Local connectivity of the Julia set of real polynomials

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1 Introduction and statements of theorems

One of the main questions in the field of complex dynamics is the question whether the Mandelbrot set is locally connected, and related to this, for which maps the Julia set is locally connected. In this paper we shall prove the following

Main Theorem *Let f be a polynomial of the form $f(z) = z^\ell + c_1$ with ℓ an even integer and c_1 real. Then the Julia set of f is either totally disconnected or locally connected.*

In particular, the Julia set of $z^2 + c_1$ is locally connected if $c_1 \in [-2, 1/4]$ and totally disconnected if $c_1 \in \mathbb{R} \setminus [-2, 1/4]$ (note that $[-2, 1/4]$ is equal to the set of parameters $c_1 \in \mathbb{R}$ for which the critical point $c = 0$ does not escape to infinity). This answers a question posed by Milnor, see [Mil1]. We should emphasize that if the ω-limit set $\omega(c)$ of the critical point $c = 0$ is not minimal then it very easy to see that the Julia set is locally connected, see for example Section 10. Yoccoz [Y] already had shown that each quadratic polynomial which is only finitely often renormalizable (with non-escaping critical point and no neutral periodic point) has a locally connected Julia set. Moreover, Douady and Hubbard [DH1] already had shown before that each polynomial of the form $z \mapsto z^\ell + c_1$ with an attracting or neutral parabolic cycle has a locally connected Julia set. As will become clear, the difficult case is the infinitely renormalizable case. In fact, using the reduction method developed in Section 3 of this paper, it turns out that in the non-renormalizable case the Main Theorem follows from some results in [Ly3] and [Ly5], see the final section of this paper.

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We should note that there are infinitely renormalizable non-real quadratic maps with a non-locally connected Julia set, see [DH] and [Mil]. Hence, the results above really depend on the use of real methods. On the other hand, Petersen has shown that quadratic polynomials with a Siegel disc such that the eigenvalue at the neutral fixed point satisfies some Diophantine condition is locally connected, see [Pe].

In principle, the methods of Yoccoz completely break down in the infinitely renormalizable case and in the case of polynomials with a degenerate critical point. The purpose of Yoccoz's methods is to solve the well-known conjecture about the local connectedness of the Mandelbrot set and therefore, some version of our ideas might be helpful in proving this conjecture. For a survey of the results of Yoccoz, see for example [Mil] and also [Ly4].

We should note also that Hu and Jiang, see [HJ] and [Ji1] have shown that for infinitely renormalizable quadratic maps which are real and of so-called bounded type, the Julia set is locally connected. Their result is heavily based on the complex bounds which Sullivan used in his renormalization results, see [Sul] and also the last chapter and in particular Section VI.5 of [MS] (cf. also [Ji2]).

In fact, our methods enable us to extend Sullivan's result to the class of all infinitely renormalizable unimodal polynomials *independently of the combinatorial type*! We should emphasize that these complex bounds form the most essential ingredient for the renormalization results of Sullivan [Sul]; in fact in McMullen's approach to renormalization, see [McM], these complex bounds play an even more central role. In the previous proofs of the complex bounds see [Sul], and also Section VI.5 of [MS], it is crucial that the renormalization is of bounded type and, moreover, the proof is quite intricate. Therefore we are very happy that our methods give a fairly easy way to get complex bounds independently of the combinatorial type of the map (i.e., only dependent of the degree of the map):

Theorem A *Let f be a real unimodal polynomial infinitely renormalizable map. Let $f^{s(n)}\colon V_n \to V_n$ be a renormalization of this map. Then there exists a polynomial-like extension of this map $f^{s(n)}\colon \Omega_n' \to \Omega_n$ such that the modulus of $\Omega_n \setminus \Omega_n'$ is bounded from below by a constant which only depends on ℓ and such that the diameter of Ω_n is at most a universally bounded constant times the diameter of V_n.*

The way we prove that such sets Ω_n exist is through cross-ratio estimates. In fact, the estimates are similar to those that were made previously in [SN]. In this way, we are able to get the 'complex bounds' of Theorem A similar to those used by Sullivan in his renormalization result. Note that our bounds are completely independent of the combinatorial type of the map. We should note that Theorem A and its proof hold for any renormalization f^s of (a maybe only finitely renormalizable map) f provided f^{2s} does not have an attracting or neutral fixed point.

In the non-renormalizable case we also have complex bounds. Firstly, for each level for which one has a high return one has a polynomial-like mapping. (Our definition of high case also includes what is sometimes called a central-high return, see the definition in the next section.)

Theorem B *Let $f(z) = z^\ell + c_1$ with ℓ an even integer and c_1 real be a non-renormalizable polynomial so that $\omega(c)$ is minimal. Assume W is the real trace of a central Yoccoz puzzle piece and $F\colon \cup V^i \to W$ is the corresponding first return map (on the real line) and assume that this map has a high return, i.e., assume that $F(V^0) \ni c$ where V^0 is the central interval. Then there exist topological discs Ω^i and Ω with $\Omega^i \cap \mathbb{R} = V^i$ and $\Omega \cap \mathbb{R} = W$ and a complex polynomial-like extension $G\colon \cup_i \Omega^i \to \Omega$ of F. The diameter of the disc Ω is comparable to the size of W.*

Moreover, one has the following result which follows from [Ly3] and [Ly5] (as was pointed out to us in an e-mail by Lyubich). Graczyk and Świątek informed us that they also have a proof of this Theorem C.

Theorem C [Lyubich] *Let $f(z) = z^\ell + c_1$ with ℓ an even integer and c_1 real be a non-renormalizable polynomial so that $\omega(c)$ is minimal. If W is the real trace of a central Yoccoz puzzle piece and $F\colon \cup V^i \to W$ is the corresponding return map (on the real line). Then after some 'renormalizations' one can obtain an iterate $\tilde{F}\colon \cup \tilde{V}^i \to \tilde{W}$ of F with $\tilde{W} \subset W$ such that there exist topological discs $\tilde{\Omega}^i$ and $\tilde{\Omega}$ with $\tilde{\Omega}^i \cap \mathbb{R} = \tilde{V}^i$ and $\tilde{\Omega} \cap \mathbb{R} = \tilde{W}$ and a complex polynomial-like extension $\tilde{G}\colon \cup_i \tilde{\Omega}^i \to \tilde{\Omega}$ of $\tilde{F}$. The diameter of the disc $\tilde{\Omega}$ is comparable to the size of $\tilde{W}$.*

Let us say a few words about our proofs. The main idea behind our proof of the Main Theorem is to construct generalized polynomial-like mappings $F_n\colon \cup_i \Omega_n^i \to \Omega_n$ which coincide on the real line with the first return maps to certain Yoccoz puzzle-pieces. To do this we first obtain real bounds to get Koebe space: these are based on a sophisticated version of the 'smallest interval' argument. They are a sharper version of those used before by Blokh, Lyubich, Martens, de Melo, Sullivan, van Strien, Świątek and others. Using those real bounds and the use of certain Poincaré domains we construct these polynomial-like mappings and show that the diameter of these domains is comparable with that of the interval $\Omega_n \cap \mathbb{R}$. Next we compare these polynomial-like maps with those from the Yoccoz puzzle because the intersection of a Yoccoz puzzle-piece with the Julia is connected. Next we show that the Julia set of the polynomial-like mappings of the Yoccoz puzzle coincides with the Julia set of the polynomial-like mappings F_n, see Section 3. Since these domains get small, we are able to conclude local connectivity of the Julia set.

The paper is organized as follows. In Section 2 some background information is given and in Sections 3 and 4 we give an abstract description of our method for proving local connectivity of the Julia set. In Section 5, 6 and 7 we develop real bounds which will enable to estimate the shape of the pullbacks of certain discs or other regions. We should emphasize that the real bounds in these sections hold for all unimodal maps with negative Schwarzian derivative. In Sections 8 to 13 we apply these estimates to several cases. The reader will observe that certain cases are proved by several methods. For example, in Section 8 the local connectivity of the infinitely renormalizable case with $\ell \geq 4$ is proved, while this case also follows from the estimates (for a more general case) in Section 12. However, the domains in Section 8 are discs and those in Section

12 are considerably more complicated. We believe that for future purposes it might be important to have good domains, and therefore even if it was sometimes not necessary for the proofs of our theorems, we have tried to treat each case in a fairly optimal way. In the final six pages of this paper – Section 14 – we prove Theorem C and complete the Main Theorem in the non-renormalizable case.

Finally, a short history of this paper since several others have partial proofs of Theorem A and the Main Theorem in the quadratic case. Firstly, we were inspired by the papers of Hu and Jiang, see [HJ] and [Ji1] where it is shown that infinitely renormalizable maps of bounded type (where Sullivan's bounds hold) have a locally connected Julia set. The first widely distributed version of our paper (dated December 31, 1994) included the proof of the Main Theorem in the quadratic case, the infinitely renormalizable case, Theorem A (without doubling) and also some non-renormalizable cases. Subsequently, Theorem B was included in the version of this paper of January 27, 1995. Graczyk and Świątek distributed a preprint with a proof of Theorem A in the quadratic case on February 3, 1995. Lyubich and Yampolsky gave an alternative proof of the Main Theorem and Theorem A in the quadratic case, in a draft dated February 22, 1995. The 'quadratic' proofs of Graczyk, Świątek, Lyubich and Yampolsky of Theorem A improve our estimates in certain cases because it sometimes allows one to obtain annuli with large moduli, but those proofs seem to heavily rely on the map being quadratic. (In view of the estimates in [SN] such large moduli cannot be expected to exist in the higher order case.) After we told Lyubich about our methods to obtain local connectivity, he realized the relevance of his methods, see [Ly3], [Ly5], for proving local-connectivity in the non-renormalizable case. In an e-mail dated February 10, 1995, he told us how to prove Theorem C using these methods, thus completing the proof of the Main Theorem in the non-renormalizable case. To make this paper self-contained we added his proof in Section 14 in our paper, in the version of April 5, 1995.

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2 Some notation and some background

Let f be a real unimodal polynomial. For example, $f(z) = z^\ell + c_1$ where ℓ is even. We find it convenient to denote the critical point by c, i.e., $c = 0$. The critical value is therefore $c_1 = f(c)$ and we shall write $c_s = f^s(c)$. When $w \neq c$ then we shall define $\tau(w)$ to be the point $\neq w$ so that $f(\tau(w)) = f(w)$. For our specific map, we have $\tau(z) = -z$ but since most results in this paper do not rely on the specific form of the map f we shall write $\tau(z)$ rather than $-z$. If A, B are intervals then we shall write $[A, B]$ for the smallest interval containing A and B. Furthermore, we shall use the following notation

$$(A, B] = [A, B] \setminus A \ , \quad [A, B) = [A, B] \setminus B \text{ and } (A, B) = [A, B] \setminus (A \cup B).$$

As usual, if $J \subset T$ are two intervals and L, R are the components of $T \setminus J$ then we define $C(T, J)$ to be the cross-ratio of this pair of intervals:

$$C(T, J) = \frac{|J||T|}{|L||R|}.$$

Here $|U|$ stands for the length of an interval U. Cross-ratios play a crucial role in all recent results in real interval dynamics. Often, it suffices to use some qualitative estimates based on the so-called Koebe Principle. In our analysis, we shall need somewhat sharper estimates, which are based on direct use of the cross-ratio. For example, we shall often use the inequality that

$$|L|/|J| \geq C^{-1}(T, J).$$

If g is a map which is monotone on T and $Sg < 0$ then

$$C(gT, gJ) \geq C(T, J), \text{ i.e., } C^{-1}(T, J) \geq C^{-1}(g(T), g(J)).$$

In our case we shall apply this to maps g of the form f^n. Since $Sf < 0$ one has also that $Sf^n < 0$ so the previous inequality applies when we take $g = f^n$ and $f^n|T$ is monotone. The Koebe Principle states that if $Sg < 0$ and $J \subset T$ are intervals so that $g: T \to g(T)$ is a diffeomorphism and so that each component of $g(T \setminus J)$ has size $\tau|g(J)|$ (i.e., $g(T)$ is a τ-*scaled neighbourhood* of $g(J)$) then $|Dg(x)|/|Dg(y)| \leq (1+\tau)^2/\tau^2$ for each $x, y \in J$. The intervals $g(T \setminus J)$ are referred to as 'Koebe space'. We shall also use the following fact: if $Sf < 0$ and if $f^n|T$ is monotone and has a hyperbolic repelling fixed point, then $f^n(T) \supset T$.

We say that W is a *symmetric* interval if it is of the form $W = [w, \tau(w)]$. The boundary point w is called *nice* if $f^i(w) \notin W$ for all $i > 0$. Note that there are plenty of nice points: each periodic orbit contains a nice point. Also, if f is not renormalizable, preimages of the orientation reversing fixed point of f can be used to find nice points. This is done in the Yoccoz puzzle, see also the proof of Theorems B and C. Nice points are also considered in, for example, the thesis of Martens [Mar], see also Section V.1 of [MS].

If f is renormalizable, then we can take for u_n the points which are in the boundary of an interval $I_n \ni c$ which is mapped into itself in a unimodal way after $q(n)$ iterates.

If f is not renormalizable then we can construct a sequence of nice points u_n as follows. Assume that f has an orientation reversing fixed point u_0. Then we define u_n inductively as follows: let $k(n)$ be the smallest integer such that

$$(u_{n-1}, \tau(u_{n-1})) \bigcap \left(\cup_{i=0}^{k} f^{-i}(u_0) \right) \neq \emptyset$$

and let $u_n, \tau(u_n)$ be the points in this intersection which are nearest to c. If f has no periodic attractor, then u_n is defined for each n. It is easy to see that each u_n is a nice point.

Let us explain why these nice points play such an important role. Let W be a symmetric interval with nice boundary points. Let

$$D_W = \{x\,;\ \text{ there exists } k > 0 \text{ such that } f^k(x) \in W\}.$$

For $x \in D_W$ let $k(x)$ be the smallest integer $k > 0$ for which $f^k(x) \in W$ and define

$$R_W(x) = f^{k(x)}(x).$$

Let V be the component of D_W which contains c and take $s' \in \mathbb{N}$ be so that $R_W|V = f^{s'}$. Because W has nice boundary points, each component – except the component V – of the domain of D_W is mapped diffeomorphically by D_W onto W. Clearly, V is symmetric and also has nice boundary points. Similarly, let U be the components containing c of the domain D_V of the first return map R_V. Take $s \in \mathbb{N}$ so that $R_V|U = f^s$. Note that $f(U), \ldots, f^s(U)$ are disjoint and that similarly $f(V), \ldots, f^{s'}(V)$ are also disjoint.

We say that R_V has a *high return* if $R_V(U) \ni c$. (We should emphasize that this situation also includes the so-called central-high return case.) This implies that $R_V(U) = f^s(U)$ contains a component of $V \setminus \{c\}$ and therefore $f^{s+i}(U) \supset f^i(U)$ for $i \geq 1$.

It is possible that $U = V = W$ is a periodic interval: in this case f is renormalizable and $s' = s$ is the period of this interval V. In this case, we certainly can assume that $R_V\colon V \to V$ (which is equal to f^s in this case and consists of one fold) has a high return: otherwise this return map has a periodic attractor and therefore we do not have to consider this case.

If f is non-renormalizable and the critical point of f is recurrent, then taking $\hat{W}_{n-1} = [u_{n-1}, \tau(u_{n-1})]$ one gets as the domain of $R_{\hat{W}_{n-1}}$ containing the critical point the interval $\hat{W}_n = [u_n, \tau(u_n)]$. In Theorem B we demand that there are infinitely many n's for which $R_{\hat{W}_{n-1}}$ has a *high return*.

Finally, as in the complex bounds of Sullivan, we shall use the Poincaré metric on a slit region in the complex plane. Given a real interval T we shall write $D_*(T)$ for the disc which is symmetric with respect to the real line and which intersects the real line exactly in T. More generally, if T is a bounded real interval and $\alpha \in (0, \pi)$ then $D(T; \alpha)$ will denote the union of two discs which are symmetric w.r.t. the real axis, intersect the real line exactly in T and which have an external angle with the real line of angle α. The reason these sets play an important role, can be explained as follows. Let $\mathbb{C}_T = \mathbb{C} \setminus (\mathbb{R} \setminus T)$. The set $\mathbb{C}_T$ with two infinite slits, carries a Poincaré metric, and with respect to this metric the set $D(T; \alpha)$ consists of all points whose distance to T is at most equal to some constant $k(\alpha)$. From this interpretation and the Schwarz contraction principle, it follows that if $\phi\colon \mathbb{C}_T \to \mathbb{C}_{T'}$ is a univalent conformal mapping sending T diffeomorphically to T', then

$$\phi(D(T; \alpha)) \subset D(T'; \alpha). \tag{2.1}$$

We shall apply this statement, in the following way:

Lemma 2.1 *Let $F: \mathbb{C} \to \mathbb{C}$ be a real polynomial whose critical points are on the real line and which maps T' diffeomorphically onto T, then there exists a set $D \subset D(T';\alpha)$ with $D \cap \mathbb{R} = T'$ which is mapped diffeomorphically onto $D(T;\alpha)$ by F.*

Often we shall use $\alpha = \pi/2$ and so we define

$$D_*(T) = D(T;\pi/2).$$

3 Method showing that the Julia set of two polynomial-like mappings coincide

We shall use the fundamental notion of polynomial-like mapping [DH] or more precisely, we need its extension due to Lyubich and Milnor from [LM]. Let $D^0, D^1, \ldots, D^i$, and D be topological discs bounded by piecewise smooth curves and such that the closures $D^0, \ldots, D^i$ are contained in the interior of D, and such that each the discs $D^0, \ldots, D^i$ are pairwise disjoint. Then we call

$$R: D^0 \bigcup D^1 \bigcup \ldots \bigcup D^i \to D$$

by *ℓ-polynomial-like* if $R|_{D^j}$ is a univalent map onto D for each $j = 1, \ldots, i$ and $R|_{D^0}$ is a ℓ-fold covering of D^0 onto D. If $i = 0$ in this definition, we obtain a polynomial-like map in the original sense of Douady-Hubbard.

The *filled Julia set* of R is said to be the set $F_R \subset \cup_{j=0}^{i} D^j$ of the points z such that $R^k(z)$ is defined for all $k = i, 2, \ldots$. The *Julia set* $J_R = \partial F_R$. An equivalent definition of the filled Julia set F_R is:

$$F_R = \bigcap_{k=1}^{\infty} R^{-k}(D).$$

We shall use an extension of the Straightening Theorem due to Douady and Hubbard, [DH]. This extension was also used in Lemma 7.1 of [LM], for the case that $i = 1$.

Lemma 3.1 *Let $R: D^0 \cup \ldots \cup D^i \to D$ be a ℓ-polynomial-like map. Then R is quasi-conformally conjugate to a polynomial in neighborhoods of the filled Julia set F_R and filled Julia set of the polynomial.*

Proof: Let us first pick a point $x_0 \in D \setminus (D^0 \cup \ldots D^i)$ and choose closed simple curves $\gamma_0, \ldots, \gamma_i: [0, 2\pi] \to \mathbb{C}$ such that $\gamma_i(0) = \gamma_i(2\pi) = x_0$, the curves γ_i only meet at x_0 and γ_i surrounds D^i. Moreover, we choose the function γ_i to be smooth and so that $\frac{d}{dt}\gamma_i(0)$ and $\frac{d}{dt}\gamma_i(2\pi)$ are two vectors based at x_0 having an angle $\pi/(i+1)$.

If, for example, $i = 1$ then $\gamma_0 \cup \gamma_1$ is a figure eight. Next pick a curve γ in $\mathbb{C} \setminus D$ and a point $x_1 \in \gamma$. Moreover, choose a smooth function ϕ defined on a neighbourhood N

of x_0 such that $\phi(x_0) = x_1$ and so that ϕ maps $\gamma_i \cap N$ diffeomorphically to $\gamma \cap \phi(N)$ for each $i = 0, 1, \ldots, i$. In local coordinates this map will have an expression of the form $z \mapsto z^{i+1}$ plus higher order terms, i.e., this map ϕ will have a critical point of order $i+1$. Now let A_j be the open annulus between γ_j and D^j and let A be the open annulus between γ and D. Moreover, find a smooth map $\tilde{R}: A_j \to A$ which extends to the closure of these sets so that it agrees with R on ∂D^j and with ϕ on the neighbourhood N of z_0. Choose this extension so that $\phi: A_0 \to A$ is a ℓ-covering and $\phi: A_j \to A$ is a diffeomorphism for $j = 1, \ldots, i$. This map $\tilde{R}$ becomes an extension of R if we define it equal to R on $D^0 \cup \ldots \cup D^i$. Next choose $r > 1$ so that the circle centered at the origin with radius $r > 1$ surrounds $A_0 \cup \ldots \cup A_0$. We can extend $\tilde{R}$ to a map $\hat{R}: \mathbb{C} \to \mathbb{C}$ so that $\hat{R}(z) = z^{\ell+i}$ for $|z| \geq r$ and so that $\hat{R}$ coincides with $\tilde{R}$ on $A_0 \cup \ldots \cup A_i$. The map $\tilde{R}$ on the annulus $\{z\,;\ |z| < r\} \setminus (A_0 \cup \ldots \cup A_i)$ is a ℓ-covering map to the annulus $\{z\,;\ |z| < (\ell + i)r\} \setminus A$.

Now we use the standard trick from the Straightening Theorem. Take a standard conformal structure (i.e., the Beltrami coefficient $\mu = 0$) on the basin of ∞ of R and extend this structure to a L^1 function $\mu: \mathbb{C} \to \{z\,;\ |z| < 1\}$ which is invariant under $\hat{R}$. Since $\hat{R}$ is conformal near infinity and on $D^0 \cup \ldots \cup D^0$, there are only a bounded number of points in each orbit of $\hat{R}$ where this map is not conformal. It follows that the supremum of $|\mu(z)|$ is bounded away from one, and by the Measurable Riemann Mapping Theorem, it follows that there exists a quasiconformal homeomorphism $h: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ with $h(\infty) = \infty$ which has μ as its Beltrami coefficient. Since μ is invariant under $\hat{R}$, it follows that

$$h \circ \hat{R} \circ h^{-1}$$

is an holomorphic $(\ell + i)$-covering. Hence $\hat{R}$ is quasiconformally conjugate to a polynomial map P (of degree $(\ell + i)$). □

A corollary is:

Corollary 3.1 *The Julia set J_R is the limit set for the preimages of any point $z \in D$ (except, in the case that $i = 0$, for the point zero where zero is the ℓ-multiple fixed point of R).*

We can use all this to show that the Julia set of two polynomial-like mappings coincide. In the applications of this we shall later on use for one of these the polynomial-like mapping of the Yoccoz puzzles.

Proposition 3.1 *(cf. [Ji1], [McM].) Let*

$$R_1: D_1^0 \bigcup D_1^1 \bigcup \ldots \bigcup D_1^i \to D_1,$$

$$R_2: D_2^0 \bigcup D_2^1 \bigcup \ldots \bigcup D_2^i \to D_2$$

be two ℓ-polynomial-like mappings, such that the critical point c of these maps coincide. That is, $c \in D_1^0 \cap D_2^0$ is the unique and ℓ-multiple critical point for both R_1 and for R_2. Moreover, assume that the following conditions hold:

1. *$R_1(z) = R_2(z)$ whenever both sides are defined, so that R_1 and R_2 are extensions of the same map R.*

2. *Let C be the component of $D_1 \cap D_2$ which contains $R(c)$. Then also $c \in C$, and there exist precisely i other points $c^1, \ldots, c^i$ so that $c^j \in D_1^j \cap D_2^j$ and $R(c^j) = R(c)$, and, furthermore, $c^1, \ldots, c^i \in C$.*

Under these conditions, the Julia sets of R_1 and R_2 coincide:

$$J_{R_1} = J_{R_2}.$$

If, additionally, $c \in J_{R_1}$, (and, hence, $c \in J_{R_2}$), then there exists a component of a preimage $R_2^{-n}(D_2)$, which contains c and is contained in D_1.

Proof: For $k = 1, 2$, let $C_k^0, \ldots, C_k^i$ be the components of $R_k^{-1}(C)$, such that $c^j \in C_k^j$ when $j \neq 0$, and $c \in C_k^0$. Firstly, $R_k\colon C_k^j \to C$ is a covering, which is just one-to-one if $j \neq 0$, and $R_k\colon C_k^0 \to C$ is a ℓ-branching covering. In particular, boundaries are mapped to boundaries. Since $R_1 = R_2$ on the common domain of definition, we get that, in fact, $C_1^j = C_2^j := C^j$, for every j. Secondly, because of 2), each component C^j has a point c^j in common with the component C. Since C^j is connected and is contained in both D_1 and D_2, it belongs to a component of $D_1 \cap D_2$ containing c^j, i.e., $C^j \subset C$. Now consider a map $R\colon C^0 \cup C^1 \cup \ldots \cup C^i \to C$, which is one-to-one on every $C^j, j \neq 0$, and ℓ-to-one on C^0. Take a point $x \in C$. Then $R^{-1}(x)$ is a subset of C and it consists of $l + i$ points (counting with multiplicities). That is,

$$\text{for any } x \in C, \text{ the sets } R_1^{-1}(x) \text{ and } R_2^{-1}(x) \text{ coincide and belong to } C. \tag{3.1}$$

Starting with $x_0 \in C$, we apply the corollary to Lemma 3.1 and (3.1) to get $J_{R_1} = J_{R_2} := J$. If $c \in J$, then consider a component K of J containing c. Since $K \subset D_1$, there exists a component of a preimage $R_2^{-n}(D_2)$, which contains K and is contained in D_1. $\square$

In the sequel we will use a particular case of Proposition 3.1. Let us state it separately:

Proposition 3.2 *Let*

$$R_1\colon D_1^0 \bigcup D_1^1 \bigcup \ldots \bigcup D_1^i \to D_1,$$

$$R_2\colon D_2^0 \bigcup D_2^1 \bigcup \ldots \bigcup D_2^i \to D_2$$

be two ℓ-polynomial-like mappings, such that the critical points of R_k coincide, this point $c \in D_1^0 \cap D_2^0$ and is a ℓ-multiple critical point of both R_1 and R_2. Moreover, we assume that the following conditions hold:

1. *$R_1(z) = R_2(z)$ whenever the both parts are defined, so that R_1 and R_2 are extensions of a map R.*

2. *For $k = 1, 2$, all topological discs $D_k, D_k^0, \ldots, D_k^i$ are symmetric w.r.t. the real line $\mathbb{R}$ and satisfy $R_k(\overline{z}) = \overline{R_k(z)}$.*

3. *Denoting $I_k = D_k \cap \mathbb{R}$ and $I_k^j = D_k^j \cap \mathbb{R}$, one has $I_2 \subseteq I_1$, $I_2^j \subseteq I_1^j$, and, for $j = 1, \ldots, i$, the (real) map $R_k: I_k^j \to I_k$ is one-to-one.*

Under these conditions, the Julia sets of R_1 and R_2 coincide. If, additionally, $c \in \mathbb{R}$ lies in the Julia set of R_1 (and, hence of R_2), then there exists a component of a preimage $R_2^{-n}(D_2)$, which contains c and is contained in D_1.

4 How to construct a polynomial-like mapping?

Let $f: \mathbb{C} \to \mathbb{C}$ be a map of the form $f(z) = z^\ell + c_1$ with c_1 real and ℓ an even positive integer. Let V be a (real) symmetric interval with nice boundary points. Let U be the component of the domain of the first return map to V containing c and let $\hat{U}$ be the component of this map containing $f(U) \ni c_1$. Take s so that $R_V|U = f^s$.

Proposition 4.1 *Let $\hat{U} \supset f(U)$ be the interval which is mapped diffeomorphically onto V by f^{s-1}. Write $v^f = f(v)$, $V = [v, \tau(v)]$ and $\hat{U} = [\hat{u}^f, u^f]$. Here $u^f = f(u)$ and $\hat{u}^f$ is a point which is not the f-image of some real point. Assume*

$$|\hat{u}^f - c_1| < |v^f - c_1|. \tag{4.1}$$

Moreover, assume that the critical point $c = 0$ of f is recurrent, i.e., all iterates of c under $R_V: D_V \to V$ remain in D_V and that $\omega(c)$ is minimal. Then there exists a ℓ-polynomial-like mapping

$$R: D^0 \cup \ldots \cup D^i \to D_*(V')$$

such that $c \in J_R$. Here $V' = V$ if $U \neq V$ (i.e. $f^s: U \to V$ is not a renormalization), and V' is equal to some ε-neighbourhood of V with $\varepsilon > 0$ small enough if $U = V$ (i.e., when $f^s: U \to U$ is a renormalization). The map R is a real polynomial on each of its components and $\mathbb{R} \cap D^i$ are the components of $D_V \cap V$ intersecting points of $\omega(c)$.

Proof: Since $f: \omega(c) \to \omega(c)$ is minimal, each point $x \in \omega(c)$ is in the domain of the map R_V. By compactness, there exists therefore a finite covering of $\omega(c)$ of disjoint intervals $I^0, \ldots, I^i$ consisting of components of R_V with $I^0 \ni c$. Let us first consider a component I^j with $j \neq 0$. Since then $I^j \notin c$ we get that R_V maps I^j diffeomorphically onto $R_V(I^j) = V$ and it follows that there is a region D^j contained in $D_*(I^j)$ which is mapped diffeomorphically onto $D_*(V)$ by R_V. So consider $I^0 = U = [u, \tau(u)]$. The map f^{s-1} sends $\hat{U} \supset f(U) = f(I^0) \ni c^1$ diffeomorphically onto V. Again there is a region $D' \ni c_1$ contained in $D_*(\hat{U})$ which is mapped diffeomorphically onto $D_*(V)$ by f^{s-1}. Because of (4.1), the f-inverse D^0 of $D' \subset D_*(\hat{U})$ is contained in $D_*(V)$.

