Iterations of rational functions: which hyperbolic components contain polynomials?[1]

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Abstract. *Let H^d be the set of all rational maps of degree $d \geq 2$ on the Riemann sphere which are expanding on Julia set. We prove that if $f \in H^d$ and all or all but one critical points (or values) are in the immediate basin of attraction to an attracting fixed point then there exists a polynomial in the component $H(f)$ of H^d containing f. If all critical points are in the immediate basin of attraction to an attracting fixed point or parabolic fixed point then f restricted to Julia set is conjugate to the shift on the one-sided shift space of d symbols.*

We give exotic *examples of maps of an arbitrary degree d with a non-simply connected, completely invariant basin of attraction and arbitrary number $k \geq 2$ of critical points in the basin. For such a map $f \in H^d$ with $k < d$ there is no polynomial in $H(f)$.*

Finally we describe a computer experiment joining an exotic example to a Newton's method (for a polynomial) rational function with a 1-parameter family of rational maps.

Introduction.

In the space Q^d of rational maps of degree $d \geq 2$ of the Riemann sphere $\bar{\mathbb{C}}$, denote by H^d the set of maps which are expanding on the Julia set. Expanding means that there exists $n > 0$ such that for every z in the Julia set $|(f^n)'(z)| > 1$ (the derivative is in the standard spherical Riemann metric). We call $z \in \bar{\mathbb{C}}$ a critical point if $f'(0) = 0$. We call v a critical value if $v = f(z)$ for a critical point z.

In Section 1 we prove following:

Theorem A. Let $f \in H^d$. Suppose that all, or all but one, of the critical values of f are in an immediate basin of attraction $B(f)$ to one attracting f-fixed point p. Then the component $H(f)$ of H^d containing f contains also a polynomial.

(Critical values are counted in Theorem A without multiplicities. However critical points everywhere in the paper are counted with multiplicities.)

Corollary B. If all critical points of f are in $B(f)$ then f restricted to the Julia set $J(f)$ is conjugate to the full one-sided shift.

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Theorem C. Let $f \in Q^d$, p be a parabolic fixed point, and let $B(f)$ be an immediate basin of attraction to p adjacent to p such that $f(B(f)) = B(f)$ (which is equivalent to $f'(p) = 1$). Suppose that all critical points of f are in $B(f)$. Then f restricted to $J(f)$, is conjugate to the full one-sided shift.

The procedure to prove Theorem C is similar to Corollary B so it will be only sketched.

It is much easier to prove that, under the assumptions of Corollary B or Theorem C, $J(f)$ is Cantor, $f|_{J(f)}$ is conjugate to a topological Markov chain and $f^n|_{J(f)}$ is conjugate to a 1-sided shift, then to prove that $f|_{J(f)}$ itself is conjugate to a full 1-sided shift.

The questions answered in Corollary B and Theorem C were asked to me by John Milnor. In the case of the basin of a sink he suggested to join critical values with the sink along trajectories of the gradient flow , Morse curves, described below. This was a fruitful idea. After proving Theorem A and Corollary B I learned that these facts were proved by P. Makienko[3] in his PhD paper but stayed unpublished. Corollary B is also stated in [GK] but proved only for $d = 2$.

Pay attention that Corollary B and Theorem C depend on holomorphic phenomena. Indeed there exists a 1-sided topological Markov chain T for which each point has 10 preimages which has the same ζ-function as the full 1-sided shift of 10 symbols S_{10} and which sufficiently high power T^n is conjugate to S_{10}^n, but T is not conjugate to S_{10} [Boyle].

If B, the immediate basin of attraction to an attracting fixed point, is simply connected, then the number of critical points of f in B is equal to $\deg(f|_B) - 1$. (Because f pulled-back to the unit disc $\mathbb{D}$ by a Riemann mapping is a Blaschke product which has $\deg(f|_B) - 1$ critical points in $\mathbb{D}$ (and the same number of them outside).)

If B is the basin of attraction to ∞ for a polynomial and B is not simply-connected, then the number of critical points of f in B is at least $\deg(f|_B)$ (including ∞ as $(\deg(f)-1)$-multiple critical point).

Surprisingly this is false in general. The "proof" that if the basin is not simply-connected than it contains at least as many critical points as the degree of f on it , given in [P], is wrong for degree larger than 2, the corresponding Lemma in [P] is false.

Here in Section 2 we prove with the use of the quasiconformal surgery technique [D] the existence of *exotic basins*:

Theorem D. There exists a rational function f of an arbitrary degree $d \geq 3$ with a completely invariant (i.e. invariant under f^{-1}) non-simply connected basin of immediate attraction to an attracting fixed point, and with an arbitrary $2 \leq k \leq 2d - 2$ number of critical points in the basin.

[3] Recently his proof appeared in the preprint [M]. It influenced our revised version of the paper. Also our Theorem C is proved in [M].

The conclusion is that the assumption all or all but one critical values are in the basin in Theorem A, is essential. Namely, due to Theorem D for $k \le d-1$, we arrive at

Corollary E. For every $d \ge 3$ there exists $f \in H^d$ with a completely invariant basin of immediate attraction to an attracting fixed point, such that $H(f)$ contains no polynomial.

Neither in Theorem A nor in Corollary E does it matter that $f \in H^d$. Only the basin $B(f)$ matters. The reader will find appropriate precise assertions in Sections 1 and 2.

I checked with the help of computer that $f(z) = z^2+c+b/(z-a)$ for $c = -3.121092, a = 1.719727, b = .3142117$ is an exotic example for $d = 3$ which existence is asserted in Corollary E. I am grateful to Ben Bielefeld and Scott Sutherland for the help in producing several computer pictures of Julia sets for such f's and related pictures in the parameter space. Ben invented the parametrization in which the pictures were done. Our 1-parameter families join exotic examples of the above type of $d = 3$, having two superattracting fixed points and a critical point of period 2, with functions having three superattracting fixed points. It is easy to see (and well-known) that the latter functions must be Newton's method rational functions for degree 3 polynomials.

(If $f \in Q^d$ has d superattracting fixed points then in appropriate holomorphic coordinates on $\bar{\mathbb{C}}$ it is Newton's for a degree d polynomial. Hint to a proof: Change first the coordinates on $\bar{\mathbb{C}}$ by a homography so that unique repelling fixed point becomes ∞.)

Section 1. Rearranging critical values. Proofs of Theorem A and Corollary B.

Let $B(f)$ be the immediate basin of attraction to an attracting fixed point p for a rational map $f \in H^d$. Suppose $f'(p) \neq 0$.

We shall make in this Section the following types of perturbations of such maps in H^d:

1. A perturbation along a curve γ. We have in mind here the following construction: Suppose there is a curve $\gamma = \gamma(t), t \in [0,1]$ embedded in the basin $B(f)$ with $p \notin \gamma$. Take a small neighbourhood U of $\gamma, U \subset B(f)$ disjoint with a neighbourhood of p. Let g_t for every $t \in [0,1]$ be a diffeomorphism of $\bar{\mathbb{C}}$ so that $g_t(\gamma(0)) = \gamma(t)$ and g_t be the identity outside U , $g_0 = \mathrm{id}$ and g_t smoothly depend on t. We obtain the homotopy $h_t = g_t \circ f$. Pay attention that though we called our perturbation "along γ" we change the map in a neighbourhood of $f^{-1}(\gamma)$.

