DYNAMICS OF QUADRATIC POLYNOMIALS.
II. RIGIDITY.

MIKHAIL LYUBICH

1. Introduction

This is a continuation of the series of notes on the dynamics of quadratic polynomials. The first part of this series [L3] will be systematically used for the reference. In particular we will assume that the reader is familiar with the background outlined in §2 of Part I: quadratic-like maps, straightening, combinatorial classes, external rays, the Mandelbrot set M, secondary limbs, puzzle, etc.

Let f be a quadratic-like map which does not have non-repelling periodic points. Let us say that f satisfies the *secondary limbs condition* if there is a finite family of truncated secondary limbs L_i of the Mandelbrot set such that the hybrid classes of all renormalizations $R^m f$ belong to $\cup L_i$. Let $\mathcal{SL}$ stand for the class of quadratic-like maps satisfying the secondary limbs condition.

Here are some examples of maps of class $\mathcal{SL}$:

• Maps which are at most finitely renormalizable and don't have non-repelling periodic points (Yoccoz class);

• Infinitely renormalizable maps of bounded type ("bounded type" means that the relative periods of all renormalizations are bounded);

• Real maps which don't have non-repelling periodic points.

Recall that a quadratic-like map f has *a priori bounds* if there is an $\epsilon > 0$ such that $\operatorname{mod}(R^m f) \geq \epsilon > 0$ for all renormalizations.

The goal of this paper is to prove the following result:

Rigidity Theorem. *Any combinatorial class contains at most one quadratic polynomial satisfying the secondary limbs condition with a priori bounds.*

We believe that the second assumption actually follows from the first one:

Conjecture. *The secondary limbs condition implies a priori bounds.*

This conjecture is supported by a few partial results (see below). Note, however, that *a priori* bounds don't hold for all quadratics: see examples of non-locally connected Julia sets [M1].

Let $\mathcal{QC}(c) \subset \mathcal{T}op(c) \subset \mathcal{C}om(c) \subset \mathbb{C}$ stand respectively for the quasi-conformal, combinatorial and topological classes of the quadratic map P_c. A map P_c is called combinatorially (respectively topologically or quasi-conformally) rigid if $\mathcal{C}om(c) = \{c\}$ (respectively $\mathcal{T}op(c) = \{c\}$ or $\mathcal{QC}(c) = \{c\}$).

Published in modified form as part of
"Dynamics of quadratic polynomials, I-II",
Acta Math. **178** (1997), 185-297.

Stony Brook IMS Preprint #1995/14
October 1995

Corollary I. *Assume that all maps of $\mathcal{C}om(c)$ (respectively $\mathcal{T}op(c)$) satisfy the secondary limbs condition with a priori bounds. Then P_c is combinatorially (respectively topologically) rigid.*

The corresponding quasi-conformal rigidity problem is settled by McMullen's Rigidity Theorem [McM2] which asserts that any quadratic polynomial with *a priori* bounds is quasi-conformally rigid.

The strongest, combinatorial, rigidity of a map P_c turns out to be equivalent to the local connectivity of the Mandelbrot set M at c (see [DH1, Sch]). This property of M was conjectured by Douady and Hubbard under the name "MLC". Prior to this work it was established in the following cases:

- Parabolic points (Douady and Hubbard [DH1]);
- Boundaries of the hyperbolic components of M (Yoccoz, see Hubbard [H]);
- At most finitely renormalizable maps (Yoccoz, see Hubbard [H], Kahn [K]).

The following Corollary adds a pool of infinitely renormalizable maps to this list. In Part I of this paper *a priori bounds* have been proven for all maps of class $\mathcal{SL}$ with sufficiently big type (in the sense of Theorems IV and IV$'$ of Part I). Thus we have:

Corollary II [L3]**.** *A quadratic polynomial $P_c \in \mathcal{SL}$ of a sufficiently big type is rigid, so that the Mandelbrot set is locally connected at c.*

In particular, this gives first examples of infinitely renormalizable parameter values $c \in M$ of *bounded type* where MLC holds (though one does not need the full capacity of Corollary II to produce some examples of such kind).

Remark. It is easy to construct some infinitely renormalizable parameter values of unbounded type where MLC holds (oral communication by A. Douady). First find arbitrary small copies M_n of the Mandelbrot set near $c = -2$. Then for an appropriate subsequence $n(k)$, the tuned Mandelbrot copies $M_{n(1)} * M_{n(2)} * \cdots * M_{n(l)})$ shrink to a single point.

One might wonder of how big is the set of infinitely renormalizable parameter values satisfying the assumptions of Corollary II. We can show that this set has Lebesgue measure zero and Hausdorff dimension at least 1 (in preparation). Note that 1=(1/2)2 where $2 = \mathrm{HD}(\partial M)$ by Shishikura's Theorem [Sh].

Let us now dwell on the case of real parameter values. In this case, "sufficiently big type" means sufficiently big *essential period* (see [LY] for the precise definition). For maps with "small" essential period, the MLC problem is still open. However, *a priori* bounds have been established for all infinitely renormalizable real quadratics (see [S, MvS, L2, GS, LS, LY]). Let us say that a parameter value $c \in \mathbb{R}$ (or the corresponding quadratic polynomial P_c) is *rigid on the real line* if $\mathcal{C}om(c) \cap \mathbb{R} = \{c\}$. Thus we have:

Corollary III. *Any quadratic polynomial P_c without attracting cycles is rigid on the real line.*

By the Milnor-Thurston kneading theory [MT], Corollary III implies:

Corollary IV. *Hyperbolic quadratics are dense on the real line.*

The last two Corollaries were first announced by Swiatek [Sw] who approached them by methods of real dynamics. The methods of holomorphic dynamics presented in this paper were developed in [L2].

Another application of the above Rigidity Theorem is a construction of the unstable manifolds for the renormalization operator at infinitely renormalizable points of bounded type (in preparation).

Let us now outline the structure of this paper. In §2 we show that the secondary limbs condition and *a priori* bounds yields a definite space between the bouquets of little Julia sets. This provides us with special disjoint neighborhoods of little Julia bouquets with bounded geometry (called "standard"). Also, together with the work of Hu & Jiang [HJ, J] and McMullen [McM3] this yields local connectivity of the corresponding Julia set (Theorem I).

We start §3 with a discussion of reductions which boil the Rigidity Theorem down to the following problem: Two topologically equivalent maps (satisfying the assumptions of the theorem) are Thurston equivalent. Then we set up an inductive construction of a sequence of approximations to the Thurston conjugacy. In particular, we adjust an approximate conjugacy in such a way that it respects the standard neighborhoods of little Julia bouquets.

The last section, §4, which follows §4 of [L2], presents the proof of the Main Lemma. This lemma gives a uniform bound on the Techmüller distance between the generalized renormalizations of two combinatorially equivalent quadratic-like maps (the bound depends only on the selected secondary limbs and *a priori* bounds). The main geometric ingredient which makes this work is the linear growth of the principal moduli proved in Part I of this paper.

In the Appendix we collect necessary background material in the theory of quasi-conformal maps.

In conclusion let us make a couple of remarks on history and some related results and methods. The origin of our approach to the rigidity problem can be tracked back to the proof of Mostow Rigidity: from topological to quasi-conformal equivalence, and then (by means of ergodic theory) from quasi-conformal to conformal equivalence. This set of ideas were brought to the iteration theory by Sullivan and Thurston.

The passage from quasi-conformal to conformal equivalence in our setting is settled by McMullen's Rigidity Theorem [McM3]. Our main task is to pass from topological to quasi-conformal equivalence. A way to do this called "pull-back argument" is to start with a quasi-conformal map respecting some dynamical data, and to pull it back so that it will respect more and more data on every step. In the end it will become (with some luck) a quasi-conformal conjugacy. This method was introduced by Thurston (see [DH3] and also [McM1]) for postcritically finite maps, and exploited by Sullivan [S, MvS] for real infinitely renormalizable maps of bounded type. These first applications dealt with maps with rather simple combinatorics.

For more complicated combinatorics, a certain real version of this method based on the so called "inducing" was suggested by Jacobson & Swiatek [JS, Sw]. (Roughly speaking, "inducing" means building out of f an expanding map with a definite range.) On the other hand, by means of a purely complex pull-back argument in the puzzle framework, Jeremy Kahn [K] proved removability of non-renormalizable Julia sets (which yields the Yoccoz Rigidity Theorem) .

Our way is different from all the above, though it has some common features with them. We believe that holomorphic dynamics is the right framework for the rigidity problem, and our method is purely complex. Rather than building an induced expanding map, we pass consecutively from bigger to smaller scales by means of the generalized renormalization [L1], and carry out the pull-back using growth of moduli and complex *a priori* bounds [L2, L3].

Let us note that there is a different approach to rigidity problems, by comparing the dynamical and parameter planes. This method was used by Branner & Hubbard [BH] to prove rigidity of cubic maps with one escaping critical point and "non-periodic tableaux" (which corresponds to non-renormalizable quadratics). It was also used by Yoccoz to prove rigidity of at most finitely renormalizable quadratics. In the forthcoming notes we will discuss this approach in our setting.

Let us also note that the MLC problem is closely related to the problem of landing of parameter rays at points $c \in \partial M$. MLC certainly yields landing of all rays, but, on the other hand, landing of some special rays has been a basis for progress in the MLC problem. The first results in this direction (landing at parabolic and Misiurewicz points) were obtained by Douady & Hubbard (see [DH1, M2, Sch]). Recently Anthony Manning [Ma] has estimated the Hausdorff dimension of the set of rays landing at infinitely renormalizable points.

Notations and terminology. Throughout the paper f will stand for a quadratic-like map with critical point at 0.

Saying that a modulus of some annulus A is *definite* means that $\mod A \geq \epsilon > 0$, where ϵ depends only on the selected truncated secondary limbs and *a priori* bounds. Saying that some quantity is *bounded* has an analogous meaning.

Given a family of compact subsets $X_i \subset U$, we say that there is *a definite space* (at least $\mu > 0$) in between them (in a domain U) if for any i, there exists an annulus $A_i \subset U \setminus \cup X_i$ with a definite modulus (at least μ) which goes around X_i but does not go around other sets X_j, $j \neq i$. If U is not specified, then $U = \mathbb{C}$.

We will use the following notations:

$\mathbb{D}_r = \{z : |z| < r\}$ is the standard disk of radius r, $\mathbb{D} \equiv \mathbb{D}_1$ is the unit disk;
$\mathbb{T}_r = \partial\mathbb{D}_r$ is the standard circle of radius r, $\mathbb{T} \equiv \mathbb{T}_1$ is the unit circle;
$\mathbb{A}(r, R) = \{z : r < |z| < R\}$ is a standard annulus; similar notation is used for a closed annulus $\mathbb{A}[r, R]$ (or a semi-closed one).
Let $P_c : z \mapsto z^2 + c$.
As usual, $\omega(z) \equiv \omega(f, z)$ stands for the limit set of the forward orbit $\{f^n z\}_{n=0}^{\infty}$. The set $\omega(0)$ is called *postcritical.*
$R^m f$ is the m-fold renormalization of f.

Acknowledgement. I would like to thank Curt McMullen and Yair Minsky for useful discussions. I also thank MSRI for their hospitality: Part of this work was done during the Complex Dynamics and Hyperbolic Geometry spring program 1995. This work has been partially supported by NSF grants DMS-8920768 and DMS-9022140, and the Sloan Research Fellowship.

2. Space between Julia bouquets.

2.1. **Space and unbranching.** Let J_i^m denote the little Julia sets of level m, that is, $J^m \equiv J_0^m = J(R^m f)$ and $J_i^m = f^i J^m$, $i = 0, \dots, r_m - 1$. They are organized in the pairwise disjoint bouquets $B_j^m = B_j^m(f)$ of the Julia sets touching at the same periodic point. Namely, if level $m-1$ is immediately renormalizable with period l then each B_j^m consists of l little Julia sets J_i^m touching at their β-fixed points. Otherwise the bouquets B_j^m just coincide with the little Julia sets J_j^m. By $B^m \equiv B_0^m$ we will denote the *critical* bouquet containing the critical point 0. Let $\mathbb{J}^m = \mathbb{J}^m(f) = \cup_i J_i^m = \cup_j B_j^m$. Finally let K_i^m be little filled Julia sets.

We will use the notation F_m for the quadratic-like map f^{p_m} near any little Julia set J_i^m (it should be clear from the context which one is considered). In particular, $F_m = R^m f$ near the critical Julia set $J^m \ni 0$.

Recall that $\mathcal{Q}(\mu)$ stands for the space of quadratic-like maps f with $\mod (f) \geq \mu > 0$ supplied with the Caratheodory topology (see [McM2] and §5.6 of Part I). Take a little copy $M' \subset M$ of the Mandelbrot set with root at b. Let $\mathcal{Q}(\mu, M')$ denote the subspace of $\mathcal{Q}(\mu)$ consisting of renormalizable quadratic-like maps f such that the hybrid class of Rf belongs to $M' \setminus \{b\}$.

Let us have a family $\mathcal{F}$ of sets $X_a \subset \mathbb{C}$ depending on some parameter a ranging over a topological space $\mathcal{T}$. This dependence is said to be (sequencially) *upper semi-continuous* if for any $a(i) \to a$, the Hausdorff limit of $X_{a(i)}$ is contained in X_a. For example it is easy to see that the filled Julia set $K(f)$ of a quadratic-like map f depends upper semi-continuously on f. Let us say that a family $\mathcal{F}$ of sets $X_f \subset \mathbb{C}$ is (upper) *semi-compact* if any sequence X_n of these sets contains a subsequence $X_{n(i)}$ converging in Hausdorff topology to a subset of some $X \in \mathcal{F}$.

Lemma 2.1. *The little filled Julia sets $K_i^1(f)$ form a semi-compact family of sets as f ranges over the space $\mathcal{Q}(\mu, M')$.*

Proof. By the Compactness Lemma (see §5.6 of Part I), the space $\mathcal{Q}(\mu, M')$ is compact. Moreover the quadratic-like map F_1 depends continuously on $f \in \mathcal{Q}(\mu, M')$ near any K_i^1. In turn, the little filled Julia sets K_i^1 depend upper semi-continuously on F_1. ☐

Lemma 2.2. *Let f be a quadratic-like map of class $\mathcal{SL}$ with complex a priori bounds. Then there is a definite space in between its bouquets B_j^m.*

Proof. Let us take a bouquet B^m. Let $\mathcal{I}^m$ stand for the set of indices j such that $B_j^{m+1} \subset B^m$. We will show first that there is a definite annulus

$$T^m \subset B^m \setminus \bigcup_{j \in \mathcal{I}^m} B_j^{m+1},$$

which goes around B^{m+1} but does not go around other bouquets B_j^{m+1}, $j \in \mathcal{I}^m$.

If $R^m f$ is not immediately renormalizable, then this follows from point (ii) of Theorem II (Part I). So assume that $R^m f$ is immediately renormalizable.

If $B^m = J^m$, then it is nothing to prove as there is only one bouquet B^{m+1} inside B^m. Otherwise there are only finitely many renormalization types producing the bouquet B^m (which correspond to the little Mandelbrot sets attached to the main cardioid and belonging to the selected secondary limbs). By Lemma 2.1, the bouquets B_j^{m+1} contained in B^m belong to a compact family of sets. As they don't touch each other, there is a definite space in between them.

Let $N(L,\epsilon)$ denote an $\epsilon\cdot\operatorname{diam} L$-neighborhood of a set L (that is, the set of points on distance at most $\epsilon \operatorname{diam} L$ from L). We have shown that there is an $\epsilon > 0$ such that the neighborhood $N(B^{m+1},\epsilon)$ does not intersect other bouquets B_j^{m+1} contained in the same B^m. In particular, $N(B^1,\epsilon)$ does not intersect any other B_j^1 (as all of them are contained in $B^0 \equiv J(f)$).

Let us show by induction that

$$N(B^m,\epsilon)\cap B_k^m = \emptyset,\ k\neq 0 \tag{2.1}$$

Assuming this for m, we should show that

$$N(B^{m+1},\epsilon)\cap B_j^{m+1} = \emptyset,\ j\neq 0. \tag{2.2}$$

As we already know (2.2) for $j\in\mathcal{I}^m$, let $j\notin\mathcal{I}^m$. Then $B_j^{m+1}\subset B_k^m$ for some $k\neq 0$, while $N(B^{m+1},\epsilon)\subset N(B^m,\epsilon)$, and (2.2) follows from (2.1).

What is left, is to show that there a definite space around any bouquet B_j^{m+1} (not only around the critical one). But there is an iterate f^l which univalently maps B_j^{m+1} onto B^{m+1}. Pulling back the space around B^{m+1} we obtain the desired space about B_j^{m+1}. □

An infinitely renormalizable map f is said to satisfy an *unbranched a priori bounds* condition (see [McM3]) if for infinitely many levels m, there is a definite space in between J^m and the rest of the postcritical set, $\omega(0)\setminus J^m$.

Lemma 2.3. *A map $f\in\mathcal{SL}$ with a priori bounds satisfies an unbranched a priori bounds condition.*

Proof. We will show that the unbranched condition can fail only if the level m is not immediately renormalizable, while $m-1$ is immediately renormalizable. As the complimentary sequence of levels is infinite, the lemma will follow.

If $R^{m-1}f$ is not immediately renormalizable then the bouquet B^m coincides with the little Julia set J^m. By Lemma 2.2, there is a definite space in between J^m and $\mathbb{J}^m\setminus J^m$. As $\omega(0)\setminus J^m\subset\mathbb{J}^m\setminus J^m$, the unbranched condition holds on level m.

Assume now that both levels $m-1$ and m are immediately renormalizable. Then we will show that there is a definite space in between J^m and $\mathcal{B}^{m+1}\equiv\cup_{j\neq 0}B_j^{m+1}$.

