ON A CONJECTURE OF VARCHENKO

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Dedicated to the memory of Aldo Andreotti.

1. Introduction

This note was motivated by a problem posed by Varchenko in [7]. (See also the earlier paper by Aomoto [2]). Let $f_I(x_1, \dots, x_n)$ $(I = 1, \dots, N-1)$ be linear functions of n complex variables, and let $D_I = \{f_I = 0\} \subset \mathbb{C}^n$ be the hyperplanes defined as their corresponding zero loci. The product $\phi_\lambda = \prod_{I=1}^{N-1} f_I^{\lambda_I}$, where the λ_I are complex parameters, is a multivalued holomorphic function on the complement $Y = \mathbb{C}^n - \cup_{I=1}^{N-1} D_I$.

Varchenko's conjecture. *If the exponents λ_I are sufficiently generic, then, under certain broad conditions on the hyperplanes D_I:*

(1) *The critical set of ϕ_λ in Y is a union of isolated points;*
(2) *all critical points of ϕ_λ are nondegenerate and*
(3) *their number is equal to $|\chi(Y)|$, the topological Euler characteristic of Y made positive.*

In its original version [7], the problem arose as a necessary step for evaluating the asymptotic behaviour of certain generalized hypergeometric integrals. In the same paper, Varchenko also went on to prove the various assertions in the case where the family $\{D_1, \dots, D_{N-1}\}$ is a real arrangement, meaning that the linear functions f_I have real coefficients. Subsequently, Orlik and Terao [6] proved Varchenko's conjecture in the general situation of linear functions f_I with complex coefficients. The conditions are spelled out in [6]: the family $\{D_1, \dots, D_{N-1}\}$ should be an *essential arrangement*, i.e., the lowest dimensional multiple intersections of hyperplanes in the family should be isolated points.

The proofs in [7][6] are of a combinatorial–topological nature and thus substantially rely on the assumption that the D_I be hyperplanes in $\mathbb{C}^n$. However, the counting problem being purely topological, one would naturally expect that its answer should be found either in the evaluation of an appropriate characteristic class or, alternatively, in a suitably construed Morse theoretical argument. In this note we shall give two independent proofs of a generalization of Varchenko's conjecture to the case of hypersurfaces in an algebraic

Invent. Math. **126** (1996), 235–248

Stony Brook IMS Preprint #1995/11
October 1995

manifold X. The first and most straightforward proof is algebraic; its main step consists of identifying and evaluating the number of critical points as the top Chern class of the sheaf of logarithmic 1–forms on a blow–up of X. The second proof is an application of Morse theory; it is in part reminescent of the arguments used by Andreotti and Frankel [1] and by Bott [3] in their proofs of Lefschetz hyperplane theorem. It also bears some similarities to Aomoto's work [2] concerning a naturally related cohomology of multivalued meromorphic forms on X. In either case the assumption that the D_I be hyperplanes becomes immaterial and can be simply dispensed with.

We shall thus consider the following generalization of Varchenko's problem. Let X be a smooth projective variety of complex dimension n, that is a complex manifold which can be embedded in projective space $\mathbb{P}^m$, for some $m \geq n$, as the common zero set of homogeneus polynomials. Let also D be a hypersurface in X with (not necessarily smooth) irreducible components $D_1, \ldots, D_N$. We shall consider nowhere vanishing multivalued holomorphic sections of a flat complex line bundle on $Y = X - D$ required to have "power behaviour" near D. Concretely, such a section is a multivalued, holomorphic and nowhere vanishing function ϕ_λ on Y with the following property: If $\{f_I = 0\}$ is a local defining equation of D_I on a sufficiently small neighborhood $U \subset X$ of a smooth point $p \in D$, ϕ_λ has the local form

$$\phi_\lambda|_U = f_I^{\lambda_I}\, h,$$

where λ_I is a complex number and h is holomorphic and non–vanishing throughout U. It can be readily verifed that the number λ_I neither depends on the chosen smooth point p nor on the choice of the local defining function f_I near p; we shall call it the *order* of ϕ_λ along D_I. The space of orders $\lambda = (\lambda_1, \ldots, \lambda_N)$ is generally a homogeneus hyperplane Λ in C^N defined by a linear polynomial with positive integer coefficients.

The following preliminary proposition is proven in Section 2.

Proposition 1.1. *Let ϕ_λ be as above, and assume that there is a dense open subset $V \subset \Lambda$ such that, for $\lambda \in V$, the critical points of ϕ_λ in Y are all non–degenerate. Then there is an algebraic homogeneus subset $A \subset \Lambda$ such that the number of critical points is independent of $\lambda \in V - A \cap V$.*

Our main theorem then solves the counting problem.

Theorem 1.2. *Under the same assumption, then, for $\lambda \in V - A \cap V$, the number of critical points of ϕ_λ in Y is equal to $(-1)^n \chi(Y)$, the topological Euler characteristic of Y up to a sign.*

Remark 1.3. The subset A has the following geometric origin. By Hironaka's resolution of singularities theorem, there exists a resolution $\sigma\colon \hat{X} \to X$ in which the preimage $\hat{Y} = \hat{X} - \hat{D}$ of Y becomes the complement of a normal crossing divisor $\hat{D}$ and such that σ restricts to an isomorphism from $\hat{Y}$ to Y. Let $\{g_i = 0\}$ be a local defining equation of the component $\hat{D}_i$ of $\hat{D}$. Near a smooth point of $\hat{D}_i$, the pull–back $\sigma^*\phi_\lambda$ has the local form $g_i^{\hat{\lambda}_i}\, h$, where h is holomorphic and nowhere vanishing and the order $\hat{\lambda}_i \equiv \hat{\lambda}_i(\lambda)$ of $\sigma^*\phi_\lambda$ along $\hat{D}_i$ is a homogeneus linear polynomial in the original orders λ_I with positive integer coefficients. The subset A is then defined as the the zero set

$$A = \left\{ \lambda \in \Lambda \mid \prod_{i \in \hat{I}} \hat{\lambda}_i(\lambda) = 0 \right\},$$

where $\hat{I}$ indexes the irreducible components of $\hat{D}$.

In fact, one can easily obtain (see Section 4) a general—though less explicit—formula for the number of critical points valid for <u>any</u> $\lambda \in V$.