If the following assumption holds:

$$p \notin U, \quad h_t^n(U) \to p \tag{1}$$

then of course the basin of attraction to p is the same for h_t as for f.

Then we construct an invariant measurable L^∞ conformal structure for each h_t as follows: We take the standard structure on a small neighbourhood of p then we pull-back it by h_t^{-n} . On the complement we take the standard structure. Now we integrate this structure (we refer to Measurable Riemann Mapping Theorem [B], [AB]) and h_t in the new coordinates gives a homotopy f_t through maps in H^d. See [D] for this technique.

2. A small C^1-perturbation. If a map g is C^1 close to f on U such that $\mathrm{cl}U \subset B(f) \setminus \{p\}$, $f = g$ outside U and in neighbourhoods of critical points g differs from f only by affine parts, then clearly it is homotopic to f through also small perturbations satisfying the same conditions as g. The condition (1) holds automatically. As before we introduce new conformal structures, integrate them and obtain a perturbation homotopic through maps in $H(f)$ to f.

3. Blaschke type perturbation. Let $U \subset B(f)$ be an open topological disc containing p, with smooth boundary not containing critical points, such that $f(\mathrm{cl}U) \subset U$ $f : U \to f(U)$ is a proper map and $d' := \deg f|_U \geq 2$. Then we construct a 1-parameter family of maps joining in $H(f)$ the map f to a map having p as a d'-multiple fixed point as follows:

Let R_1, R_2 be Riemann maps from U, respect. $f(U)$, to the unit disc $\mathbb{D}$ such that $R_i(p) = 0, i = 1, 2$. Let $a_1 = 0, a_2, ..., a_{d'}$ be R_1-images of $f|_U$-preimages of p. Let $B_t = \lambda z \prod_{i=2}^{d'} \frac{z - t a_i}{1 - t\bar{a}_i z}, |\lambda| = 1, t \in [0, 1]$. We set $h_t = R_2^{-1} \circ B_t \circ R_1$. Here λ is chosen so that $h_1 = f$. It is useful to write $h_t = R_1^{-1} \circ g_t \circ R_1$, where $g_t = R_1 \circ R_2^{-1} \circ B_t$. Change h_t in U close to ∂U by a smooth isotopy so that h_t and f coincide on ∂U for all t. We extend h_t outside U by f to the whole $\bar{\mathbb{C}}$. As in the previous cases we pull-back the standard conformal structure from $R_1(f(U))$ by R_1^{-1} to $f(U)$, extend it by h_t^{-n}, complete on $\bar{\mathbb{C}} \setminus \bigcup h_t^{-n}(f(U))$ with the standard structure and integrate.

Let Φ_f conjugates f to $z \to \lambda z$ where $\lambda := f'(p) \neq 0$, in a neighbourhood of p (i.e. $\Phi_f f(z) = \lambda \Phi_f(z)$. Extend Φ_f to $B(f)$ by $\Phi_f(z) = \lambda^{-n} \Phi_f f^n(z)$. Define $G_f(z) = |\Phi_f(z)|^2$.

If $f \in H^d$ and $g \in H(f)$ then we write p_g for the point $z : g(z) = z$ such that (g, z) belongs to the component of the Cartesian product $H(f) \times \bar{\mathbb{C}}$ containing (f, p_f). There exist $B(g)$ and Φ_g (provided $g'(p_g) \neq 0$) as above.

Now we can formulate

Main Lemma. For every $f \in H^d$ there exists $g \in H(f)$ such that $g'(p_g) \neq 0$ and there exists $a > 0$ such that all critical values of critical points in $B(g)$ are in a component ∂ of $\{G_g = a\}$ which is a topological circle separating p_g from Julia set $J(g)$.

Proof. By small perturbations (types 2 and 3) we assure that the sink p_f is neither a critical point nor a critical value for a critical point in $B = B(f)$ all critical points in B are simple and their forward trajectories are pairwise disjoint.

At the end it may occur useful also to have $\frac{1}{\pi}\mathrm{Arg} f'(p)$ irrational. We assure this by type 3 perturbation where $\mathrm{Arg}\lambda$ is the parameter.

The critical points of $G = G_f$ are

1) the fixed point p and its iterated pre-images (these are minimum points with $G = 0$),

2) the critical points of f and their iterated pre-images (these are saddle points for G).

Denote the set of points in 1) by M and the set of points in 2) by S.

For every $q \in M$ let $A(q)$ denote the basin of attraction to q for the flow of the vector field $-\mathrm{grad}G$.

Denote by $r(q)$ the least non-negative integer such that $f^{r(q)}(q) = p$. For every $z \in A(q)$ or $X \subset A(q)$ write $r(z) := r(q)$ and $r(X) := r(q)$

Observe that for every $z \in B(f)$ there exists a curve γ joining z with p consisting of critical points of G and of trajectories of grad$G$ where this field is non-zero (i.e. γ goes from z to a critical point,, say a minimum, then to a saddle, then to a minimum etc. until it reaches p. Let $\gamma(z)$ denote a curve as above intersecting the minimal possible number of $A(q)$'s. Denote the number of these $A(q)$'s by $s(z)$ or $s_f(z)$ and call the curve a *Morse curve.*

Observations

1. $r(f(z)) = r(z) - 1$ if $r(z) \geq 1$
2. $s(f(z)) \leq s(z)$.

The observation 2. follows from the fact that f maps trajectories of grad$G$ to trajectories of grad$G$.

The plan is now to move all critical values to the same level $G = a$ in $A(p)$.

We shall do it for each critical point separately so that we do not move the critical values moved before to the level a in $A(p)$. We move each critical value $f(c)$ step by step so that after each step $s(f(c))$ decreases and $f(c)$ is in some $A(q)$. When $f(c) \in A(p)$ we move $f(c)$ along the trajectory of grad$G$ to the level a as described in 1.

Take c a critical point for f. By a small perturbation in a neighbourhood of c we obtain $f(c) \in A(q)$. We can assume that $q \neq p$.

This is correct because

a) The perturbation is above the G level of $f(c)$ so it does not change a part of the stable manifolds of a saddle to which $f(z)$ might belonged, in a neighbourhood of $f(c)$ and between it and the saddle).

b) The change of coordinates integrating the new invariant conformal structure is conformal there. This is so because the structure is non-standard only in some places above a. Thus the change of coordinates maps the gradient lines of the old G to the gradient lines of the new one and stable manifolds (separatrices) to separatrices.

Now let γ be the part of a Morse curve $\gamma(f(c))$ joining $f(c)$ with the first saddle $\omega \in \gamma(f(c))$.

