

Hofer's L^∞ -geometry:
energy and stability of Hamiltonian flows,
part II

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Abstract

In this paper we first show that the necessary condition introduced in our previous paper is also a sufficient condition for a path to be a geodesic in the group $\text{Ham}^c(M)$ of compactly supported Hamiltonian symplectomorphisms. This applies with no restriction on M . We then discuss conditions which guarantee that such a path minimizes the Hofer length. Our argument relies on a general geometric construction (the gluing of monodromies) and on an extension of Gromov's non-squeezing theorem both to more general manifolds and to more general capacities. The manifolds we consider are quasi-cylinders, that is spaces homeomorphic to $M \times D^2$ which are symplectically ruled over D^2 . When we work with the usual capacity (derived from embedded balls), we can prove the existence of paths which minimize the length among all homotopic paths, provided that M is semi-monotone. (This restriction occurs because of the well-known difficulty with the theory of J -holomorphic curves in arbitrary M .) However, we can only prove the existence of length-minimizing paths (i.e. paths which minimize length amongst *all* paths, not only the homotopic ones) under even more restrictive conditions on M , for example when M is exact and convex or of dimension 2. The new difficulty is caused by the possibility that there are non-trivial and very short loops in $\text{Ham}^c(M)$. When such length-minimizing paths do exist, we can extend the Bialy–Polterovich calculation of the Hofer norm on a neighbourhood of the identity (C^1 -flatness).

Although it applies to a more restricted class of manifolds, the Hofer-Zehnder capacity seems to be better adapted to the problem at hand, giving sharper estimates in many situations. Also the capacity-area inequality for split cylinders extends more easily to quasi-cylinders in this case. As applications, we generalise Hofer's estimate of the time for which an autonomous flow is length-minimizing to some manifolds other than \mathbf{R}^{2n} , and derive new results such as the unboundedness of Hofer's metric on some closed manifolds, and a linear rigidity result.

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1 Statement of main results

1.1 Geodesics

In this paper, unless specific mention is made, M will be any symplectic manifold, compact or not, with or without boundary, but always geometrically bounded at infinity in the non-compact case. When $\partial M \neq \emptyset$, all Hamiltonians have compact support in $M - \partial M$.

For the convenience of the reader, we begin by recalling some of the basic definitions from Part I ([8]). The length $\mathcal{L}(\gamma)$ of the path $\gamma = \phi_{t \in [a,b]}$ in the group $\text{Ham}^c(M)$ generated by the compactly supported Hamiltonian function H_t is defined by:

$$\mathcal{L}(\gamma) = \mathcal{L}(\phi_t) = \int_a^b \|H_t\| dt,$$

where

$$\|H_t\| = \text{Totvar}(H_t) = \sup_{x \in M} H_t(x) - \inf_{x \in M} H_t(x).$$

The Hofer norm $\|\phi\|$ of $\phi \in \text{Ham}^c(M)$ is defined to be the infimum of the lengths of all paths from the identity to ϕ .

We say that the path $\gamma = \phi_{t \in [a,b]}$ is **regular** if its tangent vector $\dot{\phi}_t$ is non-zero for all $t \in [a, b]$. A **stable geodesic** $\gamma = \phi_{t \in [a,b]}$ is a regular path which is a local minimum for \mathcal{L} on the space \mathcal{P} of all paths $\psi_{t \in [a,b]}$ homotopic to $\phi_{t \in [a,b]}$ with fixed endpoints ϕ_a, ϕ_b , where \mathcal{P} is given the usual C^∞ topology.

As in Part I, we will say that a path $\gamma = \phi_{t \in [0,1]}$ has a certain property **at each moment** if every $s \in [0, 1]$ has a closed connected neighbourhood $\mathcal{N}(s)$ in $[0, 1]$ such that, for all subintervals \mathcal{N} containing s , the subpath $\phi_{t \in \mathcal{N}}$ has this property. Very often (but not always) it will suffice to check that the property in question holds on the interval $\mathcal{N}(s)$ itself since it will then automatically hold on any subinterval. For example, this is clearly the case for the property of being a stable geodesic. We recall that a regular path γ is said to be a **geodesic** if it is a stable geodesic at each moment. Hence each $s \in [a, b]$ has a connected neighbourhood $\mathcal{N}(s)$ such that $\phi_{t \in \mathcal{N}(s)}$ is a stable geodesic. Unless specific mention is made to the contrary, we will normalise t so that it takes values in the interval $[a, b] = [0, 1]$, and will assume that $\phi_0 = \mathbb{1}$. Also, often it is convenient to talk about a path of Hamiltonians $H_{t \in [0,1]}$ rather than the corresponding path in $\text{Ham}^c(M)$.

A point P such that $H_t(P) = \sup_{x \in M} H_t(x)$ for all $t \in [0, 1]$ is called a **fixed maximum** of the corresponding path γ (or equivalently of $H_{t \in [0,1]}$.) A similar definition applies to a fixed minimum p . We often denote these by P, p , and write q for a fixed extremum. Borrowing terminology from [1], we will often call a path with a fixed maximum and minimum **quasi-autonomous**. Clearly, every autonomous (i.e. time-independent) Hamiltonian H is quasi-autonomous.

In Part I we established the following necessary condition for a regular path to be a geodesic.

Theorem 1.1 *A regular path $\phi_t, t \in I$, in $\text{Ham}^c(M)$ is a geodesic only if its generating Hamiltonian has at least one fixed maximum and one fixed minimum at each moment.*

We will prove below in §3.1 that:

Theorem 1.2 *For any manifold M , the above necessary condition is also sufficient.*

When $M = \mathbf{R}^{2n}$ the above result can be deduced from the work of Bialy–Polterovich in [1]. We refer the reader to the other papers of this series [8, 9] for the characterization of the stability of geodesics.

1.2 Length-minimizing properties of geodesics

In order to state our results on the existence of length-minimizing paths, we need to introduce the idea of the **capacity** $c(H)$ of a Hamiltonian $H_{t \in [0,1]}$. We will explain this precisely in §2 below. However, roughly speaking, it is, for each given choice c of a capacity, the minimum of the c -capacity of the regions over and under the graph of $H_{t \in [0,1]}$.

We recall that, for any symplectic manifold M , with or without boundary, compact or not, the Gromov capacity of M is

$$c_G(M) = \sup\{a : \text{there is a symplectic embedding } f : B^{2n}(a) \rightarrow (M - \partial M)\}$$

where $B^{2n}(a)$ denotes the standard closed ball of \mathbf{R}^{2n} of capacity $a = \pi r^2$. The Hofer-Zehnder capacity $c_{HZ}(M) \in [0, \infty]$ of M is the supremum over all positive real numbers $m \in [0, \infty)$ such that there is a surjective function $H : M \rightarrow [0, m]$ equal to m on ∂M and outside some compact set of M , and whose Hamiltonian flow has no non-constant closed trajectory in time less than 1.

Thus, for instance, if $c = c_G$ is the Gromov capacity, $c_G(H)$ is the maximum capacity of a symplectic ball which embeds on both sides of the graph of H . One can show that if H_t is sufficiently C^2 -small, then $c(H) \geq \mathcal{L}(H_t)$ for any choice of c .

We also need to put a condition on M in order to use the theory of J -holomorphic curves. Following the terminology of [12], we will say that (M, ω) is **weakly monotone** if for every spherical homology class $B \in H_2(M)$

$$\omega(B) > 0, \quad c_1(B) \geq 3 - n \quad \implies \quad c_1(B) \geq 0.$$

This condition is satisfied if either $\dim M \leq 6$ or M is **semi-monotone**, i.e. there is a constant $\mu \geq 0$ such that, for all spherical homology classes $B \in H_2(M)$,

$$c_1(B) = \mu\omega(B).$$

As explained in [12], weak monotonicity is exactly the condition under which the theory of J -holomorphic curves behaves well. Since we apply this theory to the product $M \times S^2$ rather than to M itself, our arguments do not work for all weakly monotone M . However they do work when $\dim M \leq 4$, since this implies that $M \times S^2$ is weakly monotone and also, by a slightly different argument, when M is semi-monotone. Thus it applies to the semi-monotone manifolds $M = \mathbf{T}^{2n}$ (the standard torus) and $M = \mathbf{C}P^n$ (complex projective space) as well as to some but not all products of the form $\mathbf{C}P^k \times \mathbf{C}P^\ell$.

Recall that a manifold M is called weakly exact if $[\omega] |_{\pi_2(M)} = 0$.

Paths minimal in a homotopy class

Theorem 1.3 (i) *Let M be have dimension ≤ 4 or be semi-monotone. Let $H_{t \in [0,1]}$ be any path with*

$$c_G(H) \geq \mathcal{L}(H_t).$$

Then $H_{t \in [0,1]}$ is length-minimizing amongst all paths homotopic rel endpoints to $H_{t \in [0,1]}$.

(ii) The same conclusion holds if $c_{HZ}(H) \geq \mathcal{L}(H_t)$ and M is weakly exact.

Remark 1.4 (i) As a consequence of the Non-Squeezing Theorem, $\mathcal{L}(H_t)$ is actually the maximal possible value of $c_G(H)$. This is also true of $c_{HZ}(H)$ when M is weakly exact as a consequence of the c_{HZ} -area inequality, and of any capacity when $M = \mathbf{R}^{2n}$. See §2. Thus the theorem states in fact that the path is length-minimizing in its homotopy class as soon as $c(H)$ reaches its maximal value.

(ii) Note that in this theorem, we do not need any other hypothesis on $H_{t \in [0,1]}$. In particular, this theorem implies that any path with $c_G(H) = \mathcal{L}(H_t)$ must be a stable geodesic and therefore must have two fixed points by the necessary condition for stability. A more elaborate argument leads to linear rigidity, see Theorem 5.9 at the end of §5.

This theorem is proved in §5. (A simplified proof is available when $\dim M = 2$: see Remark 2.3.) To apply it we need to know when $c(H) \geq \mathcal{L}(H_t)$. This seems a hard problem in general. However, the inequality $c(H) \geq \mathcal{L}(H_t)$ is always true when H is C^2 -small and, as we shall see in §3, the inequality $c_{HZ}(H) \geq \mathcal{L}(H_t)$ is also true when H is autonomous and has no non-constant closed trajectory in time < 1 . The same statement holds for c_G when M has dimension 2. Hence we find:

Corollary 1.5 *Let M be a weakly exact manifold or any surface. Let $H_{t \in [0,1]}$ be an autonomous Hamiltonian whose flow has no non-constant closed trajectory in time less than 1. Then the flow $\phi_{t \in [0,1]}$ generated by H is length-minimizing among all homotopic paths rel endpoints.*

Short loops and length-minimizing paths

Our methods do not allow us to compare the lengths of non homotopic paths in $\text{Ham}^c(M)$ for arbitrary M . However, there is a topological reason for this. Consider the length function on the fundamental group of $\text{Ham}^c(M)$:

$$\mathcal{L} : \pi_1(\text{Ham}^c(M)) \rightarrow [0, \infty), \quad \mathcal{L}([\gamma]) = \inf_{\gamma \in [\gamma]} \mathcal{L}(\gamma).$$

We will say that $\text{Ham}(M)$ has **short loops** if $\{0\}$ is not an isolated point in the image of \mathcal{L} . Thus, if M has short loops, there is for each $\varepsilon > 0$ a class $[\gamma]$ in $\pi_1(\text{Ham}^c(M))$ which can be represented by a loop of length $< \varepsilon$ but not by a loop of length $< \varepsilon/2$. When M does not have short loops we define $r_1 = r_1(M) > 0$ to be

$$r_1 = \inf (\{\text{Im } \mathcal{L} : \pi_1 \rightarrow \mathbf{R}\} \cap (0, \infty))$$

when this is not empty, and ∞ otherwise. If M has short loops we put $r_1 = 0$.