Remark 1.4. When one allows for possibly degenerate but still isolated critical points, the result still holds true with the following modification (see Section 4). For U_p a small neighborhood of a critical point p of ϕ_λ, let $\varphi_\lambda|_{U_p^*} : U_p^* = U_p - \{p\} \to \mathbb{C}^n - \{0\}$ denote the map whose components are the components of the 1–form $d \log \phi_\lambda$ on U_p. Then one has

$$\chi(Y) = (-1)^n \sum_{\{\text{critical points } p\}} \deg \varphi_\lambda|_{U_p^*},$$

where $\deg \varphi_\lambda|_{U_p^*}$ denotes the topological degree.

In this note we shall not address the question of finding "effective" geometric conditions on the hypersurfaces D_I in order for an open set $V \subset \Lambda$ satisfying the above hypotheses to exist. One can easily find, however, large classes of non–trivial examples. Here we list two.

Example 1.5. This is the case of Varchenko–Orlik–Terao. Let $X = \mathbb{P}^n$ and let D_N be the hyperplane at infinity. The remaining D_I are the hyperplanes $\{f_I = 0\} \subset \mathbb{C}^n$, where the linear functions $f_I = \sum_{i=1}^n a_{iI} x_i + b_I$ are such that the $n \times (N-1)$ constant matrix (a_{iI}) has rank equal to n. The family $D_1, \ldots, D_{N-1}$ is thus an affine essential arrangement. In this case Orlik and Terao [6: Section 4] have shown that the conditions of Proposition 1.1 and Theorem 1.2 are satisfied.

Example 1.6. This example should be contrasted with Example 1.5. It is meant to give a simple illustration of how the geometric conditions on the intersections that have to be imposed on the D_I in the case of hyperplanes (i.e., that the arrangement be essential), become superfluous when at least one of the D_I is a hypersurface of higher degree.

Let $D = D_1 \cup D_2$ be the hypersurface in $\mathbb{P}^n$ whose components are respectively the zero locus of the degree $d \geq 2$ homogeneus polynomial $F_1 = \sum_{i=0}^n X_i^d$ and of $F_2 = X_0$. Thus, D_2 being the hyperplane at infinity, $Y = \mathbb{P}^n - D$ is the complement in $\mathbb{C}^n = \mathbb{P}^n - D_2$ of the affine hypersurface $D_1 \cap \mathbb{C}^n = \{f_1 = \sum_{i=1}^n x_i^d + 1 = 0\}$, where $x_i = X_i/X_0$ are affine coordinates on $\mathbb{C}^n$. The multivalued function $\phi_\lambda = f_1^\lambda$ depends on a single parameter $\lambda \in \mathbb{C}$. The critical set C_λ of ϕ_λ, defined by the equations $\lambda x_1 = \cdots = \lambda x_n = 0$, consists of a single point, the origin $C_\lambda = \{0\}$ in $\mathbb{C}^n$, for $\lambda \in \mathbb{C}^*$, and of all of Y, $C_0 = Y$, for the special value $\lambda = 0$. From the Hessian matrix at 0,

$$\mathrm{Hess}(\phi_\lambda) = \lambda\, f_1^{\lambda-1}\, d(d-1)\ \mathrm{diag}\big(x_1^{d-2}, \ldots, x_n^{d-2}\big),$$

one sees that the critical point 0 is degenerate unless $d = 2$. On the other hand, let U_0 be a small neighborhood of the origin in $\mathbb{C}^n$; the topological degree of the map $\varphi_\lambda|_{U_0^*}\colon U_0^* = U_0 - \{0\} \to \mathbb{C}^n - \{0\}$ sending x to $\big(\partial_{x_1}\log\phi_\lambda(x), \ldots, \partial_{x_1}\log\phi_\lambda(x)\big) = \lambda d\, f_1^{-1}\,(x_1^{d-1}, \ldots, x_n^{d-1})$, is equal to $(d-1)^n$. Theorem 1.2 and the following remark say in this case that $\chi(Y) = \chi(\mathbb{C}^n) - \chi(D_1 \cap \mathbb{C}^n) = (-1)^n\,(d-1)^n$. This is in agreement with the well–known fact that the homology of the affine hypersurface $D_1 \cap \mathbb{C}^n$ is non–vanishing only in degree 0 and $n-1$, where $\dim H_0(D_1 \cap \mathbb{C}^n) = 1$, $\dim H_{n-1}(D_1 \cap \mathbb{C}^n) = (d-1)^n$.

Acknowledgements. The author was partially supported by NSF grant DMS 92–04196. He has benefitted from discussions with R. Friedman, Y. Karpishpan, J. Morgan, D.H. Phong, H. Pinkham and S. Wu. He would like to especially thank P. Orlik and H. Terao for several exchanges, and Nicholas Shepherd–Barron for extremely helpful indications.

2. A few preliminary observations

The purpose of the following observations is to show that the number of critical points of ϕ_λ can be easily identified, for almost all λ, with a topological invariant of the pair (X, D).

It will be more convenient to work with single–valued objects on Y rather than directly with ϕ_λ. We thus note first of all that, since ϕ_λ is nowhere vanishing on Y, the critical set C_λ of ϕ_λ is precisely the set of points in Y where the meromorphic 1–form $d\log\phi_\lambda = \frac{1}{\phi_\lambda} d\phi_\lambda$ vanishes. Let $\varphi_{\lambda,i} = \dfrac{\partial}{\partial x_i}\log\phi_\lambda\colon U \to \mathbb{C}$ for $i = 1, \ldots, n$ denote the components of $d\log\phi_\lambda$ on any open set $U \subset Y$ of a coordinate cover of Y with local coordinate $x = (x_1, \ldots, x_n)\colon U \to \mathbb{C}^n$. Then C_λ is the analytic subvariety of Y with local defining equations $\varphi_{\lambda,1} = \cdots = \varphi_{\lambda,n} = 0$ on U. Second, note that the Hessian matrix of ϕ_λ on U is related to the Jacobian of the map $\varphi_\lambda\colon U \to \mathbb{C}^n$, $p \mapsto \big(\varphi_{\lambda,1}(p), \ldots, \varphi_{\lambda,n}(p)\big)$:

$$\begin{aligned}\frac{\partial^2\phi_\lambda}{\partial x_j \partial x_i} &= \phi_\lambda\,\frac{\partial\varphi_{\lambda,i}}{\partial x_j} + \phi_\lambda\,\varphi_{\lambda,i}\,\varphi_{\lambda,j}\\ &= \phi_\lambda\ \mathrm{Jac}(\varphi_\lambda)_{ij} + \phi_\lambda\,\varphi_{\lambda,i}\,\varphi_{\lambda,j}.\end{aligned}$$

Again, since ϕ_λ is never zero on Y, it follows that a critical point $p \in C_\lambda$ is non–degenerate if and only if the determinant $\det\mathrm{Jac}(\varphi_\lambda)(p) \neq 0$. As usual, we say in this case that p is a *non–degenerate zero* of $d\log\phi_\lambda$.

