underline{\phi}, t)).$$

and let $H_2 : P(k+1) \times \mathbb{R} \rightarrow P(k) \times \mathbb{R}$ be defined by

$$H_2(\underline{\phi}, t) = (\pi_k(\underline{\phi}), t).$$

Let

$$V^+(k) = H_1(U^+(k)) \subset P(k) \times (0, 1].$$


 Figure 7. $R : V^+(k) \rightarrow P(k) \times \mathbb{R}$

Consider the function $R : V^+(k) \rightarrow P(k) \times \mathbb{R}$ defined by

$$R = H_2 \circ H_1 \circ \mathcal{R}_{\text{dyn}} \circ H_1^{-1}.$$

It is this function to which we would like to apply the previous Proposition. However, we do not know whether $V^+(k)$ has the right product form. It could have different components. The solution is to extend the function R continuously to $P(k) \times [0, 1]$. To be able to make this extension we have to deform R slightly.

Let $D : P(k) \times \mathbb{R} \rightarrow P(k) \times \mathbb{R}$ defined as follows

- 1) $A|_{P(k) \times [0, \infty)} = id$,
- 2) $A(P(k) \times (-\infty, 0)) \subset P(k) \times [-1, 0)$,
- 3) $A(R(\partial V^+(k))) = \{(\underline{\phi}_0, -1)\}$.

The third property is possible because of Lemma 8.2(4): $R(\partial V^+(k)) \subset P(k) \times [-1, 0)$ is compact. The function $F = A \circ R : V^+(k) \rightarrow P(k) \times \mathbb{R}$ is constant on $\partial V^+(k)$ and hence can be continuously extended to $F : P(k) \times [0, 1] \rightarrow P(k) \times \mathbb{R}$.

It has a fixed point $(\underline{\phi}, v) \in P(k) \times [0, 1]$, by Proposition 8.3. It has to be in $V^+(k)$, otherwise it would be maps into $P(k) \times \{-1\}$. In particular it is a fixed point for R . The truncation fixed point is going to be

$$(\underline{\phi}, t) = H^{-1}(\underline{\phi}, v) \in U^+(k).$$

There is $\epsilon > 0$ such that the dynamical geometry g of $(\underline{\phi}, t)$ is in G_ϵ . Because $(\underline{\phi}, v)$ is a fixed point of R we have

$$(\underline{\phi}, v) = H_2 \circ H_1(\mathcal{R}_{\text{dyn}}(\underline{\phi}, t)).$$

So

$$\underline{\phi} = \pi_k(\mathcal{R}_g(\underline{\phi})).$$

Hence

$$\underline{\phi} = \mathcal{R}_g^{k+1}(id).$$

Indeed we found a truncation fixed point in U^+ .

□ (Proposition 7.1)

9. A Priori Bounds

In this section we are going to prove the a priori bounds on the geometry of truncation fixed points, Proposition 7.2. Fix a truncation fixed point $(\Phi_k(g), t) \in U^+$, $k \geq 0$, say

$$g = (\mathcal{S}, (\mathcal{S}_\tau)_{\tau \in T})$$

with

$$\begin{aligned} \mathcal{S} &= \{S_1, S_2, \dots, S_q = [-|p|, |p|]\}, \\ \mathcal{S}_\tau &= \{S_1(\tau), S_2(\tau), \dots, S_q(\tau)\}, \text{ for } \tau \in T. \end{aligned}$$

The a priori bounds on the geometry g will follow easily when a uniform ϵ is found with

$$\mathcal{S} \in Q_\epsilon.$$

The first step is to find a uniform bounds on the position of the periodic point $p \in (-1, 1)$ of $f = O(\Phi_k(g)) \circ q_t$.