As an example, observe that $\text{Ham}^c(\mathbf{R}^{2n})$ does not have short loops, since any given loop may be homotoped to one which is arbitrarily short by conjugating it by an appropriate rescaling factor. A similar argument shows:

Lemma 1.6 *Suppose that (M, ω) is exact and is convex in the sense that it admits a contracting Liouville vector field whose flow exists for all positive time. Then $\text{Ham}^c(M)$ does not have short loops.*

Another situation in which M obviously cannot have short loops is when $\pi_1(\text{Ham}^c(M))$ is finite or cyclic. This applies to surfaces: it is well-known that when M has dimension 2, $\text{Ham}^c(M)$ is contractible, except in the case of the 2-sphere where

$$\pi_1\text{Ham}(S^2) = \pi_1(SO(3)) = \mathbf{Z}/2.$$

It also applies to certain 4-manifolds, for example to $\mathbf{C}P^2$ with its standard form and to $S^2 \times S^2$, provided that the latter manifold has a product form $\sigma_0 \oplus \sigma_1$ which has the same integral on both sphere factors. This follows from Gromov's calculation of the homotopy type of $\text{Ham}^c(M)$ in these cases: see [3].

When (M, ω) does not have short loops, it is sometimes possible to calculate the invariant r_1 . For example, when $M = \mathbf{C}P^2$ or is any surface other than the sphere, $\pi_1(\text{Ham}^c(M)) = 0$ and so $r_1 = \infty$. Less trivially, in §5.2 we use Theorem 1.3 to show that:

Lemma 1.7 *Let $\phi_{t \in [0,1]}$ be an essential loop in $\text{Ham}(S^2)$, where S^2 has area A . Then $\mathcal{L}(\phi_{t \in [0,1]}) \geq A$.*

Observe that the path generated by a multiple of the height function, which rotates the sphere about the north-south axis through a complete turn, is an essential loop with length exactly A . Hence

Corollary 1.8 *If (S^2, ω) has area A , then $r_1(S^2) = A$.*

The following theorem collects together our main results on the existence of length-minimizing paths.

Theorem 1.9 *Let M have dimension ≤ 4 or be semi-monotone, and have no short loops. In the non-compact case, assume also that it has bounded geometry at infinity. Then the path $H_{t \in [0,1]}$ is length-minimizing amongst all paths with the same endpoints in the following three cases:*

- (i) *when $c_G(H) = \mathcal{L}(H_t) \leq r_1/2$;*
- (ii) *when $c_{HZ}(H) = \mathcal{L}(H_t) \leq r_1/2$ and M is weakly exact;*
- (iii) *when $M = \mathbf{R}^{2n}$ and there is some capacity for which*

$$c(H) \geq \mathcal{L}(H_t).$$

Proof: By Theorem 1.3, any path γ with $c_G(H) = \mathcal{L}(H_t)$ is length-minimizing among all homotopic paths. Suppose its length is $\leq r_1/2$, and let γ' be some other path with the same endpoints and with $\mathcal{L}(\gamma') < \mathcal{L}(\gamma)$. Then the loop $-\gamma' * \gamma$ has length $< r_1$ and so represents a loop which has an arbitrarily short representative. We may choose a representative λ of length ε where

$$\mathcal{L}(\gamma') + \varepsilon < \mathcal{L}(\gamma).$$

The $\gamma' * \lambda$ is homotopic to γ and is shorter: a contradiction. This proves (i). Essentially the same proof applies in the other two cases. \square

Corollary 1.10 *Let M be an exact and convex manifold (\mathbf{R}^{2n} for instance) or any surface, not necessarily compact, and let H be an autonomous Hamiltonian with time-1 map ϕ . Suppose that H has no non-constant closed trajectory in time less than 1, and if M is a sphere, that $\|H\| \leq \frac{1}{2}\text{area}(M)$. Then*

$$\|\phi\| = \|H\|.$$

This generalises Hofer's criterion in [4]. See Richard [15] for further results in this direction. It is possible that our methods can be used in the non-autonomous case too: then the above theorem would be true for any quasi-autonomous Hamiltonian satisfying the above hypothesis.

Diameter of $\text{Ham}^c(M)$

Much of our work in studying $\text{Ham}^c(M)$ for a closed orientable surface was prompted by the question of whether this group has bounded diameter. If M is open and of finite volume, the existence of the Calabi invariant implies that $\text{Ham}^c(M)$ has infinite diameter. However, in view of Sikorav's result in [16] that the subgroup $\text{Ham}^c(B^{2n}(1))$ has bounded diameter in $\text{Ham}^c(\mathbf{R}^{2n})$, it is not clear what the answer should be for compact M . We have not been able to decide this when $M = S^2$. When the genus is > 0 , one can use the energy-capacity inequality to show:

Proposition 1.11 *Any closed orientable surface of genus > 0 supports an autonomous Hamiltonian H whose flow $\{\phi_t^H\}_{t \geq 0}$ is length minimizing for all t .*

Corollary 1.12 *For these manifolds M , the group $\text{Ham}(M)$ has unbounded diameter with respect to the Hofer norm.*

This corollary can be extended to all manifolds of the form $M \times \Sigma$, where Σ is any closed orientable surface of genus at least 1 and M any manifold. This uses holomorphic methods and will be developed elsewhere.

Local flatness

If $H_{t \in [0,1]}$ has both a fixed maximum and minimum and is also sufficiently small, it follows from a version of the Hamilton-Jacobi equation that $\mathcal{L}(H_t)$ may be measured by the difference between the actions of its fixed maximum and minimum. We therefore can extend the local flatness result of Bialy–Polterovich [1] in this case:

Corollary 1.13 *If (M, ω) is semi-monotone and has no short loops, there is a C^1 -neighbourhood \mathcal{U} of the identity in $\text{Ham}^c(M)$ which, when given the Hofer metric, is isometric to a neighbourhood of $\{0\}$ in a Banach space.*

Properties holding at each moment

The above results concern the length of whole paths. One can also look at properties of pieces of paths. By definition, a geodesic is stable at each moment, and it is natural to wonder how much this property can be strengthened. Ideally, one would like geodesics to be length-minimizing at each moment. However, we have not been able to prove this for arbitrary M . The next result is an easy consequence of Theorem 1.3 and the equality $c_G(H) = \mathcal{L}(H_t)$ for C^2 -small Hamiltonians.

Proposition 1.14 *Suppose that (M, ω) is semi-monotone. Then every geodesic minimizes length among homotopic paths at each moment. Moreover, if in addition (M, ω) has no short loops, every geodesic is length-minimizing at each moment.*

This means that for such M our geodesics behave like geodesics in Riemannian geometry. In particular, any quasi-autonomous Hamiltonian isotopy is absolutely length-minimizing for some interval of time $[0, t_0]$, $t_0 > 0$, a result proved by Bialy–Polterovich in [1] when $M = \mathbf{R}^{2n}$ using other methods.

The final result we will mention here concerns rational manifolds (M, ω) , that is to say those which have a positive **index of rationality** r . Here r is defined to be the largest positive element of the extended reals $(0, \infty]$ such that $\omega(\pi_2(M)) \subset r\mathbf{Z}$, if this exists, and to be 0 otherwise.

Proposition 1.15 *Suppose that (M, ω) has bounded geometry at infinity, and that it has index of rationality $r > 0$. Then, at each moment, a geodesic minimizes length among paths which are homotopic to it by paths of length $< 2r$. More precisely, for each $s \in [0, 1]$ there is an interval $\mathcal{N}(s)$ such that, whenever $s \in \mathcal{N} \subset \mathcal{N}(s)$, the path $\phi_{t \in \mathcal{N}}$ has minimal length among all paths which are smoothly homotopic to it through paths of length $< 2r$.*

A version of the above Proposition which applies to smooth deformations of certain long initial paths is given in Theorem 5.2.

1.3 Non-squeezing results for quasi-cylinders

In order to establish the length-minimizing properties of geodesics we need an extension of the non-squeezing theorem to “quasi-cylinders”. We will state our basic results on quasi-cylinders in this subsection, and show in §2 how they are related to Hofer’s geometry.

Given a capacity c , a symplectic manifold (M, ω) , and $a \in (0, \infty)$, we say that the cylinder $M \times D^2(a)$ endowed with the split form, satisfies the c -area inequality if

$$c(M \times D(a)) \leq a.$$

Note that, by the definition of a capacity, the space $\mathbf{R}^{2n} \times D^2(a)$ satisfies the c -area inequality for any c and any $a > 0$ (see [6] for instance for the definition of a (intrinsic) capacity).

We now state the capacity-area inequalities. The first one, due to Gromov and Lalonde-McDuff, applies to all manifolds. The second one, due to Floer, Hofer and Viterbo, applies to rational manifolds in a specific range.

Theorem 1.16 (Non-Squeezing Theorem, [3, 7]) *The split cylinder $M \times D^2(a)$ satisfies the c_G -area inequality for any (M, ω) and any $a > 0$.*

As explained in [7], there is a direct proof using J -holomorphic curves when M is “nice” (eg if M has dimension ≤ 4 or is semi-monotone), and a much more elaborate indirect proof for arbitrary M .

Theorem 1.17 (HZ-capacity-area inequality, [2, 5]) *Let M be a rational manifold of index of rationality $r \in (0, \infty]$. Then $M \times D^2(a)$ satisfies the c_{HZ} -area inequality whenever $a \leq r$.*

Definition 1.18 Let (M, ω) be a symplectic manifold and D a set diffeomorphic to a disc in $(\mathbf{R}^2, \sigma = du \wedge dv)$. Then the manifold $Q = (M \times D, \Omega)$ endowed with a symplectic form Ω is called a **quasi-cylinder** if

- (i) Ω restricts to ω on each fiber $M \times pt$;
- (ii) Ω is the product $\omega \oplus \sigma$ near the boundary $M \times \partial D$, and, in the case when M is non-compact, outside a set of the form $X \times D$, for some compact subset X in M .

A quasi-cylinder Q is said to be **split** if there is a symplectomorphism

$$(M \times D, \Omega) \rightarrow (M \times D, \omega \oplus \sigma)$$

which is the identity near the boundary. Its **area** is the number A such that

$$\text{vol}(X \times D, \Omega) = A \text{vol}(X, \omega),$$

where X is a compact set as in (ii) above.

We will see in Lemma 2.4 below that the area of Q does not depend on the choice of X . It is not known whether every quasi-cylinder is split, though by the results of [11] on the structure of ruled surfaces this is the case when $\dim M = 2$.

We will see in §2 that every pair of paths from $\mathbb{1}$ to ϕ gives rise to two quasi-cylinders, whose areas are related to the lengths of the paths. When the paths are sufficiently C^2 -close, these quasi-cylinders split and we can use the usual capacity-area inequalities to obtain results valid for arbitrary (M, ω) . This is the method used to prove Theorem 1.2, for example. In general, we must deal with arbitrary quasi-cylinders, and our results will hold only for certain (M, ω) .

For ease of exposition, we now state slightly simplified versions of the results in §4. We will say that the c -area inequality holds for a quasi-cylinder Q of area A if $c(Q) \leq A$ (when $c = c_G$, we will also call this a non-squeezing theorem).

Proposition 1.19 *Let M be a symplectic manifold with bounded geometry and $Q = (M \times D, \Omega)$ be a quasi-cylinder of area A .*

- (i) *Then the non-squeezing theorem holds for Q whenever M has dimension ≤ 4 or is semi-monotone.*
- (ii) *The c_{HZ} -area inequality holds for Q whenever M is weakly exact.*

The proof of (i) is a somewhat tricky extension of the usual proof of the non-squeezing theorem (as given for example in [7]), and involves a non-embedding result for a pair consisting of a ball and a cylinder. But, not surprisingly, the proof of (ii) is much easier: indeed the relation between c_G and c_{HZ} is somewhat similar to the relation between handle decompositions and Morse functions. The c_{HZ} -capacity is more manageable since functions are easier to homotope than their level sets. Since this proposition is the fundamental result which underlies the proof of Theorem 1.3 it holds for the same M as does that theorem: see Remark 4.2.

The following result on manifolds with positive index of rationality is a deformation result analogous to [7, Lemma 3.3].