By Lemma 2.2, there is a definite space in between $B^m\supset J^m$ and $\mathcal{B}^{m+1}\setminus B^m$. So we should check that there is a definite space in between J^m and $\mathcal{B}^{m+1}\cap B^m$ (that is, the union of non-critical bouquets B_j^{m+1} contained in B^m). But J^m does not touch any such B_j^{m+1}. Indeed, the only point where they can touch could be the β-fixed point β_m of J^m. But one can easily see that the little Julia sets of level $m+1$ never contain β_m. By Lemma 2.1 there is a desired space.

Finally, as $\omega(0)\setminus J^m\subset\mathcal{B}^{m+1}$, the statement follows. □

Remark. If $R^m f$ is not immediately renormalizable, while $R^{m-1}f$ is immediately renormalizable, then the unbranched condition can fail. Indeed in this case there are several Julia sets J_i^m which touch at the common fixed point $\beta_m\in J^m$. But the postcritical set $\omega(0)\cap J_i^m$ can come arbitrarily close to β_m (when $R^m f$ is a small perturbation of a map whose critical orbit eventually lands at β_m).

2.2. **Local connectivity of Julia sets.** Hu and Jiang [HJ] proved that the Feigenbaum quadratic polynomial has locally connected Julia set. The proof makes use of Sullivan's *a priori* bounds (see [MvS, S]). Then a more general result of this kind was worked out: Any infinitely renormalizable quadratic map with unbranched *a priori* bounds has locally connected Julia set (see [J, McM3]). Together with Lemma 2.3 this yields (compare Theorem V of Part I):

Theorem I. *Let $f \in \mathcal{SL}$ be an infinitely renormalizable quadratic polynomial with a priori bounds. Then the Julia set $J(f)$ is locally connected. In particular, all maps from Theorems IV and IV′ of Part I have locally connected Julia sets.*

For the sake of completeness, we will give a proof of this result.

Proof. *A priori* bounds imply that the "little" Julia sets J^m shrink down to the critical point. Indeed let $f_m \equiv R^m f \equiv f^{p_m} : U'_m \to U_m$ where $\bmod (U_m \setminus U'_m) \geq \epsilon > 0$, with an ϵ independent of m. Clearly U_m does not cover the whole Julia set.

Let $\Gamma_m \subset U_m \setminus U'_m$ be a horizontal curve in the annulus $U_m \setminus U'_m$ which divides it into two subannuli of modulus at least $\epsilon/2$, and $\Gamma'_m \subset U'_m$ be its pull-back by f_m. By the Koebe Theorem, these curves have a bounded eccentricity about 0 (with a bound depending on ϵ). Since the inner radius of curve Γ'_m about 0 tends to 0 as $m \to \infty$ (it follows from the fact that the sufficiently high iterates of any disk intersecting $J(f)$ cover the whole $J(f)$), the $\operatorname{diam} \Gamma'_m \to 0$ as well. All the more, $\operatorname{diam}(J_m) \to 0$ as $m \to \infty$.

Let us take a $\delta > 0$, and find an m such that J_m is contained in the $\mathbb{D}_\delta$.

Let us now inscribe into $\mathbb{D}_\delta$ a domain bounded by equipotentials and external rays of the original map f. Let α_m denote the dividing fixed point of the Julia set J^m, and $\alpha'_m = -\alpha_m$ be the symmetric point. Let us consider a puzzle piece $P^{m,0} \ni 0$ bounded by any equipotential and four external rays of the *original map* f landing at α_m and α'_m. This is a "degenerate" domain of the renormalized map F_m (see §2.5 of Part I). By definition of the renormalized Julia set, the preimages $P^{m,k} \equiv F_m^{-k} P^{m,0}$ shrink down to J^m. Hence there is a puzzle piece $P^{m,l}$ contained in the $\mathbb{D}_\delta$. As $J(f) \cap P^{m,l}$ is clearly connected, the Julia set $J(f)$ is locally connected at the critical point.

Let us now prove local connectivity at any other point $z \in J(f)$. This is done by a standard spreading of the local information near the critical point around the whole dynamical plane. Let us consider two cases.

Case (i). Let the orbit of z accumulates on all Julia sets J^m. Let m be an unbranched level. Then there is an $l = l(m)$ such that the puzzle piece $P^{m,l}$ is well inside $\mathbb{C} \setminus (\omega(0) \setminus J^m)$.

Take now the first moment $k = k(m) \geq 0$ such that $f^k z \in P^{m,l}$. Let us consider the pull-backs $Q^{m,l} \ni z$ of $P^{m,l}$ along the orbit $\operatorname{orb}_k(z) = \{z, ..., f^k z\}$. By Lemma 3.3 of Part I, this pull-back is univalent. Moreover, it allows a univalent extension to a definitely bigger domain.

By the Koebe Theorem, $Q^{m,l}$ has a bounded eccentricity about z. Since the inner radius of this domain about z tends to 0 as $m \to \infty$, the $\operatorname{diam} Q^{m,l} \to 0$ as well. As $Q^{m,l} \cap J(f)$ are connected, the Julia set is locally connected at z.

Case (ii). Assume now that the orbit of z does not accumulate on some J^m. Hence it accumulates on some point $a \notin \omega(0)$. Let us consider the puzzle associated with the periodic point α_m (so that the initial configuration consists of a certain equipotential and the external rays landing at α_m). Since the critical puzzle pieces shrink to J^m, the puzzle pieces Y_i^l of sufficiently big depth l containing a are disjoint from $\omega(0)$ (there are several such pieces if a is a preimage of α_m). Take such an l, and let X be the union of these puzzle pieces. It is a closed topological disk disjoint from $\omega(0)$ whose interior contains a.

Consider now the moments $k_i \to \infty$ when the orbit of z lands at $\operatorname{int} X$, and pull X back to z. By the same Koebe argument as in case (i) we conclude that these pull-backs shrink to z. It follows that $J(f)$ is locally connected at z. □

2.3. **Standard neighborhoods.** In this section we will construct some special fundamental domains near little Julia bouquets. Let us consider first the non-immediately renormalizable case when the construction can be done in a particularly nice geometric way.

Lemma 2.4. *Let f be m times renormalizable quadratic map. Assume that the space in between the little Julia sets J_i^m is at least $\mu > 0$. Then there are disjoint fundamental annuli A_i^m around little Julia sets J_i^m, with* $\mod A_i^m \geq \nu(\mu) > 0$.

Proof. Let us consider the Riemann surfaces $S = \mathbb{C} \setminus \mathbb{J}^n$ and $S' = \mathbb{C} \setminus f^{-1}\mathbb{J}^n \subset S$. Then $f : S' \to S$ is a double branched covering. Let us uniformize S, that is represent it as the quotient $\mathcal{H}^2/\Gamma$ of the hyperbolic plane modulo the action of a Fuchsian group. In this conformal representation S admits a compactification $S \cup \partial S$ to a bordered Riemann surface, with the components ∂S_i^m of the "ideal boundary" ∂S corresponding to the little Julia sets J_i^m.

Let $\hat{S} = S \cup \partial S \cup \bar{S}$ be the double of S, that is $(\mathbb{C} \backslash \Lambda(\Gamma))/\Gamma$, where $\Lambda(\Gamma) \subset S^1$ is the limit set of Γ. The boundary components ∂S_i^m are geodesics in $\hat{S}$. Moreover, these geodesics have hyperbolic length bounded by a constant $L = L(\mu)$ independent of m.

Let $\sigma : S \to S$ be the natural anti-holomorphic involution of S. Let $\bar{S}' = \sigma S'$ and $\hat{S}' = S \cup \partial S \cup \bar{S}' \subset \hat{S}$ be the double of S' inside S. Then f allows an extension to a holomorphic double branched covering $\hat{f} : \hat{S}' \to \hat{S}$ commuting with the involution σ. Its restrictions $\hat{f}|\partial S_i^m$ are the double branched coverings of the topological circles ∂S_i^m.

Let $C_i^m(r) \supset \partial S_i^m$ stand for the hyperbolic r-neighborhood of the geodesic ∂S_i^m. By the Collar Lemma (see [Ab]), there is an $r = r(L)$ (independent of the particular Riemann surface and geodesics) such that the collars $C_i^m \equiv C_i^m(r)$ are pairwise disjoint. Moreover, $\mod (C_i^m) \geq \mu(L) > 0$.

Let us now take such a collar $C = C_i^m$, and let $\Gamma = \partial S_i^m$. Let $C' \subset S' \cap C$ be the component of $\hat{f}^{-1}S$ containing Γ. Then $\hat{f} : C' \to C$ is a double covering preserving Γ. As we have in the hyperbolic metric of S:

$$\int_\Gamma \|D\hat{f}\| = 2l(\Gamma),$$

there is a point $z \in \Gamma$ such that $\|Df(z)\| \geq 2$. This easily implies that $\|D\hat{f}^{-1}(\zeta)\| \leq q(a) < 1$ if the hyperbolic distance between z and ζ does not exceed a. In particular, $\|D\hat{f}^{-1}\|(\zeta) \leq q = q(L, r) < 1$ for all $\zeta \in C$.

It follows that C' is contained in the hyperbolic r/q-neighborhood of Γ, and hence the annulus $\mathrm{mod}\,(C\backslash C') \geq \rho(r,q) = \rho(\mu)$. Let now $A_i^m = (C\backslash C') \cap S$. $\square$

Note that in the above lemma we don't assume *a priori* bounds but just a definite space between the Julia sets (which thus implies *a priori* bounds). Assuming *a priori* bounds, let us now give a different construction which works in the immediately renormalizable case as well.

Let us consider a bouquet $B_j^m = \cup_i J_i^m$ of level m, where J_i^m touch at point α_{m-1}. Let $b_{m,i} \in J_i^m$ be the points F_m-symmetric to α_{m-1}, that is, $F_m b_i^m = \alpha_{m-1}$ ("co-fixed points"). Let us consider the domain Υ_j^m bounded by the pairs of rays landing at these points (defined via a straightening of F_{m-1}), and p_m arcs of equipotentials. Let us then thicken this domain near the points b_i^m as described in §2.5 of Part I (that is, replace the rays landing at b_i^m by nearby rays and little circle arcs around b_i^m). Denote the thickened domains by U_j^m (see Figure 1). We also require that these domains are naturally related by dynamics so that $f\Upsilon_j^m = \Upsilon_k^m$ and $fU_j^m = U_k^m$ whenever $fB_j^m = B_k^m$ and B_j^m is non-critical. Let us call U_j^m a *standard neighborhood* of the bouquet B_j^m. Let $\mathbb{U}^m = \cup U_j^m$.

Lemma 2.5. *Let f be an m times renormalizable quadratic map of class $\mathcal{SL}$ with a priori bounds. Then there exist disjoint standard neighborhoods U_j^m of B_j^m with bounded geometry, and such that the annulus* $\mathrm{mod}\,(U_i^m \backslash B_i^m)$ *have a definite modulus.*

Proof. By the Straightening Theorem, the renormalization $R^{m-1}f : J_k^{m-1} \to J_k^{m-1}$ is K-qc conjugate to a quadratic polynomial $P_c : z \mapsto z^2 + c$, with K dependent only on *a priori bounds*.

Let $B \subset J(P_c)$ be the critical bouquet of little Julia sets of RP_c. Let $\Omega(\epsilon)$ be its neighborhood bounded by arcs of equipotentials of level $1-\epsilon$, circle arcs of radius ϵ, and rays with arguments $\theta + t(\epsilon)$ (see §2.4 of Part I). Here θ are the arguments of the rays landing at the co-fixed points, and $t(\epsilon) \in (-\epsilon, \epsilon)$ is selected in such a way that $\Omega(\epsilon)$ is a renormalization domain for any P_c from selected truncated secondary limbs.

The geometry of these domains depends only on the selected limbs and ϵ. Also, the Hausdorff distance $d_c(\epsilon)$ of $\partial\Omega(\epsilon)$ to B tends to 0 as $\epsilon \to 0$ uniformly over c belonging to the selected truncated limbs. Indeed, this is clearly true for a given parameter value c. Take a little $\delta > 0$, and find an $\epsilon = \epsilon_c$ such that $d_c(\epsilon_c) < \delta$. Then for all b sufficiently close to c, $d_b(\epsilon_c) < 2\delta$. Compactness of the truncated limbs completes the argument.

It follows that for all sufficiently small ϵ (depending only on the selected limbs and *a priori* bounds), $\Omega(\epsilon)$ belongs to the range of the straightening map. Hence these neighborhoods can be transferred to the dynamical f-plane. We obtain neighborhoods $U(\epsilon)$ of the corresponding bouquet B with bounded geometry (depending on parameter ϵ).

Moreover, as quasi-conformal maps are quasi-symmetric (see Appendix), the Hausdorff distance from $\partial U(\epsilon)$ to the bouquet B is at most $\rho(\epsilon)\cdot \mathrm{diam}\, B$, where $\rho(\epsilon) \to 0$ as $\epsilon \to 0$. Hence for all sufficiently small ϵ, the neighborhood $U(\epsilon)$ is well inside the domain $\mathbb{C} \setminus \cup_{j\neq 0} B_j^m$.

Let us now pull this neighborhood back by dynamics to obtain standard neighborhoods $U_j^m(\epsilon)$ of other bouquets B_j^m. Since $U(\epsilon)$ is well inside $\mathbb{C} \setminus \cup_{j\neq 0} B_j^m$, these pull-backs have a bounded distortion. Hence the Hausdorff distance from $\partial U_j^m(\epsilon)$ to the bouquet B_j^m is at most $\rho(\epsilon)\cdot \mathrm{diam}\, B$, where $\rho(\epsilon) \to 0$ as $\epsilon \to 0$.

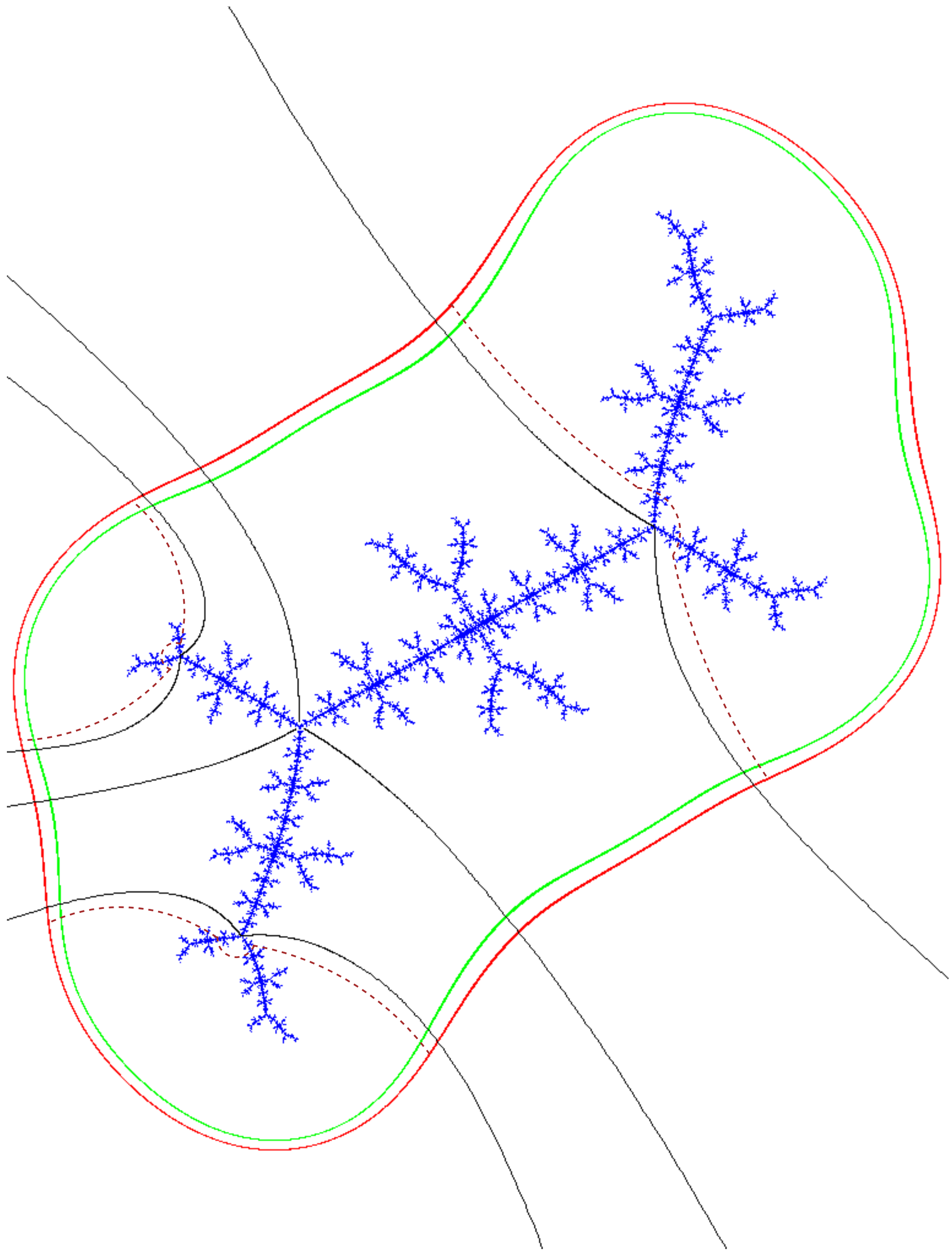

Figure 1. Standard neighborhood of a Julia bouquet (made by B. Yarrington).

Since by Lemma 2.2 there is a definite space between the bouquets, there is also a definite space between the neighborhoods $U_j^m(\epsilon)$, for for all $\epsilon \in (0, \epsilon_*]$ (with ϵ_* depending only on the selected limbs and *a priori* bounds). Also, the moduli of $U_j^m(\epsilon) \setminus B_j^m$ depend only on the limbs, *a priori* bounds and ϵ. So they are definite, for instance in the range $\epsilon \in (0.01\epsilon_*, \epsilon_*]$. □

We keep using the notations B_j^m, Υ_j^m etc. introduced before Lemma 2.5, and we also assume that the standard neighborhoods U_j^m satisfy the conclusions of Lemma 2.5. We will define a special qc map

$$S_m : (U_j^m \setminus B_j^m) \to \mathbb{A}(1,4). \tag{2.3}$$

with bounded dilatation. This map will be called a *standard straightening*, or a *standard local chart* near the bouquet B_j^m.

