A Note on Carnot Geodesics in Nilpotent Lie Groups

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Abstract

We show that strictly abnormal geodesics arise in graded nilpotent Lie groups. We construct such a group, for which some Carnot geodesics are strictly abnormal and, in fact, not normal in any subgroup. In the 2-step case we also prove that these geodesics are always smooth. Our main technique is based on the equations for the normal and abnormal curves, that we derive (for any Lie group) explicitly in terms of the structure constants.

1 Introduction

Our work is motivated by the problem of differentiability of Carnot geodesics (or minimizers) in nilpotent Lie groups.

A Carnot (or sub-Riemannian) structure on a manifold G is given by a smoothly varying distribution D (i.e. a field of tangent subspaces) and a smoothly varying inner product on this distribution. A *horizontal curve* is an absolutely continuous curve on G which is tangent to D wherever it is differentiable. The inner product on D enables one to define the length of a horizontal curve, and it is then natural to study Carnot geodesics, or minimizers for this length. A *minimizer* is an absolutely continuous horizontal curve x in G which is such that, for each t, there exists $\epsilon > 0$ such that x minimizes the length between $x(t_0)$ and $x(t_1)$ whenever t_0, t_1 are in $(t-\epsilon, t+\epsilon)$.

Until recently, it was not understood that minimizers could be of two different, *but non mutually exclusive* types. One type is given by the projections of solutions of a Hamiltonian system (see Section 2.2), which is in a sense the Legendre transform of the inner product on D. This generalizes the Riemannian situation. Such curves are called *normal*. Normal curves are known to be differentiable minimizers [5].

The other type of minimizer belongs to a category of horizontal curves called *abnormal* or *singular*. Although originally given by the Maximum Principle, Hsu [6] (see also [10]) shows that they are projections onto G of characteristic curves (in the symplectic sense) of the annihilator of D in T^*G. See Definition 2.1.

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Contrary to the normal curves, the abnormal ones need not be minimizing. If they are minimizing, they are called *abnormal minimizers.* That such curves exist was only proven recently by Montgomery [11]. For rank-2 distributions, Liu and Sussmann [8] point to a generic class of abnormal curves that also minimize. They are called *regular abnormal extremals*, and are used in Section 2.3. These curves were also studied by Bryant and Hsu in their paper on rigid curves [2]. Again, let us emphasize that the same curve can be both normal and abnormal: it can have a lift in the cotangent bundle that makes it abnormal and another one that makes it normal. If a curve is abnormal but not normal, we say that it is *strictly abnormal.*

In this paper we study Carnot minimizers in nilpotent Lie groups. This context is important because, under natural conditions, any Carnot manifold, viewed as a metric space, asymptotically looks like some nilpotent Lie group. More precisely, given any metric space (M,d), Gromov ([4] Section 1.4.B) defines the tangent cone to M at $q \in M$ to be the Hausdorff limit of the metric spaces (M, λd), with base point at q, when $\lambda \to \infty$. Mitchell [9] proves that if a Carnot manifold (M, D), is regular at q, then the tangent cone at q is isometric (as a metric space) to a Carnot graded nilpotent Lie group (called the nilpotentization), defined using the bracket relations on TM (see Section 2 below).

Our most surprising result (see Theorem 3.2) is the construction of a graded nilpotent Lie group, where we find many abnormal minimizers that are not normal, i.e. we find *strictly abnormal minimizers.* In light of Mitchell's theorem, this shows that the analysis of abnormal minimizers cannot be avoided in general Carnot spaces.

That strictly abnormal minimizers exist is relevant to a natural problem posed by Hamenstädt: *are Carnot minimizers always differentiable ?* (see [5],[11]). Indeed, since normal minimizers are automatically differentiable, Hamenstädt's question reduces to: *are abnormal minimizers always differentiable?* There are examples of non differentiable abnormal curves, even in the nilpotent Lie group situation studied here, but none that are minimizers. To solve this problem in Lie groups, Hamenstädt suggested to try to prove that any minimizer is normal in a subgroup, say $H < G$, with the Carnot structure given by the distribution $D_H = D \cap TH$ and the restricted inner product. The existence of such an H for each minimizer would obviously imply that any minimizer is differentiable. This program was carried out successfully by Montgomery in [10] in the case where G is a compact connected Lie group and D is the left invariant distribution orthogonal to its maximal torus.

