Polynomial maps with a Julia set of positive Lebesgue measure: Fibonacci maps*

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Abstract

In this paper we shall show that there exists ℓ_0 such that for each even integer $\ell \geq \ell_0$ there exists $c_1 \in \mathbb{R}$ for which the Julia set of $z \mapsto z^\ell + c_1$ has positive Lebesgue measure. This solves an old problem.

Editor's note: In 1997, it was shown by Xavier Buff that there was a serious flaw in the martingale argument (section 7), leaving a gap in the proof. Currently (1999), the question of positive measure Julia sets remains open.

Contents

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1 Introduction and statement of results

Since the work of Julia and Fatou from the 1920's there has been a continuous interest in the dynamics of rational maps. One of the main objects of study is the Julia set. This is the closure of the set of repelling periodic points or – equivalently – the complement of the set of points which have neighbourhoods on which the iterates of the map form a normal family. It was shown back in the 20's that the Julia set of a polynomial map is nowhere dense and that its Julia set is the boundary of the set of points whose iterates do not tend to infinity. Thus it was natural to conjecture that the Julia set of such maps have Lebesgue measure zero. In this paper we shall show that this conjecture is false. Given a map $f: \mathbb{C} \to \mathbb{C}$, let $\omega(z)$ be the set of accumulation points of the sequence $z, f(z), f^2(z), \ldots$.

Main Theorem
For each sufficiently large even integer ℓ there exists $c_1 \in \mathbb{R}$ such that the map $f(z) = z^\ell + c_1$ has the following properties:

- *the set $\omega(0)$ is a Cantor set with zero Lebesgue measure;*
- *the set of points $z \in \mathbb{C}$ for which $\omega(z)$ is contained in $\omega(0)$ has positive Lebesgue measure;*
- *the set of points whose forward iterates remain bounded has no interior.*

In particular, the Julia set of $z \mapsto z^\ell + c_1$ has positive Lebesgue measure. This map has the Fibonacci dynamics (to be defined in the next section).

In other words, the Julia set from our example is 'thin but heavy'. In the picture below we have drawn the Julia set of the unimodal polynomial $z \mapsto z^\ell + c_1$ when $\ell = 16$ and $c_1 = -1.04710851003600355\ldots \in \mathbb{R}$ is chosen so that this map has the Fibonacci map dynamics. To determine c_1, we have used a program of Tangerman which determines, given ℓ, the corresponding coefficient c_1 to any required precision, [Tan]. The Julia set is drawn, using a well-known program of Milnor.

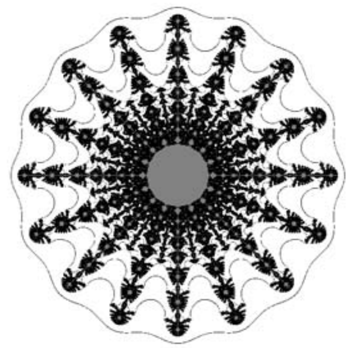

Figure 1: The Julia set of the Fibonacci map $z \mapsto z^\ell + c_1$ where $\ell = 16$ and $c_1 = -1.04710851003600355\ldots$ (with some equipotentials drawn in as well). We do not know whether this value of $\ell = 16$ is large enough for our theorem to hold.

We should point out that – as far as we know – this is the only example of a rational mapping with a 'heavy but thin' Julia set, but that several numerical studies and mathematical results already indicated that some rational maps should have a nowhere dense Julia set with positive Lebesgue measure, see [D] and [Je]. We did not make precise estimates on how large we have to take ℓ but one should think of ℓ as being pretty large.

For entire functions 'thin but heavy' Julia sets where constructed before by McMullen, [McM1], see also Eremenko and Lyubich, [EL]. Shishikura, [Sh1], has shown that there exist (non-real) quadratic maps whose Julia set has Hausdorff dimension two. From this he is able to conclude that the boundary of the Mandelbrot set has also Hausdorff dimension two. We expect that our methods may be helpful in improving this result by showing that the Lebesgue measure of the Julia set of such a map is positive. We also believe that this should imply that either the boundary of the Mandelbrot set has positive Lebesgue measure or that there are queer domains in the Mandelbrot set (i.e., open sets in the parameter plane of conjugate non-hyperbolic maps).

The inspiration for the Main Theorem came from the analogous result on interval mappings which was proved by the authors jointly with Gerhard Keller and Henk Bruin:

Theorem (The real case [BKNS])
For each sufficiently large $\ell \in \mathbb{R}$ there exists $c_1 \in \mathbb{R}$ such that for the map $f(x) = |x|^\ell + c_1$ the set $\{z\ ;\ \omega(z) \subset \omega(c)\}$ is a set of positive Lebesgue measure in $\mathbb{R}$.

The maps we consider have Fibonacci-type dynamics. Such maps were introduced by Hofbauer and Keller in the real context as examples with very slow recurrence, see

[HK]. In the complex setting these maps came up in [BH]. In that paper Branner and Hubbard study cubic maps with one critical point escaping to infinity. They associate a tableau to each map and the Fibonacci map has again the worst possible behaviour w.r.t. the tableau rules. We believe that a related cubic-like map also has a Julia set with positive Lebesgue measure: consider a polynomial map of degree $\ell + 1$ which preserves the real line and with two critical points. One of these critical points has order two and escapes to infinity and the other has degree ℓ. One can take this map so that its kneading sequence is the same as that of the real cubic map Branner and Hubbard considered. Note that the Julia sets of these maps are Cantor sets. A preliminary investigation suggests that for ℓ large enough, these Julia sets have positive Lebesgue measure also.

Let us now give a short outline of the ideas needed for the proof of the Main Theorem. Certain real estimates form one of the main ingredients for the proof of the Main Theorem. Several of these real estimates were proved already in [BKNS], and extend the estimates made in [KN]. They follow from cross-ratio distortion results for interval maps which were developed in the mid 1980's – for an extensive overview of these tools, see the monograph [MS]. We would like to emphasize that some of the estimates in this paper are really much stronger than those from [BKNS]. These cannot be derived by applying Koebe and follow from two ideas. Firstly, some estimates which show that maps, which are not like Moebius transformations, satisfy improved Koebe estimates, see for example Proposition 10.1. Secondly, estimates which show that if one has a converging sequence of maps with an 'almost neutral point' then – up to a map with very small distortion – one can compare their composition with the solution of a particular differential equation, see Theorems 10.1 and 10.2. For an abstract statement of this type of result, see Theorem 10.3. In this way we get an asymptotic expression for some high iterate of f even though this limit is extremely non-linear and one has extremely little Koebe space. (Presumably similar estimates should also work in a more general context.)

The second type of ingredients come from complex analysis: using the Koebe Lemma and the Schwarz Lemma we are able to show that the real estimates imply that some return maps are polynomial-like in the sense defined by Lyubich and Milnor [LM]. This is an extremely important step because with this, and because of the renormalization theory of Sullivan [S2] and of McMullen [McM2], we can improve the real estimates. The idea to apply renormalization theory also in this case is due to Lyubich, see for example [Ly4] and [Ly5]. We should emphasize, that many of the estimates on this paper rely on this idea.

The third ingredient is that of a certain induced map: this induced map is a very natural consequence of the real analogue of Yoccoz's puzzle construction, see Martens [Mar]. This induced map is applied to the so-called Fibonacci map which was 'invented' in the real context by Hofbauer and Keller [HK] and in the complex context by Branner and Hubbard [BH]. (For results on this maps, see [LM], [KN] and [BKNS].) Interestingly, rather than using the Yoccoz partition of the Julia set we found it advantageous in our proof to define a dynamical partition which is based on two circular curves (rather than by curves formed by equipotentials and rays). In fact, this was inspired by a kind of procedure which was originated by Misha Lyubich and was first published in [LM]. Something a little similar is done in [Sw2] and [GJ], see also [GS]. Because of this choice of domains, the proof that we get a polynomial-like map requires some careful real estimates. To get good estimates on the shape of these curves we combine the real bounds (including the improved estimate referred to above) with complex tools such as the Koebe and Schwarz Lemma combined with more delicate estimates which use the renormalization results. In some sense this is the key part of this paper. To get good estimates on the distortion of the induced map we also use results which are better than those following from Koebe, see Section 10. More precisely, we will decompose a high iterate of f into a composition of maps ϕ_i. These maps ϕ_i send some preimage z_i of c into another preimage z_{i-1}. By translating z_i and z_{i-1} to the origin and rescaling small intervals to unit size, we can pretend that such a map has a fixed point which is almost neutral and in fact is close to a map of the type $z \mapsto z - z^3$. Then, using a method which is reminiscent of Ecalle cylinders, see [Sh1], we obtain a very good estimate for the composition of the maps ϕ_i. Putting all this together will give that for n and ℓ sufficiently large, certain iterates f^{S_n-1} of f map a neighbourhood of the critical value c_1 to a neighbourhood of the critical point $c = 0$ approximately as

$$c_1 + z \mapsto \sqrt{M_\ell(z^2)},$$

where M_ℓ is a Moebius transformation which becomes more and more degenerate as ℓ tends to infinity. The precise non-linearity obtained from the composition of the Moebius transformation with the quadratic map will give us our improved Koebe bounds.

The fourth and last ingredient in our proof is that of a probabilistic (random walk) analysis of the behaviour of typical points: this tool uses the language of martingales. In fact, the result we use to apply these ideas was proved by Gerard Keller, is stated in Section 7, and is also one of the essential ingredients in [BKNS]. Many people have though of the idea to use such random walk arguments. For example, Guckenheimer and Johnson, use this terminology in [GJ]. Martens and van Strien discussed this idea extensively in 1989, and this approach became the motivation for the main result in [Mar]. Also, Luybich, Sutherland and Tangerman performed computer experiments several years ago to check the likelihood of our Main Theorem using a random walk on the Yoccoz puzzle, [LST]. The first papers in which this idea was successfully applied were [KN] and [BKNS].

Because this 'random walk' approach is not so usual in this subject, we would like to explain these ideas by stating a simple version of the result we use. Consider $r \in (0, 1)$ and the map $F\colon [0, 1) \to [0, 1)$ which for $n \geq 1$ sends the interval $[r^{n+1}, r^n)$ in an affine way to $[0, r^{n-1})$ and which on $[r, 1)$ is equal to the identity map. Then it is not hard to show that there exists $r_0 > 0$ such that for each $r \in (r_0, 1)$ there exists a set D of positive Lebesgue measure such that $x \in D$ implies that $F^n(x) \to 0$ as $n \to \infty$. This is not surprising: when r_0 is close to one, a point in $[r_n, r_{n+1})$ moves with probability $1-r$ to the right and with probability r to the left. More precisely, the chance to move $i \in \{-1, 0, 1, \ldots\}$ states from a given interval $[r^{n+1}, r^n)$ is equal to $(1 - r)r^{i+1}$ and so the expected drift is

$$\sum_{i\geq -1} i\cdot(1-r)\cdot r^{i+1}$$

which tends to infinity as $r\uparrow 1$. With a random walk argument this implies that points typically move to the states with higher index, i.e., to the origin.

In this paper, we will have a similar random walk model. Here F will be some iterate of $f(z)=z^\ell+c_1$ and the role of the intervals $[r^n,r^{n+1})$ will be replaced by some nested sequence of annuli in the complex plane surrounding $0\in\mathbb{C}$. If we are able to show that there is a set $D\subset\mathbb{C}$ of positive Lebesgue measure such that $z\in D$ implies that $F^n(z)\to 0$ as $n\to\infty$, then it follows that points in D are not in the basin of ∞. Since the map f will be chosen in such a way that it has no periodic attractors, we obtain that $D\subset J(f)$. Hence $J(f)$ has positive Lebesgue measure! As we shall see, however, the random walk model is in this case considerably more subtle then in the previous one-dimensional model. One important difference – which also explains the difficulty to get conclusive evidence from the numerical experiments – is that the probability to go 'further away from zero' is certainly not small. However, as we shall see at the last section of this paper, the 'probability' to move i 'states' closer to 0 (in one step of the induced map F) is roughly of the order $\frac{1}{i^2}e^{-i/\ell}$, where ℓ is the order of the critical point. This implies that the expected drift is equal to

$$\sum_i i\cdot\frac{1}{i^2}e^{-i/\ell}.$$

This sum is of the order $\log(\ell)$ and therefore grows relatively slowly with ℓ. Therefore one might have to take an extremely large ℓ to offset all types of constants and get a proper drift towards 0. We shall elaborate on this issue in the last section of this paper.

One of the main reasons why one is interested in the Lebesgue measure of the Julia set is the Measurable Riemann Mapping Theorem. Indeed, as became apparent through Sullivan's work, one way to solve the well-known stability conjecture that the set of polynomials which are structurally stable form a dense set is through the Measurable Riemann Mapping Theorem. The real quadratic version of this conjecture was solved by Świątek ([Sw2]) using this idea of Sullivan. Świątek shows that any two real quadratic polynomials P and Q which are conjugate are quasi-symmetrically conjugate on the real line. Using Sullivan's pullback method this implies that they are quasiconformally conjugate on the Riemann sphere. By considering the Beltrami coefficient of the conjugacy and by using the Measurable Riemann Mapping Theorem one obtains a path of polynomial maps $[0,1]\ni t\mapsto P_t$ with $P_0=P$ and $P_1=Q$ and where t is defined on a neighbourhood of $[0,1]$ in $\mathbb{C}$. It is easy to show that this is only possible if P is structurally stable.

Now if the Julia set of these quadratic maps would have zero Lebesgue measure, then one could substantially simplify this proof: in this case the conjugacy would not need to be quasi-symmetric on the real line in order to obtain a quasiconformal extension.

More precisely, using the λ-Lemma, see [MSS] and also [McM2][Theorem 4.7], there would be a quasiconformal conjugacy which is conformal outside the Julia set (provided $J(f)$ does not disconnect the plane). This was – of course – one of the motivations for Lyubich and Shishikura's result that non-renormalizable quadratic maps have a Julia set with zero Lebesgue measure, see [Ly2] and [Sh2]. Lyubich's method is based on the combinatorial pattern of the Yoccoz puzzle [Y] and of the moduli of annuli argument of [BH]. This last argument states that the modulus of the preimage by a quadratic map of an annulus (perhaps of higher genus) is either equal to or otherwise half the size of the modulus of the original annulus.

This last method breaks down entirely if we consider polynomials with critical points of higher order. As follows from the next result, in our specific example, other methods can be used to show that the Fibonacci maps from the Main Theorem do not form counter examples to the stability conjecture:

Theorem A *For each even $\ell \geq 4$ one has the following properties.*

- *For each ℓ there exists a unique parameter $c_1 \in \mathbb{R}$ such that $f(z) = z^\ell + c_1$ has Fibonacci dynamics.*
- *There are no measurable invariant linefields on $J(f)$.*
- *There exists a nested sequence of discs D_n centered at the origin and disjoint topological discs D_n^0, D_n^1 which are compactly contained in D_n such that the maps*

$$R_n\colon (D_n^0 \cup D_n^1) \to D_n$$

defined for $z \in D_n^0 \cup D_n^1$ by

$$R_n(z) = \{f^i(z) \ ; \ i > 0 \text{ is minimal with } f^i(z) \in D_n\}$$

converge – up to scaling – as $n \in 4\mathbb{N}$ tends to infinity. The discs D_n are chosen as in Section 6.

Let us clarify the last part of this theorem. Fix $i \in \{1, 2, 3, 4\}$. Then the sequence $R_n\colon (D_n^0 \cup D_n^1) \to D_n$ converges as $n \in 4\mathbb{N} + i$ tends to infinity. To say that such a sequence of maps converges is perhaps unclear because the domains of the maps vary. However, D_n is a Euclidean disc and – as we shall see in Section 6, $R_n|D_n^0 \to D_n$ is a branched covering onto (with a single ℓ fold branching point) and $R_n|D_n^1 \to D_n$ is a diffeomorphism. So if we take Λ_n the scaling map from D_n to the unit disc, the two inverses $\Lambda_n \circ R_n^{-1} \circ \Lambda_n^{-1}$ become maps defined on the unit disc (the inverse of the first map is ℓ-valued). If these two inverses converge then we say that the above sequence of maps R_n converge.

In Theorem A we use the pullback method of Sullivan [S2] and McMullen's results on renormalization, see [McM2]. We would like to thank Misha Lyubich for suggesting

this result to us. The first two statements of Theorem A are standard, see Section 6. We would like to thank Jacek Graczyk for some very useful discussions on the third part of this result.

For simplicity we shall denote the Lebesgue measure of a set A in $\mathbb{R}^n$ by $|A|$. Moreover, if $a, b \in \mathbb{R}$ then (a, b) denotes the interval connecting a and b. For $a \in \mathbb{R}$ we define

$$\hat{a} = -a$$

and $a^{\#}$ will denote either a or $\hat{a}$ depending on the context (for example depending on the parity of some integer n). Finally, given two sequences of positive real numbers u_n and v_n (depending also on some parameter ℓ), we write $u_n \leq C v_n$ if for each sufficiently large ℓ there exists $n_0(\ell)$ so that this inequality holds for $n \geq n_0(\ell)$. The same letter C will be used throughout this paper for several such universal constants.

The authors would like to thank Gerard Keller for allowing us to use his result on random walks from Section 7 which was also one of the essential ingredients in [BKNS]. We also would like to thank Misha Lyubich for some very helpful discussions on the renormalization theory of polynomial-like maps. It is a pleasure to thank Adrien Douady and Jean-Christoff Yoccoz for helpful discussions. During a discussion with Yoccoz a mistake was found in the last part of a previous version of this paper. To fix this, we had to develop the improved Koebe estimates from Section 10. Folkert Tangerman's notes on a method of McMullen's to get renormalization results were very useful. Discussions with Jacek Graczyk on this and other aspects of the paper were very much appreciated. We have included some computer generated pictures of the Julia set and the Yoccoz puzzle. These were made using programs written by Folkert Tangerman, Scott Sutherland and Misha Lyubich.

2 Combinatorial properties

As is well-known, see [HK] or [LM] and also [MS], the Fibonacci map is a non-renormalizable unimodal interval map for which the closure of the forward orbit of the critical point c is a minimal Cantor set $\omega(c)$. In this section we want to construct 'by hand' this Fibonacci map. The main reason for doing this, is that it also gives a nice sequence of induced maps, and a good covering of $\omega(c)$. This covering will be used in Section 5 to show that $\omega(c)$ is a Cantor set of 'bounded geometry', provided ℓ is large.

2.1 Construction of the Fibonacci map

Rather than giving the kneading invariant of the map, or its kneading map, and check that it satisfies some admissibility conditions, we shall construct by hand a topological version of the map. This will be done inductively, and at the same time we shall construct a partition of the interval which is the real analogue of the Yoccoz puzzle, see [Y] and also see [LM] in this context. This partition was also used by Martens [Mar] and Keller and Nowicki [KN]. Something similar was done in [GJ]. A complex extension of the return maps which we construct in this section are crucial in the remainder of this paper.

Let us first introduce some notation. If $I \subset \mathbb{R}$ is a bounded closed interval then we say that $f\colon I \to I$ is a unimodal map if f is continuous, f has a unique extremal point $c \in I$ (which is a minimum) and $f(\partial I) \subset \partial I$. If I is unbounded then we require that $I = \mathbb{R}$ and replace the last condition by $f(x) \to \infty$ as $|x| \to \infty$. Hence for each $x \neq c$ there exists $\hat{x} \neq x$ such that $f(x) = f(\hat{x})$. We define x_n to be $f^n(x)$ and for simplicity we shall often assume that c is equal to 0.

Let us start by taking a unimodal map $f_0\colon I \to I$, and assume that f_0 has an orientation reversing fixed point q and a minimum at $c \in \mathrm{int}(I)$ (and so $c_1 < c$). Assume that $c_2 \in (\hat{q}, 1)$ and $c_3 \in (c, \hat{q})$. We shall modify f_0 in $(q, \hat{q})$ repeatedly to suit our needs.

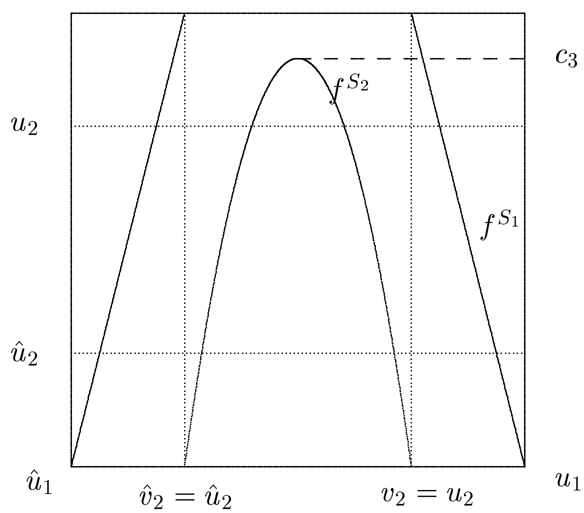

Figure 2: The return map $R_2\colon U_2 \to U_2$. Here $u_0 = q$, $u_1 = \hat{q}$, $S_1 = 2$ and $S_2 = 3$.

Define $S_0 = 1$, $S_1 = 2$ and $S_k = S_{k-1} + S_{k-2}$ for $k \geq 2$. Let us define

$$u_0 = q \text{ and } u_1 = \hat{q}$$

and consider the first return map R_2 of f_0 to

$$U_2 = (q, \hat{q}) = (\hat{u}_1, u_1).$$

Since $c_3 \in U_2$ this first return map consists of three branches: two diffeomorphic ones U_2^1, U_2^2 where the return time is equal to $S_1 = 2$ (these intervals are symmetric) and one which is defined on a 'central' interval U_2^0 containing c on which the map has a fold and on which the return time is equal to $S_2 = 3$.

Now we modify f_0 on the central interval U_2^0 (i.e., we keep f_0 the same outside this interval) and call the new map f_1. We do this so that $R_2(U_2^0)$ strictly contains the closure of U_2^0. The reason this can be done, is because there exists a neighbourhood around c_1 which is disjoint from U_2^0 and which is mapped homeomorphically onto U_2^0 by f^{S_2-1}. Because R_2 is a first return map, this modification does not affect $R_2|(U_2 \setminus U_2^0)$. This implies that $f_1^{S_2}(U_2^0)$ contains one component of $U_2 \setminus U_2^0$ and that $f_1^{S_2}(c)$ is contained in the other component (which we will call U_2^1). We should emphasize that we have complete freedom where inside U_2^1 to choose $f_1^{S_2}(c)$. Now we let

$$U_2^1 = (u_2, x_2) \text{ and } U_2^0 = (\hat{v}_2, v_2)$$

where we make the choices so that u_2 is the endpoint of U_2^1 which is closer to c and so that u_2 and v_2 are equal. Note that $R_2(c) \in U_2^1$ and that $R_2(u_2) = u_1$.