Observe that for every $j > 0$ we have $f^j(\gamma) \cap \gamma = \emptyset$.

Indeed, we have $f^j(\gamma \cap A(q)) \cap \gamma = \emptyset$ by Observation 1. . By the same observation $f^j(\omega) \in \mathrm{cl}A(f^j(q))$ cannot be in $\gamma \cap A(q)$. Finally $f^j(\omega) \to p$ so $f^j(\omega) \neq \omega$.

As $f^j(\gamma) \to p$ we can find U a neighbourhood of γ so that $f^j(U) \cap U = \emptyset$ for all $j > 0$. Take in U a curve γ' joining $f(c)$ with a point z close to ω, $z \in A(q')$, $s(q') = s(q) - 1$. Take care about γ' not containing critical values under iterates f^j of other critical points.

Now perturb f along γ' as described in 1. Do it with g_t different from identity only in a neighbourhood of γ' small enough not to contain critical values of other critical points, under iterates f^j . The condition (1) is of course satisfied.

Now let us explain why the new map f_1 has the property

$$s_{f_1}(f_1(c_{f_1}) = s_f(f(c)) - 1. \tag{2}$$

Here c_{f_1} is the old c in new coordinates, it is a critical point for f_1. We use the fact that a part of the domain we changed f to h_t is the basins $A(f^{-1}(q))$ where by Observation 2. $s \leq s(q)$. So we did not change G in the basins, the part of the Morse curve γ beyond ω goes through, where $s < s(q)$. We changed f also in a neighbourhood of $f^{-1}(\omega)$. This does not change G below a neighbourhood of ω because $G(f^{-j}(\omega) > G(\omega), j = 1, 2, ...$ (Compare this with the argument (a).) This does not change G below the part of γ beyond a neighbourhood of ω neither, because this change is close to the set $s \geq s(q)$.

The change of h_t to f_t does not hurt (2). Indeed also by the above arguments the new measurable conformal structure is the standard one below ω and the part of γ beyond ω. So the change of coordinates maps the gradient lines of the old G to the gradient lines of the new one as in (b).

When $f(c) \in A(p)$ we first make a small perturbation so that the gradient line of G passing through $f(c)$ does not intersect forward trajectories of other critical values which are already in $A(p)$. Next we move $f(c)$ along γ which is a piece of the gradient line of G passing through $f(c)$ joining it with $\{G = a\}$ in $A(p)$. We succeed because $\frac{1}{\pi}\mathrm{Arg}(f^n)'(p)$ is irrational so all the curves $f^n(\gamma), n \geq 0$ are pairwise disjoint and the conformal structure does not change below these curves.

After a sequence of consecutive perturbations as above we obtain a rational mapping g with all the critical values on one level $\{G = a\}$, more precisely its component ∂ intersecting $A(p)$.

Now denote the domain of $\bar{\mathbb{C}} \setminus \partial$ containing p by D_a. To finish Proof of Main Lemma take $b > 0$ so close to 0 that a component ∂' of $\{G = b\}$ is in the domain around p where g is linearizable. So ∂' is a topological circle. Then each $x \in \partial'$ can be mapped to the point of intersection of ∂ with the trajectory of $\mathrm{grad}G$ starting from x. This gives a homeomorphism between ∂' and ∂.

Otherwise $\mathrm{cl}D_a$ contains an S-type critical point x for G_g. Then there exists $n > 0$ such that $y = g^n(x)$ is a critical value for g. Hence $G_g(y) = |g'(p_g)|^n G_g(x) < G_g(x) \leq a$. This contradicts just achieved $G_g(y) = a$.

Proof of Main Lemma is finished. ♣

Proof of Theorem A. Let f be already as g in Main Lemma. By perturbing along curves one obtains additionally all critical values of critical points from $f^{-1}(B(f)) \setminus B(f)$ also in ∂. (*A posteriori* we will see that under the assumptions of Theorem A we have $f^{-1}(B(f)) \setminus B(f) = \emptyset$ i.e. $B(f)$ is completely invariant.)

Denote again the domain of $\bar{\mathbb{C}} \setminus \partial$ containing p by D_a. Denote the complementary open topological disc $\bar{\mathbb{C}} \setminus \mathrm{cl} D_a$ by D'_a.

Observations

3. There is at most one critical value for f in D'_a. So the components of $f^{-1}(D'_a)$ are topological discs $D'^j, j = 1, ..., \hat{d}$ where $\hat{d} \leq d$, with closures in D'_a. (In particular $f^{-1}(\mathrm{cl} D_a)$ is connected hence $B(f)$ is completely invariant.)

4. Closures of D'^j intersect or "self-intersect" only at critical points of f and $\mathrm{cl} f^{-1}(D'_a)$ is connected , see Fig 1..

If the latter were false then $f^{-1}(D_a)$ would contain a nonsimply-connected component V . But f maps maps V onto the disc D_a so it V would contain a critical point hence D_a would contain a critical value. This would contradict the assumption that all critical values are on the level a.

After a small perturbation moving (exposing) critical values of some $d-1$ critical points towards below a, the set $\mathrm{cl} f^{-1}(D'_a)$ consists of $\hat{d}$ closed discs intersecting one another at at most 1 point, which union is connected and simply-connected, Fig 2.

So for $\varepsilon > 0$ small enough the set $f^{-1}(\partial D_{a-\varepsilon})$ (where $D_{a-\varepsilon} := D_a \cap \{G < a - \varepsilon\}$), is a topological circle ∂_ε bounding a topological disc $U \ni p$ and under f it winds d times onto $\partial D_{a-\varepsilon}$. Of course $f(\mathrm{cl} U) = \mathrm{cl} D_{a-\varepsilon} \subset U$.

After performing Blaschke type perturbation and a holomorphic change of coordinates on $\bar{\mathbb{C}}$ we arrive at a polynomial. ♣

Figure 1.

Figure 2. Critical points c_1 and c_4 are exposed towards p.

Proof of Corollary B. This Corollary follows from Theorem A because its assertion is an open property in H^d and it is true and easy for polynomials.

Indeed if f is a polynomial, then with the help of a small perturbation we guarantee that for each critical value $v_j, j = 1, ..., d-1$ (different from ∞) the trajectory γ_j for grad$G$ where G is Green's function in the basin of attraction to ∞, pole at ∞, goes from v_j up to ∞. In other words γ_j does not go to any critical point for G. Then there are no critical values for f in the topological disc

$$U = \bar{\mathbb{C}} \setminus A \quad \text{where} \quad A = \bigcup_{j=1,\dots,d-1,\ n\geq 0} f^n(\gamma_j).$$

Because $f(A) \subset A$, we have a collection of branches $g_j : U \to U$ of f^{-1}. Denote $g_j(U)$ by U_j. Then each $z \in J(f)$ is coded by the sequence of symbols $j_n, n = 0, 1, ...$ where $f^n(z) \in U_{j_n}$.