In the case of renormalization, we replace V above by its ε-neighborhood V', with $\varepsilon > 0$ so small that (4.1) holds for the new points $\hat{u}^f, v^f$, and so that the new interval U

is strictly inside V' (this is possible since in this case the point u is a repelling periodic point of f^s). □

Remark 4.1 *As we will show in Section 8, one can apply this proposition for any renormalizable unimodal polynomial f of degree $\ell \geq 4$. In addition, we shall give a specific bound for the modulus of the corresponding annuli in Section 8. For the degree $\ell = 2$ we will need a modification of the above domains: see Section 9.*

5 Real bounds if R_V has a high return

As before, let W be a symmetric interval with nice boundary points, let R_W be the first return map to W and let V be the domain of R_W containing c. Similarly, let R_V be the first return map to V and U the component of the domain of R_V which contains c. Let $\hat{U}$ (resp. $\hat{V}$) be the component of R_V (resp. of R_W) containing the critical value c_1. Let s, s' be so that $R_V|\hat{U} = f^{s-1}$ and $R_W|\hat{V} = f^{s'-1}$.

In this section we will assume that R_V has a high return and derive some conditions which - when satisfied - will imply that the component of the $f^{-s}(D_*(V))$ which contains c is contained in $D_*(V)$.

Let j be the component of $\hat{U} \setminus c_1$ which is outside $[c_1, c_2]$, i.e., $j = \hat{U} \setminus f(U)$. If R_V has a high return, $f^{s-1}|f(U)$ contains c and so we can define r to be the interval in $f(U)$ which contains c_1 and such that $f^{s-1}(r) \ni c$. Furthermore, let l be the maximal interval having a unique common point with the boundary point of $\hat{U}$ outside $[c_1, c_2]$ on which f^{s-1} is monotone. We also write,

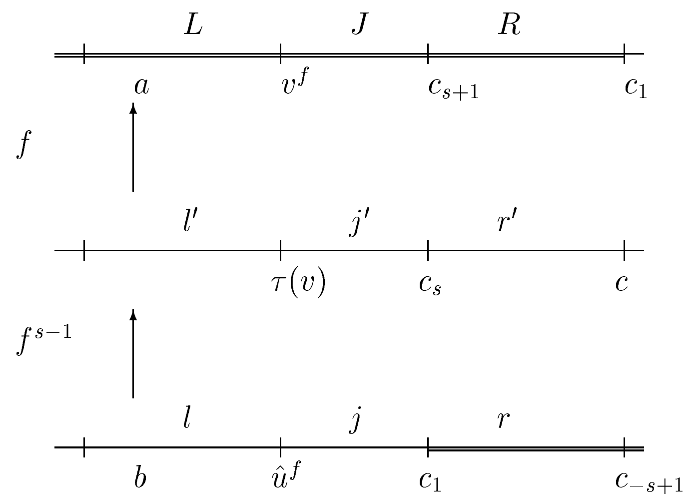
$$U = [u, \tau(u)], \quad \hat{U} = [\hat{u}^f, f(u)] \ni c_1, \quad t = l \cup j \cup r, \quad V = [v, \tau(v)],$$

$$l' = f^{s-1}(l), \quad j' = f^{s-1}(j), \quad r' = f^{s-1}(r), \quad t' = l' \cup j' \cup r'$$

and

$$L = f^s(l), \quad J = f^s(j), \quad R = f^s(r) \text{ and } T = L \cup J \cup R.$$

Mark the typographical difference between the degree ℓ and the interval l. The situation is drawn below. (The fat lines denote the part near c_1 which is inside the interval $[c_1, c_2]$; note that the map $f^s|t$ is orientation reversing.)



The intervals of Lemma 5.1.

Given V as above, take $a \in L$ (including possibly v^f), choose $b \in l$ so that $f^s(b) = a$ and define

$$K_\ell(a) = \frac{|b - c_1|}{|a - c_1|}.$$

In the case that $a = v^f$ and $b = \hat{u}^f$ this becomes

$$K_\ell(v^f) = \frac{|\hat{u}^f - c_1|}{|v^f - c_1|} = \frac{|j|}{|J \cup R|}.$$

This number is important for our question. Indeed, if

$$K_\ell(v^f) < 1 \tag{5.1}$$

then, if $f(z) = z^\ell + c_1$ then we get that $f^{-1}(D_*(\hat{u}^f, u^f)) \subset D_*(v, \tau(v))$. As in Lemma 2.1 this allows us to get a polynomial-like extension of $R_V \colon D_V \to V$. In this section we shall derive a condition for (5.1). Define

$$t = \frac{|c_1 - c_{s+1}|}{|T|} = \frac{|R|}{|T|},$$

$$y = \frac{|a - c_1|}{|T|}.$$

This last quantity measures the amount of 'extendability' around $[a, c_1]$. For example, if $a = v^f$ then $y = \frac{1}{1+|L|/(|J \cup R|)}$ where $|L|/(|J \cup R|)$ is the 'space' which exists around $f(V) = J \cup R$.

Lemma 5.1 *(See also Proposition 3.2 in [SN].) Assume that R_V has a high return and that f^{2s} has no neutral or attracting fixed point. Then*

$$K_\ell(a) \le \frac{t(y^{1/\ell} - t^{1/\ell})}{t^{1/\ell} y(1 - y^{1/\ell})}.$$

Proof: Denote $\overline{J} = [a, c_{s+1}]$, $\overline{L} = T \setminus (R \cup \overline{J})$. Then instead of j', l' we can choose some intervals $\overline{j}', \overline{l}'$ so that $f(\overline{j}') = \overline{J}$, $f(\overline{l}') = \overline{L}$), and $\overline{l}, \overline{j}$ replace the intervals l, j, i.e., $\overline{l} = f^{-(s-1)}(\overline{l}')$, $\overline{j} = f^{-(s-1)}(\overline{j}')$). If we do this, then the intervals r', r, R do not change. Write

$$\alpha = |r'|, \overline{\beta} = |r' \cup \overline{j}'|, \gamma = |r' \cup \overline{j}' \cup \overline{l}'|,$$

and

$$Q = \frac{|\overline{J}| \cdot |T|}{|\overline{L}| \cdot |R|} \Big/ \frac{|\overline{j}'| \cdot |t'|}{|\overline{l}'| \cdot |r'|}.$$

Then, from the expansion of the cross-ratio's

$$\frac{|\overline{J}| \cdot |T|}{|\overline{L}| \cdot |R|} \ge Q \cdot \frac{|\overline{j}| \cdot |t|}{|\overline{l}| \cdot |r|} \ge Q . \frac{|\overline{j}|}{|r|},$$

and using this inequality we get

$$1/K_l(a) = |\overline{J} \cup R|/|\overline{j}| \ge (|\overline{J} \cup R|/|r|) \cdot Q \cdot (|\overline{L}| \cdot |R|)/(|\overline{J}| \cdot |T|)$$

$$\ge \frac{\frac{|\overline{J}| \cdot |T|}{|\overline{L}| \cdot |R|}}{\frac{|\overline{j}'| \cdot |t'|}{|\overline{l}'| \cdot |r'|}} \cdot \frac{|\overline{J} \cup R| \cdot |\overline{L}|}{|\overline{J}| \cdot |T|} = \frac{|\overline{J} \cup R| \cdot |\overline{l}'| \cdot |r'|}{|R| \cdot |\overline{j}'| \cdot |t'|}$$

$$= \frac{\overline{\beta}^l \cdot (\gamma - \overline{\beta}) \cdot \alpha}{\gamma \cdot (\overline{\beta} - \alpha) \cdot \alpha^l}.$$

Here we have used in the last inequality that $|r| \le |R|$ which holds because $f^{2s}|r$ has no periodic attractor. Now writing $\overline{\beta}^l/\gamma^l = y$ and $\alpha^l/\gamma^l = t$ the lemma follows. □

Corollary 5.1

$$K_\ell(a) \le K_\ell^*(y) = \frac{(1 - 1/\ell)^{\ell - 1}}{\ell \cdot (1 - y^{1/\ell})},$$

so that

$$K_\ell^*(y) \to 1/(e \cdot \log(1/y))$$

as $\ell \to \infty$.

Example 5.1 .
(a) If the extendability space is 0.6*, i.e.,* $y_1 = 1/(1+0.6) = 0.625$ *then*

$$K_2^*(y_1) = 1.19371...,\ K_4^*(y_1) = 0.951366... < 1.$$

(b) If the extendability space is $1/2$*, i.e., if* $y_2 = 1/(1+1/2) = 2/3$ *then*

$$K_2^*(y_2) = 1.36237...,\ K_4^*(y_2) = 1.0941..,\ K_6^*(y_2) = 1.02502..,\ K_8^*(y_2) = 0.993...$$

(c) If the extendability space is $1/3$*, i.e.,* $y_3 = 1/(1+1/3) = 3/4$ *then*

$$K_2^*(y_3) = 1.8660... \text{ and } \lim_{\ell\to\infty} K_\ell^*(y_3) \to 1.2788....$$

This means that with these estimates for the extendability space, we can apply the method suggested by Proposition 4.1 respectively for $\ell \geq 4$*,* $\ell \geq 8$ *and not at all in the last case. (In fact, if we can prove the space is more than* $e^{1/e} - 1 = 0.44466..$ *then we could apply this method for each* ℓ *sufficiently large.) In the last section of this paper we shall use a slightly different method (using different Poincaré neighbourhoods) which also works when the space is equal to* $1/3$*.*

In the next two section we shall derive estimates for the number y from above.

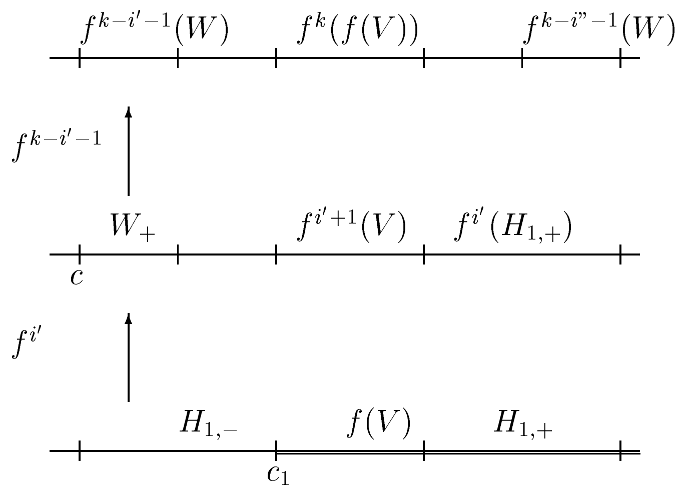
6 Lower bounds for 'space' in the renormalizable case

In this and the next section we shall derive lower bounds for the number y, i.e., find lower bounds for 'space' by looking for a 'smallest' interval among a finite number of intervals. This idea is used in a large number of results in one-dimensional dynamics. In particular we were inspired by the thesis of Martens [Mar] or, specifically, by Lemma 1.2 in Section V.1 of [MS]. In this section we shall obtain quite sharp bounds, which will enable to deal with all real infinitely renormalizable maps $z \mapsto z^\ell + c_1$ of degree $\ell \geq 2$. Unfortunately, the proof splits in quite a few subcases. The main result in the section is Lemma 6.4. In the next section, we shall obtain weaker bounds which work in a more general context; these weaker bounds only apply to the case that $\ell \geq 4$.

Let $\hat{V} \supset f(V)$ be the interval which is mapped monotonically onto W by $f^{s'-1}$.

Lemma 6.1 *Let* $2 \leq k < s'$ *and let* H_1 *be the maximal interval containing* $f(V)$ *such that* $f^k|H_1$ *is monotone. Then* $f^k(H_1)$ *contains* $f^k(f(V))$ *and on each side of this interval also an interval of the form* $f^i(V)$ *with* $i \leq k$*.*

Proof: Let $H_{1,-}, H_{1,+}$ be the components of $H_1 \setminus f(V)$. From the maximality of H_1 it follows that there exists $i' < k$ such that $f^{i'}(H_{1,+})$ contains c. Since $f^{i'+1}(V)$ is outside W it follows that $f^{i'}(H_{1,+})$ contains one component W_+ of $W \setminus \{c\}$. It follows that $f^k(H_{1,+})$ contains $f^{k-i'}(W) \supset f^{k-i'}(V)$. Since the same holds for $H_{1,-}$, the lemma follows. $\square$



The proof of Lemma 6.1.

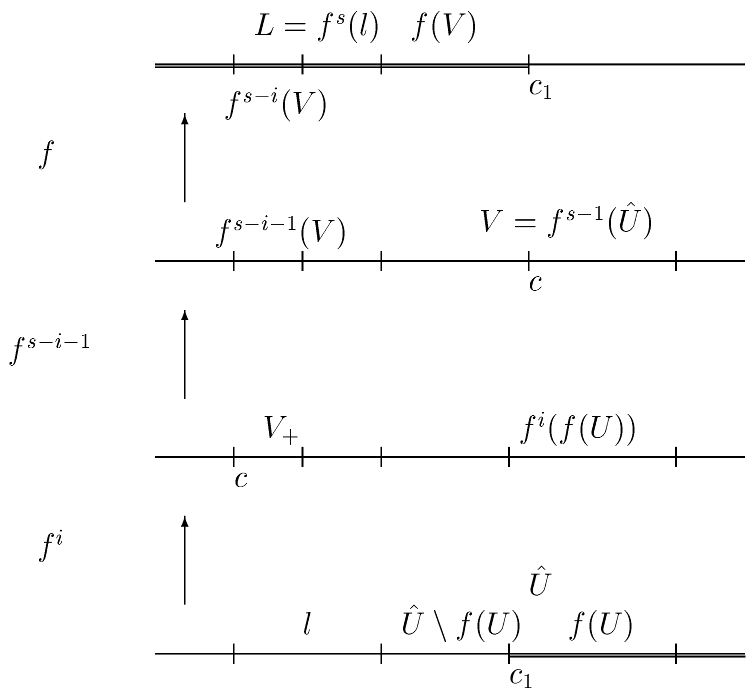
Let $\hat{U} \supset f(U)$ be the interval which is mapped monotonically onto V by f^{s-1}.

Lemma 6.2 *Assume that R_W has a high return and let l be one of the two maximal intervals outside $\hat{U}$ for which $f^s|l$ is monotone and which has a unique common point with $\hat{U}$. (If we take the interval which is outside $[c_1, c_2]$ then it is equal to the interval l from Lemma 5.1.) Then $L := f^s(l)$ contains an interval of the form $f^i(V)$, $1 \le i \le s'$. If $U = V$ (so f is renormalizable with period s) and f is not also renormalizable of period $s/2$ then $L := f^s(l)$ contains two distinct intervals of the form $f^{i+2}(V), f^{s-i}(V)$, with $1 \le i+2, s-i < s'$ and $i+2 \ne s-i$.*

Proof: Let $H = l \cup \hat{U}$. By maximality of H there is i with $0 < i < s$ such that $f^i(H)$ contains c in its boundary. Choose i maximal with this property. Since $f^i(\hat{U})$ is outside V it follows that $f^i(H)$ contains one component V_+ of $V \setminus \{c\}$. Hence $f^{i+1}(l)$ contains $f(V)$ and therefore $f^{s-1}(H)$ contains $f^{s-i-1}(V)$ (and also a point in $f^{s-1}(\hat{U}) = V$). Since R_W has a high return and $f^{s'-1}(f(V))$ contains c in its interior, and since by definition $f^s|l$ is monotone, it follows that $s - i < s'$. Hence $f^{s-1}(H)$ contains one of the intervals $f(V), \ldots, f^{s'-1}(V)$.

Now assume that $U = V$ and take $\tilde{H} = f^i(H) = [c, f^i(\hat{U})]$. If $f^{s-i-1}(\tilde{H}) = f^{s-1}(H)$ only contains $f^{s-i-1}(V)$ from the collection $f(V), \ldots, f^{s'}(V)$, then $f^{s-i-1}(V) = f^{s-i-1}(U)$ is contained in the interval $f^i(\hat{U})$ (i.e., $f^{i+1}(U) = f^{s-i-1}(U)$). Hence $s - i - 1 = i + 1$, i.e., $i + 1 = s - i - 1 = s/2$. It follows that $f^{s-i-1} = f^{s/2}$ maps $[\tilde{H}, \tau(\tilde{H})]$ inside itself. Since, by assumption, this interval only contains two of the intervals of the orbit $V, \ldots, f^{s-1}(V)$, it follows that f is also renormalizable of period $s/2$. Therefore

$f^{s-1}(H)$ contains $f^{s-i-1}(V)$ and $f^{i+1}(V)$. □



The proof of Lemma 6.2.

Lemma 6.3 *Assume that $U = V$ has period s and f is not also renormalizable of period $s/2$. Consider the disjoint intervals $f(V), \ldots, f^s(V)$ and assume that $f(V)$ and $f^2(V)$ are both smaller than their neighbours. Then there exists an integer $k \geq 2$ such that $f^k(V)$ is shorter than its two neighbours from the collection $f(V), \ldots, f^{k-1}(V)$. Take k maximal with respect to this property. Let $1 \leq i_0, i_1 < k$ be so that $f^{i_0}(V), f^{i_1}(V)$ are the neighbours of $f^k(V)$ from the collection $f(V), \ldots, f^{k-1}(V)$. Let*

$$Q_k = [f^{i_0}(V), f^{i_1}(V)]$$

and define $H_1 \supset f(V)$ to be the maximal interval on which f^{k-1} is monotone. Then $H_k = f^{k-1}(H_1) \supset Q_k$. Let $Z_k \subset H_k$ be the maximal interval such that each component of $Z_k \setminus Q_k$ contains at most one interval of the form $f^j(V)$ with $k < j \leq s$. If we define $Z_1 \subset H_1$ so that $Z_k = f^{k-1}(Z_1)$ then

$$C^{-1}(Z_1, f(V)) \geq 0.6.$$

Proof: Such an integer k exists because otherwise $\{2, \ldots, s'\} \ni k \mapsto |f^k(V)|$ would be increasing, contradiction our assumption that $f^2(V)$ is smaller than its neighbour. Let i_0, i_1 be the intervals as in the statement of the lemma. By the choice of k these neighbours are longer than $f^k(V)$. Let

$$Q_k = [f^{i_0}(V), f^{i_1}(V)] \supset f^k(V).$$

Throughout the remainder of the proof we shall consider the case that $f^{i_0}(V)$ lies to the left of $f^{i_1}(V)$. Notice that the fact that $f^{i_0}(V)$ and $f^{i_1}(V)$ are neighbours implies that Q_k only contains intervals of the form $f^j(V)$ with $k < j \le s$. From the maximality of k this implies that each such interval $f^j(V) \subset Q_k$ is longer than the intervals $f^{i_0}(V)$, $f^{i_1}(V)$ and $f^k(V)$. Lemma 6.1 gives $f^{k-1}(H_1) \supset Q_k$. Write

$$H_k = f^{k-1}(H_1).$$

Let $Z_1 \subset H_1$ be as in the statement of the lemma. Let $Q_1 \supset f(V)$ be the subset of H_1 for which $f^{k-1}(Q_1) = Q_k$ and let

$$Q_{i_j} = f^{i_j-1}(Q_1)\,,\ Z_{i_j} = f^{i_j-1}(Z_1) \text{ and } H_{i_j} = f^{i_j-1}(H_1) \text{ for } j = 0, 1.$$

Since $f^{i_0}(V)$ and $f^{i_1}(V)$ are longer than $f^k(V)$ we at least have

$$C^{-1}(Z_1, f(V)) \ge C^{-1}(Q_1, f(V)) \ge C^{-1}(Q_k, f^k(V)) \ge 1/3.$$

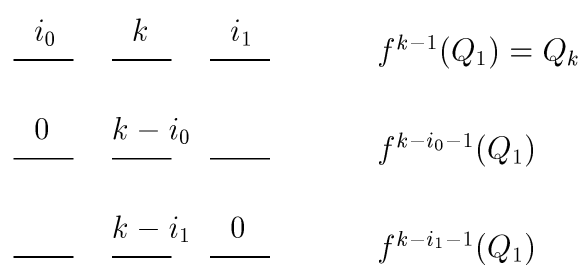
We shall now improve this estimate, by pulling back the interval $Q = Q_k$ either to Q_{i_0} or to Q_{i_1}. In this way we shall either find another interval $f^j(V)$ inside the interval Q_k or find a lower bound for the space between the intervals $f^j(V)$ in Q_k. For this we shall distinguish between several cases depending on whether or not $Q_k = H_k$ and depending on the position of Q_{i_0} and of Q_{i_1} relative to Q_k. Often we shall even show that

$$C^{-1}(Q_k, f^k(V)) \ge 0.6.$$

Since $Z_1 \supset Q_1$ this suffices:

$$C^{-1}(Z_1, f(V)) \ge C^{-1}(Q_1, f(V)).$$

Case I. Assume that $Q_1 = H_1$. By maximality of H_1 this implies that there exist i_0 and i_1 such that $f^{k-i_0}(H_1)$ and $f^{k-i_1}(H_1)$ contain c in their boundary, see the figure below.



Case I.

If $f^{k-i_0}(V)$ lies closer to c then $f^{k-i_0+i_1}(V)$ lies between k and i_1. Similarly, if $f^{k-i_1}(V)$ lies closer to c then $f^{k-i_1+i_0}(V)$ lies between k and i_0. Therefore, since there is no $f^j(V) \subset Q_k$ with $j < k$ and $j \ne i_0, i_1$, this implies that the first possibility occurs if $i_1 > i_0$ and the second one if $i_1 < i_0$. In order to be definite, we shall assume (in this case) that $i_0 < i_1$. This implies that the situation inside Q_k is as drawn below.

$$\underline{\;i_0\;} \quad \underline{\;k\;} \quad k + \underline{i_1} - i_0 \quad \underline{\;i_1\;} \qquad\qquad Q_k$$

Case I: The next interval in Q_k.

Since each of these intervals $f^{i_0}(V)$, $f^{i_1}(V)$ and $f^{k+i_1-i_0}(V)$ is at least as long as $f^k(V)$, it follows that

$$C^{-1}(Q_k, f^k(V)) = \frac{\text{left} \cdot \text{right}}{\text{middle} \cdot \text{total}} \geq \frac{2 \cdot 1}{1 \cdot 4} = \frac{1}{2}.$$

Since

$$C^{-1}(Q_{i_0}, f^{i_0}(V)) \geq C^{-1}(Q_k, f^k(V)) \geq \frac{1}{2}$$

this implies that each of the components of $Q_{i_0} \setminus f^{i_0}(V)$ has length 1/2 times the length of $f^{i_0}(V)$. Therefore, if Q_{i_0} does not contain any points of $f^k(V)$, then one of these components of $Q_{i_0} \setminus f^{i_0}(V)$ is contained in the gap between $f^{i_0}(V)$ and $f^k(V)$. Hence

$$C^{-1}(Q_k, f^k(V)) \geq \frac{2 \cdot 3/2}{3/2 + 1 + 2} = \frac{6}{9} > 0.6 \quad .$$

So we are finished in this case. If there exists an interval $f^j(V)$ between $f^{i_0}(V)$ and $f^k(V)$ then we also are finished, because this interval then has length $\geq |f^k(V)|$ and therefore

$$C^{-1}(Q_k, f^k(V)) \geq \frac{2 \cdot 2}{2 + 1 + 2} = \frac{4}{5} > 0.6 \quad .$$

So we shall consider the case that Q_{i_0} contains some points of $f^k(V)$ and that there exists no interval $f^j(V)$ between $f^{i_0}(V)$ and $f^k(V)$. Therefore the map $f^{k-i_0}\colon Q_{i_0} \to Q_k$ is orientation preserving. Indeed, otherwise the interval $[f^{i_0}(V), f^k(V)]$ would be mapped inside itself by the map f^{k-i_0}. Since there is no interval $f^j(V)$ contained in this interval $[f^{i_0}(V), f^k(V)]$, this implies that f is also renormalizable with half the period $s/2$. By assumption this is not the case. So $f^{k-i_0}\colon Q_{i_0} \to Q_k$ is orientation preserving. If Q_{i_0} contains (some points of) $f^k(V)$ and no points to the right of $f^k(V)$ then the gap between $f^{i_0}(V)$ and $f^k(V)$ is mapped onto $f^{k+i_1-i_0}(V)$. So if we define $W_1 \subset H_1$ so that $W_k = f^{k-1}(W_1) = [f^{i_0}(V), f^{k+i_1-i_0}(V)]$ then

$$C^{-1}(W_{i_0}, f^{i_0}(V)) \geq C^{-1}(W_k, f^k(V)) \geq \frac{1}{3}.$$

Hence the component of $W_{i_0} \setminus f^{i_0}(V)$ which is between $f^{i_0}(V)$ and $f^k(V)$ has at least length 1/3 times the length of $|f^{i_0}(V)|$. This implies that

$$C^{-1}(Q_k, f^k(V)) \geq \frac{4/3 \cdot 2}{4/3 + 1 + 2} = \frac{8}{13} > 0.6.$$

So we finally have to consider the case that Q_{i_0} strictly contains a neighbourhood of $f^k(V)$. Then Q_k contains $f^{k+k-i_0}(V)$ and therefore this entire interval. Since $k+k-i_0 >$

$k + i_1 - i_0 > i_1$, Q_k contains three intervals of the form $f^j(V)$ to the right of $f^k(V)$ and therefore

$$C^{-1}(Q_k, f^k(V)) \geq \frac{1 \cdot 3}{1+1+3} = \frac{3}{5} = 0.6.$$

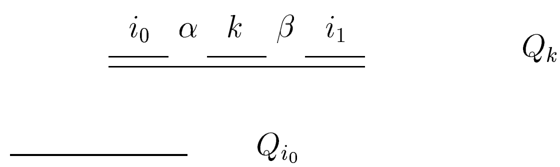
This completes case I.

Case II and Case III. Assume that Q_1 is strictly contained in H_1. In order to be specific, let us assume that $f^{k-1}(H_1)$ contains a neighbourhood of $f^{i_1}(V)$. This information is useful since $f^j(V)$ cannot be mapped monotonically onto $f^{i_i}(V)$ by an iterate of f when $j > i_1$. In particular, f^{k-i_0} cannot map $f^k(V)$ to $f^{i_1}(V)$. Therefore there are only two possibilities: II) Q_{i_0} lies to the left of $f^k(V)$ or III) Q_{i_0} contains some points to the right of $f^k(V)$. (Remember that we had assumed that $f^{i_0}(V)$ lies to the left of $f^{i_1}(V)$.)

Case II. Q_{i_0} lies to the left of $f^k(V)$.

Now we shall analyze the situation near $f^{i_1}(V)$. We shall subdivide several cases:

Case II.a. Q_{i_1} lies to the right of $f^k(V)$. Let $\alpha \cdot |f^{i_0}(V)|$ be the size of the gap between $f^{i_0}(V)$ and $f^k(V)$. Similarly, let $\beta \cdot |f^{i_1}(V)|$ be the size of the gap between $f^{i_1}(V)$ and $f^k(V)$.



Case II: Q_{i_0} lies to the left of $f^k(V)$.

Since Q_{i_0} lies to the left of $f^k(V)$,

$$\alpha \geq C^{-1}(Q_{i_0}, f^{i_0}(V)) \geq C^{-1}(Q_k, f^k(V)) \geq \frac{(1+\alpha)(1+\beta)}{3+\alpha+\beta} \geq 1/3.$$

In this case II.a, we also have a similar inequality as above for β, i.e., we have

$$\alpha, \beta \geq \frac{(1+\alpha)(1+\beta)}{3+\alpha+\beta}.$$

Since the right hand side of the above inequalities is increasing in both α and in β, it follows that $\alpha, \beta \geq \kappa$ where

$$\kappa = \frac{(1+\kappa)(1+\kappa)}{3+\kappa+\kappa}, \text{ i.e., } \kappa^2 + \kappa = 1.$$

Hence

$$\alpha, \beta \geq \kappa = \sqrt{(5/4)} - 1/2 > 0.6 \quad .$$

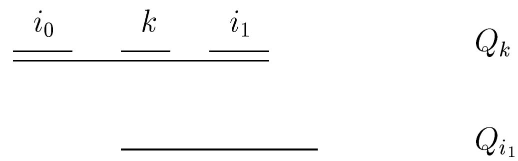
This implies that

$$C^{-1}(Q_k, f^k(V)) \geq \frac{(1+a)(1+b)}{1(3+a+b)} \geq \kappa \geq 0.6 \quad .$$

Case II.b. Q_{i_1} contains some points of $f^k(V)$ but no point to the left of $f^k(V)$. As we remarked above, the fact that we are in Case II or Case III, implies that $f^k(V)$ cannot be mapped homeomorphically onto $f^{i_1}(V)$. It follows that f^{k-i_1} cannot map Q_{i_1} in an orientation reversing way homeomorphically onto Q_k. Hence $f^{k-i_1}: Q_{i_1} \to Q_k$ is orientation preserving and this map sends $f^k(V)$ to $f^{i_0}(V)$. Writing $r = k - i_0$, this gives $k + (k - i_1) = i_0 + s$, i.e., $k - i_1 = s - (k - i_0) = s - r$. So

$$k = i_0 + r \text{ and } i_1 = k + (r - s) = i_0 + 2r - s. \tag{6.1}$$

Let α be so that the gap $(f^{i_0}(V), f^k(V))$ has length $\alpha|f^{i_0}(V)|$ and similarly, let β be so that the gap between $f^k(V)$ and $f^{i_1}(V)$ has size $\beta|f^{i_1}(V)|$. Let Z_k^r be the right component of $Z_k \setminus f^{i_1}(V)$ and define γ so that the size of Z_k^r is equal to $\gamma|f^{i_1}(V)|$. Similarly, let H_k^r be the right component of $H_k \setminus f^{i_1}(V)$.



Case II.b. The map $f^{k-i_1}: Q_{i_1} \to Q_k$ is orientation preserving.