Proposition 1.20 *Let $\Omega_{s \in [0,1]}$ be a smooth 1-parameter family of symplectic forms on $M \times D$ such that each manifold $Q_s = (M \times D, \Omega_s)$ is a quasi-cylinder. Assume that Q_0 is split.*

- (i) *If $\text{area}(Q_s) < 2r$ for all s , then the non-squeezing theorem holds for Q_1 ;*
- (ii) *If there is a smooth symplectic isotopy*

$$g_s : B^{2n+2}(c) \rightarrow Q_s, \quad 0 \leq s \leq 1$$

which at time $s = 0$ has the form

$$B^{2n+2}(c) \hookrightarrow B^{2n}(c) \times D(c) \xrightarrow{f \times g} M \times D$$

where $c \geq 2r$, and if

$$\text{area } Q_s < c + 2r, \quad 0 \leq s \leq 1,$$

then $\text{area } Q_1 \geq c$.

1.4 Organization of the paper

This paper is organised as follows. In §2 we explain the connection between Hofer’s geometry and quasi-cylinders and work out the simplest properties of quasi-cylinders. In §3 we describe some methods of embedding balls, and apply them to prove Theorem 1.2 and results in dimension 2 such as Lemma 1.7. We also construct Hamiltonians which realise the Hofer-Zehnder capacity. §4 is the heart of the paper, and concerns non-squeezing theorems for quasi-cylinders. All applications to Hofer’s geometry are proved in §5.

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2 Hofer’s geometry and quasi-cylinders

In this section we first explain how the lengths of paths are related to the capacities of quasi-cylinders. In §2.2 we develop the basic properties of quasi-cylinders.

2.1 Gluing monodromies

The basic geometric construction needed consists in defining a symplectic space (a “quasi-cylinder”) made from two Hamiltonian isotopies which are homotopic rel endpoints. If $H_{t \in [0,1]}, K_{t \in [0,1]}$ are the corresponding Hamiltonians, we attach the lower part of the graph of one of these maps to the upper part of the graph of the other by the identification of the graphs which match their characteristic foliations.

Up until now, we have insisted that geodesics γ be regular paths, which means that their time-derivatives $\frac{\partial}{\partial t} \gamma(t)$ vanish nowhere. In this section it will be convenient for technical reasons to consider paths which are constant for t in some intervals $[0, \varepsilon], [1 - \varepsilon, 1]$ near the endpoints, and have non-vanishing time-derivative elsewhere. Note that any regular path can be reparametrized so that this is the case, without changing its length. Further, since we will also only consider paths which have at

least one fixed maximum and minimum, we may normalise their generating Hamiltonians $H(x, t)$ by assuming that $\inf_x H(x, t) = 0$ for all t .¹

We will write Γ_H for the graph of H :

$$\Gamma_H = \{(x, H(x, t), t) \in M \times \mathbf{R} \times [0, 1]\}.$$

Observe that, if $M \times \mathbf{R} \times [0, 1]$ is provided with the symplectic form $\Omega = \omega \oplus ds \wedge dt$, then the lines of the characteristic flow induced on the hypersurface Γ_H are precisely

$$(\phi_t(x_0), H(\phi_t(x_0)), t), \quad 0 \leq t \leq 1.$$

We write R_H^-, R_H^+ respectively for the region under and over the graph:

$$\begin{aligned} R_H^- &= \{(x, s, t) \in M \times \mathbf{R} \times [0, 1] : 0 \leq s \leq H(x, t)\} \\ R_H^+ &= \{(x, s, t) \in M \times \mathbf{R} \times [0, 1] : H(x, t) \leq s \leq \max_x H(x, t)\} \end{aligned}$$

For a small parameter $\nu > 0$, we denote by $R_H^-(\nu), R_H^+(\nu)$ the thickened regions:

$$\begin{aligned} R_H^-(\nu/2) &= \{(x, s, t) \in M \times \mathbf{R} \times [0, 1] : \lambda(t) \leq s \leq H(x, t)\} \\ R_H^+(\nu/2) &= \{(x, s, t) \in M \times \mathbf{R} \times [0, 1] : H(x, t) \leq s \leq \mu_H(t)\} \end{aligned}$$

where $\lambda(t)$ is a smooth function taking values in $[-\delta, 0]$ which equals $-\delta$ except near $0, 1$, and where $\mu_H(t)$ is a smooth function such that $m_H(t) = \max_x H(x, t) \leq \mu_H(t) \leq \max_x H(x, t) + \delta$. We assume that $\lambda(t)$ and $\mu_H(t)$ are chosen so that

$$U_H = \{(s, t) : \lambda(t) \leq s \leq \mu_H(t)\} \subset \mathbf{R}^2$$

has a smooth boundary. Then ν refers to the area added in the thickening:

$$\nu/2 = \int_0^1 -\lambda(t) dt = \int_0^1 (\mu_H(t) - \max_x H(x, t)) dt.$$

Of course,

$$R_H(\nu) = R_H^-(\nu/2) \cup R_H^+(\nu/2) = M \times U_H$$

is the direct product of M with a disc of area $\|H\| + \nu$. Note that the two halves $R_H^\pm(\nu/2)$ of $R_H(\nu)$ have corners at $t = 0, 1$: their boundaries are the union of two smooth pieces joined along $M \times \{0\}$ and $M \times \{1\}$. One of these boundary pieces is a product and has trivial holonomy, and the other is Γ_H with holonomy ϕ_1 .

Suppose that $\{\phi_t\}$ and $\{\psi_t\}$ are two isotopies with $\phi_1 = \psi_1$, and let H_t, K_t be the corresponding Hamiltonians. Then we may join $R_K^+(\nu/2)$ to $R_H^-(\nu/2)$ by the map $\Gamma_K \rightarrow \Gamma_H$ given by:

$$(x, s, t) \mapsto (\phi_t \circ \psi_t^{-1}(x), s - K(x) + H(\phi_t \circ \psi_t^{-1}(x)), t).$$

Observe that this map is a symplectomorphism on the whole of $R_K^+(\nu/2)$. Indeed, if f_t is the isotopy generated by F_t , then

$$(y, s, t) \mapsto (f_t(y), s + F_t(f_t(y)), t)$$

¹Note that in general the function $t \mapsto \inf_x H(x, t)$, though continuous, need not be smooth.

is a symplectomorphism. Here, we have

$$f_t = \phi_t \circ \psi_t^{-1}, \quad F_t(y) = H_t(y) - K_t(\psi_t \circ \phi_t^{-1}(y)).$$

Thus

$$R_{H,K}(\nu) = R_H^-(\nu/2) \cup R_K^+(\nu/2) = \{(x, s, t) : \lambda(t) \leq s \leq \mu_K(t) + F_t(x)\}$$

is a manifold with smooth boundary.

Because $\phi_1 = \psi_1$, the monodromy round $\partial R_{H,K}(\nu)$ is trivial. Therefore, there is a symplectomorphism

$$\Phi_K : M \times \text{nbhd}(\partial U_H) \rightarrow \text{nbhd}(\partial R_{H,K}(\nu))$$

given by

$$(x, u, v) \mapsto (\phi_v \circ \psi_v^{-1}(x), u - \mu_H(v) + F_v(\phi_v \circ \psi_v^{-1}x), v)$$

near the part of the boundary where $u = \mu_H(v)$, and by the identity elsewhere. In fact, this formula makes sense on the whole of the thickening $R_{H,K}(\nu) - R_{H,K}$ of $\partial R_{H,K}$. Because the loop $\{\phi_t \circ \psi_t^{-1}\}$ contracts in $\text{Ham}(M)$, Φ_K extends to a diffeomorphism

$$\Phi_K : M \times U_H \rightarrow R_{H,K}(\nu)$$

of the form

$$(x, u, v) \mapsto (f_{u,v}(x), s(x, u, v), v),$$

where each $f_{u,v} \in \text{Ham}^c(M)$. It follows easily that $(M \times U, \Phi_K^*(\Omega))$ or, equivalently $(R_{H,K}(\nu), \Omega)$, is a quasi-cylinder.

The following very simple lemma is the key to our results.

Lemma 2.1 *Suppose that $\mathcal{L}(K_t) < \mathcal{L}(H_t) = A$. Then, for sufficiently small $\nu > 0$, at least one of the quasi-cylinders $(R_{H,K}(\nu), \Omega)$ and $(R_{K,H}(\nu), \Omega)$ has area $< A$.*

Proof: Choose $\nu > 0$ so that

$$\mathcal{L}(K_t) + 2\nu < \mathcal{L}(H_t),$$

and suppose first that M is compact. Evidently,

$$\begin{aligned} \text{vol}(R_{H,K}(\nu)) + \text{vol}(R_{K,H}(\nu)) &= \text{vol}(R_H(\nu)) + \text{vol}(R_K(\nu)) \\ &= (\text{vol } M) \cdot (\mathcal{L}(H_t) + \mathcal{L}(K_t) + 2\nu) \\ &< 2(\text{vol } M) \cdot \mathcal{L}(H_t). \end{aligned}$$

If M is non-compact, we may restrict to a large compact piece X of M and then take the volume. \square

In order to use this proposition, we consider the following. Let c be any capacity and define the capacity $c(H)$ of the Hamiltonian H_t as the minimum of $\inf_{\nu>0} c(R_H^-(\nu/2))$ and $\inf_{\nu>0} c(R_H^+(\nu/2))$.¹ Suppose now that

$$c(H) \geq \mathcal{L}(H_t).$$

¹This means that, for instance, the c_{HZ} -capacity of $H_{t \in [0,1]}$ is defined by looking at autonomous Hamiltonians defined under (or over) the graph of a non-autonomous Hamiltonian.

(Actually, $c_G(H)$ cannot be greater than $\mathcal{L}(H_t)$ by Theorem 1.16, since $R_H^-(\nu/2) \cup R_H^+(\nu/2)$ is a split cylinder of area $\mathcal{L}(H_t) + \nu$ for all ν . The same remark applies for c_{HZ} when M is weakly exact by Theorem 1.17). Then for any Hamiltonian $K_{t \in [0,1]}$ which generates an isotopy homotopic rel endpoints to ϕ_t , there are symplectic embeddings

$$R_H^-(\nu/2) \hookrightarrow R_{H,K}(\nu) \quad \text{and} \quad R_H^+(\nu/2) \hookrightarrow R_{K,H}(\nu).$$

Therefore $c(R_{H,K}(\nu)) \geq \mathcal{L}(H_t)$ and $c(R_{K,H}(\nu)) \geq \mathcal{L}(H_t)$ for all $\nu > 0$. If we can find conditions on M, H, K such that the capacity-area inequality holds for these quasi-cylinders, we could then conclude that both areas must be greater or equal to $\mathcal{L}(H_t)$, and hence, by Lemma 2.1, that

$$\mathcal{L}(K_t) \geq \mathcal{L}(H_t).$$

The above argument shows:

Proposition 2.2 *Let M be any symplectic manifold and $H_{t \in [0,1]}$ a Hamiltonian generating an isotopy ϕ_t from $\mathbb{1}$ to ϕ . Suppose that there exists a capacity c such that the following two conditions hold:*

- (i) $c(H) = \mathcal{L}(H_t)$ and
- (ii) *there exists a class \mathcal{C} of Hamiltonian isotopies homotopic rel endpoints to $\phi_{t \in [0,1]}$, which is such that the capacity-area inequality holds (with respect to the given capacity c) for all quasi-cylinders $R_{H,K}(\nu)$ and $R_{K,H}(\nu)$ corresponding to Hamiltonians $K_{t \in [0,1]} \in \mathcal{C}$.*

Then the length of the path ϕ_t is minimal among all paths in \mathcal{C} .

Here the class \mathcal{C} might be a small neighbourhood of $\phi_{t \in [0,1]}$, or the set of all paths from $\mathbb{1}$ to ϕ which are homotopic rel endpoints to $\phi_{t \in [0,1]}$.

In §3 we calculate $c_G(H)$ and $c_{HZ}(H)$ by geometric methods, postponing to §4 consideration of the conditions needed on M and \mathcal{C} so that condition (ii) of Proposition 2.2 holds.

