ERGODIC PROPERTIES OF ERDÖS MEASURE, THE ENTROPY OF THE GOLDENSHIFT, AND RELATED PROBLEMS*

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November 12, 1996

To the memory of Paul Erdös

Abstract. We define a two-sided analog of Erdös measure on the space of two-sided expansions with respect to the powers of the golden ratio, or, equivalently, the Erdös measure on the 2-torus. We construct the transformation (goldenshift) preserving both Erdös and Lebesgue measures on $\mathbb{T}^2$ which is the induced automorphism with respect to the ordinary shift (or the corresponding Fibonacci toral automorphism) and proves to be Bernoulli with respect to both measures in question. This provides a direct way to obtain formulas for the entropy dimension of the Erdös measure on the interval, its entropy in the sense of Garsia-Alexander-Zagier and some other results. Besides, we study central measures on the Fibonacci graph, the dynamics of expansions and related questions.

0. Introduction

Among numerous connections between ergodic theory and metric theory of numbers, the questions related to algebraic irrationalities, expansions associated with them and ergodic properties of arising dynamical systems, are of a special interest. The simplest case, i.e. the golden ratio, the Fibonacci automorphism etc., serves as a deep source of problems and conjectures until now.

In 1939 P. Erdös [Er] proved in particular the singularity of the measure on the segment which is defined as the one corresponding to the distribution of the random variable $\sum_1^\infty \varepsilon_k \lambda^{-k}$ with λ being the larger golden ratio and independent ε_k taking the values 0 and 1 (or ± 1) with probabilities $\frac{1}{2}$ each. We think it is natural to call this measure the *Erdös measure*. This work gave rise to extensive publications and numerous generalizations (see, e.g., [AlZa] and references therein). Nevertheless, little attention was paid to dynamical properties of this natural measure. *The aim of this paper is to begin studying dynamical properties of Erdös measure and its two-sided extension.* We

(1) define two-sided generalization of Erdös measure (Section 1);
(2) introduce a special automorphism ("goldenshift") which preserves Erdös measure and which is a Bernoulli automorphism with a natural generator with respect to Erdös and Lebesgue measures (Section 2);

*Revised version of the preprint N 367 of Laboratoire de Probabilité de l'Université Paris VI

Supported in part by the INTAS grant 93-0570. The second author was partially supported by the Institute of Mathematical Science of the SUNYSB during his stay in Stony Brook

Published in modified form:
Monatsh. Math. **126** (1998), 215–261.

Stony Brook IMS Preprint #1996/14
December 1996

(3) find the entropy of this automorphism and prove that it coincides with the entropy considered in [AlZa] (Section 3);
(4) discuss the connection with some properties of the Fibonacci graph, its central measures and the adic transformation on it (see Appendix A);
(5) define some new kind of expansions corresponding to the goldenshift (see Appendix B).

We will describe all this in more detail below.

Several years ago certain connections between symbolic dynamics of toral automorphisms and arithmetic expansions associated with their eigenvalues were established. The first step in this direction was also related to the golden ratio (see [Ver5]) and led to a natural description of Markov partition in terms of the arithmetics of the 2-torus and homoclinic points of the Fibonacci automorphism. The main idea was to consider the natural extension of the shift in the sense of ergodic theory and the adic transformation on the space of one-sided arithmetic expansions and in identifying the set of two-sided expansions with the 2-torus.

In the present paper we use the same idea for a detailed study of the Erdös measure. Namely, we define the *"two-sided" Erdös measure* as a measure on the space of expansions infinite to both sides (= a *measure on the 2-torus*) and study the properties of the ordinary shift and the *goldenshift* as a transformation of the space of expansions introduced by means of the notion of *block*. The goldenshift turns out to preserve both Lebesgue and Erdös measures, both being Bernoulli in the natural sense with respect to the goldenshift; this is one of the main results of the paper (Theorem 2.7). By the way, this immediately yields a proof the Erdös theorem on the singularity of Erdös measure. Moreover, the two-sided goldenshift is an induced automorphism for the Fibonacci automorphism of the torus. Other important consequences of our approach follow from the fact that the entropy of the goldenshift is directly related to the entropy of Erdös measure in the sense of Garsia-Alexander-Zagier, i.e. to the entropy of the random walk with equal transition measures on the Fibonacci graph (Theorem 3.1).

In [AlZa] it was attempted to compute the entropy of Erdös measure as the infinite convolution of discrete measures, which was in fact introduced by A. Garsia [Ga] in more general situation. Note that it proved to be the entropy of a random walk on the Fibonacci graph. In a recent work [LePo] the authors compute the dimension of the Erdös measure on the interval in the sense of L.-S. Young [Y] and relate a certain two-dimensional dynamics to it.

Finally, making use of a version of Shannon's theorem for random walks (see [KaVe]) yields the value of the dimension of the Erdös measure in the sense of Young (Theorem 3.4).

Thus, the dynamical viewpoint for arithmetic expansions and for measures related to them, provides new information and an essential simplification of computations of invariants invloved. One may expect that methods of this paper are applicable to more general algebraic irrationalities and also to some nonstationary problems.

The contents of the present paper is as follows. In Section 1 we present auxiliary notions (canonical expansions and others) and give main definitions (Erdös measure on the interval and the 2-torus, normalization, the Markov measure corresponding to Lebesgue measure etc.). Besides, we deduce some preliminary facts on the one-sided and two-sided Erdös measure. In Section 2 we study the combinatorics of blocks in terms of canonical expansions, introduce the notion of the goldenshift (both one-sided and two-sided) and prove its Bernoulli property with respect to Lebesgue and Erdös measures. Section 3 contains main results on the entropy and dimension of the Erdös measure and relationship to the random walk on the Fibonacci graph.

In appendices we consider some related problems. Namely, in Appendix A the combinatorial and algebraic theory of the Fibonacci graph is presented. In particular, we describe the ergodic central measures on this graph and the action of the *adic* transformation which is defined as the transfer to the immediate successor in the sense of the natural lexicographic order (in our case it is just the next expansion of a given real in the sense of the natural ordering of the expansions). We study the metric type of the adic transformation with respect to the ergodic central measures. In Appendix B we consider arithmetic block expansions of almost all points of the interval. The interest to them is caused by the fact that the "digits" of the block expansions are independent with respect both to Erdös and Lebesgue measures. Note that there are some pecularities influenced by the difference between the one-sided and two-sided shifts. For instance, the one-sided Erdös measure is only quasi-invariant under the one-sided shift, while the two-sided measure in shift-invariant. In Appendix C the densities of the Erdös measure with respect to the shift and to the rotation by the golden ratio are computed by means of blocks. Finally, in Appendix D another proof of Alexander-Zagier's formula for the entropy is given. It is worthwhile because of its connection with the geometry of the Fibonacci graph.

1. Erdös measure on the interval and on the 2-torus.

1.1. Canonical expansions. Let $\lambda = \frac{\sqrt{5}+1}{2}$, let $X = \{(\varepsilon_1\varepsilon_2\dots) \in \prod_1^\infty \{0,1\} : \varepsilon_i\varepsilon_{i+1} = 0,\ i \ge 1\}$, i.e. the *Markov compactum* with the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $L : X \to [0,1]$ be the mapping acting by the formula

$$L(\varepsilon_1\varepsilon_2\dots) := \sum_{k=1}^{\infty} \varepsilon_k \lambda^{-k}. \tag{1.1}$$

It is well-known that L is one-to-one, except for a countable number of points. The inverse mapping L^{-1} is specified with the help of the *greedy algorithm*. Namely, let $\tau x = \{\lambda x\}$, and

$$\varepsilon_k = [\lambda \tau^{k-1} x], \quad k \ge 1.$$

We call the constructed sequence $(\varepsilon_1(x)\varepsilon_2(x)\dots)$ the *canonical expansion* of x. This expansion is easily shown to be the unique one lying in X, except for a countable number of x. For those, from two possible expansions we choose the one whose tail is 0^∞. Note that usually the canonical expansions are called *β-expansions* (for $\beta = \lambda$). They were introduced in [Re] and thoroughly studied in [Pa].

1.2. The Markov measure on X. The transformation $\tau:(0,1)\to(0,1)$ is transferred by L to the Markov compactum X and acts as the one-sided shift (we will denote it by the same letter):

$$\tau(\varepsilon_1\varepsilon_2\varepsilon_3\dots):=\varepsilon_2\varepsilon_3\dots$$

The transformation $\tau:(0,1)\to(0,1)$ is well-studied (see, e.g., [Pa]), and the measure m' with the density

$$\rho(x)=\begin{cases}\lambda^2/\sqrt{5}, & 0<x\le\lambda^{-1}\\ \lambda/\sqrt{5}, & \lambda^{-1}<x\le 1\end{cases}$$

is known to be invariant under τ. The corresponding Markov measure $L^{-1}m'$ on X is the one with the stationary initial distribution $\begin{pmatrix}\lambda/\sqrt{5}\\ \lambda^{-1}/\sqrt{5}\end{pmatrix}$ and the transition probability matrix $\begin{pmatrix}\lambda^{-1} & \lambda^{-2}\\ 1 & 0\end{pmatrix}$. The L-preimage of the Lebesgue measure on X differs from this stationary Markov measure only by its initial distribution $\begin{pmatrix}\lambda^{-1}\\ \lambda^{-2}\end{pmatrix}$. Note that the adic transformation on X (for definition see [Ver2] or Appendix A) with the alternating ordering on the paths preserves the latter measure, as it turns into the rotation by the angle λ^{-1} under the mapping L (for more details see [VerSi]).

1.3. Erdös measure and Erdös theorem. Let us define the *Erdös* measure. By definition, the continuous Erdös measure μ on the unit interval is the infinite convolution $\vartheta_1*\vartheta_2*\dots$, where $\operatorname{supp}\vartheta_n=\{0,\lambda^{-n-1}\}$, and $\vartheta_n(0)=\vartheta_n(\lambda^{-n-1})=\frac12$ (see [Er]).

We are going to specify this measure more explicitly. Let L_0 denote the extension of L to the full compactum $\prod_1^\infty\{0,1\}$ and let $\pi(x_1x_2\dots):=\lambda^{-1}L_0(x_1x_2\dots)$ be the projection of $\prod_1^\infty\{0,1\}$ onto $[0,1]$ (clearly, if $(x_1x_2\dots)\in\prod_1^\infty\{0,1\}$, then $0\le\pi(x_1x_2\dots)\le1$). Let p denote the product measure with the equal multipliers $(1/2,1/2)$ on the compactum $\prod_1^\infty\{0,1\}$. Then it is easy to see that $\mu=\pi(p)$.

We are also interested in the specification of the Erdös measure on the Markov compactum X. Of course, it is just $L^{-1}\mu$, however it is worthwhile to introduce a direct mapping. Namely, we recall that the *normalization* of an arbitrary 0-1 sequence $(x_1x_2\dots)$ is, by definiton, the sequence $(\varepsilon_0\varepsilon_1\varepsilon_2\dots)\in X$ such that $x=\sum_1^\infty x_k\lambda^{-k}=\sum_0^\infty\varepsilon_k\lambda^{-k}$, where the digits $\varepsilon_0,\varepsilon_1,\dots$ are obtained by the greedy algorithm for x with regard to the fact that $x\in[0,\lambda]$ (i.e. we treat ε_0 as the integral part in the canonical expansion). If $\varepsilon_0=0$, we will write simply $(\varepsilon_1\varepsilon_2\dots)$, and it will be the case we will be interested in below. Let $\mathfrak{n}:\prod_1^\infty\{0,1\}\to X$ denote the mapping of normalization. Then $\mu=\mathfrak{n}(p)$ on the Markov compactum X.

Remark. The mapping $\mathfrak{n}$ can be specified directly on $\prod_1^\infty\{0,1\}$, i.e. without addressing the interval. The simplest algorithm is as follows. Given a 0-1 sequence $x_1x_2\dots$, we start from the zero coordinate $x_0:=0$ and look for the first occurence of the triple 011, after which we replace it by 100. The next step is the same, i.e. we return to the zero coordinate and start from there until we meet again 011, etc. It is easy to see that the process leads to stabilization of a normalized sequence. Note that this algorithm is rather rough, as it is known that there exists a finite automaton carring out the process of normalization faster (see, e.g., [Fr]).

Consider the Erdös measure on the unit interval in more detail. We first prove that μ is quasi-invariant with respect to the transformation τ, and present the corresponding invariant measure ν equivalent to μ. To do this, we begin with the self-similar relation for μ.

Lemma 1.1. *The Erdös measure μ on the interval $[0,1]$ satisfies the following self-similar relation:*

$$\mu E = \begin{cases} \frac{1}{2}\mu(\lambda E), & E \subset [0,\lambda^{-2}) \\ \frac{1}{2}(\mu(\lambda E)+\mu(\lambda E-\lambda^{-1})), & E \subset [\lambda^{-2},\lambda^{-1}) \\ \frac{1}{2}\mu(\lambda E-\lambda^{-1}), & E \subset [\lambda^{-1},1] \end{cases}$$

for any Borel set E.

Proof. Let $F_1 = 1, F_2 = 2, \dots$ be the sequence of Fibonacci numbers. Let $f_n(k)$ denote the number of representations of a nonnegative integer k as a sum of not more than n first Fibonacci numbers. We first show that for $n \geq 3$,

$$f_n(k) = \begin{cases} f_{n-1}(k), & 0 \leq k \leq F_n - 1 \\ f_{n-1}(k) + f_{n-1}(k-F_n), & F_n \leq k \leq F_{n+1} - 2 \\ f_{n-1}(k-F_n), & F_{n+1} - 1 \leq k \leq F_{n+2} - 2. \end{cases} \tag{1.2}$$

To prove this, we represent $f_n(k)$ as $f_n(k) = f'_n(k) + f''_n(k)$ for each $k < F_{n+2} - 1$, where $f'_n(k)$ is the number of representations with $\varepsilon_n = 0$, and $f''_n(k)$ is the one with $\varepsilon_n = 1$. Obviously, if $k \leq F_n - 1$, then $k = \sum_1^n \varepsilon_j F_j = \sum_1^{n-1} \varepsilon_j F_j$, whence $f_n(k) = f'_n(k)$. If $F_{n+1} - 1 \leq k \leq F_{n+2} - 2$, then $f_n(k) = f''_n(k)$. In the case $F_n \leq k \leq F_{n+1} - 2$, obviously, $f'_n(k) > 0$, $f''_n(k) > 0$. It remains to note only that $f'_n(k) = f_{n-1}(k)$, and $f''_n(k) = f_{n-1}(k - F_n)$.