Since we are in Case II, the interval H_k^r contains at least some interval of the form $f^m(V)$ (with in fact $m < k$). If $Z_k^r \neq H_k^r$ then Z_k^r contains an interval $f^{j_1}(V)$ with $k < j_1 < s$. So in any case Z_k^r contains an interval $f^n(V)$. If the right component of $Q_{i_1} \setminus f^{i_1}(V)$ is not contained in Z_k^r, then Q_{i_1} contains a neighbourhood of $f^n(V)$ and therefore then Q_k contains an interval $f^j(V)$ between $f^k(V)$ and $f^{i_1}(V)$. Since $\alpha \geq 1/3$, this implies that $C^{-1}(Q_k, f^k(V)) \geq (1 + 1/3)2/(4 + 1/3) = 8/13 > 0.6$. Therefore, we may assume that the right component of $Q_{i_1} \setminus f^{i_1}(V)$ is contained in Z_k^r. Define $W_1 \subset Q_1$ so that $W_k = f^{k-1}(W_1) = (f^{i_0}(V), f^{i_1}(V)]$. Since we are in Case II.b, we have that f^{k-i_1} maps $f^k(V)$ to $f^{i_0}(V)$. Hence one component of $W_{i_1} \setminus f^{i_1}(V)$ is contained in the gap $(f^k(V), f^{i_1}(V))$ corresponding to β and the other in the interval Z_k^r corresponding to γ. Therefore

$$\frac{\beta\gamma}{1 + \beta + \gamma} \geq C^{-1}(W_{i_1}, f^{i_1}(V)).$$

Since

$$C^{-1}(W_{i_1}, f^{i_1}(V)) \geq C^{-1}(W_k, f^k(V)) \geq \frac{\alpha(1 + \beta)}{2 + \alpha + \beta}$$

this gives,

$$\frac{\beta\gamma}{1 + \beta + \gamma} \geq \frac{\alpha(1 + \beta)}{2 + \alpha + \beta}.$$

Hence

$$\beta(2 + \alpha + \beta) \geq \alpha(1 + \beta) + \frac{\alpha(1 + \beta)^2}{\gamma},$$

or

$$\beta^2 + 2\beta \geq \alpha + \frac{\alpha(1+\beta)^2}{\gamma},$$

i.e.,

$$\beta \geq \sqrt{1 + \alpha + \frac{\alpha(1+\beta)^2}{\gamma}} - 1.$$

Now we study the situation on the other side, around $f^{i_0}(V)$. Define $\hat{Z}_k = [f^{i_0}(V), Z_k^r]$ and let $\hat{Z}_1$ be so that $f^{k-1}(\hat{Z}_1) = \hat{Z}_k$. Then $\hat{Z}_{i_0}$ is to the left of $f^k(V)$. Indeed, $f^r = f^{k-i_0}$ maps $f^k(V)$ to $f^{k+(k-i_0)}(V) = f^{k+r}(V)$. Because of (6.1) we have $k+r > s$, that $f^r|f^k(V)$ is not monotone and $f^{k+r}(V) = f^{i_1}(V)$. Since f^r maps Z_{i_0} monotonically to Z_k (which strictly contains $f^{i_1}(V)$), we finally get that Z_{i_0} lies to the left of $f^k(V)$. Hence,

$$\alpha \geq C^{-1}(\hat{Z}_{i_0}, f^{i_0}(V)) \geq C^{-1}(\hat{Z}_k, f^k(V)) \geq \frac{(1+\alpha)(1+\beta+\gamma)}{3+\alpha+\beta+\gamma}$$

which gives that

$$\alpha^2 + 2\alpha \geq 1 + \beta + \gamma \text{ i.e., } \alpha \geq \sqrt{2+\beta+\gamma} - 1.$$

If $\gamma \geq 0.56$ then $\alpha \geq \sqrt{2.56} - 1 = 0.6$ and so we have

$$C^{-1}(Z_k, f^k(V)) \geq \frac{(1+\alpha)(1+\gamma)}{3+\alpha+\gamma} \geq \frac{1.6 \cdot 1.56}{4.16} = 0.6 \quad .$$

If $\gamma \leq 0.56$ then we have that

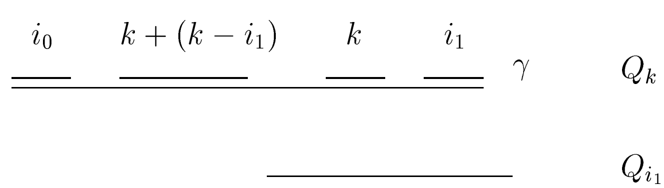
$$\frac{\alpha}{\gamma} \geq \frac{\sqrt{2+\gamma} - 1}{\gamma} \geq \frac{\sqrt{2.56} - 1}{0.56} > 1.$$

Therefore

$$\beta \geq \sqrt{1 + \alpha + 1} - 1.$$

Since we also have $\alpha \geq \sqrt{2+\beta} - 1$, and since the function $(0,\infty) \ni x \mapsto \sqrt{2+x} - 1$ has a unique attracting fixed point $\sqrt{5/4} - 1/2 > 0.61$ it follows that $\alpha, \beta > 0.61$. Again this is sufficient and this completes case II.b.

Case II.c. Q_{i_1} contains $f^k(V)$ and also some point between $f^{i_0}(V)$ and $f^k(V)$. As before, $f^{k-i_1}\colon Q_{i_1} \to Q_k$ is orientation preserving in this case, because otherwise f^{k-i_1} maps $[f^k(V), f^{i_1}(V)]$ monotonically into itself and since f has no periodic attractor, this is impossible. Hence the interval $f^{k+(k-i_1)}(V)$ lies between $f^{i_0}(V)$ and $f^k(V)$.



Case II.c.

Note that Z_k contains another interval $f^j(V)$ to the right of $f^{i_1}(V)$ because we have assumed in Cases II and III that H_k contains a neighbourhood of $f^{i_1}(V)$. Therefore we may assume that Q_{i_1} is contained in Z_k, because otherwise Q_k contains another (i.e., a fifth) interval $f^{j'}(V)$ inside Q_k and so $C^{-1}(Q_k, f^k(V))$ is at least 0.6. Now let γ be so that the length of the component $Z_k \setminus Q_k$ to the right of $f^{i_1}(V)$ is equal to $\gamma|f^{i_1}(V)|$. Since we have assumed that Q_{i_1} is contained in Z_k,

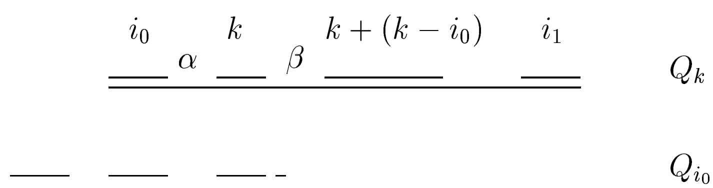
$$\gamma \geq C^{-1}(Q_{i_1}, f^{i_1}(V)) \geq C^{-1}(Q_k, f^k(V)) \geq \frac{2 \cdot 1}{4} = 0.5 \quad .$$

Hence

$$C^{-1}(Z_k, f^k(V)) \geq \frac{2(1+\gamma)}{2+1+1+\gamma} \geq \frac{2(1+0.5)}{4+0.5} = 2/3 > 0.6 \quad .$$

This completes the proof of Case II.

Case III. Q_{i_0} contains a neighbourhood of $f^k(V)$ and, moreover, H_k contains a neighbourhood of $f^{i_1}(V)$. The first assumption implies as before that $f^{k-i_0}\colon Q_{i_0} \to Q_k$ is orientation preserving. The last assumption implies that Q_{i_0} cannot have its right endpoint in some interval $f^j(V)$ with $j > i_1$ since then f^{k-i_0} cannot map $f^j(V)$ monotonically onto $f^{i_1}(V)$. Hence $f^{k+(k-i_0)}(V)$ is contained between $f^k(V)$ and $f^{i_1}(V)$. If Q_{i_0} contains points from $f^{k+(k-i_0)}(V)$, then Q_{i_0} contains this interval in its interior and therefore Q_k contains three intervals $f^j(V)$ to the right of $f^k(V)$ and so we get the required estimate. Therefore we can (and will) assume in the remainder of the proof of Case III that Q_{i_0} is to the left of $f^{k+(k-i_0)}(V)$.



Case III.

Let $\alpha > 0$ be so that the gap $(f^{i_0}(V), f^k(V))$ has size $\alpha|f^{i_0}(V)|$. Define β so that $(f^k(V), f^{k+(k-i_0)}(V))$ has size $\beta|f^k(V)|$. The gap corresponding to α is mapped to the gap corresponding to β by f^{k-i_0}. Defining $W_k = [f^{i_0}(V), f^{k+(k-i_0)}(V))$ and $W_1 \subset H_1$ so that $f^{k-1}(W_1) = W_k$, we have that

$$\alpha \geq C^{-1}(W_{i_0}, f^{i_0}(V)) \geq C^{-1}(W_k, f^k(V)) \geq \frac{(1+\alpha)\beta}{2+\alpha+\beta}.$$

This means that

$$\alpha \geq \sqrt{1+\beta} - 1.$$

Now let σ be so that $|f^{k+(k-i_0)}(V)| = \sigma|f^k(V)|$; one has $\sigma \geq 1$. Since one component of $Q_{i_0} \setminus f^k(V)$ is contained in the gap corresponding to β, and using the definition of

σ, we have that

$$\beta \geq C^{-1}(Q_{i_0}, f^k(V)) \geq C^{-1}(Q_k, f^{k+(k-i_0)}(V)) \geq \frac{(2+\alpha+\beta)1}{\sigma(3+\sigma+\alpha+\beta)}.$$

This means that

$$\beta^2 + \beta(3+\sigma+\alpha-1/\sigma) \geq 2/\sigma + \alpha/\sigma. \tag{6.2}$$

Our aim is to prove that

$$C^{-1}(Q_k, f^k(V)) \geq \frac{(1+\alpha)(\beta+\sigma+1)}{3+\alpha+\beta+\sigma} \geq 0.6 \quad .$$

If $\sigma \geq 2$ or if $\alpha \geq 1/3$ then this holds. So assume that $1 \leq \sigma \leq 2$ and that $\alpha \leq 1/3$. If $\sigma \in [3/2, 2]$ then $3+\sigma+\alpha-1/\sigma \leq 5$ and so (6.2) implies that $\beta^2 + 5\beta \geq 1$, i.e., $\beta \geq 0.19$. Hence $\alpha \geq \sqrt{1.19} - 1 \geq 0.09$ and hence

$$\frac{(1+\alpha)(\beta+\sigma+1)}{3+\alpha+\beta+\sigma} \geq \frac{1.09 \times 2.69}{4.78} > 0.6 \quad .$$

If $\sigma \in [5/4, 3/2]$ then $3+\sigma+\alpha-1/\sigma \leq 4.2$ and therefore $\beta^2 + 4.2\beta \geq 4/3$ and so $\beta \geq 0.29$. Therefore, $\alpha \geq \sqrt{1.29} - 1 \geq 0.13$ and

$$\frac{(1+\alpha)(\beta+\sigma+1)}{3+\alpha+\beta+\sigma} \geq \frac{1.13 \times 2.54}{4.67} > 0.6 \quad .$$

If $\sigma \in [1, 5/4]$ then $\beta^2 + 3.8\beta \geq 8/5$ and so $\beta \geq 0.38$. Therefore, $\alpha \geq \sqrt{1.38} - 1 \geq 0.17$ and

$$\frac{(1+\alpha)(\beta+\sigma+1)}{3+\alpha+\beta+\sigma} \geq \frac{1.17 \times 2.38}{4.55} > 0.6 \quad .$$

Thus we get the required estimate in each case. This completes the proof of this lemma. □

The above lemma allows us to show that the interval $L = f^s(l)$ from Lemma 6.2 is not too short compared to $f(V)$:

Lemma 6.4 *Assume that R_W has a high return. As in Lemma 6.2, let l be one of the two maximal intervals for which $f^s|l$ is monotone and which has a unique common point with $\hat{U}$. (If we take the interval which is outside $[c_1, c_2]$ then it is equal to the interval l from Lemma 5.1.) Write $L = f^s(l)$. Assume that $U = V$ has period s and assume that f is not renormalizable of period $s/2$. Then*

$$|L| \geq 0.6 \cdot |f(V)|.$$

Proof: From the previous lemma, $L = f^s(l)$ contains at least two intervals of the form $f^2(V), \dots, f^{s'}(V)$. We shall consider the disjoint intervals $f(V), \dots, f^{s'}(V)$. Of course, $f(V)$ and $f^2(V)$ have just one neighbour in this collection, and all other have two.

First consider the case that $f(V)$ is shorter than its neighbour. Because $f^s(l)$ contains at least this neighbour, one gets $|f^s(l)| \geq |f(V)|$ which gives the required estimate. So we may assume that $f(V)$ is longer than its neighbour.

Similarly, let us consider the case that $f^2(V)$ is shorter than its nearest neighbour $f^j(V)$. Then let G be the interval containing $f(V)$ which is mapped diffeomorphically onto $[f^2(V), f^j(V)]$. Because $z \mapsto |Df(z)|$ is monotone on G and takes its maximum on $f(V)$ it follows that $G \setminus f(V)$ is longer than $f(V)$. Now since $f^s(l)$ contains a neighbour, it certainly contains $G \setminus f(V)$. In particular, as in the previous case we get $|f^s(l)| \geq |f(V)|$ and the proof is complete in this situation.

So we may assume that $f(V)$ and $f^2(V)$ are both smaller than their neighbours. This implies that, as in the previous lemma, there exists a maximal integer $k \geq 2$ with $k \leq s'$ and such that $f^k(V)$ is shorter than its two neighbours from the collection $f(V), \dots, f^{k-1}(V)$. As before, $Q_k = [f^{i_0}(V), f^{i_1}(V)] \supset f^k(V)$ contains only intervals of the form $f^j(V)$ with $k < j \leq s$ which are all longer than the intervals $f^{i_0}(V)$, $f^{i_1}(V)$ and $f^k(V)$. For simplicity assume again that $f^{i_0}(V)$ lies to the left of $f^{i_1}(V)$. Let $H_1 \supset f(V)$ be the maximal monotone interval and, as in the previous lemma, let Z_1 be the maximal interval in H_1 such that $Z_k = f^{k-1}(Z_1)$ contains at most one interval of the form $f^j(V)$ with $k < j \leq s$ on each side of $f^k(V)$. Then $f^{k-1}(Z_1) \supset Q_k$ by Lemma 6.1. Write

$$H_k = f^{k-1}(H_1).$$

Let $Z_{1,-}$ and $Z_{1,+}$ be the components of $Z_1 \setminus f(V)$ and for simplicity take $Z_{1,+}$ be the component which lies on the same side of $f(V)$ as L (so it lies in $[c_1, c_2]$). Label i_0 and i_1 so that $f^{k-1}(Z_{1,+})$ contains $f^{i_1}(V)$. If $L \supset Z_{1,+}$ then we have that

$$\frac{|L|}{|f(V)|} \geq \frac{|Z_{1,+}|}{|f(V)|} \geq C^{-1}(Z_1, f(V)) \geq C^{-1}(Z_k, f^k(V)) \geq 0.6,$$

where in the last inequality we used the previous lemma.

Therefore we may assume that L is a proper subset of $Z_{1,+} \subset H_1$. Hence $f^{k-1}|L$ is monotone and therefore, because of Lemma 6.2, $f^{k-1}(L)$ contains at least two neighbours $f^j(V)$ with $k < j \leq s$ of $f^k(V)$ (on the same side as $f^{i_1}(V)$). These intervals $f^j(V)$ are to the left of $f^{i_1}(V)$ because otherwise $f^{k-1}(L) \subset H_k$ would contain an interval $f^j(V)$ to the right of $f^{i_1}(V)$ and so $f^{k-1}(L) \supset Z_{k,+}$. Hence $L \supset Z_{1,+}$, a contradiction. If $f^{k-1}(L)$ contains three or more intervals $f^j(V)$ then $C^{-1}(W_k, f^k(V)) \geq 4/5 > 0.6$ and we are done. Here $W_1 = H_{1,-} \cup f(V) \cup L$ and $W_j = f^{j-1}(W)$. So we may assume that $f^{k-1}(L)$ contains precisely two neighbours $f^{j_1}(V)$ and $f^{j_2}(V)$ and assume for simplicity that $f^{j_1}(V)$ is to the left of $f^{j_2}(V)$. Of course, this implies that we may also assume that L contains precisely two intervals of the form $f^j(V)$. Hence, it suffices to show that

$$C^{-1}(W_k, f^k(V)) \geq 0.6.$$

We have that $f^{i_0}(V) \subset f^{i_0}(H_{1,-})$ lies to the left of $f^k(V)$. There are two cases.
Case 1. W_{i_0} lies to the left of $f^k(V)$. In this case choose $\alpha > 0$ so that the gap between $f^{i_0}(V)$ and $f^k(V)$ has size $\alpha|f^{i_0}(V)|$. Then

$$\alpha \geq C^{-1}(W_{i_0}, f^{i_0}(V)) \geq C^{-1}(W_k, f^k(V)) \geq \frac{(1+\alpha)2}{4+\alpha}$$

and this implies that $\alpha \geq 1/2$ and that $C^{-1}(W_k, f^k(V)) \geq 3/5 = 0.6$.
Case 2. W_{i_0} contains some points of $f^k(V)$. Then, as before, $f^{k-i_0}: W_{i_0} \to W_k$ is order preserving. If the image of the gap between $f^{i_0}(V)$ and $f^k(V)$ under f^{k-i_0} contains one of the intervals $f^j(V)$ in $f^{k-1}(L) \subset W_k$ then define $\hat{W}_k = [f^{i_0}(V), f^j(V)]$, $\hat{W}_1 \subset H_1$ so that $f^{k-1}(\hat{W}_1) = \hat{W}_k$ and we get that

$$\alpha \geq C^{-1}(\hat{W}_{i_0}, f^{i_0}(V)) \geq C^{-1}(\hat{W}_k, f^k(V)) \geq \frac{(1+\alpha)1}{3+\alpha},$$

i.e., $\alpha \geq 1/3$ and $C^{-1}(W_k, f^k(V)) > 0.6$. Since, by assumption, $f^{k-1}(L)$ contains no more than two intervals $f^j(V)$ and since $f^{k-i_0-1}(L)$ also contains two interval of the form $f^m(V)$, the only remaining possibility is that f^{k-i_0} maps $f^k(V)$ to $f^{j_1}(V)$ and $f^{j_1}(V)$ to $f^{j_2}(V)$. (Here we use that W_{i_0} cannot contain W_k because otherwise f would have a periodic attractor.) Hence

$$j_1 = k + (k - i_0) \text{ and } j_2 = k + 2(k - i_0).$$

But now we use a more precise statement from Lemma 6.2: the intervals $f^j(V)$ in L are of the form $f^{i+2}(V)$ and $f^{s-i}(V)$. Hence

$$j_1 = (i+2) + (k-1) \text{ and } j_2 = (s-i) + (k-1)$$

for some i. Writing $r = k - i_0$, combining all this gives

$$r = j_1 - k = i + 1 \text{ and } 2r = j_2 - k = s - i - 1.$$

Hence $3r = s$ and L contains the intervals $f^{r+1}(V)$ and $f^{2r+1}(V)$. But then the map $f^{k-1} = f^{r+i_0-1}$ cannot be monotone on $f^{2r+1}(V)$ (because $3r = s$ and $f^s|f(V)$ is not monotone). Therefore we get a contradiction with the assumption that $f^{k-1}|L$ is monotone. □

Finally, we shall also give in this section an estimate for the case that a map is renormalizable of period s and also of period $s/2$ (this case was not covered by the previous lemma).

Lemma 6.5 *Assume that R_W has a high return. As in Lemma 6.2, let l be one of the two maximal intervals for which $f^s|l$ is monotone and which has a unique common point with $\hat{U}$. Write $L = f^s(l)$. Assume that $U = V$ has period s and assume that f* **is** *renormalizable of period $s/2$. Then*

$$|L| \geq (1/2) \cdot |f(V)|.$$

Proof: Let R be the renormalizable interval of period $r := s/2$ containing both U and $f^{s/2}(U)$. Let $k = 0, 1, \dots, r-1$ be so that $|f^k(R)| \le |f^i(R)|$ for each $i = 0, 1, \dots, r-1$. There are two cases: $|f^k(U)| \le |f^{k+r}(U)|$ or $|f^{k+r}(U)| \le |f^k(U)|$. In the former case, let $m = k$ and define $f^{m\pm r}(U) = f^{m+r}(U)$ and in the latter case we take $m = k + r$ and define $f^{m\pm r}(U) = f^{m-r}(U)$. If $m = 1, 2$ then one has, just like in the proof of Lemma 6.4, that

$$|f^{r+1}(U)| \ge |f(U)|. \tag{6.3}$$

(Note that $f^{r+1}(U)$ is the nearest neighbour of $f(U)$ from the collection of disjoint intervals $U, f(U), \dots, f^{s-1}(U)$ because by assumption f is also renormalizable of period $r = s/2$.) So assume that $m > 2$. Then define Q_m to be the smallest interval containing $f^{m\pm r}(U)$ on one side of $f^m(U)$ and containing also the nearest neighbour from the collection $R, \dots, f^{r-1}(R)$ on the other side of $f^m(U)$. Let H_1 be the maximal interval containing $\hat{U}$ so that $f^{m-1}|H_1$ is monotone. We claim that $f^{m-1}(H_1) \supset f^m(U)$ contains Q_m. Indeed, let $H_1^{1,2}$ be the components of $H_1 \setminus \hat{U}$. By maximality, there exist $i_1, i_2 < s$ with $i_1 \ne i_2$ so that $f^{i_1}(H_1^1)$, $f^{i_2}(H_1^2)$ contains c in its boundary. If $i_j \ne r-1$ then $f^{i_j}(U) \cap R = \emptyset$ and therefore $f^{i_j}(H_1^j)$ contains one component of $R \setminus \{c\}$. Therefore $f^{m-1}(H_1^j)$ contains a neighbour of $f^m(R)$ from the collection $R, \dots, f^{r-1}(R)$. If $i_j = r-1$ then $f^{i_j}(H_1^j)$ merely contains one component of $U \setminus \{c\}$ and then $f^{m-1}(H_1^j)$ contains $f^{m-r}(U)$. Since either i_1 or i_2 is different from $r-1$ the claim follows. By the choice of m we therefore get

$$C^{-1}(Q_m, f^m(U)) \ge \frac{1 \cdot 2}{1+1+2}.$$

Hence the interval $Q_1 \supset \hat{U}$ for which $f^{m-1}(Q_1) = Q_m$ satisfies

$$C^{-1}(Q_1, f(U)) \ge 1/2.$$

In particular,

$$\text{both components of } Q_1 \setminus f(U) \text{ have length } \ge (1/2)|f(U)|. \tag{6.4}$$

From the first part of Lemma 6.2 it follows that $L = f^s(l)$ contains at least one of neighbours of $f(U)$ from the collection $U, f(U), \dots, f^{s-1}(U)$. The nearest neighbour of $f(U)$ is $f^{r+1}(U)$. So if $m = 1, 2$ then from (6.3) it follows that $|L| \ge |f(U)|$. If $m > 2$ then we have that either $f^{m-1}(L)$ contains either at least two intervals from the collection $U, f(U), \dots, f^{s-1}(U)$ or it contains $f^{r+m}(U)$. Hence from the definition of Q_m and since $f^{m-1}|Q_1$ is monotone we get that L contains one component of $Q_1 \setminus f(U)$. In particular, from (6.4), $|L| \ge (1/2)|f(U)|$. □

7 Lower bounds for 'space' when R_V has a high return

Let $\hat{V} \supset f(V)$ be the interval which is mapped monotonically onto W by $f^{s'-1}$. Let s'' be the smallest integer such that $f^{s''-1}(V) \ni c$. In this section we assume that R_V has a high return. Hence $f^{s-1}(\hat{U}) \ni c$ and so $s'' \leq s$.

Proposition 7.1 *Assume that R_V has a high return. Let T_0 be the smallest interval containing $f(V)$ and another disjoint interval from the collection $f^2(V), \ldots, f^{s''}(V)$. Write $L_0 = T_0 \setminus f(V)$. Then*

$$|L_0| \geq \frac{1}{3}|f(V)|.$$

Moreover, if we define l to be one of the two maximal intervals outside $\hat{U}$ for which $f^s|l$ is monotone and which has a unique common point with $\hat{U}$, then $f^s(l)$ contains L_0.

For the proof of this proposition we need two lemmas.

Lemma 7.1 *Assume that R_V has a high return. Let $2 \leq k < s''$ and let H_1 be the maximal interval containing $f(V)$ such that $f^k|H_1$ is monotone. Then $f^k(H_1)$ contains $f^k(f(V))$ and on each side of this interval also an interval of the form $f^i(V)$ with $i \leq k$.*

Proof: Let $H_{1,-}, H_{1,+}$ be the components of $H_1 \setminus f(V)$. From the maximality of H_1 it follows that there exists $i' < k$ such that $f^{i'}(H_{1,+})$ contains c. Since $i' < k \leq s'' - 1 \leq s-1$, the interval $f^{i'+1}(U)$ (which is contained in $f^{i'+1}(V) \subset f^{i'}(H)$) is outside V. Since $f^{i'+1}(U)$ and $f^{i'+1}(V)$ have $f^{i'+1}(c)$ as one common endpoint and the other endpoint of $f^{i'+1}(V)$ is certainly outside V (because the endpoints of V are nice), it follows that $f^{i'+1}(V)$ is outside V. Hence $f^{i'}(H_{1,+})$ contains one component of $V \setminus \{c\}$. It follows that $f^k(H_{1,+})$ contains $f^{k-i'}(V)$. □

Let $\hat{U} \supset f(U)$ the interval which is mapped monotonically onto V by f^{s-1}. In the next lemma we prove the second part of the statement of Proposition 7.1.

Lemma 7.2 *Let l be one of the two maximal intervals outside $\hat{U}$ for which $f^s|l$ is monotone and which has a unique common point with $\hat{U}$. Then $L := f^s(l)$ contains at least one interval of the form $f^i(V)$, $1 \leq i \leq s''$ which is disjoint from $f^s(\hat{U}) = V$.*

Proof: By maximality of l there is i with $0 < i < s$ such that $f^i(l)$ contains c in its boundary. Choose i maximal with this property. Since $f^i(\hat{U})$ is outside V it follows that $f^i(l)$ contains one component V_+ of $V \setminus \{c\}$. Hence $f^s(l)$ contains $f^{s-i}(V)$ (and also a point in $f^{s-1}(\hat{U}) = V$). Since $f^{s''-1}(f(V))$ contains c in its interior, and since by definition $f^s|l$ is monotone, it follows that $s - i < s''$. Hence $f^{s-1}(H)$ contains one of the intervals $f(V), \ldots, f^{s''-1}(V)$. □

Proof of Proposition 7.1 Let L_0 be defined as in the proposition and consider the intervals $f(V), \ldots, f^{s''}(V)$. Since s'' might be larger than s', we cannot be sure that these intervals are disjoint. Evenso, there exists k such that

$$|f^k(V)| \leq |f^i(V)| \text{ for each } i = 1, 2, \ldots, s''.$$

If $k = 1$ then in particular the length $f(V)$ is at most equal to the length of its nearest disjoint neighbour from the collection $f(V), \ldots, f^{s''}(V)$. Since L_0 contains an interval $f^i(V)$, $1 \le i \le s''$ which is disjoint from $f(V)$ it follows that $|L_0| \ge |f(V)|$. If $k = 2$ then a similar argument applies: again the length $f^2(V)$ is at most equal to the length of its nearest disjoint neighbour from the collection $f(V), \ldots, f^{s''}(V)$. Since L_0 contains an interval $f^i(V)$, $1 \le i \le s''$ which is disjoint from $f(V)$, and since $z \mapsto |Df(z)|$ increases monotonically as z moves away from $c = 0$, it follows again that $|L_0| \ge |f(V)|$. So we may assume that $k > 2$. Because of Lemma 7.1 we can find an interval Z_1 around $\hat{V}$ such that $f^{k-1}|Q_1$ is monotone and so that $Z_k = f^{k-1}(Z_1)$ contains on each side of $f^k(V)$ an interval of the form $f^i(V)$ with $i < k$ (and which is disjoint with $f^k(V)$). Let $Z_{1,\pm}$ be two components of $Z_1 \setminus \hat{U}$ marked so that $Z_{1,+}$ intersects L_0. If $Z_{1,+} \subset L_0$, then

$$\frac{|L_0|}{|f(V)|} \ge \frac{|Z_{1,+}|}{|f(V)|} \ge C^{-1}(Z_1, f(V)) \ge C^{-1}(Z_k, f^k(V)) \ge \frac{1}{3}$$

because Z_k contains on each side of $f^k(V)$ an interval $f^i(V)$ with $i \le s''$ which is at least as long as $f^k(V)$ and because of the choice of k.