It follows from *a priori bounds* assumption that for any Julia set J_i^l there exist Jordan disks $\Omega_i^l \supset \Pi_i^l \supset J_i^l$ such that $F_l : \Pi_i^l \to \Omega_i^l$ is a quadratic-like map, and there exists a qc map

$$\Psi_{l,i} : (\Omega_i^l \setminus J_i^l,\ \Pi_i^l \setminus J_i^l) \to (\mathbb{A}[1,4],\ \mathbb{A}[1,2]) \tag{2.4}$$

with bounded dilatation conjugating $F_l : \Pi_i^l \to \Omega_i^l$ and $P_0 : \mathbb{A}[1,2] \to \mathbb{A}[1,4]$, $P_0 : z \mapsto z^2$.

Moreover, if J_i^m does not touch other Julia sets of level m (that is, F_{m-1} is not immediately renormalizable) then one can select the standard neighborhood U_i^m as Ω_i^m. In this case let us define the standard straightening (2.4) as $\Psi_{m,i}$.

If F_{m-1} is immediately renormalizable, then let us consider a family of little Julia sets and bouquets:

$$\bigcup_i J_i^m = B_j^m \subset J_k^{m-1}. \tag{2.5}$$

Let us cut Υ_j^m by the rays landing at the fixed point α_{m-1} into components $\Xi_i^m \supset J_i^m$. Since the hybrid class of F_{m-1} may belong to a bounded number of little Mandelbrot sets (attached to the main cardioid and intersecting the selected secondary limbs), the domains Ξ_i^m have a bounded geometry. Hence the maps $\Psi_{m,i}$ can be selected in such a way that they have bounded dilatation and

$$\Psi_{m,i} | \bigcup_i \Xi_i^m = \Psi_{m-1,k}.$$

Thus they glue together into a single qc map (2.3).

By the rays and equipotentials near the bouquet we will mean the S_m-preimages of the vertical intervals and horizontal circles in the cylinder $\mathbb{A}(1,4)$. This will be also referred to as the *standard coordinate system* near B_j^m.

Let us show in conclusion that the little Julia bouquets and corresponding standard neighborhoods exponentially decay. Let $\operatorname{diam}(X)$ stand for the Euclidean diameter of a set X.

Lemma 2.6. *Let $f \in \mathcal{SL}$ be a quadratic-like map with a priori bounds. Then there exist constants $\lambda < 1$ and $l_0 > 0$ depending on the choice of limbs and a priori bounds such that for any two Julia bouquets $B_j^{m+l} \subset B_i^m$,*

$$\operatorname{diam} B_j^{m+l} \le \lambda^l \operatorname{diam} J_i^m, \quad l \ge l_0$$

Proof. Let us straighten the renormalization $R^m f$ near J_i^m to a quadratic polynomial P_c. The dilatation K of the straightening depends only on the *a priori* bounds, and K-qc maps are Hölder continuous with exponent $1/K$ (see [A]). Hence it is enough to show that for the quadratic map P_c, there exist constants $\lambda < 1$ and $l_0 > 0$ depending on the choice of limbs and *a priori* bounds such that

$$\operatorname{diam} B_j^l \le \lambda^l. \tag{2.6}$$

(Now B_j^m, $\mathbb{J}^m$ etc. stand for the objects associated to P_c).

Note that $J(P_c) \subset \mathbb{D}_2$. Let ρ_l be the hyperbolic metric on $\mathbb{D}_3 \setminus \mathbb{J}^l$. Let γ_i^m be the hyperbolic geodesic in $\mathbb{D}_3 \setminus \mathbb{J}^l$ homotopic to a curve $\Gamma_i^m \subset \mathbb{C} \setminus \mathbb{J}^m$ going once around B_i^m but not going around other Julia bouquets B_k^m, $k \neq i$.

By Lemma 2.2, there are annuli $A_i^n \subset \mathbb{D}_3 \setminus \mathbb{J}^m$ in the homotopy class of Γ_i^m with a definite modulus, $\mod (A_i^m) \ge \nu$. Let us pick Γ_i^m as the hyperbolic geodesic in A_i^m. Then the hyperbolic length of this geodesic in A_i^m is at most π/ν. All the more, the hyperbolic length of γ_i^m in $\mathbb{D}_3 \setminus \mathbb{J}^l$ is bounded by the same constant.

By the Collar Lemma (see [Ab]), there are exist disjoint annuli $\Lambda_i^m \subset \mathbb{D}_3 \setminus \mathbb{J}^l$ in the homotopy class of γ_i^m with $\mod (\Lambda_i^m) \ge \eta = \eta(\nu) > 0$. By the Grötzsch inequality, $\mod (\mathbb{D}_3 \setminus B_j^l) \ge l\eta$. Hence there is an absolute constant C such that $\operatorname{diam} B_j^l \le Ce^{-l\eta}$ (see Appendix A1 in Part I), and (2.6) follows. ☐

Corollary 2.7. *Under the assumptions of Lemma 2.6, there exist constants $\lambda < 1$ and $l_0 > 0$ such that for the standard neighborhoods $U_j^m \subset U_i^m$ the following estimates holds:*

$$\operatorname{diam} U_j^{m+l} \le \lambda^l \operatorname{diam} U_i^m, \quad l \ge l_0.$$

Proof. Indeed, the standard neighborhoods U_i^m are commensurable with the corresponding Julia sets J_i^m. ☐

2.4. **Removable sets.** The reader is referred to the Appendix for the definition and a discussion of removability.

Lemma 2.8 (McM2)**.** *Under the assumptions of Lemma 2.6, the post-critical set $\omega(0)$ is a removable Cantor set coinciding with $\cap \mathbb{J}^m$.*

Proof. It was shown in the proof of Lemma 2.6 that for any $z \in \omega(0) \subset \cap \mathbb{J}^m$, there is a nest of disjoint annuli around z with a definite modulus. Thus the first statement follows from the Removability Condition (see Appendix).

Clearly, $\omega(0) \subset \cap \mathbb{J}^m \subset \cap \mathbb{U}^m$. Vice versa, by Lemma 2.6, $\cap \mathbb{J}^m$ is covered by the uniformly shrinking bouquets B_i^m. As every B_i^m contains a postcritical point, $\omega(0)$ is dense in $\cap \mathbb{J}^m$. ☐

Let us finish this section with stating a standard fact on removability of expanding Cantor sets. Let $\{U_i\}$ be a finite family of closed topological disks with disjoint closures. Let us consider a Markov map $g : \cup U_i \to \mathbb{C}$ satisfying the following property: If $\operatorname{int}(gU_i \cap U_j) \neq \emptyset$ then $gU_i \supset U_j$. As usual, let

$$K(g) = \{z : \ g^n z \in \cup U_i, \ n = 0, 1, \dots\}$$

stand for the filled Julia set of g.

Lemma 2.9. *For a Markov map as above, the filled Julia set $K(g)$ is removable.*

Proof. Let us select a family of annuli $A_j \subset gU_j \setminus \cup U_i$ homotopic to $\partial(gU_i)$ in $gU_j \setminus \cup U_i$. Let consider cylinder sets $U^m_{i(0),i(1),\dots,i(m-1)}$ defined by the following property:

$$g^k U^m_{i(0),i(1),\dots,i(m-1)} \subset U_{i(k)},\ k = 0, 1, \dots, m-2; \quad g^{m-1} U^m_{i(0),i(1),\dots,i(m-1)} = U_{i(m-1)}.$$

The pull-back of the annulus A_j to $U^m_{i(0),i(1),\dots,i(m-1)} \setminus \bigcup_i U^{m+1}_{i(0),i(1),\dots,i(m-1),i}$ by the univalent map $g^m : U^m_{i(0),i(1),\dots,i(m-1)} \to gU_{i(m-1)}$ has the same modulus as $A_{i(m-1)}$. This provides us with a nest of disjoint annuli with definite modulus about any $z \in K(g)$. The Removability Condition concludes the proof. ☐

3. Rigidity: beginning of the proof

3.1. **Reductions.** In this section we begin to prove the Rigidity Theorem stated in the Introduction. Since quadratic polynomials label hybrid classes of quadratic-like maps, this theorem can be stated in the following way:

Rigidity Theorem (equivalent statement). *Let $f, \tilde{f} \in \mathcal{SL}$ be two quadratic-like maps with a priori bounds. If f and $\tilde{f}$ are combinatorially equivalent then they are hybrid equivalent.*

The proof is split into three steps:

Step 1. f and $\tilde{f}$ are topologically equivalent;

Step 2. f and $\tilde{f}$ are qc equivalent;

Step 3. f and $\tilde{f}$ are hybrid equivalent.

The first step (passage from combinatorial to topological equivalence) follows from the local connectivity of the Julia sets (Theorem I). Indeed, a locally connected Julia set is homeomorphic to its combinatorial model (see [D]). Since the combinatorial model is the same over the combinatorial class, the conclusion follows.

The last step (passage from qc to hybrid equivalence) is taken care of McMullen's Rigidity Theorem [McM2]. Indeed, it asserts that an infinitely renormalizable quadratic-like map with *a priori* bounds does not have invariant line fields on the Julia set. It follows that if h is a qc conjugacy between f and $\tilde{f}$ then $\partial\bar{h} = 0$ almost everywhere on the Julia set. Thus h is a hybrid conjugacy between f and $\tilde{f}$.

So, our task is to take care of Step 2:

Theorem II. *Let $f, \tilde{f} \in \mathcal{SL}$ be two quadratic-like maps with a priori bounds. If f and $\tilde{f}$ are topologically equivalent then they are qc equivalent.*

In what follows we will mark with tilde the objects for $\tilde{f}$ corresponding to those for f (unless another meaning is explicitly assumed). When we introduce some objects for f, we assume that the corresponding tilde-objects are automatically introduced as well.

3.2. **Thurston's equivalence.** Let $f : U' \to U$ and $\tilde{f} : \tilde{U}' \to \tilde{U}$ be two topologically equivalent quadratic-like maps. Let us say that f and $\tilde{f}$ are *Thurston equivalent* if for appropriate choice of domains $U, U', \tilde{U}, \tilde{U}'$, there is a qc map $h : (U, U', \omega(0)) \to (\tilde{U}, \tilde{U}', \omega(0))$ which is homotopic to a conjugacy $\psi : (U, U', \omega(0)) \to (\tilde{U}, \tilde{U}', \omega(0))$ relative $(\partial U, \partial U', \omega(0))$. Note that h conjugates $f : \omega(0) \cup \partial U' \to \omega(0) \cup \partial U$ and $\tilde{f} : \omega(0) \cup \partial \tilde{U}' \to \omega(0) \cup \partial \tilde{U}$. A qc map h as above will be called a *Thurston conjugacy.*

Remark. It is enough to assume that h is homotopic to ψ rel postcritical sets. Then one can extend it to a qc map $U \to \tilde{U}$ which is homotopic to ψ rel the bigger set as required above.

The following result comes from the work of Thurston (see [DH2, McM1]) and Sullivan (see [MvS, S]). It originates the "pull-back method" in holomorphic dynamics.

Lemma 3.1. *If two quadratic-like maps are Thurston equivalent then they are qc equivalent.*

Proof. We will use the notations for the domains and maps preceding the statement of the lemma. Let U^n be the preimages of U under the iterates of f, and let $c = f(0)$. Let h has dilatation K.

Since $h(c) = \tilde{c}$, we can lift h to a K-qc map $h_1 : U^1 = \tilde{U}^1$ homotopic to ψ rel $(\partial U^1, \partial U^2, \omega(0))$. (Note that the dilatation of h_1 is the same as the dilatation of h, since the lift is analytic). Hence $h_1 = h$ on these sets, and we can extend h_1 to $U \setminus U^1$ as h (keeping the same notation h_1). By the Gluing Lemma from the Appendix this extension has the same dilatation K. Moreover, this map is homotopic to ψ rel $(\omega(0), \cup_{1 \le k \le 2} \partial U^k)$. Also, it conjugates $f : \omega(0) \cup (U^1 \setminus U^2) \to \omega(0) \cup (U^0 \setminus U^1)$ to the corresponding tilde-map (notice that h_1 is a conjugacy on a bigger set than h).

Let us now replace h with h_1 and repeat the procedure. We will construct a K-qc map $h_2 : U \to \tilde{U}$ homotopic to ψ rel $(\omega(0), \cup_{1 \le k \le 3} \partial U^k)$ and conjugating $f : \omega(0) \cup (U^1 \backslash U^3) \to \omega(0) \cup (U \backslash U^2)$ to the corresponding tilde-map.

Proceeding in this way we construct a sequence of K-qc maps h_n homotopic to ψ rel $(\omega(0), \cup_{1 \le k \le n+1} \partial U^k)$ and conjugating $f : \omega(0) \cup (U^1 \backslash U^{n+1}) \to \omega(0) \cup (U \backslash U^n)$ to the corresponding tilde-map. By the Compactness Lemma from the Appendix, we can select a converging subsequence $h_{n(l)} \to h$. The limit map h is a desired qc conjugacy. □

The method used in the above proof is called "the pull-back argument". The idea is to start with a qc map respecting some dynamical data, and then pull it back so that it will respect some new data on each step. In the end it becomes (with some luck) a qc conjugacy.

Remark. For infinitely renormalizable maps of bounded type with *a priori* bounds, McMullen proved that the postcritical set $\omega(0)$ has bounded geometry [McM3]. It easily follows that there is a qc map $h : (\mathbb{C}, \omega(f, 0)) \to (\mathbb{C}, \omega(\tilde{f}, 0))$ conjugating f to $\tilde{f}$ on their postcritical sets. This is close to being a Thurston conjugacy but not quite the same, as h may be in a wrong homotopy class.

3.3. **Approximating sequence of homeomorphisms.** So we need to construct a Thurston conjugacy. We will construct it as a limit of an appropriate sequence of maps. Take a sufficiently small $\epsilon > 0$, and consider the corresponding sequence of standard neighborhoods $\mathbb{U}^m = \cup_i U_i^m \equiv$

$U_i^m(\epsilon)$ (see §2.3). By Corollary 2.7 there is an l such that $\mathbb{U}^m$ is well inside U^{m-l} (that is, the annulus $U^{m-l} \setminus U^m$ has a definite modulus). Moreover, by Lemma 2.8 $\cap \mathbb{J}^m = \omega(0)$.

We will consecutively construct a sequence of homeomorphisms

$$h_m : (\mathbb{C}, \mathbb{U}^m, \mathbb{J}^m) \to (\mathbb{C}, \tilde{\mathbb{U}}^m, \tilde{\mathbb{J}}^m) \tag{3.1}$$

such that

(i) h_0 is a topological conjugacy;

(ii) h_{m+1} is homotopic to h_m rel $(\mathbb{J}^{m+1} \cup (\mathbb{C} \setminus \mathbb{U}^{m-l}))$. In particular $h_{m+1}|\mathbb{J}^{m+1} = h_m|\mathbb{J}^{m+1}$ and $h_{m+1}|(\mathbb{C} \setminus \mathbb{U}^{m-l}) = h_m|(\mathbb{C} \setminus \mathbb{U}^{m-l})$.

(iii) The h_m are K_*-qc on $\mathbb{U}^{m-1} \setminus \mathbb{J}^m$, with dilatation K_* depending only on the choice of limbs and *a priori* bounds;

(iv) $\text{Dil}(h_m|U^{m-l}) \le 4K_*^4 \, \text{Dil}(h_{m-1}|U^{m-l})$.

Such a sequence will do the job:

Lemma 3.2. *A sequence h_m satisfying the above three properties uniformly converges to a Thurston conjugacy.*

Proof. By the second property, this sequence eventually stabilizes outside $\cap \mathbb{J}^m$ and thus it pointwise converges to a homeomorphism $h : (\mathbb{C}, \cap \mathbb{J}^m) \to (\mathbb{C}, \cap \tilde{\mathbb{J}}^m)$. By the last two properties, the dilatation of h_m on $\mathbb{U}^{m-l} \cap \mathbb{J}^m$ is at most $4^l K_*$. Hence h is quasi-conformal on $\mathbb{C} \setminus \cap \mathbb{J}^m$. But by Lemma 2.8 $\cap \mathbb{J}^m = \omega(0)$ is a removable Cantor sets. Hence h admits a qc extension across $\omega(0)$.

Further, h is homotopic to h_0 rel $\omega(0)$. Indeed, let h^t, $1 - 2^{-m} \le t \le 1 - 2^{-(m+1)}$, be a homotopy between h_m and h_{m+1} given by (iii). Let $\epsilon_m = \max_i \text{diam}\, U_i^m$. As the $\mathbb{U}^m$ shrink to a Cantor set, $\epsilon_m \to 0$. As $h(U_i^{m-l}) = h^t(U_i^{m-l}) = \tilde{U}_i^{m-l}$, $1 - 2^{-m} \le t < 1$, the uniform distance between h and h^t is at most ϵ_{m-l}. It follows that the h^t uniformly converge to h as $t \to 1$. Hence h is homotopic to h_0 rel $\omega(0)$.

Since h_0 is a topological conjugacy by (i), h is a Thurston conjugacy. ☐

3.4. **Construction of h_0.** Let us supply the exterior $\mathbb{C} \setminus \text{cl}\, \mathbb{D}$ of the unit disk, with the hyperbolic metric ρ. The hyperbolic length of a curve γ will be denoted by $l_\rho(\gamma)$, while it Euclidean length will be denoted by $|\gamma|$.