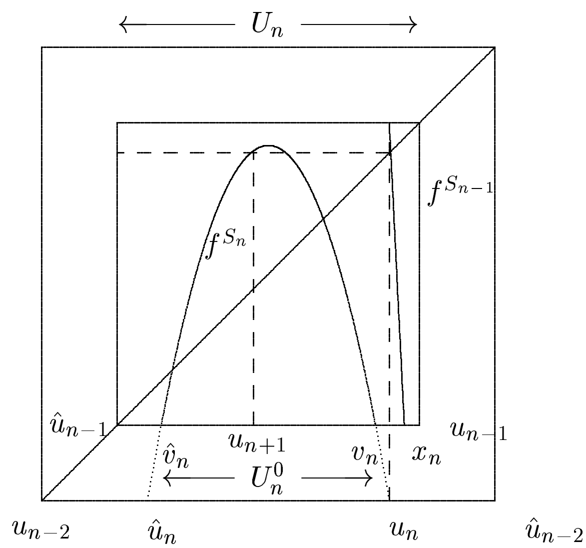

Figure 3: R_n on the central branch $U_n^0 = (v_n, \hat{v}_n)$ can be extended to $U_{n+1} = (u_n, \hat{u}_n)$ as shown. R_n is a surjection from $U_n^1 = (v_n, x_n)$ onto U_n and can also be extended to a monotone map onto U_{n-1}. Moreover, u_{n+1} is defined to be the point in U_n^0 for which $f^{S_n}(u_{n+1}) = u_n$ and which is on the same side of c as $c_{S_{n+1}}$.

We continue with the construction inductively. So assume that f_n, $U_n = (u_{n-1}, \hat{u}_{n-1})$, $U_n^1 = (x_n, u_n)$, $U_n^0 = (v_n, \hat{v}_n)$, $R_n\colon (U_n^0 \cup U_n^1) \to U_n$ are already constructed for $n < N$. Here we label these points so that u_n and v_n are on the same side of c and so that $u_n \in (x_n, c)$. Also assume that

a) $R_n\colon U_n^1 \to U_n$ is a homeomorphism, $R_n\colon U_n^0 \to U_n$ has one extremum and $R_n(\partial U_n^0) \subset \partial U_n$;

b) $R_n(U_n^0) \supset U_n^0$;

c) Moreover, $R_n(c)$ is contained in the other set U_n^1 and we have complete freedom where to place $R_n(c)$ in U_n^1 by changing f_n inside U_n^0; but, we shall choose it so that

d) $R_n \circ R_n(c) \in U_n^0$.

In Figure 3 the graph of $R_n: U_n \to U_n$ is drawn over the intervals U_n^0 and U_n^1 (and the extension of the central interval is also depicted). Now we can proceed the construction inductively as follows by taking $U_N = (u_{N-1}, \hat{u}_{N-1})$ and letting R_N to be the first return map to U_N. Then define $U_N^{1,2} \subset U_{N-1}^0$ to be the two intervals on which R_N coincides with R_{N-1} and which are mapped diffeomorphically by R_N onto U_N. The map R_N has a 'unimodal' branch over a central interval U_N^0 (containing c), and we modify f_{N-1} in such a way that $R_N(U_N^0) \supset U_N^0$. This proves properties a) and b) for $n = N$. Of course, b) implies that $R_N(U_N^0)$ contains one of the sets U_N^i; so let us call this set U_N^2. By the inductive hypothesis c) we can even modify f_{N-1} on U_N^0 such that $R_N(c)$ is contained in the other set U_N^1 and we have complete freedom where to place $R_N(c)$ in U_N^1. However, we shall choose it so that $R_N \circ R_N(c) \in U_N^0$. This proves statements c) and d) for $n = N$. Let us call the modified function f_N and write

$$U_N^1 = (x_N, u_N) \text{ and } U_N^0 = (v_N, \hat{v}_N),$$

where u_N and v_N lie on the same side of c and $u_N \in (x_N, c)$. We have $U_N^1 \cup U_N^0 \subset U_{N-1}^0 \subset U_N$.

In this way we get sequences of points, intervals return maps and modified functions f_n. Without loss of generality, we may assume that $|U_n| \to 0$ as $n \to \infty$, and as f_n will only be modified on U_k for $k \geq n$, there exists a unimodal limit function f. In this way we have shown how to construct a topological version of a *Fibonacci interval map.*

Of course, this construction merely gives a continuous map. However, using a general fullness result from the theory of interval maps, in any reasonable family of unimodal maps one can find maps with the same combinatorial properties:

Lemma 2.1 *Consider a family of C^1 unimodal maps $g_t: [-1,1] \to [-1,1]$ such that g_0 has no periodic points of period > 1 and g_1 is surjective. Then there exists a parameter t' such that $g = g_{t'}$ is a Fibonacci map.*

In particular, there exists for any $\ell \in 2\mathbb{N}$ a Fibonacci map in the family type $z \mapsto z^\ell + t$, $t \in \mathbb{R}$.

Proof: This follows immediately from the fullness of such families, see for example Section II.4 in [MS]. Indeed, this fullness result implies that there exists such a parameter t' such that $g_{t'}$ has the same kneading invariant as the Fibonacci map constructed above. $\square$

2.2 Topological properties of the Fibonacci map

Let $\{S_k\}$ be the Fibonacci numbers, i.e. $S_0 = 1$, $S_1 = 2$ and $S_k = S_{k-1} + S_{k-2}$. We prove the following properties of a Fibonacci map. Define $U_n = (u_{n-1}, \hat{u}_{n-1})$, $U_n^0 = (v_n, \hat{v}_n)$, $U_n^1 = (u_n, x_n)$ and $R_n: U_n^0, U_n^1 \to U_n$ be the two branches of the return map as above. In the next lemma we show how these return maps are related to f and what the orbit of the intervals U_n^i look like.

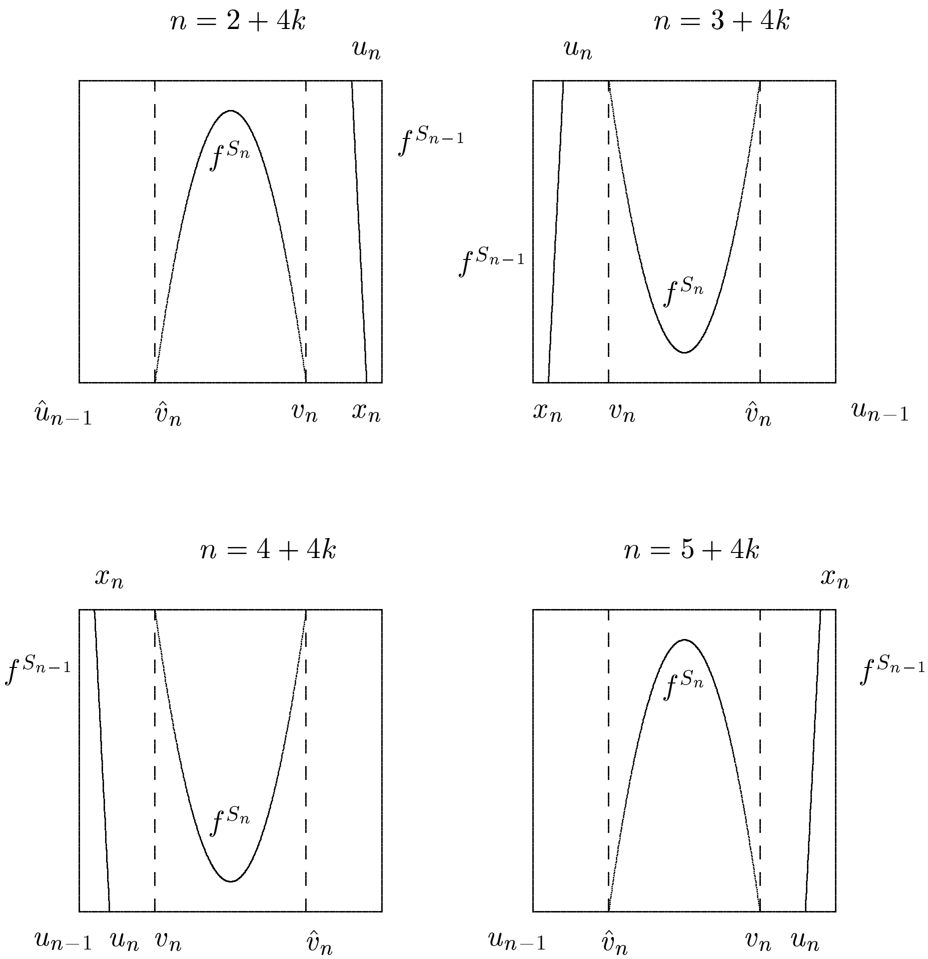

Figure 4: The successive first return maps $R_n: U_n \to U_n$ for $n \geq 2$ (i.e., for $k \geq 0$; note however that $v_2 = u_2$).

Lemma 2.2 *Let f be a Fibonacci map and take $n \in \mathbb{N}$. Then one has the following properties.*

1. *$R_n|U_n^0$ coincides with $f^{S_n}|U_n^0$, and $R_n|U_n^{1,2}$ coincides with $f^{S_{n-1}}|U_n^{1,2}$.*
2. *$R_n(u_n) = f^{S_{n-1}}(u_n) = u_{n-1}$.*
3. *$c_{S_n} \in (c_{S_{n-1}}, \hat{c}_{S_{n-1}})$ and $c_k \notin (c_{S_{n-1}}, \hat{c}_{S_{n-1}})$ for $0 < k < S_n$.*
4. *Let c_{-k} and $\hat{c}_{-k}$ be the points in $f^{-k}(c)$ which are closest to c, then $c_{-S_n} \in (c_{-S_{n-1}}, \hat{c}_{-S_{n-1}})$ and $c_{-k} \notin (c_{-S_{n-1}}, \hat{c}_{-S_{n-1}})$ for $0 < k < S_n$.*
5. *$f^{S_{n-1}}(U_n^0) \subset U_{n-1}^1$.*
6. *For every $n \geq k \geq 2$, $f^i(U_n^0) \cap U_k \subset U_k^0 \cup U_k^1$ for each $i = 0, 1, \ldots, S_n$ and $f^i(U_n^1) \cap U_k \subset U_k^0 \cup U_k^1$ for $i = 0, 1, \ldots, S_{n-1}$, except if $n = k$ in which case $f^{S_n}(U_n^0) \subset U_n$ $f^{S_{n-1}}(U_n^1) \subset U_n$.*
7. *$\omega(c) \cap U_n \subset U_n^0 \cup U_n^1$ for every $n \geq 2$.*

Proof: By the above construction these properties hold for $n = 2$. Since, by property b), $R_n|U_n^{1,2} = R_{n-1}|U_{n-1}^0$ and $R_n|U_n^0 = R_{n-1}|U_{n-1}^1 \circ R_{n-1}|U_{n-1}^0$, the first statement follows immediately from induction. Statement 2) follows from the choice of u_n and surjectivity of $R_n|U_n^1$. Since R_n is a first return map, u_n is a preimage of q and since $R_n|U_n^0 = f^{S_n}|U_n^0$, it follows that c_{S_n} are the successive closest returns to c. So let us prove 4). The intervals $U_n^{1,2}$ contain precritical points in $f^{-S_{n-1}}(c)$, because $R_n|U_n^{1,2} = f^{S_{n-1}}|U_n^{1,2}$ is surjection onto U_n. Both branches are part of the central branch of the first return map R_{n-1}. As $R_{n-1}(u_{n+1}, \hat{u}_{n+1})$ does not contain c, the interval $U_n = (u_{n+1}, \hat{u}_{n+1})$ contains no point in $\cup_{i=1}^{S_{n-1}} f^{-i}(c)$. So $c_{-S_{n-1}} \cap U_n^{1,2}$ is a closest precritical point. This proves 4).

In order to prove 5), observe that since $U_n^0 \subset U_{n-1}^0$, property c) implies $f^{S_{n-1}}(U_n^0) \cap U_{n-1}^1 \neq \emptyset$. Therefore $f^{S_{n-1}}(U_n^0) \subset U_{n-1}^1$ since $f^{S_{n-2}}$ maps a neighbourhood of U_{n-1}^1 homeomorphically onto a neighbourhood of U_{n-1} and so if $f^{S_{n-1}}(U_n^0)$ is not completely contained in U_{n-1}^1 then $f^{S_n}(U_n^0) = f^{S_{n-2}} \circ f^{S_{n-1}}(U_n^0) \not\subset U_{n-1}$, contradicting that $f^{S_n}|U_n^0$ is a branch of the first return map to U_n. This proves 5). Let us now prove statement 6) for $2 \leq k \leq n < N$ by induction on N. For $N = 3$ this statement is obvious. Assume statement 6) holds for $2 \leq k \leq n < N$ and let us show it also holds for $2 \leq k \leq n \leq N$. Because $U_N^0 \subset U_{N-1}^0$, the induction assumption implies $f^i(U_N^0) \cap U_k \subset U_k^0 \cup U_k^1$ for $0 \leq i \leq S_{N-1}$. However $f^{S_{N-1}}(U_N^0) \subset U_{N-1}^1$, hence $f^i(U_N^0) \cap U_k \subset f^{i-S_{N-1}}(U_{N-1}^1) \subset U_k^0 \cup U_k^1$ for $S_{N-1} < i < S_N$ (because $S_N - S_{N-1} = S_{N-2}$ and using the the second part of the induction hypothesis fot the last inclustion). The other part of statement 6) is proved similarly.

Since U_n^0 and U_n^1 are contained in the interior of U_{n-1}^0, statement 7) follows immediately. $\square$

Now we will discuss the ordering of some crucial dynamically defined points. Firstly, we let $T_n \ni c_1$ be the maximal interval for which $F^{S_n-1}|T_n$ is a diffeomorphism and define

$$y_n = f^{S_n}(c_{S_{n+2}}) \ , \quad y_n^f = f(y_n).$$

Also define w_n^f, r_n^f to be the points in T_n to the left of c_1 so that

$$f^{S_n-1}(w_n^f) = \hat{u}_{n-1} \text{ and } f^{S_n-1}(r_n^f) = \hat{u}_{n-2}$$

(note that w_n^f is not the image of a point $w_n \in [-1,1]$ so the notation is only to suggest that w_n^f lies near c_1). As before, let x_n^f be the point in the interval T_{n-1} for which $f^{S_{n-1}-1}(x_n^f) = \hat{u}_{n-1}$. For simplicity we also write

$$d_n = c_{S_n} \text{ and } d_n^f = f(d_n).$$

Moreover, we shall write z_n for one of the two points in $f^{-S_n}(c)$ closest to c.

Proposition 2.1 *The points u_n^f, d_n^f, x_n^f, y_n^f, w_n^f and z_n^f are ordered as in the picture below (we state the ordering near c_1 rather than near c so that we do not need to be careful about on which side of c these points lie).*

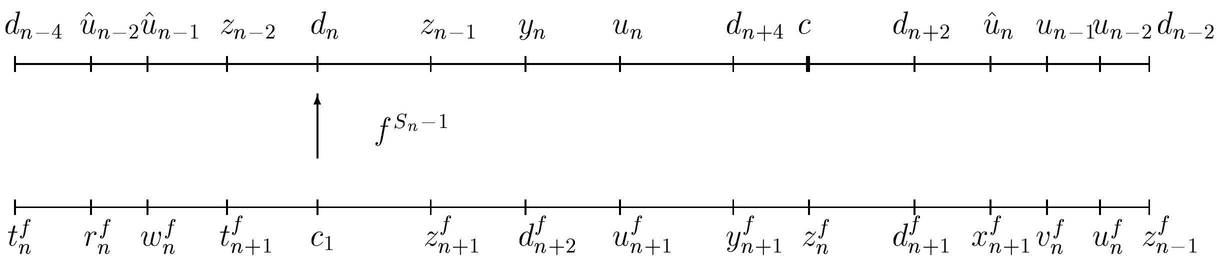

Figure 5: Points and their images under f^{S_n-1}. Note that c_1 is the mimumum of $f\colon \mathbb{R} \to \mathbb{R}$. The points u_n, w_n, x_n are in the full orbit of the fixed point u_0 whereas $d_n = f^{S_n}(c)$ and $y_n = f^{S_n}(d_{n+2})$ are forward iterates of c. The point z_n is a point in $f^{-S_n}(c)$ nearest to c. We should note that the position of u_{n-1} and $\hat{u}_{n-1}$ should be interchanged for n even (in that case $f^{S_n}(v_n) = \hat{u}_{n-1}$).

Proof: The proof of these statements can be found in [KN]. □

Next we shall show that the set $\omega(c)$ of accumulation points $c, f(c), f^2(c), \ldots$ is a minimal Cantor set. This means that each point $f^k(c)$ is the limit of some sequence $f^{n(k)}(c)$ with $n(k) \to \infty$. Moreover, we shall show that this Cantor set can be covered in a very natural way. In the next section this covering shall be used to show that this Cantor set has 'bounded geometry' provided the critical point of f has order $\ell > 2$.

Lemma 2.3 *The union*

$$\bigcup_{i=0}^{S_n-1} f^i(U_n^0) \cup \bigcup_{i=0}^{S_{n-1}-1} f^i(U_n^1) \tag{2.1}$$

is a cover of $\omega(c)$ with mutually disjoint intervals (the closures of the intervals are disjoint if $n \geq 3$). Moreover, $\omega(c)$ is a minimal Cantor set.

Proof: For $n = 2$ statement (2.1) is easily verified. Because of 1) in the previous lemma, $f^{S_n}(U_n^0) \subset U_n$ and $f^{S_{n-1}}(U_n^1) \subset U_n$. But due to 7) and since $\omega(c)$ is forward invariant, $f^{S_n}(U_n^0 \cap \omega(c))$ and $f^{S_{n-1}}(U_n^1 \cap \omega(c))$ are both contained in $U_n \cap \omega(c) \subset U_n^0 \cup U_n^1$. This proves the covering property. To show that the covering consists of disjoint intervals, mark that $f^i(U_n) \cap U_{n-1} = \emptyset$ for $0 < i < S_{n-1}$. This is easily verified by similar arguments as in the previous lemma. In fact, $f^{S_{n-2}}(U_n)$ is adjacent to U_{n-1}, and $f^{S_{n-1}}(U_n) = (d_{n-1}, u_{n-1}) \supset U_{n-1}$. It follows that

$$U_n, f(U_n), \ldots, f^{S_{n-1}-1}(U_n)$$

are mutually disjoint. The interval U_n^0 is symmetric, so $f(U_n^0) \cap f(U_n^1) = \emptyset$. Hence

$$U_n^0, f(U_n^0), \ldots, f^{S_{n-1}-1}(U_n^0) \text{ and } U_n^1, f(U_n^1), \ldots, f^{S_{n-1}-1}(U_n^1)$$

are all mutually disjoint. $f^{S_{n-1}}(U_n^0) \subset U_{n-1}^1$ and using induction, $f^{S_{n-1}+i}(U_n^0) \subset f^i(U_{n-1}^1)$ is disjoint from $f^i(U_n^0) \supset f^i(U_n^0 \cup U_n^1)$ for $0 \leq i < S_{n-2}$. This proves that also the intervals $f^i(U_n^0)$, $S_{n-1} \leq i < S_n$, are mutually disjoint and disjoint from the other intervals.

The fact that $\omega(c)$ is covered by this union implies that $orb(x) \cap U_n \neq \emptyset$ for every $x \in \omega(c)$ and every $n \geq 2$. So $\omega(c)$ is a minimal Cantor set: for each $x \in \omega(c)$ one has $\omega(x) \ni c$. □

Next we define a sequence of nested sets F_n, each consisting of 2^{n-1} intervals, which generates a Cantor set $\cap_n F_n$ such that $\cap_n F_n \supset \omega(c)$. Let W be the interval containing c_1 such that f maps W diffeomorphically onto U_2. Take

$$F_2^1 = \{U_2^0, U_2^1\}$$

and

$$F_2 = F_2^1 \vee ((f|W)^{-1}(F_2^1)).$$

By the previous lemma F_2^1 is a covering of $\omega(c) \cap U_2$ and since $\omega(c) \subset U_2 \cup W$ this implies F_2 is a covering of $\omega(c)$. F_3 is defined in three steps:

$$F_3^1 = \{U_3^0, U_3^1\},$$

$$F_3^2 = F_3^1 \vee (f^{S_1}|U_2^1)^{-1}(F_2)$$

and

$$F_3 = F_3^2 \vee ((f|W)^{-1}(F_3^2)).$$

In general, we define $F_n = F_n^{n-1} \vee ((f|W)^{-1}(F_n^{n-1}))$ where

$$\begin{array}{rcl} F_n^1 & = & \{U_n^0, U_n^1\}, \\ F_n^2 & = & F_n^1 \vee (f^{S_{n-2}}|U_{n-1}^1)^{-1}(F_n^1), \\ .. & = & .. \\ F_n^i & = & F_n^{i-1} \vee (f^{S_{n-i}}|U_{n-i+1}^1)^{-1}(F_n^{i-1}), \\ .. & = & .. \\ F_n^{n-1} & = & F_n^{n-2} \vee (f^{S_1}|U_2^1)^{-1}(F_n^{n-2}). \end{array}$$

Clearly all intervals in F_n are disjoint.