In contrast, the strictly abnormal minimizers that we exhibit in Theorem 3.2, are not normal in any subgroup. However, we can carry out Hamenstädt's program in the case where G is a 2–step nilpotent Lie group (i.e. its Lie algebra $\mathfrak{g}$ satisfies $[\mathfrak{g}, [\mathfrak{g},\mathfrak{g}]]= 0$) and D is a left invariant distribution such that $D \oplus [\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ (Theorem 4.1). Note that the latter condition is satisfied for graded nilpotent Lie algebras. This extends known results on the so-called Gaveau-Brockett problem [1],[3].

Our methods rely on deriving the equations for the normal and abnormal curves purely in terms of the structure constants. These equations appear in Section 2. We also found the methods in [7] useful in our investigations.

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2 Abnormal and Normal equations

Let G be a Lie group, and D a left invariant distribution. We identify D with a left translation of a subspace of the Lie algebra $\mathfrak{g}$, that we will also denote by D. Choose a left invariant frame $\{e_1, \ldots, e_r\}$ for D, and complete it to a basis $\{e_1, \ldots, e_n\}$ of the whole Lie algebra. We give D a metric that makes $\{e_1, \ldots, e_r\}$an orthonormal basis. Let $\{\theta_1, \ldots, \theta_n\}$ be the dual co-frame to $\{e_1, \ldots, e_n\}$. We write a vector field in TG as $\sum_1^n \gamma_i e_i$, where $\gamma_1, \ldots, \gamma_n$ are the coordinate functions on the fiber of TG. Likewise, a covector is written $\sum_1^n \lambda_i \theta_i$. A vector in D (resp. a covector in $D^\perp$) can be written as $\sum_1^r \gamma_i e_i$ (resp. $\sum_{r+1}^n \lambda_i \theta_i$).

The structure constants α_{ijk} of $\mathfrak{g}$ with respect to the basis $\{e_1, \ldots, e_n\}$ are defined by:

$$[e_i, e_j] = \sum_{k=1}^n \alpha_{ijk} e_k \ .$$

Note that $\alpha_{ijk} = -\alpha_{jik}$. Since e_i is the Hamiltonian vector field for λ_i, we also have:

$$\{\lambda_i, \lambda_j\} = -\sum_{k=1}^n \alpha_{ijk} \lambda_k \ ,$$

with respect to the standard Poisson brackets of functions on $\mathfrak{g}^*$.

Denote $D^1 = D$, $D^{i+1} = D^i + [D, D^i]$ (if this sum is direct, one says that $\mathfrak{g}$ is *graded*). If there exists r for which $D^r = \mathfrak{g}$ we say that D is *bracket-generating.* Define also $V^1 = D^1$, $V^i = D^i/D^{i-1}$ and $Gr\ \mathfrak{g} = V^1 \oplus \ldots \oplus V^r$. The latter is a graded nilpotent Lie algebra and the associated simply connected Lie group is called the *nilpotentization* of G, which is also endowed with a Carnot metric.

Remark In the case of a general Carnot manifold G, the above makes sense locally, at a point $q \in G$, if one assumes, in addition, that $r(q)$ is locally constant at that point. The theorem of Mitchell [9] alluded to in the introduction relates the local metric properties of a Carnot manifold with those of its nilpotentization.

2.1 The Abnormal Equations

We begin by giving a rigorous definition of abnormal curves and minimizers:

2.1 Definition: *An abnormal curve is a horizontal curve which is the projection onto G of an absolutely continuous curve in the annihilator $D^\perp \subset T^*G$ of D, with square integrable derivative, which does not intersect the zero section and whose derivative, whenever it exists, is in the kernel of the canonical symplectic form restricted to $D^\perp$. An abnormal minimizer is an abnormal curve which is a minimizer, in the sense given in the introduction. A strictly abnormal curve (resp. minimizer) is an abnormal curve (resp. minimizer) which is not normal, in the sense of Definition 2.4.*

This definition makes it clear that being abnormal is independent of the parameterization. In [10], Proposition 1, Montgomery shows that the above definition of abnormal curve is equivalent to three other ones, which we will not use in this paper, but will state for the further confusion of the reader. Let x be a curve in G and ζ be an absolutely continuous curve in T^*G which does not intersect the zero section and whose derivative is square integrable. Suppose that $x = \pi(\zeta)$ and x and ζ satisfy the above definition. Then, the following are equivalent to the above definition.

1. x is an abnormal extremal in the sense of the Pontryagin maximum principle of control theory (this is the original definition of abnormal curves).

2. ζ annihilates the image of the differential $d(end(x(t)))$ at each t, where *end* is the map associating to a curve its endpoint.