Lemma 9.1. *There exist constants $K > 0$ and $p_0 > 0$, independent of k with the following property. If $|p| < p_0$ then*

$$|g| \leq K \alpha^q \sqrt[q]{|p|},$$

and

$$\sum_{\tau \in \bigcup_{j \geq \alpha q + 1} L_j} |\phi_\tau| \leq K.$$

Proof. The geometry is the dynamical geometry of $(\Phi_k(g), t)$. It is obtained by pulling back the interval $S_q = [-|p|, |p|]$. All the maps in $\Phi_k(g)$ have negative Schwarzian derivative which implies immediately the first statement. To prove the second statement observe that

$$\sum_{i=1}^{q-1} |a_i| \leq (\alpha - 1) \frac{1}{p} \cdot (q - 1).$$

To recover $\Phi_k(g)$ we have to apply repeatedly the geometric renormalization operator \mathcal{R}_g . The bound on $|g|$ implies that the geometrical renormalization operator contracts with a constant $K \alpha^q \sqrt[q]{p}$. So for all $j \geq 0$

$$\sum_{\tau \in L_j} |\phi_\tau| \leq (\alpha - 1) \frac{1}{p} \cdot (q - 1) \cdot (K \alpha^q \sqrt[q]{p})^j.$$

In particular

$$\begin{aligned} \sum_{\tau \in L_{\alpha q+1}} |\phi_\tau| &\leq (\alpha - 1) \frac{1}{p} \cdot (q - 1) \cdot (K \sqrt[\alpha q]{p})^{\alpha q+1} \\ &= (\alpha - 1) \cdot (q - 1) \cdot K^{\alpha q+1} \cdot p \cdot (K \sqrt[\alpha q]{p})^1 \leq 1, \end{aligned}$$

for p small enough. Because the contraction constant $K \sqrt[\alpha q]{p}$ is small, we get a uniform bound on the total norm of the diffeomorphisms in levels deeper than $\alpha q + 1$.

□ (Lemma 9.1)

Lemma 9.2. *There exist $K > 0$ and $p_0 > 0$, independent of k , with the following property. If $|p| < p_0$ then*

$$(O(\Phi_k(g)))'(x) \leq K,$$

for all $x \in [-1, 1]$.

Proof. Observe that $O(\Phi_k(g))$ is obtained by composing the maps in the levels $L_0, L_1, \dots, L_{\alpha q}$, which are finitely many rescaled restrictions of the canonical folding map, and diffeomorphisms in the deeper levels. These diffeomorphisms in the deeper levels are controlled by Lemma 9.1. Unfortunately the maps in the first αq levels can be highly non-linear. However, there is a uniform bound on the derivative of each of them.

The finite number of maps in the first αq levels have uniform bounded derivative and the maps in the deeper levels are uniformly bounded, $O(\Phi_k(g))$ has uniform bounded derivative. □ (Lemma 9.2)

The next Lemma states the a priori lower bound on the position of the periodic point p .

Lemma 9.3. *There exists $\epsilon > 0$, independent of k , such that*

$$|p| > \epsilon.$$

Proof. Assume that the periodic point p of f is very close to 0. Because $(\Phi_k(g), t)$ is a truncation fixed point we have

$$|f^q(S_q)| \geq \frac{1}{2} |S_q| = |p|.$$

But

$$|f^q(S_q)| \leq (2\alpha K)^{q-1} \cdot K \cdot 2t |p|^\alpha,$$

where $(2\alpha K)^{q-1}$ is the bound obtained from Lemma 9.2 for the derivative of f . These two estimates are impossible for $|p|$ very small.

□ (Lemma 9.3)

Lemma 9.4. *There exists $\epsilon > 0$, independent of k , such that*

$$\mathcal{S} \subset Q_\epsilon.$$

Proof. The previous Lemma states that the periodic point is not too close to 0. Left is to show that for some uniform $\epsilon > 0$

$$\bigcup_{i=1}^q S_i \subset (-1 + \epsilon, 1 - \epsilon).$$

Assume the contrary. Every interval S_i lies between S_1 and $-S_1$. Hence the assumption implies that the right most interval in \mathcal{S} , that is S_1 , is very close to the boundary point 1. Use the notation $\phi = O(\Phi_k(g))$ and $\psi = O(\Phi_{k+1}(g))$ and $f = \phi \circ q_t$.