Remark 2.3 It is possible that the Non-Squeezing Theorem holds for all quasi-cylinders. For example, this would be the case if all quasi-cylinders were split, that is symplectomorphic to a product $M \times D^2$. When M has dimension 2 it follows from the structure theory of ruled surfaces that any quasi-cylinder splits: see Lemma 2.9. Therefore, in this case Theorem 1.3 is an immediate consequence of Proposition 2.2. However, what happens in higher dimensions is still unclear. Therefore, our arguments are more indirect and work only under certain assumptions on the behaviour of holomorphic spheres in M .

2.2 Properties of quasi-cylinders

Lemma 2.4 *The area of a quasi-cylinder Q is independent of the choice of the compact set X in Definition 1.18. It is equal to the area*

$$\int_{D_x} \Omega$$

of the disc $D_x = \{x\} \times D$ for any $x \in M$.

Proof: First observe that the integral $A_x = \int_{D_x} \Omega$ is independent of the choice of x . This holds because the circles $\{x\} \times \partial D$ are tangent to the characteristic flow of Ω on ∂Q . Thus, given any path α in M , the form Ω restricts to zero on $\alpha \times \partial D$. We will write A for the common value of the A_x .

When M is compact, we may take $X = M$ and there is nothing further to prove. Therefore we now assume that M is noncompact.

We wish to show that for any X as in Definition 1.18,

$$\text{vol}(X \times D) = A \text{vol}(X),$$

that is to say

$$\int_{X \times D} \Omega^{n+1} = (n+1) \int_x \omega^n \int_{D_x} \Omega.$$

But the definition of quasi-cylinder implies that

$$\Omega = (\omega \oplus \sigma) + \tau,$$

where σ is the standard area form on D , and where τ is a closed 2-form vanishing near the boundary of $X \times D$ and whose restriction to each fiber $M \times \{pt\}$ vanishes. Thus τ has the form $\tau = \alpha \wedge du + \beta \wedge dv$, where α, β are 1-forms depending on the point $p = (x, u, v) \in M \times D$ but with values in $T_x^* M \subset T_p^*(M \times D)$. Therefore,

$$\int_{X \times D} \Omega^{n+1} = (n+1) \int_{X \times D} \omega^n \wedge \sigma + C \int_{X \times D} \omega^{n-1} \wedge \tau^2,$$

where C is some integer. Now, the integral of τ over any element of $H_2(M; \mathbf{R}) = H_2(M \times D; \mathbf{R})$ is zero because its restriction to $M \times \{pt\}$ vanishes. It is therefore exact, say $\tau = d\rho$. But then $\omega^{n-1} \wedge \tau^2 = \omega^{n-1} \wedge d(\rho \wedge \tau)$, and so, by Stokes' theorem, the second term in the right hand side of the above equation must vanish. \square

The next lemmas establish simple criteria for quasi-cylinders to split.

Lemma 2.5 *Let $(M \times U, \tau)$ be a quasi-cylinder such that the restriction of τ to each disc $pt \times U$ is non-degenerate. Then $(M \times U, \tau)$ is split.*

Proof: By hypothesis τ has the form $\omega + \rho\sigma + du \wedge \alpha + dv \wedge \beta$, where the function ρ is > 0 , and $\alpha = \alpha_{u,v}$ and $\beta = \beta_{u,v}$ are 1-forms on M for each $u, v \in U$. Further, near the boundary $\rho = 1$ and $\alpha = \beta = 0$. Thus

$$\begin{aligned} \tau^{n+1} &= \rho(n+1)\omega^n \wedge \sigma - \frac{n(n+1)}{2} \omega^{n-1} (du \wedge dv \wedge \alpha \wedge \beta) \\ &= (n+1)\omega^n \wedge \sigma(\rho + \nu), \quad \text{say,} \end{aligned}$$

where $\nu > -\rho$. Let

$$\tau_t = \omega + (1-t+t\rho)\sigma + t(du \wedge \alpha + dv \wedge \beta).$$

Then, it is easy to check that

$$\begin{aligned} \tau_t^{n+1} &= (n+1)\omega^n \wedge \sigma(1-t+t\rho+t^2\nu) \\ &> (n+1)\omega^n \wedge \sigma(1-t+t\rho-t^2\rho) \\ &= (1-t)(1+t\rho)(n+1)\omega^n \wedge \sigma. \end{aligned}$$

Since $(1-t)(1+t\rho) \geq 0$ when $0 \leq t \leq 1$, τ_t is a symplectic form for all t . Since $\tau_t = \tau_0$ near the boundary, the usual Moser argument provides an isotopy h_t which is the identity near the boundary and is such that $h_t^*(\tau_t) = \tau_0$. \square

Lemma 2.6 *When the Hamiltonian K_t is sufficiently close to H_t in the C^2 -topology, the quasi-cylinders $R_{H,K}(\nu), R_{K,H}(\nu)$ constructed above split for all ν .*

Proof: By symmetry, it suffices to prove this for $R_{H,K}(\nu)$. Recall that the group $\text{Ham}^c(M)$ is locally contractible: there is a contractible neighbourhood \mathcal{U} of $\mathbb{1}$ which consists of all Hamiltonian diffeomorphisms ψ whose graph lies close enough to the diagonal *diag* in $(M \times M, -\omega \oplus \omega)$ to correspond to an exact 1-form in T^*M . (As usual, this correspondence is induced by a symplectomorphism from a neighbourhood of the Lagrangian submanifold *diag* in $(M \times M, -\omega \oplus \omega)$ to a neighbourhood of the zero section in the cotangent bundle.) Thus, if K_t is so close to H_t that the corresponding paths $\{\psi_t\}, \{\phi_t\}$ satisfy $\phi_t \circ \psi_t^{-1} \in \mathcal{U}$ for all t , there is a canonical choice of the retracting homotopy $f_{u,v}$. Hence, one can choose the diffeomorphism $\Phi_K : M \times U_H \rightarrow R_{H,K}$ to vary continuously (in the C^∞ -topology) with respect to K_t . Observe also, that by choice of Φ_K ,

$$\Phi_K^*(\Omega) = \omega \oplus \sigma$$

on $R_{H,K}(\nu) - R_{H,K}$. Therefore, by choosing K_t sufficiently close to H_t , we may ensure that the form $\Phi_K^*(\Omega)$ satisfies the conditions of Lemma 2.5 for all ν . \square

The results which follow will be needed in §4.

Let $Q = (M \times D^2, \Omega)$ be a quasi-cylinder of area A . Then there exists a small neighbourhood of the boundary of the form $M \times C$ for a round annulus C of radii $r_2 > r_1 > 0$, over which Ω restricts to a split form $\omega \oplus (du \wedge dv)$. Let $C' \subset C$ be the annulus of radii $r_2, r_1 + \frac{r_2 - r_1}{2}$, and write $Q(\kappa)$, $\kappa > 0$ for the quasi-cylinder whose underlying space is the same as Q , and symplectic form is $\Omega_\kappa = \Omega + \kappa \pi^*(\rho)$ where $\pi : Q \rightarrow D^2$ is the projection and ρ is a 2-form supported in $\text{Int } C'$ which is everywhere a non-negative multiple of the area form. We normalise ρ so that $\int \rho = 1$. Then $\text{area } Q(\kappa) = A + \kappa$. We will denote by $Q^*(\kappa)$ the obvious S^2 -compactification of $Q(\kappa)$, which has same area as $Q(\kappa)$. Since Ω is a product in $\pi^{-1}C'$, the forms Ω_κ are all symplectic.

We claim that:

Lemma 2.7 *$Q(\kappa)$ is symplectomorphic to a product for large κ .*

Proof: We may assume that $Q(\kappa)$ is the manifold

$$Q(\kappa) = M \times (U \cup L_\kappa) = M \times [0, A + \kappa] \times [0, 1]$$

provided with the form Ω_κ which equals $\Phi^*(\Omega)$ in $M \times U$ and equals the product $\omega \oplus \sigma$ over L_κ , where $\sigma = du \wedge dv$. This manifold is embedded as a submanifold of $V = M \times \mathbf{R}^2$ with the split symplectic form outside U .

Our aim is to construct a family of disjoint transverse symplectic discs in $Q(\kappa)$ and then apply Lemma 2.5. In order that the diffeomorphism be the identity near the boundary, it is essential that these discs be flat there, that is they should coincide with the discs $\{(x, u, v) : (u, v) \in U \cup L_\kappa\}$ near the boundary. We will first construct these discs over U and then extend them over L_κ .

For each $x \in M$, let D_x be the surface obtained by restricting to $M \times U$ all characteristic lines of the hypersurfaces $v = \text{constant}$ which pass through the line segment $\{(x, 0, v) : 0 \leq v \leq 1\}$. Because the form is split near the boundary of U , D_x is flat there, and because Ω restricts to ω on all M -fibers, these surfaces are everywhere transverse to the fibers. Hence they are symplectic surfaces transverse to M which foliate $M \times U$, and are standard outside a compact subset of M as well as near the three sides $u = 0$, $v = 0$ and $v = 1$ of U . Of course the map $f_{u,v} : M \rightarrow M$ defined by the identification of $M \times \{(0, v)\}$ with $M \times \{(u, v)\}$ given by the foliation, is a ω -symplectic diffeomorphism for all (u, v) . (For $A' > A$ large enough, it is not difficult to see that the isotopy $f_{A',v \in [0,1]}$ is actually Hamiltonian: the graph of its generating Hamiltonian is the image by ψ_T of $M \times \{0\} \times [0, 1]$ where ψ_t is the Hamiltonian generated by $F = v$ and T is large enough so that this image lies entirely outside U .)

Let $\beta : [A, A + \kappa] \rightarrow [0, A]$ be a decreasing surjective function which is constant near the endpoints. We extend D_x to the disc

$$D'_x = D_x \cup \{(f_{\beta(u),v}(x), u, v) : (u, v) \in L_\kappa\}.$$

Clearly, if κ is sufficiently big, we may choose β to have so small derivative that the disc D'_x is symplectic for all x . Further, if M is non-compact, we are only concerned with x in a compact subset of M since everything is constant near infinity. Finally, we can use the family of symplectic diffeomorphisms $f_{u,v}$ to define a diffeomorphism Ψ of $M \times [0, A + \kappa] \times [0, 1]$ which sends the surfaces D_x to flat surfaces. The image of the symplectic form by Ψ is then still a quasi-cylinder which now satisfies the condition of Lemma 2.5. \square

When $\dim M = 2$, we can give a direct proof that the manifold $(Q(0), \Omega)$ (or equivalently $(R_{H,K}, \Omega)$) is a product. The reason why this case is special is that in this dimension we can construct the family of disjoint embedded discs or spheres which is needed in Lemma 2.5 as a family of J -holomorphic curves. In higher dimensions there is nothing which forces these curves to be disjoint and embedded, but in dimension 4 (the dimension of $M \times S^2$) we can use positivity of intersections.

We will need the following lemma from [11]:

Lemma 2.8 *Suppose that Ω is a symplectic form on $M \times S^2$ which does not vanish on one fiber $M \times \{z_0\}$ and on one section $\{x_0\} \times S^2$. Then, if $\dim M = 2$, Ω is symplectomorphic to a product form, by a symplectomorphism which is the identity on $M \times \{z_0\}$ and on $\{x_0\} \times S^2$.*

Lemma 2.9 *When $\dim M = 2$ the manifold (Q, Ω) is a product.*

Proof: (Sketch) Since Ω does not vanish on any of the fibers of Q , it suffices by the above lemma to produce one section on which Ω does not vanish. We do this by showing that for a generic Ω -tame almost complex structure J on Q^c the class $[pt \times S^2]$ has an embedded J -holomorphic representative. This is certainly true for J which are Ω_κ -tame, for large κ , and it remains true as J deforms to be Ω -tame. For more details, see [10, 11]. \square

3 Construction of symplectic embeddings and Hamiltonians

This section begins by constructing functions defined under and over the graph of autonomous Hamiltonians H in order to estimate $c_{HZ}(H)$. We then describe situations in which it is possible to calculate

$c_G(H)$ by embedding balls symplectically under and over the graph of H . (This gives lower bounds of $c(H)$ for any other capacity because, as an obvious consequence of its definition, c_G is the smallest capacity.) The first case is a local result, where we embed a small ball near an extremum of H , and is enough to prove Theorem 1.2 characterizing geodesics. The second is a global embedding method which works when M is 2-dimensional, and allows us to calculate the Hofer norm of certain elements of $\text{Ham}^c(M)$.