3. x is horizontal, $\zeta \in D^{\perp}$ and $\zeta(t) = (D\Phi_t^T)^{-1}\zeta(0)$ where Φ_t is any time dependent flow which generates the curve x.

We now follow the derivation of the abnormal equations in ([10] Section 4). Remember that the canonical 1-form on T^*G, call it η, is defined by $\eta_\alpha(v) = \alpha(\pi_* v)$, where $\alpha \in T^*G$ is the base point of the vector $v \in T_\alpha(T^*G)$. We claim that $\eta_\alpha = \sum_{i=1}^n \lambda_i(\alpha)\theta_i$, where θ_i are viewed as 1-forms on T^*G. Let $v = \sum \gamma_i e_i + \sum h_i \frac{\partial}{\partial \lambda_i} \in T(T^*G)$, then by the definition of η:

$$\eta_\alpha(v) = \alpha(\pi_* v) = \alpha(\sum \gamma_i e_i) = \sum \lambda_i(\alpha)\theta_i(\sum \gamma_i e_i) = \sum \lambda_i(\alpha)\gamma_i \ ,$$

which coincides with $\sum \lambda_i(\alpha)\theta_i(v)$. In particular, $\eta_{|_{D^\perp}} = \sum_{i=r+1}^n \lambda_i(\alpha)\theta_i$, and, since $\omega = d\eta$,

$$\omega_{|_{D^\perp}} = \sum_{i=r+1}^{n} (d\lambda_i \wedge \theta_i + \lambda_i d\theta_i) \ .$$

Let $(x(t), \lambda(t)) \in T^*G$ be such that $x(t)$ is an abnormal curve. Then

$$x' = \frac{dx}{dt} = \sum_{i=1}^{r} \gamma_i e_i \quad , \quad \lambda' = \frac{d\lambda}{dt} = \sum_{i=r+1}^{n} {\lambda_i}' \frac{\partial}{\partial \lambda_i} \quad , \quad \lambda \neq 0 \quad ,$$

and

2.2

$$0 = \omega_{(x,\lambda)} \ ((x', \lambda'), \cdot) = \sum_{i=r+1}^{n} ({\lambda_i}'\theta_i - \gamma_i d\lambda_i) + \sum_{k=r+1}^{n} \lambda_k d\theta_k((x', \lambda'), \cdot) \quad .$$

The Maurer-Cartan equations are:

$$d\theta_k + \frac{1}{2}\sum_{i,j=1}^{n} \alpha_{ijk}\theta_i \wedge \theta_j = 0 \ .$$

Therefore

$$d\theta_k((x', \lambda'), \cdot) = -\frac{1}{2}\sum_{i,j=1}^{n} \alpha_{ijk}(\gamma_i\theta_j - \gamma_j\theta_i) = \sum_{i,j=1}^{n} \alpha_{ijk}\gamma_j\theta_i \ .$$

Plug this in (2.2), using $\gamma_i = 0$ for $i > r$:

$$0 = \sum_{i=r+1}^{n} \left(\ {\lambda_i}' + \sum_{j=1}^{n}\sum_{k=r+1}^{n} \alpha_{ijk}\gamma_j \ \right) \theta_i + \sum_{i=1}^{r} \left(\ \sum_{j=1}^{n}\sum_{k=r+1}^{n} \alpha_{ijk}\gamma_j \ \right) \theta_i \ .$$

Since the θ_i's are linearly independent we get

2.3 *The abnormal equations:*

$$\sum_{j=1}^{r}\sum_{k=r+1}^{n}\alpha_{ijk}\gamma_j\lambda_k = 0\ ,\quad \text{for}\ \ i=1,\ldots,r\ .$$

$$\lambda_i{}' + \sum_{j=1}^{r}\sum_{k=r+1}^{n}\alpha_{ijk}\gamma_j\lambda_k = 0\ ,\quad \text{for}\ \ i=r+1,\ldots,n\ .$$

$$\gamma_{r+1} = \ldots = \gamma_n = 0$$
$$\lambda_1 = \ldots = \lambda_r = 0$$

Remark These equations are mixed algebraic-differential equations. Unlike the normal equations (which, as we will see, are defined by a single vector field), the abnormal ones cannot be expressed as ordinary differential equations. Note also that the notion of abnormal curve only depends on D, and not on the metric.

2.2 The Normal Equations

To distinguish the cotangent lifts of normal and abnormal curves, we denote the covector frame coordinates in the normal case by $h_1,\ldots,h_n$ (instead of $\lambda_1,\ldots,\lambda_n$).