The decompositions $\Phi_k(g)$ and $\Phi_{k+1}(g)$ differ only in level $k + 1$. This implies that the map ψ is obtained from ϕ by Sandwicing the maps in the level $k + 1$ of $\Phi_{k+1}(g)$ into ϕ . From this we can not conclude general distortion statements. However, we can compare $\psi'(1)$ and $\phi'(1)$. They differ by the product of the derivatives in 1 of the maps in this level $k + 1$.

$$\psi'(1) = \phi'(1) \times \prod_{\tau \in L_{k+2}} \psi'_\tau(1).$$

The last factor is bounded. The reason for this is that the diffeomorphisms in this level $k + 1$ are obtained by repeatedly applying the geometrical renormalization operator \mathcal{R}_g . So, their total norm is bounded by

$$\sum_{\tau \in L_{k+1}} |\psi_\tau| \leq \sum_{i=1}^{q-1} |a_i| \leq (\alpha - 1) \frac{1}{p} \cdot (q - 1) \leq (\alpha - 1) \frac{1}{\epsilon} \cdot (q - 1),$$

where $\epsilon > 0$ is the minimal distance of the periodic point p to 0 given in Lemma 9.3. This bound is uniform. Now apply Lemma 10.3 and we get a uniform bound

$$\psi'(1) \leq K \cdot \phi'(1).$$

The next part compares these two derivatives dynamically. The interval $\hat{S}_1 = \phi^{-1}(S_1)$ has two boundary points: $q_t(p)$ and b_0 . Let $b_1 = \phi(b_0)$ be the corresponding boundary point of S_1 . Then

$$\psi'(1) = \frac{|\hat{S}_1|}{|S_q|} \cdot (f^{q-1})'(b_1) \cdot \phi'(b_0).$$

The first factor is a normalization factor. Compare the dynamical picture in Figure 6. We are going to show a uniform bound

$$(f^{q-1})'(b_1) \leq B.$$

This will be done in a few steps.

First we claim

$$|S_1| \approx 0.$$

If S_1 is big then the periodic point p has to be away from the boundary and the interval $S_q = [-|p|, |p|]$ has bounded hyperbolic length. The other intervals S_i are obtained by pulling back by ϕ and q_t . This maps has negative Schwarzian derivative and we get a uniform bound on the hyperbolic length of all intervals S_i , $i = 1, 2, \dots, q$. But S_1 is very close to the boundary (by assumption) and $|S_1|$ is big. Contradiction.

Next we claim a uniform lower bound

$$|\hat{S}_1| \geq \delta.$$

The critical value of $(\Phi_k(g), t)$ is denoted by $v \in [-1, 1]$. Also use the notation $\rho = \rho(\Phi_k(g), t)$. Because $(\Phi_k(g), t)$ is a truncation fixed point we have

$$2t - 1 = \phi^{-1}(v) \text{ and } 2\rho - 1 = \psi^{-1}(v).$$

From Lemma 10.6 (see appendix) we get a universal constant K such that the hyperbolic distance between $2t - 1 \in [-1, 1]$ and $2\rho - 1 \in [-1, 1]$ is bounded by

$$K \sum_{\tau \in L_{k+1}} |\phi_\tau| \leq K \sum_{i=1}^{q-1} |a_i| \leq K \cdot (\alpha - 1) \cdot \frac{1}{\epsilon} \cdot (q - 1),$$

where $\epsilon > 0$ is given by Lemma 9.3. We get universal bound on the hyperbolic distance between $t, \rho \in [0, 1]$. In particular if t is very small we get

$$\rho \leq \text{const} \cdot t.$$

Now we can finish the lower estimate for

$$\left| \hat{S}_1 \right| = \frac{|q_t(S_q)|}{\rho} = \frac{2t \cdot p^\alpha}{\rho} \geq \frac{t}{\rho} \cdot 2\epsilon^\alpha,$$

where $\epsilon > 0$ is given again by Lemma 9.3. In case t is not too small there will be a lower bound because $\rho \leq 1$. For small t we use the universal estimate $\rho \leq \text{const} \cdot t$ to obtain a universal lower bound for $|\hat{S}_1|$.