Lemma 2.4 *$F_n \supset \omega(c)$ for every n. Moreover, F_n consists of 2^n components and each component of F_n contains exactly two components of F_{n+1}.*

Proof: Because $\omega(c) \cap U_n \subset U_n^0 \cup U_n^1$, in order to prove that F_n^2 is a covering of $\omega(c)$ it suffices to prove that

$$x \in \omega(c) \cap U_{n-1} \text{ implies } f^{S_{n-2}}(x) \in U_n. \tag{2.2}$$

In fact, (2.2) also implies inductively that F_n^i covers $\omega(c) \cap U_{n-i+1}$ (by replacing in (2.2) n by $n-i$ it follows that F_n^i covers U_{n-i+1} if the previous collection F_n^{i-1} already covers U_{n-i+2}.) To prove (2.2), note that Lemma 2.2 implies that $f^{S_{n-1}}(U_n^0)$ is the first return to U_{n-1}, and $f^{S_{n-1}}(U_n^0) \subset U_{n-1}^1$. Similarly, $f^{S_{n-2}}(U_{n-1}^1)$ is the first return of U_{n-1}^1 to U_{n-1}. In particular, $f^{S_{n-2}}(f^{S_{n-1}}(U_n^0))$ is the first return of $f^{S_{n-1}}(U_n^0)$ to U_{n-1}. But

$$f^{S_{n-2}}(f^{S_{n-1}}(U_n^0)) = f^{S_n}(U_n^0) \subset U_n,$$

because $R_n|U_n^0 = f^{S_n}|U_n^0$. For the same reason $f^{S_{n-1}}(U_n^1)$ is the first return of U_n^1 to U_{n-1}, and $f^{S_{n-1}}(U_n^1) \subset U_n$. It follows from (2.1) that $x \in U_{n-1}^1 \cap \omega(c)$ implies $x = f^{S_{n-1}}(y)$ for some $y \in U_n^0 \cap \omega(c)$, and therefore that $f^{S_{n-2}}(x) = f^{S_n}(y) \subset U_n$. □

3 Real bounds for smooth Fibonacci maps

In this section we shall state and prove some results on the metric properties of a smooth Fibonacci map. First we shall quickly state the main tool needed for these estimates.

3.1 The cross-ratio tool and the Koebe Principle

Let $j \subset t$ be intervals and let l, r be the components of $t \setminus j$. Then the cross-ratio of this pair of intervals is defined as

$$C(t,j) := \frac{|t|}{|l|}\frac{|j|}{|r|}.$$

Let f be a smooth function mapping t, l, j, r onto T, L, J, R diffeomorphically. Define

$$B(f,t,j) = \frac{|T|\,|J|}{|t|\,|j|}\,\frac{|l|\,|r|}{|L|\,|R|} = \frac{C(T,J)}{C(t,j)}.$$

It is well known that if the *Schwarzian derivative* of f, i.e., $Sf = f'''/f' - 3(f''/f')^2/2$, is negative then $B(f,t,j) \geq 1$. It is easy to check that our map $f(z) = z^\ell + c_1$ satisfies $Sf(x) < 0$ for $x \in \mathbb{R}$.

We say that a set $t \subset \mathbb{R}^k$ contains a τ-*scaled* neighbourhood of a disc $j \subset \mathbb{R}^k$ with midpoint x and radius r if t contains the ball around x with radius $(1+\tau)r$.

Proposition 3.1 (Real Koebe Principle) *Let $Sf < 0$. Then for any intervals $j \subset t$ and any n for which $f^n|t$ is a diffeomorphism one has the following. If $f^n(t)$ contains a τ-scaled neighbourhood of $f^n(j)$ then*

$$\frac{|Df^n(x)|}{|Df^n(y)|} \leq \left[\frac{1+\tau}{\tau}\right]^2 \tag{3.1}$$

for each $x, y \in j$. Moreover, there exists a universal function $K(\tau) > 0$ which does not depend on f, n and t such that

$$|l|, |r| \geq K(\tau) \cdot |j|.$$

3.2 The bounds

Bounds on the relative position of the points u_n and $d_n = c_{S_n}$ are essential in this paper. They are given in the following theorem. (All the results in this section also hold if f is a C^2 Fibonacci map using the disjointness statements as in [BKNS].)

Theorem 3.1 (The real bounds) *There exists $\ell_0 \geq 4$ such that if f is a real unimodal Fibonacci map with a critical point of order $\ell \geq \ell_0$ with $Sf < 0$ then one there exist universal constants $0 < \lambda < \mu \in (0,1)$ such that the ratio between two consecutive terms*

$$|d_{n+1}^f - c_1| < |u_n^f - c_1| < |z_{n-1}^f - c_1| < |d_n^f - c_1|$$

is between λ and μ for all n sufficiently large. In fact, all the distances in the bottom part of Figure 3.2 are of the same order. From this it follows that the distances near c as stated in the caption of this figure. Moreover,

$$\frac{|d_{n-2}^f - c_1|}{|d_n^f - c_1|} \geq 3.85$$

and therefore

$$\frac{|d_{n-4}^f - c_1|}{|d_n^f - c_1|} \geq 14$$

for all n sufficiently large.

Proof: The last two inequalities can be found in [KN] and also in Lemma 3.3 in [BKNS]. In Theorem 3.1 of [BKNS] it is shown that

$$\frac{|d_n^f - c_1|}{|u_n^f - c_1|}, \frac{|d_n^f - c_1|}{|d_{n+1}^f - c_1|} \text{ and } \frac{|u_n^f - c_1|}{|u_{n+1}^f - c_1|}$$

are bounded and bounded away from one. Hence there exists uniform constants C_1, C_2 such that

$$\frac{C_1}{\ell} \leq \frac{|d_n - u_n|}{|u_n - c|}, \frac{|d_n - c| - |d_{n+1} - c|}{|d_n - c|}, \frac{|u_n - c| - |u_{n+1} - c|}{|u_n - c|} \leq \frac{C_2}{\ell}$$

for all n large. From this, by considering the map drawn in Figure 3.2 and by the Koebe Principle one obtains that all distances are comparable is size. For example, these inequalities imply that $[d_{n-2}, c]$ is a uniformly scaled neighbourhood of $[u_{n-2}, d_{n+2}]$ and by Koebe it follows that $[z^f_{n-1}, z^f_n]$ is also a scaled neighbourhood of $[u^f_n, d^f_{n+1}]$. Hence

$$\frac{|z^f_{n-1} - c_1|}{|u^f_n - c_1|} \text{ and } \frac{|d^f_{n+1} - c_1|}{|z^f_n - c_1|}.$$

are both bounded away from one. Continuing in this way the proposition follows. $\square$

In fact, we should remark that the last theorem holds for $\ell_0 = 4$. We shall not need this however, and since the necessary real bounds are only proved in [BKNS] for ℓ_0 sufficiently large we only claim the existence of such an integer ℓ_0.

We should point out that the previous theorem is false if $\ell = 2$. In that case, $|u^f_n - c_1|/|u^f_{n+1} - c_1|$ goes exponentially fast to infinity, see [LM] and [KN].

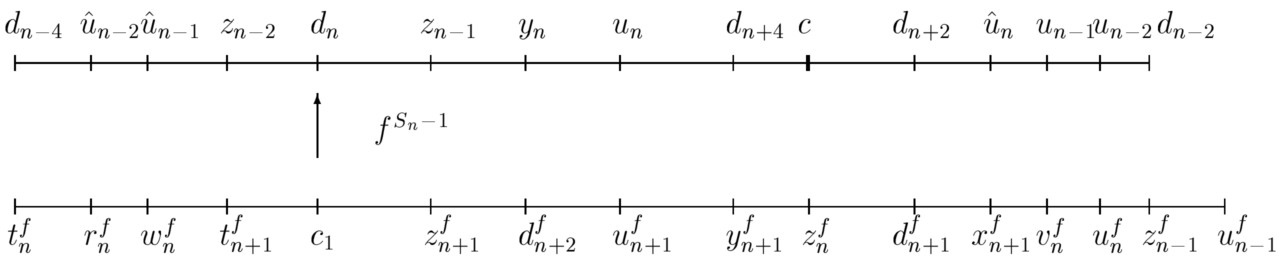

Figure 6: In the top figure the actual scaling is completely different for large ℓ: $|d_{n+2} - c|/|d_{n+4} - c|$ is of order $1 - C\frac{1}{\ell}$ whereas the mutual distance of all points in the top figure on one component of $\mathbb{R} \setminus \{c\}$ is of order $(C/\ell)|d_{n+2} - c|$. All the distances between the marked points in the bottom figure (which shows the situation near c_1) are of the same order.

Let $T_n = (z^f_{n-1}, t^f_{n-1})$ be the maximal interval containing c_1 on which f^{S_n-1} is a diffeomorphism and let $w^f_n \in T_n$ be so that $f^{S_n}(w^f_n) = u^f_{n-1}$. Then we have the following estimate, see also Figure 10. This estimate will be needed in Section 6.

Proposition 3.2 (Bounds near c_1) *There exists $\ell_0 \geq 4$ such that if f is a real unimodal Fibonacci map with a critical point of order $\ell \geq \ell_0$ and $Sf < 0$ then*

$$\frac{|u^f_{n-1} - c_1|}{|w^f_n - c_1|} \geq \frac{4}{3}$$

for all n sufficiently large.

Proof: To prove this proposition we use the following lemma.

Lemma 3.1 *Let $J' \subset J \subset T$ be intervals on which f is a diffeomorphism and assume that $Sf < 0$. Then*

$$B(f,T,J) \geq B(f,T,J'). \tag{3.2}$$

Furthermore, if $f(x) = x^\ell$, $T = [0,\gamma]$ and $J = [\alpha,\beta] \subset T$ then

$$B(f,T,J) \geq \ell(1 - \frac{\alpha}{\gamma}).$$

Proof: We may assume that one boundary of J' coincides with one boundary of J (by applying the lemma twice in this situation we get the lemma also for general intervals J). Let L' and R' be the components of $T \setminus J'$ which are labeled so that R' and R both lie on the right hand side of J and J'. In order to be definite, assume that the left endpoints of J' and J coincide. This means that $L' = L$. It follows that (3.2) is equivalent to

$$\frac{|f(J)||f(R')|}{|f(R)||f(J)|} \geq \frac{|J||R'|}{|R||J|}.$$

If we define $\hat{T} = J \cup R$, $\hat{L} = J'$, $\hat{J} = J \setminus J'$ and $\hat{R} = R'$ then this last inequality becomes

$$\frac{|f(\hat{L} \cup \hat{J})||f(\hat{J} \cup \hat{R})|}{|f(\hat{L})||f(\hat{R})|} \geq \frac{|\hat{L} \cup \hat{J}||\hat{J} \cup \hat{R}|}{|\hat{L}||\hat{R}|}$$

which is equivalent to the usual cross-ratio expansion:

$$\frac{|f(\hat{J})||f(\hat{T})|}{|f(\hat{L})||f(\hat{R})|} \geq \frac{|\hat{J}||\hat{T}|}{|\hat{L}||\hat{R}|}.$$

This completes the proof of the first part of the lemma.

It follows from the first part that we may assume that $\beta = \alpha$. Since $f(x) = x^\ell$,

$$B(f,(0,\gamma),\{\alpha\}) = \frac{\gamma^\ell}{\gamma} \cdot \ell\alpha^{\ell-1} \cdot \frac{\alpha}{\alpha^\ell} \cdot \frac{\gamma - \alpha}{\gamma^\ell - \alpha^\ell} = \ell(1 - \frac{\alpha}{\gamma}) \cdot \frac{\gamma^\ell}{\gamma^\ell - \alpha^\ell} \geq \ell(1 - \frac{\alpha}{\gamma}).$$

This completes the proof of this lemma. $\square$

Proof of Proposition 3.2: Now we can prove the previous proposition.

$$\begin{aligned}
&B\left(f^{S_n}, (t_n^f, z_n^f), (c_1, w_n^f)\right) \\
&\quad = B\left(f^{S_n - 1}, (t_n^f, z_n^f), (c_1, w_n^f)\right) \cdot B\left(f, (d_{n-4}, c), (d_n, \hat{u}_{n-1})\right) \\
&\quad \geq 1 \cdot \ell(1 - (\frac{|d_n^f - c_1|}{|d_{n-4}^f - c_1|})^{1/\ell}) \geq \ell(1 - (\frac{1}{14})^{1/\ell}) \geq 4(1 - (\frac{1}{14})^{1/4}) > 1.9
\end{aligned}$$

where we have used the previous lemma, the inequality from Theorem 3.1 and $\ell \geq 4$. Now $f^{S_n}(t_n) = d_{n-4}$, $f^{S_n}(z_n) = c_1$, $f^{S_n}(c_1) = d_n^f$, $f^{S_n}(w_n^f) = u_{n-1}^f$. Rewriting this last inequality and using the order structure of the points on the real line, gives

$$\begin{aligned}\frac{|u^f_{n-1}-c_1|}{|w^f_n-c_1|} &\geq 1.9\cdot\frac{|d^f_{n-4}-u^f_{n-1}|}{|d^f_{n-4}-c_1|}\cdot\frac{|u^f_{n-1}-c_1|}{|u^f_{n-1}-d^f_n|}\cdot\frac{|d^f_n-c_1|}{|z^f_n-c_1|}\cdot\frac{|t^f_n-z^f_n|}{|t^f_n-w^f_n|}\\ &\geq 1.9\cdot\frac{|d^f_{n-4}-d^f_{n-2}|}{|d^f_{n-4}-c_1|}\cdot 1\cdot 1\cdot 1\geq 1.9\cdot\left(1-\frac{1}{3.85}\right)\geq\frac{4}{3}.\end{aligned}$$

□

The next bounds require that we already know the map satisfies some renormalization properties, and is used in Section 8 to prove that certain discs really lie nested.

Proposition 3.3 (Improved bounds near c_1 if renormalization holds) *If $\ell\geq\ell_0$, f is as above and*

$$\lim_{n\to\infty}\frac{|d_n-c|/|d_{n-2}-c|}{|d_{n-2}-c|/|d_{n-4}-c|}\to 1, \tag{3.3}$$

then we have the following property. If $z^f_{n-1}<l^f_n<c_1<s^f_n<t^f_n$ are so that

$$|d_n-c|<|f^{S_n-1}(l^f_n)-c|=|f^{S_n-1}(s^f_n)-c|$$

then

$$\liminf_{n\to\infty}\frac{|l^f_n-c_1|}{|s^f_n-c_1|}\geq 1$$

Moreover, (3.3) implies that if we take $l^f_n=u^f_n$ and $r^f_n\in(c_1,t^f_n)\subset T_n$ so that $f^{S_n}(r^f_n)=\hat{u}_{n-2}$, then $|f^{S_n}(r^f_n)-c|=|u_{n-2}-c|=|f^{S_n-1}(u^f_n)-c|$ and

$$\liminf_{n\to\infty}\frac{|u^f_n-c_1|}{|r^f_n-c_1|}>1.$$

Proof: Consider f^{S_n-1} on $t=(l^f_n,t^f_n)$ and let $j=(c_1,s^f_n)$, $l=(l^f_n,c_1)$ and $r=(s^f_n,t^f_n)$.

Write $a=|f^{S_n-1}(l^f_n)-c|=|f^{S_n-1}(s^f_n)-c|$. Then $|T|=|d_{n-4}-c|+a$, $|L|=a$, $|J|=a$ and $|R|=|d_{n-4}-c|-a$. Using the cross-ratio inequality gives

$$\begin{aligned}\frac{|l^f_n-c_1|}{|r^f_n-c_1|} &= \frac{|j|}{|l|}\geq\frac{|L|}{|T|}\frac{|R|}{|J|}=\left(\frac{|d_n-c|+a}{|d_{n-4}-c|+a}\right)\left(\frac{|d_{n-4}-c|-a}{a-|d_n-c|}\right)\\ &\geq \left(\frac{|d_n-c|+|d_{n-2}-c|}{|d_{n-4}-c|+|d_{n-2}-c|}\right)\left(\frac{|d_{n-4}-c|-|d_{n-2}-c|}{|d_{n-2}-c|-|d_n-c|}\right)\to 1 \text{ as } n\to\infty.\end{aligned}$$

Here we have used that the fourth expression is decreasing in $a\in(0,|d_{n-2}-c|)$ and in the last limit that (3.3) holds. To prove the last assertion of the proposition, note that because of Proposition 3.1, $\limsup_{n\to\infty}\frac{|u_{n-2}-c|}{|d_{n-2}-c|}<1$. Hence in the second inequality above one has in fact a gain by a factor which is uniformly strictly larger than one. □

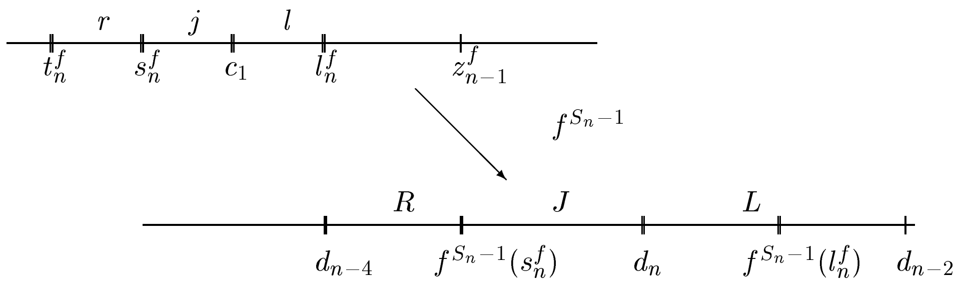

Figure 7: The proof of Proposition 3.3

4 Background in complex analysis and hyperbolic geometry

4.1 Applications of the Schwarz Lemma

First we shall review some results from hyperbolic geometry. Define

$$\mathbb{C}_J = (\mathbb{C} \setminus \mathbb{R}) \cup J$$

where $J \subset \mathbb{R}$ is an interval. This set is the complex plane slitted in two infinite rays on the real line. It is easy to show $\mathbb{C}_J$ is conformally equivalent to the upperhalf plane and that

$$D_k(J) = \{z;\ \text{the hyperbolic distance to } J \text{ is at most } k\}$$

consists of the intersection with the upper and lower half plane of two Euclidean discs which are symmetric to each other with respect to the real line and whose boundaries intersect the boundary points of J, see [MS, pages 485-486]. Moreover, k is determined by the external angle α at which the discs intersect the real line. We also denote this set by

$$D(J;\alpha).$$

For later use, we define $D_*(J)$ to be the disc symmetric w.r.t. the real-line and which intersects the real line in ∂J with angle $\pi/2$.

Lemma 4.1 (Schwarz Lemma) *Let $I, J \subset \mathbb{R}$ be two intervals. If $G\colon \mathbb{C}_J \to \mathbb{C}_I$ is a univalent map which maps I diffeomorphically onto J then $G(D_*(J)) \subset D_*(I)$.*

In particular, let $F\colon \mathbb{C} \to \mathbb{C}$ be a real polynomial map whose critical points are on the real line and which maps I diffeomorphically onto J then there exists a set $D \subset D_*(I)$ with $D \cap \mathbb{R} = I$ which is mapped diffeomorphically by F onto $D_*(J)$.

Proof: The first statement follows immediately from the Lemma of Schwarz, which states that any univalent map between hyperbolic Riemann surfaces strictly contracts the Poincaré metric.

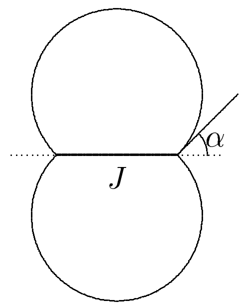

Figure 8: A Poincaré neighbourhood of J in $\mathbb{C}_J$.

Since F is a real polynomial and F has no critical values in $\mathbb{C}_J$, the inverse $G = F^{-1}: \mathbb{C}_J \to \mathbb{C}_I$ is a well defined univalent map. So let D be the inverse of $D_*(J)$ under G and apply the first part of this lemma. □

4.2 The Koebe Lemma

As before, we say that a set $t \subset \mathbb{R}^k$ contains a τ-*scaled* neighbourhood of a disc $j \subset \mathbb{R}^k$ with midpoint x and radius r if t contains the ball around x with radius $(1+\tau)r$. (Here we take the standard metric on $\mathbb{R}^k$.) Then we get the following classical annalogue of the real Koebe Principle:

Lemma 4.2 (Koebe Lemma) *Suppose that $D' \subset \mathbb{C}$ contains a τ-scaled neighbourhood of the disc $D \subset \mathbb{C}$. Then for any univalent function $f: D' \to \mathbb{C}$ one has*

$$\frac{|f'(x)|}{|f'(y)|} \leq \left[\frac{1+\tau}{\tau}\right]^2 \text{ for all } z, y \in D.$$

Proof: This result is well known and can be found in for example [Ahl1] and [Ahl2] or in [Bieb]. □

When J is a real interval then take $D(J;\alpha)$ as in the beginning of this section.

Proposition 4.1 *Assume that $D \subset D'$ and $f: D' \to \mathbb{C}$ are as in the previous lemma and assume that f maps the real line to the real line. For each $\alpha \in (\pi/2, \pi)$ there exists $\alpha' \in (\alpha, \pi)$ such that if J is a real interval in D then*

$$f(D(J;\alpha)) \supset D(f(J);\alpha').$$

(Note that $D(J;\alpha)$ is convex since $\alpha \in (\pi/2, \pi)$.)

Proof: Follows quite easily from the Koebe Lemma. □

5 Quasisymmetric rigidity on the real line

As a preparation for the proof of Theorem A we shall prove in this section the following theorem. (This theorem also holds for C^2 maps if we use the disjointess and distortion results of [BKNS].)

Theorem 5.1 *There exists an integer $\ell_0 \geq 4$ with the following property. Let f be a real unimodal Fibonacci map with $Sf < 0$ and with a critical point of order $\ell \geq \ell_0$. Then $\omega(c)$ has bounded geometry (for the definition see below). Moreover, there exists $K < \infty$ and n_0 such that for each $n, m \geq n_0$ with $n - m \in 2\mathbb{Z}$, there exists a quasiconformal homeomorphism $h: \mathbb{C} \to \mathbb{C}$ which conjugates the first return map of $R_n: \omega(c) \cap U_n \to \omega(c) \cap U_n$ to the first return map of $R_m: \omega(c) \cap U_m \to \omega(c) \cap U_m$. This map h is symmetric w.r.t. the real line and h preserves the orientation on the real line iff $n - m \in 4\mathbb{Z}$.*

Again, using the statement below Theorem 3.1, the above theorem also holds for $\ell_0 = 4$. We shall prove this result by constructing a suitable covering of $\omega(c) \cap U_n$.

Firstly, we define a *presentation* of a Cantor set C to be a decreasing collection $F_n \supset F_{n+1}$ of closed sets such that

- each F_n is a finite union of closed intervals whose boundary points are in C;
- each connected component of F_n contains the same number a_n of connected components of F_{n+1} and
- $\cap_{n=0}^{\infty} F_n = C$.

Each component of F_n is called an *interval* of generation n and each component of $F_n \setminus F_{n+1}$ is called a *gap* of generation $n+1$. Of course, there are many presentation of a Cantor set.