We now turn to the proof of of Proposition 1.1. If $\hat{X} \xrightarrow{\sigma} X$ is the blow–up of Remark 1.2, let $\hat{D} = \sigma^{-1}(D)$ and $\hat{Y} = \sigma^{-1}(Y)$. Since the Jacobian of σ is a holomorphic non–singular matrix on $\hat{Y}$, one may easily verify that the 1–form $d\log\sigma^*\phi_\lambda \in H^0\big(\hat{X}, \Omega^1_{\hat{X}}(*\hat{D})\big)$ has a non–degenerate zero at $\hat{p} \in \hat{Y}$ if and only if $\hat{p} = \sigma^{-1}(p)$, where $p \in Y$ is a non–degenerate zero of $d\log\phi_\lambda$. Thus clearly, if $\lambda \in V$, the cardinality of C_λ is given by

$$\big(\#\text{ of critical points of } \phi_\lambda \text{ on } Y\big) = \big(\#\text{ of zeroes of } d\log\sigma^*\phi_\lambda \text{ on } \hat{Y}\big).$$

Let us recall that the sheaf of logarithmic 1–forms $\Omega^1_{\hat{X}}(\log\hat{D})$ is the the sheaf of those meromorphic 1–forms ω on $\hat{X}$ which are holomorphic on $\hat{X} - \hat{D}$ and have the following local property near $\hat{D}$: For any small open neighborhood $U \subset \hat{X}$ of $\hat{D}$ on which $\hat{D}$ has a local defining equation $g = 0$, both $g\,\omega$ and $g\,d\omega$ are holomorphic throughout U. Also, recall the algebraic subset $A \subset \Lambda$ introduced in Remark 1.2.

Lemma 2.1. *Let $\lambda \in \Lambda$. Then: (i) The 1–form $d\log\sigma^*\phi_\lambda$ is an element of $\Gamma\big(\hat{X}, \Omega^1_{\hat{X}}(\log\hat{D})\big)$. (ii) For $\lambda \in \Lambda - A$, $d\log\sigma^*\phi_\lambda$ has a pole along <u>every</u> component of $\hat{D}$.*

Proof. The assertions being local, it suffices to consider the pull–back $\sigma^*\phi_\lambda$ of ϕ_λ on an arbitrarily small neighborhood $U \subset \hat{X}$ of a point where exactly m components of $\hat{D}$, say $\hat{D}_1, \ldots, \hat{D}_m$, intersect. If $\{g_i = 0\}$ is a local defining equation of $\hat{D}_i$ on U, $\sigma^*\phi_\lambda$ has the local form $g_1^{\hat{\lambda}_1} \cdots g_m^{\hat{\lambda}_m}\, h$, where h is holomorphic and nowhere vanishing throughout U. Assertion *(i)* is thus self–evident. Moreover, if none of the $\hat{\lambda}_i$ is zero—i.e., if $\lambda \notin A$—then $d\log\sigma^*\phi_\lambda|_U$ has a logarithmic pole along $\hat{D}_i$ for all $i = 1, \ldots, m$, and *(ii)* is also clear.

If $\lambda \in V - A \cap V$, the number of zeroes of $d\log\sigma^*\phi_\lambda$ is therefore a topological invariant of the vector bundle $\Omega^1_{\hat{X}}(\log\hat{D})$ on $\hat{X}$. In particular, it is obviously independent of λ.

3. Gauss–Bonnet for the complement of a divisor (First proof of Theorem 1.2)

For $\lambda \in V - A \cap V$, the number of zeroes of $d\log\sigma^*\phi_\lambda$—and hence the number of critical points of ϕ_λ in Y—has just been identified with the top Chern number, $\int_X c_n\big(\Omega^1_{\hat{X}}(\log\hat{D})\big)$, of $\Omega^1_{\hat{X}}(\log\hat{D})$. (For this standard interpretation of the top Chern class of a holomorphic vector bundle see e.g. [4: section 3 of Chapter 3]). By Theorem 4.1 below, this coincides with the topological Euler characteristic of $\hat{Y}$ up to a sign factor of $(-1)^n$. Since $\hat{Y}$ is isomorphic to Y, then $\chi(\hat{Y}) = \chi(Y)$, which concludes our first proof of Theorem 1.2.

Theorem 4.1[1]**.** *Let $D = \sum_{I=1}^N D_I$ be a normal crossing divisor in a smooth projective variety X of complex dimension n and let $\Omega^1_X(\log D)$ be the rank n holomorphic vector bundle on X whose sections are the 1–forms on X with logarithmic poles along D. Then the top Chern number of $\Omega^1_X(\log D)$ is given by the Euler characteristic of the complement, $\chi(X - D) = \sum_{i=1}^n \dim_{\mathbb{C}} H^i(X - D, \mathbb{C})$, up to a sign,*

$$\int_X c_n\big(\Omega^1_X(\log D)\big) = (-1)^n\, \chi(X - D).$$

Proof. We shall first express the Chern classes of $\Omega^1_X(\log D)$ in terms of the Chern classes of the holomorphic cotangent bundle $\Omega^1_X = T^*_X$ and of the line bundles $[D_I]$ associated with the various components D_I. For E a holomorphic vector bundle on X, we shall denote

[1]The author is indebted to N. Shepherd–Barron for having pointed out this version of the Gauss–Bonnet formula to him. However, the author has been unable to locate a proof anywhere in the literature.

its total Chern class by $c(E) = \sum_{i\geq 0} c_i(E)$ with the usual convention $c_0(E) \equiv 1$. The Poincaré residue map gives the exact sequence of sheaves on X