At last we claim

$$\phi'(b_0) \geq \phi'(1).$$

The interval $q_t(S_1)$ is very small. Because S_q is not small there has to be a point x between -1 and \hat{S}_1 with $\phi'(x) \gg 1$. The Minimal Principle for maps with negative Schwarzian

derivative (see [MS]) applied to $\phi : [x, b_0] \rightarrow [-1, 1]$ implies that the average slope on \hat{S}_1 satisfies

$$\phi'(x) \gg 1 \gg \frac{S_1}{\hat{S}_1} \geq \min\{\phi'(x), \phi'(b_0)\}.$$

So

$$\phi'(x) > \phi'(b_0).$$

The minimal Principle applied to $\phi : [x, 1] \rightarrow [-1, 1]$ implies

$$\phi'(x) > \phi'(b_0) \geq \min\{\phi'(x), \phi'(1)\}.$$

The claim follows.

These estimates together give

$$K\phi'(1) \geq \psi'(1) \geq \text{const} \cdot (f^{q-1})'(b_1) \cdot \phi'(1).$$

Indeed we get a uniform bound for $(f^{q-1})'(b_1)$.

The third part of the proof will produce the contradiction. Let $H_i \subset [-1, 1]$, $i = 1, 2, \dots, q-1$, be the connected component of $[-1, 1] \setminus S_i$ not containing 0. We claim

$$\frac{|H_i|}{|S_i|} \ll 1$$

for all $i = 1, 2, \dots, q-1$. Assume the contrary: there is an S_i with bounded hyperbolic length. This interval can be pulled back to show that also \hat{S}_1 has bounded hyperbolic length. One step more shows that also S_q has bounded hyperbolic length. Continue to pull back and it turns out that every interval S_j has bounded hyperbolic length. This implies that

$$f^{q-1} : S_1 \rightarrow S_q$$

has bounded distortion. Because $\frac{|S_q|}{|S_1|}$ is very big we get that $(f^{q-1})'(b_1)$ is also very big. Contradiction.

This Claim implies that for every $i = 2, 3, \dots, q-1$ we have $S_1 \subset H_i \cup -H_i$ and

$$|S_1| \leq |H_i| \ll |S_i|.$$

Let $A = (b_1, a]$ be the maximal interval on which f^{q-1} is monotone. The map f^{q-1} has a very big average slope on S_1 but the derivative in b_1 is uniformly bounded by B . The Minimal Principle implies that

$$(f^{q-1})'(z) \leq B,$$

for every point $z \in A$. We will construct a point in A with a very big derivative and so produce a contradiction.

Observe

$$|f^{q-1}(A)| \leq B \cdot |A| \ll |S_1|,$$

$f^{q-1}(A)$ does not contain S_1 (or $-S_1$). This cannot happen in the period doubling case $q = 2$. So $q \geq 3$. Moreover, there has to be some $j \leq q - 2$ with

$$f^j(a) = 0.$$

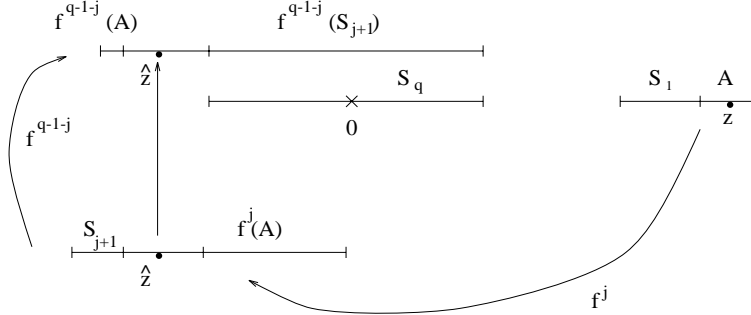


Figure 8.