Lemma 3.3. *Let A and $\tilde{A}$ be two (open) annuli whose inner boundary is the circle $\mathbb{T}$. Let $\omega : A \to \tilde{A}$ be a homeomorphism commuting with $P_0 : z \mapsto z^2$ near $\mathbb{T}$. Then ω admits a continuous extension to a map $A \cup \mathbb{T} \to \tilde{A} \cup \mathbb{T}$ identical on the circle.*

Proof. Given a set $X \subset A$, let $\tilde{X}$ denote its image image by ω. Let us take a configuration consisting of a round annulus $L^0 = \mathbb{A}[r, r^2]$ contained in A, and an interval $I_0 = [r, r^2]$. Let $L^n = P_0^{-n} L^0$, and I_k^n denote the components of $P_0^{-n} I^0$, $k = 0, 1, \dots, 2^n - 1$. The intervals I_k^n subdivide the annulus L^n into 2^n "Carleson boxes" Q_k^n.

Since the (multi-valued) square root map P_0^{-1} is infinitesimally contracting in the hyperbolic metric, the hyperbolic diameters of the boxes $\tilde{Q}_k^n$ are uniformly bounded by a constant C.

Let us now show that ω is a hyperbolic quasi-isometry near the circle, that is, there exist $\epsilon > 0$ and $A, B > 0$ such that

$$A^{-1}\rho(z,\zeta) - B \le \rho(\tilde z, \tilde\zeta) \le A\rho(z,\zeta) + B, \tag{3.2}$$

provided $z, \zeta \in \mathbb{A}(1, 1+\epsilon)$, $|z-\zeta| < \epsilon$.

Let γ be the arc of the hyperbolic geodesic joining z and ζ. Clearly it is contained in the annulus $\mathbb{A}(1, r)$, provided ϵ is sufficiently small. Let $t > 1$ be the radius of the circle $\mathbb{T}_t$ centered at 0 and tangent to γ. Let us replace γ with a *combinatorial geodesic* Γ going radially up from z to the intersection with $\mathbb{T}_t$, then going along this circle, and then radially down to ζ. Let N be the number of the Carleson boxes intersected by Γ. Then one can easily see that

$$\rho(z,\zeta) = l_\rho(\gamma) \asymp l_\rho(\Gamma) \asymp N,$$

provided $\rho(z,\zeta) \ge 10\log(1/r)$ (here $\log(1/r)$ is the hyperbolic size of the boxes Q^n_k).

On the other hand

$$\rho(\tilde z, \tilde\zeta) \le l_\rho(\tilde\Gamma) \le CN,$$

so that $\rho(\tilde z, \tilde\zeta) \le C\rho(z,\zeta)$, and (3.2) follows.

But quasi-isometries of the hyperbolic plane admit continuous extensions to $\mathbb{T}$ (see, e.g., [Th]). Finally, it is an easy exercise to show that the only homeomorphism of the circle commuting with P_0 if identical. □

Lemma 3.4. *Let f be a quadratic-like map. Let A and $\tilde A$ be two (open) annuli whose inner boundary is $J(f)$. Let $\omega : A \to \tilde A$ be a homeomorphism commuting with f near $J(f)$. Then ω admits a continuous extension to a map $A \cup J(f) \to \tilde A \cup J(\tilde f)$ identical on the Julia set.*

Proof. By the Straightening Theorem, we can assume without loss of generality that $f = P_c : z \mapsto z^2 + c$ is a quadratic polynomial. Let $R : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus \mathrm{cl}\,\mathbb{D}$ be the Riemann mapping normalized by $R(z) \sim z$ near infinity. It conjugates P_c to $P_0 : z \mapsto z^2$.

Let $\omega^\# = R \circ h \circ R^{-1} : \mathbb{C} \setminus \mathrm{cl}\, D \to \mathbb{C} \setminus \mathrm{cl}\, D$. Then $\omega^\#$ commutes with with P_0 in an open annulus attached to the circle $\mathbb{T}$. By Lemma 3.3, $\omega^\#$ continuously extends to $\mathbb{T}$ as id. Hence for any $\epsilon > 0$ there is an $r > 1$ such that $|\omega^\#(z) - z| < \epsilon$ for $z \in \mathbb{A}(1, r)$.

Let us show that the hyperbolic distance $\rho(\omega^\#(z), z)$ is bounded if $|z| < 2$. Clearly $\rho(\omega^\#(z), z) \le C(r)$, provided $1 < r \le |z| < 2$. Let $r^{\frac12} \le |z| \le r$, $\zeta = \omega^\#(z)$. Let us consider the hyperbolic geodesic γ joining z and ζ. Clearly $|\gamma| < O(\epsilon)$. Then $P_0^{-1}\gamma$ consists of two symmetric curves σ and $-\sigma$ of Euclidean length $O(\epsilon)$. One of these curves, say σ, joins z with a preimage u of $P_0(\zeta)$. Then $|z + u| > 2 - O(\epsilon) > \epsilon$, so that $-u \ne \zeta$. Thus $u = \zeta$.

As the square root map P_0^{-1} is infinitesimally contracting in the hyperbolic metric,

$$\rho(z,\zeta) \le l_\rho(\sigma) \le l_\rho(\gamma) = \rho(P_0(z), P_0(\zeta)) \le C(r).$$

Take now any point z in the annulus $\mathbb{A}(r^{\frac14}, r^{\frac12}\}$. Using the same argument we conclude that $\rho(z, \omega^\#(z)) \le C(r)$ (with the same $C(r)$. By induction, the same bound holds for all z.

Now we can complete the proof. Since the Riemann mapping R is a hyperbolic isometry, the hyperbolic distance between $\omega(z)$ and z in $\mathbb{C} \setminus J(P_c)$ is also bounded near $J(P_c)$. Hence the Euclidean distance $|z - \omega(z)|$ goes to 0 as $z \to J(f)$. It follows that the extension of ω as the identity on the Julia set is continuous. □

Corollary 3.5. *Let f and $\tilde{f}$ be two topologically equivalent quadratic-like map, and let ψ be a topological conjugacy between them. Let A and $\tilde{A}$ be two open annuli whose inner boundaries are $J(f)$ and $J(\tilde{f})$ respectively. Let $h : A \to \tilde{A}$ be a homeomorphism conjugating f and $\tilde{f}$ on these annuli. Then h matches with ψ on the Julia set, that is h admits a continuous extension to a map $A \cup J(f) \to \tilde{A} \cup J(\tilde{f})$ coinciding with ψ on the Julia set.*

Proof. Apply Lemma 3.4 to the homeomorphism $\omega = \psi^{-1} \circ h$ commuting with f. ☐

Lemma 3.6 ([DH2])**.** *If quadratic-like maps f and $\tilde{f}$ are topologically conjugate then there is conjugacy h_0 which is quasi-conformal outside the Julia sets.*

Proof. Given an annulus A, let $\partial_o A$ and $\partial_i A$ stand for its outer and inner boundary components. Let us select a closed fundamental annulus A for f with smooth boundary, and let $A^n = f^{-n}A$. Let $\tilde{A}$ and $\tilde{A}^n$ be similar objects for $\tilde{f}$. Then there is a diffeomorphism $\phi : A \to \tilde{A}$ such that

$$\phi(fz) = \tilde{f}(\phi z), \quad z \in \partial_i A.$$

This diffeomorphism can be lifted to a diffeomorphism $\phi_1 : A^1 \to \tilde{A}^1$ with the same qc dilatation and such that

$$\phi_1(z) = \phi(z),\ z \in \partial_o A^1, \quad \textit{and} \quad \phi_1(fz) = \tilde{f}(\phi_1 z),\ z \in \partial_i A^1.$$

In turn, A^1 can be lifted to a diffeomorphism $\phi_2 : A^2 \to A^2$ with the same dilatation, which matches with A^1 on $\partial_o A^2$ and respects dynamics on $\partial_i A^2$, etc.

By the Gluing Lemma from the Appendix, these diffeomorphisms glue together into a single quasi-conformal map $h_0 : A \setminus J(f) \to \tilde{A} \setminus J(\tilde{f})$ conjugating f and $\tilde{f}$.

On the other hand, let ψ be a topological conjugacy between f and $\tilde{f}$ near the Julia sets. Then by Corollary 3.5, h_0 matches with ψ on $J(f)$. ☐

3.5. **Adjustment of h_m.** Recall that p_m is the period of the little Julia sets J^m_j, and $F_m = f^{p_m}$ is the corresponding quadratic-like map near J^m_j. Let $\mathbb{U}^m = \cup U^m_j$ be a standard neighborhood of the little Julia orbit $\mathbb{J}^m = \cup B^m_j = \cup J^m_i$, with a definite space in between the U^m_j and definite annuli $U^m_j \setminus B^m_j$, and let $S_m : \mathbb{U}^m_j \setminus \mathbb{B}^m_j \to \mathbb{A}(1,4)$ be the standard straightenings (2.3). Its dilatation is bounded by a constant K_* depending only on the choice of secondary limbs and *a priori* bounds. Let $U^m_j(t) = S^{-1}_m \mathbb{A}(1,t)$ (note that $U^m_j \equiv U^m_j(4)$). The notation $\mathbb{U}^m(t)$ is self-evident.

We say that a homeomorphism $\phi : U^m_j(2) \setminus B^m_j \to \tilde{U}^m_j(2) \setminus \tilde{B}^m_j$ is *standard* near the bouquet B^m_j if it is identical in the standard coordinates on $U^m_j(2)$, that is,

$$\tilde{S}_m \circ \phi | U^m_j(2) = S_m. \tag{3.3}$$

The dilatation of such a map is bounded by K^2_*. Note also that by Corollary 3.5, the standard map admits a homeomorphic extension across the Julia bouquet.

We will now adjust the map h_m so that it will become standard near $\mathbb{J}^m$.

Lemma 3.7. *Take an l as in §3.3. Let a homeomorphism $h_m : (\mathbb{C}, \mathbb{J}^m) \to (\mathbb{C}, \tilde{\mathbb{J}}^m)$ be a conjugacy on $\mathbb{J}^m$ and be K_m-qc on $\mathbb{U}^{m-l} \setminus \mathbb{J}^m$. Then there is a homeomorphism*

$$\hat{h}_m : (\mathbb{C}, \mathbb{U}^m, \mathbb{J}^m) \to (\mathbb{C}, \tilde{\mathbb{U}}^m \tilde{\mathbb{J}}^m)$$

homotopic to h_m *rel* $(\mathbb{J}^m \cup (\mathbb{C} \setminus \mathbb{U}^{m-l}))$*, such that* $\mathrm{Dil}(\hat{h}^m|(\mathbb{U}^{m-1} \setminus \mathbb{J}^m)) \leq 4K_*^4 \cdot K_m$*, and* $h_m : \mathbb{U}^m(2) \setminus \mathbb{J}^m \to \tilde{\mathbb{U}}^m(2) \setminus \tilde{\mathbb{J}}^m$ *is standard.*

Proof. Let us consider a retraction $\psi_j^t : U_j(4) \setminus B_j \to U_j(4) \setminus U_j(2)$ which is the affine vertical contraction in the standard coordinates. Its dilation is bounded by $2K_*^2$. Let us extend the ψ_j^t to a homeomorphism $\psi : \mathbb{C} \setminus \mathbb{J} \to \mathbb{C} \setminus U(t)$ by identity on $\mathbb{C} \setminus \mathbb{U}(4)$. By the Gluing Lemma from the Appendix, ψ is also $2K_*^2$-qc.

Let us now define a homeomorphism $h^t : (\mathbb{C}, \mathbb{U}^t, \mathbb{J}) \to (\mathbb{C}, \tilde{\mathbb{U}}^t, \tilde{\mathbb{J}})$ as follows:

$$h^t|(\mathbb{C} \setminus \mathbb{U}^t) = \tilde{\psi}^t \circ h \circ (\psi^t)^{-1},$$

while $h^t : \mathbb{U}^t \to \tilde{\mathbb{U}}^t$ is standard. Then h^1 is a desired adjusted map (homotopic to $h^0 = h$ via the $\{h^t\}$). □

In what follows we will assume that h_m is adjusted as in Lemma 3.7, and will skip the "hat" in the notation for the adjusted map.

3.6. **Beginning of the construction of h_{m+1}.** Let p_m denote the combinatorial rotation number of the α-fixed of the Julia sets J_i^m. Consider the configurations $\mathcal{R}_i^m$ of $2p_m$ rays landing at the α-fixed and co-fixed points of the J_i^m. Let $\Omega_s^{m,0} \equiv \Omega_s^m$ stand for the component of $U_j^m \setminus \mathcal{R}_i^m$ containing J_s^{m+1}, and let $\Omega_s^{m,1} \subset \Omega_s^m$ be the component of $F_m^{-p_m}\Omega_s^m$ containing J_s^{m+1}, so that

$$G_m \equiv F_m^{\circ p_m} : \Omega_s^{m,1} \to \Omega_s^m \tag{3.4}$$

is a double branched covering. The boundaries of these domains are naturally marked with the standard coordinates. (*Marking* of a curve means its preferred parametrization.) As the map

$$h_m : (\mathbb{C}, U_j^m, \Omega_s^m, \Omega_s^{m,1}, J_s^{m+1}) \to (\mathbb{C}, \tilde{U}_j^m, \tilde{\Omega}_s^m, \tilde{\Omega}_s^{m,1}, \tilde{J}_s^{m+1})$$

is standard on the U_j^m, it respects this marking.

Since the configurations $(\cup\mathcal{R}_s^m, \cup\partial\Omega_s^{m,1})$ have bounded geometry (see §4 of Part I), there is a qc map with a bounded dilatation

$$\Psi_m : (\mathbb{C}, U_j^m, \Omega_s^m, \Omega_s^{m,1}) \to (\mathbb{C}, \tilde{U}_j^m, \tilde{\Omega}_s^m, \tilde{\Omega}_s^{m,1}) \tag{3.5}$$

coinciding with h_m on $\mathbb{C} \setminus \Omega^m$ and respecting the boundary marking (in particular, it conjugates $F_m : \partial\Omega_s^{m,1} \to \partial\Omega_s^m$ and $\tilde{F}_m : \partial\tilde{\Omega}_s^{m,1} \to \partial\tilde{\Omega}_s^m$). Moreover Ψ_m is homotopic to h_m rel $((\mathbb{C} \setminus \mathbb{U}^m) \cup \partial\Omega_s^m \cup \partial\Omega_s^{m,1})$, since all regions complementary to this set are simply connected Jordan domains.

Note however that unlike h_m, the map Ψ_m does not respect dynamics on the little Julia sets. We need to pay temporarily this price in order to make Ψ_m globally quasi-conformal.

3.7. **Construction of h_{m+1} in the immediately renormalizable case.** Let us consider the double covering (3.4). In the immediately renormalizable case,

$$G_m^{\circ n} 0 \in \Omega_s^{m,1},\ n = 0, 1, 2, \ldots$$

Moreover, there is a nest of topological disks

$$\Omega_s^{m,0} \supset \Omega_s^{m,1} \supset \Omega_s^{m,2} \supset \ldots$$

shrinking to the little Julia set J_s^{m+1}, and such that $G_m : \Omega_s^{m,n} \to \Omega_s^{m,n-1}$ is a branched double covering. The complement $Q_s^{m,n} = \Omega_s^{m,n-1} \setminus \Omega_s^{m,n}$ consists of 2^n quadrilaterals.

As $G_m : Q_s^{m,n} \to Q_s^{m,n-1}$ is an unbranched covering, the map $\Psi : Q_s^{m,1} \to \tilde{Q}_s^{m,1}$ can be lifted to a qc map

$$\Psi_{m,n} : Q_s^{m,n} \to \tilde{Q}_s^{m,n}$$

with the same dilatation homotopic to h_m rel the boundary. Hence all these maps glue together in a single qc map with the same dilatation

$$h_{m+1} : \Omega_s^m \setminus J_s^{m+1} \to \tilde{\Omega}_s^m \setminus \tilde{J}_s^{m+1} \tag{3.6}$$

equivariantly homotopic to h_m rel $\cup_n \partial \Omega^{m,n}$.

Let ψ^t be the corresponding homotopy, and ρ be the hyperbolic metric in $\tilde{\Omega}^m \setminus \tilde{J}^{m+1}$. Then by equivariancy $\rho(\psi^t(z), h_m(z)) \leq C$. Hence $|\psi^t(z) - h_m(z)| \to 0$ as $z \to J_s^{m+1}$ uniformly in t. It follows that the homotopy ψ^t extends across the little Julia set J_s^{m+1}. Thus the map (3.6) is extends across J_s^{m+1} to a homeomorphism homotopic to h_m rel $(\partial \Omega_s^m \cup J_s^{m+1})$.

Outside the $\bigcup_s \Omega_s^m$ let h_{m+1} coincide with h_m. This provides us with the desired map h_{m+1}.

4. Through the principal nest

In what follows we will assume that $R^m f \equiv F_m$ is not immediately renormalizable.

4.1. **Teichmüller distance between the configurations of puzzle pieces.** Let us make a choice of a standard neighborhood U^m of the Julia bouquet B^m and the corresponding standard straightening S_m, see (2.3). When F_{m-1} is not immediately renormalizable, this provides us with a family $\mathcal{Y}$ of puzzle pieces $Y_i^{(k)}$, see §2.6 of Part I.

In the immediately renormalizable case let us start the puzzle in a slightly different way. Namely, let us consider a *degenerate* domain of F_m (see §2.5 of Part I) bounded by external rays landing at fixed and co-fixed points $\alpha_{m-1} \equiv \beta_m, -\beta_m$, and two pieces of standard equipotentials of F_{m-1}. Then play the puzzle by cutting this domain with external rays of F_{m-1} landing at α_m, and pulling them back. One can easily see that this beginning is equally suitable for the puzzle game as the usual one.

As the puzzle pieces $Y_i^{(k)}$ are bounded by equipotentials and rays, they bear the *standard boundary marking*, i.e. the parametrization S_m^{-1} by the corresponding straight intervals or circle arcs.