It remains to consider the case when $L_0 \subset Z_{1,+}$, i.e., when f^{k-1} is monotone on L_0. As we have seen above, L_0 contains some $f^j(V)$ with $j \le s''$. Hence, the interval $f^{j+k-1}(V)$ lies in Z_k and $k \le j + k - 1 \le s''$. By choice of k, the interval $f^{j+k-1}(V)$ is longer than $f^k(V)$. Hence $|f^{k-1}(L_0)| \ge |f^k(V)|$. Moreover, again by Lemma 7.1, $f^{k-1}(Z_{1,-})$ also contains an interval of the $f^i(V)$ with $i \le s''$ and again by the choice of k this implies that $|f^{k-1}(Z_{1,-})| \ge |f^k(V)|$. Let $W_1 = Q_{1,-} \cup \hat{U} \cup L_0$ and $W_k = f^{k-1}(W_1)$. Then both components of $W_k \setminus f^k(V)$ are at least as long as $f^k(V)$ and therefore

$$\frac{|L_0|}{|f(V)|} \ge C^{-1}(W_1, f(V)) \ge C^{-1}(f^{k-1}(W_1), f^k(V)) \ge \frac{1}{3}.$$

This completes the proof of this proposition. □

8 The proof of the Main Theorem in the infinitely renormalizable case for $\ell > 2$

In this section we consider an infinitely renormalizable $f(z) = z^\ell + c_1$ map with $\ell \ge 4$. Such a map has renormalizations of period $q(n)$ where $q(n+1) = q(n) \cdot a(n)$ where $a(n) \ge 2$ is an integer. If $a(n) = 2$ for all n larger than some n_0 than some renormalization has Feigenbaum dynamics, and then local connectedness immediately follows from Hu and Jiang's result [HJ]. In fact, we shall prove the Main Theorem and Theorem A for this case separately in Section 13 because the bounds obtained in Lemma 6.4 do not hold at the n-th renormalization if $a(n) = 2$ (in that case the weaker bounds obtained in Lemma 6.5 will be used in Section 13.) So assume in this section that $a(n) > 2$ for infinitely many n. Then we can find a sequence of n's tending to infinity and a sequence of periodic intervals $U_n = V_n$ of period $s(n)$ such that f is not also renormalizable of period $s(n)/2$. Because f has no wandering intervals [MS], it follows that $|U_n| \to 0$.

Let us pick such an n and write $U_n = V_n = [-u_n, u_n]$ and let $s(n)$ be the period (note that for the map we consider $\tau(z) = -z$). For convenience, let us for the moment suppress the subscript n and write s for $s(n)$. Let $\hat{U} \supset f(U)$ be the interval from before and consider the diffeomorphism $f^{s-1}: \hat{U} \to V$. Let F be the inverse of this map. Since F is a diffeomorphism, it induces a univalent map

$$F: \mathbb{C}_V \to \mathbb{C}_{\hat{U}}.$$

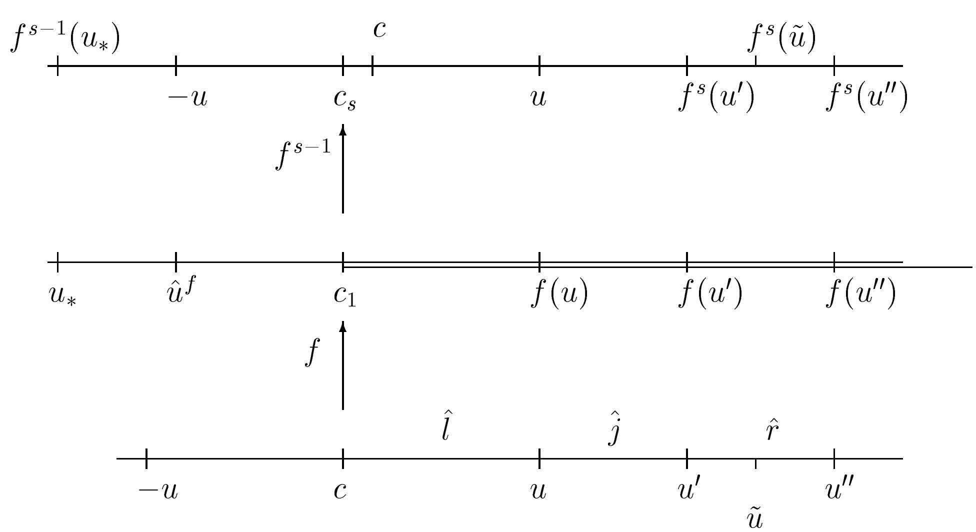
Hence $F(D_*(V)) \subset D_*(\hat{U})$. Now we use that the space from Lemma 5.1 is at least 0.6, because of the estimates of Lemma 6.4. (In fact, in Section 12, another proof of the Main Theorem is given in the infinitely renormalizable case - this proof is based on the space 1/3 but using domains which are not Euclidean discs.) Hence, see the estimates below Lemma 5.1, one gets that $f^{-1}(D_*(\hat{U})) \subset D_*(V)$, i.e.,

$$f^{-1} \circ F(D_*(V)) \subset D_*(V).$$

From this we get that

$$f^s: D'_*(V) \to D_*(V),$$

where $D'_*(V) = f^{-1} \circ F(D_*(V))$, is a proper degree ℓ map. This is still not a polynomial-like mapping since $\partial D'_*(V) \cap \partial D_*(V) = \partial V$, so these regions intersect in the repelling periodic point u (with $f^s(u) = u$) and its symmetric counterpart $-u$. Of course, this problem can be easily amended by adding to $D_*(V)$ some discs containing u because this point is a repelling. In this way we will get a polynomial-like mapping $f^s: \Omega'_n \to \Omega_n$. In fact, we even want a lower bound for the modulus of the annulus $\Omega_n \setminus \Omega'_n$. That such a lower bound exists, is not surprising since $|Df^s(u)| - 1$ is bounded from below, see Theorem B in Chapter IV of [MS] or Theorem B in [MMS]. In the Lemma below we shall give use related estimates to give lower bounds for the modulus of the annulus $\Omega_n \setminus \Omega'_n$.



Lemma 8.1 *There are universal constants $C_0, C_1, C_2 > 0$ with the following property. Let $T_1 \supset f(U)$ be the maximal interval on which f^{s-1} is monotone. Then,*

$$\text{each component of } f^{s-1}(T_1) \setminus V \text{ has length} \geq \tfrac{C_0}{\ell}|u - c|. \tag{8.1}$$

Let T be the component of $f^{-1}(T) \setminus \{c\}$ which contains u. Then there exists $\tilde{u} \in T$ such that

$$\frac{C_1/2}{\ell}|u - c| \leq |f^s(\tilde{u}) - u| \leq \frac{C_1}{\ell}|u - c| \tag{8.2}$$

such that

$$|f^s(\tilde{u}) - c| \geq (1 + C_2/\ell^2) \cdot |\tilde{u} - c|. \tag{8.3}$$

When $\ell \geq 4$, there exists also $u_* \in T$ such that

$$f^{s-1}(u_*) = -f^s(\tilde{u}) \text{ and } |u_* - c_1| < |f(u) - c_1|. \tag{8.4}$$

Proof: Let L_1 be a maximal interval on which f^s is monotone with a unique common point with $f(U)$. By Lemma 6.4, one has

$$|f^s(L_1) \setminus f(V)| \geq 0.6 \cdot |f(V)| = 0.6 \cdot |c_1 - f(u)|. \tag{8.5}$$

Since, $f(z) = (z - c)^\ell + c_1$ (where we usually take $c = 0$), the first inequality (8.1) follows because there exists C_0 such that

$$0.6^{1/\ell} \geq 1 - \frac{C_0}{\ell}. \tag{8.6}$$

So we can take $u', u'' \in T$ with

$$|f^s(u') - u| = \frac{C_1}{2\ell}|c - u| \text{ and } |f^s(u'') - u| = \frac{C_1}{\ell}|c - u| \tag{8.7}$$

provided $C_1 < C_0$. We shall now show in the remainder of the proof that there exists $\tilde{u} \in [u', u'']$ for which (8.3) holds. Let us now show that this would complete the proof, i.e. that (8.4) automatically would also hold. Indeed, because of (8.1) one can take $u_* \in T_1$ so that $f^{s-1}(u_*) = -f^s(\tilde{u})$. By Lemma 5.1 there exists a universal constant $K_* < 1$ such that provided $\ell \geq 4$,

$$|\hat{u}^f - c_1| < K_*|f(u) - c_1|. \tag{8.8}$$

Let us show that, provided we choose C_1 sufficiently small, in (8.7), the inequality in (8.4) holds. For this define $j' = [u_*, \hat{u}^f]$ and t' the interval between $f(u)$ and the endpoint of T_1 outside $[c_1, c_2]$. Let l', r' be the components of $t' \setminus j'$ (with l' the component outside $[c_1, c_2]$. One has

$$\frac{|c_1 - \hat{u}^f|}{|u_* - \hat{u}^f|} = \frac{|l'|}{|j'|} \geq C^{-1}(t', j') \geq C^{-1}(f^{s-1}(t'), f^{s-1}(j')) \geq \frac{|f^{s-1}(l')|}{|f^{s-1}(j')|}\frac{|f^{s-1}(r')|}{|f^{s-1}(t')|}. \tag{8.9}$$

The first ratio in the last term will tend to infinity if we choose C_1 small (because $|f^{s-1}(l' \cup j')|$ has size $\geq 0.6^{1/\ell} \cdot |V|$ and $|f^{s-1}(j')|$ has size $(C_1/\ell) \cdot |V|$). The second factor in the last term of (8.9) is of order one. Hence, (8.9) implies that $|u_* - \hat{u}^f|/|c_1 - \hat{u}^f|$ goes to zero provided C_1 goes to zero. Combined with (8.8) one has that $|u_* - c_1| < |f(u) - c_1|$ for some universal choice of C_1.

Thus it remains to prove (8.3). Let $\hat{l} = [c, u]$, $\hat{j} = [u, u']$, $\hat{r} = [u', u'']$ and $\hat{t} = \hat{l} \cup \hat{j} \cup \hat{r}$. Let us compare the size of $\hat{j} \cup \hat{r}$ with that of $\hat{l}$. Define $\tau = |\hat{j} \cup \hat{r}|/|\hat{l}|$. If $\tau \leq C_1/(2 \cdot \ell)$, then $|f^s(u'') - c|/|u'' - c| = (1 + C_1/\ell)/(1 + \tau) \geq (1 + C_1/\ell)/(1 + C_1/(2\ell)) \geq 1 + C_2/\ell$, and (8.3) follows with $\tilde{u} = u''$. So we may assume that

$$\tau = \frac{|\hat{j} \cup \hat{r}|}{|\hat{l}|} = \frac{|u - u''|}{|u - c|} \geq C_1/(2\ell). \tag{8.10}$$

Let us show that it suffices to show that there exists a universal constant C_3 and some $\tilde{u} \in \hat{r} = [u', u'']$ for which

$$|f^s(u) - f^s(\tilde{u})|/|u - \tilde{u}| \geq (1 + C_3/\ell^2). \tag{8.11}$$

Indeed, then

$$\frac{|c - f^s(\tilde{u})|}{|c - \tilde{u}|} \geq \frac{1 + \tau(1 + C_3/l^2)}{1 + \tau}.$$

Because of (8.10), $\tau \geq C_1/(2\ell)$ and therefore the last expression is bounded from below by $1 + C_7/\ell^2$. Therefore (8.4) follows and the proof of the lemma is complete once we have shown that (8.11) holds.

In fact, we may also assume that there exists $C_3' \in (0, 1)$ with

$$\frac{|f^s(\hat{r})|}{|\hat{r}|} \leq 1 + C_3'/\ell^2. \tag{8.12}$$

To see this, assume that (8.12) fails. Then, by the Mean Value Theorem,

$$|Df^s(\xi)| \geq 1 + C_3'/\ell^2 \text{ for some } \xi \in \hat{r} = [u', u''].$$

Hence we have the following cross-ratio inequality

$$\frac{(|f^s(u) - f^s(\xi)|/|u - \xi|)^2}{|Df^s(u)||Df^s(\xi)|} \geq 1$$

(this is just the cross-ratio inequality $C(f^s t, f^s j) \geq C(t, j)$ where we let t shrink to $j = [u, \xi]$). Hence, since u is a repelling periodic point,

$$\frac{|f^s(u) - f^s(\xi)|}{|u - \xi|} \geq \sqrt{|Df^s(u)||Df^s(\xi)|} \geq \sqrt{1 \cdot (1 + C_3'/\ell^2)}$$

which shows that (8.11) holds with $\tilde{u} = \xi$ and where we take $C_3 = C_3'/4$. Hence we are also finished if (8.12) does not hold. Of course, for the same reason we may assume $\frac{|f^s(\hat{r})|}{|\hat{r}|} \leq 2$ and with (8.7) we get

$$\frac{|\hat{r}|}{|\hat{l}|} \geq C_4/\ell \tag{8.13}$$

where $C_4 = C_1/4 > 0$ is again a universal constant. Moreover, we have

$$\frac{|\hat{l}|}{|\hat{j}|} \geq C^{-1}(\hat{t},\hat{j}) \geq C^{-1}(f^s(\hat{t}), f^s(\hat{j})) \geq \frac{1\cdot(C_1/(2\ell))}{(1+C_1/\ell)(C_1/(2\ell))} \geq C_4'. \tag{8.14}$$

Now, given intervals $j \subset t$ for which $t \setminus j$ has components l, r, define the cross-ratio operator

$$B(t,j) = \frac{|t||j|}{|l\cup j||r \cup j|}.$$

As with the cross-ratio operator C we defined before, one has $B(ft, fj) \geq B(t,j)$ if $f|t$ is monotone and has negative Schwarzian derivative. Because of (8.13) and (8.14), and using the expression $f(z) = (z-\ell)^\ell + c_1$, one can easily check that

$$\frac{B(f(\hat{t}), f(\hat{j}))}{B(\hat{t},\hat{j})} \geq 1 + C_5/\ell^2$$

where $C_5 > 0$ is a universal constant. This implies that

$$\frac{B(f^s(\hat{t}), f^s(\hat{j}))}{B(\hat{t},\hat{j})} \geq 1 + C_5/\ell^2. \tag{8.15}$$

Therefore, suppose by contradiction that (8.11) is false for $C_3 \leq \min(C_5/10, 1)$. Since f^s is a repelling fixed point at u and $\hat{j}$ has u in its boundary, we have that $|f^s(\hat{j}\cup\hat{r})|/|\hat{j}\cup\hat{r}| \geq 1$. Because (8.11) is false, this implies

$$\frac{|f^s(\hat{j})|/|\hat{j}|}{|f^s(\hat{j}\cup\hat{r})|/|\hat{j}\cup\hat{r}|} \leq 1 + C_3/\ell^2. \tag{8.16}$$

Now we also have

$$1 \leq \frac{|f^s(\hat{t})|}{|\hat{t}|}, \frac{|f^s(\hat{j}\cup\hat{l})|}{|\hat{j}\cup\hat{l}|} \tag{8.17}$$

and

$$\frac{|f^s(\hat{t})|}{|\hat{t}|} \leq \max\left(\frac{|f^s(\hat{j}\cup\hat{l})|}{|\hat{j}\cup\hat{l}|} \quad , \quad \frac{|f^s(\hat{r})|}{|\hat{r}|}\right). \tag{8.18}$$

Here (8.17) follows from the fact that $f^s|\hat{t}$ has no attracting fixed point (and u is its fixed point), and (8.18) is a general fact about mean slopes of a monotone map. Because we assumed that (8.12) holds we get from (8.17) and (8.18) that

$$\frac{|f^s(\hat{t})|/|\hat{t}|}{|f^s(\hat{j}\cup\hat{l})|/|\hat{j}\cup\hat{l}|} \leq (1 + C_3'/\ell^2). \tag{8.19}$$

Combining (8.16) and (8.19) and using $C_3' = 4C_3$ gives,

$$\frac{B(f^s(\hat{t}), f^s(\hat{j}))}{B(\hat{t},\hat{j})} \leq (1 + 6C_3/\ell^2).$$

On the other hand, by (8.15) we get that the left hand side of this expression is bounded from below by $1+C_5/\ell^2$. From this we get a contradiction since $C_3 \leq \min(C_5/10, 1)$. □

Consider domain

$$\Omega_n = D((-f^s(\tilde{u}), f^s(\tilde{u})) = D((f^{s-1}(u_*), f^s(\tilde{u})).$$

The inverse F of f^{s-1} extends in a univalent way to $\mathbb{C}_{[-f^s(\tilde{u}), f^s(\tilde{u})]}$ and therefore

$$F(D(-f^s(\tilde{u}), f^s(\tilde{u}))) \subset D_*(u_*, f(\tilde{u}))$$

where we use that $f^{s-1}(u_*) = -f^s(\tilde{u})$. Because of (8.3) and (8.4) we get that

$$f^{-1}(D_*(u_*, f(\tilde{u}))) \subset D_*(-f^s(\tilde{u}), f^s(\tilde{u})) = \Omega_n$$

and that, moreover, the difference set is an annulus with a modulus which is bounded away from zero by a constant which only depends on ℓ. In particular, writing $\Omega_n' = f^{-1} \circ F(\Omega_n)$, the map $f^{s(n)}: \Omega_n' \to \Omega_n$ is a polynomial-like mapping. Moreover, since f has no wandering intervals and $s(n) \to \infty$ we have that $|\Omega_n \cap \mathbb{R}| \to 0$. By construction the diameter of Ω_n is at most twice that of $V_n = [-u_n, u_n]$. Hence, Theorem A in the introduction follows immediately from all this.

The conclusion of the proof of the Main Theorem for the infinitely renormalizable case when $\ell \geq 4$. For simplicity, let T_n be the image under $f^{s(n)-1}$ of the maximal interval of monotonicity around c_1. By (8.1), there exists a constant $C = C(\ell) > 0$ such that $C(T_n, \Omega_n \cap \mathbb{R}) \leq C$. To end the proof, we follow an argument similar to [Ji1]. Given n, consider so-called the maximal renormalization, $g_n: W_n' \to W_n$, where W_n is a slitted complex plane $\mathbb{C}_{T_n}$ without a fixed neighborhood of infinity, so that $\Omega_n \subset W_n$ and $f_n = g_n = f^s$ on Ω_n'. By Proposition 3.2, the Julia set J_n of g_n is contained in Ω_n. On the other hand, J_n is equal to the intersection of critical Yoccoz pieces started from a piece based on the points $u, \hat{u}$, so there is a piece P_n in Ω_n containing c. Since $P_n \cap J(f)$ is connected, we have proved the local connectivity of $J(f)$ at the critical point c.

Let $z \in J(f)$ be any other point. If the forward orbit of z avoids some neighborhood of c, then f is expanding along this orbit, and the local connectivity at z follows.

So assume that the orbit of z hits any neighborhood P_n of c. Then we use the fact that the renormalizations f_n are 'unbranched' ([McM]). More precisely, there exists a domain M_n, such that $M_n \subset W_n$, the annulus $A_n = M_n \setminus \overline{P_n}$ has a modulus $\geq m = m(l) > 0$, and A_n does not contain any iteration of c. To see this, denote by t_n and T_n', where $t_n \subset T_n' \subset T_n$, the intervals, such that $g_n(t_n) = \Omega_n \cap \mathbb{R}$ and $g_n(T_n') = T_n$. Since $C(T_n, \Omega_n \cap \mathbb{R}) \leq C$, we have $C(T_n', t_n) \leq C'$, where $C' > 0$ depends only on ℓ. Furthermore, the interval T_n' is disjoint from the all iterations of c which are outside of the renormalization interval V (we use that f is not $s/2$-renormalizable). On the other hand, by the construction, the domain Ω' is inside of a domain of a definite shape based on the interval t_n (if $\ell \geq 4$, this is just a disc). This implies the existence of M_n as above. Let some iterate $f^k(z)$, $k = k_n$, of the orbit of z hit P_n the first time.

We can pull back the domain P_n along $z, f(z), \dots, f^{k-1}(z)$ by a branch G_n of f^{-k}, since P_n contains only iterations of c corresponding to the renormalization. Indeed, otherwise some $P' = f^{-i}(P_n)$ covers c for the first time. Hence, $f^i\colon P' \to P_n$ is an iteration of the renormalization f_n, i.e., $P' \subset P_n$ contradicting the choice of the iterate $f^k(z)$. By the unbranching property, the pullback G_n extends to the domain M_n. Let $P_n(z) = G_n(P_n)$. We want to show, of course, that the Euclidean diameters of $P_n(z)$ tend to zero. For this, let us consider a domain $M_n' \subset M_n$ bounded by a core curve of the annulus A_n. Then $\max_{y\in\partial M_n'} |f^k(z) - y| / \min_{y\in\partial M_n'} |f^k(z) - y| \le C(m)$,$n = 1, 2, \dots$ (see e.g. [McM]). Introduce domains $E_n = G_n(M_n')$. Since the modulus of annulus $M_n \setminus M_n'$ is $m/2$, by Koebe distortion theorem, $\max_{y\in\partial E_n} |z - y| / \min_{y\in\partial E_n} |z - y| \le C_1(m)$, $n = 1, 2, \dots$. If we assume that $\mathrm{diam} P_n \ge d > 0$ for $n = 1, 2, \dots$, then $\min_{y\in\partial E_n} |z - y| \ge d/2C_1 = r > 0$, i.e., the disc $D_z(r) \subset E_n$. Hence, $f^{k_n}(D_z(r)) \subset M_n'$, for $n \to \infty$ and $k_n \to \infty$. This is a contradiction with the non-normality of the family f^n at $z \in J(f)$. Thus, $\bigcap_{n>0} P_n(z) = \{z\}$ and so $J(f)$ is again locally connected at z.

9 The proof of the Main Theorem in the infinitely renormalizable case when $\ell = 2$

In this section we consider a renormalizable $f(z) = z^2 + c_1$ with a periodic interval V of the period s. As before, we may assume that R_V has a high return because otherwise f has a periodic attractor. Again we shall delay dealing with the case that f is also renormalizable of period $s/2$ until Section 13. Hence, because of Lemma 6.4, the space from Lemma 5.1 is at least 0.6. Note however, that the bound we obtain for $K_2^*(0.6)$ is equal to $1.19371... > 1$ and we cannot use the method of the previous section in the quadratic case. Therefore, we construct a domain of renormalization Ω for f^s which is different from $D_*(V)$, but with a diameter depending on $|V|$ only, so that the renormalizations shrink to zero together with V.

As before, denote $D(J;\theta)$ the Poincaré neighbourhood of a real interval J, with external angle $\theta \in (0, \pi/2]$. We define the domain $\Omega = \Omega(\theta)$ as the Poincare neighborhood $D(V;\theta)$ of the periodic interval V (with the angle θ to be specified later on), united with two discs: $D_*(I)$ and $D_*(-I)$ with $I = (0, 6/5 \cdot u)$, and u is the periodic endpoint of V, (i.e., $f^s(u) = u$). We may and shall assume that $u > 0$. Since R_V has a high return, one has that $c_s \in [-u, 0]$.

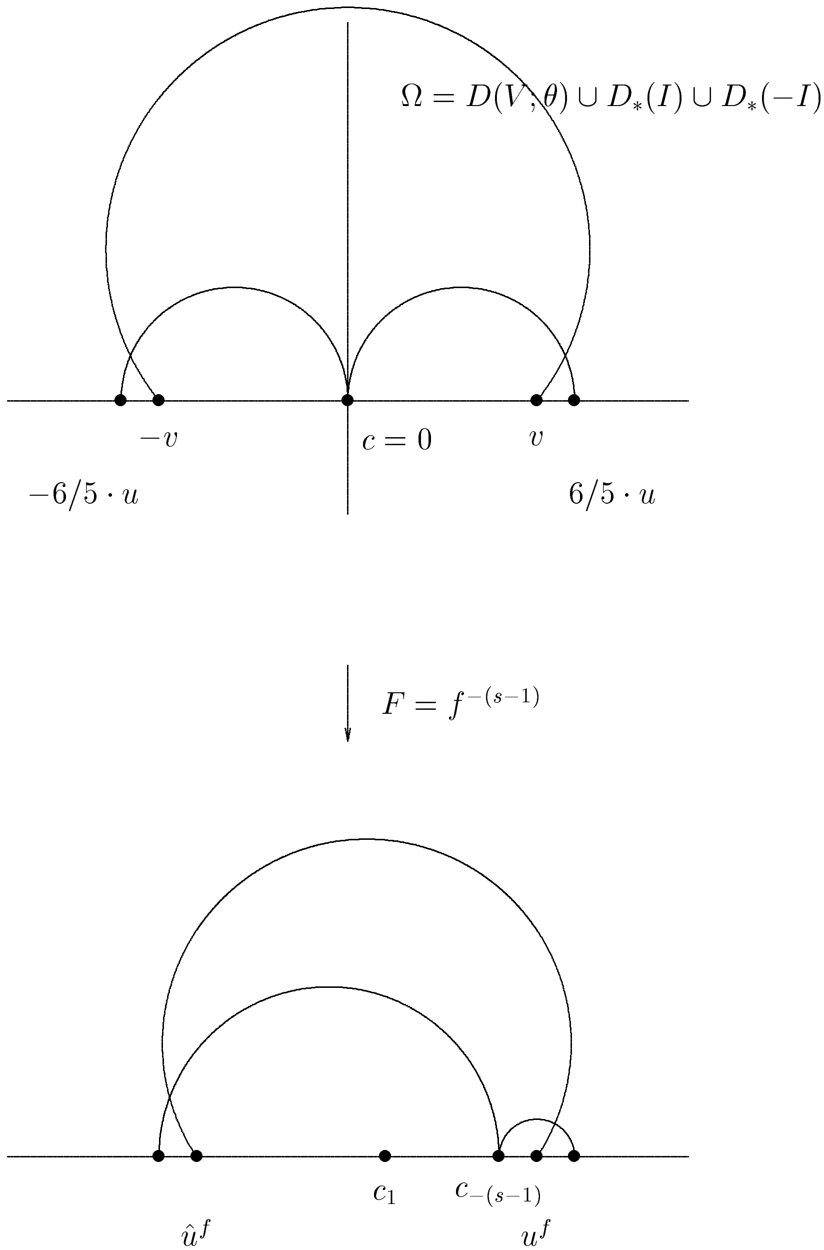
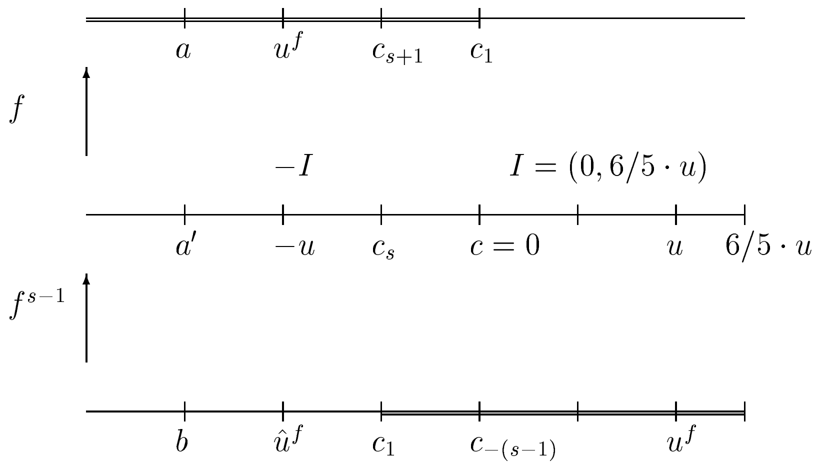


Figure 1: The region Ω and its preimage.



The intervals $I, -I$.

Remark 9.1 *Consider the map $f^{s-1}: \hat{U} \to V$. As in Lemma 6.4, let l be the maximal interval with a unique common endpoint with $\hat{U}$ and outside $[c_1, c_2]$ such that $f^{s-1}|l$ is monotone. From Lemma 6.4 we have that $|f^{s-1}(l)| \geq \sqrt{1+0.6} \cdot |V_+|$ where V_+ is one component of $V \setminus \{c\}$. The number $6/5$ is chosen for the following reason: $6/5 > K_2^*(0.6) = 1.19371..$, but $6/5 < \sqrt{1+0.6}$. Because of Lemma 6.4, the latter inequality shows that the pullback F of $f^{s-1}: \hat{U} \to V$ has a monotone extension to $\widetilde{I} = I \cup V \cup (-I)$. Let F also denote the extension of this pullback to $\mathbb{C}_{\widetilde{I}} = (\mathbb{C} \setminus \mathbb{R}) \cup \widetilde{I}$. The first inequality we shall need in the proof of Corollary 9.1 below.*

Our aim is to prove that $f^{-1} \circ F(\Omega)$ is a proper domain inside of Ω. For this, it is enough to prove

Proposition 9.1 *$f^{-1} \circ F(\partial\Omega) \subset \Omega$.*

Proof: First of all, we observe that $f^{-1} \circ F(D_*(I))$ consists of two components D^1 and $-D^1$, where D^1 is a proper domain in $D_*(I)$. Note that there exists an interval containing c_1 which is mapped diffeomorphically by f^{s-1} onto $\widetilde{I}$. Therefore, $f^{-1} \circ F$ maps I homeomorphically into itself, because u is a repelling fixed point of f^s (here we use that f^s has negative Schwarzian derivative). Moreover, $c_s = f^{s-1}(c_1)$ is not in I and therefore $F(D_*(I))$ does not contain c_1. Hence $f^{-1} \circ F(D_*(I))$ consists of two components D^1 and $-D^1$, where D^1 is a proper domain in $D_*(I)$. Thus, the corresponding branch of $f^{-1} \circ F(\Omega)$ maps $\mathbb{C}_I$ univalently into itself. By symmetry, we get also $-D^1 \subset D_*(-I)$. Thus we have shown that

$$f^{-1} \circ F(D_*(I)) \subset \Omega. \tag{9.1}$$

Next we want to show that the pullback of $D(V;\theta)$ is inside $\Omega(\theta)$ provided we choose θ conveniently. For this we shall first consider the pullbacks through the map $P(z) = z^2$. Fix $K \geq 1$. In the next lemma we are going to compare $P(D((-1,1);\theta))$ with the Poincaré disc $D((-K,1);\theta)$. The latter disc is a 'scaled-up' version of $D((\hat{u}^f, u^f);\theta) \ni c_1$.

Lemma 9.1 *Let $K > 1$. There exists $\theta_0 = \theta_0(K) > 0$ such that, for all $\theta \in (0,\theta_0)$, the boundaries of $P(D((-1,1);\theta))$ and $D((-K,1);\theta)$ intersect each other in $Z(K,\theta)$ and its complex conjugate. Furthermore,*

$$Z(K,\theta) \to K^2 \in \mathbb{R},$$

as $\theta \to 0$. Hence, the difference $\Delta(K,\theta) = D((-K,1);\theta) \setminus P(D((-1,1);\theta))$ tends to the interval $[1,K^2]$, as $\theta \to 0$.