For each sequence (j_n) the family of maps $\Psi_n = g_{j_0} \circ g_{j_1} \circ ... \circ g_{j_n}$ is a normal family of maps on U. It is easy to find a slightly smaller topological disc $U' \subset U$ so that $U' \supset J(f)$ and each

$$g_j \text{ maps } \mathrm{cl}U' \text{ into } U'. \tag{3}$$

Hence the sequences $\Psi_n|_{U'}$ converge uniformly to points in $J(f)$. This proves that the coding is one-to-one. ♣

Remark 1.1. In fact a proof of Corollary B is contained in Proof of Theorem A. Indeed already Observation 3. gives a partition of the Julia set J into $J \cap D'^j$ and $J \ni z \mapsto (j_n)$ such that $f^n(z) \in D'^{j_n}$ gives a conjugacy to the one-sided shift space of d symbols.

Remark that for this proof of Corollary B, there was no need to refer to Measurable Riemann Mapping Theorem. Namely there was no need to integrate every time a new conformal structure to obtain a rational mapping. One could work just with smooth maps until the properties described in Observation 3. are reached.

Remark 1.2. One can prove Theorem A by perturbing f along a curve $\hat{\gamma}$ close to the Morse curve, dragging the critical value to a small neighbourhood of p just in one step instead of doing it step by step decreasing $s(f(c))$. The property (1) is satisfied by Observations 1. and 2. (Every point of $\hat{\gamma}$ can come close to $\hat{\gamma}$ under f^j only further, i.e. if $f^j(\hat{\gamma}(\tau))$ is close to $\hat{\gamma}(\tau')$ then $\tau' > \tau$. The more so for f^j replaced by h_t^j .)

Remark 1.3. Makienko [M] proved the following Proposition which corresponds to our Main Lemma:

If all critical values $v_1, ..., v_m$ for critical points in the basin $B(f)$ of immediate attraction to an attracting fixed point p_f have disjoint forward orbits then there exists a topological disc $U \subset B(f)$ containing p_f such that $f|_U$ is injective, $f(\mathrm{cl}U) \subset U$ and all $v_j, j = 1, ..., m$ belong to the annulus $U \setminus f(U)$.

One can easily change U so that all v_j belong to $\partial f(U)$. There is a quasiconformal conjugacy Φ of f on U to $z \mapsto \lambda z$ on the unit disc $\mathbb{D}$, $|\lambda| < 1$. So as usually one can pull-back the standard conformal structure on $\mathbb{D}$ by Φ^{-1} to U spread it by $\Phi^{-n}, n = 1, 2, \ldots$ to the whole basin of ∂_f and complete on the complement of the basin by the standard structure. After integration of this structure the new map satisfies the properties asserted in Main Lemma.

Thus: Makienko's Proposition (preceded by a small perturbation) + Measurable Riemann Mapping Theorem, give Main Lemma.

Conversely, g from the assertion of Main Lemma is conjugate to f (provided f is already after the first perturbation in Proof of Main Lemma). So Makienko's $f(U)$ can be defined as the image under the conjugacy of the disc bounded by ∂. Observe that for that we did not need to construct the new holomorphic structures along the proof. (Compare this with Remark 1.1.)

Thus Makienko's Proposition is a topological heart of Main Lemma.

Remark 1.4. Neither in Main Lemma nor in Theorem A one needs to assume $f \in H^d$. Proofs work for an arbitrary rational map $f, \deg f \geq 2$ if one replaces in the statements $H(f)$ by Teich_J.

The latter denotes the set of all such rational maps g that there exists
1) a connected open domain $\mathcal{P} \ni 0$ in the complex plane,
2) a family of quasiconformal homeomorphisms $h_\lambda : \bar{\mathbb{C}} \to \bar{\mathbb{C}}, \lambda \in \mathcal{P}$ with $h_0 = \mathrm{id}$ such that for each z the point $h_\lambda(z)$ depends holomorphically on z, (such a family is usually called a holomorphic motion).
3) a family of rational maps $f_\lambda, \lambda \in \mathcal{P}$ with $f_0 = f$ and $f_{\lambda_0} = g$ for some $\lambda_0 \in \mathcal{P}$ such that for every λ the map h_λ conjugates f with f_λ between ε-neighbourhoods of their Julia sets (ε not depending on λ).

Proof of Theorem C. One proves first an analog to Main Lemma:

If $f(z) = z + a(z-p)^{t+1} + o((z-p)^{t+1})$, $a \neq 0$ then $H_1(z) = \frac{\lambda}{z^t}$ for some $\lambda \neq 0$ conjugates f on a "petal" $P \subset B(f)$ to $F(z) = z + 1 + o(1)$ for z with large real part. (See [DH] for the precise description.) Next conjugate smoothly F to $z \mapsto z+1$ by H_2. Define $\Phi_f = H_2 \circ H_1$ on P and extend it to $B(f)$ by $\lim_{n\to\infty} \Phi_f \circ f^n(z) - n$. Define $G_f(z) := \frac{1}{\exp \Re \Phi_f(z)}$ on $B(f)$.

Then after a small perturbation such that all critical points in B are simple and their forward orbits are disjoint, one can find a quasiconformally conjugate $g \in Q^d$ such that all critical values v_j of g-critical points in $B(g)$ are in the component of $\{G_g = a\}$ bounding a "petal". (See Fig. 3.)

A proof of this parabolic version of Main Lemma is the same as that of Main Lemma except there are no M- critical points for G_f in $B(f)$. One can think of M critical points as belonging to $\partial B(f)$, being precisely the set $\partial B(f) \cap \bigcup_{n\geq 0} f^{-n}(\{p\})$. So the perturbation is not along Morse curves in basins $A(q)$ but along curves in $B(f) \cap A(q)$ joining directly consecutive S-critical points in $\partial A(q)$.

Repeat now Proof of Corollary B:

(The assumptions imply $t = 1$.)

For f already as g above we take curves $\gamma_j, j = 1, ..., 2d-2$ joining v_j to p. We can take as γ_j's the Φ_f^{-1}-preimages of horizontal lines in $\bar{\mathbb{C}}$. Then $f(\gamma_j) \subset \gamma_j$. So for $A = \bigcup_{j=1}^{d-1} \gamma_j$ the set $\bar{\mathbb{C}} \setminus A$ is a topological disc and we have for the branches of f^{-1} $g_j(U) \subset U$, $j = 1, ..., d$.

We find $U' \subset U$ such that (3) holds, except $\partial U' \cap \partial U = \{p\}$ and $g_{j_0}(p) = p$ for the branch $g_{j_0} : g_{j_0}(p) = p$, see Figure 3 By uniform $(g^n_{j_0})' \to 0$ on clU' we obtain $\bigcap_n g^n_{j_0}(U')$ is only one point p. $\clubsuit$

Figure 3.

Remark 1.5. As we mentioned in Introduction the fact that under the assumptions of Corollary B or Theorem C $J(f)$ is a Cantor set is much easier than the fact $f|_{J(f)}$ is conjugate to the one-sided shift.