Since $h_m : U^m \to \tilde{U}^m$ is the standard conjugacy (see (3.3)), it maps the pieces $Y_i^{(k)}$ to the corresponding tilde-pieces $\tilde{Y}_i^{(k)}$ respecting the boundary marking. Given some family of puzzle pieces $P_i \in \mathcal{Y}$ contained in some $Y \in \mathcal{Y}$, let us say that a homeomorphism

$$\phi : (Y, \cup P_i) \to (\tilde{Y}, \cup \tilde{P}_i)$$

is a *pseudo-conjugacy* if it is homotopic to h_m rel the boundary $(\partial Y, \cup \partial P_i)$. Note that if $f^l : P_i \to Y$ (or $f^l : P_i \to P_j$) for some iterate of f and some puzzle pieces of our family, then the pseudo-conjugacy ϕ is a true conjugacy between the boundary maps $f^l : \partial P_i \to \partial Y$ and $\tilde{f}^l | \partial \tilde{P}_i \to \partial \tilde{Y}$ (correspondingly ∂P_j instead of ∂Y).

In particular, the above terminology will be applied to the principal nest of puzzle pieces (see §3 of Part I):

$$Y^{(m,0)} \supset V^{m,0} \supset V^{m,1} \supset \dots, \qquad V_0^{m,n} \equiv V^{m,n}, \qquad \cap_n V^{m,n} = J^{m+1}, \tag{4.1}$$

and the corresponding generalized renormalizations $g_{m,n} : \cup_i V_i^{m,n} \to V^{m,n-1}$.

Teichmüllere distance dist_T between $(V^{m,n+1}, V_i^{m,n})$ and $(\tilde{V}^{m,n+1}, \tilde{V}_i^{m,n})$ is defined as $\inf_\phi \log K_\phi$ as ϕ runs over all qc pseudo-conjugacies $(V^{m,n+1}, \cup_i V_i^{m,n}) \to (\tilde{V}^{m,n+1}, \cup_i \tilde{V}_i^{m,n})$.

Main Lemma [[L2], §4)]. *The configurations $(V^{m,n+1}, V_i^{m,n})$ and $(\tilde{V}^{m,n+1}, \tilde{V}_i^{m,n})$ stay bounded Teichmüller distance away (independently of m and n).*

The rest of this section, except the final subsection,§4.10), will be occupied with the proof of this lemma which follows [L2], §4. As the level m is fixed, *in what follows we will skip the label m in the notations of $V_i^{m,n} \equiv V_i^n$, $g_{m,n} \equiv g_n$ etc.* (unless it may lead to a confusion). In what follows referring to a qc-map, we will mean that it has a definite dilatation (depending only on the selected limbs and *a priori* bounds).

4.2. **A point set topology lemma.** In the statement below, the objects involved need not have any dynamical meaning.

Lemma 4.1. *Let P_i be a family of closed Jordan disks with disjoint interiors contained in a domain Y, such that* $\text{diam}\, P_i \to 0$. *Let $\tilde{P}_i$, $\tilde{Y}$ be another family of disks with the same properties.*
• *Let $h : (Y, \cup P_i) \to (\tilde{Y}, \cup \tilde{P}_i)$ be a one-to-one map, which is a homeomorphism on $\cup P_i$ and on $X \equiv Y \setminus (\cup \operatorname{int} P_i)$. Then h is a homeomorphism.*
• *Let $h^i : (Y, \cup P_i) \to (\tilde{Y}, \cup \tilde{P}_i)$, $i = 0, 1$, be two homeomorphisms coinciding on $Y \setminus \cup \operatorname{int} P_i$. Then h^i are homotopic rel $Y \setminus \cup \operatorname{int} P_i$.*

Proof. Given an $\epsilon > 0$, there exists an N such that $\text{diam}(\tilde{P}_n) < \epsilon$ for $n > N$. Let $T = \cup_{1 \le i \le N} P_i$. Note that h is continuous on $X \cup T$.

Given a point $z \in Y$, let us show that h is continuous at it. This is certainly true if $z \in \cup \operatorname{int} P_i$, so let $z \in X$. We will show that

$$|h(z) - h(\zeta)| < 2\epsilon \tag{4.2}$$

for any nearby point $\zeta \in Y$. Indeed, if $\zeta \in X \cup T$ it follows from the above remark. Otherwise $\zeta \in P_j$ for some $j > N$, and there is point $u \in [z, \zeta] \cap \partial P_j$. Then

$$|h(z) - h(\zeta)| \le |h(z) - h(u)| + |h(u) - h(\zeta)|.$$

If ζ is sufficiently close to z then the first term is at most ϵ by continuity of $h|X$. As the second term is bounded by $\text{diam}(P_j) < \epsilon$, and (4.2) follows.

Let us now prove the second statement. As each P_i is simply connected, $h^0|P_i$ is homotopic to $h^1|P_i$ rel ∂P_i. Let $h^t : \cup P_i \to \tilde{P}_i$ be a corresponding homotopy. Extend it to the whole domain Y as h^0. We should check that this extension $h^t(z) : (Y, \cup P_i) \to (\tilde{Y}, \cup \tilde{P}_i)$ is continuous in two variables.

Note first that for $z \notin \cup_{1 \le i \le N} P_i \equiv T$,

$$|h^t(z) - h^0(z)| < \epsilon. \tag{4.3}$$

Given a pair (z,t), we will show that $|h^t(z) - h^\tau(\zeta)| < 3\epsilon$ as (ζ,τ) is sufficiently close to (z,t). To this end let us consider a few cases:

• If $z \in \operatorname{int} \cup P_i$, it is true since $h^t|P_i$ is a homotopy.
• If $z, \zeta \in T$, it is true since $h^t|T$ is a homotopy.
• If $z \in \partial T$ but $\zeta \notin T$, then for ζ sufficiently close to z,

$$|h^t(z) - h^\tau(\zeta)| = |h^0(z) - h^\tau(\zeta)| \leq |h^0(z) - h(\zeta)| + |h^\tau(\zeta) - h^0(\zeta)| < 2\epsilon$$

by continuity of h and (4.3).
• Let $z \notin T$. Then sufficiently close points ζ don't belong to T either. Hence by (4.3) and continuity of h,

$$|h^t(z) - h^t(\zeta)| \leq |h^0(z) - h^0(\zeta)| + |h^t(z) - h^0(z)| + |h^t(\zeta) - h^0(\zeta)| < 3\epsilon.$$

□

4.3. **Expanding sets.** Let us consider Yoccoz puzzle pieces $Y_i^{(N)}$ of depth N (see §2.6 of Part I), and let $\mathcal{Y}^{(N)}$ denote the family of puzzle pieces $Y_j^{(N+l)}$ such that

$$f^k Y_j^{(N+l)} \cap Y_0^{(N)} = \emptyset, \; k = 0 \ldots, l-1.$$

Let $K^{(N)} = \{z : F^k z \notin Y^{(N)}, \; k = 0, 1 \ldots\}$ Recall that an invariant set K is called *expanding* if there exist constants $C > 0$ and $\rho \in (0,1)$ such that

$$|DF^k(z)| \geq C\rho^k, \; z \in K, \; k = 0, 1, \ldots$$

Lemma 4.2. *For a given N, $\operatorname{diam} Y_s^{(N+l)} \to 0$ as $Y_s^{(N+l)} \in \mathcal{Y}^{(N)}$ and $l \to \infty$. Moreover, the set $K^{(N)}$ is expanding.*

Proof. Let us consider thickened puzzle pieces $\hat{Y}_i^{(N)}$ as in Milnor [M1] or §2.5 of Part I. Then $\operatorname{int}(F\hat{Y}_i^{(N)})$ contains $\hat{Y}_j^{(N)}$ whenever $FY_i^{(N)} \supset Y_j^{(N)}$ (recall that the $Y^{(N)}$ are closed). Hence the inverse map $F^{-1} : \hat{Y}_j^{(N)} \to \hat{Y}_i^{(N)}$ is contracting by a factor $\lambda < 1$ in the hyperbolic metrics of the pieces under consideration.

Let $Y_s^{(N+l)} \subset Y_i^{(N)}$. It follows that the hyperbolic diameter of $Y_s^{(N+l)}$ in $Y_i^{(N)}$ is at most λ^l, and the statement follows. □

4.4. **First landing maps.** Let us have a family of puzzle pieces P_i with disjoint interiors contained in a puzzle piece X, where as usual $P_0 \ni 0$ stands for the critical puzzle piece. Let us also have a Markov map $G : \cup P_i \to X$ which is univalent on all non-critical pieces P_i, $i \neq 0$, and the double branched covering on the critical one, P_0. The Markov property means that if $\operatorname{int}(GP_i \cap P_j) \neq \emptyset$ then $GP_i \supset P_j$. Let A be the corresponding Markov matrix: $A_{ij} = 1$ if $\operatorname{int}(GP_i \cap P_j) \neq \emptyset$, and $A_{ij} = 0$ otherwise.

Let $P \equiv P^0$. A string of labels $\bar{i} = (i(0), \ldots, i(l-1))$ is called *admissible* if $A_{i(k),i(k+1)} = 1$ for $k = 0, \ldots, l-2$), and $i(k) \neq 0$ for $k < l-1$. The length l of the string will be denoted by $|\bar{i}|$. To any admissible string corresponds a *cylinder* of rank l defined by the following property:

$$G^k P_{\bar{i}}^l \subset P_{i(k)}, \; k = 0, \ldots l-2, \;\; G^{l-1} P_{\bar{i}}^l = P_{i(l-1)}. \tag{4.4}$$

Note that G^{l-1} univalently maps $P_{\bar{i}}^l$ onto $P_{i(l-1)}$.

Let us denote by $\Omega_{\bar{i}} \equiv P^l_{\bar{i}}$ the cylinders mapped onto the critical puzzle piece (so that $i(l-1)=0$). The *first landing map*

$$T : \cup\Omega_i \to P_0 \tag{4.5}$$

is defined as follows: $Tz = G^{l-1}z$ for $z \in \Omega_{\bar{i}}$, $|\bar{i}| = l$. This map is univalent on all pieces Ω_i (identical on the critical piece Ω_0).

Lemma 4.3. *Let us have a K-qc pseudo-conjugacy $H : (X, \cup P_i) \to (\tilde{X}, \cup\tilde{P}_i)$ between G and $\tilde{G}$. Then there is a K-qc pseudo-conjugacy $\phi : (X, \cup\Omega_j) \to (\tilde{X}, \cup\tilde{\Omega}_j)$ which conjugates the first landing maps T and $\tilde{T}$.*

Proof. Let us pull H back to the pieces P_i, $i \neq 0$, that is, let us consider the map

$$H_1 : (P_i, \bigcup_j P^l_{ij}) \to (\tilde{P}_i, \bigcup_j \tilde{P}^l_{ij})$$

such that $\tilde{G} \circ H_1 | P_i = h \circ G | P_i$. Since H is a pseudo-conjugacy, H_1 matches with H on $\cup_{i\neq 0} \partial P_i$. Hence these maps glue together into a single map K-qc map equal to H_1 on $\cup\partial P_i$, and equal to H outside of it. We will keep notation H_1 for this map.

Let us do the same pull-back with H_1. We will obtain a K-qc pseudo-conjugacy

$$H_2 : (P, \cup P^1_i, \cup P^2_{ij}, \cup P^3_{ijk}) \to (\tilde{P}, \cup\tilde{P}^1_i, \cup\tilde{P}^2_{ij}, \cup\tilde{P}^3_{ijk}).$$

Repeating this procedure over again, we obtain a sequence of K-qc pseudo-conjugacies

$$H_s : \bigcup_{l\leq s} \bigcup_{|\bar{i}|=l} P^l_{\bar{i}} \to \bigcup_{l\leq s} \bigcup_{|\bar{i}|=l} \tilde{P}^l_{\bar{i}}.$$

By the Compactness Lemma from the Appendix we can pass to a limit K-qc map

$$\phi : \bigcup_{l,\bar{i}} P^l_{\bar{i}} \to \bigcup_{l,\bar{i}} \tilde{P}^l_{\bar{i}}.$$

By Lemma 4.1 this map is homotopic to h rel $(\partial X \cup \partial\Omega_j)$, and hence is a desired pseudo-conjugacy. □

Let us now do a bit more (assuming a bit more). Let us consider the generalized renormalization of G on P_0, that is, the first return map $g : \cup V_j \to P_0$. Let $b = g(0) = G^t 0$ be its critical value.

Lemma 4.4. *Let us have two K-qc pseudo-conjugacies $H_0 : (X, \cup P_i) \to (\tilde{X}, \cup\tilde{P}_i)$ and $H_1 : (P_0, b) \to (\tilde{P}_0, \tilde{b})$. Then there exist a K-qc pseudo-conjugacy $\psi : (P_0, \cup V_i) \to (\tilde{P}_0, \cup\tilde{V}_i)$ between g and $\tilde{g}$.*

Proof. As H and H' match on ∂P_0, they glue together into a singe K-qc pseudo-conjugacy $H : (X, \cup P_i, b) \to (\tilde{X}, \cup\tilde{P}_i, \tilde{b})$ coinciding with H_1 on P_0 and coinciding with H_0 on $X \setminus P_0$ (see the Gluing Lemma in the Appendix). By Lemma 4.3, there is a K-qc map $\phi : (X, \cup\Omega_j) \to (\tilde{X}, \cup\tilde{\Omega}_j)$ homotopic to h rel $(\partial X \cup \partial\Omega_j)$, and conjugating the first landing maps. As $H : b \mapsto \tilde{b}$, we have: $\phi : G^k 0 \mapsto \tilde{G}^k 0$, $k = 1, \dots, t$. In particular, ϕ respects the G-critical values: $G(0) \mapsto \tilde{G}(0)$.

Recall that the domains V_i are the pull-backs of the Ω_j by $G : P_0 \to X$, that is, the components of $(G|P_0)^{-1}\Omega_j$. It follows that ϕ can be lifted to a K-qc map $\psi : (P_0, \cup V_i) \to (\tilde{P}_0, \cup\tilde{V}_i)$ homotopic to h rel $(\partial P_0 \cup \partial V_i)$. (This lift is uniquely determined by the diagram $\tilde{G} \circ \psi|P_0 = \phi \circ G|P_0$ and the homotopy condition.)

This map ψ is the desired pseudo-conjugacy. □

4.5. **Initial constructions.** Now the reader should consult §3.2 of Part I of this paper [L3], where the initial Markov partition (3-3) of the Yoccoz puzzle piece $Y^{(0)}$ is constructed. We will apply it to the renormalized map F. Let us recall some notations. The first piece of the partition, $Y \equiv Y^{(0)}$, is bounded by the external rays landing at the fixed point α, and the equipotential E. The central piece of this partition, V^0, is the first piece of the principal nest. It is obtained by pulling back a puzzle piece $Z_\nu^{(1)}$ attached to the co-fixed point α' (that is, $F(\alpha') = F(\alpha)$). There is a double branched covering $F^s : V^0 \to Z_\nu^{(1)}$. All the puzzle pieces of the initial partition intersecting the Julia set $J(F)$ are univalent pull-backs of either Y or V^0. Let us denote the pieces of this partition by P_i, in such a way that $P_0 \equiv V^0$, $P_i \equiv Z_i^{(1)}$, $i = 1, \dots p-1$, where p is the number of external rays of F landing and α. With these notations,

$$Y \cap J(F) = \bigcup (P_i \cap J(F)) \cup K, \tag{4.6}$$

where K is the residual Cantor set (of the points whose orbits never land at $\cup_{0\leq i\leq p-1} P_i$).

Lemma 4.5. *In the decomposition (4.6),* $\operatorname{diam} P_i \to 0$ *and the set K is a removable Cantor set.*

Proof. The first statement follows from Lemma 4.2. To prove removability of K, let us consider the domains Q_1 and Q_2 introduced in §3.2 of Part I. Then $F^p Q_i$ covers $Q_1 \cup Q_2$, and K is the set of points which never escape $Q_1 \cup Q_2$. By a little thickening of these domains, we obtain a Bernoulli map $F^p : \hat{Q}_1 \cup \hat{Q}_2 \to \mathbb{C}$ (so that $\operatorname{int}(F^p \hat{Q}_i)$ contains $\hat{Q}_i$). By Lemma 2.9, the Julia set $\hat{K}$ of this map is removable. All the more, $K \subset \hat{K}$ is removable (one can actually see that $K = \hat{K}$). □

Let us now go back to §4.2 of Part I where the fundamental domain Q near the fixed point α is constructed. Recall that $\gamma \in Y^{(1)}$ is the periodic point of period p, $\gamma' = -\gamma$ is the "co-periodic" point, and $\mathcal{R}(\gamma')$ is the family of rays landing at γ'. Also, let $X = Y^{(0)} \cup_{1\leq i\leq p-1} P_i$. This domain is bounded by the rays landing at α and equipotential $F^{-1}E$.

Furthermore D is the connected component of $Y^{(1)} \setminus \mathcal{R}(\gamma')$ attached to α, and $F^{-p} : D \to F^{-p}D$ is the branch of the inverse map fixing α.