Now from (1.2), and from the definition of the Erdös measure it follows that if, say, $E \subset [0, \lambda^{-2})$, then

$$\mu E = \lim_{n\to\infty} \sum_{k:\frac{k}{F_{n+2}}\in E} \frac{f_n(k)}{2^n} = \frac{1}{2}\lim_{n\to\infty} \sum_{k:\frac{k}{F_{n+1}}\in \lambda E} \frac{f_{n-1}(k)}{2^{n-1}} = \frac{1}{2}\mu(\lambda E).$$

The other cases are studied in the same way. □

Remark. The Erdös measure μ (as a Borel measure) is completely determined by the above self-similar relation. Indeed, by induction one can determine its values for any interval (a,b) with $a, b \in \mathbb{Z} + \lambda\mathbb{Z} \cap [0,1]$.

Corollary 1.2. $\mu(0,\lambda^{-2}) = \mu(\lambda^{-2},\lambda^{-1}) = \mu(\lambda^{-1},1) = \frac{1}{3}$.

Let now the Borel measure ν be defined by the formula

$$\nu E = \begin{cases} \frac{2}{3}\mu E + \frac{1}{3}\mu(E+\lambda^{-2}) + \frac{1}{6}\mu(E+\lambda^{-1}), & E \subset [0,\lambda^{-2}) \\ \frac{2}{3}\mu E + \frac{1}{3}\mu(E+\lambda^{-2}), & E \subset [\lambda^{-2},\lambda^{-1}) \\ \frac{1}{2}\mu E + \frac{1}{3}\mu(E-\lambda^{-1}), & E \subset [\lambda^{-1},1]. \end{cases}$$

The direct computation shows that $\nu(0,\lambda^{-2}) = \frac{4}{9}$, $\nu(\lambda^{-2},\lambda^{-1}) = \nu(\lambda^{-1},1) = \frac{5}{18}$, hence ν is a well defined probabilistic measure.

Proposition 1.3. *The Erdös measure μ is quasi-invariant with respect to the transformation $\tau x = \{\lambda x\}$, and ν is τ-invariant and equivalent to μ.*

Proof. The τ-invariance of ν is checked directly. Let, say, $E \subset (0, \lambda^{-2})$. Then $\tau^{-1}E = \lambda^{-1}E \cup (\lambda^{-1}E + \lambda^{-1})$. Hence

$$
\begin{aligned}
\nu(\tau^{-1}E) &= \nu(\lambda^{-1}E) + \nu(\lambda^{-1}E + \lambda^{-1}) \\
&= \frac{2}{3}\mu(\lambda^{-1}E) + \frac{1}{3}\mu(\lambda^{-1}E + \lambda^{-2}) + \frac{1}{6}\mu(\lambda^{-1}E + \lambda^{-1}) + \frac{1}{2}\mu(\lambda^{-1}E + \lambda^{-1}) \\
&+ \frac{1}{3}\mu(\lambda^{-1}E) \\
&= \frac{1}{2}\mu E + \frac{1}{3}\mu(E + \lambda^{-2}) + \frac{1}{6}\mu(E + \lambda^{-1}) + \frac{1}{6}\mu E \quad \text{(by Lemma 1.1)} \\
&= \frac{2}{3}\mu E + \frac{1}{3}\mu(E + \lambda^{-2}) + \frac{1}{6}\mu(E + \lambda^{-1}) \\
&= \nu E.
\end{aligned}
$$

The cases $E \subset (\lambda^{-2}, \lambda^{-1})$ and $E \subset (\lambda^{-1}, 1)$ are studied in the same way.

Thus, ν is τ-invariant and clearly $\mu \prec \nu$. Hence, as it is well-known, there exists a τ-invariant measure equivalent to μ. Therefore, μ is quasi-invariant under τ, and since

$$
\mu(\tau^{-1}E) = \frac{1}{2}\big(\mu E + \mu(E + \lambda^{-2} \mod 1)\big)
$$

(by Lemma 1.1), μ is also quasi-invariant under the rotations of the circle by the angles λ^{-1} and λ^{-2}, which implies, in view of the above definition of ν, that $\mu \approx \nu$. The proof is complete.

Remark. It is not hard to show that similarly to the measure m' described above, $\nu = \lim_n \tau^n \mu$.

Corollary 1.4. (Erdös theorem, see [Er]) *The Erdös measure μ is singular with respect to Lebesgue measure m.*

Proof. We have just shown that $\tau\nu = \nu$, and above it was noted that $\tau m' = m'$, hence, by the ergodic theorem, $\nu \perp m'$. Since $\mu \approx \nu,\ m \approx m'$, we are done.

Remark 1. Note that the initial proof of Erdös followed traditions of those times and was based on the study of the Fourier transform of μ.

Remark 2. There was a gap in the proof of this statement in the previous joint paper by the authors [VerSi]. Namely, when deducing the Erdös theorem from the ergodic theorem for τ, we used by mistake the measure μ instead of ν. Another proof is given in Corollary 2.8 (see below).

Remark 3. The problem of computing the densities $\frac{d\nu}{d\mu}$ and $\frac{d(\tau\mu)}{d\mu}$ will be solved in Appendix C.

Proposition 1.5. *The Erdös measure is quasi-invariant under the rotation R by the angle λ^{-1}, and*

$$\frac{d(\tau\mu)}{d\mu}(x) = \frac{1}{2}\left(1 + p^{-1}(R^{-1}x)\right),$$

where $p = \frac{d(R\mu)}{d\mu}$.

1.4. Two-sided Erdös measure. Consider now the two-sided Markov compactum $\widetilde{X} = \{(\varepsilon_k)_{-\infty}^{\infty} : \varepsilon_k \in \{0,1\},\ \varepsilon_k\varepsilon_{k+1} = 0,\ k \in \mathbb{Z}\}$. Any $x \in \mathbb{R}_+$ can be expanded as

$$x = \sum_{k=-\infty}^{\infty} \varepsilon_k \lambda^{-k} \tag{1.3}$$

with $(\varepsilon_k)_{-\infty}^{\infty} \in \widetilde{X}$ and $\varepsilon_i \equiv 0, i \le i_0(x)$, i.e. with a sequence finite to the left. We complete this expansion in the following way. Let $x \mapsto (\{x\}, \{\lambda^{-1}x\}) \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Thus, (1.3) is in fact a one-to-one mod 0 correspondence between the subset of all sequences in $\widetilde{X}$ finite to the left, and a certain *irrational winding* of the 2-torus. Completing these sets in their natural topologies (respectively product and Euclidean), we obtain a one-to-one mod 0 correspondence between $\widetilde{X}$ and $\mathbb{T}^2$. So, the mapping $\widetilde{L} : \widetilde{X} :\to \mathbb{T}^2$ acting by the formula

$$\widetilde{L}(\ldots\varepsilon_0\varepsilon_1\varepsilon_2\ldots) = \lim_{n\to\infty}\left(\sum_{k=-n}^{\infty}\varepsilon_k\lambda^{-k}, \sum_{k=-n}^{\infty}\varepsilon_k\lambda^{-k-1}\right) \mod 1$$

is well defined, a.e. one-to-one and $\widetilde{L}\widetilde{\tau} = \tau_\Phi\widetilde{L}$, where $\widetilde{\tau}$ is the two-sided shift on $\widetilde{X}$ (i.e. $(\widetilde{\tau}x)_k = x_{k+1}$) and τ_Φ is the *Fibonacci automorphism*, i.e. the toral automorphism with the matrix $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$. The mapping $\widetilde{\tau}$ was considered for the first time in [Ber] for another reason. Independently two-sided expansions were defined in [Ver3] in the framework of the study of adic transformations and a natural coding of hyperbolic systems (the author did not know about the work [Ber]).

Note that under this isomorphism the *finite* sequences in $\widetilde{X}$ (i.e. the ones with $\varepsilon_i \equiv 0,\ |i| \ge i_0$ for some $i_0 \in \mathbb{Z}$) together with the *cofinite* sequences (the ones finite to the left whose tail is of the form $(01)^\infty$) turn into the set of homoclinic points of the Fibonacci automorphism τ_Φ (see [Ver3]).

Following the one-sided framework, we are going to define the transformation of two-sided normalization (as far as we know, this has not been done so far). Let $\widetilde{L}_0 : \prod_{-\infty}^{\infty}\{0,1\} \to \mathbb{T}^2$ be the extension of $\widetilde{L}$, and let $\widetilde{\mathfrak{n}} := \widetilde{L}^{-1}\widetilde{L}_0 : \prod_{-\infty}^{\infty}\{0,1\} \to \widetilde{X}$ be, by definition, the *two-sided normalization*.

Remark. The two-sided normalization can be also specified explicitly. Namely, let for simplicity, $(\varepsilon_k) \in \widetilde{X}$ be such that $\varepsilon_k = \varepsilon_{k+1} = 0$ for some $k \in \mathbb{Z}$. We split the sequence $(\varepsilon_k)_{-\infty}^{\infty}$ into $(0\varepsilon_{k+2}\varepsilon_{k+3}\ldots)$ and $(0\varepsilon_{k-1}\varepsilon_{k-2}\ldots)$, both belonging to X. The two-sided normalization acts independently on these pieces, and for the first of them it is the ordinary

$\mathfrak{n}$, while for $(0\varepsilon_{k-1}\varepsilon_{k-2}\dots)$ we use the "mirror symmetric" one-sided normalization, i.e. the one acting by the rule "110"→ "001". Formally it is obtained in the same way as $\mathfrak{n}$ but for some other expansion of the points of $[0,1]$ considered in [VerSi]:

$$x = \lambda^{-1} + \sum_{j=1}^{\infty} \varepsilon_j (-1)^{j+1} \lambda^{-j}.$$

Definition. The two-sided Erdös measure $\widetilde{\nu}$ on the Markov compactum $\widetilde{X}$ is, by definition, the image of the measure $\widetilde{p}$, which is the product of infinite factors $(1/2, 1/2)$ on the full compactum $\prod_{-\infty}^{\infty}\{0,1\}$ under the mapping $\widetilde{\mathfrak{n}}$.

Let, as above, $\widetilde{\tau}$ denote the two-sided shift on $\widetilde{X}$, and let $\widetilde{\sigma}$ denote the two-sided shift on $\prod_{-\infty}^{\infty}\{0,1\}$, i.e. $(\widetilde{\sigma}x)_k = x_{k+1}$.

Propostion 1.6. *The two-sided Erdös measure $\widetilde{\nu}$ is invariant under the two-sided shift, i.e. $\widetilde{\tau}\widetilde{\nu} = \widetilde{\nu}$ (cf. the one-sided case, where it does not take place).*

Proof. From the above specification of the mapping $\widetilde{\mathfrak{n}}$ it follows that

$$\widetilde{\mathfrak{n}}\widetilde{\sigma} = \widetilde{\tau}\widetilde{\mathfrak{n}}, \tag{1.4}$$

hence $\widetilde{\nu}(\widetilde{\tau}^{-1}E) = \widetilde{p}(\widetilde{\mathfrak{n}}^{-1}\widetilde{\tau}^{-1}E) = \widetilde{p}(\widetilde{\sigma}^{-1}\widetilde{\mathfrak{n}}^{-1}E) = (\widetilde{\sigma}\widetilde{p})(\widetilde{\mathfrak{n}}^{-1}E) = \widetilde{p}(\widetilde{\mathfrak{n}}^{-1}E) = \widetilde{\nu}(E)$ for any Borel set $E \subset \widetilde{X}$.

Proposition 1.7. *For any cylinder $C = (\varepsilon_1 = i_1, \dots, \varepsilon_r = i_r) \subset \widetilde{X}$, its measure $\widetilde{\nu}$ is strictly positive.*

Proof. It follows from the direct specification of the two-sided normalization described above that for the cylinder $C' = (\varepsilon_0 = 0, \varepsilon_1 = i_1, \dots, \varepsilon_r = i_r, \varepsilon_{r+1} = 0, \varepsilon_{r+2} = 0) \subset \prod_{-\infty}^{\infty}\{0,1\}$, we have $\widetilde{\mathfrak{n}}^{-1}(C) \supset C'$, whence, by definition of the Erdös measure, $\widetilde{\nu}(C) \geq 2^{-r-3}$.

Corollary 1.8. *The shift $\widetilde{\tau}$ is ergodic with respect to the Erdös measure.*

Let, by definition, the two-dimensional Erdös measure on the 2-torus be defined as $\widetilde{L}(\widetilde{\nu})$. Note that the $\widetilde{L}$-preimage of the two-dimensional Lebesgue measure on $\widetilde{X}$ is just the two-sided stationary Markov measure $\widetilde{m}$ with the initial distribution and the transition matrix described in item 1.2.

Theorem 1.9. *The two-sided Erdös measure on $\mathbb{T}^2$ is singular with respect to Lebesgue measure and is positive on all open subsets of $\mathbb{T}^2$.*

Proof. An application of the ergodic theorem to the transformation $\widetilde{\tau}$ and the measures $\widetilde{m}$ and $\widetilde{\nu}$ and of Proposition 1.7.

Remark. It is interesting to prove that this measure is a Gibbs measure for a certain natural potential.

Proposition 1.10. *The two-sided shift $\widetilde{\tau}$ on the Markov compactum $\widetilde{X}$ with the two-sided Erdös measure is a Bernoulli automorphism.*

Proof. We note that this dynamical system is a factor of the Bernoulli shift $\widetilde{\sigma} : \prod_{-\infty}^{\infty}\{0,1\} \to \prod_{-\infty}^{\infty}\{0,1\}$ with the product measure $(1/2,1/2)$ (see relation (1.4)) and apply the celebrated theorem due to D. Ornstein [Or] on the Bernoullicity of all Bernoulli factors.

Remark. We would like to emphasize that the measure $\widetilde{\nu}$ is not Markov on the compactum $\widetilde{X}$.

In Section 2 we will show that the dynamical system in question is the natural extension of (X,ν,τ), which explains the choice of the notation for the two-sided Erdös measure (see Theorem 2.11 below).

Remark. It is appropriate, following the well-known framework of the baker's transformation which serves as a model for the full two-sided shift on $\prod_{-\infty}^{\infty}\{0,1\}$, to represent the two-sided shift on $\widetilde{X}$ as the *Fibonacci-baker's* transformation.

Namely, we split a sequence $(\varepsilon_k) \in \widetilde{X}$ into the two one-sided sequences, i.e. into $(\varepsilon_1\varepsilon_2\dots) \in X$ and $(\varepsilon_0\varepsilon_{-1}\dots) \in X$ with regard to the fact that $\varepsilon_0\varepsilon_{-1} = 0$. This last condition leads to the space $Y = ([0,1]\times[0,1]) \setminus ([\lambda^{-1},1]\times[\lambda^{-1},1])$.