We say that the presentation $\{F_n; n = 0, 1, 2, \ldots\}$ of C has *bounded geometry* by $\mu \in (0,1)$ such that for any interval or gap I of generation n and any interval or gap $J \subset I$ of generation $n+1$,

$$0 < (1 - \mu) < \frac{|J|}{|I|} < \mu < 1.$$

It follows from the above definition that if the presentation $\{F_n; n = 0, 1, \ldots\}$ of C has bounded geometry then it has *bounded combinatorics*, namely, the number a_n of components of F_{n+1} in each component of F_n is bounded independently of n.

We need the following result.

Lemma 5.1 *For each $\mu \in (0,1)$ there exists $K < \infty$ with the following properties. Let $\{F_n^{(j)}; n = 0,1,2,\ldots\}$ be presentations with geometry bounded by $\mu < 1$ of the Cantor sets $C^{(i)} \subset \mathbb{R}$, $j = 1,2$. Suppose that these presentations have the same combinatorics, i.e., the number of components of $F_{n+1}^{(j)}$ in each component of $F_n^{(j)}$ does not depend on j. Then there exists a K-quasiconformal homeomorphism $h\colon \mathbb{C} \to \mathbb{C}$ which is symmetric w.r.t. the real line and maps $F_n^{(1)}$ onto $F_n^{(2)}$ (and therefore $C^{(1)}$ onto $C^{(2)}$.*

Proof: See [MS][Section VI.3]. □

Proof of Theorem 5.1: From the real bounds in Theorem 3.1 it follows that the size of the intervals U_k^0, U_k^1 and also of the components of $U_{k-1} \setminus (U_k^0 \cup U_k^1)$ are the same up to a multiplicative constant. Now take $0 \le i < k$. Since the map $f^{S_{k-i}}\colon U_{k-i+1}^1 \to U_{k-i}$ extends to a diffeomorphism onto (d_{k-i-2}, d_{k-i-4}); since (again by Theorem 3.1) (d_{k-i-2}, d_{k-i-4}) contains a uniformly scaled neighbourhood of U_{k-i}, the Koebe Principle implies that the map $f^{S_{k-i}}\colon U_{k-i+1}^1 \to U_{k-1}$ has uniformly bounded distortion. Hence the size of the components of F_k and of $F_{k-1} \setminus F_k$ are all of the same order as the component of F_{k-1} which contains them. Finally, each component of F_k contains exactly two components of F_{k-1} and $F_{i+k} \cap U_k$ consists of 2^i components.

Therefore and because of the previous lemma, it follows that there exists a quasiconformal homeomorphism h which is symmetric w.r.t. the real line which for each $i \ge 0$ sends the j-th component of $F_{i+n} \cap U_n$ (say from the left) to the j-th component of $F_{i+m} \cap U_m$ (from the left). If $n - m \in 4\mathbb{Z}$ then $R_n\colon U_n^0 \cup U_n^1 \to U_n$ is conjugate to $R_n\colon U_m^0 \cup U_m^1 \to U_m$ in an orientation preserving way and so this conjugacy also sends the j-th component of F_{i+n} to the j-th component of F_{i+m}. It follows that the homeomorphism h is a conjuagacy from $U_n \cap \omega(c)$ to $U_m \cap \omega(c)$. If $n - m \in 2\mathbb{Z} \setminus 4\mathbb{Z}$ then $R_n\colon U_n^0 \cup U_n^1 \to U_n$ and $R_n\colon U_m^0 \cup U_m^1 \to U_m$ are still conjugate but the orientation is reversed; so this conjugacy sends the j-th component of F_{i+n} to the $(2^i - j)$-th component of F_{i+m}. However, we can also impose that the homeomorphism h from the above lemma reverses orientation (because $a_n = 2$) and then with this choice h again becomes a conjugacy from $U_n \cap \omega(c)$ to $U_m \cap \omega(c)$. □

6 Quasiconformal rigidity of the return maps; renormalization and the proof of Theorem A

In this section we want to prove Theorem A. Moreover, we shall prove a renormalization result: up to rescaling the return maps $R_n\colon U_n^0 \cup U_n^1 \to U_n$ has at most four limits. This last property will be needed in the proof of the Main Theorem. In fact, it is needed in order to apply Proposition 3.3. As we mentioned before, we believe that the proof of the Main Theorem should be independent of this renormalization result (and of

Theorem A); but so far we have not been able to prove an analogue of Proposition 3.3 by real methods which is sufficient for our purposes.

Let $f: [0,1] \to [0,1]$ be a unimodal Fibonacci map. Let $U_n = [u_{n-1}, \hat{u}_{n-1}]$, $U_n^0 = [v_n, \hat{v}_n]$ and $U_n^1 = [u_n, x_n]$ as before. Moreover, let $R_n: U_n^0 \cup U_n^1 \to U_n$ be the restriction of the first return map. In this section we want to show that these first return maps converge.

Theorem 6.1 (Renormalization result for Fibonacci maps) *There exists an integer $\ell_0 \geq 4$ with the following property. Fix $\ell \in 2\mathbb{N}$ with $\ell \geq \ell_0$. Let $f: [0,1] \to [0,1]$ be a polynomial unimodal Fibonacci map with critical point c of order ℓ, let $R_n: U_n^0 \cup U_n^1 \to U_n$ be as above and let $\phi_n: U_n \to [0,1]$ be the affine rescaling map. Then for each $i \in \{0,1,2,3\}$, the sequence $\{\phi \circ R_{i+4k} \circ \phi^{-1}\}_{k \geq 0}$ converges in the C^1 topology.*

Again, using the statement below Theorem 3.1, the above theorem also holds for $\ell_0 = 4$. Moreover, using the distortion results of [BKNS] and a shuffling lemma as in Lemma 6.4 of [LM] the above results also holds for C^2 maps. We should note that in [LM] a similar result is proved for the case that $\ell = 2$ but then $|u_n - c|/|u_{n-1} - c|$ and therefore $|U_n^0|/|U_n|$ tends to zero. It follows that $|U_{n-1}|/|U_n| \to \infty$ and that the renormalization map $R|U^0 \to U_n$ tends to a unimodal quadratic map. In our case (when $\ell > 2$) the situation is more subtle, but on the other hand in our case of the Cantor set $\omega(c)$ has bounded geometry (as was shown in the previous section). This bounded geometry (which does not hold if $\ell = 2$) will help us also a great deal (compare this section with [Ly5]).

As we shall also explain in this section, the above renormalization result is related to Sullivan's [S2] and McMullen's [McM2] result. These results will also imply

Theorem A There exists $\ell_0 \geq 4$ with the following property. *For each even $\ell \geq \ell_0$ one has the following properties.*

- *For each ℓ there exists a unique parameter $c_1 \in \mathbb{R}$ such that $F(z) = z^\ell + c_1$ has Fibonacci dynamics.*
- *There are no measurable invariant linefields on $J(F)$.*
- *There exists a sequence of discs D_n and relatively compact topological discs D_n^0, D_n^1 in D_n defined in Proposition 6.2 below, such that the maps*

$$R_n: (D_n^0 \cup D_n^1) \to D_n$$

defined for $z \in D_n^0 \cup D_n^1$ by

$$R_n(z) = \{f^k(z) \ ; \ k > 0 \text{ is minimal with } f^k(z) \in D_n\}$$

converge – up to scaling – as $n \in 4\mathbb{N} + i$ tends to infinity, where $i \in \{0,1,2,3\}$.

Remarks

1. In fact, the limits for $i = 0$ and $i = 2$ in Theorem 6.1 are the same up to orientation and, similarly, the limits for $i = 1$ and $i = 3$ are also equal up to orientation. Similarly, in the last statement of Theorem A.

2. We do not make any claims about the rate of convergence in Theorem 6.1.

3. For the proofs of Theorem 6.1 it would be sufficient to assume that f is C^2. In this case, we proceed as in [LM, Lemma 6.4] or as in [MS, Theorem VI.2.3] and show that any limit of C^2 maps is an Epstein map. We shall not discuss this here.

The proof of Theorem 6.1 is very similar to the proof of D. Sullivan of the convergence of the renormalizations of Feigenbaum and more general infinitely renormalizable maps, see [S1] and [S2]. We shall refer to the exposition of these results given in [MS]. We should note that McMullen has given an alternative proof of a substantial part of Sullivan's results, see [McM2]. We shall use McMullen's approach to this result, in order not to have to develop Sullivan's theory of Riemann surface laminations for Fibonacci-like maps. The difference between Sullivan's case (of renormalizable maps) and ours (of maps which are not renormalizable in the classical sense) is that in the renormalizable case the return maps have connected Julia sets whereas in our case the relevant return maps have Julia sets which are totally disconnected.

To start with the proof of Theorem 6.1 we first state

Proposition 6.1 *There exists $\ell_0 \geq 4$ with the following property. Let $f\colon [0,1] \to [0,1]$ be a C^2 unimodal Fibonacci map with critical point of order $\ell \geq \ell_0$. Let $R_n\colon U_n \to U_n$ be the sequence of return maps and $\phi_n\colon U_n \to [0,1]$ be the affine rescaling maps. Then the closure of $\{\phi \circ R_n \circ \phi^{-1}\}_{n\geq 0}$ forms a compact family in the C^1 topology.*

Proof: This follows from the following considerations.

1. From the bounds in the previous section, the relative size of U_n^0, U_n^1 and of the components of $U_n \setminus U_n^0, U_n^1$ as subsets of U_n are bounded from above and below. (Note that the bounds are only claimed to be uniform in n for each fixed ℓ).

2. The diffeomorphism $R_n|U_n^1\colon U_n^1 \to U_n$ can be extended to a diffeomorphism onto $(d_{n-3}, d_{n-5}) \supset (d_{n-2}, d_{n-4})$; moreover, (d_{n-2}, d_{n-4}) contains a $\tau(\ell)$-scaled neighbourhood of U_n; in particular, by the Koebe Principle the distortion of $R_n|U_n^1$ is uniformly bounded and because of 1) the derivative of $R_n|U_n^1$ is bounded from above and below.

3. the unimodal map $R_n|U_n^0: U_n^0 \to U_n$ can be written as a composition of f and a map from a neighbourhood of $f(U_n^0)$ onto (d_{n-2}, d_{n-4}), Therefore, R_n^0 is a composition of $f: U_n^0 \to f(U_n^0)$ and a map whose derivative is bounded from above and below.

All this together implies the proposition. □

Now we will show that the maps $R_n: U_n^0 \cup U_n^1 \to U_n$ have a polynomial-like extension. This notion is due to Douady and Hubbard [DH], see also [MS], which was extended to be suitable for the present situation in [LM]. We shall not give the general definition, but just that of the case we will need.

Definition. Let D^0, D^1, D be topological discs bounded by smooth curves and such that the closures D^i are disjoint and contained in the interior of D. Then

$$R: (D^0 \cup D^1) \to D$$

is *ℓ-polynomial-like* if $R|D^1$ is a univalent map onto D and if $R|D^0: D^0 \to D$ is a ℓ-fold covering map, i.e., $R|D^0 \to D$ is surjective and the composition of a map of the type $z \mapsto z^\ell$ (up to translation) and a conformal map onto I. The map R is called *unbranched* (using the terminology of McMullen [McM2]) if all R-iterates of the critical point of $R|D^0$ are contained in $D^0 \cup D^1$.

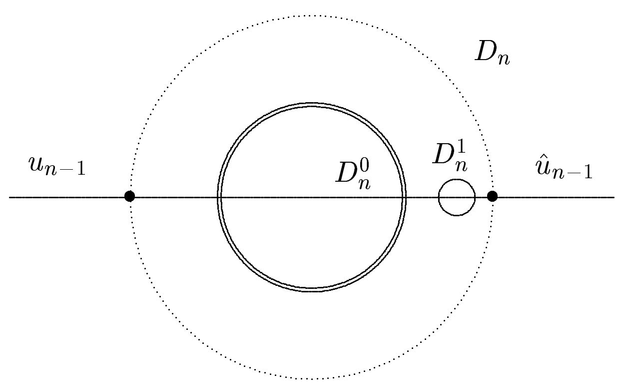

Figure 9: The complex extension of the first return map $R_n: U_n^0 \cup U_n^1 \to U_n$ is a polynomial-like map $R_n: D_n^0 \cup D_n^1 \to D_n$ if $\ell = 4$. The solid disc D_n^0 is mapped in a ℓ-fold way by f^{S_n} onto the disc D_n with dotted boundary. The smaller disc D_n^1 is mapped by $f^{S_{n-1}}$ univalently onto D_n. Furthermore, $D_n \cap \mathbb{R} = U_n^0 = (u_{n-1}, \hat{u}_{n-1})$, $D_n^0 \cap \mathbb{R} = U_n^0 = (v_n, \hat{v}_n)$ and $D_n^1 \cap \mathbb{R} = U_n^1 = (x_n, u_n)$. The domain of the first return map to D_n^0 has infinitely many components but all iterates of the critical point under the return map are contained in $U_n^0 \cup U_n^1 \subset D_n^0 \cup D_n^1$.

Proposition 6.2 *There exists $\ell_0 \geq 4$ with the following property. Let f be a polynomial Fibonacci map with a critical point of even order $\ell \geq \ell_0$ and let $R_n: U_n^0 \cup U_n^1 \to U_n$*

be the corresponding return maps for $n \geq 2$. Let R_n also denote the complex extension of this return map to the disc $D_n = D_*(U_n)$. Then R_n is polynomial-like for n sufficiently large: there exists topological discs $D_n^i \subset D_n$ which are symmetric w.r.t. the real line, with $D_n^i \cap \mathbb{R} = U_n^i$ such that there exists a complex extension

$$R_n \colon (D_n^0 \cup D_n^1) \to D_n$$

of $R_n \colon U_n^0 \cup U_n^1 \to U_n$ which is ℓ-polynomial-like and unbranched. Moreover, the modulus of the disc $D_n \setminus (D_n^0 \cup D_n^1)$ with two holes is bounded from above and below: for each $\ell \geq \ell_0$ there exist universal constants $C_1(\ell), C_2(\ell)$ such that for all n sufficiently large,

$$C_1 \, diam(D_n) \leq dist(\partial D_n, \partial D_n^0), dist(\partial D_n, \partial D_n^1), dist(\partial D_n^0, \partial D_n^1), \leq C_2 \, diam(D_n).$$

Moreover, $R_n | D_n^0 = f^{S_n}$, $R_n | D_n^1 = f^{S_{n-1}}$ and the distortion of

$$f^{S_n - 1} | D_n^0 \text{ and of } f^{S_{n-1}} | D_n^1$$

is uniformly bounded. So the boundary of D_n^i is smooth and its shape is not too far from 'round'.

Proof: Let W_n be the interval containing c_1 such that $f^{S_n - 1} \colon W_n \to U_n$ is a diffeomorphism. Since R_n is a polynomial, the inverse of $f^{S_n - 1} \colon W_n \to U_n$ extends to an analytic univalent map

$$f^{-(S_n - 1)} \colon \mathbb{C}_{U_n} \to \mathbb{C}_{W_n} \ni c_1.$$

By the Lemma of Schwarz this univalent map contracts the Poincaré metrics on these spaces, and therefore

$$D_n^{0,f} := f^{-(S_n - 1)}(D_*(U_n)) \subset D_*(W_n) \ni c_1.$$

Similarly, let $W_n' = f(U_n^1)$; this means that $f^{S_{n-1} - 1} \colon W_n' \to U_n$ is also a diffeomorphism. By the same argument, the inverse of this map has a univalent holomorphic extension and therefore

$$D_n^{1,f} := f^{-(S_{n-1} - 1)}(D_*(U_n)) \subset D_*(W_n'). \tag{6.1}$$

So the inverses of the ball D_n are both inside Euclidean balls $D_*(W_n)$ and $D_*(W_n')$. Now we have that $W_n = (v_n^f, w_n^f) \ni c_1$ and $W_n' = (x_n^f, u_n^f)$. Moreover, by Theorem 3.1 these intervals and the gap between them are of the same order. Furthermore, it was shown in Proposition 3.2, that

$$|w_n^f - c_1| \leq \frac{3}{4} |u_{n-1}^f - c_1| \tag{6.2}$$

for all n sufficiently large provided ℓ_0 is sufficiently large. Since $f(D_*(U_n))$ is equal to a Euclidean disc centered at c_1 and with radius $|c_1 - u_{n-1}^f|$ it follows that the closures of both $D_*(W_n)$ and $D_*(W_n')$ are contained in the interior of $f(D_*(U_n))$. In fact, the modulus of the difference set – a disc with two discs taken out – is bounded and bounded away from zero (in fact, these bounds can be taken to be independent of n and ℓ because of the real bounds from Theorem 3.1). Now take

$$D_n^0 := f^{-1}(D_n^{0,f}) = R_n^{-1}(D_*(U_n)) \subset D_*(U_n^0)$$

and let D_n^1 be the component of

$$f^{-1}(D_n^{1,f}) = R_n^{-1}(D_*(U_n)) \subset D_*(U_n^1)$$

which contains (u_n, x_n). So R_n maps D_n^1 univalently onto D_n and D_n^0 as an ℓ-cover onto D_n. Then the modulus of the set

$$D_n \setminus (D_n^0 \cup D_n^1)$$

is bounded from below and above (in the sense mentioned above). Since the inverses $f^{-(S_n-1)}$ and $f^{-(S_{n-1}-1)}$ even extend univalently to $\mathbb{C}_{(d_{n-2},\hat{d}_{n-2})}$ it follows that the maps $f^{S_n-1}|f(D_n^0)$ and $f^{S_{n-1}-1}|f(D_n^1)$ have uniformly bounded distortion as in the previous proposition. Since $f\colon D_n^1 \to f(D_n^1)$ has bounded distortion (since $|u_n^f - c_1|/|x_n^f - c_1|$ is bounded) this implies the last sentence of the proposition. $\square$

Now we want to show that all the return maps associated to a polynomial Fibonacci map are quasiconformally conjugate. For this we shall use the pullback argument of Sullivan, see [S2] and also Chapter VI of [MS].

Theorem 6.2 *There exists $\ell_0 \geq 4$ such that for any unimodal polynomial Fibonacci map f with a critical point of order $\ell \geq \ell_0$, there exists a constant $K(\ell) < \infty$ with the following properties. Assume that $R_n\colon D_n^0 \cup D_n^1 \to D_n$ are the polynomial-like mappings from the previous proposition. Then for any n, m larger than some sufficiently large $n_0(\ell)$, these maps R_n, R_m are K-quasiconformally conjugate.*

Proof: For simplicity, let us denote R_n, D_n, D_n^0, D_n^1 by R, D, D^0, D^1 and similarly R_m, D_m, D_m^0, D_m^1 by $\tilde{R}$, $\tilde{D}$, $\tilde{D}^0$, $\tilde{D}^1$. First a warning that we should be careful. Indeed, as we will show below the filled Julia set of $R\colon D^0 \cup D^1 \to D$,

$$K(R) = \{z; R^i(z) \in D^0 \cup D^1 \text{ for all } i \geq 0\}$$

has positive Lebesgue measure. (In fact, the filled Julia set is equal to the Julia set because the critical point of the map is recurrent.) In particular, the moduli of the set $A_N = D_n \setminus K_N(R_n)$, where $K_N(R)$ is the filled Julia set, i.e.,

$$K_N(R) = \{z; R^i(z) \in D^0 \cup D^1 \text{ for } i = 0, 1, 2 \ldots, N\}$$

will not tend to infinity. So we cannot use the method of [BH] or rather that of Kahn, see also [Ly2], to extend the quasiconformal conjugacy across $K(R)$.

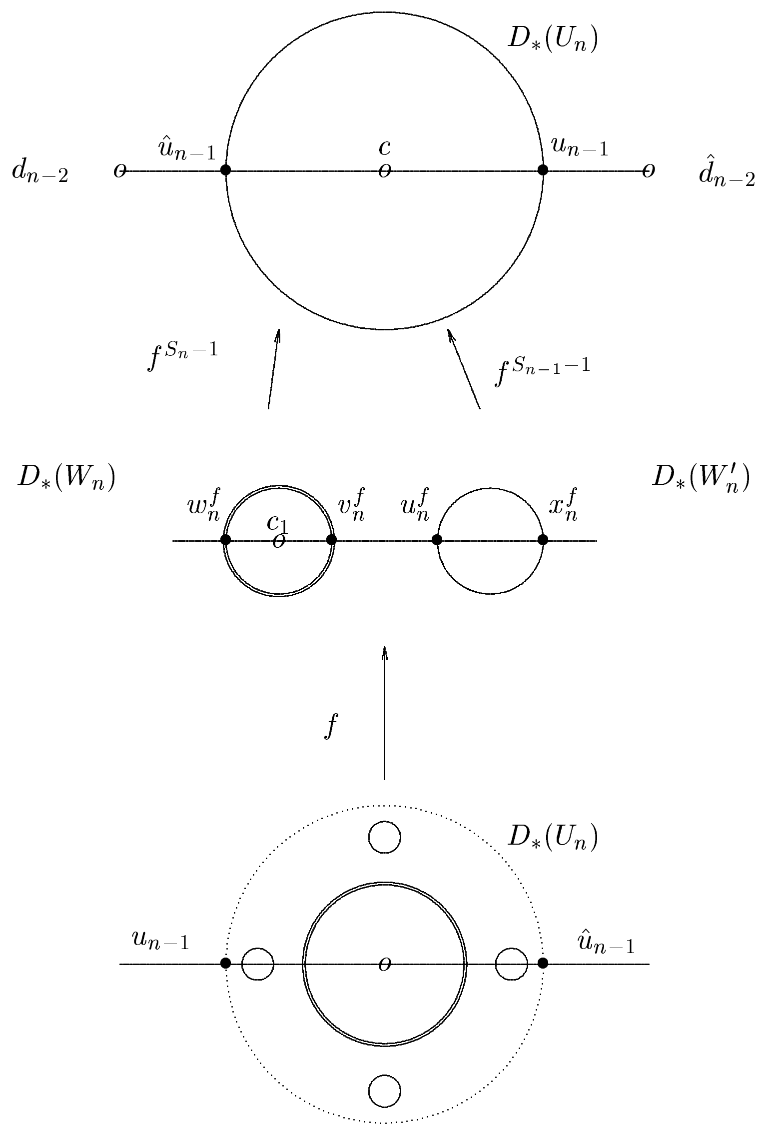

Figure 10: The polynomial-like map if $\ell = 4$. The preimages of $D_*(U_n)$ under f^{S_n-1} and $f^{S_{n-1}-1}$ are contained in the discs $D_*(W_n)$ and $D_*(W_n')$. To see that the points pull-back on the real line are as shown, we refer to Figure 5 (and the corresponding figure if we replace there n by $n-1$). In the bottom picture the fat circle represents the component of the preimage of $D_*(U_n)$ under f^{S_n} containing c. The inverse of $D_*(W_n')$ consists of ℓ (topological) discs. Even though the f-inverse of $D_*(W_n)$ need not be convex, it is contained in the disc $D_*(U_n)$ because $|w_n^f - c_1| \leq \frac{3}{4}|u_{n-1}^f - c_1|$ by Proposition 3.2.