Let us also consider quadrilaterals $D^* = D \cap Y^{(1+p)}$ and $Q^* = Q \cap Y^{(1+p)}$ obtained by cutting D and Q with the equipotential $F^{-p-1}E$. Note that $D \setminus D^* = Q \setminus Q^*$ consists of two quadrilaterals which don't contain points of the Julia set $J(F)$. Let $Q^*_{-k} = F^{-pk}Q^*$, $k = -1, 0, 1, \dots$, and $Q^*_{-2} = X \setminus F^p D$ (see Figure 2). Note that $J(F) \cap X$ is tiled into the pieces Q^*_{-k}, $k = -2, -1, \dots$

Lemma 4.6. *The hyperbolic diameter of the domains Q^*_{-k}, $k = -2, -1, 0, \dots,$ in Y is bounded. Moreover, if $|k - j| > 1$ then there is a definite space in between Q_{-k} and Q_{-j} in Y.*

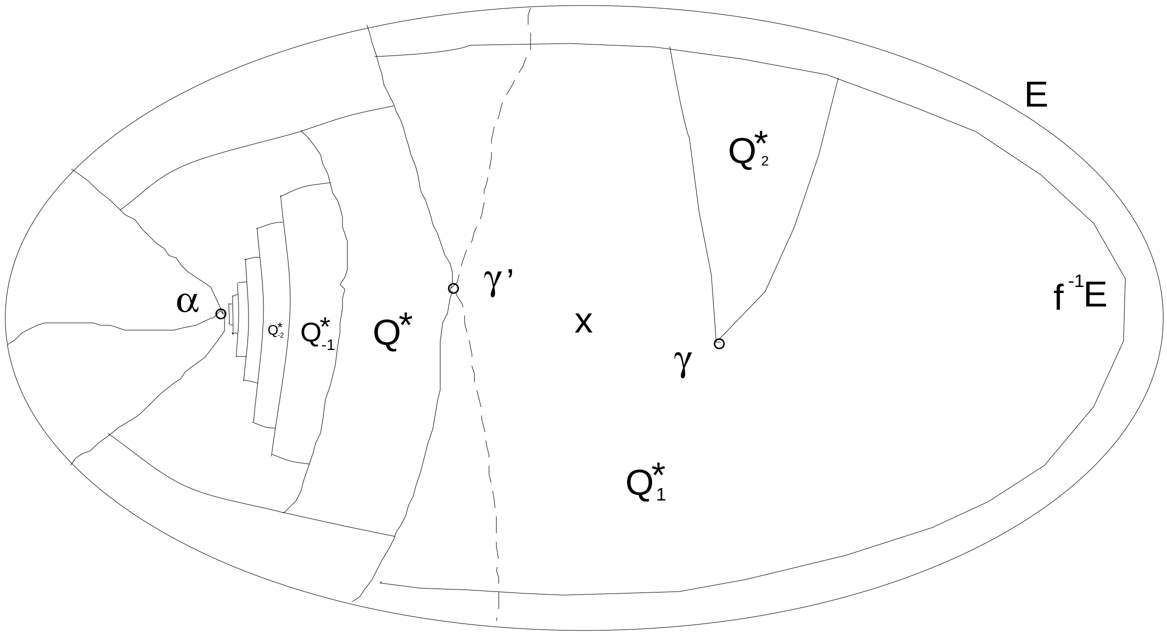

Figure 2. Initial tiling.

Proof. By the secondary limbs and *a priori* bounds assumptions, geometry of the configuration $(Y, Y^{(1)}, Y^{(1+p)}, \mathcal{R}(\gamma), \mathcal{R}(\gamma'))$ is bounded (see §4.1 of Part I). Hence Q^*_{-2} and Q^*_1 have a bounded hyperbolic diameter in Y. For the same reason, Q^* has a bounded hyperbolic diameter in F^pD^*. As $F^{-p} : F^pD^* \to F^pD^*$ is a hyperbolic contraction, the diameters of Q^*_{-k} in F^pD are bounded by the same constant. All the more, they are bounded in a bigger domain Y.

To prove the second statement, note that by bounded geometry of the initial ray-equipotential configurations, there is a definite space in between Q^*_{-1} and the Q^*_{-k}, $k = 0, 1 \dots$ For the same reason, there is a definite annulus $T_0 \subset F^pD^*$ about Q^*_0 which does not intersect Q^*_{-k}, $k = 2, 3 \dots$. Then $T_{-i} = F^{-ip}T_0 \subset F^pD^*$ is the annulus with the same modulus going around Q^*_{-i} and disjoint from Q^*_{-k} with $|k - i| > 1$. □

Our first essential step towards the Main Lemma is the following:

Lemma 4.7. *The Teichmüller distance between the configurations* $(Y, \cup P_i, \cup Q^*_{-k})$ *and* $(\tilde{Y}, \cup \tilde{P}_i, \cup \tilde{Q}^*_{-k})$ *is bounded.*

Proof. Recall that $F^s(V^0) = P_\nu$, and $F(P_i)$ univalently covers Y. Let us consider a point $a = F^{s+1}0 \in X$. We will construct a qc map $(Y, a) \to (\tilde{Y}, \tilde{a})$ respecting the boundary marking.

By §4.1 of Part I, geometry of the configuration $(Y, Y^{(1)}, Y^{(1+p)}, \mathcal{R}(\gamma), \mathcal{R}(\gamma'))$ (and the corresponding tilde one) is bounded , so that there is a qc pseudo-conjugacy

$$\phi : (Y, Y^{(1)}, Y^{(1+p)}, \mathcal{R}(\gamma), \mathcal{R}(\gamma')) \to (\tilde{Y}, \tilde{Y}^{(1)}, Y^{(1+p)}, \tilde{\mathcal{R}}(\gamma), \tilde{\mathcal{R}}(\gamma')).$$

In particular, this map conjugates $F^p : Q^* \to F^pQ^*$ to the corresponding tilde map.

As F^p univalently maps Q^*_{-k-1} onto Q^*_{-k}, ϕ can be re-defined on the Q^*_{-k}, $k \geq 0$, in such a way that it becomes the pseudo-conjugacy between the configurations

$$\phi : (Y, Y^{(1)}, \cup Q^*_{-k}) \to (\tilde{Y}, \tilde{Y}^{(1)}, \cup \tilde{Q}^*_{-k}) \tag{4.7}$$

with the same dilatation. (Just let $\phi(z) = \tilde{F}^{-kp} \circ \phi \circ F^{kp}(z)$ for $z \in Q^*_{-k}$).

It follows that $\phi(a)$ and $\tilde{a}$ belong to the same piece of the family $\{Q^*_{-k}\}_{k=-2}^{\infty}$ By Lemma 4.6 the hyperbolic distance between $\phi(a)$ and $\tilde{a}$ in $\tilde{Y}$ is bounded.

By the Moving Lemma from the Appendix, there is a qc map $\psi : \tilde{Y} \to \tilde{Y}$ identical on the boundary and carring $\phi(a)$ to $\tilde{a}$. Hence $\phi_1 = \psi \circ \phi : (Y, a) \to (\tilde{Y}, \tilde{a})$ is a qc map (with a definite, though bigger, dilatation) respecting the boundary marking.

Consider now the double branched covering $F^{s+1} : (V^0, 0) \to (Y, a)$ with the critical point at 0, and the corresponding tilde map. As $\phi_1 : (Y, a) \to (\tilde{Y}, \tilde{a})$ respects the critical values for these maps, it can be lifted to a map $\phi_2 : V^0 \to \tilde{V}^0$ with the same dilatation respecting the boundary marking.

Let us now construct a qc pseudo-conjugacy ϕ_3 between corresponding non-critical puzzle pieces P_i and $\tilde{P}_i$. It is easy as any non-central puzzle piece P_i under some iterate F^{l_i} is univalently mapped onto either Y or V^0. In the first case let ϕ_3 be the pullback of $\phi : Y \to \tilde{Y}$; in the second let it be the pull-back of ϕ_2. This pull-back preserves the dilatation and respects the boundary marking. This provides us with a qc map $\phi_3 : \cup P_i \to \cup \tilde{P}_i$ respecting the boundary marking of the puzzle pieces.

The latter property means that ϕ_3 matches with h on $\cup \partial P_i$. By the first part of Lemma 4.5 and Lemma 4.1, these maps glue together into a single homeomorphism coinciding with ϕ_3 on $\cup P_i$ and with h outside, homotopic to h rel $\partial Y \cup \partial P_i$ (we will still denote it ϕ_3).

By the Gluing Lemma from the Appendix, this homeomorphism is qc on $Y \setminus K$. By the second part of Lemma 4.5, the residual set K is removable, and thus ϕ_3 is automatically quasi-conformal across it (with the same dilatation). □

The next step towards the Main Lemma is the following:

Lemma 4.8. *The configurations $(V^0, \cup V^1_i)$ and $(\tilde{V}^0, \cup \tilde{V}^1_i)$ stay bounded Teichmüller distance away.*

Proof. Let us consider the first return $b = g_1 0$ of the critical point back to V^0. We will construct a qc map

$$H : (V^0, b) \to (\tilde{V}^0, \tilde{b}) \tag{4.8}$$

respecting the boundary marking.

Let $u = F^{s+1} b \in X$ (where F^s maps V^0 onto P_ν). Let ϕ be a pseudo-conjugacy given by Lemma 4.7.Then $\phi(u)$ and $\tilde{u}$ belong to the same piece of the tiling $X \cap J(F) = \cup_{-\infty < k \leq 2} (Q^*_{-k} \cap J(F))$. By Lemma 4.6, the hyperbolic diameters of these pieces in Y (and the corresponding tilde-pieces) are bounded by a constant ρ dependent only on the selected limbs and *a priori* bounds. Hence $\rho_{\tilde{Y}}(\phi(u), \tilde{u}) \leq \rho$.

Let $a = F^{s+1} 0$, as in the proof of Lemma 4.7. Assume that $a \in Q_k$, $u \in Q_j$. Let us consider two cases:

• Let $|k-j| \le 1$. Then $\rho_{\tilde{Y}}(u,a) \le 2\rho$. Hence there is an annulus $C \subset Y$ going around a and u with $\mod C \ge \mu(\rho) > 0$. As $F^{s+1} : (V^0, 0, b) \to (Y, a, u)$ is a double branched covering with critical point at 0, the pull-back C_0 of this annulus to V_0 has modulus at least $\mu(\rho)/2$. Hence $\mod (\phi(C_0)) \ge K^{-1}\mu(\rho)$, where $K = \mathrm{Dil}(\phi)$ depends only on the selected limbs and *a priori* bounds. Hence $\rho_{\tilde{V}^0}(\phi b, 0)$ is bounded. For the same reason, $\rho_{\tilde{V}^0}(\tilde{b}, 0)$ is bounded, and hence $\rho_{\tilde{V}^0}(\phi(b), \tilde{b})$ is bounded.

By the Moving Lemma from the Appendix, there is a qc map $\psi : (\tilde{V}^0, \phi(b)) \to (\tilde{V}^0, \tilde{b})$, identical on the boundary. Then $H = \psi \circ \phi$ is a desired map (4.8).

• Let now $|k-j| > 1$. Then by Lemma 4.6, there is a definite space in between Q_k^* and Q_j^* (and between the corresponding tilde-sets). By the Moving Lemma, there is a qc map $\psi : (\tilde{Y}, \phi(a), \phi(u)) \to (\tilde{Y}, \tilde{a}, \tilde{u})$, identical on $\partial\tilde{Y}$. This map lifts to a qc map (4.8) (with the same dilatation).

So, we have constructed a qc map (4.8) which carries the critical value $b = g_1(0)$ to the critical value $\tilde{b} = \tilde{g}_1$. Lemma 4.4 completes the proof. ☐

4.6. **Inductive step (non-central case).** Let us now inductively estimate the Teichmüller distance between the configurations $(V^{n-1}, \cup V_i^n)$ and $(\tilde{V}^{n-1}, \cup \tilde{V}_i^n)$. Let τ_n stand for the maximum of this Teichmüller distance and $\log \mathrm{Dil}(h)$, where as above, h stands for the conjugacy between F and $\tilde{F}$). Recall that $\mu_n = \mod (V^{n-1} \setminus V^n)$ denote the principal moduli.

The following lemma is the main step of our construction.

Lemma 4.9. *Let $\mu_n \ge \bar{\mu} > 0$ and $\tau_n \le \bar{\tau}$. Assume that $g_n(0) \in V_k^n$ with $k \ne 0$, that is, the return to level $n-1$ is non-central. Then $\tau_{n+1} \le \tau_n + O(\exp(-\mu_n/4))$, with a constant depending only on $\bar{\mu}$.*

Remark. We don't assume that the non-critical puzzle-pieces V_i^n, $i \ne 0$, are well inside V^{n-1}, since this is not the case on the levels which immediately follow long central cascades (see Theorem II of Part I). to be degenerate which actually happens in the beginning.

Proof. Let us skip n in the notations of the objects of level n, so that $V_i^n \equiv V_i$, $g_n \equiv g$, $\mu_n \equiv \mu$, etc. Also, let $V^{n-1} \equiv \Delta$ and $g(0) \equiv c_1$. As above, the corresponding objects for $\tilde{f}$ are marked with tilde. Thus we have two generalized polynomial-like maps $g : \cup V_i \to \Delta$ and $\tilde{g} : \cup \tilde{V}_i \to \tilde{\Delta}$, which are pseudo-conjugate by a $K = e^{\tau}$-qc map

$$\phi : (\Delta, \cup V_i) \to (\tilde{\Delta}, \tilde{V}_i). \tag{4.9}$$

The objects on the next level, $n+1$, will be marked with prime: $V^{n+1} \equiv V'$, $g' \equiv g_{n+1}$ etc. (where g' is *not* the derivative of g). So $g' : \cup V_j' \to \Delta'$ is the generalized renormalization of g, $\Delta' \equiv V_0$.

Let $\lambda(\nu)$ be the maximal hyperbolic distance between the points in the hyperbolic plane enclosed by an annulus of modulus ν. Note that $\lambda(\nu) = O(e^{-\nu})$ as $\nu \to \infty$ (see Appendix A1 in Part I). Set $\lambda = \lambda(\mu)$.

Our goal is to lift ϕ to a $K(1 + O(\lambda))$-qc pseudo-conjugacy

$$\phi' : (\Delta', \cup V_i') \to (\tilde{\Delta}', \tilde{V}_i'). \tag{4.10}$$

The problem is that ϕ need not respect the positions of the critical values: $\phi(c) \neq \tilde{c}$.

Let us consider the first landing map $T : \cup\Omega_j \to V^0$. By Lemma 4.3, the pseudo-conjugacy ϕ admits the pull-back to a K-qc pseudo-conjugacy

$$\phi_1 : (\Delta, \cup\Omega_j) \to (\tilde{\Delta}, \cup\tilde{\Omega}_j). \tag{4.11}$$

This *localizes* the positions of the critical values in the sense that $\phi_1(c_1)$ and $\tilde{c}_1$ belong to the same domain $\Omega_s \subset V_k$ (see Figure 3) and hence the hyperbolic distance between these points in $\tilde{V}_k$ is at most λ.

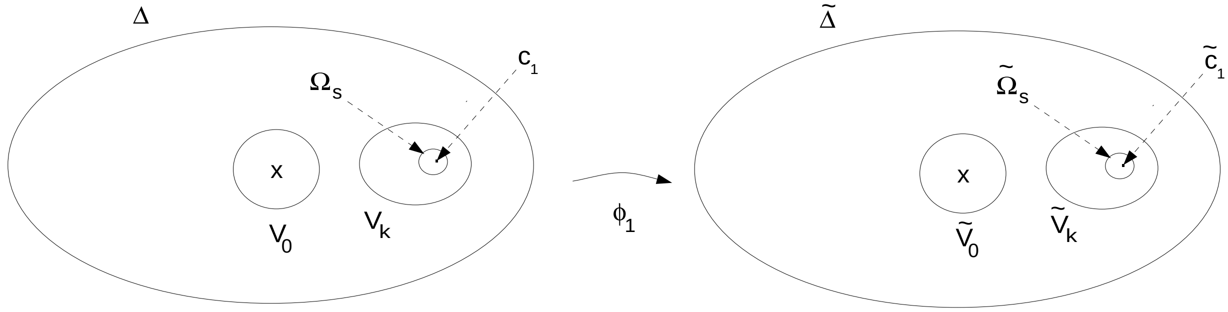

Figure 3. Localization of the critical values.

By the Moving Lemma from Appendix, we can find a $(1+O(\lambda))$-qc map

$$\psi : (\tilde{V}_k, \phi_1(c_1)) \to (\tilde{V}_k, \tilde{c}_1) \tag{4.12}$$

identical outside $\tilde{V}_k$. Then the map

$$\phi_2 = \psi \circ \phi_1 : (\Delta, \cup V_i, c) \to (\Delta, \cup V_i, \tilde{c})$$

is a $K(1+O(\lambda))$-qc pseudo-conjugacy respecting the critical values.

Let $\{U'_j\}$ be the family of the components of the $\{(g|\Delta')^{-1}V_i\}$. The the map ϕ_2 can be lifted to a $K(1+O(\lambda))$-qc pseudo-conjugacy

$$H : (\Delta', U'_i) \to (\tilde{\Delta}', \tilde{U}'_i). \tag{4.13}$$

However U'_i are not the same as V'_j (components of $\{(g|\Delta')^{-1}\Omega_i\}$), so we have to do more: We will localize the positions of the critical values $a = g'c$ and $\tilde{a}$ in Δ', and construct a $K(1+O(\lambda))$-qc map

$$\phi'_0 : (\Delta', a) \to (\tilde{\Delta}', \tilde{a}) \tag{4.14}$$

respecting the boundary marking. The argument depends on the position of a-points. Let $a_1 = g(a) \in V_j$.

Case (i). Assume V_j is non-critical and different from V_k. Let $a_1 \in \Omega_l$. Then the annuli $V_j \setminus \Omega_l \subset V_j$ and $V_k \setminus \Omega_s \subset V_k$ are disjoint (recall that $c_1 \in \Omega_s$). Hence by the Moving Lemma, there is a $1+O(\lambda)$-qc map

$$\psi_1 : (\tilde{\Delta}, \phi_1(c_1), \phi_1(a_1)) \to (\tilde{\Delta}, \tilde{c}_1, \tilde{a}_1)$$

identical outside $(\tilde{V}_k \cup \tilde{V}_j)$ (where ϕ_1 is the map (4.11)). With this map instead of (4.12), the above construction leads to a map (4.13) which already respects the critical values: $H(a) = \tilde{a}$. Then we can let $\phi_0' = H$.

Case (ii). Assume that $V_j = V_k$.