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \xrightarrow{\text{residue}} \mathcal{O}_{\tilde{D}} = \bigoplus_{I=1}^{N} \mathcal{O}_{D_I} \to 0,$$

where the $\mathcal{O}_{D_I}$ are to be viewed as the sheaves on X extending the structure sheaves $\mathcal{O}_{D_I}$ by zero outside the divisors D_I. The resulting identity among total Chern classes, $c\big(\Omega^1_X(\log D)\big) = c(\Omega^1_X)\, c(\mathcal{O}_{\tilde{D}}) = c(\Omega^1_X)\, \prod_{I=1}^{N} c(\mathcal{O}_{D_I})$, gives

$$\begin{aligned} c_n\left(\Omega^1_X(\log D)\right) &= \sum_{i=0}^{n} \sum_{j_1+\cdots+j_N=i} c_{n-i}(\Omega^1_X)\, c_{j_1}(\mathcal{O}_{D_1}) \cdots c_{j_N}(\mathcal{O}_{D_N}) \\ &= c_n(\Omega^1_X) + \sum_{i=1}^{n} \sum_{j_1+\cdots+j_N=i} c_{n-i}(\Omega^1_X)\, c_{j_1}(\mathcal{O}_{D_1}) \cdots c_{j_N}(\mathcal{O}_{D_N}). \end{aligned}$$

Moreover, one has for every I an exact sequence of sheaves on X,

$$0 \to \mathcal{O}_X\big([-D_I]\big) = [-D_I] \to \mathcal{O}_X \xrightarrow{\text{restriction}} \mathcal{O}_{D_I} \to 0,$$

implying $c(\mathcal{O}_X) = 1 = c(\mathcal{O}_{D_I})\, c\big([-D_I]\big) = c(\mathcal{O}_{D_I})\ \big(1 + c_1\big([-D_I]\big)\big)$. Thus

$$c(\mathcal{O}_{D_I}) = \sum_{j\geq 0} (-1)^j c_1\big([-D_I]\big)^j = \sum_{j\geq 0} c_1\big([D_I]\big)^j,$$

and the Chern classes of $\mathcal{O}_{D_I}$ are $c_j(\mathcal{O}_{D_I}) = c_1\big([D_I]\big)^j$. Now, on the one hand we have the Gauss–Bonnet formula $\int_X c_n(T^*_X) = (-1)^n \int_X c_n(T_X) = (-1)^n \chi(X)$ for the top Chern class of Ω^1_X; on the other hand, by a standard excision argument, we have the addition formula $\chi(X - D) = \chi(X) - \chi(D)$. Hence

$$\int_X c_n\left(\Omega^1_X(\log D)\right) = (-1)^n\, \chi(X) + \sum_{i=1}^{n} \sum_{j_1+\cdots+j_N=i} \int_X c_{n-i}(\Omega^1_X)\, c_1\big([D_1]\big)^{j_1} \cdots c_1\big([D_N]\big)^{j_N}$$

and the sought for result is equivalent to the

Claim. *Let $D_1, \ldots, D_N$ be smooth and normally intersecting divisors in a n–dimensional smooth projective variety X, and let $D = \bigcup_{I=1}^{N} D_I$. Then the following identity holds*

$$(1) \qquad (-1)^{n-1}\chi(D) = \sum_{i=1}^{n} \sum_{j_1+\cdots+j_N=i} \int_X c_{n-i}(\Omega^1_X)\, c_1\big([D_1]\big)^{j_1} \cdots c_1\big([D_N]\big)^{j_N}.$$

In order to prove the claim, let us first consider a smooth divisor D_I in X. Since D_I has complex codimension 1 in X, the normal bundle $N_{D_I/X}$ is a line bundle on D_I equivalent to the restriction $[D_I]|_{D_I}$. The C^∞ decomposition $T_X|_{D_I} = T_{D_I} \oplus N_{D_I/X} = T_{D_I} \oplus [D_I]|_{D_I}$ implies the identity $c(T_X)|_{D_I} = c\big(T_X|_{D_I}\big) = c(T_{D_I})\, c\big([D_I]|_{D_I}\big) = c(T_{D_I})\, c([D_I])|_{D_I}$ of Chern polynomials. The resulting equalities of cohomology classes on D_I,

$$c_i(T_X)|_{D_I} = c_i(T_{D_I}) + c_{i-1}(T_{D_I})\, c_1([D_I])|_{D_I} \qquad \text{for } i = 1, \ldots, n-1$$

together with the general relations $c_i(E) = (-1)^i c_i(E^*)$ between the Chern classes of a vector bundle and of its dual, give

$$c_i(T_X^*)|_{D_I} = c_i(\Omega_X^1)|_{D_I} = c_i(T_{D_I}^*) - c_{i-1}(T_{D_I}^*)\, c_1([D_I])|_{D_I} \qquad \text{for } i = 1, \ldots, n-1. \tag{2}$$

Our proof of the claim now proceeds by induction on the number N of divisors.

Step 1. For $N = 1$, let $D = D_1$. The claim follows at once from (2):

$$\begin{aligned}
\sum_{i=1}^{n} \int_X c_{n-i}(\Omega_X^1)\, c_1([D])^i &= \sum_{i=1}^{n} \int_D c_{n-i}(\Omega_X^1)\, c_1([D])^{i-1} \\
&= \sum_{i=1}^{n} \int_D c_{n-i}(T_D^*)\, c_1([D])^{i-1} - \sum_{i=1}^{n-1} \int_D c_{n-1-i}(T_D^*)\, c_1([D])^i \\
&= \sum_{i=0}^{n-1} \int_D c_{n-1-i}(T_D^*)\, c_1([D])^i - \sum_{i=1}^{n-1} \int_D c_{n-1-i}(T_D^*)\, c_1([D])^i \\
&= \int_D c_{n-1}(T_D^*) = (-1)^{n-1} \chi(D),
\end{aligned}$$

where in the first step we have used the fact that $c_1([D])$ is Poincaré dual to the fundamental class of D.