Proof: Consider a possible intersection point

$$z_1 \in P(\partial D((-1,1);\theta)) \cap \partial D((-K,1);\theta).$$

That is, $z_1 = P(z_2) = z_2^2$ with

$$z_2 \in \partial D((-1,1);\theta)).$$

Since these sets are symmetric with respect to the real axis and since $D((-1,1);\theta)$ is also symmetric with respect to the imaginary axis, we may consider the case that z_1 is in the upper half plane, and that z_2 is in the first quadrant.

Since $z_2 \in \partial D((-1,1);\theta)$,

$$z_2 = 1 + \frac{i\exp(i\theta)}{\sin(\theta)}(1 - \exp(i\alpha)) \tag{9.2}$$

where $\alpha \in (0, 2\pi - \theta)$ is the angle between the vectors $z_2 - C, 1 - C$, with C the centre of the circle $D((-1,1);\theta)$. Similarly, since $z_1 \in \partial D((-K,1);\theta)$,

$$z_1 = 1 + \frac{K+1}{2} \cdot \frac{i \cdot \exp(i\theta)}{\sin(\theta)} \cdot (1 - \exp(i\phi)). \tag{9.3}$$

Taking the square of (9.2),

$$z_2^2 = 1 + \frac{2\exp(i\theta)}{\sin^2(\theta)}(1 - \exp(i\alpha)) \left\{ i\sin(\theta) - \frac{1}{2}\exp(i\theta)(1 - \exp(i\alpha)) \right\}.$$

We have in the brackets $\{...\}$ term:

$$i\sin(\theta) - \frac{1}{2}\exp(i\theta)(1 - \exp(i\alpha)) =$$

$$\frac{1}{2}(\exp(i\theta) - \exp(-i\theta) - \exp(i\theta) + \exp(i(\theta+\alpha))) =$$
$$\frac{\exp(i\alpha)}{2}(\exp(i\theta) - \exp(-i(\theta+\alpha))).$$

So,

$$z_2^2 = 1 + \frac{2\exp(i\theta)}{\sin^2(\theta)}(1-\exp(i\alpha))\frac{\exp(i\alpha)}{2}(\exp(i\theta) - \exp(-i(\theta+\alpha))) =$$
$$1 + \frac{2\exp(i\theta)}{2\sin^2(\theta)}\exp(i\alpha)\{\exp(i\theta) - \exp(-i(\theta+\alpha)) - \exp(i(\theta+\alpha)) + \exp(-i\theta)\} =$$
$$1 + \frac{\exp(i(\theta+\alpha))}{\sin^2(\theta)}\{2\cos(\theta) - 2\cos(\theta+\alpha)\}.$$

If $z_2^2 = z_1$, then we compare the last expression with (9.3), cancel 1 in both hand-sides and then divide them by $\exp(i\theta)$, and multiply by $\sin(\theta)$, and after that separate Re and Im parts:

$$\frac{2(\cos(\theta) - \cos(\theta+\alpha))}{\sin(\theta)}\cos(\alpha) = \frac{K+1}{2}\sin(\phi),$$
$$\frac{2(\cos(\theta) - \cos(\theta+\alpha))}{\sin(\theta)}\sin(\alpha) = \frac{K+1}{2}(1-\cos(\phi)).$$

Now divide the second equality to the first one:

$$\tan(\alpha) = \tan(\phi/2).$$

Here $0 < \phi/2 < \pi$, so either $\alpha = \phi/2$, or $\alpha = \phi/2 + \pi$. The latter case is impossible, since z_2 is in the first quarter, i.e. $0 < \alpha + \theta < \pi/2$.

Thus,

$$\alpha = \phi/2.$$

Now we substitute $\phi = 2\alpha$ in the equality, say, for the real parts:

$$\frac{2(\cos(\theta) - \cos(\theta+\alpha))}{\sin(\theta)}\cos(\alpha) = \frac{K+1}{2}\sin(2\alpha),$$

or

$$4\sin(\alpha/2)\sin(\theta+\alpha/2)\cos(\alpha) = \frac{K+1}{2}2\sin(\alpha)\cos(\alpha)\sin(\theta),$$

or

$$\sin(\theta+\alpha/2) = \frac{K+1}{2}\cos(\alpha/2)\sin(\theta),$$

or

$$\cos(\theta)\sin(\alpha/2) = \frac{K-1}{2}\cos(\alpha/2)\sin(\theta),$$

or, at last,

$$\tan(\alpha/2) = \frac{K-1}{2}\tan(\theta).$$

If θ here is small, we find a unique α in the admissible interval $(0, \pi - \theta/2)$ for the angle α (this is because $\alpha = \phi/2 \in (0, \pi - \theta/2)$). So there is the unique point of intersection of the curves $\partial P(D((-1,1);\theta))$ and $\partial D((-K,1);\theta)$ in the upper halfplane. If $\theta \to 0$, then $\phi \sim 2(K-1)\theta$ and $z_1 \sim K^2$. The rest of the lemma follows easily. $\square$

Corollary 9.1 *If θ is small enough, then*

$$f^{-1} \circ F(\partial D(V;\theta)) \subset \Omega(\theta). \tag{9.4}$$

Proof: Take $K = K_2^*(0.6) = 1.19371....$ By the Schwarz contraction principle, see Lemma 2.1, $F(D(V;\theta))$ is contained in $D(\hat{I};\theta)$, where the interval $\hat{I} = (\hat{u}^f, u^f)$ is around c_1, with $|c_1 - \hat{u}^f| \le K|u^f - c_1|$. By rescaling and the previous lemma, for all θ less than some positive absolute constant θ_1 the closure of the domain $F(D(V;\theta))$ is inside $f(\Omega(\theta))$. Here we have used that the previous lemma implies that $f^{-1}(F(D(V;\theta))) \setminus D(V;\theta)$ converges to the interval $\pm(u, K \cdot u)$ as $\theta \to 0$. Since $K < 6/5$, see Remark 9.1, this implies that this difference set is contained in $D_*(I) \cup D_*(-I)$.

$\square$

Conclusion of the proof of Proposition 9.1. To complete the proof, it remains to show that for all angles θ less than some other positive constant θ_2 the domain $F(D_*(-I))$ is contained properly inside $P(\Omega(\theta))$. Note that $P(-I) = P(I) = (c_1, a)$, where $|c_1 - a| = (6/5)^2|c_1 - u^f| < (1 + 0.6) \cdot |c_1 - u^f|$, i.e. $a \in L$. Hence, by Lemma 5.1, the interval $F(-I)$ is inside an interval (b, u^f), where

$$|b - c_1|/|u^f - c_1| = (|b - c_1|/|a - c_1|) \cdot (5/6)^2 < K_2^*(y_0) \cdot (5/6)^2 := K_1 < 1$$

where, using the notation of Lemma 5.1, $y_0 = |a - c_1|/|T| = (6/5)^2/(1 + 0.6) \in (0.6, 1)$. It follows, $F(D_*(-I))$ is inside of the ball $D_*(b, u^f)$. By rescaling, we need to show that $P(D((-1,1);\theta))$ contains a fixed ball $D_*(-K_1, 1)$. If θ is small, it is a not difficult exercise. Hence

$$f^{-1} \circ F(D_*(-I)) \subset f^{-1}(D_*(b, u^f)) \subset D_*(U;\theta).$$

Together with (9.1) and (9.4) this implies Proposition 9.1. $\square$

With the constructed sequence of the domains of renormalizations $\{\Omega\}$ we end the proof of the Main Theorem in the infinitely renormalizable case for degree two, simply repeating the proof of this theorem for the larger degrees (see the end of the previous section).

Let us now prove Theorem A for $\ell = 2$. For this we make use of Lemma 8.1 (or rather its proof): for every small enough $\varepsilon > 0$ there exists a constant $C, 0 < C < 1$

(depending only on ε), and a point $\tilde{u}$ in the interval $I \setminus V$ such that the image $f^s(\tilde{u})$ lies in the ε-neighbourhood of the point $\frac{6}{5}u$ and

$$|\tilde{u} - c| \leq C|f^s(\tilde{u}) - c|. \tag{9.5}$$

With this ε small enough (but fixed) and the corresponding point $f^s(\tilde{u})$, which replaces the previous point $\frac{6}{5}u$, we can construct the domain Ω (see Remark 9.1), with the same angle θ_0. Then the modulus of the annulus $\Omega \setminus f^{-1} \circ F(D(V;\theta_0) \cup D_*(-I))$ is bounded from below by a positive absolute constant. On the other hand, by (9.5), two preimages $f^{-1} \circ F(D_*(I))$ are also on a proportionally definite distance from the boundary of Ω, and we obtain Theorem A. □

10 The proof of the Main Theorem when $\omega(c)$ is not minimal and in the Fibonacci case

In the remainder of the paper we shall deal with the non-renormalizable case (except in Section 13, where we shall finish the proof of Theorem A in the infinitely renormalizable case when period doubling occurs). Firstly, if the ω-limit set $\omega(c)$ of the critical point $c = 0$ is not minimal then it very easy to see that the Julia set is locally connected. To see this, note that if $\omega(c)$ is not minimal then it contains a point x whose forward orbit stays away from the critical point c. Hence this forward orbit lies in a hyperbolic set. Therefore the Yoccoz puzzle-pieces $P_n(x)$ containing x shrink down in diameter to zero. Since $x \in \omega(c)$, the forward orbit of c enters these pieces and it follows that all the puzzle-pieces tend to zero in diameter. Since the intersection of the Julia set with puzzle-pieces is connected the result follows.

Now we shall prove that the Julia set of a Fibonacci map of the form $f(z) = z^\ell + c_1$ with c_1 is real is locally connected. We should emphasize that this result also follows from Theorem B. However, since the Fibonacci map is often thought of as the 'bad case', we want to show explicitly that the careful estimates obtained in [SN] imply that the proof of local connectivity in this case is in fact very easy. Let us write as before $\tau(z) = -z$. Let us remind that a Fibonacci map is a map defined by the following property: For $i \geq 0$ and $x \in \mathbb{R}$, let $x_i = f^i(x)$ and choose $x_{-i} \in f^{-i}(x)$ so that the interval connecting this point to $c = 0$ contains no other points in the set $f^{-i}(x)$. Note that if c is not a periodic point there are always precisely two such points c_{-i} (which are symmetric with respect to each other). Let $S_0 = 1$ and define S_i inductively by

$$S_i = \min\{k \geq S_{i-1};\, c_{-k} \in (c_{-S_{i-1}}, \hat{c}_{-S_{i-1}})\}.$$

f is called a *Fibonacci* map if the sequence S_i coincides with the Fibonacci numbers: $S_0 = 1$, $S_1 = 2$ and $S_{k+1} = S_k + S_{k-1}$, i.e., the sequence $1, 2, 3, 5, 8, \ldots$. Moreover, let us define inductively a sequence of points u_n as follows. Let u_0 be the orientation reversing fixed point q of f and let us define u_{n+1} to be the nearest point to c with

$$u_{n+1} \in f^{-S_n}(u_n)$$

so that u_{n+1} is on the same side of c as $c_{S_{n+1}}$. In particular, $u_1 = \hat{u}_0 = \hat{q}$. We shall use

Proposition 10.1 *[SN] For each even integer $\ell \geq 2$, there exists a sequence of standard discs D_n centred at the critical point* 0 *and relatively compact topological discs D_n^0, D_n^1 in D_n, such that:*

- *A trace of D_n on the real line is ended by two symmetric preimages $u_{n-1}, \tau(u_{n-1})$ of an orientation reversing fixed point q of f.*
- *The sequence of discs D_n shrinks to zero.*
- *For each n big enough, the map*

$$R_n : (D_n^0 \bigcup D_n^1) \to D_n$$

  defined by

$$R_n(z) = \{f^k(z)\,;\ k > 0 \textit{ is minimal with} f^k(z) \in D_n\}$$

  is l-polynomial-like with l-multiple critical point zero.
- *Moreover, the critical point of R_n lies in the Julia set of R_n.*

Proof: For the proof see [SN] (this theorem is proved there for each even $\ell \geq 2$). For $\ell = 2$ this result is also proved in [LM]. □

Proof of the Main Theorem in the Fibonacci case. There is a critical Yoccoz piece P_n containing $u_{n-1}, \tau(u_{n-1})$ in its boundary. Let Q^{-1} be the extension of the R_n^{-1} from $D_n \bigcap P_n$ to P_n. Let P_n^0, P_n^1 be the images of P_n under this map Q^{-1}. Then $Q\colon (P_n^0 \bigcup P_n^1) \to P_n$ is again l-polynomial-like map. Observe that $R_n = Q$ on the real line. Hence, the critical point zero belongs to both Julia sets J_{R_n} and J_Q. As we have proved in Proposition 3.2, we have $F_{R_n} = F_Q$ and that there exists a component of some preimage of $Q^{-i}(P_n)$ which contains c and which is contained in D_n. By construction, $J(f) \cap P_n$ is connected. Hence $J(f) \cap Q^{-i}(P_n)$ is connected and for some i and some component $C_n \ni c$ of $Q^{-1}(P_n)$ is contained in D_n. Hence C_n is an open neighbourhood of c which is contained in D_n and such that $C_n \cap J(f)$ is connected. Since the sequence u_n tends to $c = 0$, the diameter of D_n tends to zero and we get that the diameter of C_n tends to zero also. Hence $J(f)$ is locally connected in $c = 0$.

To prove the local connectivity at any other point $z \in J_f$, we can repeat arguments for quadratic case (see [Mil]), which work in our case as well. If the orbit of z does not hit a critical piece, then the pieces around z shrink to z by contraction principle. Let now zero be an accumulation point of the orbit of z. Consider the annuli given by the Yoccoz pieces. First, note that the sum over the all depths $d = 1, 2, \ldots$ of the moduli of the annuli $A_d(0)$ around zero is infinite, just because the critical pieces tend to the point. Fix d and find the first iterate z_j of z that hits the critical piece at depth

$d+1$. Then an annulus of the puzzle around z at depth $d+j$ is isomorphic to $A_d(0)$. Furthermore, distinct values of d give distinct values of $d+j$. Hence, the sum of the moduli of annuli around z is infinite, as we needed. □

We should note that the Julia set of the Fibonacci polynomial $z \mapsto z^\ell + c_1$ with c_1 real has positive Lebesgue measure when ℓ is large, see [SN]. It follows that there exists Julia sets which are locally connected but have positive Lebesgue measure.

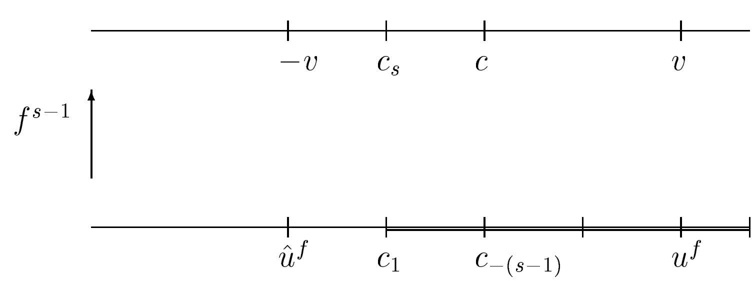
11 The proof of the Theorem B for ℓ large

In this section we prove Theorem B and the Main Theorem in some non-renormalizable cases when ℓ is large.

Theorem 11.1 *There exists ℓ_0 as follows. Let $f(z) = z^\ell + c_1$ with ℓ an even integer and c_1 real be a non-renormalizable polynomial such that the limit set $\omega(c) \ni c$ is minimal and f has infinitely many times a high return in the partition given by the Yoccoz puzzle on the real line. Then the Julia set of f is locally connected provided $\ell > \ell_0$.*

For a definition of the notion of a high return, see the end of the introduction. We should note that the proof of this theorem also holds for every infinitely renormalizable f with ℓ is large enough (and thus giving an alternative proof of Main Theorem in the infinitely renormalizable case when ℓ is large).

As before, let W be a symmetric interval with nice boundary points, let R_W be the first return map to W and let $V = [v, \tau(v)]$ be the domain of R_W containing c. Similarly, let R_V be the first return map to V and $U = [u, \tau(u)]$ the component of the domain of R_V which contains c. Let $\hat{U} = [\hat{u}^f, u^f]$ be the component of R_V containing the critical value c_1. Let s be so that $R_V|\hat{U} = f^{s-1}$. In this section we will assume that R_V has a high return, i.e., $R_V(U) = f^s(U) \ni c$.



We cannot use Lemmas 6.3-6.4 since f above is not renormalizable, however we can Proposition 7.1 (which also holds for renormalizable f). Let us state it quickly again.

Let T_0 be a minimal interval containing $f(V)$ and its immediate neighbour among the disjoint intervals $f^2(V), \dots, f^{s'}(V)$. Write $L_0 = T_0 \setminus f(V)$. By Lemma 6.2, there exists an interval l on either side of $\hat{U}$ which has a unique common point with $\hat{U}$ such that $f^s: l \to L_0$ is one-to-one and by Proposition 7.1:

Lemma 11.1

$$|L_0| \geq \frac{1}{3}|f(V)|.$$

Given the interval $V = (-v, v)$ we construct an ℓ-polynomial-like map sitting inside the domain $\Omega = \Omega(\ell, V)$ defined as

$$\Omega = D(V;\theta) \bigcup D(I;\theta) \bigcup D(-I;\theta),$$

where $\theta = \theta_0$ is some absolute constant (angle) to be determined later on, the interval

$$I = (0, v + \frac{\log(11/10)}{l} v).$$

Let $F: V \to \hat{U}$ be the inverse to the map $f^{s-1}: \hat{U} \to V$. Then F extends to a univalent map on a maximal interval T containing V and then to domain $\mathbb{C}_T$. The first observation is that for all sufficiently large degrees ℓ, the interval

$$\widetilde{I} = I \cup V \cup (-I) = \Omega \cap \mathbb{R}$$

is inside the interval T. This is because

$$(1 + \frac{\log(11/10)}{\ell})^\ell \sim \frac{11}{10} < 1 + 1/3,$$

that is $f(\widetilde{I}) \subset (c_1, v^f) \cup L_0$, by Lemma 11.1. In the formula above the symbol $\sim$ means that the left hand side converges to the right hand side as $\ell \to \infty$. We shall use this convention throughout this section.

We are going to prove

Proposition 11.1 *There exists $\theta_0 > 0$, such that for all sufficiently big ℓ,*

$$f^{-1} \circ F(\partial\Omega) \subset \Omega.$$

Before proving this proposition we show

Proposition 11.2 *Proposition 11.1 implies the Main Theorem in the non-renormalizable high case when ℓ is sufficiently large.*

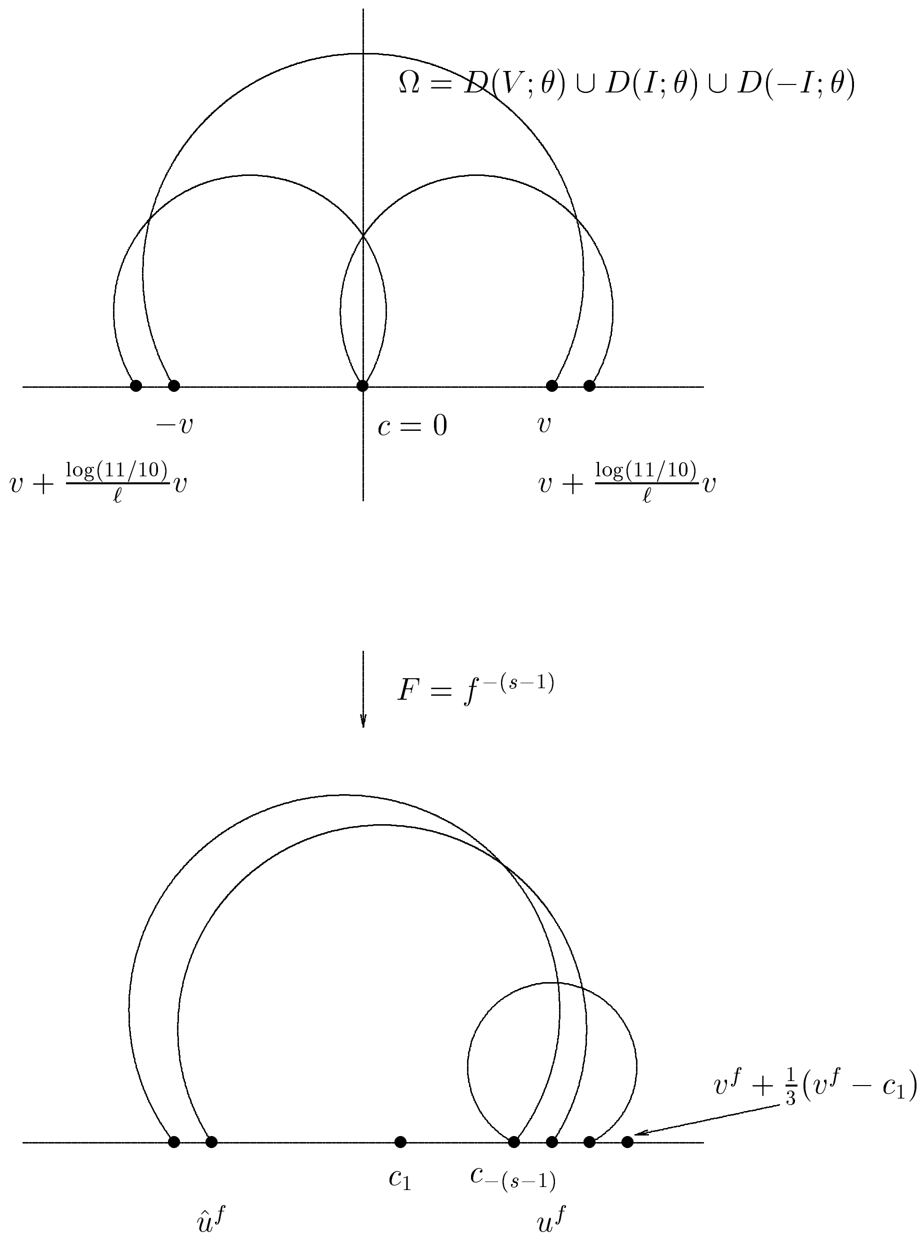


Figure 2: $\frac{c_1 - \hat{u}^f}{v^f - c_1} < 1.3$ when $\ell >> 1$.

Proof: It is enough to construct the ℓ-polynomial-like mapping inside the domain Ω. Since $f:\omega(c) \to \omega(c)$ is minimal, each point in $\omega(c)$ eventually is mapped into V and therefore is in the domain of definition of the map R_V. There exists a finite collection of disjoint intervals $I^0 = U$ and $I^1, \dots, I^i$ in V, which form the domain of definition of the map R_V (see Proposition 4.1 and its proof). More precisely, every $x \in \omega(c) \cap V$ belongs to some I^j or to U, the map $R_V: I^j \to V$ is a diffeomorphism for $j = 1, 2, \dots, i$, and $U \in c$ with $R_V|_U(U) = f^s(U) \ni c$ (the high return). Given $j = 1, \dots, i$, there exists an interval $\hat{I}^j$ containing I^j such that $R_V: I^j \to V$ extends to a diffeomorphism from $\hat{I}^j$ onto the interval $\widetilde{I}$ (from the definition of the domain Ω). Indeed, if $l^j \supset I^j$ is a maximal interval on which R_V is monotone, then $R_V(l^j)$ contains V and its immediate neighbour (from the collection of intervals $f^j(V)$) on either side of V. Hence, $R_V(l^j) \subset f^{-1}(T_0) \cap \mathbb{R}$. Thus, $\hat{I}^j \subset l^j$. Since f has no attracting periodic orbit, $\hat{I}^j$ is a subset of either the right part I of $\widetilde{I}$ or its left part $-I$. Let, for example, $\hat{I}^j \subset I$. By Lemma 2.1, there exists a domain Ω_j inside $D(I^j;\theta) \subset D(I;\theta) \subset \Omega$ which is mapped diffeomorphically by a map R^j (an iteration of f) onto Ω. We have constructed Ω_j for each $j = 1, 2, .., i$, and each $\theta \in (0/\pi/2)$. Assuming Proposition 11.1, we fix the angle θ_0 and find a domain $\Omega_0 \subset \Omega$ such that $R^0 = f^s: \Omega_0 \to \Omega$ is a proper ℓ-cover. Since the domains $\Omega_j, j = 0, 1, \dots, i$ may intersect each other, we modify so that they become ℓ-polynomial-like. To do this, let us consider a Yoccoz piece P_V containing the ends of the interval V on its boundary. Let D be a component of $P_V \cap \Omega$ containing V. For every $j = 0, 1, \dots, i$, there exists a domain $D_j \subset \Omega_j$ such that $R_j: D_j \to D$ is a diffeomorphism if $j > 0$, and an ℓ-cover if $j = 0$. Since all D_j lie in different Yoccoz pieces, we obtain the ℓ-polynomial-like map. □

Proof of Proposition 11.1: We prove this proposition using three lemmas and their corollaries.

Lemma 11.2 *If $a \in L_0$ such that $0 < a - c_1 \leq (11/10)(v^f - c_1)$, then $K_\ell(a) < 2$, for all l big enough.*

Proof: If $y \leq (3/4)\cdot(11/10)$, then, by Corollary 5.1, $K_\ell^* \sim 1/(e\cdot\log(1/y)) \leq 1.9123... < 2$. □

Let $P_\ell(z) = z^\ell$. The next lemma gives information about the asymptotic shape of $P_\ell(D((-1,1);\theta))$ as $\ell \to \infty$. We need even something more general. Fix K between $+\infty$ and -1, some $C > 0$, and $\theta \in (0, \pi/2)$, and consider a Poincaré neighbourhood $D = D((-(1+\frac{C}{\ell})K, 1+\frac{C}{\ell});\theta)$ of the interval $(-(1+\frac{C}{\ell})K, 1+\frac{C}{\ell})$ (here $K \geq -1$). Take a real $\Lambda > 0$ and consider a point $z_\ell(\Lambda)$ of the boundary of the above D with $\arg(z) = \Lambda/\ell$.

Lemma 11.3 *$P_\ell(z_\ell(\Lambda))$ tends, as $\ell \to \infty$, to a point*

$$\exp(C)\exp\left(\Lambda\frac{\cos(\theta)}{\sin(\theta)}\right)\exp(i\Lambda)$$

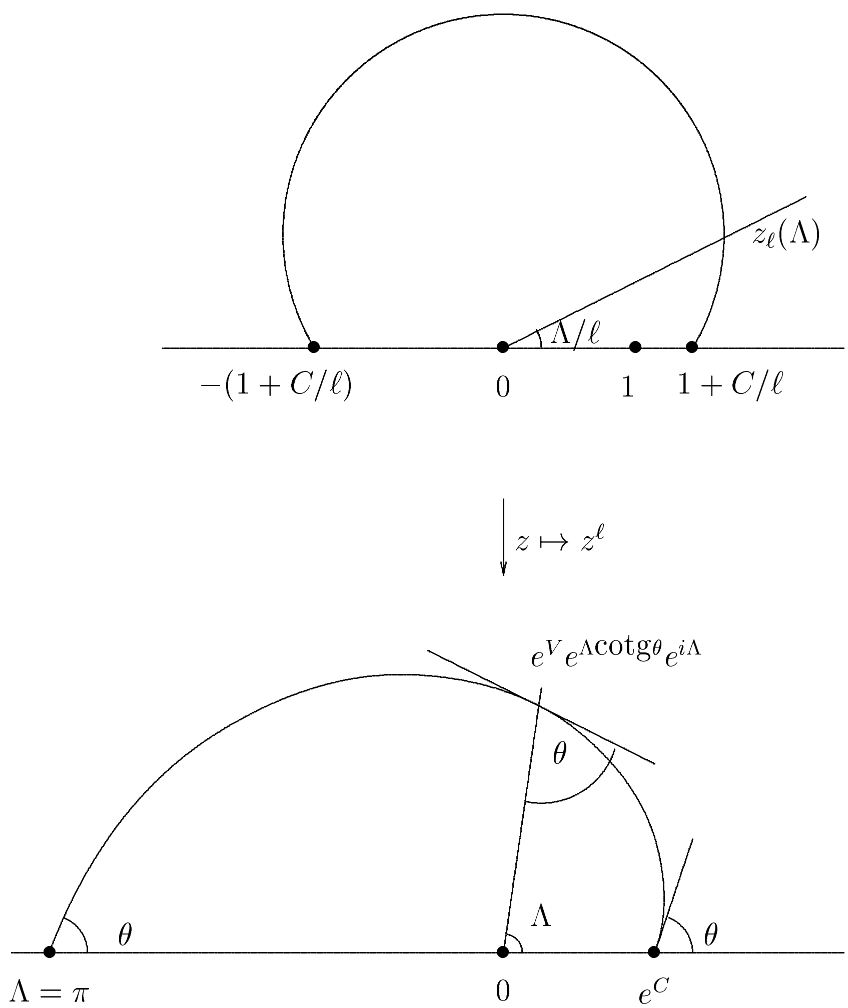


Figure 3: The image of a Poinacaré disc under the map $z \mapsto z^\ell$.

of a logarithmic spiral, and the convergence is uniform in Λ on every compact of $(0, +\infty)$.