• Assume first that the hyperbolic diameter of the set of four points $\{\tilde{c}_1, \tilde{a}_1, \phi_1(c_1), \phi_1(a_1)\}$ in $\tilde{V}_k$ does not exceed $\sqrt{\lambda}$. Then the hyperbolic distance between the points $\tilde{a}_1$ and $H(a_1)$ in $\tilde{\Delta}'$ is $O(\sqrt{\lambda})$ (where H is the map (4.13)). Hence there is a $(1+O(\lambda^{1/4}))$-qc map $\psi_2 : (\Delta', H(a_1)) \to (\Delta', \tilde{a})$ identical on $\partial\Delta'$. Define now the map (4.14) as $\psi \circ H$.

• Otherwise the hyperbolic distance between the pairs $(\phi_1(a_1), \tilde{a}_1)$ and $(\phi_1(c_1), \tilde{c}_1)$ in $\tilde{V}_k$ is greater than $\sigma\sqrt{\lambda}$ (since these is an annulus of modulus μ separating one pair from another). Then there are separating annuli S_i about these pairs with $\mod (S_i) \geq q\sqrt{\lambda}$ (where $\sigma > 0$ and $q > 0$ depend only on the choice of limbs and *a priori* bounds). By the Moving Lemma, we can simultaneously move these points to the right positions by a $(1 + O(\sqrt{\lambda}))$-qc map

$$\psi_2 : (\tilde{\Delta}, \tilde{V}_k, \phi(a_1), \phi(c_1)) \to (\tilde{\Delta}, \tilde{V}_k, \tilde{(}a_1, \tilde{c}_1)),$$

identical on $\tilde{\Delta} \setminus \tilde{V}_k$. Using this map instead of (4.12) we come up with a $(1 + O(\sqrt{\lambda}))$-qc map (4.13) respecting the critical values of g: $H(a) = \tilde{a}$.

Case (iii). Let us finally assume that $V_j = V_0$ is critical. Then a belongs to a pre-critical puzzle-piece $V_t' \subset \Delta'$. Since $\mod(\Delta' \setminus V_t') \geq \mu/2$, the hyperbolic distance between $H(a)$ and $\tilde{a}$ in Δ' is $O(\sqrt{\lambda})$ (where H is the map (4.13)). By the Moving Lemma, there is a $(1+O(\sqrt{\lambda}))$-qc map

$$\psi_3 : (\Delta', \phi(a)) \to (\Delta', \tilde{a}).$$

Let us now define a map (4.14) as follows: $\phi_0' = \psi_3 \circ H$.

So in all cases we have constructed a $(1 + O(\lambda^{1/4}))$-qc map (4.14). It is still not the desired map (4.10), though. Now Lemma 4.4 completes the proof. □

4.7. **Through a central cascade.** Let $V^m \supset V^{m+1} \supset \ldots \supset V^{m+N}$ be a cascade of central returns, so that the critical value $g_{m+1}0$ belongs to V_k, $k = m+1, \ldots, m+N-1$, but escapes V^{m+N}.

Lemma 4.10. *Let $\mu_m \geq \bar{\mu} > 0$ and $\tau_m \leq \bar{\tau}$. Then for $k \leq N+1$, $\tau_{m+k} \leq \tau_m + O(\exp(-\mu_m/4))$, with a constant depending only on $\bar{\mu}$.*

Proof. We will adjust the proof of Lemma 4.9 to this situation. Let $g = g_{m+1}$, $\mu = \mod (V^m \setminus V^{m+1})$, etc. By definition, there is a $K = e^{\tau}$-qc pseudo-conjugacy:

$$\phi : (V^m, \cup V_i^{m+1}) \to (\tilde{V}^m, \cup \tilde{V}_i^{m+1}).$$

Let us consider the first landing map $T : \cup\Omega_j \to V^{m+1}$ corresponding to g, $\Omega_0 = V^{m+1}$. By Lemma 4.3, T and $\tilde{T}$ are pseudo-conjugate by a K-qc map

$$\phi_1 : (V^m, \cup\Omega_j) \to (\tilde{V}^m, \cup\tilde{\Omega}_j).$$

Let us take a family of puzzle pieces $V_i^{m+1} \subset A^{m+1} = V^m \backslash V^{m+1}$ and pull them back to the annuli $A^{m+2}, \dots, A^{m+N}$. We obtain a family of puzzle pieces W_i^{m+k}, together with a Bernoulli map

$$G : V^{m+N} \bigcup_{k,i} W_i^{m+k} \to V^m \tag{4.15}$$

(see §3.6 of Part I). Similarly let Ω_l^{m+k} stand for the pull-backs of the $\Omega_j \equiv \Omega_j^{m+1}$, $j \neq 0$, to the A^{m+k}, $k = 1, \dots N$. If $W_i^{m+k} \supset \Omega_l^{m+k}$ then

$$\text{mod } (W_i^{m+k} \setminus \Omega_l^{m+k}) \geq \mu,$$

so that the dynamically defined points are well localized by these puzzle pieces.

Let us now lift ϕ_1 to the annuli $A^{m+k} \to \tilde{A}^{m+k}$, $k = 2, \dots, N$. We obtain a K-qc map

$$\phi_2 : (V^m \setminus V^{m+N}, \cup W_i^{m+k}, \cup \Omega_l^{m+k}) \to (\tilde{V}^m \setminus \tilde{V}^{m+N}, \cup \tilde{W}_i^{m+k}, \cup \tilde{\Omega}_l^{m+k}) \tag{4.16}$$

respecting the boundary marking.

Let $c_1 \equiv g(0) \in P_l^{m+N} \subset V_k^{m+N}$. By the Moving Lemma, there is a $(1 + O(e^{-\mu}))$-qc map

$$\psi : (\tilde{V}^m, \tilde{V}_k^{m+N}, \phi_2(c_1)) \to (\tilde{V}^m \tilde{V}_k^{m+N}, \tilde{c}_1),$$

identical outside $\tilde{V}_k^{m+N}$. Then the map

$$\phi_3 = \psi \circ \phi_2 : (V^m \setminus V^{m+N}, \bigcup_{1 \leq k \leq N,\, i \neq 0} W_i^{m+k}, c_1) \to (\tilde{V}^m \setminus \tilde{V}^{m+N}, \bigcup_{1 \leq k \leq N,\, i \neq 0} \tilde{W}_i^{m+k}, \tilde{c}_1) \tag{4.17}$$

is $K(1 + O(e^{-mu}))$-qc, respects the boundary marking and positions of the critical values.

Consider now the topological disks Q_1 and Q_2 in V^{m+N} univalently mapped by g onto V^{m+N}. The Bernoulli map $g : Q_1 \cup Q_2 \to V^{m+n}$ produces a family of cylinders $Q_{\bar{i}}^t$, $\bar{i} = (i(0), i(1), \dots, i(t-1))$, such that

$$g^j Q_{\bar{i}}^t \subset Q_{i(j)}, \quad g^t Q_{\bar{i}} = V^{m+N}.$$

Let $\mathbb{Q}^t = \bigcup_{\bar{i}} Q_{\bar{i}}^t$, $Q^0 \equiv V^{m+N}$. Moreover, by Lemma 2.9, the residual set $K = \cap \mathbb{Q}^t$ is removable.

The map ϕ_3 can be consecutively lifted to the maps

$$\omega_t : \mathbb{Q}^{t-1} \setminus \mathbb{Q}^t \to \tilde{\mathbb{Q}}^{t-1} \setminus \tilde{\mathbb{Q}}^t, \ t = 1, 2, \dots$$

with the same dilatation respecting the boundary marking. By the Gluing Lemma, they are organized in a single qc map

$$\omega : V^{m+N} \setminus K \to \tilde{V}^{m+N} \setminus \tilde{K}$$

with the same dilatation. As K is removable, this map automatically extends across K:

$$H : (V^{m+N}, \cup U_i^{m+N+1}, Q_1, Q_2) \to (\tilde{V}^{m+N}, \cup \tilde{U}_i^{m+N+1}, \tilde{Q}_1, \tilde{Q}_2), \tag{4.18}$$

where $U_i^{m+N+1} \subset V^{m+N}$ are the components of $g^{-1} W_j^{m+N}$, $U_0^{m+N+1} \equiv V_0^{m+N+1}$. Note that mod $(V^{m+N} \setminus U^{m+N+1}) \geq \mu/2$.

The maps (4.18) and (4.17) glue together into a single $K(1+O(e^{-mu}))$-qc map

$$\phi_4 : (V^m, \bigcup_{1\le k\le N,\, i\neq 0} \bigcup W_i^{m+k}, V^{m+N}) \to (\tilde V^m, \bigcup_{1\le k\le N} \bigcup_{i\neq 0} \tilde V_i^{m+N}, \tilde V^{m+N}).$$

Take now a family of cylinders $W_{\bar i}^{m+k}$ of the Bernoulli map (4.15) (where $\bar i$ are finite strings of symbols). The map ϕ_4 is naturally lifted to a qc pseudo-conjugacy Φ with the same dilatation which respects this family of cylinders. Moreover, every $W_{\bar i}^{m+k}$ contains a piece $V_{\bar i}^{m+k}$ such that

$$G^l V_{\bar i}^{m+k} = V^{m+k-1}, \;\; where \;\; l = |\bar i|,$$

and all puzzle pieces V_j^{m+k} are obtained in such a way. As ϕ_4 respects the V^{m+k-1}-pieces, $k \le N$, the new map Φ respects the V_j^{m+k}-pieces. Thus Φ is a $K(1+O(e^{-\mu}))$-qc pseudo-conjugacy between g_{m+k} and $\tilde g_{m+k}$, so that $\tau_{m+k} \le \log K + O(e^{-\mu})$, $k = 1, \dots m+N$.

Let us proceed with the estimate of τ_{m+N+1}. Take the first return a of the critical point back to V^{m+N}, and construct a $K(1+O(e^{-\frac{\mu}{4}}))$-qc map

$$\phi_0' : (V^{m+N}, a) \to (\tilde V^{m+N}, \tilde a) \tag{4.19}$$

To this end let us go through Cases (i), (ii), (iii) of the proof of Lemma 4.9 using the $\{W_i^{m+N}\}$ in place of $\{V_i\}$ and V^{m+N} in place of $V^{m+1} \equiv \Delta'$.

In the first two cases the argument is the same as above. However, the last case is different since the pre-critical puzzle-pieces Q_1 and Q_2 are not necessarily well inside of V^{m+N}. To take care of this problem let us consider the first "escaping moment" t when $b \equiv g^t a \notin Q_1 \cup Q_2$. Then $b \in U_i^{m+N+1}$ for some U-domain from (4.18). Then there is a domain $\Lambda \subset Q_1 \cup Q_2$ containing a which is univalently mapped onto U_i^{m+N+1} by g^t. Moreover

$$\text{mod } (Q \setminus \Lambda) = \quad \text{mod } (V^{m+N} \setminus U_i^{m+N+1}) \ge \mu.$$

By means of $g : Q_1 \cup Q_2 \to V^{m+N}$, the map (4.18) can be turned into a qc map (with the same dilatation)

$$H_1 : (V^{m+N}, \Lambda) \to (\tilde V^{m+N}, \tilde\Lambda)$$

(coinciding with H outside $Q_1 \cup Q_2$). This gives us an appropriate localization of the a-points. The Moving Lemma turns H_1 into (4.19).

Lemma 4.4 completes the proof. □

4.8. **Proof of the Main Lemma.** Let $\{i(k)\}$ be the sequence of non-central levels in the principal nest $V^0 \supset V^1 \supset \dots$ Let $i(n-1)+1 < m \le i(n)+1$. By Lemma 4.10,

$$\tau_m \le \log K^* + O(\sum_{k=0}^{n-1} \exp(-\frac{1}{4}\mu_{i(k)+1})). \tag{4.20}$$

But by Theorem III from Part I [L3], the principal moduli $\mu_{i(k)+1}$ grow at linear rate: $\mu_{i(k)+1} \ge Bk$, where the constant B depends only on μ_1. Hence the sum (4.20) is bounded by $\log K_* + C(\mu_1)$.

In turn, by Theorem I of Part I the modulus μ_1 is bounded by a constant depending only on the selected limbs and *a priori* bounds. Hence $\tau_n \le \log K_* + A$, where A depends only on the choice of limbs and *a priori* bounds. The Main Lemma is proved. □

4.9. **Last cascade.** If the map $F \equiv F_m = R^m f$ is not renormalizable then the principal nest consists of infinitely many central cascades, and the Main Lemma gives a uniform bound on the Teichmüller distance between the corresponding generalized renormalizations.

Otherwise the principal nest ends up with an infinite central cascade $V^{n-1} \supset V^n \supset \dots$ shrinking to the little Julia set J^{m+1} of the next renormalization $g_n = F_{m+1} \equiv R^{m+1} f$. All levels $n, n+1, \dots$ of this final cascade are called the *renormalization levels*.

Lemma 4.11. *Let n be a renormalization level and $H : (V^{n-1}, V^n) \to (\tilde{V}^{n-1}, \tilde{V}^n)$ be a K-qc pseudo-conjugacy between g_n and $\tilde{g}_n$. Then there is a homeomorphism $\phi : (V^{n-1}, J^{m+1}) \to (\tilde{V}^{n-1}, \tilde{J}^{m+1})$ homotopic to h rel $(J^{m+1} \cup \partial V^{n-1})$, and K-qc on $V^{n-1} \setminus J^{m+1}$.*

Proof. Recall that $k^n = V^{k-1} \setminus V^k$. The map $H : A^n \to \tilde{A}^n$ admits a lift to qc maps (with the same dilatation) $H_k : A^{n+k} \to \tilde{A}^{n+k}$ homotopic to h rel the annuli boundary. These maps match to a single qc map $\phi : V^{n-1} \setminus J^{m+1} \to \tilde{V}^{n-1} \setminus \tilde{J}^{m+1}$ with the same dilatation conjugating F_{m+1} to $\tilde{F}_{m+1}$. By Corollary 3.5, this map (and the whole homotopy between it and h) matches with h on $J(F_{m+1})$. ☐

4.10. **Spreading around.** Let us consider the pieces $P_j \subset Y \equiv Y^{(0)}$ of the initial partition (4.6), and the Markov map $G : \cup P_i \to Y$ (see §3.2 of Part I). Let us consider the *first landing map* to V^0, $T_0 : \cup \Omega_i^0 \to P_0$. By Lemma 4.7 and Lemma 4.3, there is a qc pseudo-conjugacy $\phi_0 : (Y, \cup \Omega_i^0) \to (\tilde{Y}, \cup \tilde{\Omega}_i^0)$. Let us also consider the following maps:

- The first landing maps to V^n corresponding to the generalized renormalization $g_n : \cup V_i^n \to V^{n-1}$:

$$T_n : \cup \Omega_i^n \to V^n, \quad \Omega_i^n \subset V^{n-1};$$

- The first landing maps to V^n corresponding to G:

$$S_n : \cup O_i^n \to V^n, \quad O_i^n \subset Y.$$

Clearly

$$S_0 = T_0 \ \textit{and} \ S_n = T_n \circ S_{n-1}. \tag{4.21}$$

By the Main Lemma and Lemma 4.3, there is a sequence of qc pseudo-conjugacies

$$\phi_n : (V^{n-1}, \cup \Omega_i^n) \to (\tilde{V}^{n-1}, \cup \tilde{\Omega}_i^n), \quad n < N+1,$$

where N is the first DH renormalizable level (if F is non-renormalizable then $N = \infty$). Let us turn it inductively into a sequence of pseudo-conjugacies

$$H_n : (Y, \cup O_i^n) \to (\tilde{Y}, \cup \tilde{O}_i^n) \tag{4.22}$$

between S_n and $\tilde{S}_n$ (with the same dilatation). Indeed, using (4.21), we can define it as follows:

$$H_n | O_i^{n-1} = (\tilde{S}_{n-1} | \tilde{O}_i^{n-1})^{-1} \circ (\phi_n | V^{n-1}) \circ S_{n-1} | O_i^{n-1}.$$

As these maps match with H_{n-1} on the boundaries ∂O_i^{n-1}, the glue together into single qc conjugacy (4.22) with the same dilatation.

If F is non-renormalizable, we obtain an infinite sequence of qc pseudo-conjugacies H_n (with uniformly bounded dilatation). As the pieces V_i^n shrink as $n \to \infty$, there is the limit qc map

$$H : (Y, J(F) \cap Y) \to (\tilde{Y}, \tilde{J}(F) \cap \tilde{Y}) \tag{4.23}$$

homotopic to $h : J(F) \cap Y \to \tilde{J}(F) \cap \tilde{Y}$ rel $\partial Y \cup J(F)$.

Assume F is renormalizable. Let $\mathcal{I}$ be the family of little Julia sets J_i^{m+1} contained in Y, $J^{m+1} \equiv J_0^{m+1}$. Let us consider the last pseudo-conjugacy (4.22) on the renormalization level N. Let us replace it on V^{n-1} by the pseudo-conjugacy

$$\phi_N : (V^N, J^{m+1}) \to (\tilde{V}^N, \tilde{J}^{m+1})$$

constructed in Lemma 4.11. Spread it around by the landing map S_N, that is, set

$$H|O_N = (\tilde{S}_N|\tilde{O}_N)^{-1}(\phi_N|V_N) \circ S_N|O_N.$$

As these maps match on the ∂O_N with H_N, they glue together into a homeomorphism

$$H : (Y, \bigcup_{i \in \mathcal{I}} J_i^{m+1}) \to (\tilde{Y}, \bigcup_{i \in \mathcal{I}} \tilde{J}_i^{m+1}), \tag{4.24}$$

quasi-conformal on $Y \setminus \bigcup_{i \in \mathcal{I}} J_i^{m+1}$ (with dilatation depending only on the choice of limbs and *a priori* bounds), and homotopic to h rel $\partial Y \bigcup_{i \in \mathcal{I}} J_i^{m+1}$.