There has to be a point $z \in A$ such that $\hat{z} = f^j(z)$ (or $-\hat{z}$) is periodic with period $q - 1 - j < q$, see Figure 8. Such a point can not be in the orbit of S_1 because this orbit has period q . In particular it does not attract the critical orbit. The map f has negative Schwarzian derivative which implies that the periodic orbit is expanding

$$|(f^{q-1-j})'(\hat{z})| > 1.$$

Let $A' = f^{-j}(S_q) \cap A$ and estimate

$$\begin{aligned} |(f^{q-1})'(z)| &= |(f^{q-1-j})'(\hat{z})| \cdot |(f^j)'(z)| \\ &\geq 1 \cdot |(f^j)'(z)| \\ &\geq \min\left\{\frac{|f^j(S_1)|}{|S_1|}, \frac{1}{2} \cdot \frac{|S_q|}{|A'}\right\} \\ &\geq \min\left\{\frac{|S_{j+1}|}{|S_1|}, \frac{1}{2} \cdot \frac{|S_q|}{|S_1|}\right\} \gg 1. \end{aligned}$$

Contradiction.

□ (Lemma 9.4)

Proof of the A Priori Bounds, Proposition 7.2. The previous Lemma gives uniform bound on the elementary geometry $\mathcal{S} \in Q_\epsilon$. It is contained in an interval with bounded hyperbolic length. The Schwarzian derivative of all the diffeomorphisms $O^\tau(\Phi_k(g))$ is negative we can

pull back this elementary geometry \mathcal{S} and recover the whole geometry g and see that all the elementary geometries \mathcal{S}_τ are contained in an interval with uniformly bounded hyperbolic length. In particular they have also Euclidean length uniformly bounded away from 2,

$$|g| \leq 1 - \delta.$$

The geometric renormalization operator \mathcal{R}_g has uniformly bounded contraction constant. This implies that the decomposition $\Phi_k(g)$ lies in a uniformly bounded set in X_C . We get a uniform bound on the derivatives of the diffeomorphisms $O^\tau(\Phi_k(g))$. Hence the dynamical intervals $S_i(\tau)$ with $i = 1, 2, \dots, q$ and $\tau \in T$ can also not be too close to the boundary of $[-1, 1]$. The a priori bounds on the geometry of any truncation fixed point are shown.

□ (Proposition 7.2)

10. Appendix

In this appendix we discuss some Lemmas describing the relation between the C^2 topology on $\mathcal{D} = \text{Diff}_+^2([-1, 1])$ and the non-linearity norm used throughout the text.

Lemma 10.1 (Chain-Rule for Non-linearities). *Let $\phi, \psi \in \mathcal{D}$ then*

$$\eta_{\psi \circ \phi}(x) = \eta_\psi(\phi(x)) \cdot \phi'(x) + \eta_\phi(x),$$

for all $x \in [-1, 1]$.

Lemma 10.2. *If $\phi, \psi \in \mathcal{D}$ then*

$$|\psi(x) - \phi(x)| \leq 2(e^{4|\psi - \phi|} - 1)$$

for all $x \in [-1, 1]$.

Proof. Use the inverse of the non-linearity:

$$\begin{aligned} \psi(x) &= 2 \frac{\int_{-1}^x e^{\int_{-1}^s \eta_\psi ds}}{\int_{-1}^1 e^{\int_{-1}^s \eta_\psi ds}} - 1 \\ &= 2 \frac{\int_{-1}^x e^{\int_{-1}^s (\eta_\psi - \eta_\phi + \eta_\phi) ds}}{\int_{-1}^1 e^{\int_{-1}^s (\eta_\psi - \eta_\phi + \eta_\phi) ds}} - 1 \\ &\leq e^{4|\psi - \phi|} \cdot \phi(x) + e^{4|\psi - \phi|} - 1. \end{aligned}$$

So

$$\begin{aligned} \psi(x) - \phi(x) &\leq (e^{4|\psi - \phi|} - 1) \cdot \phi(x) + e^{4|\psi - \phi|} - 1 \\ &\leq 2(e^{4|\psi - \phi|} - 1). \end{aligned}$$

The above estimate is symmetric in ψ and ϕ : the Lemma follows.