Indeed, it is easy to prove that $f^N|_{J(f)}$ is conjugate to a one-sided shift, for an integer $N > 0$.

Just take a small topological disc $D \subset B(f)$ so that $D \ni p$ (or a small petal in the parabolic case) such that $f(\mathrm{cl}D) \subset D$. ,

For each critical value v_j and every $n \geq 0$ such that $f^n(v_j) \notin D$ join $f^n(v_j)$ to D by an embedded curve $\gamma_{j,n}$ so that these curves are mutually disjoint. Take N such that for each j,n $\quad f^N(\gamma_{j,n}) \subset D$. Consider now $U = \bar{\mathbb{C}} \setminus A$ where $A = D \cup \bigcup_{j,n} \gamma_{j,n}$. Next proceed as in Proof of Corollary B where f is replaced by f^N.[4]

Remark 1.6. It is also easy to prove that $f|_{J(f)}$ is conjugate to a one-sided topological Markov chain.

Indeed, let $D_0 = D$ as in Remark 1.5 . Let D_n, $n = 1, 2, ...$ be defined recursively: D_n is the component of $f^{-1}(D_{n-1})$ containing D_{n-1}.

Let $N \geq 0$ be large enough that there are no critical values in $\bar{\mathbb{C}} \setminus D_N$.

Let $U_1, ..., U_K$ denote all the components of $\bar{\mathbb{C}} \setminus \mathrm{cl} D_N$. These are topological discs because D_N is connected. Consider the family of topological discs $g_j(U_k)$ for all branches $g_j, j = 1, ..., d$ of f^{-1} and $k = 1, ..., K$. Then for every sequence of pairs $(j_n, k_n), n = 0, 1, ...$ such that $g_{j_n}(U_{k_n}) \subset U_{k_{n+1}}$ there exists precisely one point $z \in J(f)$ such that for every $n \geq 0$ $\quad f^n(z) \in g_{j_n}(U_{k_n})$.

Remark 1.7.[5] Considering the situation in Theorem A such that p is already not critical (say all f-critical points belong to $B(f)$ and $\mathrm{Arg} 2\pi f'(p)$ is rational), John Milnor asked whether it is possible to find a set A which is union of parts of trajectories of grad$G$ contains all critical values is compact in $B(f)$ is f-forward invariant and is connected simply-connected (a tree).

This would immediately allow to prove Corollary B with the set A as here.

Unfortunately the answer is negative. We can modify an exotic example in Section 2 (the case $d = 3$, Fig. 5) so that $f(c_4) = f^2(c_3) = a$ the pole. Then the set of saddles for G is $S = \bigcup_{n \geq 0} f^{-n}(c_2)$. So every curve Γ built from the pieces of trajectories of grad$G$ joining $c_4 = f(c_4)$ to ∞ passes a point of $f^{-n}(\{c_2\})$. Hence $f^n(\Gamma)$ joins ∞ with ∞ passing through c_2, hence it is a loop, i.e. A is not a tree.

(Even the assumption $f(c_4) \neq a \neq f^2(c_3)$ does not help if $f(c_4), f^2(c_3)$ are close to a. Γ must still leave the basin $A(a)$ for grad$G$ passing through a point in $\bigcup_{n \geq 0} f^{-n}(\{c_2\})$.)

Remark 1.8.[5] Though in Main Lemma we can arrange all critical values of critical points in $B(f)$ on one component of a level of G it sometimes is not so for critical points (unless all critical points for f or all but one, are in $B(f)$ as in Theorem A, see Fig 2.). Again we modify an exotic example from Section 2. Here degree of f is 5. Start with a cubic polynomial P which has degree 1 on ∂A_1, degree 2 on ∂A_2, see Fig 3, Section 2, and P maps the critical point $a \in A_2$ to the critical point at the self-intersection of the figure 8 line $\partial A_1 \cup \partial A_2$, which escapes to ∞.

Consider now the function $z \mapsto P(z) + \frac{b}{(z-a)^2}$ with b small real positive number. We obtain the picture as on Figure 4. Do the surgery as in Section 2 to have it holomorphic. For a final example split a into two different poles which gives an f-critical point c_8 between them, Fig. 4. Make Blaschke type perturbation close to ∞ to have ∞ not critical and move $f(c_8), f(c_1), f(c_2)$ to one level with $f(c_3)$.

[4] I owe this proof to K. Barański.

[5] Before reading this Remark the reader is advised to read Section 2.

Figure 4. $f^2(c_4) = c_4, f^2(c_6) = c_6, f(c_5) = c_5, f(c_7) = c_7$.

Section 2. Exotic basins.

Proof of Theorem D, case $d = 3$. We start with a geometric description of an exotic example of degree 3, illustrated on Fig. 5 .

Start with a quadratic polynomial $P(z) = z^2 + c$ with the critical point $c_2 = 0$, escaping to $c_1 = \infty$ a attracting fixed point of multiplicity 2. The level $\partial = \{G = t\}$ of Green's function of the basin of attraction to ∞ with the pole at ∞, containing c_2, is figure eight. Now we change the map on B_2, one of the two discs B_1, B_2 bounded by ∂ as follows:

Draw two little discs D_1, D_2 in B_2, intersecting one another. Let $D_1 \cap D_2$ be maped 1-to-1 onto $\bar{\mathbb{C}} \setminus (B_1 \cup B_2)$. Let $D_1 \setminus D_2$ goes onto B_1 and $D_2 \setminus D_1$ goes onto B_2 both both proper maps with degree 2. So there are critical points $c_3 \in D_1 \setminus D_2$ and $c_4 \in D_2 \setminus D_1$. On D_2 this map f is quadratic-like so we can do anything there, for example $f(c_4) = c_4$. On D_1 the map f^2 is quadratic-like so we can assume $f^2(c_3) = c_3$.

Figure 5.

The rational function is obtained out of this topological picture by the quasi-conformal surgery technique [D]. We shall explain it closer now:

We need following Lemma which generalizes Douady-Hubbard's theorem that a polynomial-like mapping is quasiconformally conjugate to a polynomial [DH1]:

Lemma 2.1. Let $U \subset \bar{\mathbb{C}}$ be an open set (not necessarily connected or simply-connected) with boundary being a family of smooth Jordan curves. Let $F_1 : U \to U$ be a holomorphic map such that its $F_1(\mathrm{cl}U) \subset U$. (We denote the continuous extension of F_1 to $\mathrm{cl}U$ by the same symbol F_1.)

Let $V \subset \bar{\mathbb{C}}$ be homeomorphic to $\bar{\mathbb{C}} \setminus \mathrm{cl}U$ by a homeomorphism h_1 which extends orientation preserving to a homeomorphism of $\bar{\mathbb{C}}$. (Again we do not assume V is connected or simply-connected.) Let $F_2 : V \to \bar{\mathbb{C}}$ be a holomorphic map. We also suppose that the boundary of V is smooth and denote by F_2 the continuous extension of th original F_2 to $\mathrm{cl}V$.