Step 2. For general $N > 1$ we decompose the sum in (1) into the terms with $j_N = 0$ and those with $j_N \geq 1$:

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{j_1 + \cdots + j_N = i} c_{n-i}(\Omega_X^1)\, c_1\big([D_1]\big)^{j_1} \cdots c_1\big([D_N]\big)^{j_N} = \\
&\qquad \sum_{i=1}^{n} \sum_{j_1 + \cdots + j_{N-1} = i} c_{n-i}(\Omega_X^1)\, c_1\big([D_1]\big)^{j_1} \cdots c_1\big([D_{N-1}]\big)^{j_{N-1}} + \\
&\qquad\qquad \sum_{i=1}^{n} \sum_{\substack{j_1 + \cdots + j_N = i \\ j_N \geq 1}} c_{n-i}(\Omega_X^1)\, c_1\big([D_1]\big)^{j_1} \cdots c_1\big([D_N]\big)^{j_N}.
\end{aligned}$$

The second sum on the right hand side can be computed using (2), $c_i(\Omega_X^1)|_{D_N} = c_i(\Omega_{D_N}^1) -$

$c_{i-1}(\Omega^1_{D_N})\, c_1([D_N])|_{D_N}$:

$$\sum_{i=1}^{n} \sum_{\substack{j_1+\cdots+j_N=i\\ j_N\geq 1}} \int_X c_{n-i}(\Omega^1_X)\, c_1\big([D_1]\big)^{j_1}\cdots c_1\big([D_N]\big)^{j_N} =$$

$$\sum_{i=1}^{n} \sum_{\substack{j_1+\cdots+j_N=i\\ j_N\geq 1}} \int_{D_N} c_{n-i}(\Omega^1_X)\, c_1\big([D_1]\big)^{j_1}\cdots c_1\big([D_N]\big)^{j_N-1} =$$

$$\sum_{i=1}^{n} \sum_{\substack{j_1+\cdots+j_N=i\\ j_N\geq 1}} \int_{D_N} c_{n-i}(\Omega^1_{D_N})\, c_1\big([D_1]\big)^{j_1}\cdots c_1\big([D_N]\big)^{j_N-1} -$$

$$\sum_{i=1}^{n-1} \sum_{\substack{j_1+\cdots+j_N=i\\ j_N\geq 1}} \int_{D_N} c_{n-1-i}(\Omega^1_{D_N})\, c_1\big([D_1]\big)^{j_1}\cdots c_1\big([D_N]\big)^{j_N} =$$

$$\int_{D_N} c_{n-1}(\Omega^1_{D_N}) + \sum_{i=1}^{n-1} \sum_{j_1+\cdots+j_{N-1}=i} \int_{D_N} c_{n-1-i}(\Omega^1_{D_N})\, c_1\big([D_1]\big)^{j_1}\cdots c_1\big([D_{N-1}]\big)^{j_{N-1}}.$$

We have thus found the recursion

$$\sum_{i=1}^{n} \sum_{j_1+\cdots+j_N=i} \int_X c_{n-i}(\Omega^1_X)\, c_1\big([D_1]\big)^{j_1}\cdots c_1\big([D_N]\big)^{j_N} =$$

$$\sum_{i=1}^{n} \sum_{j_1+\cdots+j_{N-1}=i} \int_X c_{n-i}(\Omega^1_X)\, c_1\big([D_1]\big)^{j_1}\cdots c_1\big([D_{N-1}]\big)^{j_{N-1}} + (-1)^{n-1}\chi(D_N) +$$

$$\sum_{i=1}^{n-1} \sum_{j_1+\cdots+j_{N-1}=i} \int_{D_N} c_{n-1-i}(\Omega^1_{D_N})\, c_1\big([D_1]\big)^{j_1}\cdots c_1\big([D_{N-1}]\big)^{j_{N-1}}.$$

Let us now assume that the claim is true for a number $N-1$ of smooth divisors with transverse intersections. This means that the first term above is given by

$$\sum_{i=1}^{n} \sum_{j_1+\cdots+j_{N-1}=i} \int_X c_{n-i}(\Omega^1_X)\, c_1\big([D_1]\big)^{j_1}\cdots c_1\big([D_{N-1}]\big)^{j_{N-1}} = (-1)^n \chi\left(\cup_{I=1}^{N-1} D_I\right);$$

moreover, since the $N-1$ divisors $D_1\cap D_N,\ \ldots,\ D_{N-1}\cap D_N$ in the compact $(n-1)$–dimensional smooth projective variety D_N are smooth and have transverse intersections, by the induction hypothesis we also have

$$\sum_{i=1}^{n-1} \sum_{j_1+\cdots+j_{N-1}=i} \int_{D_N} c_{n-1-i}(\Omega^1_{D_N})\, c_1\big([D_1]\big)^{j_1}\cdots c_1\big([D_{N-1}]\big)^{j_{N-1}} =$$

$$(-1)^{n-2}\chi\left(\cup_{I=1}^{N-1} D_I\cap D_N\right).$$

But the Mayer–Vietoris cohomology exact sequence of the pair $\big(\cup_{I=1}^{N-1} D_I,\, D_N\big)$ implies the relation

$$\chi(D) = \chi\left(\cup_{I=1}^{N} D_I\right) = \chi\left(\cup_{I=1}^{N-1} D_I\right) + \chi(D_N) - \chi\left(\cup_{I=1}^{N-1} D_I \cap D_N\right)$$

among Euler characteristics. It follows that

$$\sum_{i=1}^{n} \sum_{j_1+\cdots+j_N=i} \int_X c_{n-i}(\Omega^1_X)\, c_1\big([D_1]\big)^{j_1} \cdots c_1\big([D_N]\big)^{j_N} =$$
$$(-1)^{n-1}\chi\left(\cup_{I=1}^{N-1} D_I\right) + (-1)^{n-1}\chi(D_N) + (-1)^{n-2}\chi\left(\cup_{I=1}^{N-1} D_I \cap D_N\right) =$$
$$(-1)^{n-1}\chi(D),$$

and the proof of the induction step is complete.

4. Variants

In this brief section we discuss the variants mentioned in Remark 1.3 and Remark 1.4.

4.1. Even if $\lambda \in A \cap V$ one can still in principle compute the number of critical points as follows. For $\lambda \in \Lambda$, let us thus introduce the divisor $\hat{D}(\lambda)$ given by those irreducible components of $\hat{D}$ along which the order of $\sigma^*\phi_\lambda$ is non-zero,

$$\hat{D}(\lambda) = \bigcup_{\hat{\lambda}_i \neq 0} \hat{D}_i \;\subset \hat{D}.$$

Note that $\hat{D}(\lambda) = \hat{D}$ if and only if $\lambda \in \Lambda - A \cap V$. One can immediately deduce the following sharper version of Lemma 2.1:

Let $\lambda \in \Lambda$. Then the 1–form $d\log\sigma^*\phi_\lambda$ is an element of $\Gamma\big(\hat{X}, \Omega^1_{\hat{X}}(\log \hat{D}(\lambda))\big)$ having a pole along every component of $\hat{D}(\lambda)$.