□ (Lemma 10.2)

Lemma 10.3. *Let ψ be a composition of finitely many $\phi_i \in \mathcal{D}$, $i = 1, 2, \dots, s$. Let $|\underline{\phi}| = \sum |\phi_i|$. Then*

$$e^{-2|\underline{\phi}|} \leq \psi'(x) \leq e^{2|\underline{\phi}|}$$

and

$$e^{-|\underline{\phi}|e^{2|\underline{\phi}|} \cdot |x-y|} \leq \frac{\psi'(x)}{\psi'(y)} \leq e^{|\underline{\phi}|e^{2|\underline{\phi}|} \cdot |x-y|}$$

for all $x, y \in [-1, 1]$.

Proof. Let $\psi = \phi_s \circ \dots \circ \phi_2 \circ \phi_1$. Take $x, y \in [-1, 1]$ and let x_i and y_i be the images of x and y under the partial composition $\phi_{i-1} \circ \dots \circ \phi_2 \circ \phi_1$. Then

$$\begin{aligned} |\ln \psi'(x) - \ln \psi'(y)| &= \left| \sum_i \ln \phi'_i(x_i) - \ln \phi'_i(y_i) \right| \\ &\leq \sum_i |\ln \phi'_i(x_i) - \ln \phi'_i(y_i)| \\ &\leq \sum_i |\phi_i| |x_i - y_i| \\ &\leq |\underline{\phi}| \cdot 2. \end{aligned}$$

Because there is some point where ψ' equals 1 we get the first estimate. Moreover we can use this estimate to get a bound

$$|x_i - y_i| \leq e^{2|\underline{\phi}|} |x - y|.$$

This gives us immediately the second estimate. □ (Lemma 10.3)

Lemma 10.4. *For every bounded set $B \subset \mathcal{D}$ there exists a constant K such that for any pair $\psi, \phi \in B$*

$$|y - x| \leq K \cdot |\psi(y) - \phi(x)| + K \cdot |\psi - \phi|$$

for all $x, y \in [-1, 1]$.

Proof.

$$\begin{aligned} \psi(y) &= \psi(x) + \psi'(\theta)(y - x) \\ &= \phi(x) + \psi'(\theta)(y - x) + \psi(x) - \phi(x) \end{aligned}$$

So

$$\begin{aligned} |y - x| &= \left| \frac{1}{\psi'(\theta)} \cdot \{\psi(y) - \phi(x) + (\phi(x) - \psi(x))\} \right| \\ &\leq K \cdot |\psi(y) - \phi(x)| + K \cdot |\phi - \psi| \end{aligned}$$

where we used Lemma 10.2 and Lemma 10.3. □ (Lemma 10.4)

The Sandwich Lemma 10.5. *For all $b > 0$ and $C > 0$ there exists a Sandwich constant K , such that the following holds. Let ψ_1, ψ_2 be compositions of finitely many $\phi, \phi_i \in \mathcal{D}$, $i = 1, 2, \dots, s$:*

$$\psi_1 = \phi_s \circ \dots \circ \phi_t \circ \phi_{t-1} \circ \dots \circ \phi_2 \circ \phi_1$$

and

$$\psi_2 = \phi_s \circ \dots \circ \phi_t \circ \phi \circ \phi_{t-1} \circ \dots \circ \phi_2 \circ \phi_1.$$

If $\sum_i |\phi_i| + |\phi| \leq b$ and for $i = 1, \dots, s$ $|\eta'_{\phi_i}(x)| \leq C|\eta_{\phi_i}(x)|$ then