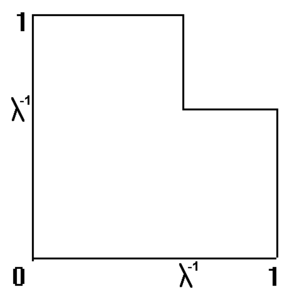

Fig. 1. The natural domain for the Fibonacci-baker's transformation

Thus, the shift $\widetilde{\tau}$ on the two-sided Markov compactum $\widetilde{X}$ is isomorphic to the transformation F on the space Y with

$$F(x,y) = \begin{cases} \big(\{\lambda x\}, \lambda^{-1}y\big), & x \in [0,\lambda^{-1}] \\ \big(\{\lambda x\}, \lambda^{-1}y + \lambda^{-1}\big), & x \in (\lambda^{-1},1]. \end{cases}$$

We call F the *Fibonacci-baker's* transformation on the set Y (see Fig. 1).

2. Symbolic dynamics of expansions

In this section we will study in detail the combinatorics of all possible representations of a real x of the form (1.1) with $\varepsilon_k \in \{0,1\}$ for all k.

2.1. Blocks. Let us give an important technical definition.

Definition. A finite 0-1 sequence without pairs of adjacent unities starting from 1 and ending by an even number of zeroes, will be called the *block*, if it does not contain any piece "$1(00)^l1$" with $l \geq 1$.

Let us make some remarks. Note first that a block has an odd length; the simplest example of a block is "100". Next, there are exactly 2^{n-1} blocks of length $2n+1$. This assertion follows from the fact that a block B can be represented in the form $1(00)^{a_1}(01)^{a_2}(00)^{a_3}\dots(01)^{a_{t-1}}(00)^{a_t}$ for t odd or $1(01)^{a_1}(00)^{a_2}\dots(01)^{a_{t-1}}(00)^{a_t}$ for t even. Thus, any block B is naturally parametrized by means of a finite sequence of positive integers $a_1,\dots,a_t$, and we will write $B=B(a_1,\dots,a_t)$.

Let $X_0:=[\lambda^{-1},1)$, i.e. the interval corresponding to the cylinder $(\varepsilon_1=1)\subset X$.

Definition. Let x lie in the interval X_0, and let the canonical expansion of x have infinitely many pieces "$1(00)^l1$" with $l\ge 1$. We call such a point x *regular*.

Remark. Almost every point x in X_0 with respect to the Lebesgue measure is regular.

Now we split the canonical expansion of a regular x into blocks as follows. Since $x\in X_0$, its canonical expansion starts from 1. It is just the beginning of the first block $B_1=B_1(x)$. The first block ends, when an even number of zeroes followed by 1 appears for the first time. This unity begins the second block $B_2=B_2(x)$ of the canonical expansion of x, etc. We defined thus a one-to-one mapping Ψ acting from the set of all regular points of $(\lambda^{-1},1)$ into the space of block sequences.

Definition. The sequence $(B_1(x),B_2(x),\dots)=\Psi(x)$ will be called the *block expansion* of a regular x.

Let $\mathfrak{B}$ denote the set of all blocks; by the above, any block except "100" can be uniquely specified with the help of a rational number r and one extra bit of information showing, which of two types of the continued fraction expansion of r is chosen (with $a_t=1$ or with $a_t\ge 2$). So, $\mathfrak{B}$ is naturally isomorphic to $((\mathbb{Q}\cap(0,1))\times\mathbb{Z}_2)\cup\{1\}$. Let next $\mathfrak{X}:=\prod_1^\infty\mathfrak{B}$. Thus, Ψ is a one-to-one correspondence mod 0 between the interval $(\lambda^{-1},1)$ and the noncompact space $\mathfrak{X}$ of block sequences. We call a cylinder $\{B_1=B_1',\dots,B_k=B_k'\}\subset\mathfrak{X}$ the *multiblock*.

Remark. It is necessary to distinguish multiblocks and cylinders in X. For instance, the cylinders "100" and "10000" in X intersect, while treated as blocks, they correspond to the intervals $(\lambda^{-1}+\lambda^{-4},\lambda^{-1}+\lambda^{-3})$ and $(\lambda^{-1}+\lambda^{-6},\lambda^{-1}+\lambda^{-5})$ respectively. To avoid confusion, we introduce the following definition.

Definition. We will say that a multiblock is *closed*, if it coincides with a multiblock as a cylinder in X in the above sense with extra "1" at the end. For example, the shortest closed block is "1001".

Now we can consider the mapping Ψ^{-1} acting from the set of all blocks into $(\lambda^{-1},1)$ from the viewpoint of the images of the multiblock cylinders. Namely, for any multiblock $B_1'B_2'\dots B_k'$ we construct the closed multiblock $B_1'\dots B_k'1$ and project it (as the cylinder in X) onto X_0 by the mapping (1.1). It will be just $\Psi^{-1}(\{B_1=B_1',\dots,B_k=B_k'\})$.

In the same way we define the two-sided space of block sequences $\widetilde{\mathfrak{X}}=\prod_{-\infty}^\infty\mathfrak{B}$ implying that B_1 begins with the first coordinate of $\widetilde{X}$ (i.e. with $\varepsilon_1=1$). The corresponding subset of $\widetilde{X}$ (analogous to $(\varepsilon_1=1)\subset X$ for the one-sided case) is $\widetilde{X}_0:=\bigcup_{k=1}^\infty(x_{-2k}=1,\ x_{-2k+1}=\dots=x_0=0,\ x_1=1)$.

2.2. The cardinality of a 0-1 sequence and its properties. We are going to define an equivalence relation on the set of all finite 0-1 sequnces.

Definition. Two 0-1 sequences (finite or not) $(x_1x_2\dots)$ and $(x_1'x_2'\dots)$ are called *equivalent* if $\sum_k x_k\lambda^{-k} = \sum_k x_k'\lambda^{-k}$ (or, equivalently, if their normalizations coincide — see Section 1). Let for a finite 0-1 sequence x, $\mathcal{E}(x)$ denote the set of all 0-1 sequences equivalent to x. This set is always finite, and let $f(x) := \#\mathcal{E}(x)$. We call f the *cardinality* of a finite sequence (or the cardinality of an equivalence class).

Note that this function (of positive integers) was considered in [Ca], [AlZa] and recently in [Si] and [Pu].

Below assertions answer the question about the cardinality of a block and explain the purpose of the introduction of blocks as natural structural units in this theory.

Lemma 2.1. *Let $p/q = [a_1,\dots,a_t]$ be a finite continued fraction. Then $f(B(a_1,\dots,a_t)) = p+q$.*

Proof. Note first that $f(100) = 2 = p+q$. The desired relations for the blocks 10000 and 10100 are direct inspection. Next, let $\varkappa_k = \varkappa_k(a_1,\dots,a_k) = f(B(a_1,\dots,a_k))$. We need to show that similarly to the numerators and denominators of the convergents, $\varkappa_t = a_t\varkappa_{t-1} + \varkappa_{t-2}$, whence the required assertion will follow.

Let, say, $B = 1(00)^{a_1}\dots(00)^{a_{t-2}}(01)^{a_{t-1}}(00)^{a_t}$. We will present all 0-1 sequences equivalent to B but not ending by $(00)^{a_t}$ and see that their number is $a_t\varkappa_{t-1}$. Namely,

$$
\begin{aligned}
B^{(1)} &= 1(00)^{a_1}\dots(00)^{a_{t-2}}(01)^{a_{t-1}-1}0011(00)^{a_t-1},\\
B^{(2)} &= 1(00)^{a_1}\dots(00)^{a_{t-2}}(01)^{a_{t-1}-1}0(01)^{2}1(00)^{a_t-2},\\
&\dots\\
B^{(a_t)} &= 1(00)^{a_1}\dots(00)^{a_{t-2}}(01)^{a_{t-1}-1}0(01)^{a_t}1.
\end{aligned}
$$

Besides, the number of 0-1 sequences equivalent to B and ending by $(00)^{a_t}$, is just $\varkappa_{t-2}$, as they in fact should end by $(01)^{a_{t-1}}(00)^{a_t}$. All $B^{(j)}$ contain $1(00)^{a_1}\dots(00)^{a_{t-2}}(01)^{a_{t-1}-1}00$, hence $\varkappa_t = a_t\varkappa_{t-1} + \varkappa_{t-2}$, as $[a_1,\dots,a_{t-1}-1,1] = [a_1,\dots,a_{t-1}]$. $\square$

Remark. For any rational $r \in (0,1)$ there are exactly two blocks $B = B(a_1,\dots,a_t)$ and $B' = B'(a_1',\dots,a_t')$ with $r = [a_1,\dots,a_t] = [a_1',\dots,a_t']$, and the unique block "100" corresponds to $r = 1$.

Lemma 2.2. $f(B_1\dots B_k) = \prod_1^k f(B_i)$, *i.e. the cardinality is blockwise multiplicative.*

Proof. We need to show in fact that $\mathcal{E}(B_1\dots B_k) = \mathcal{E}(B_1)\dots\mathcal{E}(B_k)$. Let us restrict ourselves by the case $k=2$ (the general one is studied in the same way). We will see that there is no sequence in $\mathcal{E}(B_1B_2)$ containing a triple 011 or 100 which cross the "border" between the first $|B_1|$ digits and the last $|B_2|$. In other words, we need to show that any sequence equivalent to B_1B_2 can be constructed as the concatenation of a sequence equivalent to B_1 and a sequence equivalent to B_2.

Let "|" below denote the border in question. First, any sequence from $\mathcal{E}(B_2)$ must begin either with 10 or with 01, hence, the situation (1|00) or (0|11) is impossible. Next, a sequence in $\mathcal{E}(B_1)$ ends by either 00 or 11 (see the proof of the previous lemma), neither leading to (10|0) or (01|1). □

This simple result shows that the space of all equivalent infinite 0-1 sequences for a given regular x splits into the direct infinite product of spaces, the k'th space consisting of all finite sequences equivalent to the block $B_k(x)$. So, we see that the notion of block, initially arised in terms of the canonical expansion, can be naturally extended to all representations.

Remark. Note that this block partition appeared for the first time in [Pu] in somewhat different terms and for algebraic and combinatorial purposes. Namely, let the partial ordering on a space $\mathcal{E}(x)$ for some finite sequence x be defined as follows. We set $x \prec x'$ if there exists $k \ge 2$ such that $x_{k-1} = 0, x_k = 1, x_{k+1} = 1, x'_{k-1} = 1, x'_k = 0, x'_{k+1} = 0$, and $x_j = x'_j,\ j \ge 2$. Next, one extends this ordering by transitivity. It was shown in [Pu] that any equivalence class has the structure of a distributive lattice in the sense of this order.

2.3. Goldenshift. Let $\mathcal{S} : \mathfrak{X} \to \mathfrak{X}$ be the one-sided shift in the space of block sequences, i.e. $\mathcal{S}(B_1B_2B_3\dots) = (B_2B_3\dots)$. By the above, it is well defined also on the set of regular points of $(\lambda^{-1}, 1)$. We call this transformation the *goldenshift*. Let us describe the action of $\mathcal{S}$ on the interval $(\lambda^{-1}, 1) \bmod 0$. The goldenshift $\mathcal{S}$ treated as a mapping acting from the interval $(\lambda^{-1}, 1) \bmod 0$ into itself, is piecewise linear. Moreover, if $(\lambda^{-1}, 1) = \bigcup_r \Delta_r$ is the partition of $(\lambda^{-1}, 1) \bmod 0$ into the intervals corresponding to that of $\mathfrak{B}$ into the states of the first block, then $\mathcal{S}$ is linear inside $\Delta_r =: [\alpha_r, \beta_r)$, and $\mathcal{S}(\alpha_r) = \lambda^{-1},\ \mathcal{S}(\beta_r) = 1$.

Remark. We have
$$\mathcal{S}x = \tau^{n(x)}x, \quad x \text{ is regular},$$
where $n(x)$ is just the length of the first block of the block expansion of x. Thus, $\mathcal{S}$ is a *generalized power* of τ in the sense of Dye (see, e.g., [Bel]). In other words, the goldenshift is a random power of the ordinary shift, as the number of shifted coordinates depends on the length of the first block. Note that the goldenshift is not an induced endomorphism for τ but for the two-sided case it is (see Proposition 2.3 and Theorem 2.13 below).

Let $\widetilde{\mathfrak{X}}$, $\widetilde{\tau}$ and $\widetilde{X}_0$ be defined as above, and let $\widetilde{\mathcal{S}} : \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{X}}$ be the *two-sided goldenshift*, i.e. the shift by 1 in the space $\widetilde{\mathfrak{X}}$. In order to specify $\widetilde{\mathcal{S}}$ on the Markov compactum $\widetilde{X}$, we introduce a $\widetilde{\tau}$-invariant set $\widetilde{X}^{\mathrm{reg}} \subset \widetilde{X}$ which is defined as the one consisting of all sequences containing pieces "$10^{2l}1$" with $l \ge 1$ infinitely many times both to the left and to the right with respect to the first coordinate. Clearly, $\widetilde{\nu}\big(\widetilde{X}^{\mathrm{reg}}\big) = 1$, as by Proposition 1.7, the measure $\widetilde{\nu}$ of any cylinder in $\widetilde{X}$ is positive, hence, it suffices to apply the ergodic theorem to the dynamical system $(\widetilde{X}, \widetilde{\nu}, \widetilde{\tau})$. Let $\widetilde{X}_0^{\mathrm{reg}} = \widetilde{X}^{\mathrm{reg}} \cap \widetilde{X}_0$. Then $\widetilde{\mathcal{S}} : \widetilde{X}_0^{\mathrm{reg}} \to \widetilde{X}_0^{\mathrm{reg}}$ on the Markov compactum $\widetilde{X}$ is the shift by the length of B_1.

Proposition 2.3. *The two-sided shift $\widetilde{\tau}$ on the set $\widetilde{X}^{\mathrm{reg}}$ is a special automorphism under the goldenshift $\widetilde{\mathcal{S}}$ on the set $\widetilde{X}_0^{\mathrm{reg}}$. The number of steps over a sequence $(\varepsilon_k) \in \widetilde{X}_0^{\mathrm{reg}}$ is equal to the length of the block beginning with $\varepsilon_1 = 1$.*

Proof. It suffices to present the steps of the corresponding tower. Let, by definition, $\widetilde{X}_1^{\mathrm{reg}} = \widetilde{\tau}\widetilde{X}_0^{\mathrm{reg}}$, and $\widetilde{X}_{2j}^{\mathrm{reg}} = \widetilde{\tau}\widetilde{X}_{2j-1}^{\mathrm{reg}}$, $\widetilde{X}_{2j+1}^{\mathrm{reg}} = \widetilde{\tau}\widetilde{X}_{2j}^{\mathrm{reg}} \setminus (\varepsilon_1 = 1)$, $j \ge 1$. This completes the proof, as $\widetilde{X}_{\mathrm{reg}} = \bigcup_0^\infty \widetilde{X}_j^{\mathrm{reg}}$, the union being disjoint.