Proof: If $z \in \partial D$, then, by 9.2,

$$z = (1 + \frac{C}{\ell}) \cdot \left[1 + \frac{K+1}{2} \cdot \frac{i \cdot \exp(i\theta)}{\sin(\theta)} \cdot (1 - \exp(i\phi)) \right],$$

where ϕ is the angle between the vectors $z - z_0, 1 - z_0$, with z_0 the centre of the circle D. As $\arg(z) = \Lambda/\ell \to 0$ with $\ell \to \infty$, then $\phi \to 0$ uniformly in Λ on every compact of $(0, +\infty)$ (remember that θ is fixed). So,

$$z^\ell \sim \exp(C) \exp \left\{ \frac{K+1}{2\sin(\theta)} \exp(i\theta) \ell \phi \right\},$$

as $\ell \to \infty$. Let us prove that $\ell\phi \to \Lambda \frac{2}{K+1}$. Indeed, by the expression for z and using the notation $\alpha = \Lambda/\ell$,

$$\tan(\alpha) = \frac{\frac{K+1}{2\sin(\theta)}(\cos(\theta) - \cos(\theta + \phi))}{1 + \frac{K+1}{2\sin(\theta)}(\sin(\theta + \phi) - \sin(\theta))}.$$

Then

$$\cos(\theta + \phi - \alpha) = \cos(\theta) - 2\sin(\alpha/2)(\cos(\theta)\sin(\alpha/2) - \frac{K-1}{K+1}\sin(\theta)\cos(\alpha/2)).$$

It follows, as $\alpha \to 0$ (and θ =const),

$$\theta + \phi - \alpha \sim \theta + \frac{\alpha(-\frac{K-1}{K+1}\sin(\theta))}{\sin(\theta)},$$

i.e., $\phi \sim 2\alpha/(K+1)$. The uniform convergence also follows, and the statement is proved. □

Given $A \geq 1$, $\theta \in (0, \pi/2)$, and $0 \leq \Lambda_1 < \Lambda_2 \leq \infty$, we denote

$$\Gamma(A, \theta; \Lambda_1, \Lambda_2) = \{ z = A \exp \left\{ \Lambda \frac{\cos(\theta)}{\sin(\theta)} \right\} \exp(i\Lambda) \, ; \; \Lambda_1 < \Lambda < \Lambda_2 \}$$

a part of the logarithmic spiral. We have proved in the lemma above that an arc of $\partial D((-1,1);\theta)$ of the points z with $0 < \arg(z) < \Lambda/\ell$, where $0 < \Lambda < \infty$, is mapped by P_ℓ asymptotically onto $\Gamma(1, \theta; 0, \Lambda)$, and an arc of $\partial D((0, 1 + (1/\ell)\log(11/10)); \theta)$ of the points with $0 < \arg(z) < \Lambda/\ell$ is mapped by P_ℓ asymptotically onto $\Gamma(11/10, \theta; 0, \Lambda)$. Let us note that the arc $\Gamma(1, \theta; 0, \Lambda)$ is inside $D((-K, 1); \theta)$, $K > 1$, if $\Lambda > 0$ is small enough, because these curves are tangent at 1, but the curvature of the logarithmic spiral at 1 is less than the curvature of the curve (a part of a circle) $\partial D((-K, 1); \theta)$ at 1. On the other hand, it is clear that for given K and for A big enough the spiral $\Gamma(A, \theta; 0, \infty)$ is already outside of $D((-K, 1); \theta)$. We are going to find a lower bound for A.

Fix $K > 1$, and $A > 1$.

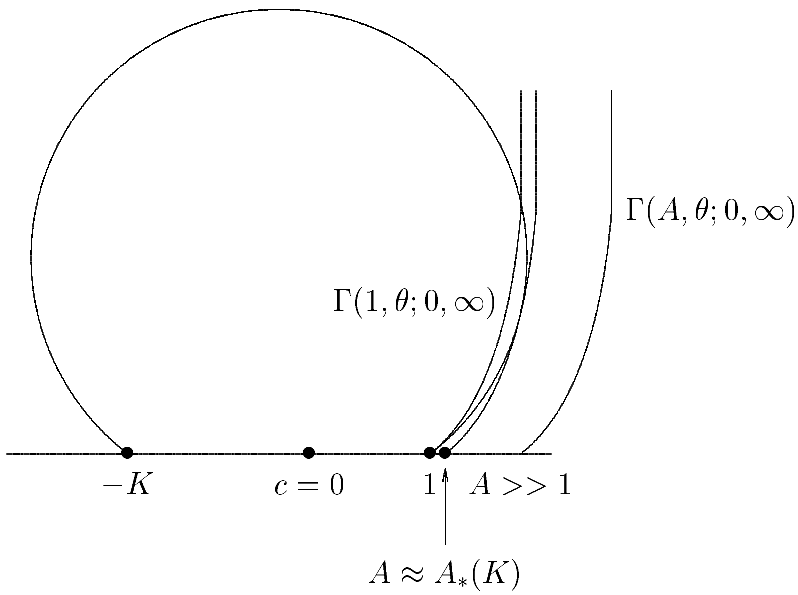


Figure 4: $D((-K, 1); \theta)$ when $\theta << 1$

Lemma 11.4 *If for some sequence of θ tending to zero, the (open) curve $\Gamma(A, \theta; 0, \pi)$ intersects the curve $\partial D((-K, 1); \theta)$ at a point $Z(\theta)$, then $Z(\theta)$ tends to $1 + x$, where x is a positive (real) solution of the equation*

$$A \exp\left\{\frac{x(1 + \frac{x}{K+1})}{1 + x}\right\} - (1 + x) = 0. \tag{11.1}$$

Remark 11.1 *It can be seen from the proof below that this condition is also 'only if'.*

Proof: We have at the point of intersection $Z(\theta)$ (of argument α):

$$A \exp\left\{\alpha \frac{\cos(\theta)}{\sin(\theta)}\right\} \exp(i\alpha) = 1 + \frac{K + 1}{2} \cdot \frac{i. \exp(i\theta)}{\sin(\theta)} \cdot (1 - \exp(i\phi)), \tag{11.2}$$

and a consequence is the equality for arguments:

$$\tan(\alpha) = \frac{\frac{K+1}{\sin(\theta)} \sin(\frac{\phi}{2}) \sin(\theta + \frac{\phi}{2})}{1 + \frac{K+1}{\sin(\theta)} \sin(\frac{\phi}{2}) \cos(\theta + \frac{\phi}{2})}. \tag{11.3}$$

Remember that some sequence of $\theta \to 0$. A priori the following cases are possible for some subsequence:

I. $\phi/\theta \to \infty$. We are going to prove that this case is, in fact, impossible.

II. $\phi/\theta \to t < \infty$.

Case I is divided into three subcases.

Ia. $\phi \to \pi$. Then (11.3) and $\phi/\theta \to \infty$ gives $\tan(\alpha) \to \infty$, i.e., $\alpha \to \pi/2$. Now we compare the left-hand side (LHS) and the right-hand side (RHS) of (11.2). The modulus of the LHS is equal to

$$A \exp\{\alpha \frac{\cos(\theta)}{\sin(\theta)}\} \sim A \exp(\frac{\pi}{2\theta})$$

while for the modulus of the RHS we can write

$$|1 + \frac{K+1}{2} \cdot \frac{i \cdot \exp(i\theta)}{\sin(\theta)} \cdot (1 - \exp(i\phi))| \sim |1 + i(K+1)\frac{1}{\theta}|,$$

so the equality (11.2) cannot hold in this case.

Ib. $\phi \to 2\pi$. Then $\alpha > \pi/2$, i.e.,

$$A \exp\{\alpha \frac{\cos(\theta)}{\sin(\theta)}\} > A \exp(\frac{\pi \cos(\theta)}{2\sin(\theta)}) \sim A \exp(\frac{\pi}{2\theta})$$

while the modulus of the RHS of (11.2) can be at most

$$1 + \frac{K+1}{\theta},$$

as $\theta \to 0$. This is again a contradiction.

Ic. ϕ tends neither to $\pi/2$ nor to π (but $\phi/\theta \to \infty$). Then, from (11.3), $\alpha \sim \theta + \phi/2$, and the modulus of the LHS of (11.2) is at least

$$A \exp \frac{3}{4}(1 + \frac{\phi}{2\theta})$$

while the modulus of the RHS of (11.2) is less than

$$1 + (K+1)\frac{\phi}{2\theta}.$$

Since $\phi/\theta \to \infty$, this is impossible again.

Case II: $\phi/\theta \to t < \infty$ (as $\theta \to 0$ along a sequence). Then

$$(K+1)\frac{\sin(\phi/2)}{\sin(\theta)} \to x = (K+1)t/2 < \infty$$

and $\tan(\alpha) \to 0$. If $t = 0$, then the RHS tends to 1, but $|LHS| > A > 1$. Thus, t and x are not zero, and, from (11.3),

$$\alpha \sim \frac{x(1 + \phi/(2\theta))}{1+x}\theta \sim \frac{x(1 + \frac{x}{K+1})}{1+x}\theta.$$

Then the RHS of (11.2) tends to

$$1 + x.$$

Substituting these in (11.2), we obtain the equation (11.1) for x. Moreover, the point of intersection tends to $1 + x \in \mathbb{R}$. The lemma is proved. $\square$

Corollary 11.1 *Given $K > 1$, if*

$$A > A_*(K) = \frac{K}{\exp(2\frac{K-1}{K+1})}, \tag{11.4}$$

then, for all θ close enough to zero, the arc $\Gamma(A, \theta; 0, \pi)$ of the logarithmic spiral does not intersect the domain $D((-K, 1); \theta)$.

Remark 11.2 *The condition (11.4) is also 'only if'.*

Proof: It is enough to prove only that, with this particular choice of A, equation (11.1) does not have positive solutions. If $A > 1$ is close enough to 1, equation (11.1) has at least two positive solutions. On the other hand, since the second derivative of the left-hand side of (11.1),

$$A \exp\left\{\frac{x(1+\frac{x}{K+1})}{1+x}\right\} \frac{1}{(K+1)^2}\left[1 + \frac{2K}{(1+x)^2} - \frac{2K(K+1)}{(1+x)^3} + \frac{K^2}{(1+x)^4}\right],$$

has exactly one positive root, the number of positive roots of (11.1) is at most two. If A is large, there are no positive roots at all. Hence, there exists some $A_* > 1$, such that with $A < A_*$ there are two roots, and with $A > A_*$ there are no roots, and A_* can be defined by a condition that with $A = A_*$, the equation has one multiple positive root x. So,

$$A_* \exp\{\frac{x(1+\frac{x}{K+1})}{1+x}\} = 1 + x,$$

and

$$A_* \left[\frac{1}{K+1} + \frac{K}{(K+1)(1+x)^2}\right] \exp\left\{\frac{x(1+\frac{x}{K+1})}{1+x}\right\} = 1.$$

Then

$$\frac{1}{K+1} + \frac{K}{(K+1)(1+x)^2} = \frac{1}{1+x},$$

i.e., x is either 0, or $K-1$. The zero corresponds to the trivial value $A_* = 1$. Substituting $x = K - 1$, we obtain the formula (11.4). □

Corollary 11.2 *Given an arbitrary interval $J \subset \mathbb{R}$, for all θ small enough, the curve $\Gamma(1, \theta; \pi, 2\pi)$ is outside the closure of the domain $D(J; \theta)$.*

Proof: Obviously, the curve $\Gamma(1, \theta; \pi, 2\pi)$ is a curve $\Gamma(A_\theta, \theta; 0, \pi)$ rotated by the angle π, where $A_\theta = \exp(\pi \frac{\cos(\theta)}{\sin(\theta)})$ tends to ∞ as θ tends to the zero. □

Conclusion of the proof of the Proposition 11.1. Remember that we chose the domain of renormalization

$$\Omega = D(V; \theta) \bigcup D(I; \theta) \bigcup D(-I; \theta),$$

where $V = (-v, v)$ and v is the boundary point of V so that $F(v) = u^f \in (c_1, v^f)$ (we assume that $v > 0$), θ is some absolute constant (angle) to be determined later on, and $I = (0, v + \frac{\log(11/10)}{\ell} v)$.

We need to find $\theta = \theta_0 > 0$ and ℓ_0, such that for every $\ell > \ell_0$,

$$f^{-1} \circ F(\partial\Omega) \subset \Omega.$$

To do this, consider the pullback $F(\Omega)$. Let us rescale Ω such that the interval $V = (-v, v)$ turns into the interval $(-1, 1)$. We call the obtained domains by Ω^*. Let us rescale also $F(\Omega)$ by shifting first by $-c_1$, and then rescaling it so that the interval (c_1, v^f) turns into the interval $(0, 1)$. We call the obtained domains by $\widetilde{\Omega}^*$. It is convenient to introduce also the scaled map F^* corresponding to the map F (the pullback of f^{s-1}):

$$F^*(z) = \frac{F(vz) - c_1}{v^f - c_1}.$$

So $\widetilde{\Omega}^* = F^*(\Omega^*)$. It is enough to find $\theta = \theta_0$ and ℓ_0, such that for every $\ell > \ell_0$ we have that the closure of $P_\ell^{-1}(\widetilde{\Omega}^*)$ is inside Ω^*. (As above, $P_\ell(z) = z^\ell$.) By our choice,

$$\Omega^* = D((-1, 1); \theta) \cup D(I^*; \theta) \cup D(-I^*; \theta),$$

where $I^* = (0, 1 + \frac{\log(11/10)}{\ell})$.

Let us look at the all parts of $P_\ell^{-1} \circ F^*$. Given $z \neq 0$, we let $E_i(z)$ be a unique point w such that $P_\ell(w) = w^\ell = z$ and $\arg(w) \in [(2i-1)\pi/\ell, (2i+1)\pi/\ell)$, $i = 0, 1, \ldots, l-1$. Because of Lemma 11.1, the restriction of F^* to the real axis is defined on the interval $(-(\frac{4}{3})^{1/\ell}, (\frac{4}{3})^{1/\ell})$. Moreover, the real map $E_0 \circ F^*\colon (0, (\frac{4}{3})^{1/\ell}) \to \mathbb{R}$ is a homeomorphism and it sends the interval $(0, (\frac{4}{3})^{1/\ell})$ into itself (since R_V has a high return and f has no attracting periodic orbit). It follows that

$$E_0 \circ F^*(D(I^*; \theta)) \text{ is a proper subset of } D(I^*; \theta), \tag{11.5}$$

for any angle $\theta \in (0, \pi/2]$.

Let us consider the rest of $E_0 \circ F^*(\Omega^*)$, i.e., the set $E_0(W)$, where

$$W = F^*\left(D((-1, 1); \theta) \cup D(-I^*; \theta)\right).$$

The trace of W on the real axis is contained in the interval $(-K_1, 1)$, where for $K_1 > 0$ we have a bound controlled by Lemma 11.2:

$$\frac{K_1}{11/10} < 2,$$

for all big ℓ. It follows, that the set W is covered by $D((-K_1, 1); \theta)$, for any θ.

Observe that by (11.4),

$$A_*(K_1) < A_*(2.2) = 1.04... < 11/10. \tag{11.6}$$

Applying Lemma 11.3 and Corollary 11.1, we find an angle $\theta_1 > 0$ and a degree ℓ_1, such that, for all $\theta \le \theta_1$, and for all $\ell \ge \ell_1$, the set $P_\ell(\Omega^* \cap \{z;\ \arg(z) \in [-\pi/\ell, \pi/\ell]\})$ contains W, that is $E_0(W)$ is inside Ω^*. Therefore, we have proved that $E_0 \circ F^*(\Omega^*)$ is inside Ω^*. By the symmetry of Ω^*, $E_{\ell/2} \circ F^*(\Omega^*)$ is inside Ω^* too, for the same θ and ℓ.

Now we consider $E_1 \circ F^*(\Omega^*)$. The domain $F^*(\Omega^*)$ is contained in a domain $D((-K_1, 4/3); \theta)$. So, we can apply Corollary 11.2 (together with Lemma 11.2) to conclude that, for some $\theta_2 > 0$ and ℓ_2, if $\theta \le \theta_2$ and $\ell \ge \ell_2$, then $E_1 \circ F^*(\Omega^*)$ is contained in the Poincaré neighbourhood $D((-1,1);\theta)$. Essentially, this is the end of the proof. Indeed, each other $E_i \circ F^*(D((-1,1);\theta))$, $(i \neq 0, \ell/2)$, is contained in $D((-1,1);\theta)$, since $D((-1,1);\theta)$ is invariant under the rotation $z \mapsto \exp(i \cdot 2\pi/\ell)z$, for z in the first quarter.

Thus, for $\theta = \theta_0 = \min\{\theta_1, \theta_2\}$, and for every $\ell > \ell_0 = \max\{\ell_1, \ell_2\}$, we have that $\widetilde{\Omega}^*$ is inside of Ω^*. □

Thus we have completed the proof of Proposition 11.1 and of Theorem B for the case when ℓ is large.

12 The proof of Theorem B for all degrees

In this section we complete the proof of Theorem B and of the Main theorem in some non-renormalizable cases:

Theorem 12.1 *Let $f(z) = z^\ell + c_1$ with ℓ an even integer and c_1 real be a non-renormalizable polynomial such that the limit set $\omega(c) \ni c$ is minimal and f has infinitely many high returns in the partition given by the Yoccoz puzzle on the real line. Then the Julia set of f is locally connected.*

Of course, we may assume in the proof below that $\ell \ge 4$ because when $\ell = 2$ then the result holds (even without the assumption about high returns) by the result of Yoccoz [Y], see the Introduction. So let us fix $\ell \ge 4$. In the previous section we have proved the above result already for ℓ sufficiently large. Since the estimates in this section for ℓ 'small' are more delicate and since the proof in the previous section shows that the shape of the domain (i.e., θ) can be chosen uniformly in ℓ, we have dealt with the asymptotic case separately in the previous section. We should note that the proof of this theorem also holds for every infinitely renormalizable f with $\ell \ge 4$. Since we use only the 'easy space' 1/3, one might hope to extend this result to certain non-real polynomials.

Given the interval $V = (-v, v)$ such that R_V has a high return we construct an ℓ-polynomial-like map sitting inside the domain $\Omega = \Omega(\ell, V)$, where Ω is either the disc $D_*(V)$ based on the diameter V, or

$$\Omega = D(V;\theta) \bigcup D(I;\theta) \bigcup D(-I;\theta),$$

where $\theta = \theta_0$ is some absolute constant (angle) to be determined later on, and

$$I = (0, 1.07^{1/\ell} v).$$

Here $F(v) = u^f \in (c_1, v^f)$ and we may assume that $v > 0$.

As before, it is enough to prove

Proposition 12.1 *Given $\ell \geq 4$, there exists $\theta_0 > 0$, such that*

$$f^{-1} \circ F(\partial\Omega) \subset \Omega.$$

The proof of the Proposition 12.1 is somewhat similar to the proof of the Main Theorem in the infinitely renormalizable case for degree 2 and the proof of the Theorem B for sufficiently large degrees. The main new ingredient is contained in the following lemma:

Lemma 12.1 *Either the disc $D_*(V)$ is a domain of the ℓ-polynomial-like mapping, i.e., $f^{-1} \circ F(D_*(V)) \subset D_*(V)$, or otherwise $F(D(-I;\theta))$ lies inside $D(I^f;\theta)$, where I^f is an interval around c_1:*

$$I^f = (c_1 - 2.12|v^f - c_1|, c_1 + 0.68|v^f - c_1|).$$

Proof: Remember that the constant $K_\ell(v^f) = |\hat{u}^f - c_1|/|v^f - c_1|$ depends not only on the extendability space (which is 1/3), but on the parameter $t = |c - c_{s+1}|/|T|$ as well (see Lemma 5.1). If $t \geq 0.51$, then

$$K_\ell(v^f) \leq \frac{0.51((3/4)^{1/\ell} - 0.51^{1/\ell})}{0.51^{1/\ell}(3/4)(1-(3/4)^{1/\ell})} \leq \frac{0.51((3/4)^{1/4} - 0.51^{1/4})}{0.51^{1/4}(3/4)(1-(3/4)^{1/4})} \leq 0.991818... < 1,$$

so that we apply Proposition 4.1. Thus, we can assume $t < 0.51$. The right end of the interval $F(-I)$ is just the point c_{-s+1}, which belongs to the interval (c_1, c_{s+1}) (since we have a high return). Hence,

$$\frac{|c_1 - c_{-s+1}|}{|c_1 - v^f|} \leq \frac{|c_1 - c_{s+1}|}{|c_1 - v^f|} \leq 0.51 \times (4/3) = 0.68.$$

The left end b of the interval $F(-I)$ is obtained from Corollary 5.1, where we put $y = 1.07/(4/3) = 0.8025$, so that $K^*_\ell(y) \leq K^*_4(y) = 1.97063... < 1.98$. Since $a = c_1 + 1.07|v^f - c_1|$ and $K_\ell(a) = |b - c_1|/|a - c_1| < 1.971$, indeed, $|b - c_1|/|v^f - c_1| < 1.98 \times 1.07 < 2.12$. □

If the first alternative in the lemma holds then Proposition 12.1 holds. So we will assume in the remainder of the proof that the second alternative holds. The next lemma will allow us to apply Lemma 9.1 for any degree $\ell \geq 4$ (and not just for $\ell = 2$). Let us denote for simplicity $D(\theta) = D((-1,1);\theta)$. Set

$$\Pi_\ell(\theta) = \{z \in D(\theta) : 0 \leq \arg z \leq \pi/\ell\}.$$

As before, $P_\ell(z) = z^\ell$.

Lemma 12.2 *Fix $0 < \theta < \pi/2$. Then*

$$P_\ell(\Pi_\ell(\theta)) \subset P_{\ell+2}(\Pi_{\ell+2}(\theta)), \quad \ell = 2, 4, \ldots.$$

Proof: Assume the contrary. Then the boundaries of $P_\ell(\Pi_\ell(\theta))$ and $P_{\ell+2}(\Pi_{\ell+2}(\theta))$ have a common non-real point, i.e., $z^\ell = u^{\ell+2}$, for some $z \in \partial D(\theta)$, $0 \le \arg z \le \pi/\ell$, and $u \in \partial D(\theta)$, $0 \le \arg z \le \pi/(\ell+2)$. Hence, $u = z^t$, with $t = \ell/(\ell+2)$ between zero and 1. The point z^t belongs to an arc $\Gamma(1, \theta_1; 0, \Lambda_0)$ of a logarithmic spiral starting at the points 1 and ending at $z \in \partial D(\theta)$ and crossing the circle $\partial D(\theta)$ at the other point u. If $\theta_1 \le \theta$, it is clearly impossible (see Section 11). Consider the case $\theta_1 > \theta$. Then $\Gamma(1, \theta_1; 0, \Lambda)$ is inside $D(\theta)$, if Λ is small. Hence, there are two points z_1, z_2 of the intersection of the arc $\Gamma(1, \theta_1; 0, \Lambda_0)$ with $\partial D(\theta)$, such that $\arg z_1 < \arg z_2$, and this arc leaves the disc $D(\theta)$ at z_1 and again enters it at z_2. By the geometry of the logarithmic spiral, the angle between the vector z_1 and the circle $D(\theta)$ is at least θ_1, and the angle between the vector z_2 and the circle $D(\theta)$ is at most θ_1. This is a contradiction with the fact that the angle between a vector $w \in \partial D(\theta)$ and the tangent to $\partial D(\theta)$ at w is increasing as $w \in \partial D(\theta)$ moves from 1 to -1 (in fact, it increases from θ to $2\pi - \theta$). $\square$

In order to prove Proposition 12.1, we need to find for any $\ell \ge 4$ some $\theta = \theta_0 > 0$ such that $f^{-1} \circ F(\partial\Omega) \subset \Omega$. To do this, consider the pullback $F(\Omega)$. Let us rescale Ω such that the interval $V = (-v, v)$ turns into the interval $(-1, 1)$. We call the obtained domain Ω^*. Let us rescale also $F(\Omega)$ by shifting first by $-c_1$, and then rescaling it so that the interval (c_1, v^f) turns into the interval $(0, 1)$. We call the obtained domain $\widetilde{\Omega}^*$. It is convenient to introduce also the scaled map F^* corresponding to the map F (the pullback of f^{s-1}):

$$F^*(z) = \frac{F(vz) - c_1}{v^f - c_1}.$$

So $\widetilde{\Omega}^* = F^*(\Omega^*)$. It is enough to find $\theta = \theta_0$ such that the closure of $P_\ell^{-1}(\widetilde{\Omega}^*)$ is inside Ω^*. (As above, $P_\ell(z) = z^\ell$.) By our choice,

$$\Omega^* = D((-1, 1); \theta) \cup D(I^*; \theta) \cup D(-I^*; \theta),$$

where $I^* = (0, 1.07^{1/\ell})$. Let us look at each piece of $P_\ell^{-1} \circ F$. Given $z \ne 0$, define $E_i(z)$ to be the unique point w such that $P_\ell(w) = w^\ell = z$ and $\arg(w) \in [(2i-1)\pi/\ell, (2i+1)\pi/\ell)$, $i = 0, 1, \ldots, l-1$. Because of Lemma 11.1, the restriction of F^* to the real axis is defined on the interval $(-(\frac{4}{3})^{1/\ell}, (\frac{4}{3})^{1/\ell})$. Moreover, the real map $E_0 \circ F^* \colon (0, (\frac{4}{3})^{1/\ell}) \to \mathbb{R}$ is a homeomorphism and it sends the interval $(0, (\frac{4}{3})^{1/\ell})$ into itself (since R_V has a high return and f has no attracting periodic orbit). It follows that

$$E_0 \circ F^*(D(I^*; \theta)) \text{ is a proper subset of } D(I^*; \theta), \tag{12.1}$$

for any angle $\theta \in (0, \pi/2]$.

Let us now show that

$$E_0(F^*(D((-1, 1); \theta))) \subset \Omega, \tag{12.2}$$

in other words, that $F^*((D((-1,1);\theta)))$ is covered by the set $P_\ell(\Omega \cap \{z;\ -\pi/\ell < \arg z < \pi/\ell\})$. To see this, first note that by Corollary 5.1, $F^*((D((-1,1);\theta))) \subset D((-K_0,1);\theta)$ with the constant

$$K_0 = K_\ell^*(4/3) \leq K_4^*(4/3) = 1.51983... < 1.52.$$

By Lemma 12.2, the difference $\Delta_\ell(\theta) = D((-K_0,1);\theta)\backslash P_\ell(\Pi_\ell(\theta))$ is contained in $\Delta_2(\theta)$. Hence (12.2) follows from:

Lemma 12.3 *For all θ small enough,*

$$\Delta_2(\theta) \subset P_\ell\left(D(I^*;\theta) \cap \{z;\ 0 < \arg z < \pi/\ell\}\right), \ell = 4, 6, \ldots \quad .$$

Proof: Assume that this is not the case for some sequence $\theta \to 0$. Then $z_1 = z_2^\ell$, for some $z_1 \in \partial D((-K_0,1);\theta)$ and $z_2 \in \partial D(I^*;\theta)$. Moreover, $\arg z_2 < \pi/\ell$, and, what is crucial, since we were able to apply Lemma 9.1, z_1 tends to a point of the real interval $[1, K_0^2]$ (for some subsequence) as $\theta \to 0$. We have

$$z_1 = 1 + \frac{K_0+1}{2} \cdot \frac{i \cdot \exp(i\theta)}{\sin(\theta)} \cdot (1 - \exp(i\phi)). \tag{12.3}$$

where $\phi \in (0, 2\pi - \theta)$ is the angle between the vectors $z_1 - C, 1 - C$, with C the centre of the circle $D((-K_0,1);\theta)$. For $z_2 \in \partial D(I^*;\theta)$ we have a similar expression:

$$z_2 = A \cdot \frac{\sin(\theta+\gamma)}{\sin\theta} \cdot \exp(i\gamma), \tag{12.4}$$

where $A = 1.07^{1/\ell}$ and $\gamma \in (0, \pi/\ell)$ is an argument of z_2. Since z_1 tends to a real point in $[1, K_0^2]$, it follows from (12.3), that ϕ/θ tends to a non-negative finite constant B, as $\theta \to 0$, and $1 + \frac{K_0+1}{2}B \leq K_0^2$, i.e.,

$$0 \leq B \leq 2(K_0 - 1). \tag{12.5}$$

Hence, from the condition $z_1 = z_2^\ell$ and from (12.4), γ/θ tends to a finite $D \geq 0$. Separating now real and imaginary parts of the equality $z_1 = z_2^\ell$, we obtain the following system for B and D:

$$1 + \frac{K_0+1}{2} \cdot B = 1.07 \cdot (1+D)^\ell \tag{12.6}$$

$$\frac{K_0+1}{2} \cdot B \cdot (1 + \frac{B}{2}) = 1.07(1+D)^\ell \cdot \ell \cdot D, \tag{12.7}$$

where

$$K_0 = 1.52.$$

Dividing (12.7) by (12.6) and substituting the obtained expression for D into (12.6), we come to the equation:

$$1 + \frac{K_0+1}{2} \cdot B = A \cdot \{1 + \frac{1}{\ell} \cdot \frac{K_0+1}{2} \cdot \frac{B \cdot (1+\frac{B}{2})}{1 + \frac{K_0+1}{2} \cdot B}\}^\ell, \tag{12.8}$$

where

$$K_0 = 1.52$$

and

$$A = 1.07.$$

With these K_0 and A, this equation (12.8) has no solutions on the interval $[0, 2(K_0-1)]$ for $\ell = 4$, and, hence, for all $\ell \geq 4$. In order to see this we claim that, given $A > 1$, this equation has either exactly two non-negative solutions (maybe one multiple), or no non-negative solutions at all. Before proving this claim let us first show that this implies the lemma.

Indeed, if $B = 2(K_0 - 1)$ is a solution, for some parameter A_0, then

$$A_0 = \frac{K_0^2}{(1 + \frac{1}{4} \cdot \frac{K_0^2-1}{K_0})^4} = 1.05835....$$

On the other hand, taking the derivative of both sides of (12.8) (with $\ell = 4$) with respect to B, we obtain, of course, $D_1 = (K_0 + 1)/2$ on the left hand-side, and $D_2 = (K_0 + 1)/2 \cdot 4(K_0^2 - K_0 + 1)/(K_0^2 + 4K_0 - 1)$ on the right-hand side. Since $D_1 > D_2$, it means that $2(K_0 - 1)$ is the smallest positive solution of (12.8) (for A_0). Since $A = 1.07 > A_0 = 1.05835$, the smallest positive solution of (12.8) for $A = 1.07$ is therefore at least $2(K_0 - 1)$.