$$|\psi_2 - \psi_1| \leq K|\phi|.$$

Proof. Let $x \in [-1, 1]$. For $0 \leq i \leq s-1$ define

$$x_i = \phi_i \circ \dots \circ \phi_2 \circ \phi_1(x), \quad x_0 = x,$$

and

$$D_i = (\phi_i \circ \dots \circ \phi_2 \circ \phi_1)'(x).$$

Furthermore for $t-1 \leq j \leq s-1$ let

$$x'_j = \phi_j \circ \dots \circ \phi_t \circ \phi(x_{t-1})$$

and

$$D'_j = (\phi_j \circ \dots \circ \phi_t)'(x'_{t-1})\phi'(x_{t-1})D_{t-1}.$$

For $0 \leq j \leq t-2$ let

$$x'_j = x_j \text{ and } D'_j = D_j.$$

Then the chain rule for non-linearities gives

$$\begin{aligned} |\eta_{\psi_2}(x) - \eta_{\psi_1}(x)| &= \left| \sum_{i=0}^{s-1} \{ \eta_{\phi_{i+1}}(x'_i)D'_i - \eta_{\phi_{i+1}}(x_i)D_i \} + \eta_{\phi}(x_{t-1})D_{t-1} \right| \\ &\leq \sum_{i=t-1}^{s-1} | \eta_{\phi_{i+1}}(x'_i)D'_i - \eta_{\phi_{i+1}}(x_i)D_i | + |\eta_{\phi}(x_{t-1})D_{t-1}|. \end{aligned}$$

The last term is easy to estimate. From Lemma 10.3 we have a uniform estimate on the derivatives D_i ,

$$|\eta_{\phi}(x_{t-1})D_{t-1}| \leq K \cdot |\phi|.$$

Let us concentrate on the other terms. We will use the symbol K for all constants appearing in the estimates. It will only depend on b and C .

We need two other derivatives: let $t-1 \leq i \leq s-1$ then

$$E_i = (\phi_i \circ \dots \circ \phi_t)'(x_{t-1})$$

and

$$E'_i = (\phi_i \circ \cdots \circ \phi_t)'(x'_{t-1}).$$

Then

$$D_i = E_i \cdot D_{t-1} \text{ and } D'_i = E'_i \cdot \phi'(x_{t-1}) \cdot D_{t-1}$$

for $t-1 \leq i \leq s-1$. Take $t-1 \leq i \leq s-1$ and consider the corresponding term

$$\begin{aligned} & \left| \eta_{\phi_{i+1}}(x'_i) D'_i - \eta_{\phi_{i+1}}(x_i) D_i \right| = \\ & \left| \eta_{\phi_{i+1}}(x'_i) E'_i \phi'(x_{t-1}) - \eta_{\phi_{i+1}}(x_i) E_i \right| D_{t-1} = \\ & \left| (\eta_{\phi_{i+1}}(x_i) + \eta'_{\phi_{i+1}}(\theta)(x'_i - x_i)) \frac{E'_i}{E_i} E_i \phi'(x_{t-1}) - \eta_{\phi_{i+1}}(x_i) E_i \right| D_{t-1} = \\ & \left| \eta_{\phi_{i+1}}(x_i) \left\{ \frac{E'_i}{E_i} \phi'(x_{t-1}) - 1 \right\} E_i + \eta'_{\phi_{i+1}}(\theta)(x'_i - x_i) E_i \phi'(x_{t-1}) \right| D_{t-1} \leq \\ & |\phi_{i+1}| \cdot \left| \frac{E'_i}{E_i} \phi'(x_{t-1}) - 1 \right| \cdot K + C \cdot |\phi_{i+1}| \cdot K \cdot \phi'(x_{t-1}) \cdot |x'_i - x_i|. \end{aligned}$$

To continue we have to estimate $|x'_i - x_i|$. The composition $\phi_{i-1} \circ \cdots \circ \phi_t$ has by Lemma 10.3 a derivative bounded by K . So $|x'_i - x_i| \leq K \cdot |x'_{t-1} - x_{t-1}|$. By using Lemma 10.2 we can estimate this distance in terms of the norm of ϕ : This gives us a constant such that $|x'_{t-1} - x_{t-1}| \leq K \cdot |\phi|$. Hence we get a constant such that

$$C \cdot |\phi_{i+1}| \cdot K \cdot \phi'(x_{t-1}) \cdot |x'_i - x_i| \leq K \cdot |\phi_{i+1}| \cdot |\phi|.$$

It is left to estimate $\left| \frac{E'_i}{E_i} \phi'(x_{t-1}) - 1 \right|$. By Lemma 10.3

$$e^{-K|x'_{t-1} - x_{t-1}|} \cdot e^{-2|\phi|} - 1 \leq \frac{E'_i}{E_i} \phi'(x_{t-1}) - 1 \leq e^{K|x'_{t-1} - x_{t-1}|} \cdot e^{2|\phi|} - 1.$$