3.1 Construction of optimal functions

Proposition 3.1 *Let M be any symplectic manifold and $H : M \rightarrow \mathbf{R}$ be any compactly supported Hamiltonian with no non-constant closed trajectory in time less than 1. Then*

$$c_{HZ}(H) \geq \mathcal{L}(H).$$

Proof: Fix some small $\nu > 0$. We must show that $c_{HZ}(R_H^-(\nu/2)) \geq \max H$, where H is rescaled so that $\min H = 0$. Set $\max H = m$ and consider the following space

$$S_{H,\nu/4} = \{(x, \rho, t) \in M \times D^2(m + \nu/4) \mid 0 \leq \rho \leq H(x) + \nu/4\}.$$

Here (ρ, t) are the action-angle coordinates of the disc, and are related to polar coordinates (r, θ) by

$$\rho = \pi r^2/2, \quad \theta = 2\pi t.$$

It is very easy to see that $S_{H,\nu/4}$ can be embedded in $R_H^-(\nu/2)$ by a symplectic map which moves points only in the \mathbf{R}^2 -factor, that is to say by the restriction to $S_{H,\nu/4}$ of a map of the form $\text{id} \times \phi : M \times \mathbf{R}^2 \rightarrow M \times \mathbf{R}^2$. Thus it suffices to show that $c_{HZ}(S_{H,\nu/4}) \geq m$. Define the function $K : S_{H,\nu/4} \rightarrow \mathbf{R}$ by

$$K(x, \rho, t) = -H(x) + m + \rho = -H(x) + m + \pi r^2.$$

Then clearly K is a smooth positive function which has no non-constant closed trajectory in time less than 1, and is constant on the boundary of $S_{H,\nu/4}$. But $\|K\| = m + \nu/4$ which shows that $c_{HZ}(S_{H,\nu/4}) \geq m$, at least when M is compact. If M is non-compact, and since $\text{supp}H$ is compact, the above function K is not equal to $m + \nu/4$ outside a compact subset of $S_{H,\nu/4}$. This is because K is non-constant on $(M - \text{supp}H) \times D^2(\nu/4)$. But we can smooth out K to the constant value $m + \nu/4$ on that subset. Clearly, this can be done without creating a closed trajectory in time less than 1 as soon as ν is small enough. \square

3.2 Local embeddings

Recall that a path $H_{t \in [0,1]}$ is called quasi-autonomous if it has a fixed minimum and a fixed maximum.

Lemma 3.2 *There is a C^2 -neighbourhood \mathcal{U} of 0 in the space of Hamiltonians $M \times [0, 1] \rightarrow \mathbf{R}$ such that*

$$c_G(H) = \mathcal{L}(H_t)$$

for all quasi-autonomous paths $H_{t \in [0,1]}$ in \mathcal{U} .

Proof: Let P and p be fixed extrema of H_t . After reparametrization, we may assume that $\min H_t = 0$. As before, let $\mu_H(t) = \max_x H_t(x)$ and set

$$m = \int_0^1 \mu_H(t) dt = \mathcal{L}(H_t).$$

We will show how to embed the ball $B^{2n+2}(m - \varepsilon)$ in R_H^- for all $\varepsilon > 0$. The argument for R_H^+ is similar.

We will construct a map $\tilde{f} : B^{2n+2}(m - \varepsilon) \rightarrow R_H^-$ such that the following diagram commutes:

$$\begin{array}{ccc} B^{2n+2}(m - \varepsilon) & \xrightarrow{\tilde{f}} & R_H^\pm \subset M \times \mathbf{R}^2 \\ \pi \downarrow & & \downarrow \pi' \\ B^{2n}(m - \varepsilon) & \xrightarrow{f} & M \end{array}$$

where the vertical arrows are the obvious projections. (Such fibered embeddings were considered also in [7].) Observe that if $x \in \partial B^{2n}((1 - c)(m - \varepsilon))$ then the fiber $\pi^{-1}(x)$ is the 2-disc $B^2(c(m - \varepsilon))$, which sits inside $B^2(m - \varepsilon)$. Hence we may construct \tilde{f} as the product of a symplectic embedding f with a suitable area-preserving embedding ψ of the 2-disc $B^2(m - \varepsilon)$ into \mathbf{R}^2 .

To construct f , observe first that if H_t is C^2 -small, there clearly is a symplectic embedding $f : B^{2n}(m - \varepsilon) \rightarrow M$ which takes 0 to the fixed maximum P and is such that, for each $c \in [0, 1]$,

$$f(B^{2n}((1 - c)(m - \varepsilon))) \subset H_t^{-1}[c\mu_H(t), \mu_H(t)], \text{ for all } t \in [0, 1].$$

Then, if $x \in \partial B^{2n}((1 - c)(m - \varepsilon))$, the fiber $(\pi')^{-1}(f(x))$ contains the set

$$U_c = \{(s, t) \in \mathbf{R} \times [0, 1] : 0 \leq s \leq c\mu_H(t)\}.$$

Since the area of U_c is greater or equal to cm and $\varepsilon > 0$, it is easy to see that there is an area-preserving embedding ψ of the 2-disc $B^2(m - \varepsilon)$ into U_1 which takes $B^2(c(m - \varepsilon))$ into U_c for each $c \in [0, 1]$. This completes the proof. \square

Remark 3.3 Note that because the Gromov capacity c_G is the smallest capacity, the above lemma proves that $c_{HZ}(H) \geq \mathcal{L}(H_t)$ too.

Here is an immediate corollary.

Corollary 3.4 *Let $\{H_t\}$ be a Hamiltonian generating a regular path, which has at least one fixed minimum and one fixed maximum at each moment. Then, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$,*

$$c_G(\varepsilon H_{a+t\varepsilon}) = \mathcal{L}(\varepsilon H_{a+t\varepsilon}).$$

Here, of course, $H_{a+t\varepsilon}$ is short for $H_{a+t\varepsilon}$, $t \in [0, 1]$. We can now prove:

Corollary 3.5 *Theorem 1.2 holds.*

Proof: Suppose that $H_{t \in [0,1]}$ is a Hamiltonian which has a fixed maximum and a fixed minimum at each moment, and let ε_0 be as above. Then we claim that, whenever $0 < b - a \leq \varepsilon_0$, the path $\phi_{t \in [a,b]}$ is a stable geodesic. Since its generating Hamiltonian is just $(b - a)H_{a+t(b-a)}$, the above corollary shows that condition (i) of Proposition 2.2 holds for $c = c_G$. If we take \mathcal{C} to be a sufficiently small neighbourhood of $\phi_{t \in [a,b]}$, then condition (ii) holds also by Lemma 2.6 and Theorem 1.16. Therefore, the result follows from Proposition 2.2. \square

3.3 Global embeddings when $\dim M = 2$

We discuss the construction of symplectic embeddings for Hamiltonians defined on surfaces. Although the method of estimating $c_{HZ}(H)$ given in §3.1 is easier, one sometimes obtains sharper estimates by using embedded balls: see Remark 5.6. However, this section is not used in the applications in §5, except for the S^2 case of Theorem 5.4.

Trapezoids

It is often convenient to think of embedding linear shapes such as trapezoids instead of balls. The following lemma shows that the embedding problems for these different shapes are essentially the same.

As above, we write $B^{2n}(c)$ for the ball with capacity c . Let $T^{2n+2}(a)$ be the trapezoid:

$$T^{2n+2}(a) = \{(x, u, v) \in \mathbf{R}^{2n+2} : 0 \leq u \leq a, 0 \leq v \leq 1, x \in B^{2n}(a - u)\}.$$

We think of $T^{2n+2}(a)$ as fibered over the (u, v) rectangle $R(a)$, with suitable balls as fibers. Similarly, $B^{2n+2}(c)$ fibers over $B^2(c)$ with balls of varying size as fibers. The embeddings which we now construct will preserve this fibered structure.

Lemma 3.6 *For all $\varepsilon > 0$,*

- (i) $B^{2n+2}(a)$ embeds symplectically in $T^{2n+2}(a + \varepsilon)$,
- (ii) $T^{2n+2}(a)$ embeds symplectically in $B^{2n+2}(a + \varepsilon)$.

Proof: $B^{2n+2}(a)$ fibers over the disc $B^2(a)$. Let $h_B : B^2(a) \rightarrow \mathbf{R}$ be the function which at the point (u, v) equals the capacity of the fiber at that point. So $h_B(u, v) = a - \pi(u^2 + v^2)$. Let h_T be the similar function for $T^{2n+2}(a)$ (defined over the rectangle $R(a)$). To define an embedding $B^{2n+2}(a) \rightarrow T^{2n+2}(a + \varepsilon)$ which is the identity on the fibers, it clearly suffices to find an area preserving map $f : B^2(a) \rightarrow R(a + \varepsilon)$ such that

$$h_T(f(u, v)) \geq h_B(u, v).$$

It is not hard to see that such a map f exists. For example, one can construct f as follows. It should take $p = (-\sqrt{a/\pi}, 0)$ to the vertex $(0, 0)$ of $R(a + \varepsilon)$. Further, if γ_μ is a family of disjoint loops based at p which go close to the ray $y = 0$ then round the circle centered at the origin with radius μ and then back again close to the ray, f should take this family to an appropriate family of loops in the rectangle. These loops exist because, for all λ ,

$$\text{area} \{(u, v) : h_B(u, v) \geq \lambda\} \leq \text{area} \{(u, v) : h_R(u, v) \geq \lambda\}.$$

One also uses the fact that these sets are connected. This shows (i).

The proof of (ii) is similar. □

We now apply these ideas to calculating $c_G(H)$. We will give a prototype result here, with only a partial proof. Full details of the proof and extensions of this result may be found in [15].

Proposition 3.7 *Let (M, ω) have dimension 2, and let H be an autonomous Hamiltonian with time-1 map ϕ . Suppose that H has no non-constant t -periodic orbits for $0 < t < 1$. Then*

$$c_G(H) = \|H\|.$$

Proof: We will prove this only in the special case when there is a path β in M from an absolute maximum P to an absolute minimum p of H along which H decreases. In other words, we assume that the gradient flow of H with respect to a generic metric on M has a trajectory going from P to p . We will renormalise H so that $H(P) = m, H(p) = 0$. Thus $\|H\| = m$.

We will show that, given any $\varepsilon > 0$, we can embed the trapezoid $T = T^4(m - \varepsilon)$ under the graph of H in $X = M \times [0, m] \times [0, 1]$. By Lemma 3.6, this will imply that $c_G(R_H^-) = m$. By replacing H with $m - H$, we obtain $c_G(R_H^+) = m$. Hence

$$c_G(H) = \min\{c_G(R_H^-), c_G(R_H^+)\} = m$$

as required.

Let ξ_H be the Hamiltonian vector field on M corresponding to H . By slightly perturbing H , we may assume that ξ_H has a finite number of critical points. Let M' be M minus these critical points. Then the given area form σ on M may be written as $\alpha \wedge dH$ on M' where α is a 1-form such that $\alpha(\xi_H) = 1$. Observe that, because H has no periodic orbits of period < 1 , each level set of H has ξ_H -length at least 1. Thus, the integral of α round each level surface of H is at least 1 and so the area of Σ is at least m .

We will construct a fibered embedding of $\text{Int } T$ into X which covers an embedding $f : \text{Int } R(a) \rightarrow M'$. We define f as follows. Parametrize the path β from P to p so that

$$H(\beta(t)) = m - t, \quad t \in [0, m],$$

and define f along the axis $v = 0$ by

$$f(u, 0) = \beta(\varepsilon/2 + u), \quad u \in [0, m - \varepsilon].$$

Then define f so that it takes each arc $u = c$ into the level set $H = m - \varepsilon/2 - c$ in such a way that $\frac{\partial}{\partial v}$ goes to ξ_H . This is an embedding because the arcs $H = \text{const}$ all have ξ_H -length at least 1. Moreover, it preserves the u, v area, since dH pulls back to $-du$.