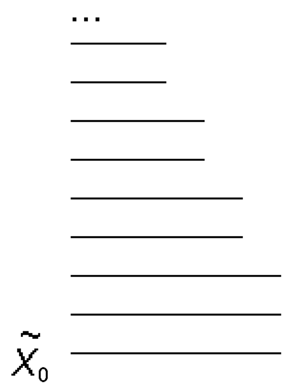

Fig. 2. The steps of the special automorphism $\widetilde{\tau}$

Below the corresponding result will be established for the metric case with the two-sided Erdös measure.

2.4. Bernoullicity of the goldenshift. In this subsection we will show that the goldenshift (one-sided or two-sided) is a Bernoulli shift in the space $\mathfrak{X}$ (respectively $\widetilde{\mathfrak{X}}$) with respect both to Lebesgue and Erdös measures and compute their one-dimensional distributions.

We denote by m the normalized Lebesgue measure on the interval $(\lambda^{-1}, 1)$; let $m_{\mathfrak{X}}$ stand for the measure Ψm in the space $\mathfrak{X}$, and, similarly, let $\mu_{\mathfrak{X}}$ denote $\Psi(\mu_{X_0})$ (this measure is well defined, as μ-a.e. point $x \in (\lambda^{-1}, 1)$ is also regular). We recall that to any block cylinder $\{B_1 = B(r)\} \subset \mathfrak{X}$ we associated the interval Δ_r defined as the image of the cylinder "$B(r)1$" in X by the mapping (1.1).

We will show that the measures $m_{\mathfrak{X}}$ and $\mu_{\mathfrak{X}}$ are Bernoulli in the space $\mathfrak{X}$ and also compute their one-dimensional distributions.

Theorem 2.4. *The measure $m_{\mathfrak{X}}$ in the space $\mathfrak{X}$ is a product measure with equal multipliers, i.e. a Bernoulli measure.*

Proof. By the above specification of $\mathcal{S}$ on the interval, for any Borel set $E \subset (\lambda^{-1}, 1)$,

$$m(\mathcal{S}^{-1}E \mid \Delta_r) = mE, \tag{2.1}$$

whence the required assertion immediately follows by virtue of the obvious $\mathcal{S}$-invariance of m, and by setting $E = \Delta_{r'}$ in relation (2.1) for any r', which yields the $m_{\mathfrak{X}}$-independence of the first and the second blocks.

So, it remains to compute the one-dimensional distribution of $m_{\mathfrak{X}}$.

Corollary 2.5. *The total measure $m_{\mathfrak{X}}$ of all block cylinders $\{B_1 = B\}$ of length $2n+1$ equals $\frac{1}{2\lambda}\left(\frac{2}{\lambda^2}\right)^n$. Any such cylinder has the measure $m_{\mathfrak{X}}$ equal to λ^{-2n-1}.*

Proposition 2.6. *The measure $\mu_{\mathfrak{X}}$ is also product on $\mathfrak{X}$ with equal multipliers.*

Proof. It suffices to establish a relation similar to (2.1) for the measure $\mu_{\mathfrak{X}}$ and for any finite block sequence $E = B_1 \dots B_k$. Note first that by virtue of Lemma 2.2, $\mathcal{E}(B_1 \dots B_k) = \mathcal{E}(B_1) \dots \mathcal{E}(B_k)$ for any blocks $B_1, \dots, B_k$.[1] Next,

$$\mathfrak{n}^{-1}(B_1 \dots B_k 1) = \mathcal{E}(B_1) \dots \mathcal{E}(B_k) \mathfrak{n}^{-1}(\varepsilon_1 = 1).$$

We are going to show that

$$\mu_{\mathfrak{X}}\{B_1 \dots B_k\} = \mu_{\mathfrak{X}}\{B_1\} \dots \mu_{\mathfrak{X}}\{B_k\} = \frac{f(B_1)}{2^{|B_1|}} \cdots \frac{f(B_k)}{2^{|B_k|}}, \quad k \geq 1. \tag{2.2}$$

To do this, we use previous remarks and the definition of the Erdös measure on X by means of the normalization (see Section 1). We have $\mu_{\mathfrak{X}}\{B_1 \dots B_k\} = \mu(B_1 \dots B_k 1)/\mu(\varepsilon_1 = 1)$, and

$$\begin{aligned}\mu(B_1 \dots B_k 1) &= p(\mathfrak{n}^{-1}(B_1 \dots B_k 1)) = p(\mathcal{E}(B_1) \dots \mathcal{E}(B_k) \mathfrak{n}^{-1}(\varepsilon_1 = 1)) \\ &= \frac{f(B_1)}{2^{|B_1|}} \cdots \frac{f(B_k)}{2^{|B_k|}} \cdot \mu(\varepsilon_1 = 1)\end{aligned}$$

(by Lemma 2.2), whence the required assertion follows.

Thus, we have proved one of the main results of the present paper.

Theorem 2.7. *The goldenshift $\widetilde{\mathcal{S}}$ is a Bernoulli automorphism with respect to Lebesgue and Erdös measures.*

Now we are ready to give the second proof of Erdös theorem (see Section 1).

Corollary 2.8. (a new proof of Erdös theorem) *The Erdös measure is singular with respect to Lebesgue measure.*

Proof. In fact, we have proved that the measures involved are mutually singular on the interval $(\lambda^{-1}, 1)$, which yields the assertion of the corollary, as any series of independent discrete measures is known to be either singular or absolutely continuous with respect to Lebesgue measure.

The one-dimensional distribution of $\mu_{\mathfrak{X}}$ is a bit more sophisticated than for $m_{\mathfrak{X}}$. It is described as follows (see formula (2.2)):

Corollary 2.9. *The measure $\mu_{\mathfrak{X}}$ of all block cylinders $\{B_1 = B\}$ of length $2n+1$ equals $\frac{1}{3} \cdot \left(\frac{3}{4}\right)^n$, and for a block $B = B(a_1, \dots, a_t)$ with $a_1 + \dots + a_t = n$,*

$$\mu_{\mathfrak{X}}\{B_1 = B\} = \frac{f(B)}{2^{|B|}} = \frac{p+q}{2^{2n+1}},$$

where, as usual, $p/q = [a_1, \dots, a_t]$.

[1] Henceforward $E = E_1 E_2$ is the concatenation of two sets of sequences, i.e. any sequence in E begins with a word from E_1 and ends by a word from E_2.

2.5. Concluding remarks on the Erdös measure. We conclude the study of ergodic properties of Erdös measures (one-sided and two-sided) and the transformations of shift and goldenshift.

2.5.1. One-sided case. Recall that the Erdös measure μ is quasi-invariant under the one-sided shift τ, and the equivalent measure ν is τ-invariant. It is worthwhile to know the behavior of ν with respect to the goldenshift $\mathcal{S}$.

Let $\nu_{\mathfrak{X}}$ be defined in the same way as $\mu_{\mathfrak{X}}$. We formulate the following claim (for more details see Appendix C).

Proposition 2.10. *The measure $\nu_{\mathfrak{X}}$ on the space $\mathfrak{X}$ of block sequences is quasi-invariant under the goldenshift $\mathcal{S}$. More precisely, any two cylinders $\{B_j = B'_j\}$ and $\{B_i = B'_i\}$ with $i \neq j$ are $\nu_{\mathfrak{X}}$-independent, and*

$$\nu_{\mathfrak{X}}\{B_k = B\} = \mu_{\mathfrak{X}}\{B_k = B\} = \frac{f(B)}{2^{|B|}}, \quad k \geq 2,$$

$$\nu_{\mathfrak{X}}\{B_1 = B\} = \begin{cases} \frac{\frac{4}{5}p+\frac{6}{5}q}{p+q}\frac{f(B)}{2^{|B|}}, & B = 100\ldots \\ \frac{\frac{6}{5}p+\frac{4}{5}q}{p+q}\frac{f(B)}{2^{|B|}}, & B = 101\ldots \end{cases}$$

2.5.2. Two-sided case. Recall that we have already defined the two-sided Erdös measure as the image of the product measure $(1/2, 1/2)$ on $\prod_{-\infty}^{\infty}\{0,1\}$ under the normalization $\widetilde{\mathfrak{n}}$ (see Section 1). Besides, we construct the following two measures: the natural extension of the normalized measure μ on X_0 by means of the two-sided goldenshift $\widetilde{\mathcal{S}}$ and the natural extension of ν by means of the two-sided shift $\widetilde{\tau}$ (we remind that, by definition, a measure in the natural extension of a shift coincides with an initial one-sided measure on each cylinder, see [Ro]).

Theorem 2.11. *The automorphism $(\widetilde{X}, \widetilde{\tau}, \widetilde{\nu})$ is the natural extension of the endomorphism (X, τ, ν) with the generator described above.*

Proof. Let within this proof $\nu_{\leftrightarrow}$ denote the natural extension of ν in the above sense. To prove that $\nu_{\leftrightarrow} = \widetilde{\nu}$, we observe that both measures are $\widetilde{\tau}$-invariant, thus, it suffices to show that they are not mutually singular and apply the ergodic theorem. We will prove that

$$\widetilde{\nu}(C) \geq \frac{1}{10}\nu_{\leftrightarrow}(C) \tag{2.3}$$

for any cylinder $C \subset X$.

We first note that without loss of generality we may prove (2.3) for the cylinders C beginning with $\varepsilon_1 = 1$, as both measures are shift-invariant, and any cylinder is a disjoint union of cylinders beginning with 1. Next, we reduce in the same way the problem of proving inequality (2.3) to a closed multiblock $C = B_1 \ldots B_k 1 \subset X$ with B_1 beginning with $\varepsilon_1 = 1$.

By definition, $\widetilde{\nu}(C) = \widetilde{p}(\widetilde{\mathfrak{n}}^{-1}C)$. It is obvious by virtue of Remark after the definition of normalization (see Section 1) that

$$\widetilde{\nu}(C) \supset (0|\mathcal{E}(B_1)\ldots\mathcal{E}(B_k)100),$$

hence,

$$\widetilde{\nu}(C) \geq \frac{1}{8} \cdot \frac{f(B_1)}{2^{|B_1|}} \cdots \frac{f(B_k)}{2^{|B_k|}}.$$

On the other hand, by definition, $\nu_{\leftrightarrow}(C) = \nu_{X_0}(B_1 \dots B_k 1) \leq \frac{6}{5} \cdot \frac{f(B_1)}{2^{|B_1|}} \cdots \frac{f(B_k)}{2^{|B_k|}}$ (see Proposition 2.10). Finally,

$$\frac{\widetilde{\nu}(C)}{\nu_{\leftrightarrow}(C)} \geq \frac{5}{48} > \frac{1}{10}. \quad \square$$

Corollary 2.12. $\widetilde{\nu}\widetilde{X}_0 = \frac{1}{9}$.

Proof. We have by the definition of the set $\widetilde{X}_0$, Proposition 2.10, Theorem 2.11 and the fact that $\nu(\varepsilon_1 = 1) = \frac{5}{18}$,

$$\widetilde{\nu}\widetilde{X}_0 = \sum_{k=1}^{\infty} \widetilde{\nu}\big(1(00)^k 1\big) = \frac{5}{18} \sum_{k=1}^{\infty} \frac{\frac{4}{5} + \frac{6}{5}k}{1+k} \cdot \frac{k+1}{2^{2k+1}} = \frac{1}{9}.$$

Finally, we prove a metric version of Proposition 2.3.

Theorem 2.13. *The two-sided shift $\widetilde{\tau}$ with the measure $\widetilde{\nu}$ is a special automorphism over the goldenshift $\widetilde{\mathcal{S}}$ with the measure $\widetilde{\mu}$ on the space $\widetilde{X}_0$. The step function is defined as the length of the block beginning with the first coordinate.*

Proof. It suffices to show that the lifting measure for $\widetilde{\mu}$ coincides with $\widetilde{\nu}$. This in turn is implied again by the ergodic theorem applied to $\widetilde{\tau}$ which preserves both measures. Since they are clearly equivalent, we are done.

3. The entropy of the goldenshift and applications

In this section we will establish a relationship between the entropy of the Erdös measure in the sense of A. Garsia and the entropy of the goldenshift with respect to μ, i.e. between two different entropies. As an application, we will reprove the formula for Garsia's entropy proved in [AlZa]. Besides, we use the random walk theory to compute the dimension of the Erdös measure on the interval.

3.1. Fibonacci graph, random walk on it and Garsia's entropy. The combinatorics of equivalent 0-1 sequences may be expressed graphically, namely, by means of the Fibonacci graph introduced in [AlZa]. Let, as in Section 1, the mapping $\pi : \prod_1^\infty \{0,1\} \to [0,1]$ be defined as

$$\pi(\varepsilon_1\varepsilon_2\dots) = \sum_{k=1}^{\infty} \varepsilon_k \lambda^{-k-1}. \tag{3.1}$$

Since the ε_n assume the values 0 and 1 without any restrictions, a typical x will have a continuum number of representations, and they all may be illustrated with the help of the *Fibonacci graph* depicted in Fig. 3. This figure appeared for the first time in the work due to J. C. Alexander and D. Zagier [AlZa]. Let us give the precise definition.

Definition. The Fibonacci graph Φ is a binary graph with the edges labeled with 0 each left and 1 each right. Any vertex at the n'th level corresponds to a certain x, for which some representation (3.1) is finite with the length n (obviously, in this case $x = \{N\lambda\}$ for some $N \in \mathbb{Z}$). The paths are 0-1 sequences treated as representations of the form (3.1).[2]

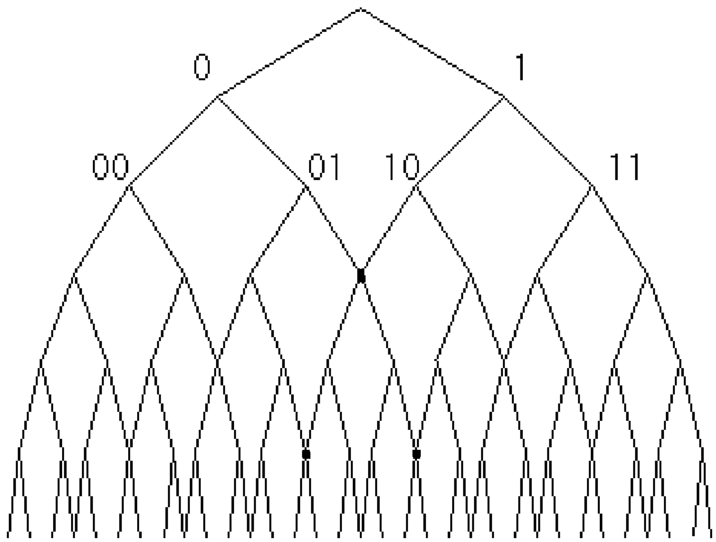

Fig. 3. The Fibonacci graph Φ

Remark. The vertices of the n'th level of the graph Φ can be treated as the nonnegative integers from 0 to $F_{n+2} - 2$. Namely, if a path $(\varepsilon_1\varepsilon_2 \dots \varepsilon_n)$ goes to a vertex k, then, by definition, $k = \sum_1^n \varepsilon_j F_{n-j}$ (obviously, this sum does not depend on the choice of a path).