Hence

$$e^{-K|\phi|} \cdot e^{-2|\phi|} - 1 \leq \frac{E'_i}{E_i} \phi'(x_{t-1}) - 1 \leq e^{K|\phi|} \cdot e^{2|\phi|} - 1.$$

The factor $\left| \frac{E'_i}{E_i} \phi'(x_{t-1}) - 1 \right|$ can be estimated by a constant times the norm of ϕ . Taking all estimates together we get

$$\left| \eta_{\phi_{i+1}}(x'_i) D'_i - \eta_{\phi_{i+1}}(x_i) D_i \right| \leq K \cdot |\phi| \cdot |\phi_{i+1}|.$$

Moreover

$$|\psi_2 - \psi_1| = \sup |\eta_{\psi_2}(x) - \eta_{\psi_1}(x)| \leq K \cdot |\phi|.$$

The Sandwich Lemma is proved. □ (Lemma 10.5)

Lemma 10.6. *Let ψ_1, ψ_2 be compositions of finitely many $\phi, \phi_i \in \mathcal{D}$ which expand hyperbolic distance, $i = 1, 2, \dots, s$:*

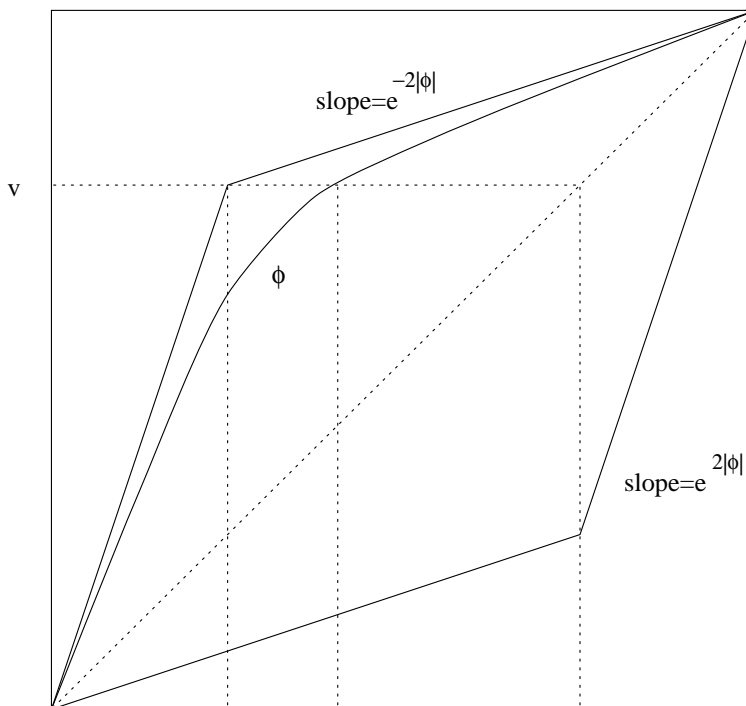
$$\psi_1 = \phi_s \circ \dots \circ \phi_t \circ \phi_{t-1} \circ \dots \phi_2 \circ \phi_1$$

and

$$\psi_2 = \phi_s \circ \dots \circ \phi_t \circ \phi \circ \phi_{t-1} \circ \dots \phi_2 \circ \phi_1.$$

There is a universal constant K such that the hyperbolic distance between $\psi_1^{-1}(v)$ and $\psi_2^{-1}(v)$, $v \in (-1, 1)$ is bounded by $K|\phi|$.

Proof. The Lemma follows immediately once it is proved that the hyperbolic distance between $\phi^{-1}(v)$ and v is proven to be bounded by $K|\phi|$. From Lemma 10.3 we get $e^{-2|\phi|} \leq \phi'(x) \leq e^{2|\phi|}$. Then we can use the figure 9. to estimate the maximal distance between v and $\phi^{-1}(v)$. A computation finishes the proof of the Lemma. \square (Lemma 10.6)



max distance

Figure 9.

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