Suppose the family of curves being the components of ∂U has the same combinatorics in $\bar{\mathbb{C}}$ as the family of curves being the components of ∂V. We mean by this, that

1. There exists a homeomorphism $h_2 : \partial U \cup \partial F_1(U) \to \partial V \cup \partial F_2(V)$ such that the boundary of each component of $\bar{\mathbb{C}} \setminus (\partial U \cup \partial F_1(U)$ is mapped to the boundary of a component of $\partial V \cup \partial F_2(V)$

2. For each component ∂ of ∂U the map h_2 maps ∂ to $h_1(\partial)$, $F_1(\partial)$ to $F_1(h_1(\partial))$ and there a continuous map (a lift) $\tilde{h}_2 : \partial \to h_1(\partial)$ such that on ∂ we have $h_2 \circ F_1 = F_2 \circ \tilde{h}_2$. (i. e. h_2 preserves orders between $F_1(\partial)$ and $F_2(h_1(\partial))$).

Then there exists a rational map $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ and an open $W \subset \bar{\mathbb{C}}$ such that f is quasiconformally conjugate to F_1 on W and quasiconformally conjugate to F_2 on $\bar{\mathbb{C}} \setminus W$.

Proof. We replace h_2 on ∂U by the lift $\tilde{h}_2$ and then extend it from $\partial U \cup \partial F_1(U)$ to a quasiconformal homeomorphism $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$. Define $F : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ by F_2 on $\mathrm{cl}V$ and by $h \circ F_1 \circ h^{-1}$ on $\bar{\mathbb{C}} \setminus V$.

Let μ_0 denote the standard conformal structure on $\bar{\mathbb{C}}$. Take $\mu_1 = h_*(\mu_0|_U)$ on $\bar{\mathbb{C}} \setminus \mathrm{cl}V$. (Think about μ_1 as a field of ellipses, up to a multiplication by a positive function.) For each $z \in V$ define $\mu_1(z)$ as a pull-back $F_*^{-n}((\mu_1)(F^n(z))$ where $n \geq 0$ is such that $F^n(z) \in \bar{\mathbb{C}} \setminus \mathrm{cl}V$. If such n does not exist take $\mu_1(z) = \mu_0(z)$. This is correct due to the crucial property $F(\bar{\mathbb{C}} \setminus \mathrm{cl}V) \subset \bar{\mathbb{C}} \setminus \mathrm{cl}V$. As F_2 is holomorphic, μ_1 is in L^∞ !

Now integrate μ_1. In the new coordinates F changes to a rational map f we looked for. ♣

Now we construct F_1 and F_2 satisfying the assumptions of Lemma. It is illustrated on Fig. 6.

Take the polynomial $P(z) = z^2 + c$, $c < -2$. Make F_1 by adding to P a term $\frac{b}{z-a}$ for $a = \sqrt{-c} \in P^{-1}(0) \cap B_2$, $(a > 0)$. Let $b > 0$ be small so that for our F_1 the level

$\hat\partial = \{\hat G = t_0\}$ containing the F_1 critical point $\hat c_2$ close to $c_2 = 0$ is figure 8 close to ∂. Here $\hat G$ is defined analogously to Green's function or to G in Section 1: on the basin of attraction to ∞ by F_1, one defines $\hat G(z) = \lim_{n\to\infty} 2^{-n}\log|F_1^n(z)|$. Denote discs bounded by adequate parts of $\hat\partial$ close to B_1, B_2 by $\hat B_1, \hat B_2$ respectively.

It is easy to compute that $\hat c_2 = \frac{b}{2a^2} + o(b)$ and for two other F_1-critical points $\hat c_3, \hat c_4$ we have $F_1(\hat c_{3,4}) = \mp 2\sqrt{2a}\sqrt{b} + o(\sqrt{b})$. So $F_1(\hat c_3) \in \hat B_1, F_1(\hat c_4) \in \hat B_2$.

Let $0 < t_2 < t_1 < t_0$ with $t_2 \approx t_1 \approx t_0$ and denote by K_2, K_1 the topological discs both in $\hat B_1$ bounded by $\{\hat G = t_2\}$, respect. $\{\hat G = t_1\}$. Denote by K_4' the topological disc in $\hat B_2$ bounded by $\{\hat G = t_1\}$. Finally denote the component of $F_1^{-1}(K_1)$ in B_2 by K_3 and denote the component of $F_1^{-1}(K_4')$ in B_2 by K_4. We have $\hat c_3 \in K_3,\ \hat c_4 \in K_4$.

Define $U := \bar{\mathbb{C}} \setminus (K_2 \cup K_3 \cup K_4)$. We have $F_1(\mathrm{cl}U) \subset U$. So F_1 and U satisfy the assumptions of Lemma. Now we need to define F_2:

Set $F_2(z) := z^2$ on a geometric disc $L_4 = \{|z| < r_4\}$, $r_4 > 1$. Take a disc $L_3 = \{|z - z_0| < r_3\} \subset F_2(L_4) \setminus \mathrm{cl}L_4$ and define $\tilde F_2(z) = (z - z_0)^2 + z_0$. One finds large r_4, r_3 such that $\tilde F_2(L_3) \supset \mathrm{cl}F_2(L_4)$. Pick in $\tilde F_2(L_3) \setminus F_2(L_4)$ two discs $L_2 \subset L_1$ of the form $L_1 = \{|z - z_1| < r_1\}$, $L_2 = \{|z - z_1| < r_2\}, r_2 < r_1$. Take an affine holomorphic map $\Psi : L_1 \to \tilde F_2(L_3)$ (onto). Define

$F_2 = \Psi^{-1} \circ \tilde F_2$ on L_3 and

$F_2 = \Psi|_{L_2}$ on L_2.

We care to have r_2 so close to r_1 that $\Psi(L_2) \supset \mathrm{cl}F_2(L_4)$.

Now take $V = L_2 \cup L_3 \cup L_4$ and F_2 defined on V as above.

The assumptions of Lemma are satisfied. So we can "glue" F_1 and F_2 in one rational mapping f.

Figure 6.

Observe finally that $J(f)$ is disconnected because $F_1^n(\hat c_2) \to \infty$ and moreover $F_1^n(l) \to \infty$ where $l = \{\Im z = \Im \hat c_2\}$. The line l separates K_2 from say K_4. Both K_1 and K_4 intersect $J(f)$ (in the coordinates after the integration of μ_1) so the intersections belong to different components of $J(f)$.

The degree of f on the basin of attraction to ∞ is 3 because such is the degree of F_1 on U. Only two critical points: ∞ and that one corresponding to $\hat c_2$ belong to the basin, because $\hat c_{3,4}$ do not escape under the iteration by F_2. Theorem D is proved for $d = 3$.

Remark 2.2. Observe that in appropriate holomorphic coordinates on $\bar{\mathbb{C}}$ we have $f(z) = z^2 + c + \frac{b}{z-a}$. Indeed after subtracting from f constructed above the principal part of the Laurent series expansion at the pole, we are left with a quadratic polynomial. By an affine holomorphic change of coordinates we arrive with teh polynomiial to $z^2 + c$.