Observe that the fiber in X over the point $f(u, v)$ is the rectangle $R_c = [0, c] \times [0, 1]$ where $c = m - u - \varepsilon/2$. It is not hard to see that there is an area preserving map

$$g : B^2(m - \varepsilon) \rightarrow R_{m-\varepsilon/2}$$

which takes $B^2(c - \varepsilon/2)$ into R_c for all c . Then the product

$$g \times f : T \rightarrow X$$

is the desired embedding. □

4 Generalised capacity-area inequalities

We now investigate conditions on M and classes \mathcal{C} so that capacity-area inequalities hold for spaces of the form $R_{H,K}(\nu)$ where $H, K \in \mathcal{C}$. When combined with Proposition 2.2 this will prove Theorem 1.3.

We begin by considering the Gromov capacity. The proof in [7] of the Non-Squeezing Theorem in all manifolds does not extend to quasi-cylinders Q , the main difficulty being that, if one considers Q lying inside $M \times \mathbf{R}^2$, it is not clear that one can disjoin Q from itself with energy equal to $\text{area}(Q)$. But, fortunately, a holomorphic argument, similar to the one contained in Proposition 3.1 of [7], can be applied here (see Proposition 4.3 below). We begin with a different argument, where we first extend the quasi-cylinder and then trivialise the extension.

Proposition 4.1 *Let $Q = (M \times D^2, \Omega)$ be a quasi-cylinder. Then the Non-Squeezing Theorem holds whenever M has bounded geometry at infinity and satisfies one of the following conditions*

- (i) $\dim(M) \leq 4$;
- (ii) $(M \times S^2(a), \omega \oplus \sigma)$ is weakly monotone for all $a > 0$.
- (iii) there are no spherical homology classes $A \in H_2(M)$ such that

$$\omega(A) > 0, \quad 2 - n \leq c_1(A) \leq 0,$$

Remark 4.2 The condition in (iii) above is clearly satisfied when M is semi-monotone. However, it is also satisfied when the minimal Chern number (the smallest positive value of c_1 on spherical homology classes) is $\geq n - 2$, for instance when $M = S^2 \times S^2 \times S^2$, and for various other manifolds.

Proof: We will begin with the proof of the sufficiency of condition (ii). Since manifolds of dimension ≤ 6 are automatically weakly monotone, this will imply the sufficiency of condition (i).

Proof of the sufficiency of condition (ii) of Proposition 4.1.

Let us assume, by contradiction, that Q contains a ball of capacity $c > A$, and consider its split expansion $Q(\kappa)$ defined in Lemma 2.7. Then, by construction, $Q(\kappa)$ contains a disjointly embedded ball $B^{2n+2}(c)$ and cylinder $M \times B^2(\kappa)$, where $c + \kappa > A + \kappa$. We now use the theory of J -holomorphic curves to show that this is impossible. We assume that the reader is familiar with the basics of this theory, as explained in [7, §3] or [12, Chapter 1], for example. This theory tells us that, given any Ω -tame almost complex structure J on the compactification $Q^*(\kappa) = M \times S^2(A + \kappa)$ and any point $x \in Q^*(\kappa)$, there is a J -holomorphic curve (or cusp-curve) C through x , which represents the homology class of $pt \times S^2$ and so satisfies

$$\int_C \Omega = A + \kappa.$$

(It is at this point that we use the hypothesis that $M \times S^2(A + \kappa)$ is weakly monotone. For general manifolds, it is not known whether the above statement is true for any J .)

Let us apply this when J restricts to the standard complex structure J_0 on the ball B and to a product structure J_{spl} on the cylinder Cyl , and let us take the point x to be the center of the ball. Then, as usual, it follows by Gromov's monotonicity argument that

$$\int_{C \cap B} \Omega \geq c.$$

We claim that

$$\int_{C \cap \text{Cyl}} \Omega \geq \kappa.$$

To see this, observe that the projections $\pi_D : \text{Cyl} \rightarrow D = B^2(\kappa)$ and $\pi_M : \text{Cyl} \rightarrow M$ are both holomorphic. Thus, if σ denotes the area form on D ,

$$\begin{aligned} \int_{C \cap \text{Cyl}} \Omega &= \int_{\pi_D(C \cap \text{Cyl})} \sigma + \int_{\pi_M(C \cap \text{Cyl})} \omega \\ &\geq \int_{\pi_D(C \cap \text{Cyl})} \sigma \\ &\geq \kappa, \end{aligned}$$

where the first inequality holds because $\pi_M(C \cap \text{Cyl})$ is a holomorphic curve in M and the second holds because $\pi_D : C \cap \text{Cyl} \rightarrow D$ is surjective. (C must intersect every fiber $M \times pt$ of $X \rightarrow S^2$ for topological reasons.)

Since J is Ω -tame, the integral of Ω over the piece of C which is outside $B \cup \text{Cyl}$ is positive. Therefore,

$$\int_C \Omega > \int_{C \cap B} \Omega + \int_{C \cap \text{Cyl}} \Omega \geq c + \kappa > A + \kappa = \int_C \Omega,$$

which is impossible.

Proof of the sufficiency of condition (iii) of Proposition 4.1. Here we need to control the degeneracies of the pseudo-holomorphic sphere. We do this by considering a special (not necessarily generic) path of almost complex structures. Let J_t be a path of Ω -tame almost complex structures on $Q^*(\kappa)$ such that J_0 is the split structure and J_1 is standard with respect to the ball and to the cylinder, and such that one fiber $M \times pt$ is J_t -holomorphic for all $t \in [0, 1]$. (It is easy to find such a path.) It suffices to show that there is a J_1 -holomorphic curve in class $\{pt\} \times S^2$ through the center p of B . We first show that along such a path of almost complex structures, only very special degeneracies can occur. Indeed, let C be any J_t -holomorphic cusp-curve

$$C = C_1 \cup \dots \cup C_k$$

in class $[\{pt\} \times S^2]$. We may decompose the homology class of C_i as

$$[C_i] = p_i [pt \times S^2] + Z_i, \quad \text{where } Z_i \in H_2(M).$$

Since every intersection of C_i with the J_t -holomorphic fiber $M \times pt$ contributes positively to the intersection number $p_i = [C_i] \cdot [M]$, we must have $p_i \geq 0$ for all i . Hence $p_1 = 1$ and $p_i = 0, i \geq 2$, so $[C_i] \in H_2(M)$ for $i \geq 2$. Further, because J_t is Ω -tame,

$$0 < \int_{C_i} \Omega = \int_{C_i} \omega.$$

for all $i \geq 2$.

We claim that a similar statement holds for any path J'_t which is sufficiently close to J_t . For otherwise, there would be a sequence of cusp-curves $C^\nu = C_1^\nu \cup \dots \cup C_{k_\nu}^\nu$ converging to a J_t -holomorphic

cuspidal curve $C = C_1 \cup \dots \cup C_k$ with $[C - i] \in H_2(M)$ for $i \geq 2$. But the compactness theorem implies that (after taking an appropriate subsequence) the components of C^ν converge to unions of components of C . Hence for large ν , all but one of the classes $[C_i^\nu]$ must belong to $H_2(M)$. Therefore, we may assume that the path J_t' is a generic path which joins the regular element J_0 to a point J_1' which is very close to J_1 . Then the hypothesis (iii) on M implies that the only classes $[C_i] \in H_2(M)$ which have J_t' -holomorphic representatives in $Q^*(\kappa)$ for some t are those with $c_1 \geq 0$ since these are the ones for which the corresponding moduli space has non-negative dimension. (The number 4 rather than 3 was used in (iii) to compensate for the fact that $\dim Q^*(\kappa) = \dim M + 2$.)

Thus none of the J_t' -holomorphic cuspidal curves contain multiply-covered components of negative Chern number. Therefore, by the remarks in [7, §3], there is a J_1' -curve in class $[pt \times S^2]$ passing through any generic point arbitrarily close to p . Using once again the compactness theorem, we conclude that there must be a J_1 -curve in class $[pt \times S^2]$ passing through p . \square

Our last global Non-Squeezing Theorem holds for rational manifolds. Let r be the index of rationality of M . For the convenience of the reader we restate Proposition 1.20.

Proposition 4.3 *Let $\Omega_{s \in [0,1]}$ be a smooth 1-parameter family of symplectic forms on $M \times D$ such that each manifold $Q_s = (M \times D, \Omega_s)$ is a quasi-cylinder. Assume that Q_0 is split.*

- (i) *If $\text{area}(Q_s) < 2r$ for all s , then the non-squeezing theorem holds for Q_1 ;*
- (ii) *If there is a smooth symplectic isotopy*

$$g_s : B^{2n+2}(c) \rightarrow Q_s, \quad 0 \leq s \leq 1$$

which at time $s = 0$ has the form

$$B^{2n+2}(c) \hookrightarrow B^{2n}(c) \times D(c) \xrightarrow{f \times g} M \times D$$

where $c \geq 2r$, and if

$$\text{area } Q_s < c + 2r, \quad 0 \leq s \leq 1,$$

then $\text{area } Q_1 \geq c$.

Proof: We begin with the proof of (ii). The idea is to use the embedded ball in the quasi-cylinders in order to control the area of the components of any possible cuspidal curve. Actually, we will combine the control given by these embedded balls to the one provided by the positivity of intersection with a given fixed J -invariant fiber.

Suppose, by contradiction, that $\text{area } Q_1 < c$. Denote by Q_s^* the S^2 -compactification of Q_s obtained by attaching a cylinder $(M \times D, \omega \oplus \sigma)$ of area a to the boundary of Q_s , where a is small enough so that $A_s = \text{area}(Q_s^*) < c + 2r$ and $A_1 < c$. We still denote by Ω_s the form on Q_s^* . Let U be a small neighbourhood of the point p_∞ , the center of D , sitting in the interior of D . It is enough to produce a J -A-curve passing through $p = g_1(0)$, where J extends the standard structure on $g_1(J_0)$ and is split on the open subset $W = M \times U$. Let J_s , $0 \leq s \leq 1$, be a smooth (not necessarily generic) one-parameter family of almost complex structures tamed by Ω_s , beginning with the split (regular) structure and

ending at $J_1 = J$, such that J_s be split on W and extends for each s the structure $g_s(J_0)$. Then any J_s - A -cusp-curve passing through $p_s = g_s(0)$ must be of type

$$A = (A - B) + B \quad \text{or} \quad A = (A - B) + B_1 + B_2$$

for some classes $B, B_1, B_2 \in H^2(M)$. Indeed, the positivity of intersection implies as before that $A = (A - B) + \sum_i B_i$, but the component that passes by p_s must have area at least c . There remains then only a total area less than $2r$ for all other components, which easily implies the above cusp-curve decomposition. Because $A - B$ is a primitive class, it cannot be a multiple covering. Consider a B -component: if it passes through p_s , and is a multiple covering, its area would be at least $2c$, and therefore would be greater than $\text{area } Q_s^*$, since $c \geq 2r$ and $\text{area } Q_s^* < c + 2r$. If it does not pass through p_s , its area is smaller than $2r$ and therefore cannot be a multiple covering either. Hence no component can be a multiple covering. As above, we conclude that there are J - A -curves passing through p .

The proof of (i) is easier and left to the reader.

We now turn to c_{HZ} -area inequalities for quasi-cylinders.