Let us consider the backward orbit $Y \equiv Y_0, Y_{-1}, \ldots, Y_{-r+1}$ of Y under f such that $Y_{-k} \ni f^{r-k}0$, where r is the first return time of the critical orbit to Y. The disks Y_{-k} have disjoint interiors. Let us pull the map H back to these disks, that is, set

$$h_{m+1}|Y_{-k} = (\tilde{f}^k|\tilde{Y}_{-k})^{-1} H \circ f^k|Y_{-k}.$$

As this map respects the boundary marking of the Y_{-k}, it extends to to the whole plane as h_m, which provides the desired next approximation to the Thurston conjugacy (see §3.3).

The Rigidity Theorem is proved.

5. Appendix: Quasi-conformal maps

5.1. There are a few Russian and English sources on the basic theory of quasi-conformal maps: [A, B, Kr, LV, V].

A homeomorphism $h : U \to V$, where $U, V \subset \mathbb{C}$, is called quasi-conformal (qc) if it has locally integrable distributional derivatives ∂h, $\bar{\partial} h$, and $|\bar{\partial} h / \partial h| \leq k < 1$ almost everywhere. As this local definition is conformally invariant, one can define qc homeomorphisms between Riemann surfaces.

One can associate to a qc map an analytic object called Beltrami differential, namely

$$\mu = \frac{\bar{\partial} h}{\partial h} \frac{d\bar{z}}{dz},$$

with $\|\mu\|_\infty < 1$. The corresponding geometric object is a measurable family of infinitesimal ellipses (defined up to dilation), pull-backs by h_* of the field of infinitesimal circles. The eccentricities of these ellipses are ruled by $|\mu|$, and are uniformly bounded almost everywhere, while the orientation of the ellipses is ruled by the $\arg \mu$. The dilatation $\mathrm{Dil}(h) \equiv K_h =$

$(1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$ of h is the essential supremum of the eccentricities of these ellipses. A qc map h is called K-qc if $\mathrm{Dil}(h) \le K$.

Weil's Lemma. *A 1-qc map is analytic.*

One of the most remarkable facts of analysis is that *any Beltrami differential with $\|\mu_\infty\| < 1$ (or rather a measurable field of ellipses with essentially bounded eccentricities) is locally generated by a qc map*, unique up to post-composition with an analytic map. Thus such a Beltrami differential on a Riemann surface S induces a conformal structure quasi-conformally equivalent to the original structure of S, Together with the Riemann mapping theorem this leads to the following statement:

Measurable Riemann Mapping Theorem. *Let μ be a Beltrami differential on $\bar{\mathbb{C}}$ with $\|\mu_\infty\| < 1$, Then there is a quasi-conformal map $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ which solves the Beltrami equation: $|\bar\partial h/\partial h| = \mu$.*

In what follows by a conformal structure we will mean a structure associated to measurable Beltrami differentials μ with $\|\mu\|_\infty < 1$. We will denote by σ the standard structure corresponding to zero Beltrami differential.

Another fundamental property of the space of qc maps is compactness:

Compactness Lemma. *The space of K-qc maps $h : \mathbb{C} \to \mathbb{C}$ normalized by $h(0) = 0$ and $h(1) = 1$ is compact in the uniform topology on the Riemann sphere.*

The following gluing property is also important:

Gluing Lemma. *Let us have two disjoint domains D_1 and D_2 with a piecewise smooth arc γ of their common boundary. Let $D = D_1 \cup D_2 \cup \gamma$. If $h : D \to \mathbb{C}$ is a homeomorphism such that $h|D_i$ is K-qc, then h is K-qc.*

One of Sullivan's leading ideas was the idea of the Teichmüller metric on the space of deformations of a conformal dynamical systems. The prototype for this metric is the classical Teichmüller metric on the space of marked Riemann surfaces. A Riemann surface (perhaps with boundary) is said to be *marked* if it is endowed with a preferred basis of the fundamental group and a preferred parametrization of the boundary components. The Teichmüller distance $\mathrm{dist}(S_1, S_2)$ between two marked Riemann surfaces is defined as the infimum of the dilatations K_h, where $h : S_1 \to S_2$ runs over qc homeomorphisms in the homotopy class respecting the marking.

Let D be a simply connected domain conformally equivalent to the hyperbolic plane $\mathbb{H}^2$. Given a family of subsets $\{S_k\}_{k=1}^n$ in D, let us say that a family of disjoint annuli $A_k \subset D \setminus \cup S_k$ is *separating* if A_k surrounds S_k but does not surround the S_i, $i \ne k$. The following lemma is used in the present paper uncountably many times:

Moving Lemma. • *Let $a, b \in D$ be two points on hyperbolic distance $\rho \le \bar\rho$. Then there is a diffeomorphism $\phi : (D, a) \to (D, b)$, identical near ∂D, with dilatation $\mathrm{Dil}(\phi) = 1 + O(\rho)$, where the constant depends only on $\bar\rho$.*

• *Let* $\{(a_k, b_k)\}_{k=1}^n$ *be a family of pairs of points which admits a family of separating annuli* A_k *with* $\mod A_k \geq \mu$. *Then there is a diffeomorphism* $\phi : (D, a_1, \dots a_n) \to (D, b_1, \dots, b_n)$, *identical near* ∂D, *with dilatation* $\mathrm{Dil}(\phi) = 1 + O(e^{-\mu})$.

Proof. As the statement is conformally equivalent, we can work with the unit disk model of the hyperbolic plane, and can also assume that $a = 0$. Also, it is enough to prove the statement for sufficiently small ρ.

There is a smooth function $\psi : [0, 1] \to [\rho, 1]$ such that $\psi(x) \equiv \rho$ near 0, $\psi(x) \equiv 0$ near 1, and $\psi'(x) = O(\rho)$, with a constant depending only on $\bar{\rho}$.

Let us define a smooth map $\phi : (\mathbb{D}, 0) \to (\mathbb{D}, b)$ as $z \mapsto z + \psi(|z|)$. Then

$$\partial\phi(z) = 1 + \psi'(|z|)\frac{\bar{z}}{2|z|} = 1 + O(\rho), \quad \bar{\partial}\phi(z) = \psi'(|z|)\frac{z}{2|z|} = O(\rho). \tag{5.1}$$

Thus

$$Jac(f) = |di\phi(z)|^2 - |\bar{\partial}\phi(z)|^2 = 1 - O(\rho).$$

Hence for sufficiently small $\rho > 0$, f is a local orientation preserving diffeomorphism. As $f : \partial\mathbb{D} \to \partial\mathbb{D}$, f is a proper map. Hence it is a diffeomorphism.

Finally, (5.1) yields that the Beltrami coefficient $\mu_f = O(\rho)$, so that the dilatation $\mathrm{Dil}(f) = 1 + O(\rho)$. □

Let $Q \subset \mathbb{C}$, $h : Q \to \mathbb{C}$ be a homeomorphism onto its image. It is called *quasi-symmetric (qs)* if for any three points $a, b, c \in Q$ such that

$$q^{-1} \leq \frac{|a - b|}{|b - c|} \leq q,$$

we have:

$$\kappa(q)^{-1} \leq \frac{|a - b|}{|b - c|} \leq \kappa(q).$$

It is called κ-quasi-symmetric if $\kappa(1) \leq \kappa$. It follows from the Compactness Lemma that any K-qc map is κ-quasi-symmetric, with a κ depending only on K.

Ahlfors-Börling Extension Theorem. *Any* κ*-quasi-symmetric map* $h : \mathbb{T} \to T$ *extends to a* $K(\kappa)$*-qc map* $H : \mathbb{C} \to \mathbb{C}$. *Vice versa: The restriction of any* K*-qc map* $H : (\mathbb{A}(r^{-1}, r), \mathbb{T}) \to (U, \mathbb{T})$ *(where* $U \subset \mathbb{C}$*) to the circle* $\kappa(K, r)$*-quasi-symmetric.*

Let us note that in the upper half-plane model, the Ahlfors-Börling extension of a qs map $\mathbb{R} \to \mathbb{R}$ is affinely equivariant (that is, commutes with the action of the complex affine group $z \mapsto az + b$).

Interpolation Lemma. *Let us consider two round annuli* $A = \mathbb{A}[1, r]$ *and* $\tilde{A} = \mathbb{A}[1, \tilde{r}]$, *with* $0 < \epsilon \leq \mod A \leq \epsilon^{-1}$ *and* $\epsilon \leq \mod \tilde{A} \leq \epsilon^{-1}$. *Then any* κ*-qs map* $h : (\mathbb{T}, \mathbb{T}_r) \to (\tilde{\mathbb{T}}, \tilde{\mathbb{T}}_{\tilde{r}})$ *admits a* $K(\kappa, \epsilon)$*-qc extension to a map* $H : A \to \tilde{A}$.

Proof. Since A and $\tilde{A}$ are ϵ^2-qc equivalent, we can assume without loss of generality that $A = \tilde{A}$. Let us cover A by the upper half-plane, $\theta : \mathbb{H} \to A$, $\theta(z) = z^{\frac{-\log ri}{\pi}}$, where the covering group generated by the dilation $T : z \mapsto \lambda z$, with $\lambda = e^{\frac{2\pi^2}{\log r}}$. Let $\bar{h} : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be the lift of h to $\mathbb{R}$ such that $\bar{h}(1) \in [1, \lambda) \equiv I_\lambda$ and $\bar{h}(1) \in (-\lambda, -1]$ (note that $\mathbb{R}_+$ covers $\mathbb{T}_r$, while $\mathbb{R}_-$ covers $\mathbb{T}$). Moreover, since $\deg h = 1$, it commutes with the deck transformation T.

A direct calculation shows that the dilatation of the covering map θ on the fundamental intervals I_λ and $-I_\lambda$ is comparable with $(\log r)^{-1}$. Hence $\bar{h}$ is $C(\kappa, r)$-qs on this interval. By equivariance it is $C(\kappa, r)$-qc on the rays $\mathbb{R}_+$ and $\mathbb{R}_-$.

It is also quasi-symmetric near the origin. Indeed, by the equivariance and normalization,

$$(1+\lambda)^{-1}|J| \leq |\bar{h}(J)| \leq (1+\lambda)|J|$$

for any interval J containing 0, which easily implies quasi-symmetry.

Since the Ahlfors-Börling extension is affinely equivariant, the map $\bar{h}$ extends to a $K(\kappa, r)$-qc map $\bar{H} : \mathbb{H} \to \mathbb{H}$ commuting with T. Hence $\bar{H}$ descends to a $K(\kappa, r)$-qc map $H : A \to A$. □

5.2. **Removability.** A compact set $X \subset \mathbb{C}$ is called *removable* if for any neighborhood $U \supset X$, any conformal map $h : U \setminus X \to \mathbb{C}$ admits a conformal extension across X. Let us show that removability is quasi-conformally invariant.

Lemma 5.1. *Let $\phi : (\mathbb{C}, X) \to (\mathbb{C}, \tilde{X})$ be a qc map. If the set X is removable then $\tilde{X}$ is removable as well.*

Proof. Let σ be the standard conformal structure on $\mathbb{C}$. Let $\tilde{U} \supset \tilde{X}$ be a neighborhood of $\tilde{X}$, and let $\tilde{h} : \tilde{U} \setminus \tilde{X} \to \mathbb{C}$ be a conformal map. Let as consider a conformal structure $\tilde{\mu}$ on $\mathbb{C}$ which is equal to $(h \circ \phi)_*(\sigma)$ on $h(\tilde{U} \setminus \tilde{X})$, and is equal to σ outside. By the Measurable Riemann Mapping Theorem, there is a qc map $\psi : \mathbb{C} \to \mathbb{C}$ such that $\tilde{\mu} = \psi_*(\sigma)$.

Let $U = \phi^{-1}\tilde{U}$. Then the function $h = \psi^{-1} \circ \tilde{h} \circ \phi : U \setminus X \to \mathbb{C}$ is conformal. As X is removable, it admits a conformal extension across X. We will use the same notation h for the extended function. Then the formula $\tilde{h} = \psi \circ h \circ \phi^{-1}$ provides us with a conformal extension of $\tilde{h}$ across $\tilde{X}$. □

Let us now show that removable sets are also qc-removable.

Lemma 5.2. *Let X be a removable set and $U \supset X$ be its neighborhood. Then any qc map h on $U \setminus X$ admits a qc extension across X.*

Proof. Let us consider a conformal structure μ equal to $h^*(\sigma)$ on $U \setminus X$, and equal to σ on the rest of $\mathbb{C}$. By the Measurable Riemann Mapping Theorem, there exists a qc map $\phi : \mathbb{C} \to \mathbb{C}$ such that $\mu = \phi^*(\sigma)$. Then the function $\tilde{h} = h \circ \phi^{-1}$ is univalent on $\tilde{U} \setminus \tilde{X} \equiv \phi U \setminus \phi X$.

By Lemma 5.1, the set $\tilde{X}$ is removable. Hence $\tilde{h}$ admits a conformal extension across $\tilde{X}$. Then the formula $h = \tilde{h} \circ \phi$ provides us with a qc extension of h across X. □

Let us finally state a simple condition for removability (see, e.g., [SN]) which is used many times in this paper.

Removability Condition. *Let X be a Cantor set satisfying the following property. There is an $\eta > o$ such that for any point $z \in X$, there is a nest of disjoint annuli $A_i(z) \subset \mathbb{C} \setminus X$ surrounding z with* $\mod (A_i(z)) \geq \eta$. *Then X is removable.*

References

[A] L. Ahlfors. Lectures on quasi-conformal maps. Van Nostrand Co, 1966.

[Ab] W. Abikoff. The real analytic theory of Teichmüller space. Lecture Notes in Math. , 820. Springer-Verlag, 1980.

[B] P.P. Belinskii. General properties of quasiconformal maps (in Russian). "Nauka", Novosibirsk 1974.

[D] A. Douady. Description of compact sets in $\mathbb{C}$. In: "Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's 60th Birthday", Publish or Perish, 1993.

[BH] B. Branner & J.H. Hubbard. The iteration of cubic polynomials, Part II, Acta Math. v. 169 (1992), 229-325.

[DH1] A. Douady & J.H. Hubbard. Étude dynamique des polynômes complexes. Publication Mathematiques d'Orsay, 84-02 and 85-04.

[DH2] A. Douady & J.H. Hubbard. On the dynamics of polynomial-like maps. Ann. Sc. Éc. Norm. Sup., v. 18 (1985), 287-343.

[DH3] A. Douady & J.H. Hubbard, A proof of Thurston's topological characterization of rational functions. Acta Math. v. 171 (1993), 263–297.

[GS] J. Graczyk & G. Swiatek. Polynomial-like property for real quadratic polynomials. Preprint 1995.

[H] J.H. Hubbard. Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. In: "Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's 60th Birthday", Publish or Perish, 1993.

[HJ] J. Hu & Y. Jiang. The Julia set of the Feigenbaum quadratic polynomial is locally connected. Preprint 1993.

[J] Y. Jiang. Infinitely renormalizable quadratic Julia sets. Preprint, 1993.

[JS] M. Jacobson & G. Swiatek. Quasisymmetric conjugacies between unimodal maps. Preprint IMS at Stony Brook, # 1991/16.

[K] J. Kahn. Holomorphic removability of Julia sets. Thesis, 1995.

[Kr] S.L. Krushkal. Quasi-conformal mappings and Riemann surfaces. John Wiley 1979.

[LS] G.Levin, S. van Strien. Local connectivity of Julia sets of real polynomials, Preprint IMS at Stony Brook, 1995/5.

[L1] M. Lyubich. On the Lebesgue measure of the Julia set of a quadratic polynomial, Preprint IMS at Stony Brook, #1991/10.

[L2] M. Lyubich. Geometry of quadratic polynomials: moduli, rigidity and local connectivity. Preprint IMS at Stony Brook #1993/9.

[L3] M. Lyubich. Dynamics of quadratic polynomials. I. Combinatorics and geometry of the Yoccoz puzzle. Preprint MSRI # 026-95.

[LY] M. Lyubich & M. Yampolsky. Complex bounds for real maps. Preprint MSRI # 034-95.

[LV] Lehto & Virtanen. Quasiconformal mappings in the plane. Springer 1973.

[Ma] A. Manning. Logarithmic capacity and renormalizability for landing on the Mandelbrot set. Preprint 1995.

[M1] J. Milnor. Local connectivity of Julia sets: expository lectures. Preprint IMS at Stony Brook #1992/11.

[M2] J. Milnor. Periodic orbits, external rays and the Mandelbrot set: An expository account. Manuscript 1995.

[MT] J. Milnor & W. Thurston. On iterated maps of the interval. "Dynamical Systems", Proc. U. Md., 1986-87, ed. J. Alexander, Lect. Notes Math., v. 1342 (1988), 465-563.

[McM1] C. McMullen. Families of rational maps, Ann. Math., **125** (1987), 467-493.

[McM2] C. McMullen. Complex dynamics and renormalization. Annals of Math. Studies, v. 135, Princeton University Press, 1994.

[McM3] C. McMullen. Renormalization and 3-manifolds which fiber over the circle. Preprint 1994.

[MvS] W. de Melo & S. van Strien. One dimensional dynamics. Springer-Verlag, 1993.

[Sw] G. Swiatek. Hyperbolicity is dense in the real quadratic family. Preprint IMS at Stony Brook 1992/10.

[S] D. Sullivan. Bounds, quadratic differentials, and renormalization conjectures. AMS Centennial Publications. **2**: Mathematics into Twenty-first Century (1992).

[Sch] D. Schleicher. The structure of the Mandelbrot set. Preprint 1995.

[Sh] M. Shishikura. The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. Preprint IMS at Stony Brook, 1991/7.

[SN] L. Sario & M. Nakai. Classification theory of Riemann surfaces, Springer-Verlag, 1970.

[Th] W. Thurston. The geometry and topology of 3-manifolds. Princeton University Lecture Notes, 1982.

[V] L.I. Volkovyski. Quasi-conformal mappings (in Russian). Lvov University, 1954.