Remark 2.3. One should be careful in the above construction because not every branched cover of $\bar{\mathbb{C}}$ preserves a conformal structure. Above, an annulus $\mathcal{A}$ in B_2 containing c_3 and c_4 is mapped in a proper way by f to the disc $D' = \{G < t'\}, t' > t$, i.e. a disc containing ∞ , outside the figure 8 level $\{G = t\}$.

Instead of mapping c_3 into D_1 so that $F^2(c_3) = c_3$ we can map $\mathcal{A}$ onto D' in a proper way so that $f(c_3) = c_3$ and $f(c_4) = c_4$. This will be a topological branched cover. However it does not allow a holomorphic invariant structure.

If it allowed, for c_3 close to c_4, for $\mathcal{A}$ small but of a definite modulus, then in the limit after rescallings of $\mathcal{A}$'s to be of a definite size we would end up with a covering map of an annulus with two punctures to a disc with a puncture (covering without branching point). This is not possible by the Euler characteristics argument. It means c_3 and c_4 cannot be too close in D'.

Another argument is that such f would have 3 superattracting fixed points. So it would be a Newton's method rational function of a degree 3 polynomial, see Introduction. But the basin of attraction to ∞ is not simply-connected. This contradicts a theorem that the basins of immediate attraction to the attracting fixed points for Newton's method are simply-connected [P].

Proof of Theorem D, the general case $d \geq 3$.

We shall realize holomorphically the picture on Fig. 7:

Figure 7.

On Fig. 7, D_j is mapped properly on B_j for $j = 1, 2$. Each D_j contains $d - 2$ critical points. The points $a_1, ..., a_{d-2}$ are poles.

We proceed similarly as in the case $d = 3$. Let

$$F_1(z) = z^2 + c + b(\sum_{m=1}^{d-2} \frac{1}{z - a_m}).$$

Take $a_m = \sqrt{-c} + imT$ for a real constant $T : 0 < T \ll 1$ in particular T small enough that all a_m are well in B_2.

For b real $b > 0$ small, there is a small annulus around each pole a_m, containing two critical points

$$\hat{c}_{m,3}, \hat{c}_{m,4} = a \mp \sqrt{b/2a} + o(\sqrt{b}).$$

The corresponding critical values

$$v_{m,3} = F_1(\hat{c}_{m,3}), \quad v_{m,4} = F_1(\hat{c}_{m,4})$$

are

$$a_m^2 + c \mp 2\sqrt{2a}\sqrt{b} + o(\sqrt{b}) = 2imT + (-m^2T^2 \mp 2\sqrt{2a}\sqrt{b} + o(\sqrt{b})).$$

(Computing $\hat{c}_{m,3(4)}$ and $v_{m,3(4)}$ it is comfortable to consider $z^2 + c + \frac{b}{z-a_m}$. Other terms $\frac{b}{z-a_t}$ have only the $O(b)$ influence.)

The F_1 critical point close to 0 is $O(b)$.

Take K_1, K_4' from the case $d = 3$ slightly modified, larger than the original ones: Let l be the line (parabola) $(2iT\tau, -T^2\tau^2)$ for $\tau > 0$. Observe that the critical values $v_{m,3}$ are to the left of l, and $V_{m,4}$ to the right of l. We extend K_1, K_4' to $\hat{K}_1, \hat{K}_4'$ almost to l to capture $v_{m,3}, v_{m,4}$ respectively. See Figure 8.

Figure 8.

Consider the topological discs

$K_{m,3} = \mathrm{Comp}F_1^{-1}(K_1)$, $K_{m,4} = \mathrm{Comp}F_1^{-1}(K_4')$, where Comp means the component containing $\hat{c}_{m,3}$, $\hat{c}_{m,4}$ respectively.

K_2 is as in the case $d = 3$, such that $F_1(K_2) \supset \mathrm{cl}K_4'$.

Finally set

$$U := \bar{\mathbb{C}} \setminus (K_2 \cup \bigcup_{m=1}^{d-2} (K_{m,3} \cup K_{m,4}).$$

The rest of the construction of f is the same as for $d = 3$. When we make quadratic-like maps $\tilde{F}_2$ on $L_{m,3}$ and F_2 on $L_{m,4}$ we have a complete freedom of which quadratic polynomials we glue in, in particular of whether we want the corresponding critical points to escape or not. (In particular if no critical point escape we have the most surprising case $k = 2$ of the assertion of Theorem D.) ♣

For the completeness of the exposition we shall prove the following simple facts (The first of them stated already in Introduction) :

Proposition 2.4. a) Let $f \in H^d$ be a polynomial with $B(f)$ the basin of attraction to ∞ not simply-connected. Then $B(f)$ contains at least one critical point different from ∞.

b) More generally, if $f \in Q^d$ and for a pair of topological discs $A, A_1 :\ \mathrm{cl}A \subset A_1$ the map $f|_A : A \to A_1$ is proper of degree $d' \leq d$, then A_1 is in the basin of attraction $B(f)$ to an attracting fixed point and if $B(f)$ is not simply-connected then it contains at least d' critical points. This concerns in particular the case $d' = d$ in which $f|_{\bar{\mathbb{C}}\setminus \mathrm{cl}A_1} : \bar{\mathbb{C}} \setminus \mathrm{cl}A_1 \to \bar{\mathbb{C}} \setminus \mathrm{cl}A$ is polynomial-like.

Proof. a) Take a topological disc $D = \{G > a\}$ around ∞ (cf. Proof of Corollary B or Remarks 1.5, 1.6). If there are no critical points in $B(f)$ (except ∞) then $f^{-n}(D)$ is an increasing sequence of topological discs, so $B(f) = \bigcup_{n\geq 0} f^{-n}(D)$ is a topological disc, hence $B(f)$ is simply-connected.

(Remark that we already used the argument, that if there is only one critical value for a proper map $f : W_1 \to W_2$ where W_2 is a topological disc , then W_1 is also a topological disc, in Remark 2.2)

b) The proof is similar. There are $d' - 1$ critical points in A_1 and there must be a critical point in $B(f) \setminus A_1$.

(One can also deduce b) from a) using Blaschke type perturbation, Section 1.) ♣

Proposition 2.5. Every non simply-connected immediate basin of attraction to an attracting or parabolic fixed point (with $f'(p) = 1$) contains at least 2 different critical values of critical points in the basin.

This complements Theorem D: the integer k cannot be less than 2.

Proof. Consider the sets D_n defined in Remark 1.6. As $B(f) = \bigcup_{n\geq 0} D_n$ is not simply-connected, there exists n such that D_n is simply-connected and D_{n+1} is not. Then D_n contains at least two different critical values of critical points in D_{n+1}. ♣

(Almost the same argument proves the above for periodic basins, period larger than 1.)