2.4 Definition: *A normal curve is the projection onto G of a solution of the Hamiltonian system in T^*G with Hamiltonian $H(x,h)=\frac{1}{2}\sum_1^r h_i^2$.*

It is known [12] that normal curves are smooth minimizers. We can write the Hamiltonian vector field in the frame coordinates as (γ, h'). It will satisfy:

2.5 *The normal equations:*

$$\begin{array}{lll}\gamma_i = h_i & \text{for} & i=1,\ldots,r\\ \gamma_j = 0 & \text{for} & j=r+1,\ldots,n\end{array}$$

$$h_i{}' + \sum_{j=1}^{r}\sum_{k=1}^{n}\alpha_{ijk}h_jh_k = 0 \quad \text{for}\ \ i=1,\ldots,n\ .$$

The first two equations are given by the fact that, in a canonical system of coordinates (q,p) on T^*G, h_i can be seen as the fiber linear function:

$$h_i(q,p) = p(e_i(q)),\ q\in G,\ p\in T_q^*G$$

To get the third equation observe that for any function f on T^*G and a solution $z(t)$ to the Hamiltonian equations:

$$\frac{d}{dt}(f(z(t)) = \{f,H\}\ .$$

In particular ${h_i}' = \{h_i, H\}$ for $i = 1, \ldots, n$ and thus

$$ {h_i}' = \{h_i, \frac{1}{2}\sum_1^r h_j^2\} = \frac{1}{2}\sum_1^r \{h_i, h_j\} h_j = -\sum_{j=1}^r \sum_{k=1}^n \alpha_{ijk} h_j h_k . $$

Consider now a *nilpotent* Lie group G and assume that the distribution D is complementary to $[\mathfrak{g},\mathfrak{g}]$, i.e. $D \oplus [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. Note that this condition is satisfied when G is graded and D is bracket-generating. In fact, the meaning of this assumption is that D is minimal in the sense that no proper subspace of D will bracket generate the full Lie algebra. This also means that $\alpha_{ijk} = 0$ for $k = 1, \ldots, r$ and thus the second normal equation reduces to:

$$ {h_i}' + \sum_{j=1}^r \sum_{k=r+1}^n \alpha_{ijk} h_j h_k = 0 \ . $$

2.6 Proposition: *Under these assumptions, the one-parameter horizontal subgroups are normal curves for any left invariant metric on D.*

Proof: Let $x(t)$ be a "left invariant" horizontal curve, i.e. $x'(t) = \sum \gamma_i e_i$, where γ_i are constants and $\gamma_{r+1} = \ldots = \gamma_n = 0$. We choose the covector part to have the following coordinates:

$$ \begin{aligned} h_i &= \gamma_i \quad i = 1, \ldots, r \\ h_i &= 0 \quad i = r+1, \ldots, n \end{aligned} $$

It is then easy to verify that the normal equations hold. □

Remarks

1. Note that one can always assume that a normal curve x is parameterized by arc-length. Indeed, $\frac{1}{2}\|x'\|^2 = H$ is constant.
2. As we have said in the introduction, being abnormal and being normal are not mutually exclusive properties. The same curve $x(t)$ in G can have lifts (x, λ) and (x, h), with the first making x abnormal, the other normal. When that happens, one can choose the arc-length parameterization for both.

Example: Take the Engel algebra, $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, with $D = Span\{e_1, e_2\}$. Then the only abnormal curves are tangent to e_2, i.e. they are one-parametric subgroups, and by the above proposition, they are also normal.

2.3 Regular abnormal extremals

Liu and Sussmann [8] show that if D is a two dimensional distribution in a Carnot manifold M of dimension ≥ 3, there is an efficient way of finding lots of abnormal geodesics, or minimizers.

Namely, they introduce the notion of *regular abnormal extremals* and prove that all such regular abnormal extremals are in fact minimizers ([8], Theorem 5). The following definition is equivalent to theirs, and can be extracted from their Proposition 6 and the beginning of their Section 6.2. As before, let D^k be the set of Lie brackets of order k or less of vector fields in D, and $(D^k)^\perp$ the annihilator of this set in the cotangent bundle.

2.7 Definition: *A curve $x(t)$ in G, parameterized by arclength, is called a regular abnormal extremal (or regular abnormal, in short) if it has a lift $(x(t), \lambda(t))$ which satisfies the abnormal equations 2.3 and such that*

$$\lambda(t) \in (D^2)^\perp - (D^3)^\perp.$$

Another interesting property of regular abnormals, which we will not use here, is that they are projections of integral curves of a certain vector field $\chi(D)$ in $(D^2)^\perp - (D^3)^\perp$. This is in fact the definition of a regular abnormal extremal given in [8]. Liu and Sussman have also genericity results showing that, roughly, among all lifts of abnormal curves parameterized by arclength, the regular abnormal extremals are prevalent.