Let $Y(\Phi)$ denote the set of paths in the graph Φ. Obviously, $Y(\Phi)$ is naturally isomorphic to $\prod_1^\infty \{0,1\}$, and sometimes we will not make a distinction between them. Let $(\varepsilon_1\varepsilon_2 \dots)$ be a path, and let the projection from $Y(\Phi)$ onto $[0,1]$ be also denoted by π (see formula (3.1)).

Let, as above, $f_n(k)$ denote the number of representations of a nonnegative integer k as a sum of not more than n first Fibonacci numbers. It is easy to see that $f_n(k)$ is also the *frequency* of the vertex k on the n'th level of the graph Φ.[3] Let $D_n = \{k : k = \sum_{k=1}^n \varepsilon_k F_{n-k},\ \varepsilon_k \in \{0,1\}\}$ (or, equivalently, the n'th level of the Fibonacci graph), and $D'_n = \{w : w = \sum_{k=1}^n \varepsilon_k \lambda^{-k-1},\ \varepsilon_k \in \{0,1\}\}$. These sets are clearly isomorphic ($w \leftrightarrow k$), and $\#D_n = \#D'_n = F_{n+2} - 1$. The use of D_n instead of D'_n is caused only by technical reasons. We remind that the sequence of distributions $(2^{-n} f_n(w))_{n=1}^\infty$ tends to the distribution of Erdös measure (see Section 1).

Once and for all we fix λ as the base of logarithms and denote the entropy of this discrete distribution on D'_n by $H^{(n)}$. Thus,

$$H^{(n)} = -\sum_{k=0}^{F_{n+2}-2} \frac{f_n(k)}{2^n} \log_\lambda \frac{f_n(k)}{2^n}.$$

[2]The term "Fibonacci graph" is overloaded, as the authors know several different graph also called "Fibonacci". Nevertheless, we hope that there will be no confusion with any of them.

[3]Thus, relation (1.2) completely determines the whole graph Φ.

Then, by definition,

$$H_\mu := \lim_{n\to\infty} \frac{H^{(n)}}{n}$$

(this limit is known to exist and is independent of the choice of base of logarithms, see [Ga]).

Remark 1. The quantity H_μ can be considered as the entropy of the random walk on the Fibonacci graph with the probabilities $(1/2, 1/2)$. Also, this entropy can be treated as the entropy of the Erdös measure as the one on the graph Φ. In the next item it will be shown that in fact H_μ is proportional to the entropy of the goldenshift.

Remark 2. The Erdös measure is the projection of the Markov measure $(\frac{1}{2}, \frac{1}{2})$ on the graph Φ under the mapping π. We consider the random walk on the Fibonacci graph with the equal transition measures.

The notion of the entropy in the sense of Garsia can be considered in the framework of the entropy of a random walk on the group $G = \langle a, b \mid abb = baa \rangle$ (see [KaVe] for the definition of a random walk on Cayley graphs of discrete groups and more general graphs). The only disctinction is that we consider the Fibonacci graph which, as it is easy to see, is the Cayley graph for the corresponding semigroup. Thus, by definition, H_μ is the entropy of the random walk on Φ with the transition measures equal to $\frac{1}{2}$ identically. So, the entropy in the sense of Garsia coincides with the entropy of the random walk. This fact will be used in the proof of the formula for the dimension of Erdös measure (see Theorem 3.4 below).

3.2. Main theorem. We prove an assertion being one of the central points of the present paper. Note that in [AlZa] Garsia's entropy was computed by means of generating functions. We will see that H_μ is closely connected with the entropy of the goldenshift, which gives a new simplified proof of their relation and relates it to the dynamics of the Erdös measure.

Theorem 3.1. *The following relation holds:*

$$h_\mu(\mathcal{S}) = 9H_\mu.$$

Proof. Consider the two-sided case studied in the previous sections and apply Abramov's formula for the entropy of the special automorphism (see [Ab]) to the dynamical systems $(\widetilde{X}, \widetilde{\nu}, \widetilde{\tau})$ and $(\widetilde{X}_0, \widetilde{\mu}, \widetilde{\mathcal{S}})$. By Theorem 2.13,

$$h_{\widetilde{\mu}}(\widetilde{\mathcal{S}}) = \frac{1}{\widetilde{\nu}\widetilde{X}_0} h_{\widetilde{\nu}}(\widetilde{\tau}).$$

From this relation we will deduce the required one.
1. Obviously, $h_{\widetilde{\mu}}(\widetilde{\mathcal{S}}) = h_\mu(\mathcal{S})$ (recall that the dynamical system $(\widetilde{X}, \widetilde{\mu}, \widetilde{\mathcal{S}})$ is the natural extension of $(X, \mu, \mathcal{S})$), and similarly, $h_{\widetilde{\nu}}(\widetilde{\tau}) = h_\nu(\tau)$.

2. Next, by Corollary 2.12, $\widetilde{\nu}\widetilde{X}_0 = \frac{1}{9}$.
3. The rest of the proof is devoted to establishing the validity of the relation

$$h_\nu(\tau) = H_\mu.$$

Let η denote the partition of X into the cylinders $(\varepsilon_1 = 0)$ and $(\varepsilon_1 = 1)$. Since η is a generating partition for $\widetilde{\tau}$, we have $h_\nu(\tau) = h_\nu(\tau, \eta)$ by Kolmogorov's theorem. By definition,

$$h_\nu(\tau, \eta) = \lim_{n\to\infty} \frac{1}{n} H_\nu(\eta^{(n)}),$$

where $\eta^{(n)}$ is the partition of X into F_{n+1} admissible cylinders of the form $(\varepsilon_1 = i_1, \dots, \varepsilon_n = i_n)$. We need to prove that

$$H_\nu(\eta^{(n)}) \sim H^{(n)}.$$

By virtue of the equivalence of the measures μ and ν it suffices to show this for $H_\mu(\eta^{(n)})$ instead of $H_\nu(\eta^{(n)})$. Let $\theta_n(k) = 2^{-n} f_n(k)$. Then, by relation (1.2), for $n \ge 3$,

$$\theta_n(k) = \begin{cases} \frac{1}{2}\theta_{n-1}(k), & 0 \le k \le F_n - 1 \\ \frac{1}{2}(\theta_{n-1}(k) + \theta_{n-1}(k - F_n)), & F_n \le k \le F_{n+1} - 2 \\ \frac{1}{2}\theta_{n-1}(k - F_n), & F_{n+1} - 1 \le k \le F_{n+2} - 2. \end{cases}$$

We are going to obtain almost the same recurrence relation for the distribution $\eta^{(n)}$. To do this, we return to the interval $[0, 1]$ and denote by $\eta^{(n)}$ the partiton into F_{n+1} intervals being the image of the corresponding partition of X with the help of the canonical expansion. So, let $\eta^{(n+1)} =: (\Delta_n(k))_{k=0}^{F_{n+2}-1}$ with the ordered intervals $\Delta_n(k)$. Finally, let $\mu_n(k) := \mu\Delta_n(k)$. Then by Lemma 1.1, for $n \ge 3$,

$$\mu_n(k) = \begin{cases} \frac{1}{2}\mu_{n-1}(k), & 0 \le k \le F_n - 1 \\ \frac{1}{2}(\mu_{n-1}(k) + \mu_{n-1}(k - F_n)), & F_n \le k \le F_{n+1} - 1 \\ \frac{1}{2}\mu_{n-1}(k - F_n), & F_{n+1} \le k \le F_{n+2} - 3. \end{cases}$$

Besides, $\mu_n(F_{n+2} - 2) = O(2^{-n})$, $\mu_n(F_{n+2} - 1) = O(2^{-n})$. Thus, by induction on n, there exists $C > 0$ such that

$$\frac{1}{C} \le \frac{\theta_n(k)}{\mu_n(k)} \le C, \quad 0 \le k \le F_{n+2} - 2,$$

which completes the proof of the theorem, as the entropies of the distributions $\mu_n(k)$ and $\theta_n(k)$ are clearly equivalent by the above estimate. $\square$

3.3. Alexander-Zagier's theorem. We first compute the entropy of the goldenshift with respect to the Erdös and Lebesgue measures.

Proposition 3.2. *The metric entropies of the goldenshift $\mathcal{S}$ with respect to the two measures in question are computed as follows:*

$$h_m(\mathcal{S}) = -\sum_{B\in\mathfrak{B}} m_{\mathfrak{X}}(B)\log_\lambda m_{\mathfrak{X}}(B) = 4\lambda+3 = 9.4721356\dots,$$

$$h_\mu(\mathcal{S}) = -\sum_{B\in\mathfrak{B}} \mu_{\mathfrak{X}}(B)\log_\lambda \mu_{\mathfrak{X}}(B) = -\sum_{n=1}^{\infty}\sum_{B:|B|=2n+1} \frac{p+q}{2^{2n+1}}\log_\lambda\frac{p+q}{2^{2n+1}} = 8.961417\dots$$

Proof is a direct computation using the Bernoullicity of $\mathcal{S}$ with respect to both measures and Corollaries 2.5 and 2.9.

Remark. Let $\Lambda := \log_\lambda 2$, and

$$k_n = \sum_{\substack{t\ge 1\\ (a_1,\dots,a_t)\in\mathbb{N}^t\\ a_1+\dots+a_t=n\\ p/q=[a_1,\dots,a_t]}} (p+q)\log_\lambda(p+q).$$

Then

$$h_\mu(\mathcal{S}) = 9\left(\Lambda - \frac{1}{18}\sum_{n=1}^{\infty}\frac{k_n}{4^n}\right). \tag{3.2}$$

Note that the quantity k_n appeared for the first time in [AlZa] in somewhat different notation. Namely, let k and i be positive integers, and let $e(k,i)$ denote the length of the simple Euclidean algorithm for k and i (formally: $e(i,i)=0$, $e(i+k,i)=e(i+k,k)=e(i,k)+1$). Then, obviously,

$$k_n = \sum_{\substack{k>i>0\\ \gcd(k,i)=1,\ e(k,i)=n}} k\log_\lambda k.$$

In the cited work J. C. Alexander and D. Zagier used this definition of k_n for deducing a formula for H_μ in terms of k_n. We will prove their assertion in two different ways. The first is an immediate consequence of Theorem 3.1 and the second is rather long but reveals a more essential relationship between certain structures on the Fibonacci graph (see Appendix D).

Proposition 3.3. (Alexander-Zagier, 1991). *The following relation holds:*

$$H_\mu = \Lambda - \frac{1}{18}\sum_{n=1}^{\infty}\frac{k_n}{4^n} = 0.995713\dots \tag{3.3}$$

Proof. An application of Theorem 3.1 and of relation (3.2).

3.4. The dimension of the Erdös measure. As an application of the treatment of the Erdös measure as the one being the projection of the measure of the uniform random walk on the Fibonacci graph, we will compute the dimension of μ in the sense of L.-S. Young.

We first give a number of necessary definitions (see [Y]).

Definition. Let ν be a Borel probabilistic measure on a compact space Y. The quantities

$$\begin{aligned}\dim_H \nu &= \inf\{\dim_H A : A \subset Y,\, \nu A = 1\},\\ \overline{C}(\nu) &= \limsup_{\delta\to 0} \inf\{\overline{C}(A) : A \subset Y,\, \nu A \geq 1-\delta\},\\ \underline{C}(\nu) &= \liminf_{\delta\to 0} \inf\{\underline{C}(A) : A \subset Y,\, \nu A \geq 1-\delta\}\end{aligned}$$

(where $\overline{C}(A)$ and $\underline{C}(A)$ are respectively the upper and lower capacities of A) are called the *Hausdorff dimension* of a measure ν, the *upper* and *lower capacities* of ν respectively.

Let next $N(\varepsilon,\delta)$ denote the minimal number of balls of radius $\varepsilon > 0$ which are necessary to cover a set of the ν-measure $\geq 1-\delta$.

Definition. The quantities

$$\begin{aligned}\underline{C}_L(\nu) &= \limsup_{\delta\to 0}\ \liminf_{\varepsilon\to 0} \frac{\log N(\varepsilon,\delta)}{\log(1/\varepsilon)},\\ \overline{C}_L(\nu) &= \limsup_{\delta\to 0}\ \limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon,\delta)}{\log(1/\varepsilon)}\end{aligned}$$

are called the *lower* and *upper Ledrappier capacities* of ν.

Definition. Let $H_\nu(\varepsilon) = \inf\{H_\nu(\xi) : \operatorname{diam}\xi \leq \varepsilon\}$, where $H_\nu(\xi)$ is the entropy of a finite partition ξ. The quantities

$$\begin{aligned}\overline{R}(\nu) &= \limsup_{\varepsilon\to 0} \frac{H_\nu(\varepsilon)}{\log(1/\varepsilon)},\\ \underline{R}(\nu) &= \liminf_{\varepsilon\to 0} \frac{H_\nu(\varepsilon)}{\log(1/\varepsilon)}\end{aligned}$$

are called respectively the *upper* and *lower informational dimensions* of ν (= *Rényi dimensions*).

Theorem. (L.-S. Young [Y], 1982) *If ν is a Borel probabilistic measure on an interval I, and if*

$$\alpha(x) := \lim_{h\to 0} \frac{\log \nu(x, x+h)}{\log h} \equiv \alpha$$

for ν-a.e. point $x \in I$, then

$$\dim_H \nu = \overline{C}(\nu) = \underline{C}(\nu) = \overline{C}_L(\nu) = \underline{C}_L(\nu) = \overline{R}(\nu) = \underline{R}(\nu) = \alpha.$$

Definition. If the condition of Young's theorem is satisfied, then this α is called the *dimension* of a measure ν and is denoted by $\dim(\nu)$. This notion was also proposed in [Y].

Theorem 3.4. *For the Erdös measure* μ,

$$\dim(\mu) = H_\mu.$$

Remark. When the present paper was in preparation, the authors were told that the claim of Theorem 3.4 can be obtained as a corollary of several results including the new one due to F. Ledrappier and A. Porzio whom we are grateful for this explanation. More precisely, it was shown in [AlYo] that $H_\mu = \overline{R}(\mu) = \underline{R}(\mu)$, and in [LePo] it was proved that the limit in Young's theorem does exist for the Erdös measure. This proves Theorem 3.4.