By the same argument as above, one thus deduces the more general formula

$$\big(\#\text{ of critical points of } \phi_\lambda \text{ on } Y\big) = (-1)^n\,\chi\big(\hat{X} - \hat{D}(\lambda)\big)$$

for any $\lambda \in V$. The practical usefulness of this formula for producing numerical predictions may however be limited to those concrete situations where one can easily relate the Euler characteristic of $\hat{X} - \hat{D}(\lambda)$ to the topology of the blow–down of $\hat{D}(\lambda)$.

4.2. Let $\lambda \in V$. We shall allow the section $d\log\sigma^*\phi_\lambda \in \Gamma\big(\hat{X}, \Omega^1_{\hat{X}}(\log \hat{D}(\lambda))\big)$ to have isolated but possibly degenerate zeroes p. The top Chern class $c_n\big(\Omega^1_{\hat{X}}(\log \hat{D}(\lambda)\big)$ is Poincaré dual to the degeneracy cycle $\sum_p m_p\, p$. Here the multiplicity m_p is the intersection number at p of the n divisors in $\hat{Y}$ having the local defining equations $\hat{\varphi}_{\lambda,i} = \partial_{x_i}\log\sigma^*\phi_\lambda = 0$. Equivalently, m_p is the topological degree of the map $\hat{\varphi}_\lambda|_{U_p^*} \colon U_p^* = U_p - \{p\} \to \mathbb{C}^n - \{0\}$ with components $\hat{\varphi}_{\lambda,i}|_{U_p^*}$, U_p being a small neighborhood of p. It follows that

$$(-1)^n\,\chi\big(\hat{X} - \hat{D}(\lambda)\big) = \int_{\hat{X}} c_n\big(\Omega^1_{\hat{X}}(\log \hat{D}(\lambda)\big) = \sum_p m_p.$$

If $\lambda \in V - A \cap V$, this is the formula given in Remark 1.4.

5. Morse theoretic proof of Theorem 1.2

The main idea of this second proof consists—loosely speaking—of interpreting the modulus square $|\phi_\lambda|^2 = \phi_\lambda\, \phi_\lambda^*$ as a Morse function defined on the submanifold of X obtained from this latter by deleting a tiny neighborhood of the hypersurface D. In order for $|\phi_\lambda|^2$ to be a well–defined (i.e., univalued) real function on Y, however, the monodromies of ϕ_λ must be complex numbers of modulus 1. In view of Proposition 1.1, over the open dense subset $V - A \cap V$ of Λ the critical points of ϕ_λ are all non–degenerate and constant in number. One may henceforth choose—at no loss of generality—all orders λ_I to be real, and at the same time one need make an additional assumption on the line bundle of which ϕ_λ is a multivalued section, that is:

Assumption. *The monodromy of ϕ_λ around any loop of $\pi_1(X)$ is in $U(1)$.*

We shall actually work in the blow–up $\hat{X} \xrightarrow{\sigma} X$ in which $\hat{Y} \xrightarrow{\sim} Y$ is realized as the complement of a normal crossing divisor $\hat{D}$. For simplicity, we shall denote the pull–back $\sigma^*\phi_\lambda$ by $\hat{\phi}_\lambda$. As already observed, $\hat{\phi}_\lambda$ has a non–degenerate critical point at $p \in \hat{Y}$ if and only if p is the preimage of a non–degenerate critical point of ϕ_λ in Y. Let $x_1, \ldots, x_n$ and $v_1, \ldots, v_n, w_1, \ldots, w_n$ be, respectively, analytic and real local coordinates on $\hat{X}$ with $x_i = v_i + \sqrt{-1}\, w_i$. One immediately verifies that the critical set of the function $|\hat{\phi}_\lambda|^2$, i.e., the set of points in $\hat{Y}$ where all partial derivatives $\partial_{v_i} |\hat{\phi}_\lambda|^2$, $\partial_{w_i} |\hat{\phi}_\lambda|^2$ vanish, coincides with the critical set of $\hat{\phi}_\lambda$ in $\hat{Y}$. The Hessian of $|\hat{\phi}_\lambda|^2$ at a critical point p is given by the $2n \times 2n$ matrix

$$
\begin{aligned}
\mathrm{Hess}(|\hat{\phi}_\lambda|^2)(p) &= \begin{pmatrix} \left(\partial_{v_i}\partial_{v_j} |\hat{\phi}_\lambda|^2\right)(p) & \left(\partial_{v_i}\partial_{w_j} |\hat{\phi}_\lambda|^2\right)(p) \\ \left(\partial_{w_i}\partial_{v_j} |\hat{\phi}_\lambda|^2\right)(p) & \left(\partial_{w_i}\partial_{w_j} |\hat{\phi}_\lambda|^2\right)(p) \end{pmatrix} \\
&= |\hat{\phi}_\lambda|^2 \begin{pmatrix} \mathrm{Re}\left(\partial_{x_i}\partial_{x_j} \log \hat{\phi}_\lambda\right)(p) & -\mathrm{Im}\left(\partial_{x_i}\partial_{x_j} \log \hat{\phi}_\lambda\right)(p) \\ -\mathrm{Im}\left(\partial_{x_i}\partial_{x_j} \log \hat{\phi}_\lambda\right)(p) & -\mathrm{Re}\left(\partial_{x_i}\partial_{x_j} \log \hat{\phi}_\lambda\right)(p) \end{pmatrix} \\
&= |\hat{\phi}_\lambda|^2 \begin{pmatrix} \mathrm{Re}\,\mathrm{Hess}(\log \hat{\phi}_\lambda)(p) & -\mathrm{Im}\,\mathrm{Hess}(\log \hat{\phi}_\lambda)(p) \\ -\mathrm{Im}\,\mathrm{Hess}(\log \hat{\phi}_\lambda)(p) & -\mathrm{Re}\,\mathrm{Hess}(\log \hat{\phi}_\lambda)(p) \end{pmatrix}.
\end{aligned}
$$

The following fact is elementary.