3 Strictly abnormal geodesics in nilpotent Lie groups

In this section, we produce examples of strictly abnormal curves in a graded nilpotent Lie group G. Note that such examples exist in non-nilpotent Lie groups (see for example [8], Section 9.5). Our examples also have the property that they are not normal in any proper subgroup of G.

Let $\mathfrak{g}$ be the 6 dimensional real Lie algebra spanned by $\{e_1, \ldots, e_6\}$ with the following relations:

$$[e_1, e_2] = e_3 \ , \ [e_1, e_3] = e_4 \ , \ [e_2, e_3] = e_5 \ , \ [e_1, e_4] = e_6 \ .$$

Let the distribution $D = Span\{e_1, e_2\}$. It satisfies $D \oplus [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. We put on G the Carnot metric that makes the frame e_1, e_2 of the distribution D orthonormal. We are looking for an abnormal curve, say $(x_1(t), \ldots, x_6(t))$, which cannot be normal. Writing $x' = \gamma_1 e_1 + \gamma_2 e_2$, we first find a lift of x to the cotangent bundle, which satisfies the abnormal equations. Denote (as in Section 2) the cotangent part by $\sum \lambda_i(t)\theta_i$ where $(\theta_1, \ldots \theta_6)$ is a basis of left invariant 1-forms on G dual to $\{e_1, \ldots, e_6\}$.

The abnormal equations given in (2.3) become in this case:

$$\begin{aligned} \gamma_2\lambda_3 &= 0 \\ -\gamma_1\lambda_3 &= 0 \\ \lambda_3' - \gamma_1\lambda_4 - \gamma_2\lambda_5 &= 0 \\ {\lambda_4}' - \gamma_1\lambda_6 &= 0 \\ {\lambda_5}' &= 0 \\ {\lambda_6}' &= 0 \end{aligned}$$

Remember that, for an abnormal, $\lambda_1 = \lambda_2 = 0$ and note that here $\lambda_3 = 0$ as well (otherwise x would be the trivial, constant solution), so that the λ_3' term in the third equation is actually zero. We will assume that x is parameterized by arc-length, i.e. $\gamma_1^2 + \gamma_2^2 = 1$. For our class of examples, we will seek a solution x with $\lambda_5, \lambda_6 \neq 0$.

From $\gamma_1 = \frac{\lambda_4{}'}{\lambda_6}$, $\gamma_2 = -\gamma_1 \frac{\lambda_4}{\lambda_5} = -\frac{\lambda_4 \lambda_4{}'}{\lambda_5 \lambda_6}$, and $\gamma_1^2 + \gamma_2^2 = 1$ we get an o.d.e for λ_4:

3.1

$$(\lambda_4{}')^2(\lambda_4^2 + \lambda_5^2) = \lambda_5^2\ \lambda_6^2\ .$$

3.2 Theorem: *Let λ_4 be a solution of equation (3.1), with $\lambda_5, \lambda_6 \neq 0$. Then any solution to the time dependent o.d.e $x' = \gamma_1 e_1 + \gamma_2 e_2$ with $x(0) = 0$, where γ_1, γ_2 satisfies $\gamma_1 = \lambda_4{}'/\lambda_6$, $\gamma_2 = -\lambda_4 \lambda_4{}'/\lambda_5 \lambda_6$ is a strictly abnormal minimizer. Such a curve cannot be normal in any subgroup of G either.*

Proof: That such an x is abnormal derives directly from the abnormal equations and our discussion above. To see that it is a minimizer we observe that it is a regular abnormal extremal. Indeed, λ belongs to $(D^2)^\perp$ since (x, λ) satisfies the abnormal equations, and $\lambda_5 \neq 0$ implies that $\lambda \notin (D^3)^\perp$.

We now argue that γ_1, γ_2 cannot be constant. By (3.1), $\lambda_4' \neq 0$. But then, the same equation tells us that, since λ_4 is not constant, neither is λ_4'. From that it easily follows that γ_1 and γ_2 are not constant. The following proposition tells us that, because of this, x cannot be normal in G, hence it is a strictly abnormal minimizer.