Our proof is straightforward and, which is more important, is a direct corollary of a Shannon-like theorem, so far it leads to new connections between geometric and dynamical properties of the Erdös measure.

Proof. Fix a path $\bar{\varepsilon} = (\varepsilon_1\varepsilon_2\dots) \in Y(\Phi)$, and let $x = \pi(\bar{\varepsilon})$ and $Y_n = Y_n(\bar{\varepsilon})$ be the interval whose every point x' has a path $\bar{\varepsilon}' \in Y(\Phi)$ such that $\varepsilon_i \equiv \varepsilon_i'$, $1 \le i \le n$. Clearly, $Y_n(x) = \left[\sum_1^n \varepsilon_k\lambda^{-k-1}, \sum_1^n \varepsilon_k\lambda^{-k-1} + \lambda^{-n}\right]$, We remind that $\mu = \vartheta_1 * \vartheta_2 * \dots$ (see Section 1). Let $\mu^{(n)} := \vartheta_1 * \dots * \vartheta_n$. Then by Shannon's theorem for the random walks (see Theorem 2.1 in [KaVe] and also [De]) and by H_μ being the entropy of the random walk on the semigroup $\langle a, b \mid ab^2 = ba^2\rangle$,[4]

$$\lim_{n\to\infty} \frac{\log_\lambda \mu^{(n)}(Y_n(\bar{\varepsilon}))}{n} = H_\mu$$

for μ-a.e. $x \in [0,1]$. Given $h > 0$, we choose $n = n(h)$ such that $Y_{n+1}(\bar{\varepsilon}) \subset (x, x+h) \subset Y_n(\bar{\varepsilon})$ for any $x \in \pi^{-1}(\bar{\varepsilon})$. Hence it follows that for μ-a.e. x,

$$\lim_{h\to 0} \frac{\log_\lambda \mu(x, x+h)}{\log_\lambda h} = \lim_{n\to\infty} \frac{\log_\lambda \mu Y_n(\bar{\varepsilon})}{n} = H_\mu,$$

as $h \asymp \lambda^{-n}$. $\square$

Corollary 3.5.

$$\dim_H \mu = \overline{C}(\mu) = \underline{C}(\mu) = \overline{C}_L(\mu) = \underline{C}_L(\mu) = \overline{R}(\mu) = \underline{R}(\mu) = H_\mu = 0.995713\dots$$

Remark 1. Another proof of Theorem 3.4 can be obtained by means of using the Bernoulli structure of the measure μ. More precisely, for a regular $x \in (\lambda^{-1}, 1)$ having a **normal** block expansion with respect to the measure $\mu_{\mathfrak{X}}$, as it is easy to compute, the limit in the definition of the dimension equals $\frac{1}{9}h_\mu(\mathcal{S})$. This yields also another proof of Theorem 3.1. The details are left to the interested reader.

Remark 2. In fact, we have computed the μ-typical Lipschitz exponent of the distribution function E of the Erdös measure. Note that in [Si] it was proved that the best possible Lipschitz exponent of E for all x is $\Lambda - \frac{1}{2} = 0.9404\dots$.

[4]We recall that the semigroup described above can be embedded into the group $\langle a, b \mid ab^2 = ba^2\rangle$.

Appendix A. The ergodic central measures and the adic transformation on the Fibonacci graph

In this appendix we will study in detail some properties of the space of paths $Y(\Phi)$ of the Fibonacci graph introduced in Section 3. We first give a necessary definition which is close to the defintion of canonical expansions but regards to the fact that 0 and 1 have the same rights in the graph Φ.

Definition. The *generalized* canonical expansion of a point in $(0, 1)$ is defined as follows. We construct the sequence $(\varepsilon_1\varepsilon_2\dots)$ such that relation (3.1) holds, and either $(\varepsilon_1\varepsilon_2\dots) \in X$, or $\varepsilon_1 = \dots = \varepsilon_m = 1$ for some $m \in \mathbb{N}$, and the tail is in X. The algorithm is a clear modification of the greedy algorithm.

Remark. In fact, the generalized canonical expansions lead to the normal form in the semigroup corresponding to the group G (see Section 3).

The *tail partition* $\eta(\Phi)$ of $Y(\Phi)$ is defined as follows.

Definition. Paths (ε_n) and (ε'_n), by definition, belong to one and the same element of $\eta(\Phi)$ iff

(i) $\pi(\varepsilon_1\varepsilon_2\dots) = \pi(\varepsilon'_1\varepsilon'_2\dots)$, and
(ii) there exists $N \in \mathbb{N}$ such that $\varepsilon_n \equiv \varepsilon'_n$, $n > N$.

The partial lexicographic ordering on $Y(\Phi)$ is defined for paths belonging to one and the same element of the tail partition $\eta(\Phi)$.

Definition. Let two paths $\overline{\varepsilon} = (\varepsilon_1\varepsilon_2\dots)$ and $\overline{\varepsilon}' = (\varepsilon'_1\varepsilon'_2\dots)$ belong to one and the same element of $\eta(\Phi)$. If $\varepsilon_{k-1} = 0$, $\varepsilon_k = 1$, $\varepsilon_{k+1} = 1$ and $\varepsilon'_{k-1} = 1$, $\varepsilon'_k = 0$, $\varepsilon'_{k+1} = 0$ for some $k \ge 2$, and $\varepsilon_j \equiv \varepsilon'_j$ for $k - j \ge 2$, then, by definition, $\overline{\varepsilon} \prec \overline{\varepsilon}'$. Next, by transitivity, $\overline{\varepsilon} \prec \overline{\varepsilon}'$, $\overline{\varepsilon}' \prec \overline{\varepsilon}''$ implies $\overline{\varepsilon} \prec \overline{\varepsilon}''$.

Remark. This definition is consistent, because any element of $\eta(\Phi)$ is isomorphic to a finite number of finite paths, and they all can be transferred one into another with the help of replacements $011 \leftrightarrow 100$. Note also that this linear ordering on each element of $\eta(\Phi)$ is stronger than the partial ordering introduced in [Pu] (see the end of item 2.2). Say, $(100011*) \prec (011100*)$ in the above sense but in the sense of the partial order they are noncomparable.

Definition. The *adic* transformation T juxtaposes (if possible) to a path $\overline{\varepsilon} \in Y(\Phi)$ the path $\overline{\varepsilon}'$ such that $\overline{\varepsilon}'$ is the immediate successor of $\overline{\varepsilon}$ in the sense of the lexicographic order.

It is clear that the adic transformation T is well defined not everywhere. More precisely, it is well defined on the paths $\overline{\varepsilon}$ containing at least one triple $\varepsilon_k = 0$, $\varepsilon_{k+1} = 1$, $\varepsilon_{k+2} = 1$. Let us descibe its action in more detail. Let $(\varepsilon_1\varepsilon_2\dots) \in Y(\Phi)$ be as described. After finding the first triple $\varepsilon_k = 0$, $\varepsilon_{k+1} = 1$, $\varepsilon_{k+2} = 1$ we
1) replace it by $\varepsilon_k = 1$, $\varepsilon_{k+1} = 0$, $\varepsilon_{k+2} = 0$,
2) leave the tail $(\varepsilon_{k+3}\varepsilon_{k+4},\dots)$ without changes,
3) find the minimal possible $(\varepsilon'_1\dots\varepsilon'_{k-1})$ equivalent to $(\varepsilon_1\dots\varepsilon_{k-1})$ in the sense of Section 2.

To carry out 3), we may use the algorithm of "anti-normalization", i.e. the process analogous to the ordinary normalization but with changing "100" by "011" (cf. Section 1).

So, the generalized canonical expansions are just the *maximal* paths, i.e. the ones, where T is not well defined, thus, the set of maximal paths is naturally isomorphic to the interval $[0,1]$. Geometrically a generalized canonical expansion corresponds to the rightest possible path descending to a given vertex. Similarly, the *minimal* paths (i.e. the ones, on which T^{-1} is not well defined) are just so-called *lazy* expansions (for the definition see, e.g., [ErJoKo]).

For more general definitions of adic transformation and investigation of its properties see [Ver1], [Ver2], [LivVer] and [VerSi].

Let us formulate two well-known definitions related to graded graphs (see [StVo] and [VerKe] for more details).

Definition. A Markov measure ν on the graph Φ is called *central* if any of the below equivalent conditions is satisfied:

(1) For any vertex in this graph all the paths descending to this vertex have equal conditional measures.
(2) ν is T-invariant.
(3) ν is constant on any element of the tail partition $\eta(\Phi)$.

Definition. A central measure on Φ is called *ergodic*, if any of the two equivalent conditions is satisfied:

(1) The adic transformation T is ergodic with respect to it.
(2) The tail partiton is ν-trivial, i.e. contains only sets whose measure ν is either 0 or 1.

The aim of this section is to describe
1) all ergodic central measures on Φ.
2) the action of the adic transformation T on Φ.

In below theorem we will describe occuring types of ergodic central measures and the corresponding components of the action of T. As it was noted above, T replaces the representations of one and the same x. We will see that the regularity or irregularity of the generalized canonical expansion of a given x, lead to three types of possible ergodic components of the action of T, namely, to a "full" odometer, an irrational rotation of the circle or a special automorphism over a rotation.

Theorem A.1. *1. The ergodic central measures on Φ are naturally parametrized by the points of the interval $[0,1]$. We denote by μ_x the measure corresponding to x.*
2. The measure μ_x is continuous if and only if $x \neq \{N\lambda\}$ for any $N \in \mathbb{Z}$ (or, equivalently, if x has the infinite canonical expansion).
3. The action of the adic transformation T is not transitive, and its trajectories are described as follows. Let x be as in the previous item, and let φ_x denote the space of paths in Φ such that $\varphi_x = \operatorname{supp} \mu_x$. The set φ_x is invariant under T and we have the following alternative.

a. *If the generalized canonical expansion of x contains infinitely many pieces "$1(00)^l1$"*

with $l \geq 1$ (let us call such a piece **even***), then $T|_{\varphi_x}$ is strictly ergodic and metrically isomorphic to the shift by 1 on the group of certain* **a***-adic integers (and, thus, $T|_{\varphi_x}$ has a purely discrete rational spectrum).*

b. *If the generalized canonical expansion of x does not contain even pieces at all, then $T|_{\varphi_x}$ is also strictly ergodic and metrically isomorphic to a certain irrational rotation of the circle.*

c. *Finally, if the generalized canonical expansion of x contains a finite number of even pieces, then $T|_{\varphi_x}$ is metrically isomorphic to some special automorphism over a rotation of the circle.*

Proof. (1) Let φ_x be the set of all paths projecting into $x \in [0,1]$ (a π-fiber over x). Obviously, the set φ_x is invariant under T for any x. Thus, T is not transitive, and its action splits into components, each acting in certain φ_x (below we will see that for all x, except some countable set, the action of $T|_{\varphi_x}$ is strictly ergodic).
(2) We have the following cases. If $x = 0$ or $x = 1$, then $\#\varphi_x = 1$. If $x = \{N\lambda\}$ for some $N \in \mathbb{Z}$, then it is easy to see that φ_x is countable and that the unique invariant measure for T is concentrated in a finite number of paths (see Example 1 below). Henceforward in this proof we assume that x has an infinite canonical expansion. Let $x = \sum_{j=1}^{\infty} \varepsilon_j \lambda^{-j-1}$ be the generalized canonical expansion of x. We first split it in the following way: $(\varepsilon_1\varepsilon_2\dots) = B^{(0)}B^{(1)}$, where $B^{(0)}$ is either 0^s or 1^s for some $s \geq 0$ (if $s = 0$, then $B^{(0)} = \emptyset$), and $B^{(1)}$ begins with "10". Such a splitting is caused by the trivial reason: the action of T does not touch at all the set $B^{(0)}$, as T only replaces certain triples "100" and "011". So, we have the following cases (they correspond to those enumerated in the theorem).
a. If $B^{(1)}$ contains infinitely many even pieces, then x is regular (see Section 2), hence,

$$B^{(1)} = B_1B_2B_3\dots,$$

where

$$B_j = 1(00)^{a_1^{(j)}}(01)^{a_2^{(j)}}(00)^{a_3^{(j)}}\dots(00)^{a_{t_j}^{(j)}} \quad \text{or} \quad B_j = 1(01)^{a_1^{(j)}}(00)^{a_2^{(j)}}(01)^{a_3^{(j)}}\dots(00)^{a_{t_j}^{(j)}}$$

with $a_i^{(j)} \in \mathbb{N}$ and $t_j < \infty$.
b. If $B^{(1)}$ does not contain any even piece, then, obviously,

$$B^{(1)} = 1(00)^{a_1}(01)^{a_2}(00)^{a_3}\dots \quad \text{or} \quad B^{(1)} = 1(01)^{a_1}(00)^{a_2}(01)^{a_3}\dots$$

with $a_j \in \mathbb{N}$ for any $j \geq 1$.
c. Finally, if the number of even pieces is finite (but nonzero), then

$$B^{(1)} = B_1B_2\dots B_m\widetilde{B}^{(1)},$$

where $B_1,\dots,B_m$ have the form described in the previous item, and $\widetilde{B}^{(1)}$ is an infinite block of the form described in item **b**.
(3) Consider items **a**, **b**, **c** from the viewpoint of the action of $T_x := T|_{\varphi_x}$.

a. The idea of the study of T_x in this case is based on two assertions of the previous section, namely on Lemmas 2.1 and 2.2. In particular, from Lemma 2.2 it follows that blocks B_i and B_{i+1} for all $i \in \mathbb{N}$ are replaced by any equivalent sequences independently, hence it is clear that for such a point x the transformation T_x is the shift by 1 in the group of $\mathbf{a}$-adic integers with $\mathbf{a} = (p_1 + q_1, p_2 + q_2, \dots)$. This transformation T_x is known to be strictly ergodic, i.e. there is a unique (product) measure μ_x invariant under it.
b. We recall that in this case

$$B^{(1)} = 1(00)^{a_1}(01)^{a_2}(00)^{a_3}\dots \quad \text{or} \quad B^{(1)} = 1(01)^{a_1}(00)^{a_2}(01)^{a_3}\dots$$

Let $\alpha = [1, a_1, a_2, \dots]$ denote a (regular) continued fraction. We claim that in this case the transformation T_x is strictly ergodic and metrically isomorphic to the rotation through α. The idea of the proof lies in recoding the space φ_x into the second model of the adic realization of the rotation from [VerSi] (see Section 3 of the cited work and Example 3 below). The unique invariant measure can be described with the help of Theorem 2.3 from the cited work.
c. This case in a sense is a "mixture" of the previous ones. One can easily see that if $\widetilde{B}^{(1)}$ is parametrized by the infinite sequence $(a_1, a_2, \dots)$ in the sense of the previous item, and if $B_j = B_j(a_1^{(j)}, \dots, a_{t_j}^{(j)})$, then T_x acts on φ_x as the special automorphism over the rotation through $\alpha = [1, a_1, a_2, \dots]$ with the constant step function ($\equiv 1$) and the number of upper steps equal to $\prod_1^m (p_j + q_j) - 1$. So, T_x is again strictly ergodic. The proof of the theorem is complete.