Proposition 4.4 *Let M be weakly exact. Then the c_{HZ} -area inequality holds for all quasi-cylinders $Q = (M \times D^2, \Omega)$.*

Proof: Assume by contradiction that $c_{HZ}(Q) > \text{area}(Q)$, and let $H : Q \rightarrow [0, c]$ be such that $H = c > \text{area}(Q)$ on ∂Q and outside a compact subset, vanishes somewhere inside Q and has no non-constant closed trajectory in time < 1 . Let ε be smaller than $c - \text{area}(Q)$. Then Q sits in the interior of the cylinder $Q(\kappa + \varepsilon)$ (here $Q(\kappa)$ is defined as in Lemma 2.7 and the ε -part ensures that Q is in the interior of $Q(\kappa + \varepsilon)$). Then extend H to a Hamiltonian $\tilde{H} : Q(\kappa + \varepsilon) \rightarrow [0, \infty)$ which equals $c + \kappa$ on $\partial Q(\kappa + \varepsilon)$. Because $Q(\kappa + \varepsilon) - Q$ is split and has area $\kappa + \varepsilon$, one can choose such an extension so that \tilde{H} has no non-constant closed trajectory in time < 1 . But then $c_{HZ}(Q(\kappa + \varepsilon)) \geq c + \kappa > \text{area}(Q(\kappa + \varepsilon))$, which contradicts Theorem 1.17. \square

Remark 4.5 Observe that because κ may be arbitrarily large here, we cannot say anything about manifolds with finite index of rationality unless $M = S^2$. In this case, by Lemma 2.9, the above result holds for all quasi-cylinders of area less than or equal to the area of S^2 because these cylinders all split.

5 Applications to Hofer's geometry

We now apply the results of §3 and §4 to derive the theorems on length-minimizing paths in Hofer's metric. Our tool to do this is Proposition 2.2. We first discuss general results with fixed endpoints (global with respect to time), and then give results which hold at each moment (local with respect to time). We then turn to specific results, linear rigidity, and a brief discussion of Hofer's diameter. Finally, we prove the local flatness result Proposition 5.10.

5.1 Length-minimizing paths: general results

To begin, here is a slightly more general version of Theorem 1.3.

Theorem 5.1 *Let M have dimension ≤ 4 or be semi-monotone (or, more generally, satisfy condition (iii) in Proposition 4.1). Assume further in the non-compact case that it has bounded geometry at infinity. Let $H_{t \in [0,1]}$ be any path with*

$$c_G(H) = \mathcal{L}(H_t).$$

Then $H_{t \in [0,1]}$ is length-minimizing amongst all paths homotopic (with fixed endpoints) to $H_{t \in [0,1]}$. The same conclusion holds if $c_{HZ}(H) = \mathcal{L}(H_t)$ when M is weakly exact.

This is an obvious consequence of Propositions 2.2, 4.1 and 4.4. The next result talks about standard balls in $R_H^\pm(\nu/2)$. These are balls which are embedded by a map which is the restriction to $B^{2n+2}(r)$ of a product embedding:

$$B^{2n}(r) \times B^2(r) \rightarrow M \times U_H \subset M \times \mathbf{R}^2.$$

Theorem 5.2 *Let M be a rational symplectic manifold of rationality index r with bounded geometry at infinity. Let $H_{t \in [0,1]}$ be any path with*

$$c_G(H) = \mathcal{L}(H_t).$$

Then:

- (i) *In the case $\mathcal{L}(H_t) < 2r$, $H_{t \in [0,1]}$ is length-minimizing amongst all paths which are homotopic (rel endpoints) to $H_{t \in [0,1]}$ through paths of lengths $< 2r$.*
- (ii) *In the case $\mathcal{L}(H_t) \geq 2r$, assume that, for all small $\varepsilon > 0$, there exist an embedded ball B of capacity $\geq \mathcal{L}(H_t) - \varepsilon$ in R_H^+ and a number $0 < \nu < 2r$ such that B is isotopic inside $R_H^+(\nu/2)$ to a standard ball in $R_H^+(\nu/2)$. Suppose that the same holds for R_H^- . Then $H_{t \in [0,1]}$ is length-minimizing amongst all paths which are homotopic (rel endpoints) to $H_{t \in [0,1]}$ through paths of lengths $< \mathcal{L}(H_t) + 2r$.*

Proof: (i) Let $\mathcal{L}(H) = \mathcal{L}(H_t) < 2r$ and $\psi_{t,s \in [0,1]}$ be a homotopy of paths generated by K^s , $s \in [0, 1]$ of length

$$\mathcal{L}(K^s) < 2r.$$

By Proposition 2.2, it is enough to show that the Non-Squeezing Theorem holds for the quasi-cylinders $R_{H,K^1}(\nu)$, $R_{K^1,H}(\nu)$, at least when ν is sufficiently small. We prove this using Proposition 4.3 as follows.

Suppose, by contradiction, that for all small ν , there is an embedded ball B_ν of capacity c_ν in say $R_{H,K^1}(\nu)$ with

$$\text{area}(R_{H,K^1}(\nu)) < c_\nu.$$

Choose ν small enough so that $\|K^s\| + 2\nu < 2r$ for all s , and fix once and for all that value ν . Set $c = c_\nu$, $Q_s = (R_{H,K^s}(\nu), \Omega)$ and $Q'_s = (R_{K^s,H}(\nu), \Omega)$. Up to a smoothly varying symplectomorphism, we have:

$$Q_s = (M \times D^2, \Omega_s) \quad Q'_s = (M \times D^2, \Omega'_s).$$

Since $\text{area } Q_1 < c$, Proposition 4.3 (i) implies that there exists some s such that $\text{area } Q_s \geq 2r$. Let \bar{s} be the smallest value of s for which such an inequality holds for either Q_s or Q'_s . Suppose that this happens first for Q say. This means that

$$\frac{1}{\text{vol } M} (\text{vol}(R_H^-) + \text{vol}(R_{K^{\bar{s}}}^+)) + \nu \geq 2r$$

or equivalently:

$$\lambda \mathcal{L}(H) + \mu \mathcal{L}(K^{\bar{s}}) + \nu \geq 2r \tag{1}$$

for some $\lambda, \mu \in (0, 1)$. Because \bar{s} is the smallest such value,

$$\text{area}(Q'_s) < 2r$$

for all $s < \bar{s}$. But, since R_H^+ embeds in Q'_s , there exist embedded balls of capacity arbitrarily close to $\mathcal{L}(H)$ inside Q'_s . Therefore, using again Proposition 4.3 (i), we must have

$$\text{area}(Q'_{\bar{s}}) \geq \mathcal{L}(H),$$

or equivalently:

$$(1 - \lambda) \mathcal{L}(H) + (1 - \mu) \mathcal{L}(K^{\bar{s}}) + \nu \geq \mathcal{L}(H). \tag{2}$$

Then the sum of inequalities (1) and (2) gives

$$\begin{aligned} \mathcal{L}(H) + \mathcal{L}(K^{\bar{s}}) + 2\nu &\geq \mathcal{L}(H) + 2r \\ \mathcal{L}(K^{\bar{s}}) + 2\nu &\geq 2r \end{aligned}$$

a contradiction.

(ii) This is a consequence of Proposition 1.20 (ii). The proof, similar to the proof of (i), is left to the reader. □

The results of §3 and §4 also lead to the following theorem stating that geodesics minimize the length at each moment in a strong sense.

Theorem 5.3 *Let M be a symplectic manifold with bounded geometry at infinity, and let $\phi_{t \in [0,1]}$ be a geodesic.*

(i) *Assume that M is semi-monotone. Then $\phi_{t \in [0,1]}$ minimizes length at each moment amongst all homotopic paths. More precisely, each $s \in [0, 1]$ has a closed connected neighbourhood $\mathcal{N}(s)$ such that the path $\phi_{t \in \mathcal{N}(s)}$ is length-minimizing amongst all homotopic paths.*

(ii) Assume that M is rational. Then each $s \in [0, 1]$ has a closed connected neighbourhood $\mathcal{N}(s)$ such that for all closed subintervals \mathcal{N} containing s the path $\phi_{t \in \mathcal{N}}$ is length-minimizing amongst all paths which are homotopic to $\phi_{t \in \mathcal{N}}$ through paths of lengths $< 2r$.

The statement (i) is a consequence of Theorem 1.3 and Corollary 3.4. The second one is a consequence of Theorem 5.2 (i) and Corollary 3.4.

5.2 Length-minimizing paths: specific results

We now combine the results of the previous sections to estimate the Hofer norm $\|\phi\|$ of certain elements $\phi \in \text{Ham}^c(M)$. We begin with the proof of:

Theorem 5.4 *Let M be a weakly exact manifold or any surface, and let H be an autonomous Hamiltonian with time-1 map ϕ . Suppose that H has no non-constant closed trajectory in time less than 1. Then $\phi_{t \in [0,1]}$ is length-minimizing in its homotopy class.*

Proof: This is an immediate consequence of Theorem 1.3 and Proposition 3.7 for surfaces and of Theorem 1.3 and Proposition 3.1 for the weakly exact case. \square

Proof of Corollary 1.10. When M is a surface and has genus > 0 , the group $\text{Ham}(M)$ is contractible, and when M is exact and convex, $\text{Ham}^c(M)$ has $r_1 = \infty$. Thus in these cases the result follows from Theorem 5.4.

However, $\text{Ham}(S^2)$ deformation retracts to $\text{SO}(3)$. In particular, its fundamental group has order 2, and is generated by a full rotation around a fixed axis. Thus the isotopies from $\mathbb{1}$ to ϕ divide into two homotopy classes: those which are homotopic to the flow $\phi_{t \in [0,1]}$ of H , and those which are not. Theorem 1.3 and Proposition 3.7 tell us that all isotopies in the former class have length $\geq \|H\|$. (See also Remark 4.5.) It therefore remains to consider the length of isotopies in the latter class. But, it follows from Lemma 5.5 below that, for such isotopies ψ_t ,

$$\mathcal{L}(\phi_{t \in [0,1]}) + \mathcal{L}(\psi_{t \in [0,1]}) \geq \text{area } S^2.$$

Therefore, if $\mathcal{L}(\phi_{t \in [0,1]}) = \|H\|$ is no more than half the area, it has to be minimal. \square

Lemma 5.5 *Let $\phi_{t \in [0,1]}$ be an essential loop in $\text{Ham}(S^2)$, where S^2 has area A . Then $\mathcal{L}(\phi_{t \in [0,1]}) \geq A$.*

Proof: Let $F : S^2 \rightarrow \mathbf{R}$ be the composite of the height function on S^2 with a linear map so that $\|F\| = A$. It is easy to check that its flow is exactly a full rotation about the poles. Since its orbits all have period exactly 1, Proposition 3.7 shows that, for any $\varepsilon > 0$, one can embed a ball of capacity $A - \varepsilon$ on both sides of $gr(F)$. (In fact, this is another description of the full filling of $S^2 \times D^2$ by 2 balls which was given in [17].) Thus, by Theorem 1.3,

$$\mathcal{L}(\phi_{t \in [0,1]}) \geq A. \quad \square$$

Remark 5.6 Proposition 3.7 shows that in dimension 2 the condition “ $c_G(H) = \|H\|$ ” is no weaker than the condition “ H has no non-constant periodic orbit in time < 1 ”. Actually, it is strictly stronger, even on \mathbf{R}^2 . Consider any positive (autonomous) bump function H with support of area less than half the area of M , but with $\text{area}(H^{-1}(\max H)) > \max H$. Then $c_G(H) = \|H\|$ because one can then embed a ball of capacity $\|H\|$ on both sides of the graph of H , and our argument shows that the flow $\phi_{t \in [0,1]}$ of H is length-minimizing. But H does have non-constant periodic orbits in time less than 1 as soon as the slope of the bump near the boundary of its support is high enough.

5.3 Hofer’s diameter

We briefly discuss questions concerning the diameter of $\text{Ham}^c(M)$ under the Hofer norm $\|\cdot\|$. First observe that if (M, ω) is exact and of finite volume, it follows easily from the existence of the Calabi homomorphism

$$\text{Cal} : \text{Ham}^c(M) \rightarrow \mathbf{R}$$

that this diameter is infinite.

Now the argument in Remark 5.6 shows that the diameter of $\text{Ham}^c(M)$, for a geometrically bounded semi-monotone manifold M with $r_1 = \infty$ containing two disjoint embedded copies of $B^{2n}(c)$, is at least equal to c . In some cases it is enough to know that a covering of M contains such balls. The next lemma shows that the diameter of $\text{Ham}(T^2)$ is infinite. For simplicity, the argument uses the energy-capacity inequality, but it could equally well be phrased in terms of embedded balls.