Finally, the fact that γ_1 and γ_2 are non constant also tells us that x cannot be embedded in any proper subgroup of G. If it were, the pull-backs of the tangent vectors to x would all belong to a proper subalgebra. But since γ_1, γ_2 are not constant, there exist two pull-backs that span D as a vector space, and hence generate $\mathfrak{g}$ as a Lie algebra. $\square$

3.3 Proposition: *The only normal abnormals in this Lie group are the left invariant curves, i.e. integral curves of the left invariant vector fields (for which γ_1 and γ_2 are constant).*

Proof: Let x be an abnormal curve which is also normal. As before, we assume that x is parameterized by arc-length. This implies the existence of two cotangent lifts: (x, λ) and (x, h) for which the corresponding (γ, λ) and (γ, h) satisfy the abnormal and normal equations respectively.

We first prove that if $\lambda_5 = 0$ or $\lambda_6 = 0$, then x is left invariant. Looking at (3.1), we see that if one of λ_5 or λ_6 is zero, λ_4' must also be zero, so λ_4 is constant. This gives the linear equation: $\gamma_1 \lambda_4 + \gamma_2 \lambda_5 = 0$, with constant coefficients. Since $\lambda_4 \neq 0$, the assumption of arc-length parameterization implies that γ_1 and γ_2 must be constant. We now assume that $\lambda_5, \lambda_6 \neq 0$ and get a contradiction.

We proceed by combining the abnormal equations and

3.4 *The normal equations:*

(a) $\gamma_1{}' + \gamma_2 h_3 = 0$

(b) $\quad {\gamma_2}' - \gamma_1 h_3 = 0$
(c) $\quad {h_3}' - \gamma_1 h_4 - \gamma_2 h_5 = 0$
(d) $\quad {h_4}' - \gamma_1 h_6 = 0$
(e) $\quad {h_5}' = 0$
(f) $\quad {h_6}' = 0$

We start by deriving h_3. Differentiating (3.1) we get

3.5

$$\lambda_4'' = -\lambda_4({\lambda_4}')^4/({\lambda_5}^2{\lambda_6}^2) \ ,$$

which gives, after differentiating $\gamma_2 = -\lambda_4{\lambda_4}'/\lambda_5\lambda_6$

$${\gamma_2}' = -\frac{({\gamma_4}')^2}{\lambda_5^3\lambda_6^3}(\lambda_4^2({\lambda_4}')^2 - \lambda_5^2\lambda_6^2) \quad \text{and by (3.1)} \ \ {\gamma_2}' = -\frac{({\lambda_4}')^4}{\lambda_5\lambda_6^3} \ .$$

Now, $h_3 = {\gamma_2}'/\gamma_1$, and $\gamma_1 = {\lambda_4}'/\lambda_6$. Hence

$$h_3 = -\frac{({\lambda_4}')^3}{(\lambda_5\lambda_6^2)}$$

(recall that we can divide by ${\lambda_4}'$ which is never zero by 3.1).

Differentiate this equality and substitute in (3.4c) to get

$$-\frac{3({\lambda_4}')^2{\lambda_4}''}{\lambda_5{\lambda_6}^2} = \frac{{\lambda_4}'}{\lambda_6}h_4 - \frac{\lambda_4{\lambda_4}'}{\lambda_5\lambda_6}h_5 \ .$$

Together with (3.5), it gives

$$3\lambda_4({\lambda_4}')^5/{\lambda_5}^2{\lambda_6}^3 = \lambda_5 h_4 - \lambda_4 h_5 \ .$$

Now, ${h_4}' = \gamma_1 h_6 = (h_6/\lambda_6){\lambda_4}'$ so $h_4 = (h_6/\lambda_6)\lambda_4 + c_4$, where c_4 is a constant. This implies

$$\frac{3\lambda_4({\lambda_4}')^5}{{\lambda_5}^2{\lambda_6}^2} = c_4\lambda_5\lambda_6 + \lambda_4(\lambda_5 h_6 - \lambda_6 h_5)$$

We introduce generic constants $\alpha_1, \alpha_2, \alpha_3$, which we will use liberally to denote any constants (the actual value they represent may change from one equation to the other). The above equation writes as:

$$\lambda_4(\alpha_1 + \alpha_2({\lambda_4}')^5) = \alpha_3 \qquad (\alpha_2 \neq 0) \ .$$

We differentiate, divide by ${\lambda_4}'$ and use (3.5) to get the following equation:

$$\alpha_1 + \alpha_2({\lambda_4}')^5 + \alpha_3\lambda_4^2({\lambda_4}')^7 = 0 \quad (\alpha_2, \alpha_3 \neq 0) \ .$$