Remarks. 1. It is known (see, e.g., [VerKe]) that any ergodic central measure on the Pascal graph is also parametrized by a real in $[0,1]$ but in a completely different manner, namely, by means of the first transition measure. It is appropriate to compare that situation with the Fibonacci graph. We see that in the graph Φ for any $\alpha \in [0,1]$ there exists a central ergodic measure μ such that $\mu(\varepsilon_1 = 0) = \alpha$. If α is irrational, then this measure is unique, namely $\mu = \mu_x$ for $x = \sum_j \varepsilon_j \lambda^{-j-1}$ with $\alpha = [1, a_1, a_2, \dots]$ and $(\varepsilon_1 \varepsilon_2 \dots) = 1(00)^{a_1}(01)^{a_2}\dots$ for $\alpha > \frac{1}{2}$, and $1 - \alpha = [1, a_1, a_2, \dots]$ and $(\varepsilon_1 \varepsilon_2 \dots) = 1(01)^{a_1}(00)^{a_2}\dots$ otherwise. If α is rational, then there exists the whole interval of x in $[0,1]$ such that $\mu_x(\varepsilon_1 = 0) = \alpha$.
2. A typical x from the viewpoint of Lebesgue measure, of course, corresponds to the case **a.** of the theorem.

Examples. We illustrate possible situations in the previous theorem with four examples. For a better illustration we will use the following agreement:

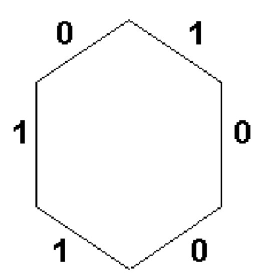

Fig. 4

We will use generalized canonical expansions writing $x \sim (\varepsilon_1\varepsilon_2\dots)$.
1. $x = \lambda^{-2} \sim (1000\dots)$. Here φ_x is countable, and the measure μ_x is concentrated on a single path $(010101\dots)$. Hence, T_x is constant (see Fig. 5 below).

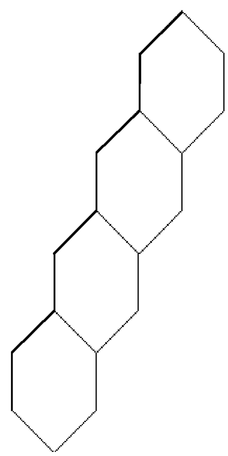

Fig. 5. The case $x = \lambda^{-2}$

2. $x = \frac{1}{2} \sim (100)^\infty$. Here $B^{(0)} = \emptyset$, $B^{(1)} = \big(1(00)^1\big)^\infty$. We have: $\varphi_x = \prod_1^\infty \{011, 100\}$, and, thus, T_x is isomorphic to the 2-adic shift, i.e. the shift by 1 in the group of dyadic integers. Therefore, T_x has the binary rational purely discrete spectrum. Below we depict the way of recoding the paths in φ_x into the full dyadic compactum by the rule: "$011 \sim 0,\ 100 \sim 1$"(see Fig. 6).

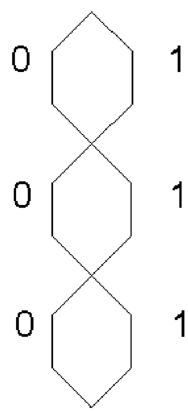

Fig. 6. Recoding the paths for $x = \frac{1}{2}$, case **a**

3. $x \sim (1(0001)^\infty)$. Here $\alpha = [1, 1, 1, \dots] = \lambda^{-1}$, and T_x acts as the rotation by the golden ratio. Fig. 8 below shows the way of recoding the paths in φ_x into the usual model for this rotation ("Fibonacci compactum"). Note that the natural ordering of these paths is alternating (see Fig. 7, 8).

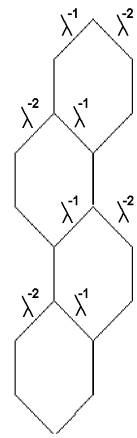

Fig. 7. Transition measures for $\alpha = \lambda^{-1}$

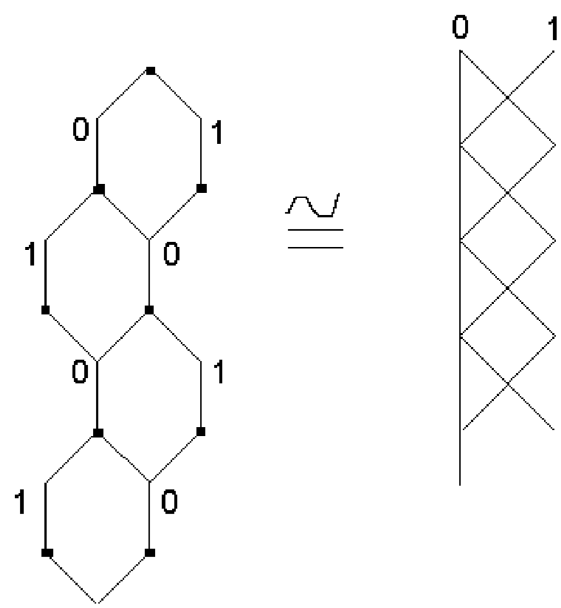

Fig. 8. Recoding the paths for $\alpha = \lambda^{-1}$, case **b**

4. $x \sim (1001(0001)^\infty)$. For this x, the transformation T_x acts as the special automorphism over the rotation by λ^{-1} with a single step equal to the base (see Fig. 9).

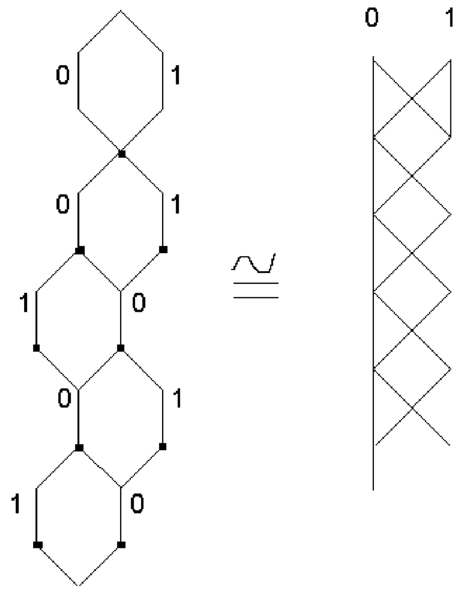

Fig. 9. Recoding the paths for case **c**

Appendix B. Arithmetic expression for block expansions

We remind that in Section 2 we have defined the mapping Ψ juxtaposing to a regular $x \in (\lambda^{-1}, 1)$ the sequence of blocks $B_1(x), B_2(x), \dots$. In this Appendix we are going to specify the mapping Ψ in an arithmetic way. To this end, we gather the canonical expansion of a given regular x blockwise.

Recall that similarly to the canonical expansion (1.1) of reals, there exists the corresponding representation of positive integers. Namely, each $N \in \mathbb{N}$ has a unique representation in the form

$$N = \sum_{i=1}^{k} \varepsilon_i F_i,$$

where $\varepsilon_i \in \{0, 1\}$, $\varepsilon_i \varepsilon_{i+1} = 0$, $\varepsilon_k = 1$ for some $k \in \mathbb{N}$. It is usually called *Zeckendorf* decomposition. We denote by $\mathcal{F}$ the class of positive integers whose Zeckendorf decomposition has $\varepsilon_1 = 1$ and $\varepsilon_i \equiv 0$ for all even i. Obviously, $\mathcal{F}$ as a subset of $\mathbb{N}$ has zero density. Let the *height* of e with a finite canonical expansion of the form $e = \sum_j \varepsilon_j \lambda^{-j}$ be, by definition, the positive integer $h(e) := \max\{j : \varepsilon_j = 1\}$.

Proposition B.1. *Each regular $x \in (\lambda^{-1}, 1)$ has a unique representation of the form*

$$x = \sum_{j=1}^{\infty} e_j(x) \lambda^{-\sum_{i=1}^{j} n_i(x)}, \tag{B.1}$$

where

(i) $e_j(x) = m\lambda - n$, $n \in \mathcal{F}$, $n = [m\lambda]$.
(ii) *n_j is odd, $n_j \geq 3$ for $j \geq 1$.*
(iii) *$n_j \geq 2h(e_j) + 1$ for all j.*

Proof. Let $\Psi(x)=B_1B_2\dots$ be the block expansion of x. Suppose $B_j=B_j(a_1^{(j)},\dots,a_{t_j}^{(j)})$. We set $n_j(x):=2\sum_{i=1}^{t_j}a_i^{(j)}+1,\ j\ge 1$, i.e. n_j is the length of the j'th block. Let $m_0^{(j)}=0$, $m_k^{(j)}=\sum_{i=1}^{k}a_i^{(j)}$, and let e_j be the "value" of B_j in the sense of formula (1.1) as if it was the first block, i.e.

$$\begin{aligned}
e_j(x)&=\lambda^{-1}+\lambda^{-1}\sum_{k=1}^{\frac12(t_j-1)}\sum_{\nu=1}^{a_{2k}^{(j)}}\lambda^{-2\left(m_{2k-1}^{(j)}+\nu\right)}\\
&=\lambda^{-1}+\lambda^{-2}\sum_{k=1}^{\frac12(t_j-1)}\left(\lambda^{-2m_{2k-1}^{(j)}}-\lambda^{-2m_{2k}^{(j)}}\right),\quad t_j\text{ odd},\\
e_j(x)&=\lambda^{-1}+\lambda^{-1}\sum_{k=1}^{\frac12 t_j}\sum_{\nu=1}^{a_{2k+1}^{(j)}}\lambda^{-2\left(m_{2k-2}^{(j)}+\nu\right)}\\
&=\lambda^{-1}+\lambda^{-2}\sum_{k=1}^{\frac12 t_j}\left(\lambda^{-2m_{2k-2}^{(j)}}-\lambda^{-2m_{2k-1}^{(j)}}\right),\quad t_j\text{ even}.
\end{aligned}$$

The uniqueness of expansion (B.1) follows from the condition (iii) and from the uniqueness of expansion (1.1) for any finite sequence (and, therefore, for any block).

Definition. We call expansion of $x\in(\lambda^{-1},1)$ of the form (B.1) satisfying the conditions (i)–(iii), the *arithmetic block expansion*.

Remark 1. n_j and e_j depend on B_j only.

Remark 2. In fact, series (B.1) is nothing but series (1.1) rewritten in a different notation. However, we will see that it has its own dynamical sense (see relation (B.2) below).

Remark 3. By Item (iii), the quantities e_j and n_j are not completely independent. Taking into consideration new quantities $s_j:=n_j-2l_j$ and representing n_j as the sum s_j and $2l_j$ in formula (B.1), we come to independent multipliers but this new form of the block representation does not seem to be natural.

Remark 4. In terms of arithmetic block expansion the goldenshift acts as

$$\mathcal{S}(x)=\sum_{j=2}^{\infty}e_j(x)\lambda^{-\sum_{i=2}^{j}n_i(x)}. \tag{B.2}$$

Appendix C. Computation of densities by means of the block expansion

We return to the subject of the first section. Recall that we have already denoted the rotation of the circle by λ^{-1} by R, and the transformation $x\mapsto\{\lambda x\}$ by τ.

C.1. Computation of densities.

Proposition C.1. *The densities* $\frac{d(R\mu)}{d\mu}$, $\frac{d(\tau\mu)}{d\mu}$ *and* $\frac{d\nu}{d\mu}$ *are unbounded and piecewise constant with a countable number of steps.*

Proof. By virtue of results of Section 1, it suffices to prove the proposition only for $\frac{d(R\mu)}{d\mu}$. Let E be a Borel subset of $(0,1)$. The idea of the study lies the fact that R does not change any block beginning with the second. As usual, we consider three cases.

I. $E \subset (0, \lambda^{-2})$. If $E \subset (\lambda^{-2k}, \lambda^{-2k+1})$, $k \geq 1$, then each point x of the set E has the canonical expansion (1.1) of the form $0^{2k-1}10*$. Hence the canonical expansion of $x + \lambda^{-1}$ is $1(00)^{k-1}10*$, and

$$\frac{\mu(E+\lambda^{-1})}{\mu E} \equiv k,$$

as $f(1(00)^{k-1}) = k$. If, on the contrary, $E \subset (\lambda^{-2k-1}, \lambda^{-2k})$, $k \geq 1$, then the situation is as follows. This interval in terms of the canonical expansion is $\bigcup_{\overline{B}} 0^{2k}\overline{B} \mod 0$, where the union runs over all closed blocks $\overline{B}$. We have two subcases.

Ia. Let in terms of the canonical expansion, $E \subset 0^{2k}1(00)^{a_1}(01)^{a_2}\dots(00)^{a_t}1$. Here $E + \lambda^{-1} \subset 1(00)^{k-1}(01)(00)^{a_1}(01)^{a_2}\dots(00)^{a_t}1$, hence $\frac{\mu(E+\lambda^{-1})}{\mu E} = \frac{f(B')}{f(B)}$, where B' is the closed block defined as $B' = B'(k-1, 1, a_1, a_2, \dots, a_t)$. So, we conclude from Lemma 2.1 that

$$\frac{\mu(E+\lambda^{-1})}{\mu E} = \frac{kp + (k+1)q}{p+q},$$

where, as usual, $\frac{p}{q} = [a_1, a_2, \dots, a_t]$.

Ib. In the same terms, $E \subset 0^{2k}1(01)^{a_1}(00)^{a_2}\dots(00)^{a_t}1$. Similarly to the above,

$$\frac{\mu(E+\lambda^{-1})}{\mu E} = \frac{(k+1)p + kq}{p+q}.$$

II. $E \subset (\lambda^{-2}, \lambda^{-1})$. This case is analogous to Case I. If $E \subset (\lambda^{-2} + \lambda^{-2k-3}, \lambda^{-2} + \lambda^{-2k-2})$, $k \geq 1$, then

$$\frac{\mu(E-\lambda^{-2})}{\mu E} \equiv \frac{1}{k}.$$

If $E \subset (\lambda^{-2} + \lambda^{-2k-2}, \lambda^{-2} + \lambda^{-2k-1})$, $k \geq 1$, then

$$\frac{\mu(E-\lambda^{-2})}{\mu E} \equiv \begin{cases} \frac{p+q}{kp+(k+1)q}, & E \subset 01(00)^{k-1}01(00)^{a_1}(01)^{a_2}\dots(00)^{a_t}1 \\ \frac{p+q}{(k+1)p+kq}, & E \subset 01(00)^{k-1}01(01)^{a_1}(00)^{a_2}\dots(00)^{a_t}1. \end{cases}$$

III. $E \subset (\lambda^{-1}, 1)$. If $E \subset (\lambda^{-1}, \lambda^{-1} + \lambda^{-4})$, then $\mu(E - \lambda^{-2}) = \mu E$. If $E \subset (\lambda^{-1} + \lambda^{-4}, 1)$, then $E - \lambda^{-2} \subset (\lambda^{-2}, \lambda^{-1})$, hence $E - \lambda^{-2} \subset 010*$.