Lemma 5.7 *There is a function H on T^2 whose flow $\{\phi_t^H\}_{t \geq 0}$ is such that*

$$\mathcal{L}(\phi_t^H) = t \text{TotVar}(H), \quad \text{for all } t \geq 0.$$

Proof: Consider a Hamiltonian of the form $H(x, y) = f(x)$, where

- (i) $f(0) = f'(0) = 0$, and
- (ii) f increases over $0 \leq x \leq 1/2$ to $f(1/2) = 1$ and then decreases to 0 at $x = 1 = 0$.

If $\phi_t, t \geq 0$, is its flow, let $\tilde{\phi}_t$ be the unique lift of ϕ_t to \mathbf{R}^2 which starts at $\mathbf{1}$. Then

$$\tilde{\phi}_t(x, y) = (x, y + tf'(x)).$$

Hence in time T this isotopy disjoins a region U_+ in the strip $0 < x < 1/2$, which is diffeomorphic to a disc and has area almost equal to $T(f(\frac{1}{2}) - f(0)) = T$ and a similar region U_- in the strip $1/2 < x < 1$.

Let K_t be any Hamiltonian on T^2 with time-1 map ϕ_T . Then its lift \tilde{K}_t to \mathbf{R}^2 has time 1-map $\tilde{\psi} = \tilde{\phi}_T + c$ for some $c \in \mathbf{Z}^2$. Since $\tilde{\phi}_T$ moves U_+ upwards and U_- downwards, $\tilde{\psi}$ must disjoin at least one of these. But the energy-capacity inequality of [4, 7] states that any map ψ which disjoins a disc of area c must have norm $\|\psi\|$ at least c . Thus

$$\mathcal{L}(\{\tilde{K}_t\}) = \mathcal{L}(\{K_t\}) \geq T,$$

as required. □

Corollary 5.8 *The norm on $\text{Ham}(T^2)$ is unbounded.*

A similar argument clearly applies to any compact Riemann surface Σ except the sphere. (Note that the universal cover of such Σ is still symplectomorphic to the plane \mathbf{R}^2 : its negative curvature is not reflected in its symplectic structure.) The argument also works for manifolds of the form $\Sigma \times M$, where M is any compact manifold whose universal cover has infinite Gromov capacity, since in this case again one can lift ϕ_t to an isotopy which disjoins an arbitrarily large ball.

In fact, the methods of [7] show that one can prove a similar result for $\Sigma \times M$ for any compact M provided one can show that there is a symplectic embedding

$$\iota : B^4(r) \times M \rightarrow B^2(R) \times \mathbf{R}^2 \times M.$$

only if $r \leq R$. This generalized non-squeezing theorem can be proved by adapting the methods of [7].

5.4 Linear rigidity

Siburg showed in [16] that when $M = \mathbf{R}^{2n}$ any path $\phi_{t \in [0,1]}$ such that ϕ_t has no non-constant closed trajectories for any $t \in [0, 1]$ is absolutely length-minimizing. (Such non-constant trajectories correspond precisely to fixed points of ϕ_t which are non-trivial in the sense that $\phi_s(x) \neq x$ for some $s \in [0, t]$.) Combining this with our necessary condition for a stable geodesic in Hofer's metric (see [8]) we obtain:

Theorem 5.9 (Linear rigidity) *Let $H_{t \in [0,1]}$ be a quasi-autonomous Hamiltonian on \mathbf{R}^{2n} , with say only one fixed minimum $p \in \mathbf{R}^{2n}$. If the linearised flow $d\phi_t(p)$ has a non-constant closed trajectory in time less than 1, so does the flow ϕ_t itself. The same conclusion still holds if $H_{t \in [0,1]}$ has a finite number of fixed extrema provided that the linearised flows at all fixed minima or at all fixed maxima have a non-constant closed trajectory in time less than 1.*

Proof: If the conclusion does not hold, then the path $\phi_{t \in [0,1-\varepsilon]}$ is length-minimizing by Siburg's criterion, for any choice of small $\varepsilon > 0$. It must then be a stable geodesic, which implies the non-existence of non-constant closed trajectories of the linearised flow in time $< 1 - \varepsilon$ by the necessary condition for stability of [8]. Since ε is arbitrarily small, this contradicts the hypothesis. \square

Note that we need no non-degeneracy condition of H_t at p . Actually, this theorem is non-trivial even in the autonomous case.

Thus the linearised flow controls the flow itself, or at least the behaviour of its closed trajectories. The above proof makes use of Siburg's criterion only to show that the given path is length-minimizing in its homotopy class. Thus a similar result would hold on any weakly exact manifold for which one can prove Proposition 3.1 for quasi-autonomous Hamiltonians.

We do not know whether the closed trajectory given by the theorem can be found in any arbitrarily small neighbourhood of p .

Of course, this linear rigidity is a symplectic phenomenon: on \mathbf{R}^2 , there are non-symplectic autonomous flows having no non-constant closed orbit and whose linearization at 0 is a pure rotation.

5.5 C^1 -flatness on manifolds without short loops

We will now calculate the Hofer norm on a neighbourhood \mathcal{U} of the identity in $\text{Ham}^c(M)$ assuming that M satisfies the conditions of the first part of Theorem 1.9 (M is semi-monotone and has no short loops). As in Lemma 2.6, we take \mathcal{U} to be a star-shaped neighbourhood of $\mathbb{1}$ in $\text{Ham}^c(M)$ consisting of Hamiltonian diffeomorphisms ψ whose graphs lie close enough to the diagonal $diag$ in $(M \times M, -\omega \oplus \omega)$ to correspond to an exact 1-form $\rho(\psi)$ in $(T^*M, -d\lambda_{\text{can}})$.² Then there is a function F_ψ on M which is unique up to a constant and such that $\rho(\psi) = dF_\psi$, and we claim:

Proposition 5.10 *Consider the family of 1-forms tdF_ψ , $0 \leq t \leq 1$, and let ψ_t be the corresponding isotopy in $\text{Ham}^c(M)$. Then, if F_ψ is sufficiently C^2 -small,*

$$\mathcal{L}(\psi_{t \in [0,1]}) = \sup_x F_\psi - \inf_x F_\psi$$

and each extremum of F is a fixed extremum of H_t . Here H_t is the Hamiltonian generating ψ_t .

Proof: Observe that the fixed points of ψ correspond precisely to the critical points of $F = F_\psi$ and hence are fixed by ψ_t for all t . Therefore, if F assumes its maximum and minimum values at P, p , these points are fixed points of the isotopy ψ_t . Let H_t be the Hamiltonian which generates ψ_t , and $\beta(s), 0 \leq s \leq 1$, be any path in M from $\beta(0) = p$ to $\beta(1) = P$. Then we claim that both $\mathcal{L}(\psi_t)$ and $\sup_x F_\psi - \inf_x F_\psi = F(P) - F(p)$ are equal to the area A swept out by the arc β under the isotopy ψ_t . More formally, this area can be written as

$$\int_{[0,1] \times [0,1]} \Psi^*(\omega),$$

where $\Psi(s, t) = \psi_t(\beta(s))$.

We will prove this in a slightly different form. Given any two critical points q_1, q_2 of F , let β be any arc in M from q_1 to q_2 and write $A(q_1, q_2)$ for the area swept out by β under the isotopy ψ_t . Then we will show that

$$F(P) - F(p) = \sup_{q_1, q_2} A(q_1, q_2) = \mathcal{L}(\psi_t).$$

To prove the statement about F observe that the path $(\beta(s), \psi_t(\beta(s)))$ on the graph of ψ_t corresponds to a path in T^*M from $(q_1, 0)$ to $(q_2, 0)$ which lies on the section of T^*M determined by the 1-form tdF_ψ . Thus as t varies in $[0, 1]$ they form a homotopy of paths (with fixed endpoints) in T^*M from the path β in the zero-section to the path $\psi(\beta)$ in graph (dF) . If we write Y' for the corresponding 2-cycle in T^*M , we find that, because $\lambda_{\text{can}} = dF$ on graph (dF) ,

$$\int_{Y'} -d\lambda_{\text{can}} = F(q_2) - F(q_1).$$

Now, look at the corresponding set Y'' traced out by the paths $(\beta(s), \psi_t(\beta(s)))$ in $(M \times M, -\omega \oplus \omega)$. Then, because Y'' lies over a 1-dimensional subset of the first factor of M , namely β , we find,

$$\int_{Y'} -d\lambda_{\text{can}} = \int_{Y''} -\omega \oplus \omega = \int_{[0,1] \times [0,1]} \Psi^*(\omega) = A(q_1, q_2).$$

²These sign conventions are the same as in [13]. More details of the arguments given here may be found in Chapter 9 of that book.

Since $F(P) - F(p) = \sup_{q_1, q_2} (F(q_2) - F(q_1))$, the result follows. (Note that this calculation shows that $A(q_1, q_2)$ is independent of the choice of β .)

To prove the statement for $\mathcal{L}(\psi_{t \in [0,1]})$, consider the graph of H_t :

$$\Gamma_H = \{(x, s, t) \in M \times \mathbf{R} \times [0, 1] \mid s = H_t(x)\}.$$

Let $Y \subset \Gamma_H$ be the union of all the characteristic lines in Γ_H starting at the points of β :

$$Y = \{(\psi_t \beta(s), H_t(\psi_t \beta(s)), t) \in \Gamma_H : s, t \in [0, 1]\}.$$

Then $\omega \oplus \sigma$ vanishes on Y because Y is a union of null lines. Hence

$$\int_Y \omega = - \int_Y \sigma.$$

But the first integral here is just $A(q_1, q_2)$, while the second is minus the area of the projection of Y onto \mathbf{R}^2 , i.e. minus the area

$$\text{area}_{t \in [0,1]} \{q_1, q_2\}$$

enclosed by the curves $H_t(q_2)_{t \in [0,1]}$ and $H_t(q_1)_{t \in [0,1]}$. Thus we have shown that for any pair of critical points of F

$$F(q_2) - F(q_1) = A(q_1, q_2) = -\text{area}_{t \in [0,1]} \{q_1, q_2\}.$$

Note that the corresponding inequality holds over any time interval $t \in [a, b] \subset [0, 1]$.

It is clear from the definition of H_t that for each t this function has the same critical points as F . (These are the infinitesimally fixed points of the isotopy ψ_t .) But the fact that the equality $(b - a)(F(q_2) - F(q_1)) = -\text{area}_{t \in [a,b]} \{q_1, q_2\}$ holds for all $0 \leq a \leq b \leq 1$ implies easily that any minimum q_t of some H_t must be a minimum of F too, and therefore a fixed minimum of the family $H_{t \in [0,1]}$. This is because the function F is independent of time and the left hand side of the above equation is maximized at the extrema of F . Since the same applies to maxima, this proves the last assertion of the Proposition. The first assertion follows at once:

$$\mathcal{L}(\phi_t) = \text{area}_{t \in [0,1]} \{q, Q\} = -A(P, p) = F(P) - F(p),$$

where q and Q are fixed minimum and maximum of H_t and P, p are the corresponding maximum and minimum of F . \square

Corollary 5.11 *If M is semi-monotone and has no short loops and if the neighborhood \mathcal{U} is sufficiently C^1 -small, then*

$$\|\psi\| = F(P) - F(p).$$

Proof: It is easy to check that if F is C^2 -small so is the generating Hamiltonian H_t of the isotopy ψ_t . By Lemma 3.2, if H_t is sufficiently C^2 -small then $\mathcal{L}(H_t) = c_G(H)$. Hence, by Theorem 1.9, the path H_t measures the length of ψ . Thus

$$\|\psi\| = \mathcal{L}(H_t) = F(P) - F(p). \quad \square$$

Remark 5.12 One of the key points in the proof of Proposition 5.10 is to understand the relation between the generating function F_ψ and the Hamiltonians H_t . This is discussed further in [13]. We have argued geometrically and somewhat indirectly here, but one could construct a proof along the lines of that in Bialy–Polterovich, using an appropriate generalization of the Hamilton-Jacobi equation.

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