Or $\ \alpha_1 + ({\lambda_4}')^5(\alpha_2 + \alpha_3{\lambda_4}^2({\lambda_4}')^2) = 0 \qquad (\alpha_2, \alpha_3 \neq 0) \ .$

From (3.1) we see that ${\lambda_4}^2({\lambda_4}')^2 = {\lambda_5}^2({\lambda_6}^2 - ({\lambda_4}')^2)$, which gives

$$\alpha_1 + \alpha_2({\lambda_4}')^5 + \alpha_3({\lambda_4}')^7 = 0 \qquad (\alpha_3 \neq 0) \; .$$

Hence ${\lambda_4}'$ must take a finite set of values, but by (3.1) so does λ_4, which is continuous. So λ_4 is constant, and ${\lambda_4}' = 0$. Contradiction. $\square$

This phenomena, where the only normal abnormals are left invariant curves, is not rare. In fact it occurs in many examples. However, one can build counter examples to this, in rank-2 distributions, whenever there exists a normal curve with a constant h_3. It is not hard to see in this case that, taking $\lambda_3 = 0$ and $\lambda_i = h_i$ for $i > 3$, gives a solution to the abnormal equations.

As an example we take the free nilpotent Lie algebra of step 4 on two generators. This Lie algebra is of dimension 8, with the following relations:

$$[e_1, e_2] = e_3 \;,\; [e_1, e_3] = e_4 \;,\; [e_2, e_3] = e_5 \;,\; [e_1, e_4] = e_6 \;,$$

$$[e_1, e_5] = e_7 \;,\; [e_2, e_4] = e_7 \;,\; [e_2, e_5] = e_8 \;.$$

Let the distribution $D = Span\{e_1, e_2\}$. The abnormal equations become:

$$\begin{aligned} \gamma_2\lambda_3 &= 0 \\ -\gamma_1\lambda_3 &= 0 \\ -\gamma_1\lambda_4 - \gamma_2\lambda_5 &= 0 \\ {\lambda_4}' - \gamma_1\lambda_6 - \gamma_2\lambda_7 &= 0 \\ {\lambda_5}' - \gamma_1\lambda_7 - \gamma_2\lambda_8 &= 0 \\ {\lambda_6}' &= 0 \\ {\lambda_7}' &= 0 \\ {\lambda_8}' &= 0 \end{aligned}$$

And the normal equations are:

$$\begin{aligned} {\gamma_1}' + \gamma_2 h_3 &= 0 \\ {\gamma_2}' - \gamma_1 h_3 &= 0 \\ {h_3}' &= \gamma_1 h_4 + \gamma_2 h_5 \\ {h_4}' &= \gamma_1 h_6 + \gamma_2 h_7 \\ {h_5}' &= \gamma_1 h_7 + \gamma_2 h_8 \\ {h_6}' &= 0 \\ {h_7}' &= 0 \\ {h_8}' &= 0 \end{aligned}$$

Any curve given by integrating

$$x'(t) = (-\sin t)e_1 + (\cos t)e_2,$$

is, on one hand, not left invariant (γ_1 and γ_2 are not constant), and on the other hand both normal and abnormal. As an abnormal lift of x to the cotangent bundle we take:

$$\lambda_1 = \lambda_2 = \lambda_3 = 0 \ , \lambda_4 = \cos t \ , \ \lambda_5 = \sin t \ , \ \lambda_6 = 1 \ , \lambda_7 = 0 \ , \lambda_8 = 1 \ .$$

And the normal lift is given by setting:

$$h_1 = \gamma_1 \ , \ h_2 = \gamma_2 \ , h_3 = 1 \ , \ h_4 = \lambda_4 \ , \ h_5 = \lambda_5 \ , \ h_6 = \lambda_6 \ , \ h_7 = \lambda_7 \ , \ h_8 = \lambda_8 \ .$$

We let the reader check that these lifts do satisfy the abnormal and normal equations respectively.

4 Smoothness of Geodesics in the 2-Step Case

In this section, we assume that G is a nilpotent lie group of 2-step. This means that $[\mathfrak{g}, [\mathfrak{g},\mathfrak{g}]] = 0$. As before we consider a Carnot metric given on a left-invariant distribution D such that $D \oplus [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

4.1 Theorem: *Under the above assumption, any minimizer through* 0 *is normal in some subgroup of G (for the induced Carnot metric) and hence any minimizer is smooth.*

Remark In the case where G is a *free* nilpotent Lie group of 2-step, one can prove that any minimizer is in fact normal, see Gaveau [3] and Brockett [1].