So instead we shall use the pullback argument from [S1] following the exposition in [MS]. The idea of this, is that the quasiconformal conjugacy between $R_n\colon U_n \cap \omega(c) \to U_n \cap \omega(c)$ to $R_m\colon U_m \cap \omega(c) \to U_m \cap \omega(c)$ from the previous section can be pulled back (stepwise for each j) to a quasiconformal conjugacies between $R_n\colon U_n \cap f^{-j}(\omega(c)) \to U_n \cap f^{-j}(\omega(c))$ to $R_m\colon U_m \cap f^{-j}(\omega(c)) \to U_m \cap f^{-j}(\omega(c))$. Indeed, from Theorem 5.1, provided $\ell \in 2\mathbb{N}$ is at least ℓ_0 the set $\omega(c) \cap U_n$ has a geometry which is bounded uniformly for $n \geq n_0$. In particular, there exists by this theorem for each $\ell \geq \ell_0$ a uniform constant $K < \infty$ such that for n, m larger than some sufficiently larger n_0 there exists a K-quasiconformal homeomorphism $h\colon \mathbb{C} \to \mathbb{C}$ which conjugates $R = R_n\colon \omega(c) \cap D_n \to \omega(c) \cap D_n$ to $\tilde{R} = R_m\colon \omega(c) \cap D_m \to \omega(c) \cap D_m$ and which is symmetric w.r.t. the real line provided $n - m$ is even. If $n - m \in 2\mathbb{Z} \setminus 4\mathbb{Z}$ then this homeomorphism reverses the orientation on the real line. Now let us change h to a map h_0 so that

- it maps ∂D, ∂D^0, ∂D^0 diffeomorphically to respectively $\partial \tilde{D}$, $\partial \tilde{D}^0$, $\partial \tilde{D}^1$;
- such that it conjugates $R\colon (\partial D^0 \cup \partial D^1) \to \partial D$ to $R\colon (\partial \tilde{D}^0 \cup \partial \tilde{D}^1) \to \partial \tilde{D}$;
- h_0 is symmetric: $h_0(\overline{z}) = \overline{h_0(z)}$.

This means that h is a conjugacy between the *fundamental domains* $D \setminus (D^0 \cup D^1)$ and $\tilde{D} \setminus (\tilde{D}^0 \cup \tilde{D}^1))$. Since the boundaries of these sets are smooth curves, we can choose h such that it is K'-quasiconformal. The number K' is finite but can be much larger than K (depending on the shape of the fundamental domains). In fact, because of the last sentence in Proposition 6.2, the shape of the boundaries of D_n^i is 'bounded' and therefore the number K' can be chosen independently of n and m provided they are sufficiently large.

Now we can define inductively a sequence of K'-quasiconformal h_i such that

$$\tilde{R} \circ h_{i+1} = h_i \circ R, \tag{6.3}$$

$$h_{i+1} = h_i \text{ on } \{x; R_n^j(x) \notin D^0 \cup D^1 \text{ for some } j = 0, 1, \ldots, i\},$$

$$h_i \text{ is symmetric w.r.t. the real axis,}$$

$$h_i \text{ conjugates } R \text{ and } \tilde{R} \text{ along the critical orbits.}$$

So assume by induction that we have already constructed h_i. Since h_i maps the critical value v of R to the critical value $\tilde{v}$ of $\tilde{R}$, there exists a unique lift of h_i to a map $h_{i+1}\colon \mathbb{C} \to \mathbb{C}$ (i.e., such that (6.3) holds) which maps D^i onto $\tilde{D}^i$, which is symmetric w.r.t. the real axis and for which $h_{i+1}|\mathbb{R}$ has the same orientation as $h_i|\mathbb{R}$. Since h_i is quasiconformal and the other maps are conformal it follows that h_{i+1} is quasiconformal with the same conformal distortion as h_i. On the other hand, since h_i coincides with h_0 on $D \setminus (D^0 \cup D^1)$ and h_0 conjugates R with $\tilde{R}$ on the boundaries of $D^0 \cup D^1$ we see that h_{i+1} coincides with h_0 on the boundary of $D^0 \cup D^1$. Hence it can be extended continuously to D by setting it equal to h_0 on $D \setminus (D^0 \cup D^1)$. This extension is quasiconformal and has the same quasiconformal distortion as h_i because the boundary of $D^0 \cup D^1$ is smooth (hence has zero Lebesgue measure). Now we claim that h_{i+1} is a conjugacy from the critical orbit of R to the critical orbit of $\tilde{R}$. This follows because, by induction, h_i has this property and because R has the same combinatorial type as $\tilde{R}$. This last statement holds because $n-m \in 2\mathbb{Z}$ and therefore the $R = R_n, \tilde{R} = R_m$ are conjugate where the conjugacy is orientation preserving precisely if $n-m \in 4\mathbb{Z}$. Therefore (6.3) and the choice which was made for the orientation of $h_{i+1}|\mathbb{R}$ implies that h_{i+1} conjugates R and $\tilde{R}$ along the critical orbits.

We claim that the sequence h_i converges uniformly to a quasiconformal homeomorphism h which is a conjugacy between R and $\tilde{R}$. Indeed, let K be the quasiconformal distortion of h_0. Since all maps h_i are K-quasiconformal, and the set of K-quasiconformal homeomorphisms is compact, we see that there are subsequences that converges uniformly. On the other hand, since h_{i+1} is equal to h_i outside of $R^{-i}(D)$ we see that any two limits of convergent subsequences must coincide in the complement of the filled Julia set of R. The claim follows because the filled Julia set of R has empty interior. (The interior components of the filled Julia set are bounded components of the Fatou set, and by Sullivan's classification theorem on wandering domains these are eventually periodic. The periodic components of the Fatou set contain iterates of the critical point on their boundary. This is impossible by the minimality of the orbit of the critical point.) $\square$

We wish to thank Misha Lyubich for pointing out that the first two statements of Theorem A can be derived easily from the Measurable Riemann Mapping Theorem, as in Section VI.4 of [MS]. In fact, it was he who convinced us that Theorem A could be useful in this context.

Proof of the first two statements of Theorem A: This is proved exactly as in Theorem 4.2a and Theorem 4.2b of Chapter VI in [MS]. The reason we can apply this argument is that because we have a quasiconformal conjugacy on the critical orbits, see Theorem 5.1. For the details we refer to Section VI.4 of [MS], but let us sketch the idea here. Firstly, the kneading invariant of $z^\ell + c_1$, depends monotonically on $c_1 \in \mathbb{R}$. So if $[c_1, \tilde{c}_1]$ is the maximal interval of parameters with this kneading invariant, then, by the Measurable Riemann Mapping Theorem, there exists a family H_u, $u \in [c_1, \tilde{c}_1]$ of quasiconformal homeomorphisms such that $H_u \circ f \circ H_u^{-1}$ is of the form $f(z) = z^\ell + w(u)$ with $w(c_1) = c_1$ and $w(c_2) = c_2$. Moreover, by the theorem of Ahlfors and Bers, $u \mapsto w(u)$ is analytic on a neighbourhood of $[c_1, c_2] \subset \mathbb{C}$ and so the image of w contains c_1 and c_2 in its interior. Hence for each $t \in \mathbb{R}$ near $[c_1, c_2]$, the map $z \mapsto z^\ell + t$ is also conjugate to $z^\ell + c_1$. This contradicts the maximality of the interval $[c_1, \tilde{c}_1]$.

The fact that one has no measurable linefield on the Julia set of f follows as in the Corollary on page 472 of [MS]. (The idea is that if an f-invariant measurable linefield on the Julia set then we would obtain a family of quasiconformal homeomorphisms $H_u\colon \mathbb{C} \to \mathbb{C}$. From the invariance of the line-field, we get that $H_u \circ f \circ H_u^{-1}$ is also holomorphic for each u. In fact, one gets that $H_u \circ f \circ H_u^{-1}(z) = z^\ell + w(u)$ where w is a *non-constant* analytic function of u (defined on a neighourhood of c_1. This would show that for each $t \in \mathbb{C}$ near c_1, the map $z \mapsto z^\ell + t$ is conjugate to $z \mapsto z^\ell + c_1$, contradicting the first part of Theorem A.

To prove the last part of Theorem A we use an idea of McMullen which is discussed in [McM3] and [McM4]. We became aware of these ideas through informal notes written by Folkert Tangerman. We would like to thank him and Jacek Graczyk for some useful discussions on this methods.

Proof of the last statement of Theorem of Theorem 6.1. Let us start by emphasizing that we shall fix ℓ in the proof of Theorem A. So we do not claim that the constants (in the proof of this theorem) are independent of ℓ.

To clarify the strategy we shall first define a sequence of renormalizations of our map related to the Yoccoz puzzle, and explain why we cannot use this sequence itself. So let us first explain that one can associate to a Fibonacci map a sequence of sets $A_n^0, A_n^1 \subset A_n$, $n \geq n_0$, where A_n^0, A_n^1 are two disjoint closed topological discs which are compactly contained in the interior of A_n. This can be done so that the return map $R_n \colon (A_n^0 \cup A_n^1) \to A_n$ of f

$$R_n(x) = \{f^i(x)\ ;\ i \text{ minimal with } f^i(x) \in A_n\},$$

is equal to f^{S_n} on A_n^0 and equal to $f^{S_{n-1}}$ on A_n^1. Indeed, by Theorem 6.2 there exists n_0 such that for $n \geq n_0$ and for $D_n = D^*(u_{n-1}, \hat{u}_{n-1})$, there exists two discs $D_n^i \subset D_n$ and a polynomial-like map

$$R_n \colon (D_n^0 \cup D_n^1) \to D_n$$

such that $R_n|D_n^0 = f^{S_n}$ is a covering map with branch point c and with $R_n(c) \in D_n^1$ and such that $R_n|D_n^1 = f^{S_{n-1}}$ is a diffeomorphism with $R_n^2(c) \in D_n^0$. So fix $n \geq n_0$ and write $A_0^0 = D_n^0$, $A_0^1 = D_n^0$, $A_0 = D_n$ and $R = R_n$. Starting with such a polynomial map

$$R \colon (A_0^0 \cup A_1^1) \to A_0 \tag{6.4}$$

we can define its renormalization $\mathcal{R}(R)$ as the polynomial-like map

$$\mathcal{R}(R) \colon (A_1^0 \cup A_1^1) \to A_1 := A_0^0 \tag{6.5}$$

where $A_1^1, A_1^2 \subset A_1$ are defined as follows. Take A_1^1 as the inverse of $A_1 \subset D$ under the ℓ-fold covering map $(R|D^0) \colon D^0 \to D$ and A_1^2 as the component containing $R^2(c)$ of the inverse of $A_1 \subset D$ under $(R|D^1) \circ (R|D^0)$.

Since f is the Fibonacci map, this procedure can be continued infinitely often. So let

$$\mathcal{R}^i(R) \colon (A_i^0 \cup A_i^1) \to A_i,$$

$i = 0, 1, 2, \ldots$ be the sequence of renormalizations. Note that in this construction we have

$$A_i = A_{i-1}^0. \tag{6.6}$$

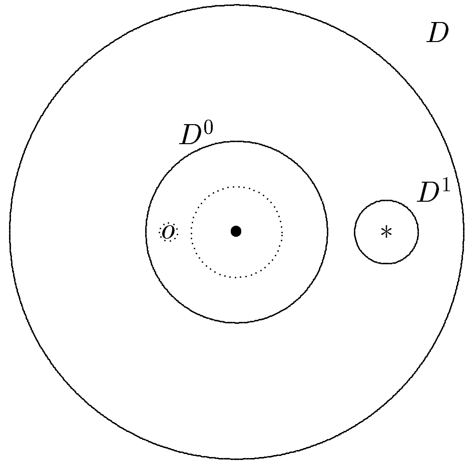

Figure 11: The renormalization of a polynomial-like mapping. The discs $A \subset D^0$ contains the critical point c which is denoted by the symbol $\bullet$. The disc D^1 contains its image $R(c)$ (denoted by $*$). The dics $B \subset D^1$ contains $R^2(c)$ (denoted by o). The map R sends A first into D^1 and its second iterate is equal to D^0. There are ℓ topological discs inside D^0 which R sends diffeomorphically onto D^0, but B is the one which contains $R^2(c)$.

Then $\mathcal{R}^i(R)$ is again the first return map from $A_i^0 \cup A_i^1 \subset A_i$ onto A_i. This gives a kind of Yoccoz puzzle, associated to the starting polynomial-like mapping (6.4). The trouble, however, is that we are not able to give lower bounds for the moduli of the annuli (with two holes) $A_i \setminus (A_i^0 \cup A_i^1)$ and also are not sure that the family $\mathcal{R}^i(R)$ is compact. Indeed, even though we have the lower bounds for the moduli because of Proposition 6.2 for $i = 0$, we do not even know for example that $A_1 = A_0^0$ lies inside the disc $D_{n_0+1} = D^*(u_{n_0}, \hat{u}_{n_0}) \subset A_0 = D_{n_0}$. If we knew this, then at least we could apply Proposition 6.2 again and get inductively that $A_i \subset D_{n_0+i}$. Unfortunately, Proposition 3.2 only allows us to conclude that $A_1 = A_0^0 \subset A_0$.

Step 1: The construction of a tower. Hence, we shall continue a little differently and drop condition (6.6). We shall do this by constructing a 'tower' of polynomial-like mappings. This tower is a limit of the polynomial-like maps from Theorem 6.2 and corresponds to an infinitely blown-up neighbourhood of the origin. Indeed, $R_n|D_n^0 = f^{S_n}$ and there exists a univalent extensions of

$$f^{S_n-1}\colon f(D_n^0) \to D_n$$

and

$$f^{S_{n-1}}\colon D_n^1 \to D_n$$

onto the disc $D^*(d_n, d_{n-2})$. By the real bounds, this last disc is a definite proportion larger than the disc D_n and so for each fixed ℓ – up to renormalization –

$$R_n\colon (D_n^0 \cup D_n^1) \to D_n \text{ is in a compact family of maps.} \tag{6.7}$$

(Note that the amount of extendability depends heavily on ℓ and so this number N might strongly depend on the choice of ℓ.) In particular, by Proposition 4.1, the set $f(D_n^0)$ contains a disc centered at c_1 with diameter $k(\ell) \in (0,1)$ times the size of $f(D_n^0) \cap \mathbb{R} = (w_n^f, v_n^f)$. So pulling back by f gives that the topological disc D_n^0 also contains at least a disc with radius $\tilde{k}(\ell) = [k(\ell)]^{1/\ell} \in (0,1)$ times $|v_n|$. Hence there exists a number $N(\ell)$ such that

$$D_n \subset D_{n-N}^0 \tag{6.8}$$

for each $n \geq n_0 + N$. Note that

$$R_n \text{ is an iterate of } R_{n-N} \tag{6.9}$$

(restricted to its domain).

By (6.7), we can take a sequence $n(j) \to \infty$, such that – up to scaling –

$$R_{n(j)}\colon (D_{n(j)}^0 \cup D_{n(j)}^1) \to D_{n(j)}$$

converges to a map

$$F_0\colon V_0 \to W_0$$

In order to fix matters, let W_0 be the unit disc and let Λ_j be the scaling map which maps the disc $D_{n(j)}$ onto W_0. This means that

$$\lim_{j\to\infty} \Lambda_j(D_{n(j)}^0 \cup D_{n(j)}^1) \to V_0,$$

where V_0 consists of two disjoint topological discs V_0^0, V_0^1 which are compactly contained in W_0 and

$$\Lambda_j \circ R_{n(j)} \circ \Lambda_j^{-1}$$

converges to F_0. Next take a subsequence of this sequence so that – up to scaling by the map Λ_j –

$$R_{n(j)-N}\colon (D_{n(j)-N}^0 \cup D_{n(j)-N}^1) \to D_{n(j)-N}$$

converges to a map

$$F_1\colon V_1 \to W_1.$$

Since, by the real bounds, there exist $1 < \kappa_0 < \kappa_1 < \infty$ (which do depend on N and so on ℓ but not on j) such that

$$1 < \kappa_0 \leq \frac{\text{radius of } D_{n(j)}}{\text{radius of } D_{n(j)+N}} \leq \kappa_1,$$

the radius of the disc W_1 is in $[\kappa_0, \kappa_1]$. So taking repeatedly subsequences of subsequences we get that for each $i \geq 0$,

$$R_{n(j)-i\cdot N}\colon (D_{n(j)-i\cdot N}^0 \cup D_{n(j)-i\cdot N}^1) \to D_{n(j)-i\cdot N}$$

converges – again up to scaling by Λ_j – as $j \to \infty$ to a map

$$F_i\colon V_i \to W_i.$$

One has

$$\kappa_0 \le \frac{\text{radius of } W_{i+1}}{\text{radius of } W_i} \le \kappa_1. \tag{6.10}$$

By Proposition 6.2, the modulus of $W_i \setminus V_i$ is bounded from above and below. By (6.7), the family of maps $F_i \colon V_i \to W_i$ are in a compact set (after rescaling so that the image becomes a unit disc). Define the *filled Julia set* of F_i as

$$J_i = \{z \in V_i \ ; \ F_i^k(z) \in V_i \text{ for all } k \ge 0\}$$

and define the *post-critical* set of F_i as

$$P_i = \text{ closure of the iterates of } c \text{ under } F_i.$$

Note that

$$\mathbb{C} = \cup_{i\ge 0} W_i = \cup_{i \ge 0} V_i$$

and observe that

$$P = \cup_{i \ge 0} P_i$$

is closed because of (6.9). Indeed, this implies that F_i is an iterate of F_{i-1} and so $P_i \subset P_{i-1}$ but since F_i the first return of F_{i-1} to W_i, one also has that $P_i \cap W_i = P_{i-1} \cap W_i$. Hence $P \cap W_i$ is equal to P_i which is compact. Thus P is closed. Moreover, J_i is the complement of an open dense set, because it is just a rescaled version of a piece of the Julia set of the Fibonacci map and since the Fibonacci set has no periodic attractors, it is the complement of an open dense set. Hence by Baire, $J = \cup J_i$ is the complement of a generic set. (Later on, we shall see that – in spite of this – J is dense.) Define the map

$$F \colon \mathbb{C} \to \mathbb{C}$$

on the 'tower' $\cup_{i \ge 0} V_i$ as follows: take $z \in \mathbb{C}$ and let $i \ge 0$ be *minimal* such that $z \in V_i$. Then define

$$F(z) = F_i(z).$$

Step 2: The Poincaré metric. Consider the Poincaré metric d_P on $S = \mathbb{C} \setminus P$. Then there exists a universal constant $C > 0$ such that

$$\frac{1}{C} \cdot \frac{|dz|}{d(z,P)} \le \varrho \le C \cdot \frac{|dz|}{d(z,P)}. \tag{6.11}$$

Moreover, the diameter of $W_i \setminus V_i \subset S$ is uniformly bounded.

Proof of Step 2: Step 2 holds because $\omega(c)$ has bounded geometry, see Theorem 5.1. One way to prove Step 2, is to use Theorem 1 of [BP], see also Theorem 2.3 of [McM2]. Let us formulate this result first. Let U be a hyperbolic region U in $\mathbb{C}$ (so $\mathbb{C} \setminus U$ consists of at least three points) and let $d(z, \partial U)$ be the Euclidean distance between z to the boundary of U. A round annulus $A = \{z \ ; \ r < |z - z_0| < s\} \subset U$ is called *essential* if it is not contractible in U; its modulus is equal to $\log |s/r|$. The *core* curve of A is the circle $|z - z_0| = \sqrt{rs}$. Next define $mod(z, U)$ as the maximal modulus of a essential round annulus in U and whose core passes through z. Then the Poincaré metric ϱ on U is comparable to

$$\varrho' = \frac{|dz|}{d(z, \partial U)(1 + mod(z, U))}. \tag{6.12}$$

This means that $1/C \leq \varrho/\varrho' \leq C$ for some universal constant $C > 0$.

Let us take $U = S = \mathbb{C} \setminus P$. Since the post-critical set P_i has bounded geometry, see Theorem 5.1, it follows that there exists a universal upperbound for the modulus of any round annulus which is essential with respect to $\mathbb{C} \setminus P_i$ and which is contained in disc of diameter comparable to W_i. Hence $mod(z, S)$ is bounded from above and (6.11) follows from (6.12). That the diameter of $W_i \setminus V_i$ (in terms of the Poincaré metric on S) is uniformly bounded, follows from (6.11) and the bounded geometry.

Step 3: F expands the Poincaré metric. We claim that there exists $\epsilon > 0$ and $\kappa > 1$ such that

$$F_i \colon (V_i \setminus F_i^{-1}(P)) \to (W_i \setminus P)$$

expands the Poincaré metric in the sense that if $x, y \in (V_i \setminus F_i^{-1}(P))$ and $d_P(x, y) \leq \epsilon$ then

$$d_P(F_i(x), F_i(y)) \geq d_P(x, y).$$

If, moreover, $F_i(x) \in W_i \setminus V_i$ then

$$d_P(F_i(x), F_i(y)) \geq \kappa d_P(x, y)$$

and therefore

$$d_P(F(x), F(y)) \geq \kappa d_P(x, y).$$

Proof of Step 3: Let $S_i = \mathbb{C} \setminus F_i^{-1}(P)$ and $S = \mathbb{C} \setminus P$. Let $d_{P,i}$ and d_P be the Poincaré metric on these sets. By Schwarz,

$$d_P(x, y) \leq d_{P,i}(x, y).$$

Since the inverse of

$$F_i \colon (V_i \setminus F_i^{-1}(P)) \to (W_i \setminus P)$$

is a holomorphic covering map (note that $W_i \cap P = P_i$), we get that F_i is a local isometry in the sense that $d_P(F_i(x), F_i(y)) = d_{P,i}(x, y)$ provided $d_P(x, y) \leq \epsilon$ where $\epsilon > 0$ is number which is independent of i. Hence $d_P(F_i(x), F_i(y)) \geq d_P(x, y)$ for $d_P(x, y) \leq \epsilon$. This implies the first statement. To prove the second statement, note that there exists a constant $C < \infty$ (which is independent of i) such that for any $z \in V_i$ with $F_i(z) \in W_i \setminus V_i$,

$$d(z, F_i^{-1}(P)) \leq C \cdot d(z, P) \tag{6.13}$$

where d is the Euclidean metric on $\mathbb{C}$. This holds because there are preimages under F_i of $P_i = P \cap W_i$ in the annulus $W_i \setminus V_i$ (for example one of the preimages of the critical point under F_i is in this annulus) and because of the previous step. Now (6.13) implies that there exists a constant $\kappa > 1$ (which is independent of i) such that

$$d_P(F_i(z), F_i(w)) \geq \kappa \cdot d_P(z, w)$$

when w is sufficiently close to z. This completes the proof of Step 3.