Lemma 5.1. *Assume that all critical points of $\hat{\phi}_\lambda$ in $\hat{Y}$ are non–degenerate. Then all critical points of $|\hat{\phi}_\lambda|^2$ in $\hat{Y}$ are also non–degenerate and have index equal to n.*

Proof. One first observes that the characteristic values of the bilinear form $\mathrm{Hess}(|\hat{\phi}_\lambda|^2)(p)$ at a critical point p occur in pairs of opposite sign. This follows from the immediate fact that, if $\binom{a}{b}$ is a characteristic vector with characteristic value α, then $\binom{-b}{a}$ is a characteristic vector with value $-\alpha$. The result is thus proven if we show that the null space of $\mathrm{Hess}(|\hat{\phi}_\lambda|^2)(p)$ is empty. We shall abbreviate $\mathrm{Hess}(\log \hat{\phi}_\lambda)(p) = H$, $\mathrm{Re}\, H = R$ and $\mathrm{Im}\, H = I$. By contradiction, let us suppose that 0 is a characteristic value of $\mathrm{Hess}(|\hat{\phi}_\lambda|^2)(p)$. Since by assumption $\hat{\phi}_\lambda$ is nowhere zero on $\hat{Y}$, this means that there exists a non–zero real

$2n$ vector $\binom{a}{b}$ so that $Ra = Ib$ and $Ia = -Rb$. Thus, in particular, $Ra + \sqrt{-1}\, Ia = Ib - \sqrt{-1}\, Rb$, i.e., $H(a + \sqrt{-1}\, b) = 0$. But by assumption p is a non-degenerate critical point of $\hat{\phi}_\lambda$, hence also of $\log \hat{\phi}_\lambda$; so H is non–singular. It follows that $a + \sqrt{-1}\, b = 0$, and—a, b being real—that $a = b = 0$, a contradiction.

Again by Proposition 1.1, one may further specialize λ to be a point in $V'_{\mathbb{Z}} = (V - A \cap V) \cap \mathbb{Z}^{N-1}$, the subset of V where all $\hat{\lambda}_i$ have vanishing imaginary part and non–zero integral real part. The hypersurface $\hat{D}$ is thus the support of the divisor of the meromorphic section $\hat{\phi}_\lambda$, $\hat{D} = \hat{D}_0 \cup \hat{D}_\infty$, where $\hat{D}_0 = \cup_{\hat{\lambda}_i > 0} \hat{D}_i$, $\hat{D}_\infty = \cup_{\hat{\lambda}_i < 0} \hat{D}_i$ are, respectively, the zero and the polar locus of $\hat{\phi}_\lambda$. Obviously, neither $\hat{D}_0$ nor $\hat{D}_\infty$ may be empty. If one were to directly apply standard Morse theory in the present set up, however, one would encounter an obstacle in the existence of points of indeterminacy for the function $|\hat{\phi}_\lambda|^2$, that is the points in $\hat{D}_0 \cap \hat{D}_\infty$, where $|\hat{\phi}_\lambda|^2$ has no limit. This difficulty admits a standard resolution which consists of recursively further blowing–up $\hat{X}$ along the components of the indeterminacy locus. There in fact exists (see e.g., [4: Section 2 of Chapter 4]) a blow–up $X' \xrightarrow{\sigma'} \hat{X}$ of $\hat{X}$ such that:

(1) The supports of, respectively, the divisors of zeroes and of poles, D'_0 and D'_∞, of the pull–back ${\sigma'}^* \hat{\phi}_\lambda$, are disjoint;
(2) $D' = {\sigma'}^{-1}(\hat{D}) = D'_0 \cup D'_\infty \cup D''$, the function ${\sigma'}^* \hat{\phi}_\lambda$ being defined and nowhere vanishing on the components of $D'' - D'' \cap \left(D'_0 \cup D'_\infty\right)$.

Next, we shall ascertain that the pull-back ${\sigma'}^* \hat{\phi}_\lambda$, and hence ${\sigma'}^* |\hat{\phi}_\lambda|^2$, has no critical points on $D'' - D'' \cap \left(D'_0 \cup D'_\infty\right)$.

Lemma 5.2. *For $\lambda \in V'_{\mathbb{Z}}$, ${\sigma'}^* |\hat{\phi}_\lambda|^2$ extends to a positive C^∞ function F on $X' - D'_\infty$ vanishing precisely along D'_0 and approaching infinity near D'_∞. Moreover, this function has no critical points on $D'' - D'' \cap \left(D'_0 \cup D'_\infty\right)$.*

Proof. The first part of the statement—which we have included for completeness—is true by definition. We only have to examine the assertion about the critical points of F. Let U be a small neighborhood in $\hat{X}$ of the intersection points of two components of $\hat{D}$ along which the orders of $\hat{\phi}_\lambda$ are opposite in sign. Then, after a finite sequence π_1 of blow–ups along the indeterminacy loci, one arrives at the local situation where the only indeterminacy points of $\pi_1^* \hat{\phi}_\lambda$ on $\pi_1^{-1}(U)$ are those lying on the intersection of two irreducible divisors $C_1 = \{z_1 = 0\}$, $C_2 = \{z_2 = 0\}$, the local form of $\pi_1^* \hat{\phi}_\lambda$ being $z_1^{-m}\, z_2^m\, h(z_1, z_2, \ldots)$, with some positive non–zero integer m and a nowhere vanishing holomorphic function h. After one more blow–up π_2 along $C_1 \cap C_2$, given by $z_1 = t_1$, $z_2 = t_1 t_2$ away from C_1, and denoting by σ' the composite $\pi_2 \pi_1$, one sees from the local form $t_2^m\, h(t_1, t_1 t_2, \ldots)$ that the pull–back ${\sigma'}^* \hat{\phi}_\lambda$ is indeed holomorphic and non–zero along the points on the exceptional divisor $E = \{t_1 = 0\}$ which do not intersect the proper transform of $C_1 \cup C_2$. Let us now consider the derivatives of ${\sigma'}^* \hat{\phi}_\lambda$. In particular, locally near E one has $\partial_{t_2} {\sigma'}^* \hat{\phi}_\lambda = m t_2^{m-1}\, h(t_1, t_1 t_2, \ldots) + t_2^m\, t_1\, \partial_{z_2} h(t_1, z_2, \ldots)$, and $\partial_{t_2} {\sigma'}^* \hat{\phi}_\lambda \Big|_E = m\, t_2^{m-1}\, h(0, 0, \ldots)$, which may vanish only if $t_2 = 0$. It follows that ${\sigma'}^* \hat{\phi}_\lambda$, hence also ${\sigma'}^* |\hat{\phi}_\lambda|^2$, has no critical points on the complement in E of the proper transform of C_2. In other words, the only critical points, if any, of F in D' necessarily lie on D'_0, and the lemma has been proven.