So it remains to prove the above claim. For this it is enough to show that the second derivative of the right-hand side of (12.8) (with $\ell = 4$) has exactly one positive root. Let us make a linear change of the variable: define $x = 1 + \frac{K_0+1}{2}B$, so that $1 \leq x < \infty$. Then

$$\frac{K_0+1}{2}\frac{B(1+\frac{B}{2})}{1+\frac{K_0+1}{2}B} = 1 + \frac{x-2}{K_0+1} - \frac{K_0}{K_0+1}\frac{1}{x} := T(x)$$

(the latter equality is just notation). Hence

$$T'(x) = \frac{1}{K_0+1}(1 + K_0\frac{1}{x^2}),$$

$$T''(x) = -\frac{2K}{K_0+1}\frac{1}{x^3}.$$

And the second derivative of the right-hand side of (12.8) w.r.t. x is (after calculations):

$$(1 + T(x)/4)^2 \frac{1}{4(K_0+1)^2}\frac{1}{x^4} \times \left\{3x^4 + 4K_0x^2 - 2K_0(5K_0+3)x + 5K_0^2\right\}.$$

The polynomial in $\{...\}$ has no more than two positive roots (because the derivative of $\{...\}$ w.r.t. x is an increasing function of $x \geq 0$). By checking the values of $\{...\}$ at $x = 0, 1, \infty$ it follows that it does have one positive root between 0 and 1 and at least one root > 1. So, it has exactly one root greater than 1 and the claim follows. $\square$

Because of (12.1) and (12.2), in order to conclude that $E_0 \circ F^*(\Omega) \subset \Omega$, we only have to show that $E_0 \circ F^*(D(-I^*;\theta)) \subset \Omega^*$, for θ small. Lemma 12.1 says that $F^*(D(-I^*;\theta)) \subset D((-2.12, 0.68);\theta)$. Therefore, by the remark below Lemma 12.2, for this it is enough to check

Lemma 12.4 *If θ is small, then*

$$D((-2.12, 0.68);\theta) \subset P_2(D((-1,1);\theta)). \tag{12.9}$$

Proof: Since $1/0.68 = 1.47059 > 1.47$ and $2.12/0.68 = 3.11765 < 3.12$, it is enough to prove (after rescaling) that

$$D((-3.12, 1);\theta) \subset P_2(D((-(1.47)^{1/2}, (1.47)^{1/2});\theta)). \tag{12.10}$$

For a possible point Z of intersection of the boundaries, we obtain an equation

$$1+\frac{3.12+1}{2}\cdot\frac{i\exp(i\theta)}{\sin\theta}\cdot(1-\exp(i\phi)) = 1.47\cdot(1+\frac{\exp(i(\theta+\alpha))}{\sin^2\theta}\cdot(2\cos\theta-2\cos(\theta+\alpha))).$$

If $\theta \to 0$ and ϕ, α is a solution of this equation, then ϕ/θ and α/θ tend to finite constants $M \geq 0$ and $N \geq 0$ respectively (proof: $P_2(D((-(1.47)^{1/2}, (1.47)^{1/2});\theta))$ certainly contains $P_2(D((-1,1);\theta))$ while the boundary of the latter domain intersects $D((-K,1);\theta)$ at a point $Z(K,\theta)$ of an angle ϕ_1 such that ϕ_1/θ is bounded as $\theta \to 0$, see Lemma 9.1.) We have the following equations for M and N:

$$1+\frac{3.12+1}{2}\cdot M = 1.47\cdot(1+4\cdot\frac{N}{2}\cdot(1+\frac{N}{2})) \tag{12.11}$$

$$\frac{3.12+1}{2}\cdot M\cdot(1+\frac{M}{2}) = 1.47\cdot 4\cdot\frac{N}{2}\cdot(1+\frac{N}{2})\cdot(1+N). \tag{12.12}$$

This system has no non-negative solutions (M,N). A way to see this is to reduce the system to a polynomial equation. For this, denote $M+1 = x$, $N+1 = y$. Then

$$\frac{x-1}{2} = \frac{1.47y^2-1}{3.12+1} \tag{12.13}$$

$$\frac{x+1}{2} = \frac{1.47(y^2-1)y}{1.47y^2-1}. \tag{12.14}$$

This implies that y is a zero of the polynomial

$$h(y) := y^4 - \frac{3.12+1}{1.47}y^3 + \frac{3.12-1}{1.47}y^2 + \frac{3.12+1}{1.47}y - \frac{3.12}{1.47^2}. \tag{12.15}$$

However, the polynomial $h(y)$ does not have solutions $y \geq 1$. Indeed, the second derivative of this polynomial $h''(y)$ is a parabola with zero's at $y = .2000905878$ and at $y = 1.201269956$. If follows that $h'(y)$ is a cubic function with a local maximum at $y = .2000905878$ and a local minimum at $y = 1.201269956$. An explicit calculation shows that $h'(1.201269956) > 0$ and therefore, one has that $h'(y) > 0$ for each $y \geq 1$. Hence $h(y) \geq h(1) > 0$ for each $y \geq 1$. □

Thus, we have proved that $E_0 \circ F^*(\Omega^*)$ is inside Ω^*. By symmetry of Ω^*, $E_{\ell/2} \circ F^*(\Omega^*)$ is inside Ω^* too, for the same θ and ℓ.

Now we consider $E_1 \circ F^*(\Omega^*)$. First note that the domain $F(\Omega^*)$ is contained in $D((-2.12, 4/3); \theta)$. So, for given $\ell \geq 4$, and for θ sufficiently small, the domain $P_\ell(D((-1,1);\theta) \cap \{z;\ \pi/\ell \leq \arg z \leq 3\pi/\ell\})$ and its complex conjugate contain $D((-2.12, 4/3); \theta)$ since the diameter of the latter domain grows as $const/\sin\theta$ while the diameter of the former domain grows as $const/\sin^\ell\theta$ as $\theta \to 0$.

This is enough to conclude the proof of the Proposition 12.1. Indeed, each other $E_i \circ F^*(D((-1,1);\theta))$, $(i \neq 0, \ell/2)$, is contained in $D((-1,1);\theta)$, since $D((-1,1);\theta)$ is invariant under the rotation $z \mapsto \exp(i \cdot 2\pi/\ell)z$, for z in the first quadrant. □

As we noted above, Proposition 12.1 implies Theorem 12.1.

13 The proof of the Main Theorem and Theorem A in the 'period doubling case'

In this section we shall modify the proof in the previous section in order to show that the Main Theorem also holds in the case of an infinitely renormalizable map of period doubling type (from some renormalization onwards). This case was not dealt with in Sections 8 and 9 because the space 0.6 which is used there, only holds in the case that f is not renormalizable of both period s and period $s/2$. In that exceptional case, the space is merely 0.5, see Lemma 6.5 and therefore we can use the method of round discs as in Section 8 only for $\ell \geq 8$, see Example 5.1. Therefore we shall use the ideas of the previous in this case when $\ell < 8$. These arguments also show that Theorem A holds in this exceptional case (that f is renormalizable of both periods $s(n)$ and $s(n)/2$).

So let us indicate the differences with the proof in the previous section. Of course, the proof of Theorem A already follows from the previous section if $\ell \geq 4$ and so we have to take $\ell = 2$ in the previous section. Firstly, define Ω as before with the difference that we take the interval $I = (0, 1.09^{1/2})$ now. Lemma 12.1 and its proof go through unchanged (replacing 1.07 by 1.09) because the actual constants for $\ell = 2$ and space 0.5 are even better than as in the proof of this lemma. Lemma 12.2 is not needed. In Lemma 12.3 we have to take $\ell = 2$ in the statement. In the proof we take $K_0 = 1.4$ and $A = 1.09$. The calculations are slightly different but it is easy to check that everything works as before. Finally, Lemma 12.4 and its proof go through unchanged. All this concludes the Main Theorem in this case. Theorem A for this case follows also in the same way as in Section 9. □

14 Proof of Theorem C

The proof in this section is an elaboration of Section 5 of [Ly3] and Lemmas 14 and 15 in [Ly5]. We wish to thank Edson Vargas for several discussions on these sections.

Let $\mathcal{E}(T^0)$ be the collection of mappings $g: \cup T_i^1 \to T^0$ where T^0 is some symmetric interval around c with nice boundary points and where T_i^1 is a finite collection of disjoint closed subintervals of T^0 for which

- for $i \neq 0$ the map $g: T_i^1 \to T^0$ is a diffeomorphism of the form $f^{j(i)}$ and the inverse map $(g|T_i^1)^{-1}$ has a univalent extension to $\mathbb{C}_{T^0}$;
- writing $T^1 = T_0^1$ we have that $g|T^1$ is a unimodal map of the form f^j and with $g(\partial T^1) \subset \partial T^0$; one can write $g|T^1 = h \circ f$ where h^{-1} has also a univalent analytic extension to $\mathbb{C}_{T^0}$;
- all iterates of the critical point $c = 0$ under g are in $\cup T_i^1$.

Assume there exists a symmetric interval T^{-1} containing T^0, so that when writing as before $g|T_i^1 = h_i \circ f$, the map h_i^{-1} has a univalent analytic extension from $\mathbb{C}_{T^{-1}}$ into $\mathbb{C}_{H_i}$ where H_i is some interval containing $f(c)$ such that $f^{-1}(H_i) \cap \mathbb{R} \subset T^0$. If this holds then we say that $g \in \mathcal{E}(T^0, T^{-1})$.

An example of a map g which is of type $\mathcal{E}(T^0, T^{-1})$ is the first return map to an interval $W_n = [u_n, \tau(u_n)]$ as in Section 2. More precisely, since we have assumed that $\omega(c)$ is minimal, we only consider the finitely many branches which contain points from $\omega(c)$. The f-image of each branch can be extended to W_{n-1} (hence the first return map is in $\mathcal{E}(W_n, W_{n-1})$). Indeed, the boundary points of W_n are nice and there are no forward iterates of ∂W_n in $W_{n-1} \setminus W_n$. So take a domain $I \cap W_n = \emptyset$ of the first return map R to W_n and a maximal interval T containing I so that h $R|I = f^i$ is monotone. By maximality of T for each component T_+ of $T \setminus I$ there exists some $j < i$ so that $f^j(T_+)$ contains c in its boundary and since R is the first return map $f^j(I) \cap W_n = \emptyset$. Hence $f^j(T_+)$ contains a boundary point of W_n and therefore $f^i(T_+)$ contains a point of ∂W_{n-1}. Since this holds for both components of $T \setminus I$ this gives $f^i(T) \supset W_{n-1}$. So on each branch $I \subset W_n$ of R one can write $R = h \circ f$ and h extends as a diffeomorphism from some neighbourhood of $f(I) \ni c_1$ onto W_{n-1}.

We say that $g \in \mathcal{E}(T^0)$ has a *low return iterate* if $g(T^1)$ does not contain the critical point c. In this case we define $\mathcal{R}g \in \mathcal{E}$ as follows. First define $\mathcal{R}g$ so that it coincides with g on $\cup_{i \neq 0} T_i^1$. Define $s_0 \geq 2$ to be minimal so that $g^{s_0-1}(c) \in T^0 \setminus T^1$ and let $s \geq s_0$ be minimal so that $g^s(c) \in T^1$. (This means that $g^s(T^1) \cap T^1 \neq \emptyset$.) Because g has a low return iterate and no periodic attractors, s_0 exists ($s_0 = 2$ if $g(T^1) \cap T^1 = \emptyset$) and since $\omega(c)$ is minimal the integer s also exists. Therefore we can define the new central domain of T^2 to be the component of the domain of g^s containing c. Note that by the choice of s we have $\mathcal{R}g(T^2) \cap T^1 \neq \emptyset$. Moreover, by the way s_0 is chosen we also have $g(T^1) \cap T^2 = \emptyset$. For $x \in T^1 \setminus T^2$ let $s(x) \leq s$ be the smallest integer for which $g^{s(x)}(x)$ and $g^{s(x)}(c)$ are in different components of $\cup_{i \neq 0} T_i^1$ and define $\mathcal{R}g(x) = g^{s(x)+1}(x)$. The domains of $\mathcal{R}g$ in $T^1 \setminus T^2$ map diffeomorphically onto T^0. In fact, we even have

$\mathcal{R}g \in \mathcal{E}$. To show this it suffices to show that if $\mathcal{R}g|T^2 = f^m$ then there exists an interval $A \supset f(T^2)$ which is mapped diffeomorphically onto T^0 by f^{m-1}. Indeed, if $g|T^1 = f^j$ then the Koebe space of $f^{j-1}|f(T^1)$ spreads over T^0 and in particular over one of the intervals connecting ∂T^0 to c. Applying $g^{s_0-2}|T^1 = f^{(s_0-2)j}$ to this it follows that the Koebe space of $g^{s_0-2} \circ f^{j-1}|f(T^1)$ spreads over one of the intervals connecting ∂T^0 to $g^{s_0-2}(c)$ (which by definition is in T^1) and in particular this space contains $\cup_{i\neq 0} T^1_i$. Since g maps such intervals onto T^0 and $s > s_0$ is minimal, the Koebe space of $f^{m-1}|f(T^2)$ spreads also over T^0. This we have proved that $\mathcal{R}g \in \mathcal{E}$. If $\mathcal{R}g$ again has a low return then we can define $\mathcal{R}^2 g \in \mathcal{E}$ and so on.

If g has a *high return iterate* (i.e., not a low return iterate) then let x be the orientation preserving fixed point of $g|T^1$ and z the boundary of T^0 on the same side of c as x. Next take preimages of $z_0 = z, z_1, z_2, \ldots$ of z along this branch of $g|T^1$ (so $[z_1, \tau(z_1)] = T^1$ and $z_k \to x$ as $k \to \infty$). Define $U_k = [z_k, \tau(z_k)]$ and choose k minimal so that $gU_k \supset U_k$. Such an integer k exists because we have assumed that f is not renormalizable. The interval $W^1 := U_k$ is the *escape interval* associated to a map g with a high return iterate. For $i = 0, 1, \ldots, k-1$ define the new map $\widetilde{\mathcal{W}}g$ on $U_i \setminus U_{i+1}$ as $g^{i+1} \circ g$ and define $\widetilde{\mathcal{W}}g$ on U_k as the first return map of g to U_k. Note that $\widetilde{\mathcal{W}}g \notin \mathcal{E}(T^0)$ because the image of the central branch T^2 is contained in the interior of T^0 (and does not stretch over to ∂T^0). However, the first return map $\mathcal{W}g$ of $\widetilde{\mathcal{W}}g$ to W^1 is contained in $\mathcal{E}(W^1, T^0)$. Note that the domain W^1 of $\mathcal{W}g$ is smaller the domain T^0 of g, and the extension associated to $\mathcal{W}g$ includes the domain of the original map g.

Lemma 14.1 *Assume that $g \in \mathcal{E}(T^0)$ is a first return map to a symmetric interval T^0 around c with nice boundary bounds. Let $g, \mathcal{R}g, \ldots, \mathcal{R}^{k-1}g$ have low returns and let $T^1, \ldots, T^{k+1}$ be the central intervals corresponding to $g, \mathcal{R}g, \ldots, \mathcal{R}^k g$. Write $\mathcal{R}^k g|T^{k+1} = f^m$ with $m \in \mathbb{N}$ and let $A \supset f(T^{k+1}) \ni c_1$ be the interval which is mapped by f^{m-1} diffeomorphically onto T^0. Then*

- *there exists a sequence of integers $k+1 \geq p_1 > p_2 > \ldots > p_r \geq 1$ and integers $n(p_1) < n(p_2) < \ldots < n(p_r)$ such that $n = n(p_i) < m$ is the largest integer such that $A^{p_i} := f^{n(p_i)}(A) \subset T^{p_i-1} \setminus T^{p_i}$. Then writing, $s(p_i) := n(p_{i+1}) - n(p_i) > 0$ for each $i = 1, \ldots, r-1$, we have that $f^{s(p_i)}$ maps A^{p_i} onto $A^{p_{i+1}}$. Moreover, the map $f^{s(p_i)-1}\colon fA^{p_i} \to A^{p_{i+1}}$ has Koebe space spread over $T^{p_{i+1}-1}$;*

- *$p_1 = k+1$ or k and the Koebe space of $f^{n(p_1)}\colon A \to A^{p_1}$ spreads over T^{p_1-1};*

- *$p_r = 1$ and the Koebe space of $f^{m-n(p_r)-1}\colon fA^1 \to T^0$ spreads over T^{-1}.*

Proof: First we observe that the boundary points of each interval T^i are nice. To see this, notice for example that T^2 is the intersection of the branches $T^{2,i}$ of g^i containing c for $i = 1, \ldots, s$. Here $g^i|T^{2,i}$ is unimodal with $g^i(\partial T^{2,i}) \subset T^0$ because $g(\partial T^{2,i}) \subset T^{2,i-1}$. By the choice of s_0 and s we then have that $g^i(\partial T^2)$ always remains outside T^2 (it is even outside T^1 for $s_0 \leq i \leq s$). Hence ∂T^2 are nice points. Similarly, ∂T^i are nice points for $i \leq k$. Secondly, from the definition of s_0 it follows that $g(T^1) \cap T^2 = \emptyset$ or more generally that $\mathcal{R}^p g(T^{p+1}) \cap T^{p+2} = \emptyset$ for $p = 0, 1, \ldots, k-1$. Hence consider the first return map R to T^{p-1}. It has a central interval $C^p \subset T^p$ and restricted to C^p one has $R|C^p = \mathcal{R}^{p-1}g$; this holds because $\mathcal{R}^{p-1}g(T^p) \cap T^{p-1} \neq \emptyset$ by the choice of s and because, as we just remarked, $\mathcal{R}^i g(T^{i+1}) \cap T^{p-1} = \emptyset$ for $i < p-1$. Moreover, at each point x its image $R(x)$ is an iterate of $\mathcal{R}^{p-1}g$. In particular, it follows that $f^i A$ does not intersect ∂T^p because $\mathcal{R}^{k-1}g(\partial T^p) \subset \partial T^0$ and because $f^{m-1}(A) = T^0$. All this implies also that $f^i(A) \cap T^{p+1} = \emptyset$ for $i < m-1$. The third observation is that if j is the smallest integer such that $f^j(c_1) \in T^{p-1}$ and $H \ni c_1$ is the largest interval on which f^j is monotone then $f^j(H) \supset T^{p-1}$. This follows from the fact that ∂T^{p-1} are nice points. The fourth observation is the following: consider the first return map R to T^{p-1}. It has a central branch C^p contained inside T^p. Now other branches are mapped by R onto T^{p-1}. So if $J \subset T^{p-1} \setminus C^p$ is so that it remains inside $T^{p-1} \setminus C^p$ for the first j iterates of R then $R^j\colon J \to T^{p-1}$ has an extension onto T^{p-1}. The first assertion follows follows from observations 3 and 4.

So let us prove the second assertion of this lemma. By definition, $\mathcal{R}^{k-1}g(T^k) \cap T^{k-1} \neq \emptyset$. Hence there exists $n < m$ such that $f^n(A) \subset T^{k-1}$ and by the first observation in the proof the Koebe space of this map contains T^{k-1}. Moreover, $f^i(T^{k+1}) \cap T^{k+1} = \emptyset$ for $i = 1, \dots, m-2$ and so it follows that $p_1 = k$ or $p_1 = k+1$. By the second observation if n is the largest integer with $f^n(A) \subset A^{p_1} \subset T^{p_1-1}$ then the Koebe space of this map still contains T^{p_1-1}.

The last assertion similarly follows from the two observations made at the beginning of this proof. We should observe that g has a low return iterate and that therefore $p_r = 1$. $\square$

Lemma 14.2 *There exists $K > 0$ and given $\lambda > 1$ there exists $\sigma > 0$ and $k_0 \in \mathbb{N}$ with the following property. Assume that $g \in \mathcal{E}(T^0, T^{-1})$ as above and that $|T^{-1}| \geq \lambda |T^0|$.*

1. *If all renormalizations $g, \mathcal{R}g, \mathcal{R}^2 g, \dots$ have low return iterates then there exists $k \leq k_0$ such that $\mathcal{R}^k g$ has a polynomial-like extension $G\colon \cup D_i \to D_*(T^0)$;*
2. *If k is minimal such that $\mathcal{R}^k g$ does* **not** *have a low return iterate then either $\mathcal{R}^k g$ has a polynomial-like extension $G\colon \cup D_i \to D_*(T^0)$ or $(|T^0|/|W^{k+1}|) \geq (1-\sigma)^{-1} \cdot (|T^{-1}|/|T^0|)$. where W^{k+1} is the escape interval associated to $\mathcal{R}^k g$.*
3. *If $|T^{-1}|/|T^0| > K$ and k is minimal such that $\mathcal{R}^k g$ does not have a low return iterate then $\mathcal{R}^k g$ has a polynomial-like extension $G\colon \cup D_i \to D_*(T^0)$.*

Proof: Suppose that $g, \dots, \mathcal{R}^{k-1}g$ have low returns so that $\mathcal{R}^k g\colon \cup I_i \to T^0$ is well defined. As before, the Schwarz Lemma implies that the pullbacks of $D_*(T^0)$ under the extensions of the monotone branches $I_i \to T^0$ of $\mathcal{R}^k g$ fit inside $D_*(T^0)$. So let us consider the pullback associated to the inverse of the map $\mathcal{R}^k g|T^{k+1} = f^m$ on the central interval T^{k+1}.

First we notice that since $g \in \mathcal{E}(T^0, T^{-1})$ there exists because of the Koebe Principle a constant $\rho > 0$ which depends on $\lambda = |T^{-1}|/|T^0| > 1$ such that the ρ-scaled neighbourhood of each domain T_i^1 is still contained in T^0. Here we use that the interval H from the definition of $\mathcal{E}(T^0, T^{-1})$ is mapped inside T^0 by f^{-1}. In particular, $|T^1| < (1+\rho)^{-1}|T^0|$. In the same way we have

$$|T^{i+1}| < (1+\rho)^{-1}|T^i| \text{ for each } i = 0, 1, \dots k. \tag{14.1}$$

Let $A \supset f(T^{k+1})$ be so that f^{m-1} maps A diffeomorphically onto T^0. Let $R^0 = \mathcal{R}^k g(T^{k+1}) = f^m(T^{k+1})$ and $I^0 = T^0 \backslash R^0$. We want to compare the sizes of the pullbacks in $I', R' \subset A$ by $f^{m-1}: A \to T^0$ of the 'real' and 'imaginary' pieces $R^0, I^0 \subset T^0$. That is, $R' = f(T^{k+1})$. Let p_i be as in the previous lemma and let $R^i = f^{n(p_i)}(R')$ and $I^i = f^{n(p_i)}(I')$ be the partition of A^i corresponding to I' and R'.

Write $\mu_{i+1} = \frac{|T_{i+1}|}{|T_i|} \in (0,1)$. If R, I are two intervals in $T_i \setminus T_{i+1}$ with a unique common point then using the fact that $f(z) = z^\ell + c_1$ it easy to see that

$$\frac{|R|}{|I|} \geq [\mu_{i+1}]^{\ell-1} \cdot \frac{|f(R)|}{|f(I)|}. \tag{14.2}$$

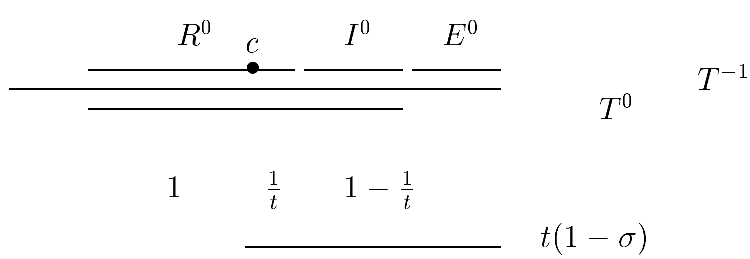
Indeed, if $T_i = (-b, b)$ and $T_{i+1} = (-a, a)$ with $0 < a < b$ and I, R are contained in (a, b) then $(|f(R)|/|R|)/(|f(I)|/|I|)$ is maximal when $I = \{a\}$ and $R = (a, b)$. So (14.2) follows from the inequality

$$\frac{b^\ell - a^\ell}{b-a}\frac{1}{\ell a^{\ell-1}} \leq \frac{b^{\ell-1}}{a^{\ell-1}}.$$

In fact, it is easy to see that because of (14.1) there exists $\tau > 0$ so that either

$$\frac{|R|}{|I|} \geq [\mu_{i+1}]^{\ell-1-\tau} \cdot \frac{|f(R)|}{|f(I)|} \quad \text{ or } \quad \frac{|R|}{|I|} \geq 1. \tag{14.3}$$

Now we will start pulling back the intervals R^0, I^0. Let $K_1 \supset f(A^1)$ be the interval which is mapped monotonically onto T^{-1} by the extension of $f^{m-n(p_r)-1}: f(A^1) \to T^0$. Let E^0 be the component of $T^{-1} \setminus T^0$ which is adjacent to I^0 (the 'extension' in the 'imaginary' direction, see the figure below).



The intervals $T^0 = I^0 \cup R^0$ *and* $T^{-1} = T^0 \cup E^0 \cup \tau(E^0)$.

If $\mathcal{R}^k g$ does not have a low return iterate, then we have that $R^0 = f^m(T^{k+1}) \supset W^{k+1}$ where W^{k+1} is the escape interval associated to $\mathcal{R}^k g$. If the inequality in assertion (2) in the statement of the lemma does not hold then, writing $t = |T^0|/|W^{k+1}|$ and defining σ as in (2), the relative size of the intervals R^0, I^0 and E^0 can be estimated as in the figure above. Therefore,

$$C^{-1}([E^0, R^0], I^0) \geq \frac{(1+1/t)\,(t(1-\sigma)-1)}{(1-1/t)\,(1+t(1-\sigma))} \geq 1 - \epsilon(\sigma) \tag{14.4}$$

where $\epsilon(\sigma)$ is some function so that $\epsilon(\sigma) \to 0$ as $\sigma \to 0$. Here we have used that $t \geq |T^0|/|T^1| > 1$ is bounded away from 1 because of (14.1). Hence, using the map which sends an interval $H \supset R^1 \cup I^1$ diffeomorphically onto $R^0 \cup I^0 \cup E^0$ we get $|f(R^1)|/|f(I^1)| \geq C^{-1}(H, I^1) \geq 1 - \epsilon(\sigma)$. By (14.2) this gives

$$\frac{|R^1|}{|I^1|} \geq (1-\epsilon(\sigma))\mu_1^{\ell-1} = (1-\epsilon(\sigma))\mu_{p_r}^{\ell-1}. \tag{14.5}$$

On the other hand, if $\mathcal{R}^k g$ has a low return then, because $\mathcal{R}^k g(T^{k+1}) = f^m(T^{k+1})$ intersects T^k, we have by the Koebe Principle some constant $\delta > 0$ which depends on $|T^{-1}|/|T^0|$ such that $|f(R^1)|/|f(I^1)| \geq \delta$ and therefore

$$\frac{|R^1|}{|I^1|} \geq \delta\mu_1^{\ell-1} = \delta\mu_{p_r}^{\ell-1}. \tag{14.6}$$

Let us now compare $|R^{p_{i+1}}|/|I^{p_{i+1}}|$ with $|R^{p_i}|/|I^{p_i}|$. Here $k+1 \geq p_1 > \ldots > p_r := 1$ are defined as in the previous lemma. There exists an interval $K_{p_i} \supset f(A_{p_i})$ which is mapped monotonically onto $T^{p_{i+1}-1}$ by $f^{s(p_i)}$. Let $E^{p_{i+1}}$ be the component of $T^{p_{i+1}-1} \setminus A^{p_{i+1}}$ containing c (again the extension). Then because $I^{p_{i+1}} \cup R^{p_{i+1}} = A^{p_{i+1}} \subset T^{p_{i+1}-1} \setminus T^{p_{i+1}}$ and $E^{p_{i+1}} \supset T^{p_{i+1}}$ and because $I^{p_{i+1}}$ is between c and $R^{p_{i+1}}$, we have

$$\frac{|f(R_{p_i})|}{|f(I_{p_i})|} \geq C^{-1}(K_{p_i}, f(I^{p_i})) \geq$$

$$\geq C^{-1}([E^{p_{i+1}}, R^{p_{i+1}}], I^{p_{i+1}}) \geq \frac{|R^{p_{i+1}}|}{|I^{p_{i+1}}|}\frac{|T^{p_{i+1}}|}{|T^{p_{i+1}-1}|} \geq \mu_{p_{i+1}}\frac{|R^{p_{i+1}}|}{|I^{p_{i+1}}|}.$$

Hence, using (14.2),

$$\frac{|R^{p_i}|}{|I^{p_i}|} \geq \mu_{p_{i+1}}\mu_{p_i}^{\ell-1}\frac{|R^{p_{i+1}}|}{|I^{p_{i+1}}|} \tag{14.7}$$

If $\mathcal{R}^k g$ has no low return iterate and the inequality in assertion (2) in the statement of the lemma is not satisfied then combining (14.5) and (14.7) we get

$$\frac{|R^{k+1}|}{|I^{k+1}|} = \frac{|R^{p_1}|}{|I^{p_1}|} \geq (1-\epsilon(\sigma))(\mu_{p_r}\cdot\ldots\cdot\mu_{p_2})^{\ell}\mu_{p_1}^{\ell-1}$$

and applying an estimate as the one above (14.7) to the map $f^{n(p_1)}\colon A \to A^{p_1-1}$ we obtain,

$$\frac{|R'|}{|I'|} \geq (1-\epsilon(\sigma))(\mu_{p_r}\cdot\ldots\cdot\mu_{p_1})^{\ell} \geq (1-\epsilon(\sigma))\left(\frac{|T^{k+1}|}{|T^0|}\right)^{\ell}. \tag{14.8}$$

In fact we can improve this: using in all the previous inequalities (14.3) instead of (14.2), we get

$$\frac{|R'|}{|I'|} \geq (1-\epsilon(\sigma)) \cdot \left(\frac{|T^{k+1}|}{|T^0|}\right)^{\ell-\tau} \tag{14.9}$$

because if the second possibility in (14.3) holds for $i = j$ with j minimal, then as above (but without using (14.5) we have $|R^k|/|I^k| \geq \mu_{p_j}(\mu_{p_{j-1}} \cdot \ldots \cdot \mu_1)^{\ell-\tau}$ which gives even a better bound than (14.9). Since $|T^{k+1}|/|T^0| \leq |T^1|/|T^0|$ is uniformly bounded away from 1, see (14.1), it follows from (14.9) that there exists $\kappa > 1$ such that provided $\sigma > 0$ is sufficiently small

$$\frac{|R'|}{|I'|} \geq \kappa \cdot \left(\frac{|T^{k+1}|}{|T^0|}\right)^{\ell} \tag{14.10}$$

Since $|R'| = |T^{k+1}|^\ell$ and the pullback under f^{m-1} of $D_*(T^0)$ is inside $D_*([R', I'])$ where c_1 is the unique common point of R' and I', the last inequality implies that the pullback of the disc $D_*(T^0)$ along the central branch fits again inside $D_*(T^0)$, showing that $\mathcal{R}^k g$ has a polynomial-like extension. This proves assertion (2).