IIIa. $E - \lambda^{-2} \subset 1(00)^{a_1}(01)^{a_2}\dots(00)^{a_t}1$, then

$$\frac{\mu(E-\lambda^{-2})}{\mu E} = 1 + \frac{p}{q}.$$

IIIb. $E-\lambda^{-2}\subset 1(01)^{a_1}(00)^{a_2}\dots(00)^{a_t}1$. Here

$$\frac{\mu(E-\lambda^{-2})}{\mu E}=1+\frac{q}{p}.$$

The proof is complete.

Remark 1. Let, as above, $p=\frac{d(R\mu)}{d\mu}$. Then, as it was mentioned in Section 1 (see Proposition 1.5), $\frac{d(\tau\mu)}{d\mu}(x)=\frac12(p^{-1}(R^{-1}(x)+1)$, and

$$\frac{d\nu}{d\mu}(x)=\begin{cases}\frac23+\frac13p^{-1}(x+\lambda^{-2})+\frac16p(x), & x\in[0,\lambda^{-2})\\ \frac23+\frac13p^{-1}(x+\lambda^{-2}), & x\in[\lambda^{-2},\lambda^{-1})\\ \frac12+\frac13p^{-1}(x-\lambda^{-1}), & x\in[\lambda^{-1},1].\end{cases}$$

Remark 2. From this relation it follows Proposition 2.10.

C.2. The polymorphism Π. Let as above $\sigma:\prod_1^\infty\{0,1\}\to\prod_1^\infty\{0,1\}$ be the one-sided shift. Let us ask the natural question: what is the image of σ on the interval $[0,1]$ under the mapping π defined by the formula (3.1)?

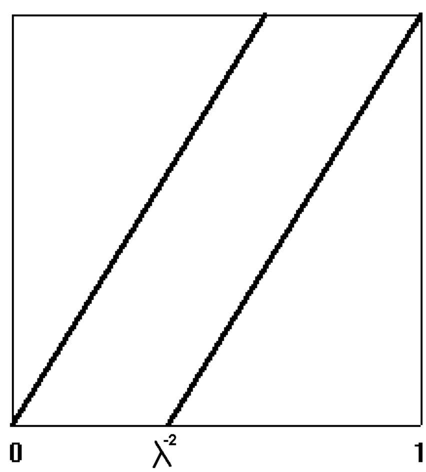

Fig. 10. The polymorphism Π

Note first that the partition into π-preimages of singletons is not invariant under σ. Indeed, if, say, $x=0110^\infty$, then $\sigma\mathfrak{n}(x)=0^\infty$, while $\mathfrak{n}\sigma(x)=110^\infty$. Thus, $\Pi:=\pi\sigma\pi^{-1}:[0,1]\to[0,1]$ is only a multivalued mapping, i.e. a polymorphism by the terminology of [Ver4]. Topologically it acts by the formula

$$\Pi(x)=\begin{cases}\lambda x, & 0\le x<\lambda^{-2}\\ \lambda x\cup\lambda x-\lambda^{-1}, & \lambda^{-2}\le x<\lambda^{-1}\\ \lambda x-\lambda^{-1}, & \lambda^{-1}\le x\le 1.\end{cases}$$

This polymorphism was considered in [VerSi]. By definition (see [Ver4]), Π preserves the Erdös measure μ, as $\mu=\pi(p)$, the latter being preserved by σ. Whence in the decomposition of the Π-preimage, i.e. $\Pi^{-1}E = \pi\sigma^{-1}\pi^{-1}E = \lambda^{-1}E \cup \lambda^{-1}E + \lambda^{-2}$, the sets involved do not have equal conditional measures, as $\lambda^{-1}E$ corresponds to $\varepsilon_1 = 0$ and $\lambda^{-1}E + \lambda^{-2}$ corresponds to $\varepsilon_1 = 1$. We state the following claim which is proved in the same way as Proposition C.1.

Proposition C.2. *Let $\gamma_1(x)$ and $\gamma_2(x)$ be the conditional measures of $\Pi^{-1}x$ with respect to $\lambda^{-1}x$ and $\lambda^{-1}x + \lambda^{-2}$ respectively, or, in other terms, of $\sigma^{-1}\pi^{-1}(x)$ with respect to the cylinders $(\varepsilon_1 = 0)$ and $(\varepsilon_1 = 1)$ in $\prod_1^\infty\{0,1\}$ respectively. Then the functions γ_1 and γ_2 are piecewise constant with a countable number of steps, and we have the following alternative.*
1. $x \in (0, \lambda^{-2})$. If $x \in (\lambda^{-2k}, \lambda^{-2k+1})$, then $\gamma_1(x) \equiv \frac{k}{k+1}$, $\gamma_2(x) \equiv \frac{1}{k+1}$. If $x \in (\lambda^{-2k-1}, \lambda^{-2k})$, then we use the terms of generalized canonical expansion (see Appendix A for the definition) and blocks. So, if $x \sim 0^{2k-1}B1*$ with B being a block corresponding to $\frac{p}{q}$, then

$$\gamma_1(x) = \begin{cases} \frac{kp+(k+1)q}{(k+1)p+(k+2)q}, & B = 100\dots \\ \frac{(k+1)p+kq}{(k+2)p+(k+1)q}, & B = 101\dots, \end{cases}$$

and

$$\gamma_2(x) = \begin{cases} \frac{p+q}{(k+1)p+(k+2)q}, & B = 100\dots \\ \frac{p+q}{(k+2)p+(k+1)q}, & B = 101\dots \end{cases}$$

2. $x \in (\lambda^{-2}, \lambda^{-1})$, hence $\gamma_1(x) = \gamma_2(x) \equiv \frac{1}{2}$.
3. $x \in (\lambda^{-1}, 1)$. If $x \sim 1^{2k+1}0*$, $k \geq 1$ in terms of generalized canonical expansion, then $\gamma_1(x) \equiv \frac{1}{k+2}$, $\gamma_2(x) \equiv \frac{k+1}{k+2}$. If, on the contrary, $x \sim 1^{2k}0*$, then

$$\gamma_1(x) = \begin{cases} \frac{p+q}{(k+2)p+(k+1)q}, & B = 100\dots \\ \frac{p+q}{(k+1)p+(k+2)q}, & B = 101\dots, \end{cases}$$

and

$$\gamma_2(x) = \begin{cases} \frac{(k+1)p+kq}{(k+2)p+(k+1)q}, & B = 100\dots \\ \frac{kp+(k+1)q}{(k+1)p+(k+2)q}, & B = 101\dots, \end{cases}$$

if $x \sim 1^{2k-1}B1*$.

Appendix D. An independent proof of Alexander-Zagier's formula

In this appendix we will present the second proof of formula (3.3) which reveals some new relations between certain structures of the Fibonacci graph Φ.

We first recall that the quantity $f_n(k)$ is nothing but the frequency of the k'th vertex on the n'th level of the Fibonacci graph which was denoted by D_n (see the beginning of Section 3). We have $\#D_n = F_{n+2} - 1$.

Consider level n of the Fibonacci graph for $n = 2N+1$. We denote the *middle* part of D_n, i.e. the segment from F_n to $F_{n+1} - 1$, by D'_n. The Erdös measure of D'_n obviously equals $\frac{1}{3} + O(\lambda^{-n})$, and we will introduce the partition of D'_n into $2^{N-1} - 1$ intervals of vertices in the following way. Let, by definition, a *Euclidean* vertex be the one, where the first block can end (they are marked in Fig. 3 for D_3 and D_5). These vertices form the *Euclidean* binary tree introduced in [AlZa]. It is easily seen that there are 2^{N-1} such vertices at level $2N+1$, and all of them lie in D'_n. Let $V_k^{(N)}$ denote the k'th Euclidean vertex from the left on level $2N+1$.

Definition. An open interval of vertices $\Omega_k^{(N)} := (V_k^{(N)}, V_{k+1}^{(N)})$ will be called the *Euclidean* interval.

So, we have divided the set of vertices D'_n into 2^{N-1} Euclidean vertices $\left(V_k^{(N)}\right)_{k=1}^{2^{N-1}}$ and $2^{N-1}-1$ open intervals $\Omega_k^{(N)}$. Now we introduce the subgraph Γ_V associated with each Euclidean vertex V. It is defined as the one containing all the successors of V in the sense of the Fibonacci graph, except any other Euclidean vertices (see Fig. 11).

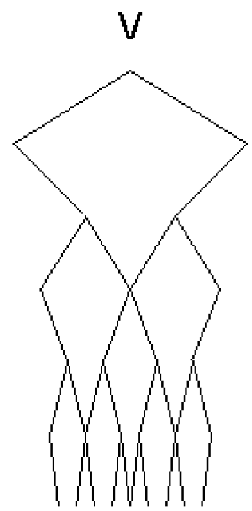

Fig. 11. The graph Γ_V

We state the straightforward lemma.

Lemma D.1. *For any Euclidean interval $\Omega_k^{(N)}$ there is a unique Euclidean vertex $V_i^{(j)}$, $j < N$ such that $\Omega_k^{(N)} \subset \Gamma_{V_i^{(j)}}$.*

So, any Euclidean interval is determined by a certain Euclidean vertex on one of the preceding odd levels of the Fibonacci graph. Moreover, in the notation of the above lemma, the entropy of $\Omega_k^{(N)}$ may be computed in terms of the frequency of $V_i^{(j)}$ and the entropy of $\widetilde{D}_{2N+1-j}$. Namely, let $H_n := \sum\limits_{k=F_n}^{F_{n+1}-1} f_n(k)\log_\lambda f_n(k)$, and let next $H_{2N+1} = \sum_{j=1}^N H_{2N+1}^{(j)}$, where $H_{2N+1}^{(j)}$ denotes the sum over the vertices $V \in \Gamma_{V_i^{(j)}}$ for all $i \le 2^{j-1}$. So, $H_{2N+1}^{(j)}$ corresponds to all Euclidean vertices of level $2j+1$.

Let next $\varphi_i^{(j)}$ denote the frequency of $V_i^{(j)}$. For instance, for $j=3$, $\varphi_1^{(3)} = \varphi_4^{(3)} = 4$, $\varphi_2^{(3)} = \varphi_3^{(3)} = 5$. In this notation $k_j = \sum_{i=1}^{2^{j-1}} \varphi_i^{(j)} \log_\lambda \varphi_i^{(j)}$.

For any $v \in (V_k^{(N)}, V_{k+1}^{(N)})$ its frequency equals $f(V_i^{(j)})$ times the frequency of the corresponding vertex of the central part of level $2N+1-j$. So, we established an essential relationship between the central part of level $2N+1$ of the Fibonacci graph and all Euclidean vertices $V_i^{(j)}$, $1 \le j \le N$, $1 \le i \le 2^{j-1}$.

Lemma D.2. *The following recurrence relation holds:*

$$H_{2N+1} = \frac{2}{3}\sum_{j=1}^{N-1} 3^j H_{2N-2j} + \frac{1}{3}\cdot 4^N \cdot \sum_{j=1}^{N} \frac{k_j}{4^j} + O\left(\sum_{j=1}^{N} k_j\right), \quad N \to \infty. \tag{D.1}$$

Proof. By the above considerations,

$$\begin{aligned} H_{2N+1}^{(j)} &= \sum_{i=1}^{2^{j-1}} \sum_{k=F_{2N-2j}}^{F_{2N-2j+1}-1} \varphi_i^{(j)} f_{2N-2j}(k) \log_\lambda \left(\varphi_i^{(j)} f_{2N-2j}(k)\right) \\ &= \sum_{i=1}^{2^{j-1}} \varphi_i^{(j)} \left(H_{2N-2j} + \frac{1}{3}\log_\lambda \varphi_i^{(j)} \cdot \left(4^{N-j} + O(1)\right)\right) \\ &= 2H_{2N-2j}\cdot 3^{j-1} + \frac{1}{3}\cdot\frac{k_j}{4^j}\cdot 4^N + O(k_j) \end{aligned}$$

(we used the fact that $\sum_{i=1}^{2^{j-1}} \varphi_i^{(j)} = 2\cdot 3^{j-1}$ easily obtained from Corollary 2.9. Hence it follows relation (D.1).

Remark. Formula (D.1) shows that the entropy of the n'th level with n odd can be computed by means of the entropies of the previous even levels and the entropy of the Euclidean tree.

Now we are ready to complete the second proof of formula (3.3). We have

$$nH_\mu \sim \sum_{k=0}^{F_{n+2}-2} \frac{f_n(k)}{2^n} \log_\lambda \frac{2^n}{f_n(k)},$$

whence

$$nH_\mu \sim 3\sum_{k=F_n}^{F_{n+1}-1} \frac{f_n(k)}{2^n} \log_\lambda \frac{2^n}{f_n(k)},$$

and

$$H_n \sim \frac{1}{3}(\Lambda - H_\mu)n2^n. \tag{D.2}$$

From relation (D.2) it follows that in the sum $\sum\limits_{j=1}^{N-1} 3^j H_{2N-2j}$ the first terms are more valuable than the last. Thus, from formulas (D.1) and (D.2) and from the fact that

$k_N = \sum_{i=1}^{2^{N-1}} \varphi_i^{(N)} \log_\lambda \varphi_i^{(N)} < 2(N-1)3^{N-1}$ it follows that

$$\frac{1}{3}(\Lambda - H_\mu) \cdot (2N+1)2^{2N+1} \sim \frac{2}{3}\sum_{j=1}^{N} 3^j \cdot \frac{1}{3}(\Lambda - H_\mu)(2N-2j)4^{N-j} + \frac{1}{3} \cdot 4^N \sum_{j=1}^{N} \frac{k_j}{4^j},$$

whence, after straightforward computations,

$$18(\Lambda - H_\mu) \sim \sum_{j=1}^{N} \frac{k_j}{4^j}, \quad N \to \infty. \quad \square$$

Remark. The Euclidean tree naturally splits into two binary subtrees (left and right) being symmetric. If we label each vertex of the left subtree with the corresponding rational p/q, then this left subtree turns out to coincide with the *Farey* tree introduced and studied in detail in [La].

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