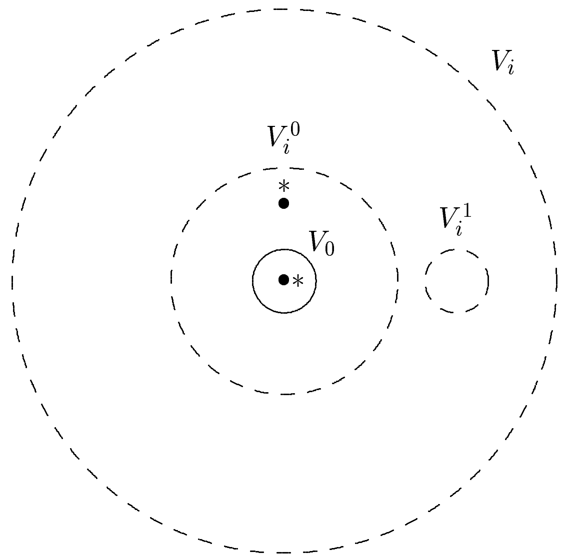

Figure 12: The points $c = 0, F(c)$ are contained in the small disc V_0; they are marked with the symbols $\bullet$ and $*$. The boundaries of the discs $V_i^0, V_i^1 \subset V_i$ are marked with a dashed curve. Since F_i wraps V_i^0 precisely ℓ times onto V_i there exists two points $x_1, x_2 \in V_i \setminus (V_{i-1} \setminus P)$ which are mapped by F_i to respectively c and $F(c)$.

Let $\Delta(x; t)$ be the hyperbolic disc based at x with radius t, where we take the Poincaré metric on $S = \mathbb{C} \setminus P$ from above.

Step 4: The set J is uniformly dense in $\mathbb{C}$. We claim that given $\epsilon > 0$ there exists $m(\epsilon)$ with the following properties. For each $x \in W_k$, $\Delta(x;\epsilon)$ contains x_0, x_1 and $k_0, k_1 \leq m(\epsilon)$ with

$$F^{k_0}(x_0) = c \text{ while } F^{k_0-1}(x_0) \notin P$$

and

$$F^{k_1}(x_1) = F(c) \text{ while } F^{k_1-1}(x_1) \notin P.$$

In particular, $\Delta(x;\epsilon) \cap J \neq \emptyset$.

Proof of Step 4: Since $c = 0 \in V_0^0$ one has that $F(c)$ is equal to $F_0(c)$ and in V_0. First we note that for each $i \geq 1$, both $c, F(c) \in V_0 \subset V_i^0$ have several preimages under maps $F_i \colon V_i \to W_i$ and we can choose these preimages outside P. (For example just take preimages which are 'near' the imaginary axis, see Figure 12.) Take two such preimages $x'_{i,0}, x'_{i,1} \in V_i \setminus V_{i-1}$. This means that

$$F(x'_{i,0}) = F_i(x'_{i,0}) = c = 0 \text{ and } F(x'_{i,1}) = F_i(x'_{i,1}) = F(c),$$

while $x'_{i,0}, x'_{i,1} \notin P$. We can choose these points so that the Euclidean distance of these points to P is of the same order as the diameter of W_i.

So let $n(1)$ be the smallest integer so that $F^{n(1)}(x) \in W_{k+1} \setminus V_{k+1}$. If such an integer $n(1)$ does not exists then $F^{n(1)}(x) \in J_{k+1}$ and so we are done. Otherwise $F^{n(1)}$ contains a disc of radius $\kappa\epsilon$. Continuing in this way we get a sequence of integers $n(1), \dots, n(m)$ so that $F^{n(i)}(x) \in W_{k+i} \setminus V_{k+i}$. for $i = 1, 2, \dots, m$ and $F^{n(m)}(B)$ contains a disc of radius $\kappa^m\epsilon$. But because of the first part of the proof of this claim, this implies that for m sufficiently large one has that $F^{n(m)}(B)$ contains a preimage of c and of $F(c)$ under the map $F_{k+m}: V_{k+m} \to W_{k+m}$. This concludes the proof of this claim.

Step 5: The polynomial-like map has no invariant linefields on its Julia set. In Part 2 of Theorem A we proved that the Julia set of f carries no measurable f-invariant linefield. Let us show that this implies that

$$R_i: (D_i^0 \cup D_i^1) \to D_i$$

also carries no measurable R_i-invariant linefield on its Julia set J_i. So suppose by contradiction that this induced map has such a linefield μ. Notice that R_i is a first return map. We will extend μ to an f-invariant measurable linefield on the subset $\cup_{k \geq 0} f^{-k}(D_i^0 \cup D_i^1)$ by $\mu = (f^k)^* \mu$ on $f^{-k}(D_i^0 \cup D_i^1)$. Of course, we have to show that μ is well defined. So assume that $x \in \cup_{k \geq 0} f^{-k}(D_i^0 \cup D_i^1)$ and there exists $k' > k$ with

$$y = f^k(x) \text{ and } y' = f^{k'}(x) = f^{k'-k}(y) \text{ are both in} (D_i^0 \cup D_i^1).$$

Since R_i is a first return map, this implies that $f^{k'-k}$ is an iterate of R_i. Hence, since μ is R_i-invariant one gets that $(f^{k'-k})^*(\mu(y')) = \mu(y)$. This implies that

$$f^k(\mu(y)) = f^{k'}(\mu(y'))$$

and hence $\mu(x)$ is well-defined. Hence, if we define μ to be zero outside the backward iterates of $D_i^0 \cup D_i^1$ then we get by construction a linefield which is f-invariant. It is measurable because the original linefield is measurable and because we have used a countable process to extend its domain. This implies that the Julia set of f would carry such a linefield, a contradiction. Therefore $R_i: (D_i^0 \cup D_i^1) \to D_i$ also carries no invariant measurable linefield. Since each of these maps R_i is uniformly quasiconformally conjugate, these maps are also quasiconformally conjugate to any limit of R_i. Therefore the Julia set of any limit of R_i also carries no invariant linefield.

Step 6: Constructing an invariant linefield if renormalization does not hold. Now we will show how to construct an invariant linefield on the tower if renormalization does not hold. Below, we shall show that such an invariant linefield cannot exist, obtaining a contradiction.

Take $n(j), \tilde{n}(j), i(j) \to \infty$ with $\tilde{n}(j) - n(j) \in 4\mathbb{N}$. Let h be a quasiconformal conjugacy h between

$$R_{n(j)-i(j)\cdot N}: (D^0_{n(j)-i(j)\cdot N} \cup D^1_{n(j)-i(j)\cdot N}) \to D_{n(j)-i(j)\cdot N}$$

and

$$R_{\tilde{n}(j)-i(j)\cdot N}\colon (D^0_{\tilde{n}(j)-i(j)\cdot N} \cup D^1_{\tilde{n}(j)-i(j)\cdot N}) \to D_{\tilde{n}(j)-i(j)\cdot N}$$

from Theorem 6.2. If the quasi-conformal distortion of h restricted to $D_{n(j)}$ tends to zero as $j \to \infty$, then taking $\Lambda_i\colon \mathbb{C} \to \mathbb{C}$ the scaling map which sends D_i onto the unit disc Δ,

$$\Lambda_{\tilde{n}(j)} \circ h \circ \Lambda^{-1}_{n(j)}\colon \Delta \to \mathbb{C}$$

tends to a scaling map. Since h conjugates

$$R_{n(j)}\colon (D^0_{n(j)} \cup D^1_{n(j)}) \to D_{n(j)} \text{ to } R_{\tilde{n}(j)}\colon (h(D^0_{\tilde{n}(j)}) \cup h(D^1_{\tilde{n}(j)})) \to h(D_{\tilde{n}(j)}),$$

it follows that if a subsequence $R_{n(j)}$ converges to some map $\hat{R}$ (after rescaling) then $R_{\tilde{n}(j)}$ also tends to the same map $\hat{R}$ after rescaling. It follows that for each $i_0 \in \{0, 1, 2, 3\}$, the sequence $\{R_{4i+i_0}\}_{i\geq 0}$ (which is contained in a compact set), has precisely one limit point (after rescaling). Hence we are done in this case.

So assume that the quasiconformal distortion of the conjugacy h does not go to zero. Then let μ be the Beltrami-coefficient of some convergent subsequence of conjugacies with quasiconformal distortion bounded away form zero (taking subsequences of the subsequences from above, so that the sequences of maps from the tower still converge). Next let $\nu = \pm\mu/|\mu|$ be the corresponding *linefield* defined on the support of μ. By assumption the support of μ has positive Lebesgue measure. Thus we get a measurable linefield ν on a set of positive Lebesgue measure in $\mathbb{C}$ which is invariant under each of the maps $F_i\colon V_i \to W_i$. We shall show that this gives a contradiction.

Step 7: Constructing a univalent linefield near c and $F(c)$.

Let us remind the reader that z is a *density point* of a set E if

$$\lim_{t\to 0} \frac{|E \cap B(z;t)|}{|B(z;t)|} = 1$$

where $B(z;t)$ is a discs with centre z and radius t and $|\cdot|$ stands for the Lebesgue measure of a set. By the Lebesgue Density Theorem, almost every z in E is a density point. Moreover, if ν on $\mathbb{C}$ is a measurable function then almost every $z \in \mathbb{C}$ is a point of *almost continuity* of ν. This means that for each $\delta > 0$ and almost every $z \in \mathbb{C}$,

$$\lim_{t\to 0} \frac{|\{y \in B(z;t) \ ; \ \ |\nu(y) - \nu(z)| < \delta\}|}{|B(x,t)|} = 1.$$

So take a point z which is both a density point of the support of the Beltrami coefficient ν as well as a point of almost continuity of ν. Since we have already shown that the Julia set of R_i (and also of any of its limits) does not carry an invariant linefield, see Step 5, ν vanishes on J and so we can choose $z \notin J$. Without loss of generality we can assume that $z \in V_1$.

So define a sequence of integers $k(i)$ so that

$$F^{k(i)-1}(z) \in V_i \text{ and } z_i := F^{k(i)}(z) \in W_i \setminus V_i.$$

Because $z \notin J$, such a sequence $k(i)$ exists. Now choose $\epsilon > 0$ such that there is a univalent pullback by $F^{k(i)}$ from $\Delta(z_i, 2\epsilon)$ to a neighbourhood of z. Let O_i be the pullback of $\Delta(z_i, \epsilon)$ by this map. By Step 3, the diameter of O_i is exponentially small in terms of i. Since z is a point of almost continuity of ν, this means that the proportion of the points $y \in O_i$ for which $|\nu(y) - \nu(z)| \geq \delta$ tends to zero. Define the constant linefield $\tilde{\nu}$ on O_i by $\tilde{\nu} \equiv \nu(z)$. Next define the linefield

$$\nu_i = (F^{k(i)})_*(\tilde{\nu})$$

on $\Delta(z_i, \epsilon)$. Observe that $F^{k(i)}$ has uniformly bounded distortion on O_i (this follows by Koebe since there exists a univalent extension to $\Delta(z_i, 2\epsilon)$). By the invariance of ν one has

$$\nu = (F^{k(i)})_*(\nu)$$

for all $y \in \Delta(z_i, \epsilon)$ and, combining all this, it follows that for any $\delta > 0$,

$$\frac{|\{y \in \Delta(z_i, \epsilon)\ ;\ |\nu(y) - \nu_i(y)| \geq \delta\}|}{|\Delta(z_i, \epsilon)|} \to 0 \tag{6.14}$$

as $i \to \infty$.

By Step 4, there exists $x'_{i,0}, x'_{i,1} \in \Delta(z_i, \epsilon)$ such that for some $k_0, k_1 \leq m(\epsilon)$ one has

$$F^{k_0}(x_0) = c \text{ while } F^{k_0-1}(x_0) \notin P$$

and

$$F^{k_1}(x_1) = F(c) \text{ while } F^{k_1-1}(x) \notin P$$

(in fact, these points even avoid some neighbourhood of P). In particular, this implies that there exists small neighbourhoods of $x_{i,0}, x_{i,1}$ which are mapped univalently and with bounded distortion to a disc centered at c respectively $F(c)$ of radius $\varrho > 0$. Let $\hat{\nu}_{i,0}$ be the linefield on $\Delta(c; \varrho)$ which is defined as the pushforward by the univalent maps $F^{k_0} \circ F^{k(i)}$ of the constant linefield $\tilde{\nu}$ (respectively $\hat{\nu}_{i,1} = (F^{k_1} \circ F^{k(i)})_*(\tilde{\nu})$ on $\Delta(F(c); \varrho)$). Since μ is invariant, and since the maps F^{k_i} have bounded distortion, (6.14) implies that the

$$\frac{|\{y \in \Delta(c, \varrho)\ ;\ |\nu(z) - \hat{\nu}_{i,0}(y)| \geq \delta\}|}{|\Delta(c, \varrho)|} \to 0 \tag{6.15}$$

as $i \to \infty$ and similarly for $\nu_{i,1}$ on $\Delta(F(c), \varrho)$. In other words, the restriction of μ to δ discs centered at c and at $F(c)$ is the limit of a sequence of linefields which are images of a constant linefield under a univalent mapping. From this it follows that the restriction of μ to these discs is actually itself the image of a constant linefield under a univalent mapping. (Such linefields are called *univalent.*)

Step 8: The final contradiction showing that renormalization holds after all.

The previous step implies that there are smooth foliations on a δ neighbourhoods of c and of $F(c)$ such that the tangent line of the leaves correspond to the linefield

μ on these neighbourhoods. However, since μ is invariant under F, the image of the foliation near c must coincide with the foliation near $F(c)$. This is impossible, because F has a critical point at c. Thus we can conclude that the assumption we made in Step 6 that renormalization fails, leads to a contradiction. $\square$

7 The random walk argument

In this section we shall state an abstract result about the evolution of typical points under a (nearly) Markov map with a kind of random walk structure. Let $(X, \mathcal{F}, m)$ be some space with probability measure m and σ-algebra $\mathcal{F}$ and $\mathcal{A} = \{A_k\colon k = 0, 1, 2, \ldots\}$ a partition of X into $\mathcal{F}$-measurable sets. $F\colon X \to X$ is a $\mathcal{F}$-measurable transformation, $\mathcal{A}_n = \bigvee_{k=0}^{n-1} F^{-k}\mathcal{A}$. Also assume that there exists $k_0 \in \mathbb{N}$ such that

$$F(A_r) \subseteq \cup_{j=0}^{\infty} A_{r-k_0+j} \text{ for all } r \geq k_0.$$

Observe that $\mathcal{A}$ is a Markov partition for F if and only if $F^k A$ is an element of $\mathcal{A}$ for each $A \in \mathcal{A}_{k+1}$ and each $k \geq 0$.

Define $\varphi : X \to \{0, 1, 2, \ldots\}$ by

$$\varphi(x) = n \text{ if } x \in A_n$$

and

$$\Delta\varphi := \varphi \circ F - \varphi\ .$$

Theorem 7.1 *Assume there are $n_0 \in \mathbb{N}$ and $M > 0$ such that for any $A \in \mathcal{A}_{k+1}$ and any $k \geq 0$ with $\varphi_{|F^k A} \geq n_0$ the following inequalities hold:*

$$\int_A (\Delta\varphi - 1) \circ F^k \, dm \geq 0 \quad \textit{and} \tag{7.1}$$

$$\int_A (\Delta\varphi)^2 \circ F^k \, dm \leq M \cdot m(A). \tag{7.2}$$

Then there exists a set $D \in \mathcal{F}$ with $m(D) > 0$ such that

$$\liminf_{n\to\infty} \frac{\varphi \circ F^j}{j}(x) \geq 1 \quad \textit{for each } x \in D,$$

and such that for every $x \in D$ the trajectory $x, Fx, F^2x, \ldots$ visits each set $A_k \in \mathcal{A}$ only finitely often.

Proof: The proof of this theorem is based on a martingale argument and is due to Gerhard Keller. This proposition and also its proof can be found as Proposition 4.1 in [BKNS]. $\square$

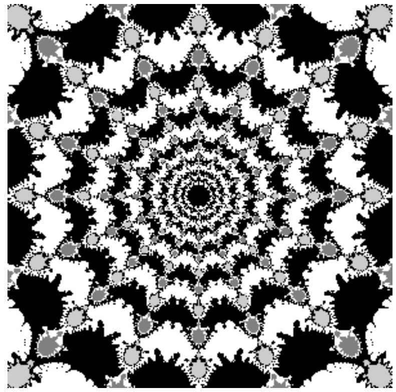

Figure 13: Some levels of the Yoccoz puzzle of the Fibonacci map $z \mapsto z^\ell + c_1$ when $\ell = 16$. One clearly sees the long spikes; these spikes are already predicted by our star-like tips of the set S_κ. Numerically, the annuli $D_n \setminus D_{n+1}$ look similar to the annular regions from the Yoccoz puzzle. This picture was made by Scott Sutherland in order to simulate a wandering walk he was studying jointly with Misha Lyubich. The purpose of this numerical study was to decide whether the Julia set might have positive Lebesgue measure. The outcome of this turned out to be quite inconclusive. For a discussions on the reasons for this, see the final section of this paper.

8 A nested sequence of discs

In this section we shall define an nested sequence of discs and give some geometric estimates on these. In the next section we shall show that these discs can be used to define an induced mapping with Markov properties. In fact, these discs are very similar to the discs constructed in the polynomial-like mapping from Figure 10. The problem with those discs is that the inner disc containing c intersects the real line in the points $v_n, \hat{v}_n$ and not again points from the sequence $u_i, \hat{u}_i$. Therefore we shall take a pull-back of f^{S_n} of a larger disc intersecting the real line in $u_{n-2}, \hat{u}_{n-2}$. That we can define inductively a sequence of topological discs follows from the real estimate near c_1 which was based on renormalization, see Proposition 3.3 and the Lemma of Schwarz. The resulting partition in annuli is related to certain annuli from the Yoccoz partition, but we are not sure whether this Yoccoz partition can be used directly in our proof. The problem is that we also need very good estimates on the shape of these annuli. Although we do not estimate the modulus of these annuli from below, we do need estimates for their area and we also need that certain discs are not too 'flat'. These estimates are again based on renormalization by analyzing a sequence of maps with an almost neutral point. We do not know whether it is possible to get similar estimates for the corresponding annuli in the Yoccoz puzzle. Therefore we prefer to use our 'cruder' partition in annuli.

First we show that one can define a nested sequence of balls D_n such that $f^{S_{n+1}}$ maps D_{n+2} as a ℓ-covering onto D_n. To state the properties of this covering more precisely, we remind the reader that

$$y_n = f^{S_n}(d_{n+2}), \quad y_n^f = f(y_n)$$

and note that $|u_n - c| < |y_n - c| < |u_{n-1} - c|$. Moreover, let

$$a_{n-1} \in \{y_{n-1}, \hat{y}_{n-1}\} \cap (u_n, \hat{u}_{n-2}).$$

Hence a_{n-1} is on the same side of c as d_n, u_n and y_n. Similarly define

$$b_{n+1}^f \in (c_1, w_n^f) \text{ so that } a_{n-1} = f^{S_n - 1}(b_{n+1}^f).$$

Then $|u_n - c| < |a_{n-1} - c| = |f^{S_n-1}(b_{n+1}^f) - c| < |u_{n-2} - c|$. Moreover, let $r_n^f \in (c_1, t_n^f)$ be so that

$$f^{S_{n-1}-1}(r_n^f) = x_{n-1}$$

and therefore so that

$$f^{S_n-1}(r_n^f) = \hat{u}_{n-2}.$$

(Note that r_n^f is not the image of a point $r_n \in \mathbb{R}$; this notation is just to emphasize that r_n^f is close to c_1. Also one should not confuse r_n^f with the previously defined point $w_n^f \in (c_1, t_n^f)$ so that $f^{S_n-1}(w_n^f) = \hat{u}_{n-1}$.)

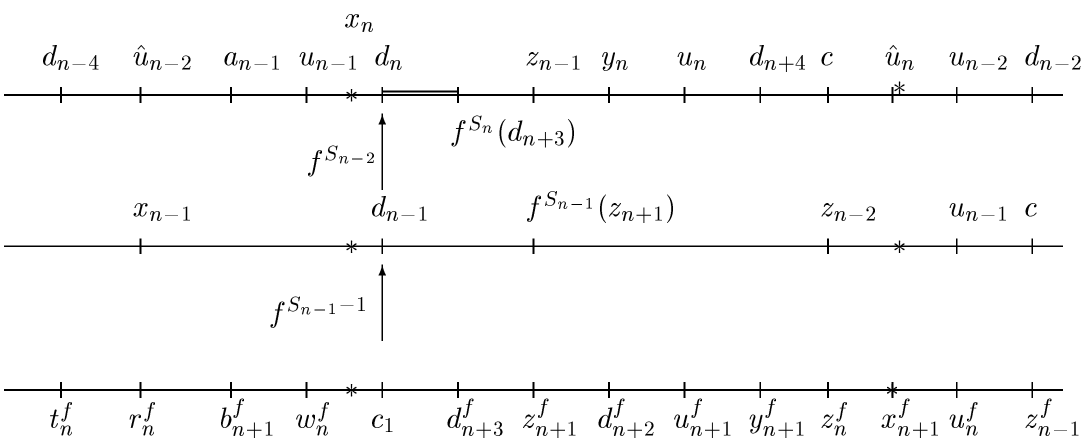

Figure 14: Points and their images under $f^{S_{n-1}-1}$ and $f^{S_{n-2}}$. The slit $Y_{n-1} = [d_n, f^{S_n}(d_{n+3})]$ which will play an important role in the next section, is marked explictly. We should emphasize that $D_{n-1} \cap \mathbb{R} = [\hat{u}_{n-2}, u_{n-2}]$ lives on the top line, $D_{n-1}^1 \cap \mathbb{R} = [x_{n-1}, u_{n-1}]$ on the middle line and $D_{n+1} \cap \mathbb{R} = [\hat{u}_n, u_n]$ on the f-preimage of the bottom line.