Proof: We need to show that any abnormal minimizer through 0 is normal in some subgroup of G. We will proceed by induction on the dimension of G. The main step of the induction is given by the following lemma, whose proof we postpone.

4.2 Lemma: *Any abnormal curve through* 0 *(if it exists) is tangent to a left invariant proper sub-distribution $K \subset D$ and hence belongs to the proper Lie subgroup H generated by the algebra $K \oplus [K, K]$.*

To start the induction, note that any nilpotent Lie group of dimension 1 or 2 is in fact abelian, and D must equal $\mathfrak{g}$. Therefore there are no abnormal curves, so every minimizer is normal (it is also easy to check that for dimension 3, the only nilpotent Lie group is the Heisenberg group, which has no abnormal curves either).

Let $x(t)$ be an abnormal minimizer with $x(0) = 0$. Let H, K be as in Lemma 4.2 . The induced Carnot metric on H is given by the induced metric on $K \subset D$ (which as before we left-translate). Since x is a minimizer in G, it is a minimizer in H (a horizontal curve of smaller length than x in H would be horizontal and of smaller length in G). Since $\dim H < \dim G$, $x(t)$ is normal, by induction. If x is a minimizer which does not pass through 0, it is as smooth as the minimizer $y(t) = x^{-1}(0)x(t)$ which does pass through 0. $\square$

We now prove Lemma 4.2.
Because $\mathfrak{g}$ is 2-step, the equations for the abnormal curves (2.3) simplify to

$$\sum_{j=1}^{r} \sum_{k=r+1}^{n} \alpha_{ijk} \gamma_j \lambda_k = 0 \quad \text{for } \ i = 1, \ldots, r$$

$$\lambda_k = 0 \quad \text{for } \ k = 1, \ldots, r \quad \text{ and } \ \lambda'_k = 0 \quad \text{for } \ k = r+1, \ldots, n$$

Indeed, in this case, $\alpha_{ijk} = 0$ whenever i or j is greater than r.

Let $(x(t), \lambda)$ be the cotangent lift of $x(t)$. In particular $x' = \gamma$ satisfies the above equations, and $\lambda \neq 0$.
Let M be the matrix whose entries are given by:

$$m_{ij} = \sum_{k=r+1}^{n} \alpha_{ijk} \lambda_k \ \ , \ i, j = 1, \ldots, r$$

4.3 Lemma: *If $\lambda \neq 0$, the space $K = \text{Ker } M$ is a proper subspace of $\mathbb{R}^r$. Therefore $\{\sum \gamma_j e_j \mid \gamma \in K\}$ is a proper subspace of D.*

Since for an abnormal minimizer $\lambda \neq 0$, this claim obviously implies Lemma 4.2, since $x' \in \text{Ker } M$.
Proof: The fact that $\mathfrak{g}$ is 2-step (that is $D \oplus [D, D] = \mathfrak{g}$) implies the existence of coefficients β_{ijl} such that:

$$e_l = \sum_{i,j \leq r} \beta_{ijl} [e_i, e_j], \quad \text{for } \ l = r+1, \ldots, n.$$

Claim: Viewing $A = \{\alpha_{ijk}\}$ and $B = \{\beta_{ijl}\}$ as $r^2 \times (n-r)$ matrices (with i, j ordered as a single index), the matrix B^T is a left inverse to A. Indeed

$$e_l = \sum_{i,j \leq r} \beta_{ijl} [e_i, e_j] = \sum_{i,j \leq r} \beta_{ijl} \sum_{k > r} \alpha_{ijk} e_k = \sum_{k > r} \left(\sum_{i,j \leq r} \alpha_{ijk} \beta_{ijl} \right) e_k,$$

which implies

$$\sum_{i,j \leq r} \alpha_{ijk} \beta_{ijl} = \delta_{kl} \quad \text{(the Kronecker symbol)} \ , \qquad \text{i.e. } A^T B = I_{n-r} \ .$$

The proof of Lemma 4.3 is now easy. We want to prove that, unless $\lambda = 0$, the matrix M is non zero. Suppose it were, i.e. $\sum_{k > r} \lambda_k \alpha_{ijk} = 0$, then

$$\left(\sum_{k > r} \lambda_k \alpha_{ijk} \right) \beta_{ijl} = 0, \quad \forall i, j, l$$

$$\Longrightarrow \qquad 0 = \sum_{k > r} \sum_{i,j \leq r} \lambda_k \alpha_{ijk} \beta_{ijl} = \sum_{k > r} \lambda_k \delta_{kl} = \lambda_l \quad \forall l \ .$$

□

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