Below is our main lemma. We shall henceforth denote by γ the number of critical points of $\hat{\phi}_\lambda$ in $\hat{Y}$ ($\lambda \in V - A \cap V$). Note that, by Lemma 5.2, this is equal to the number of critical points of F in $X' - D'_0 \cup D'_\infty$.

Theorem 5.3. *Let $\lambda \in V'_{\mathbb{Z}}$. Let also $\partial\overline{T}(D'_0)$ be the boundary of an infinitesimally small (closed) neighborhood $\overline{T}(D'_0)$ of D'_0 in X'. Then $X' - D'_0 \cup D'_\infty$ has the homotopy type of $\partial\overline{T}(D'_0)$ with a number γ of n–cells attached.*

Proof. For $0 < a < b$, let Φ denote the restriction of F to $M = F^{-1}[a, b]$. Since $D'_0 \cap \hat{D}'_\infty$ is empty, if neither a or b are critical values, M is a compact real submanifold of $X' - D'_0 \cup D'_\infty$ with smooth and disjoint boundary components $\Phi^{-1}(a)$ and $\Phi^{-1}(b)$. By Lemma 5.1 and Lemma 5.2, Φ is a Morse function all of whose critical points have index n. Let $\gamma(a, b)$ be the number of critical points of Φ whose critical values lie in the interval (a, b). In view of the basic result of Morse theory (see for example [5: Theorem 3.1 of Chapter 6]), M has the homotopy type of $\Phi^{-1}(a)$ with a number $\gamma(a, b)$ of n–cells attached. If one chooses a and b to be respectively so small and so large that (a, b) contains all critical values of $\sigma'^*|\hat{\phi}_\lambda|^2$, then, symbolically,

$$M \overset{\text{hom}}{\cong} \Phi^{-1}(a) \cup e_1 \cup \cdots \cup e_\gamma,$$

where the e_i are n–cells. But clearly $\Phi^{-1}(a)$ is homotopic to $\partial\overline{T}(D'_0)$; on the other hand M is homotopic to $X' - T(D'_0) \cup T(D'_\infty)$—where $T(D'_\infty)$ is a small open neighborhood of D'_∞—and hence to $X' - D'_0 \cup D'_\infty$.

Corollary 5.4. *One has $\gamma = (-1)^n\, \chi(X' - D'_0 \cup D'_\infty)$.*

Proof. On the level of Euler characteristics, Theorem 5.3 implies the relation

$$\chi(X' - D'_0 \cup D'_\infty) = \chi\big(\partial\overline{T}(D'_0)\big) + (-1)^n\, \gamma.$$

But $\partial\overline{T}(D'_0)$ is homotopic to the deleted neighborhood $T(D'_0) - D'_0$, whose Euler characteristic is vanishing.

In order to complete the proof of Theorem 1.2 there only remains to observe that

$$\chi(Y) = \chi(X' - D'_0 \cup D'_\infty).$$

Since Y is isomorphic to $\hat{Y}$ and $\hat{Y}$ to ${\sigma'}^{-1}(\hat{Y}) = X' - D'$, the sought for equality, $\chi(X' - D') = \chi(X' - D'_0 \cup D'_\infty)$, is equivalent—by the additivity of the Euler characteristic—to the fact that $\chi\big(D'' - D'' \cap (D'_0 \cup D'_\infty)\big) = 0$. One deduces from the explicit description of σ' given in the proof of Lemma 5.2 that D'' is a disjoint union of exceptional divisors. With the same notation used above, the component E of D'' is by definition the projectivization of the normal bundle of $C_1 \cap C_2$. Let $\tilde{C}_1$, $\tilde{C}_2$ be respectively the proper transforms of C_1, C_2 in X'. The complement $E - E \cap (\tilde{C}_1 \cap \tilde{C}_2)$ is thus a fiber bundle over $\tilde{C}_1 \cap \tilde{C}_2$ with fiber isomorphic to $\mathbb{C}^* = \mathbb{C} - \{0\}$. It follows that $\chi\big(E - E \cap (\tilde{C}_1 \cap \tilde{C}_2)\big) = 0$, and, summing over the various components, also that $\chi\big(D'' - D'' \cap (D'_0 \cup D'_\infty)\big) = 0$, as desired. This concludes our second proof of Theorem 1.2.

Example 5.5. The operation of attaching a cell e with boundary $\dot{e}$ to a topological space S consists of providing an attaching map $s\colon \dot{e} \to S$ and of identifying every $x \in \dot{e}$ with $s(x)$. The content of Theorem 5.3 is illustrated by the following simplest example of Theorem 1.2. Let $D = \{t_1, \dots, t_{N-1}, t_N = \infty\}$ be a set of distinct points in $\mathbb{P}^1$, and let $\phi_\lambda(x) = \prod_{i=1}^{N-1} (x - t_I)^{\lambda_I}$. Here $N \geq 2$ and $\lambda = (\lambda_1, \dots, \lambda_N)$ is a point on the hyperplane $\Lambda = \{\lambda_1 + \cdots + \lambda_N = 0\} \subset \mathbb{C}^N$. One easily verifies that, for a generic λ in Λ, all critical points of ϕ_λ are non–degenerate. Moreover, if $A = \{\lambda_1 \cdots \lambda_N = 0\}$ is the union of the coordinate hyperplanes in $\mathbb{C}^N$, the number of critical points for a generic $\lambda \in \Lambda - A \cap \Lambda$ is equal to $N - 2$. One may choose λ so that $D_\infty = \{\infty\}$, $D_0 = \{t_1, \dots, t_{N-1}\}$; $\partial\overline{T}(D_0)$ is the union of $N - 1$ small disjoint 1–spheres S_I centered at the points $t_I \in D_0$. We may assume the t_I are ordered as $\operatorname{Re} t_I \leq \operatorname{Re} t_{I+1}$. Theorem 4.3 says in this case that $\mathbb{P}^1 - D$ is, homotopically, the space obtained by attaching $N - 2$ open segments e_I so to connect S_I with S_{I+1} for $I = 1, \dots, N - 1$.

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