As before given a bounded real interval I, let $D_*(I)$ be the Euclidean disc which is symmetric w.r.t. the real line and which intersects the real axis in I. Moreover, if $\kappa \in [0, \pi/2)$ and $y > z > 0$ then define $S_\kappa(z,y)^0$ as follows. Let $l_{\pm\kappa}$ respectively $m_{\pm\kappa}$ be the infinite line through $z > 0$ resp. y cutting the real line with angle $\pm\kappa$. Then define $S_\kappa(z,y)$ to be the closure of the two components of

$$\{z \in \mathbb{C} \; ; \; |\arg(z)| < 2\pi/\ell\} \;\setminus (l_{\pm\kappa} \cup m_{\pm})$$

which contain points from $(0, y)$. Next let $S_\kappa(z,y)^i$ be equal to $S_\kappa(z,y)^0$ rotated over $2\pi i/\ell$ degrees and let $S_\kappa(z,y) = \cup S_\kappa(z,y)^i$.

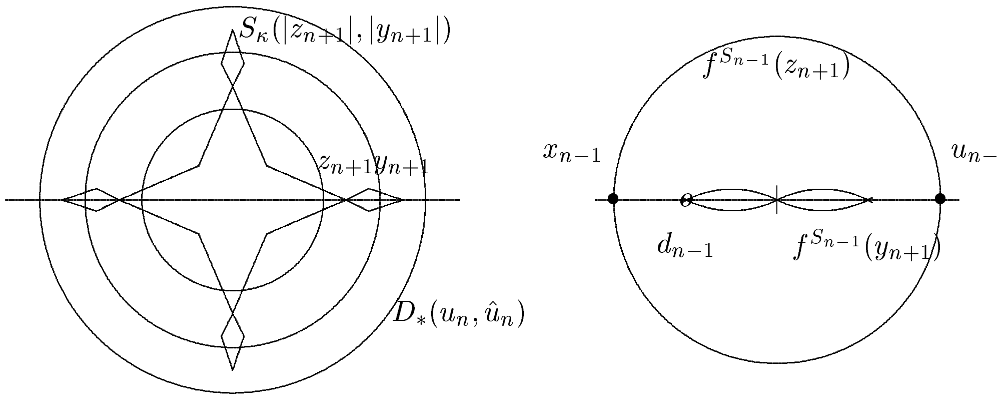

Figure 15: The topological ball D_{n+1} is between $S_\kappa(|z_{n+1}|,|y_{n+1}|)$ and the Euclidean disc $D_*(u_n,\hat{u}_n)$ on the left. (In fact, the lower bound for D_{n+1} is a pretty good bound for this set: this set is really squeezed near z_{n+1}. In particular, we should emphasize that we do not have a uniform lower bound for the moduli of the annuli A_n.) The smaller balls $D_*(u_{n+1},\hat{u}_{n+1})$ and $D_*(u_{n+2},\hat{u}_{n+2})$ are also drawn. If $\ell=4$ then this 'star' has 4 tips (consisting of 'diamonds'). The star does not necessarily contain the next Euclidean disc $D_*(u_{n+1},\hat{u}_{n+1})$ completely. The topological ball $D^1_{n-1}=f^{S_{n-1}}(D_{n+1})$ contains the interval $[x_{n-1},u_{n-1}]$ and at least the union of $D((d_{n-1},f^{S_{n-1}}(z_{n+1}));\beta)$ and $D(f^{S_{n-1}}(z_{n+1}),f^{S_{n-1}}(y_{n+1});\beta)$. It is inside the disc $D_*(x_{n-1},u_{n-1})$.

Theorem 8.1 *There exist a constant $K<\infty$ and $\ell_0\geq 4$ such that for each $\ell\geq\ell_0$ one has the following properties. There exists a nested sequence of open topological balls D_n, D^1_n for $n=k_0-2,k_0-1,\dots$ (for some large k_0) so that D_{k_0-2} and D_{k_0-1} are open Euclidean discs, centered at the critical point and with boundary through u_{k_0-3} respectively u_{k_0-2} and so that*

1. *$D_n\subset D_*(u_{n-1},\hat{u}_{n-1})$ and $D_n\cap\mathbb{R}=(u_{n-1},\hat{u}_{n-1})$;*
2. *the closure of D_n is inside D_{n-1} and D_n is invariant under a rotation of angle $2\pi/\ell$;*
3. *$D^1_n\subset D_*(u_n,x_n)$ and $D^1_n\cap\mathbb{R}=(u_n,x_n)$ (and so this set is in the annulus $A_{n-1}=D_n\setminus D_{n+1}$);*
4. *$f^{S_{n+1}-1}$ maps $(D_{n+2})^f$ diffeomorphically onto D_n;*
5. *f^{S_n-1} maps $(D_{n+2})^f$ diffeomorphically onto D^1_n;*
6. *$f^{S_{n-1}}$ maps D^1_n diffeomorphically onto D_n.*

Hence

$$f^{S_n}:\qquad D_{n+1}\xrightarrow{\quad f^{S_{n-1}}\quad} D^1_{n-1}\xrightarrow{\quad f^{S_{n-2}}\quad} D_{n-1}.$$

Moreover, we have the following estimates for the shape of the topological balls D_n and D^1_n, $n \geq k_0$: there exist universal constants $\kappa > 0$ and $\beta > 0$ such that

$$S_\kappa(|z_n|, |y_n|) \subset D_n \subset D_*(u_{n-1}, \hat{u}_{n-1})$$

for all $n \geq k_0 - 2$ and such that

$$\Big(D((d_{n-1}, f^{S_{n-1}}(z_{n+1})); \beta) \ \cup \ D((f^{S_{n-1}}(z_{n+1}), f^{S_{n-1}}(y_{n+1})); \beta)\Big) \subset D^1_{n-1} \subset D_*(x_{n-1}, u_{n-1}).$$

Here we remind the reader that given a real interval J and $\alpha \in (0, \pi)$ we defined a neighbourhood $D(J; \alpha)$ of J in section 4. This set is a hyperbolic neighbourhood of J in $\mathbb{C}_J$. We should also point out that the Properties 1 and 2 stated in this theorem imply that for $n \geq n_0$ the annulus $A_{n-1} = D_n \setminus D_{n+1}$ intersects each of the rays $\mathbb{R}^+ \ni t \mapsto te^{2\pi i/\ell} \in \mathbb{C}$ $(i = 0, 1, \ldots, \ell - 1)$ in precisely one segments and therefore these rays divide A_{n-1} into precisely ℓ components.

Corollary 8.1 *For n and ℓ sufficiently large, the Euclidean area of $D_n \setminus D_{n+1}$ is comparable to the Euclidean area of $D_*(u_{n-1}, \hat{u}_{n-1}) \setminus D_*(u_n, \hat{u}_n)$. In particular, there exists $\tau > 0$ such that $|D_n \setminus D_{n+1}| > \frac{\tau}{\ell}|D_n|$ for ℓ and n large enough. Similarly, the area of $D^{1,f}_{n-1}$ is at τ times the area of A^f_{n-2}. The area of $f^{-1}(D^{1,f}_{n-1})$ (consisting of D_{n-1} and all its ℓ rotated versions) is also at least τ times the area of A_{n-2}.*

Proof of the Corollary: From the real bounds, see Theorem 3.1,

$$\frac{|u_n| - |y_n|}{|y_{n-1}| - |u_n|}$$

is universally bounded from below and above. It follows that the part of $S_\kappa(|z_n|, |y_n|)$ which is outside $D_*(u_n, \hat{u}_n)$ (i.e., the tips) has Euclidean area which is of the same order as the size of the area of $D_*(u_{n-1}, \hat{u}_{n-1}) \setminus D_*(u_n, \hat{u}_n)$. The last statement follows since there are universal constants C_i such that

$$\frac{C_0}{\ell} \leq \frac{|u_n - u_{n-1}|}{|u_n - c|} \leq \frac{C_1}{\ell}.$$

From the real bounds, the interval $[d_{n-1}, f^{S_{n-1}}(z_n)]$ takes up a definite proportion of – for example – the interval $[\hat{u}_{n+1}, u_{n-1}]$. Hence, by the last bound of the previous theorem, the area of the topological disc D^1_{n-1} is at least a definite proportion of the area of $A_{n-2} \cap \{z \in \mathbb{C} \ ; \ \arg(z) < 1/\ell\}$. From this the last statement follows. $\square$

For later references we emphasize that this Corollary implies that

$$|D_i \setminus D_{i+i}| = C_i \cdot \frac{e^{-i/\ell}}{\ell} \cdot |D_n|, \tag{8.1}$$

for $i = \ell, \ell+1, \ldots$ and where C_i is universally bounded from below and above for $n \geq n_0(\ell)$.

Proof of Theorem 8.1: Take k_0 so large that

$$|r_n^f - c_1| < |u_n^f - c_1|$$

for all $n \geq k_0 - 2$. By Proposition 3.3 and Theorem 6.1 such a k_0 exists. (Below we shall increase k_0 even further for the last part of the theorem.) For $i = k_0 - 2, k_0 - 1$ the interval (u_i, x_i) is mapped diffeomorphically onto

$$(u_{i-1}, \hat{u}_{i-1}) = D_i \cap \mathbb{R}$$

by $f^{S_{i-1}}$. Since $f^{S_{i-1}}$ is a real polynomial, there exists by Proposition 4.1 a set $D_i^1 \subset D_*(u_i, x_i)$ with $D_i^1 \cap \mathbb{R} = (u_i, x_i)$ which is mapped by $f^{S_{i-1}}$ diffeomorphically onto D_i. Hence Properties 1-6 are satisfied for $i = k_2 - 2, k_0 - 1$.

So assume that D_i and D_i^1 satisfying Properties 1-6 already are defined by induction for $i = k_2 - 2, k_0 - 1, \ldots, n$. Since all iterates of the critical point of f are in the real line, one has that

$$c_{S_k} \in D_k \text{ and } c_i \notin D_k \text{ for } 0 < i < S_k. \tag{8.2}$$

Now we define D_{n+1} as follows. Let D_{n-1}^1 be the topological ball which is already defined and which is mapped diffeomorphically by $f^{S_{n-2}}$ onto D_{n-1}. One has $D_{n-1}^1 \cap \mathbb{R} = (u_{n-1}, x_{n-1})$ and $D_{n-1}^1 \subset D_*(u_{n-1}, x_{n-1})$. Because of the results in Section 2, $f^{S_{n-1}-1}$ maps (u_n^f, r_n^f) diffeomorphically onto (u_{n-1}, x_{n-1}). It follows by Proposition 4.1 that there exists a set $D_{n+1}^f \subset D_*(u_n^f, r_n^f)$ with $D_{n+1}^f \cap \mathbb{R} = (u_n^f, r_n^f)$ which is mapped diffeomorphically onto $D_{n-1}^1 \subset D_*(u_{n-1}, x_{n-1})$ by $f^{S_{n-1}-1}$.

Since $f^{S_{n-1}-1}$ maps $D_n^f \ni c_1$ diffeomorphically onto D_{n-2} and the set D_{n+1}^f into D_{n-1}^1 and because the closure of D_{n-1}^1 is contained in the closure of $D_{n-1} \subset D_{n-2}$, it follows that D_{n+1}^f is contained in the interior of D_n^f. Hence D_{n+1} is contained in the interior of D_n. Since $D_{n+1}^f \subset D_*(u_n^f, r_n^f) \ni c_1$ and since we have by Proposition 3.3,

$$|r_n^f - c_1| < |u_n^f - c_1|$$

(as $n \geq k_0$) we even get $D_{n+1} \subset D_*(u_n, \hat{u}_n)$. Similarly, f^{S_n} maps (u_{n+1}, x_{n+1}) diffeomorphically to $(u_n, \hat{u}_n)$. Hence by Proposition 4.1 there exists a set $D_{n+1}^1 \subset D_*(u_{n+1}, x_{n+1})$ with $D_{n+1}^1 \cap \mathbb{R} = (u_{n+1}, x_{n+1})$ which is mapped diffeomorphically onto $D_{n+1} \subset D_*(u_n, \hat{u}_n)$. This proves Properties 1-6.

Before proving the last statement we state a proposition. This proposition will imply the proof of Theorem 8.1 and the remainder of this section will be dedicated to the proof of the proposition.

It is convenient to choose z_{n+1} to be on the same side of c as a_{n+1}, d_{n+2}, u_{n+2} and y_{n+2}. Then $[z_{n+1}, a_{n+1}]$ contains d_{n+2} and this interval does not contain c.

In the next proposition we shall show that there exists a region $V_{n,\ell}$ which looks like one of the tips from $S_\kappa(|z_{n+1}|, |y_{n+1}|)$, see Figure 15, which is mapped by f^{S_n} into a similar tip associated to the next set $S_\kappa(|z_{n-1}|, |y_{n-1}|)$.

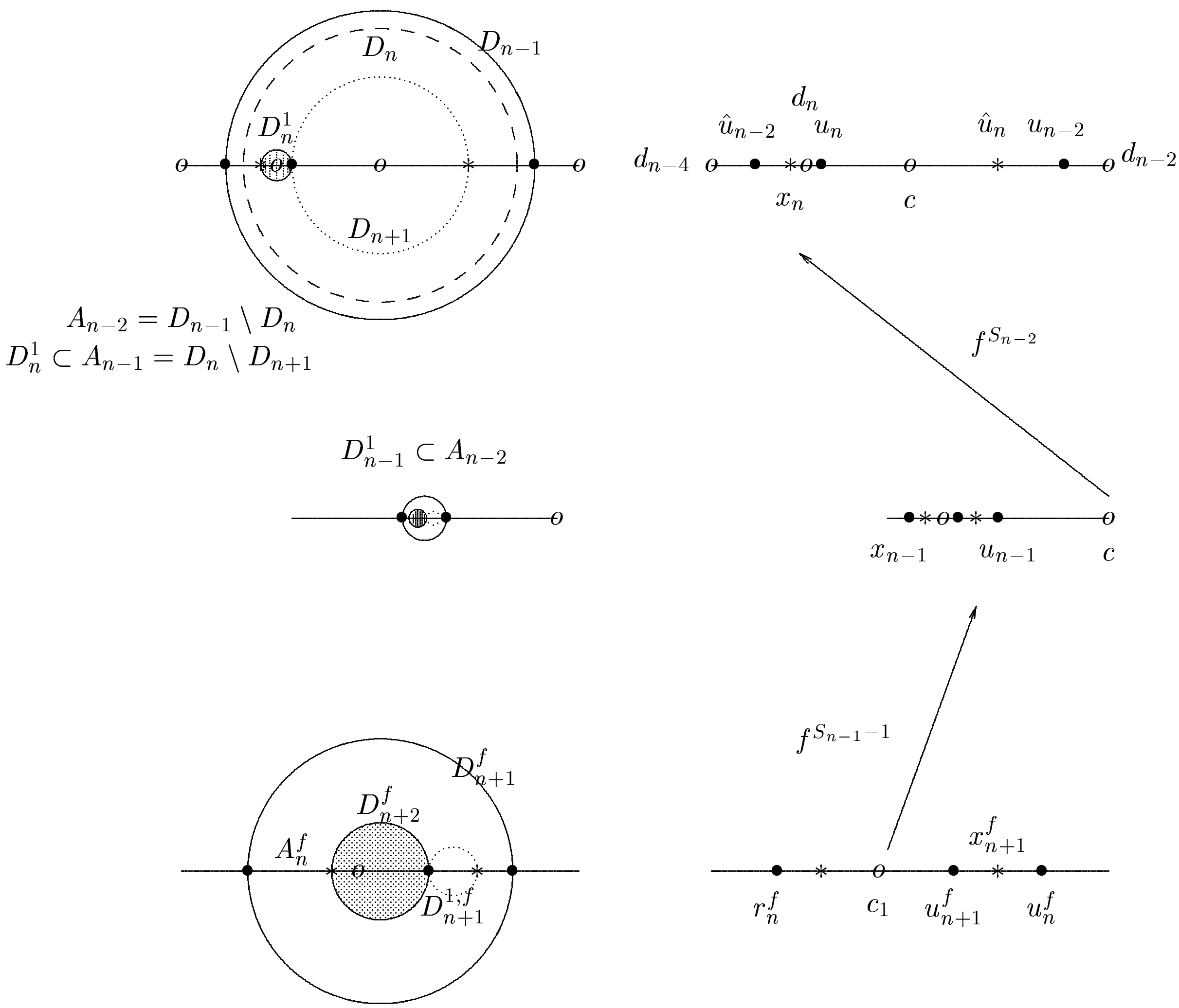

Figure 16: The construction of the new discs D_{n+1}. In the left bottom picture D^f_{n+1} is sketched (in reality the shape is not round but a smooth version of the star sets S_t from above). The 'dotted' disc in the annulus $D^f_{n+1} \setminus D^f_{n+2}$ is denoted D^f_{n+1}. The shaded region is mapped into the small shaded region inside D^1_{n-1} by $f^{S_{n-1}-1}$ and this one is mapped by $f^{S_{n-2}}$ to D^1_n. On the right the real pullback is drawn – compare this also with Figure 14, and on the left the corresponding complex pullback. If we pullback the bottom topological disc by f then we obtain the disc D_{n+1} from the top part of the figure. That this disc is inside D_n follows from Proposition 3.3. Note that $D^f_{n+1} \cap \mathbb{R} = [u^f_n, r^f_n]$, $D^{1,f}_{n+1} \cap \mathbb{R} = [x^f_{n+1}, u^f_{n+1}]$. We should emphasize that because of the real bounds, the real parts of the topological balls D^f_{n+2}, $D^{1,f}_{n+1}$ and D^f_{n+1} have more or less equal length. Since these balls are mapped to respectively D^1_n, D_{n+1} and D_{n-1} (and the first one of these is small compared to the other two) this shows that the conformal map $f^{S_n-1}\colon D^f_{n+1} \to D_{n-1}$ is very different from a Moebius transformation. We shall use this fact in a crucial way at the end of the paper to get estimates which are much better than those which would follow from Koebe. Moreover, D^1_{n-1} is in fact far from a real ball: it is squeezed at z_{n-2} as in Figure 15.

Proposition 8.1 *For each sufficiently large ℓ there exists $n_0(\ell)$ and for each $n \geq n_0(\ell)$ a closed topological disk $V_{n,\ell}$ containing $[z_{n+1}, a_{n+1}] \ni d_{n+2}$ such that*

- *f^{S_n} maps $V_{n,\ell}$ into the interior of $V_{n-2,\ell}$ for each $n \geq n_0(\ell)+1$;*
- *there exists a universal number $\alpha \in (\pi/2, \pi)$ such that $V_{n,\ell} \supset D([z_{n+1}, a_{n+1}]; \alpha)$ for each $n \geq n_0(\ell) - 1$;*
- *$V_{n,\ell} \subset D_*(u_n, \hat{u}_n)$ for $n \geq n_0(\ell) - 1$.*

Before proving this proposition let us show that it allows us to complete the proof of Theorem 8.1. That is, we shall show that D_{n+1} contains $V_{n,\ell}$ and therefore one of the tips of $S_\kappa(|z_{n+1}|, |y_{n+1}|)$. An additional argument will then show that D_{n+1} also contains the main piece of $S_\kappa(|z_{n+1}|, |y_{n+1}|)$ and also the other $\ell - 1$ tips.

Conclusion of the proof of Theorem 8.1: Let us first prove by induction that D_{n+1} contains $V_{n,\ell}$ for n sufficiently large. To do this, choose n_0 to be equal to the integer k_0 from the previous theorem. For $n = n_0 - 2, n_0 - 1$ the sets D_n are the Euclidean discs $D_*(u_{n-1}, \hat{u}_{n-1})$ which by the last property of the proposition therefore contain $V_{n+1,\ell}$. Hence the inductive statement holds for $n_0 - 2, n_0 - 1$ by the last property stated in the proposition. Now assume that $D_{n-1} \supset V_{n-2,\ell}$ for some $n \geq n_0$. Since by definition f^{S_n} maps D_{n+1} onto D_{n-1}, the second assertion of the proposition implies that $D_{n+1} \supset V_{n,\ell}$, proving the inductive step. Thus we obtain by induction that $D_{n+1} \supset V_{n,\ell}$ for all $n \geq n_0 - 2$. In other words, D_{n+1} at least contains one of the small tips of $S_\kappa(|z_{n+1}|, |a_{n+1}|)$.

Since f^{S_n} maps D_{n+1} onto $D_{n-1} \supset V_{n-2,\ell}$ we can get a better lower bound for D_{n+1}. In fact, we want to show that D_{n+1} also contains the 'big' starshaped piece of $S_\kappa(|z_{n+1}|, |a_{n+1}|)$. Indeed, consider the neighbourhood $V' = D([a_{n-1}, z_{n-1}]; \alpha) \ni d_n$ of $[a_{n-2}, z_{n-1}]$. Since $f^{S_n - 1}$ is diffeomorphism from some interval neighbourhood of c_1 to $[d_{n-2}, d_{n-4}]$ the corresponding inverse branch of f^{S_n-1} on V' has uniformly bounded distortion. It follows from Lemma 4.1 that there exists a universal number $\alpha' \in (\pi/2, \pi)$ such that the corresponding component of $f^{-(S_n-1)}(V')$ contains $D([c_1, z^f_{n+1}]; \alpha')$ (in fact, it contains $V'' = D([b^f_{n+1}, z_{n+1}]; \alpha')$.) Hence D_{n+1} contains $f^{-1}(V'')$. Note that the inverse of V'' under f contains the 'kite' component of

$$\{z \in \mathbb{C}\ ;\ |\arg(z)| < 2\pi/\ell\} \ \setminus (l_{\pm\kappa}) \tag{8.3}$$

containing $(0, z_{k+1})$. Here $l_{\pm\kappa}$ is the infinite line through $|z_{k+1}|$ with angle $\pm\kappa$ and where $\kappa > 0$ is some universal number. Moreover, as we have proved above, D_{n+1} contains $V_{n,\ell}$ and since D_{n+1} is invariant under rotation under $2\pi/\ell$ degrees, D_{n+1} also contains the rotated versions of these sets and also of the kites from (8.3). Combining this, shows that D_{n+1} contains a set of the form $S_\kappa(|z_{n+1}|, |y_{n+1}|)$.