If $\mathcal{R}^k g$ has a low return then combining (14.6), (14.7) and also the improved inequality (14.3), we get

$$\frac{|R'|}{|I'|} \geq \delta \cdot (\mu_{p_r} \cdot \ldots \cdot \mu_{p_1})^{\ell-\tau} \geq \delta \left(\frac{|T^{k+1}|}{|T^0|}\right)^{\ell-\tau}.$$

Because of (14.1) we have when k is large that this last term is $\geq 2\left(|T^{k+1}|/|T^0|\right)^\ell$ and again the central pullback is mapped inside itself. From this we get that either there exists k such that $\mathcal{R}^k g$ does not have a low return iterate or alternatively $\mathcal{R}^k g$ has a polynomial-like extension. This proves assertion (1).

Let us now prove assertion (3) of the lemma. Since the last return is high the expression in (14.4) can be replaced by $1 \times K/(2+K)$ which tends to 1 as $K \to \infty$. Hence then $|R'|/|I'| \geq K/(2+K)\left(|T^{k+1}|/|T^0|\right)^{\ell-\tau}$ becomes larger than 1 when K is large because $|T^{k+1}|/T^0|$ is bounded away from one. Again we get a polynomial-like mapping. □

Proof of Theorem C: Let $g \ni \mathcal{E}(T^0, T^{-1})$ be the first return map as in the beginning of this section. From the previous lemma it follows that $\mathcal{R}^k g$ does not have a low return iterate for some k. The new map $\mathcal{W}\mathcal{R}^k g$ (as defined above Lemma 14.1) is then defined on a smaller domain. If $\mathcal{R}^k g$ has no polynomial-like extension then because of the second assertion in the previous lemma, the corresponding Koebe space increases by a definite factor $(1-\sigma)^{-1} > 1$ (relative to the size of the new domain). Applying this idea several times, either one obtains a polynomial extension at some stage or the Koebe space becomes arbitrarily large (compared to the size of the domains). But from the last assertion of the previous lemma one then also obtains a polynomial-like extension. □

Exactly as in Section 12 one has that Theorem C implies the Main Theorem for each non-renormalizable map with a minimal critical point c. If a map is only finitely often renormalizable then again the same argument can be used: construct the Yoccoz puzzle associated to the fixed points of the last renormalization and apply Theorem C also to the last renormalization. Thus the proof of the Main Theorem is concluded. $\square$

An Extension and an Erratum

15 Theorem A holds for real analytic maps

Let us first remark that Theorem A in the paper holds for real analytic maps also. This means that the complex bounds which Sullivan proved for infinitely renormalizable Epstein maps of bounded type, even hold for arbitrary infinitely renormalizable maps which are analytic on the dynamical interval. This answers a question of W. de Melo and gives the possibility to extend certain renormalization results of Sullivan and McMullen to the class of real analytic maps.

(Let us also note that the generalized polynomial-like map in Theorems A-C have the property that the critical point does not leave the domain of definition under iterates of this polynomial-like map.)

Theorem 15.1 *Theorem A holds for a real analytic unimodal map f which is infinitely renormalizable: there exists $N(f)$ such that when V_n is a periodic central interval of f of period $s(n) \geq N(f)$, then there exists a polynomial-like extension $F_n\colon \Omega_n' \to \Omega_n$ of $f^{s(n)}\colon V_n \to V_n$ such that the modulus of $\Omega_n \setminus \Omega_n'$ is universally bounded from below by some positive number which only depends on ℓ and so that the diameter of Ω_n is of the same order as that of V_n. The number $N(f)$ is uniformly bounded when f runs over a compact space of maps.*

Proof: First we prove that the real bounds from Sections 5 and 6 still hold if f is a of class $C^{1+\text{zygmund}}$ with a non-flatness condition at c (see Section IV.2.a in [MS]). Because of Theorem IV.2.1 from [MS], this means that if $J \subset T$ are intervals such that $f^s|T$ is diffeomorphic then

$$C(f^sT, f^sJ) \geq \prod_{i=0}^{s-1} \left(1 - o(|f^i(T)|)\right) C(T, J)$$

where $o(t)$ is some function such that $o(t) \to 0$ as $t \to 0$. Now in Section 5 of [LS] let l be maximal so that as before $f^s|l$ is monotone and - this is new - $L = f^s(l)$ contains at most 5 iterates of V. Therefore, in the renormalizable case, the orbit of $T, \ldots, f^k(T)$ in the proof of Lemma 5.1 has intersection multiplicity bounded by 15. Moreover, because the map has no wandering intervals, one has $\max_{i=0,\ldots,k} |f^i(T)|$ tends to zero if the period tends to infinity. (Note also that if I_n, I_{n+1} are consecutive central interval of f then

$$|I_{n+1}| \leq \lambda |I_n| \tag{15.11}$$

where $\lambda < 1$ uniformly when f runs over a compact space of maps. This follows from the extension given by Proposition 7.1 in [LS] and the Koebe Principle.) In particular, the inequality proved in Lemma 5.1 still holds with a spoiling factor O_s such that $O_s \to 1$ as the period s tends to infinity. Now take in Lemmas 6.2-6.5 also the same definition for l. Then these lemmas still holds with a spoiling factor O_s. In Lemma 6.3 simply note that if Q_k contains more than 5 iterates of V then one simply takes $\tilde{Q}_1 \supset f(V)$ so that $\tilde{Q}_k = f^{k-1}(\tilde{Q}_1)$ contains $f^k(V)$ and precisely 5 iterates of V. Then, because the intersection multiplicity of the orbit $\tilde{Q}_1, \ldots, f^{k-1}(\tilde{Q}_1)$ is bounded by 15 and in the same way as before we get $C^{-1}(Q_1, f(V)) \geq C^{-1}(\tilde{Q}_1, f(V)) \geq O_s C^{-1}(\tilde{Q}_k, f^k(V)) \geq O_s \cdot 0.6$. If Q_k contains less than 5 intervals then we can obtain $C^{-1}(Q_1, f(V)) \geq O_s \cdot 0.6$ in the same cases as before. Only in cases II.b and II.c we used the interval Z_1. But now notice that the arguments used there also apply if replace Z_1 by the maximal interval $\tilde{Z}_1 \subset Z_1$ in H_1 so that each component of $f^{k-1}(\tilde{Z}_1) \setminus Q_k$ contains at most one iterate of V. Since the intersection multiplicity of $\tilde{Z}_1, \ldots, f^{k-1}(\tilde{Z}_1)$ is bounded by 18 we get that Lemma 6.3 still holds with a spoiling factor. In Lemmas 6.4 and 6.5 exactly the same remarks apply. Now in Lemma 8.1 we redefine T_1 as the maximal interval such that $f^{s-1}|T_1$ is monotone and such that each component of $f^{s-1}(T_1) \setminus V$ contains at most 5 iterates of V. So we can still apply Lemma 6.4 in the proof of Lemma 8.1 to this T_1 and so this lemma still holds.

All this implies that the same real bounds can be still used in Sections 8 and 9. Now of course, the Schwarz Lemma (that the pullback of some Poincaré domain with angle θ maps inside a similar region with the same angle θ) which we used in these sections does not hold anymore, because the map f is only analytic on a small neighbourhood of the dynamical interval. However, in Lemmas VI.5.2 and VI.5.3 of [MS] it is proved that we can still essentially obtain the same inclusion but with a slightly worse angle. According to Lemmas VI.5.2 and VI.5.3 the loss in the estimate tends to zero if the size of the interval tends to zero. Therefore we still get the same estimate in the proof of Theorem A.

The statement that $N(f)$ is uniformly bounded when f runs over a compact space follows from (15.11). $\square$

16 An erratum

Firstly, we should point out that the domains of the polynomial-like mapping in Theorem C is are disjoint because the f-images of these (near c_1) are based on disjoint intervals in the real line. Moreover, as Ben Hinkle pointed out, there is a mistake in the non-renormalizable case when we prove local connectivity outside the critical point (on page 42 lines 9-11 it is mistakenly argued that the sum of the moduli of some annuli is infinite in the non-renormalizable case). We like to thank Ben Hinkle for this comment and let us show how to fix the proof. We shall show that one can argue as in the proof of the local connectivity of the Julia set of infinitely renormalizable maps in Section 8. To do this we have to be a little careful since the orbit of c enters perhaps several times in $\Omega \cap \mathbb{R}$ at times which do not correspond to iterates of the polynomial-like mapping R.

So assume that f is non-renormalizable and that $\omega(c)$ is minimal. We show that the bounds from Theorem B and C imply local connectivity.

Proposition 16.1 *Let $G(j)\colon \cup_i \Omega_i(j) \to \Omega(j)$ be a sequence of polynomial-like mappings associated to a real polynomial $f(z) = z^\ell + c_1$ such that the critical point $c = 0 \in \Omega_0(j)$ does not escape the domain of $G(j)$ under iterations of $G(j)$. (As before, we assume $G(j)\colon \Omega_0(j) \to \Omega(j)$ is ℓ-to-one, and each other $G\colon \Omega_i(j) \to \Omega(j)$ is an isomorphism.) Assume moreover, that there exist interval neighbourhoods $X(j)$ of $\Omega_0(j) \cap \mathbb{R}$ so that when $x, f^i(x) \in X(j)$ then $f^i(x)$ is an iterate of x under $G(j)$ (we call this* the first return condition*) and so that the modulus of the annuli $\mathbb{C}_{X(j)} \setminus \Omega_0(j)$ is uniformly bounded away from zero. Then the Julia set of f is locally connected.*

Proof: If z is in the Julia set but $\omega(z)$ does not contain c then the Julia set is locally connected at z because of the contraction principle. So choose a point z from the Julia set of f so that $\omega(z) \ni c$. Let P_n be an open piece of the Yoccoz puzzle (corresponding to Ω) based on two preimages $v, -v$ of the orientation reversing fixed point of f so that $\Omega \cap \mathbb{R}$ is either equal to $[-v, v]$ or to a small neighbourhood of this interval. There exists a large integer N such that the full preimage $G^{-N}(P_n)$ is inside the domain of definition $\cup_i \Omega_i$ of G, see Section 3. Note that $G^{-N}(P_n)$ consists of finitely many (open) Yoccoz pieces. Let us consider the pieces of $G^{-N}(P_n)$ inside the central domain Ω_0, i.e.,

$$P'_n = G^{-N}(P_n) \cap \Omega_0.$$

Since $\omega(z) \ni c$, there exists a *minimal* k such that $f^k(z) \in P'_n$. In particular, the point $f^k(z)$ belongs to one of the Yoccoz pieces inside Ω_0. Let F be the branch of f^{-k} which maps a neighborhood of $f^k(z)$ to a neighbourhood of z. Let X be as in the statement of this lemma.

Claim 1. The map F extends to a holomorphic map in the domain $\mathbb{C}_X$. *Proof of the claim.* Assume the contrary. We then get that for some minimal $r < k$ that $f^{-r}(\mathbb{C}_X)$ (along the same orbit) meets the critical value c_1. This means that the branch f^{-r} follows the points $c_{r+1} = f^r(c_1) \in \mathbb{C}_X$, $c_r = f^{r-1}(c_1)$,..., $c_2 = f(c_1)$, c_1. Among these iterations of c_1, let us mark all those $c_{j_1}, c_{j_2}, \ldots, c_{j_m}$, where $j_1 < j_2 < \ldots < j_m$, which hit the domain $\mathbb{C}_X$ (i.e., are in X). Because of the first return assumption there exists integers $k(1) < k(2) < \ldots$ such that $c_{j_1} = G^{k(1)}(c)$, $c_{j_2} = G^{k(2)-k(1)}(c_{j_1}) = G^{k(2)}(c)$,..., $c_{j_m} = G^{k(m)}(c)$. It follows, that $f^{-r} = f^{-(s-1)} \circ G^{-(k(m)-1)}$, where $f^{-(s-1)}$ is the branch from V to $\hat{U}$ corresponding to the restriction of G on Ω_0 (so $G|\Omega_0 = f^{s-1} \circ f$). Hence, $f^{-(r+1)}(\mathbb{C}_X \cap \Omega) \subset G^{-r(m)}(\Omega) \subset \Omega_0$ and $f^{k-r-1}(z) \in f^{-(r+1)}(P'_n) = (G|_{\Omega_0})^{-1} \circ G^{-k(m)+1}(P'_n) \subset P'_n$. This contradicts the minimality of k and proves the claim.

Now apply the claim to a sequence of maps $G(j)\colon \cup \Omega_i(j) \to \Omega(j)$. This gives a sequence of annuli $\mathbb{C}_{X(j)} \setminus \Omega_0(j)$ of modulus $\geq \delta$ such that some iterate of z maps to a puzzle piece inside $\Omega_0(j)$. Since the diameter of $\Omega_0(j)$ shrinks to zero one completes the proof exactly like in the infinitely renormalizable case. $\square$

Corollary 16.1 *If f has infinitely many high first returns, then the Julia set is locally connected.*

Proof: For each high return, we have a polynomial-like mapping $G : \cup_i \Omega_i \to \Omega$ constructed in Sections 11-12, such that $G|_V$ is the first return map. Remember that Ω here is a definite complex neighborhood of the interval V so that Ω_0 is inside a definite neighborhood of the interval $U = [-u, u] \subset V$ and so that $G|_{\Omega_0} = f^s$. Denote by l' a maximal interval outside U with a common boundary point, such that $f^s|_{l'}$ is monotone, and let $f^{ks}(u)$ be the first moment when it leaves l'. Then any f-iterate of c in the interval $X = [-f^{ks}(u), f^{ks}(u)]$ is, in fact, an iterate of c under G. Hence, we can apply the Proposition 2.1 (note that the gaps $X \setminus \Omega \cap \mathbb{R}$ are not small because of Proposition 7.1 [LS]. □

Let W_n be the sequence of intervals as in Section 2 and let R_n be the corresponding first return maps. We say that the return is *low* if the image of R_n of the central component W_{n+1} does not contain c and it is called *central* if $R_n(c) \subset W_{n+1}$.

Lemma 16.1 *There exists a universal number $\lambda > 1$ (only depending on ℓ) the following property. Assume that we are in one of the following situations: 1) either R_{n-1} or R_n has a non-central low return; 2) the return of R_{n-1} is non-central high. Then $|W_n| \geq \lambda |W_{n+1}|$.*

Proof: If R_n has a non-central low return then $|W_n| \geq \lambda |W_{n+1}|$ according to the corollary on page 345 in [MS]. In the same way, if the return to W_{n-1} is non-central low then again $|W_{n-1}| \geq \lambda |W_n|$. Now let the first return map to W_n restricted to W_{n+1} be equal to f^s. There exists an interval neighbourhood T of $f(W_{n+1})$ such that f^{s-1} maps T diffeomorphically onto W_{n-1} and so that $f^{-1}(T) \cap \mathbb{R} \subset W_n$. Hence, using the Koebe Principle it follows that $f^{-1}(T) \cap \mathbb{R} \subset W_n$ is a definite neighbourhood of W_{n+1}. So in this case we are done also.

If the return map R_{n-1} to W_{n-1} is high then according to part 1 of Lemma 1.2 on page 342 in [MS] (or Proposition 7.1 of the present paper), the map on the central domain W_n is a composition of f and a map with bounded distortion. From this and the fact that R_{n-1} has a non-central high return it follows that W_{n+1} has to be a definite factor smaller than W_n also. □

Of course, there are infinitely many integers n for which the map R_{n-2} has a non-central return. Choose such an n and write $T^0 = W_n$, $T^{-1} = W_{n-1}$ and study the situation as in Section 14. So take a first return map g as in Section 14 and define $T^{2,i-1}$ to be the component of g^i containing c. Let us begin by remarking that on page 59 line -8 one better defines s minimal so that $g^s(T^{2,s-1}) \cap T^1 \neq \emptyset$ and on line -3 we should have defined $s(x)$ as the smallest integer for which $g^{s(x)}(x)$ and $g^{s(x)}(c)$ are in different components of $\cup_i T_i^1$ (so we have to also allow T_0^1.) Moreover on page 60 line 17 one should read $g^i \circ g$ instead of $g^{i+1} \circ g$. First we need the following proposition.

From the lemma above, there exists a universal $\lambda > 1$ (only depending on ℓ) such that

$$|T^{-1}| \geq \lambda |T^0|. \tag{16.12}$$

Now consider the return map g to T^0 with central domain $T^1 = W_{n+1}$.

Lemma 16.2 *Suppose that $g, \mathcal{R}g, \ldots, \mathcal{R}^k g$ exist. Then there exists a universal $\kappa > 0$ (only depending on ℓ) and a κ-scaled neighbourhood W^{k+1} of T^{k+1} such that each iterate of c inside W^{k+1} is an iterate of c under the map $\mathcal{R}^k g$.*

Proof: First notice that there exists for each component I of g an integer k such that $g|I = f^k$. Moreover, there exists $U \supset f(I)$ such that f^{k-1} maps U diffeomorphically onto T^{-1} and so that $f^{-1}(U) \cap \mathbb{R} \subset T^0$. In particular, some neighbourhood U of $f(T^1)$ is mapped diffeomorphically onto T^{-1} and because of (16.12) and by Koebe this implies that U contains a definite neighbourhood of $f(T^1)$ and since $f^{-1}(U) \cap \mathbb{R} \subset T^0$ one gets that some definite neighbourhood of T^1 is contained in T^0. Hence there exists a universal $\lambda' > 1$ (only depending on ℓ) such that

$$|T^0| \geq \lambda' |T^1|. \tag{16.13}$$

Similarly each component I of the domain of g in $T^0 \setminus T^1$ has adjacent to it an interval $I' \subset T^0$ with one point in common with I 'further away from c' (i.e., so that I lies between I' and c) so that $g|(I \cup I')$ is monotone and

$$|I'| \geq \rho |I|. \tag{16.14}$$

For $k = 0$ the result is obvious. Let us first show the lemma for $k = 1$. Let us first consider the case that g has a non-central low return. We claim that if $x, g^i(x) \in T^1$ then $g^i(x)$ is an iterate of x under $\mathcal{R}g$. To see this we first remark that

$$\mathcal{R}g|(T^0 \setminus T^1) = g. \tag{16.15}$$

Next take $x \in T^1$. If $g(x)$ is contained in a component of $\cup_{i \neq 0} T^1_i$ which is entirely contained in $g(T^1)$ (which is in $T^0 \setminus T^1$ since the return was assumed to be low) then we have $\mathcal{R}g(x) = g^2(x)$. So if $g^2(x) \in T^1$ then the required statement holds for x and if $g^2(x) \in T^0 \setminus T^1$ then by (16.15) again the required statement holds for x. If $g(x)$ is not contained in such a component then x is contained in a symmetric interval $T^{2,1} \subset T^1$ such that $g(T^{2,1})$ is inside one of the components of $\cup_{i \neq 0} T^1_i$ (in fact, $T^{2,1}$ is the component of $g^{-1}(\cup_{i \neq 0} T_i)$ containing c). In particular $g(x) \notin T^1$. So if x is not as before consider $g^2(x)$. If $g^2(x)$ is contained in a component of $\cup_{i \neq 0} T^1_i$ which is entirely contained in $g(T^{2,1})$ then $\mathcal{R}g(x) = g^3(x)$ and by (16.15) again the required statement holds for x. If $g^2(x)$ is not contained in such a component then again x is contained in a symmetric interval $T^{2,2} \subset T^{2,1}$ such that $g^2(T^{2,2})$ is inside one of the components of $\cup_{i \neq 0} T^1_i$ and in particular $g^2(x) \notin T^1$. In this way one proves the claim inductively. Now we set $W^2 = T^1$ and it suffices to show that T^1 is a definite amount larger then T^2. To see this write $g|T^1 = f^m$. The map f^{m-1} maps some neighbourhood of $f(T^1)$ diffeomorphically onto T^0 (in fact onto T^{-1} we do not need this anymore), and because the Koebe Principle and because of (16.14) it follows that a definite piece of T^1 is mapped by g outside any given component in $T^0 \setminus T^1$ of g. Hence $|T^1| > (1 + \kappa)|T^2|$ for some universal number $\kappa > 0$.

If g has a central low return then define s_0 as before and consider a (shrinking) nested sequence of intervals $T^{2,s_0-2} \subset \ldots T^{2,0} := T^1$ such that $g(\partial T^{2,i}) \subset \partial T^{2,i-1}$ for $i = 1, \ldots, s_0 - 2$ (so these intervals are associated to the saddle-cascade of the central branch; they are symmetric around c and their endpoints are preimages of ∂T^1 under the central branch of g).

Since we now assume that one has a central return we have $s_0 > 2$. Because of the corollary on page 345 of [MS], one has a universal $\lambda > 1$ (only depending on ℓ) such that $|T^{2,s_0-2}| \geq \lambda |T^{2,s_0-1}|$ and because of (16.13) and the Koebe Principle we get in the same way also

$$|T^{2,0}| \geq \lambda |T^{2,1}|. \tag{16.16}$$

(If one has a long saddle-cascade a similar uniform comparison between $T^{2,i}$ with $T^{2,i-1}$ is certainly not true.) Now s_0 is by definition the minimal integer such that $g^{s_0-1}(c) \notin T^1$. As in (16.16) one has that $|g^{s_0-1}(c) - g^{s_0-2}(c)|$ is comparible to size of $T^{2,0} \setminus T^{2,1}$, i.e., to the size to $T^0 \setminus T^1$. Write $g^{s_0-2}|T^{2,s_0-2} = f^m$. Then it follows from this that some neighbourhood of $f(T^{2,s_0-2})$ is mapped by f^{m-1} onto a definite neighbourhood of $f^m(T^{2,s_0-2})$. So

$$f^{m-1}|f(T^{2,s_0-2}) \text{ has uniformly bounded distortion.} \tag{16.17}$$

Now define $W^2 \supset T^2$ to be equal to T^{2,s_0-2}. One has by construction that $g(T^1) \cap T^{2,s_0-2} = \emptyset$ and by definition of $\mathcal{R}g$ if $x, g^i(x) \in T^{2,s_0-2}$ then $g^i(x)$ is an iterate of x under $\mathcal{R}g$. So it suffices to show that V^2 is a definite amount larger than T^2. But this follows from (16.14) and (16.17).

Now if $g, \ldots, \mathcal{R}^k g$ are defined then we get $|T^i| \geq \lambda |T^{i+1}|$ for $i = 0, \ldots, k$. If $\mathcal{R}^{k-1}g$ has a non-central low return then and this we can set $V^{k+1} = T^k$ and we are done. If $\mathcal{R}^{k-1}g$ has a central low return then we argue as above with intervals $T^{k+1,i}$. $\square$

If $g, \mathcal{R}g, \mathcal{R}^2 g, \ldots$ all exist then there exists $\mathcal{R}^k g$ with central domain T^{k+1} which has a polynomial-like with central piece $\Omega_0(k) \supset T^{k+1}$ extension and Lemma 16.2 gives an interval $W^{k+1} \supset T^{k+1}$ such that the modulus of $\mathbb{C}_{W^{k+1}} \setminus \Omega_0(k)$ is uniformly bounded from below and a first return condition is satisfied. Because of Proposition 16.1 above the Julia set is locally connected if this case happens infinitely often. If $g, \ldots, \mathcal{R}^k g$ exist but $\mathcal{R}^k g$ has a high return then according to Lemma 14.2 either $\mathcal{R}^k g$ has a polynomial-like extension and we can again use Lemma 16.2 above or one considers $g_1 = \mathcal{W}\mathcal{R}^k g$ which has a better extension domain. Remark that $\mathcal{W}\mathcal{R}^k g$ is by construction a first return map to its domain. So we can apply Lemma 14.2 again when $g_1, \ldots, \mathcal{R}^{k_1} g$ are well defined (with again the first return condition and even better extension scale). By assertions 2 and 3 of Lemma 14.2 we must reach a situation which has a polynomial-like extension. In this way we get eventually a map which a polynomial-like extension and satisfies the required conditions of Proposition 16.1 above. Thus we get

Corollary 16.2 *The Julia set of f is locally connected.*

References

[BH] B. Branner and J.H.Hubbard, *The iteration of cubic polynomials I,* Acta Math. **160**, (1988), 143-206, *The iteration of cubic polynomials II,* Acta Math. **169**, (1992), 229-325.

[BL] A.M. Blokh and M.Yu. Lyubich, *Measurable dynamics of S-unimodal maps of the interval*, Ann. Sc. E.N.S. 4*e* série, **24**, 545–573, (1991).

[BKNS] H. Bruin, G. Keller, T. Nowicki and S. van Strien, *Absorbing Cantor sets in dynamical systems: Fibonacci maps*, Stonybrook IMS preprint 1994/2. To appear in the Annals of Math.

[DH] A. Douady and J.H. Hubbard, *On the dynamics of polynomial-like mappings,* Ann.Sc.E.N.S.4e série, **18**, (1985), 287–343

[DH1] A. Douady and J.H. Hubbard, *Etude dynamique des polynomes complexes I, II,* Publication Mathematiques D'Orsay, no. 84-02 (1984), no. 85-04 (1985).

[GS] J. Graczyk and G. Świątek, *Polynomial-like property for real quadratic polynomials*, February 3, 1995

[HJ] J. Hu and Y. Jiang, *The Julia set of the Feigenbaum quadratic polynomial is locally connected,* Preprint 1993.

[Ji1] Y. Jiang, *Infinitely renormalizable quadratic Julia sets*, Preprint ETH, Zürich (1993).

[Ji2] Y. Jiang, *Renormalization in quadratic-like mappings*, Preprint November 1994.

[KN] G. Keller and T. Nowicki, *Fibonacci maps re(aℓ)visited,* Preprint (1992).

[Ly1] M.Yu. Lyubich, *Ergodic theory for smooth one-dimensional dynamical systems.* Stonybrook IMS Preprint 1991/11.

[Ly2] M.Yu. Lyubich, *On the Lebesgue measure of the Julia set of a quadratic polynomial.* Stonybrook IMS Preprint 1991/10.

[Ly3] M.Yu. Lyubich, *Combinatorics, geometry and attractors of quasi-quadratic maps.* Annals of Math. **140**, (1994), 347-404

[Ly4] M.Yu. Lyubich, *Milnor's attractors, persistent recurrence and renormalization.* In: "Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's 60th Birthday", Publish or Perish (1992).

[Ly5] M.Yu. Lyubich, *Geometry of quadratic polynomials: moduli, rigidity and local connectivity,* Stonybrook IMS Preprint 1993/9.

[LM] M. Lyubich and J. Milnor, *The unimodal Fibonacci map,* Journal of the A.M.S. **6**, 425-457 (1993).

[LY] M. Lyubich and M. Yampolsky, *Dynamics of quadratic polynomials: complex bounds for real maps*, draft Februari 22, 1995.

[Mar] M. Martens, *Interval dynamics,* Thesis, Delft, (1990).

[MS] W. de Melo and S. van Strien, *One-dimensional dynamics,* Ergebnisse Series **25**, Springer Verlag, (1993).

[MMS] M. Martens, W. de Melo and S. van Strien, *Julia-Fatou-Sullivan theory for real one-dimensional dynamics*, Acta Math. **168**, 273-318 (1992).

[McM] C. McMullen, *Complex dynamics and renormalization,* Princeton University Press, to appear.

[Mil] J. Milnor, *Local connectivity of Julia sets; expository lectures.* Stonybrook IMS Preprint 1992/11.

[Mil1] J. Milnor, Questions in: *Problems in Holomorphic Dynamics*, Stony Brook IMS Preprint 1992/7, (editors: B. Bielefeld and M. Lyubich).

[Pe] C. Petersen, *Local connectivity of some Julia sets containing a circle with an irrational rotation*, Preprint I.H.E.S. /M/94/26 (1994).

[Sh] M. Shishikura, *Unpublished.*

[SN] S. van Strien and T. Nowicki, *Polynomial maps with a Julia set of positive Lebesgue measure: Fibonacci maps* Stonybrook IMS Preprint 1994/3.

[Sul] D. Sullivan, *Bounds, quadratic differentials, and renormalization conjectures, 1990,* In AMS Centennial Publications. **2**: Mathematics into Twenty-first Century.

[Y] J.C. Yoccoz, *MLC*, Manuscript.