Proof of Corollary E. Let $f \in H^d$ be as in Theorem D for $d \geq 3, k = 2$. Consider an arbitrary $g \in H(f)$. Then there exist a real continuous 1-parameter family of homeomorphisms $h_t : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ and a real 1-parameter family of maps $f_t \in H(f)$ having the same properties as h_λ and f_λ in Remark 1.4. (There exist complex families precisely as in Remark 1.4, but we do not need them here.) Then $\sharp(\mathrm{Crit}(f_t) \cap B(f_t))$ is constant because critical points cannot be too close to $J(t)$, where $|f'| > 0$ uniformly, they move continuously with t, so they cannot jump between components of $\bar{\mathbb{C}} \setminus J(f_t)$. So by Proposition 2.4, $B(g)$ cannot be the basin of attraction to ∞ for a polynomial. But degree of f hence g on each other invariant basin B_1 is less than d. (Otherwise $\partial B_1 = J(f)$ would be connected and it is not because $B(f)$ is not simply-connected.) So g cannot be a polynomial. ♣

Remark that it follows from Proposition 2.4,b) and above Proof that none $g \in H(f)$ has a polynomial-like restriction of degree d.

Section 3. A 1-parameter family of functions joining an exotic $z \mapsto z^2 + c + \frac{b}{z-a}$ to Newton's method rational function.

Let $f(z) = z^2 + c + \frac{b}{z-a}$. Then $f'(z) = 2z - \frac{b}{(z-a)^2}$. The equation for the critical points in $\bar{\mathbb{C}}$ is

$$2z(z-a)^2 = b$$

Suppose that $w = c_4$ is an f-fixed critical point, see Fig 5, Section 2. (This restricts the number of parameters to 2.) We obtain

$$(w^2 - w + c)(w - a) = -b$$

$$2w(w-a)^2 = b$$

Let $a = kw$. We parametrize f by k and w. We obtain:

$$a = kw, \quad b = 2w^3(1-k)^2, \quad c = w^2(2k-3) + w$$

The critical points are $u = c_2, v = c_3, w$, where

$$u, v = w(-\frac{1}{2} + k \mp \frac{1}{2}\sqrt{4k-3})$$

Given a parameter $k < 1$ sufficiently close to 1, one finds w such that $f^2(v) = v$ and the trajectory of u escapes to ∞. For $k = .85$ one finds $w \approx 1.88053$. This is an exotic example as in Figure 5, Sec.2. The reason is that the geometry is as in Fig. 6, Sec.2, so the basin B of attraction to ∞ is connected, i.e. the immediate basin is completely invariant. The picture is similar to that in Fig 10d.

(It is not clear to me whether just the escape of u to ∞ proves the connectedness of B. One should be careful because for f being only a topological branched cover this is not so, see the example in Remark 2.3.)

In the rest of this Section we discuss the change of dynamics for varying parameter w.

For $k = .85$ the number $w = 1.88053$ is in the principal part of a Mandelbrot-like set $M(c_3)$, symmetric with respect to real w's, pronged to the left. For $w \in M(c_3)$, Julia set for quadratic-like $f^2|_{D_1}$ (see Fig. 5, Section 2) is connected and we still have exotic maps.

Now let us decrease w. It leaves $M(c_3)$ at $w \approx 1.86874$ and below that w the trajectory of v escapes from D_1. It need not escape to ∞. There is a sequence of intervals where $f^{2n}(v)$ hits B_w the basin of immediate attraction to w, n decreases to 2. Later on, after escape, again $f^4(v) \in B_w$ but $f^2(v) < u$ (before, it was between w and v). This happens at $w \approx 1.63045$. See Fig. 9.

Figure 9. $k = .85, w = 1.63045$.

At some parameter w the trajectory of $u = c_2$ stops to escape to ∞. It hits B_w. But next with further decrease of w it can again escape to ∞.

Starting from $w \approx 1.541549$ the trajectory of u neither escapes to ∞ nor to w. The parameter w is in a Mandelbrot-like set $M(c_2)$ pronged towards right. In fact at this parameter $f^2(u) \in (v, a)$. Only after some further decrease of w we arrive at $f^2(u) \in (u, v)$, so that one has a unimodal map $f : (f(u), v) \to (f(u), v)$.

$w \approx .7136114$ is in the principal part of $M(c_2)$ and f is Newton's. Then $f^n(c_3) \to c_2 = f(c_2)$. The number $w \approx .301$ is still in the principal part of $M(c_2)$ and f is Newton's but now $f^n(c_2) \to c_3 = c_3$.[6]

Let us present now pictures from this experiment for $k = .81$.

On Fig.10, $k = .81$, white is the basin of attraction to w, grey the basin of ∞, black is the complement. For Newton's, Fig 10a, black contains both c_2 and c_3, so it has a connected interior and accesses the only repelling fixed point in two channels. Let w grow. For $w \approx 1.37$ black Newton's basin has bifurcated to period 4, Fig 10b.

For $w \approx 1.4961$, Fig 10c., w is already in $M(c_3)$ but $u = c_2$ does not escape to ∞. It is in the basin of w. The basin of ∞ is not connected. This is so because the immediate basin (and the whole basin too) contains only 1 critical point: ∞. So it is simply-connected, see Prop. 2.5. Hence f has only degree 2 on this immediate basin.

For $w \approx 1.51545$ u escapes to ∞. The basin of ∞ becomes connected. This is one of our exotic examples: see Fig. 10d.

Figure 10a. $k = .81, w = .63$, window $-2-2i, 2+2i$.

Iteration of Newton's method rational map for a polynomial. Black, white and grey are basins of attraction to the zeros of the polynomial.

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[6] This description comes out of a computer made picture in 9 colors, showing whether c_2, c_3 escape to ∞ to w or make something else.

Figure 10b. $k = .81, w = 1.37$, window $-2-2i, 2+2i$.

Black Newton's basin has bifurcated to period 4 immediate basin and its pre-images.

Figure 10c. $k = .81, w = 1.49$, window $-2-2i, 2+2i$.

The map is still not exotic because the trajectory of the critical point u is attracted to w.

Figure 10d. $k = .81, w = 1.51545$, window $-i, 2+i$.

This is an exotic map. The pattern is as in Figures 5,6. The union of black and white does not separate plane anymore, u escapes to ∞.

Question 3.1. In the set of Newton's method rational functions NP_λ for the polynomials $P_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$ there exist Mandelbrot-like sets where the critical point different from the zeros of P_λ converges to a periodic attracting orbit different from these zeros, [CGS]. Do these sets move to $M(c_3)$ sections of the set exotic maps when we change parameters from Newton's to the exotic ones?

Question 3.2. Describe precisely how does the dynamics bifurcate (what is the limit behaviour of the trajectories of c_2 and c_3) for real parameters k, w. This is the question on the iteration of the real map having 2 critical point, namely our f restricted to $(-\infty, a)$. (The right branch from a to ∞ does not take part in the recurrence because for $z > a$ for every $n \geq 0$ $f^n(z) \geq a$.)

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