Hence D^f_{n+1} contains at least a set of the form

$$D((c_1, z^f_{n+1}); \beta') \cup D((z^f_{n+1}, y^f_{n+1}); \beta')$$

where $\beta > 0$ is some universal number. Since D^1_{n-1} is inside $D_*(x_{n-1}, u_{n-1})$ and the map $f^{S_{n-1}-1}$ has uniformly bounded distortion on D^f_{n+1} (one has uniform Koebe space around $D_*(x_{n-1}, u_{n-1})$), it follows that D^1_{n-1} contains

$$\Big(D((d_{n-1}, f^{S_{n-1}}(z_{n+1})); \beta) \;\cup\; D((f^{S_{n-1}}(z_{n+1}), f^{S_{n-1}}(y_{n+1})); \beta)\Big).$$

□

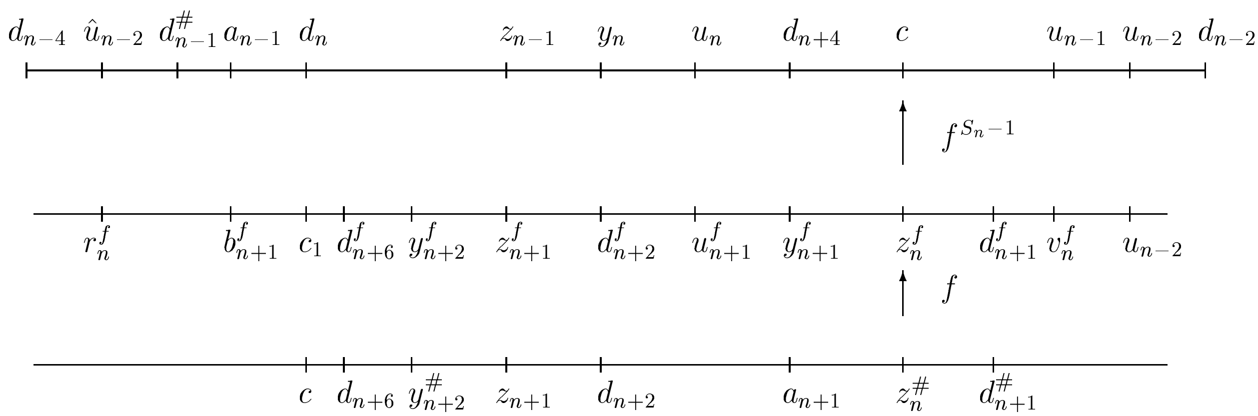

Figure 17: Points and their images under f^{S_n}. Here $p^{\#}$ is either p or $\hat{p}$. Note that $D_{n+1} \cap \mathbb{R} = (\hat{u}_n, u_n)$, $D_{n+2} \cap \mathbb{R} = (\hat{u}_{n+1}, u_{n+1})$. The region R_{n+1} is contained in D_{n+1} by the proof below the statement of Proposition 8.1. Note that $R_{n+1} \cap \mathbb{R} \supset [\hat{z}_n, z_n]$. In the complex plane, R_{n+1} will contain a diamont-shaped neighbourhood of (z_{n+1}, a_{n+1}) and also a neighbourhood of the half-open interval $[c, z_{n+1})$. Of course, the scales are quite different for ℓ large; for example $|\hat{u}_n - d_{n+6}|/|u_n - c|$ of order $1/\ell$.

In the remainder of this section we shall prove Proposition 8.1.

Proof of Proposition 8.1: First note that

$$Df^{S_{n+1}}(z_{n+1}) = Df^{S_{n-1}}(z_{n-1}) Df^{S_n}(z_{n+1})$$

and that $f^{S_{2k}}$ and $f^{S_{2k+1}}$ both converge (up to scaling and orientation) in the C^1 topology by the renormalization results from Section 6, see Theorem 6.1. It follows that the left and middle terms of the last inequality have the same limit (in fact, up to a minus sign as the maps $f^{S_{n+1}}$ and $f^{S_{n-1}}$ have opposite orientations but the points z_{n+1} and z_{n-1} are on opposite sides of c) and therefore that

$$Df^{S_n}(z_{n+1}) \to -1.$$

Notice that $[a_{n+1}, z_{n+1}] \ni d_{n+2}$ and $[a_{n-1}, z_{n-1}] \ni d_n$ lie on opposite sides of $c = 0$. Define h_n to be the (orientation reversing) affine map with

$$h_n(z_{n-1}) = z_{n+1} \text{ and with } h_n(a_{n-1}) = a_{n+1}.$$

By the renormalization result, h_n converges to a scalar contraction map (with contraction factor of the order $-(1 - C/\ell)$).

Before continuing with the proof of the proposition we shall investigate some properties of $\psi_n = h_n \circ f^{S_n}: [c, a_{n+1}] \to \mathbb{R}$. Note that ψ_n also depends on ℓ.

Lemma 8.1 *The map $\psi_n = h_n \circ f^{S_n}$ is an orientation reversing diffeomorphism from $[c, a_{n+1}]$ into $[c, a_{n+1}]$ with fixed point z_{n+1}. Moreover, for each ℓ there exists $n_0(\ell)$ such that for $n \geq n_0(\ell)$,*

- *There exists a sequence $C_n > 0$ converging to a positive constant as $n \to \infty$ such that*
$$D\psi_n(z_{n+1}) = -1 + C_n/\ell \ ;$$
- *the basin of z_{n+1} under the map ψ_n contains at least the interval $[c, a_{n+1}]$;*
- *all points in $[c, a_{n+1}]$ are mapped by a uniformly bounded number of iterates of ψ_n in a neighbourhood of z_{n+1} of size C/ℓ. More precisely, given $\epsilon > 0$ there exists $m(\epsilon)$ (not depending on n and ℓ) such that $|\psi_n^{m(\epsilon)}(p) - z_{n+1}| \leq \epsilon |z_{n+1} - a_{n+1}|$ for each $p \in [c, a_{n+1}]$.*

Note that the choice of $|z_{n+1} - a_{n+1}|$ in the last part of this lemma is more or less arbitrary: we could also take $|z_{n+1} - d_{n+2}|$ (or something else) because of the real bounds.

Proof of Lemma 8.1: The first assertion follows since f^{S_n} maps $[a_{n+1}, c]$ onto $[d_n, d_{n+4}]$ and because this last interval is 'well inside' $[a_{n-1}, c]$ in view of the real bounds from Theorem 3.1. Therefore ψ_n maps $[c, a_{n+1}]$ inside $[c, a_{n+1}]$. In fact, the image of this interval has length of order $1/\ell$. Since $Df^{S_n}(z_{n+1}) \to -1$, the orientation reversing diffeomorphism ψ_n has a contracting fixed point in z_{n+1}. Since $|a_{n-1} - c|/|a_{n+1} - c|$ is of order $1 + C/\ell$ because of the real bounds, this implies that $D\psi_n(z_{n+1}) = -1 + C_n/\ell$ where $C_n > 0$ is uniformly bounded and bounded away from zero for all large n and ℓ. Because of the renormalization result C_n converges to a positive constant as n tends to infinity.

Let us show that ψ_n attracts all points in $[c, a_{n+1}]$. Since ψ_n is orientation reversing, ψ_n would otherwise have a periodic two orbit in this interval. But the Schwarzian derivative of $f: \mathbb{R} \to \mathbb{R}$ is negative. Hence the Schwarzian derivative of ψ_n is also negative. Hence

$$[c, a_{n-1}] \ni z \mapsto |D\psi_n^2(z)|$$

has no positive local minima. It follows that ψ_n^2 cannot have three fixed points and that each point in $[c, a_{n+1}]$ is in the basin of z_{n+1}.

To prove the last assertion of the lemma, let us assume by contradiction that such a bound $m(\epsilon)$ did not exist. Then there exists a sequence of integers $n, \ell \to \infty$ and a sequence of points $p_n \in [c, a_{n+1}]$ with $|p_n - z_{n+1}| \geq \epsilon |z_{n+1} - a_{n+1}|$ (depending on n and ℓ) which are almost saddle-nodes:

$$|D\psi_n^2(p_n)| \to 1 \text{ and } \frac{|\psi_n^2(J_n)|}{|J_n|} \to 1$$

where $J_n = [z_{n+1}, p_n]$. (The last statement is illustrated in Figure 18 below.) To show this is impossible we consider the following cross-ratio:

$$B(\psi_n^2; J_n) := \frac{\left[\frac{|\psi_n^2(J_n)|}{|J_n|}\right]^2}{D\psi_n^2(z_{n+1}) D\psi_n^2(p_n)}. \tag{8.4}$$

Since $\psi_n(z_{n+1}) = z_{n+1}$ and $|D\psi_n(z_{n+1})| = 1 - C_n/\ell$, the previous properties of p_n imply that the previous cross-ratio gets arbitrarily close to one for some sequence of integers n and ℓ tending to infinity. However, since $\psi_n = h_n \circ f^{S_n}$,

$$B(\psi_n^2; J_n) = B(\psi_n; \psi_n(J_n)) \cdot B(h_n; f^{S_n}(J_n)) \cdot B(f^{S_n - 1}; f(J_n)) \cdot B(f; J_n).$$

Since h_n is affine (and therefore preserves cross-ratios), and since $Sf < 0$ (and f therefore expands cross-ratios), the previous cross-ratio is bounded from below by the cross-ratio

$$B(f; J_n) = \frac{\left[\frac{|f(J_n)|}{|J_n|}\right]^2}{Df(z_{n+1}) Df(p_n)}$$

where $J_n = [z_{n+1}, p_n] \subset \mathbb{R}_\pm$. Of course, this cross-ratio is at least one because $Sf < 0$, but in fact more holds: write $p_n = t_n z_{n+1}$ and $t_n = 1 + \kappa_n/\ell$. Then the last cross-ratio is equal to

$$\frac{\left[\frac{|t_n^\ell - 1|}{|t_n - 1|}\right]^2}{\ell^2 t_n^{\ell - 1}}.$$

From the choice of p_n (i.e., from $|p_n - z_{n+1}| \geq \epsilon |z_{n+1} - a_{n+1}|$) and from the real bounds one has $\kappa_n \geq C'\epsilon$ for some universal constant $C' > 0$. Hence the limit of the previous expression as $\ell \to \infty$ is equal to $\frac{(e^{\kappa_n} - 1)^2}{{\kappa_n}^2 e^{\kappa_n}}$ which is at least $1 + C_0 \kappa_n^2$ where $C_0 > 0$ is a uniform number. It follows that

$$\frac{\left[\frac{|f(J_n)|}{|J_n|}\right]^2}{Df(z_{n+1}) Df(p_n)} \geq 1 + C_0 \kappa_n^2 \geq 1 + C_0' \epsilon^2$$

where κ_n is as above. But as we had shown before (8.4) tends to one if the last property of the lemma does not hold, contradicting the last inequality. $\square$

Now we will continue with the proof of the proposition. For this we will analyze the map ψ_n considered as a map on a complex neighbourhood of z_{n+1}.

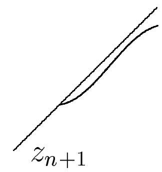

Figure 18: An almost saddle-node point implies that the cross-ratio expansion on some interval is close to one.

Lemma 8.2 *For each sufficiently large ℓ there exists $n_0(\ell)$ and a set $\hat{V}_{n,\ell}$ for $n \geq n_0(\ell)$. This set intersects the real line in a closed interval containing z_{n+1} and*

- *ψ_n maps $\hat{V}_{n,\ell}$ into the interior of $\hat{V}_{n,\ell}$ for each $n \geq n_0(\ell)$;*
- *there exists $\alpha \in (\pi/2, \pi)$, $\epsilon > 0$ and two real intervals J_1, J_2 of length $\geq \epsilon|z_{n+1} - a_{n+1}|$ with unique common point z_{n+1} such that*

$$\hat{V}_{n,\ell} \supset D(J_1, \alpha) \bigcup D(J_2, \alpha)$$

  provided $n \geq n_0(\ell)$;
- *$\hat{V}_{n,\ell}$ is contained in $D_*(\hat{V}_{n,\ell} \cap \mathbb{R})$.*

Proof of Lemma 8.2: Consider the affine map $sc_n : \mathbb{C} \to \mathbb{C}$ which sends $[z_{n+1}, a_{n+1}]$ to $[0, 1]$. Note that $|Dsc_n|$ is of order $\ell/|z_n - c|$ because of the real bounds and, in fact, $Dsc_n(c) \to -\infty$ as n or ℓ tends to infinity. Now define

$$\Psi_n = sc_n \circ \psi_n \circ sc_n^{-1}.$$

Then $\Psi_n(0) = 0$, $D\Psi_n(0) = (-1 + C/\ell)$ and by renormalization Ψ_n converges as $n \to \infty$ to some function Ψ. (Of course, Ψ might depend on ℓ.) Also $S\Psi_n < -\delta < 0$. Indeed,

$$S(g_1 \circ g_2) = Sg_1(g_2)(Dg_2)^2 + Sg_2. \tag{8.5}$$

Moreover, an explicit calculation gives that there exists $C > 0$ with

$$Sf(z) \leq -C\frac{\ell^2}{|z_n - c|^2}$$

and, therefore,

$$Sf^{S_n}(z) \leq -C\frac{\ell^2}{|z_n - c|^2}$$

for each $z \in [u_{n+4}, u_{n-4}]$ (or in fact, any similar interval). Since h_n is an affine map close to the identity, also

$$S\psi_n \leq -C\frac{\ell^2}{|z_n - c|^2}.$$

By the real bounds, the length of $[z_{n+1}, a_{n+1}]$ and therefore $|D\, sc_n^{-1}|$ is of the order of $|z_n - c|/\ell$. It follows by (8.5) and the definition of Ψ_n that there exists $C > 0$ such that

$$S\Psi_n(z) \le -C$$

for all z in, say, $[-1, 1]$.

By the renormalization result,

$$\Psi_n(z) = (-1 + \alpha'_n/\ell)z + \beta'_n z^2 + \gamma'_n z^3 + O(z^4) \tag{8.6}$$

where the coefficients $\alpha'_n, \beta'_n, \gamma'_n$ do depend on ℓ but for each fixed ℓ converge to constants as $n \to \infty$. In fact, α'_n converges to a positive constant.
Claim: the coefficients as well as the remainder term is bounded uniformly at the origin: $|O_n(z^4)| \le C|z^4|$ for some universal C for all z in, say, $sc_n([d_{n+6}, a_{n+1}]) \ni 0$ (notice that the distance of each of the endpoints of the interval $sc_n([d_{n+6}, a_{n+1}]) \ni 0$ to the origin is by the real bounds of order one).

To prove this claim, note that the coefficients and the remainder term can be estimated uniformly in n and ℓ because of the Taylor theorem. Indeed, this theorem shows that the remainder term in (8.6) is bounded by the fourth derivative of Ψ_n in the $sc_n([d_{n+6}, a_{n+1}])$. Now f^{S_n} has a univalent extension from a neighbourhood V of $[d_{n+6}, a_{n+1}]$ onto a (bounded) disc $\hat{W}$ containing the disc

$$W = D_*(f^{S_n}(d_{n+6}), f^{S_n}(a_{n+1})] = D_*(f^{S_n}(d_{n+6}), d_{n+4})$$

and with the same centre. Due to the real bounds, see Figures 16 and 17, we can make choose $\hat{W}$ so that its radius is a definite factor larger than the radius of W. Hence by the Koebe Lemma, the distance of the boundary of W is of the same order as the size of, say, the interval $[d_{n+6}, a_{n+1}]$. Hence, by applying the scaling map sc_n, all these discs and intervals get a size of unit order and the distance of the boundary of $\tilde{V} = sc_n(V)$ to $sc_n([d_{n+6}, a_{n+1}])$ is also of the order one. Moreover, $\tilde{V}$ is mapped by Ψ_n into a uniformly bounded disc. By the Cauchy integral formula,

$$\Psi_n^{(k)}(z) = \frac{k!}{2\pi i} \oint \frac{\Psi_n(t)}{(t-z)^{k+1}}\, dt.$$

Because the distance of $\tilde{V}$ to a point in $sc_n([d_{n+6}, a_{n+1}])$ is of order one, we can take a circle around z inside $\tilde{V}$ with a radius which is uniformly bounded from below. Since Ψ_n is uniformly bounded on $\tilde{V}$, it follows that $\Psi_n^{(i)}$, $i = 1, 2, 3, 4$ are bounded on $\tilde{V} \supset sc_n([d_{n+6}, a_{n+1}])$. This completes the proof of the claim.

Because of (8.6), by an explicit calculation,

$$\begin{aligned} \Psi_n^2(z) &= (-1 + \alpha'_n/\ell)^2 z + (\beta'_n \alpha'_n/\ell)z^2 - \gamma'_n z^3 + O(z^4) \\ &= (1 - \frac{\alpha_n}{\ell})z + \frac{\beta_n}{\ell} z^2 - \gamma_n z^3 + O(z^4) \end{aligned}$$

where $\alpha_n > 0$ and γ_n converge to positive constants and β_n also converges to a constant for each fixed ℓ as $n \to \infty$. (That γ_n converges to a *positive* constant as $n \to \infty$ provided

ℓ is large, follows from the fact that $S\Psi_n \le -\delta < 0$ and from the fact that $\frac{\alpha_n}{\ell} \to 0$ and $\frac{\beta_n}{\ell} \to 0$ as $n, \ell \to \infty$.)

This family of maps seem similar as $\ell \to \infty$ to the well-known map

$$z \mapsto z - z^3$$

which has a neutral fixed point at 0, having a basin containing two petals attached to 0. Now we will study our family of maps Ψ_n^2 in the same way as is commonly done for this 'limit' map and show that the basin of the fixed point is also not too small.

For this we will introduce new coordinates $w = 1/z^2$ and send the origin in a two-fold way to infinity. In these new w coordinates our map becomes

$$\Theta(w) = w + \frac{\tilde{\alpha}_n}{\ell} w - \frac{\tilde{\beta}_n}{\ell}\sqrt{w} + \tilde{\gamma}_n + O(\sqrt{w^{-1/2}}). \tag{8.7}$$

Here $\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_n$ are constants with $\tilde{\gamma}_n$ and $\tilde{\alpha}_n$ converging to positive limits (which are uniformly bounded and bounded away from zero for all ℓ large). It follows from (8.7) that if γ_1 is sufficiently large, then there exists δ_1 such that

$$|w| \ge \gamma_1 \ell \text{ implies } |\Theta(w)| \ge |w| + \delta_1 \tag{8.8}$$

for ℓ sufficiently large (here γ_1 and $\delta_1 > 0$ are universal constants). This holds because for such w, the second term on the right hand side of (8.7) dominates the last three terms since $\liminf \tilde{\alpha}_n > 0$ (provided γ_1 and ℓ, n are large). On the other hand, (8.7) also implies that if $|w| \le \gamma_1 \ell$ and $Re(w) = \gamma_2$ then there exists a universal constant $\delta_2 > 0$ such that

$$Re(\Theta(w)) \ge Re(w) + \delta_2 \tag{8.9}$$

provided γ_2 and ℓ are sufficiently large. Indeed, for $|w| \le \gamma_1 \ell$ and $Re(w) = \gamma_2 > 0$, for the second, third and last term on the right hand side of (8.7) one has

$$Re\left(\frac{\tilde{\alpha}_n}{\ell} w\right) > 0 \ , \ Re\left(\frac{\tilde{\beta}_n}{\ell}\sqrt{w}\right) \to 0 \text{ and } Re\left(O(w^{-1/2})\right) \le \text{Const}|\gamma_2|^{-1/2} \tag{8.10}$$

for n and ℓ large. Because the third term in (8.7) is uniformly bounded away from zero, one gets (8.9). Combining (8.8) and (8.9) one gets that the basin of ∞ contains points outside a big circle with radius ℓ (and centered at the origin) and also the points to the right of the vertical line l given by $Re(w) = \gamma_1$. Hence the basin of the origin under the map Ψ_n is not too small. In fact, it is the union of a disc of radius of the order $1/\sqrt{\ell}$ (a strongly related and crucial estimate will reappear in Section 10!!) and a rotated figure eight region (which does not depend on ℓ), see Figure 19. The figure eight is the inverse of the vertical line under the transformation $z \mapsto 1/z^2$. Let us call this set $\hat{V}$.

Now $sc_n^{-1}(\hat{V})$ satisfies by construction the first and second property announced in the lemma. The remaining task is show that $\hat{V} \subset D_*(\hat{V} \cap \mathbb{R})$. But this can be seen as follows. The inverse of the region to the right of the line l (so the region $Re(w) \geq \gamma_1$) under the map $z \mapsto 1/z$ is a symmetric disc through 0 and $1/\gamma_1$. Similarly, the inverse of the region outside the circle with radius $\gamma_1 \ell$ (centered at 0) under the map $z \mapsto 1/z$ is a disc centred in 0 and with radius $1/\ell$. Hence the union of the inverses under $z \mapsto 1/z^2$ of these regions is contained in $D_*(\hat{V} \cap \mathbb{R})$. □

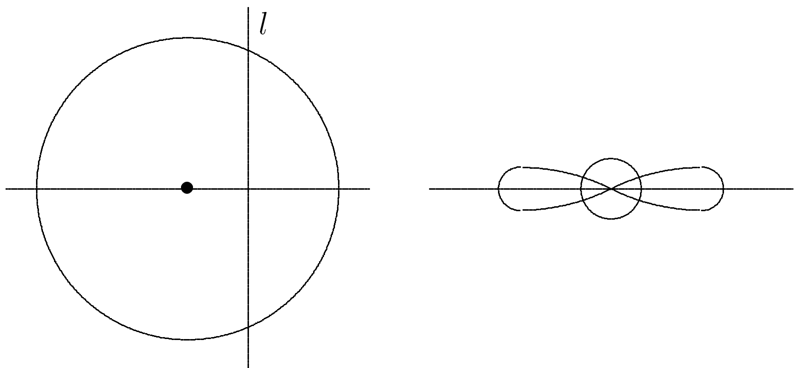

Figure 19: On the left, the big circle $|z| = \gamma_1 \ell$ and the line l are drawn. The the right, the images under the map $w = 1/z^2$ of these regions are drawn schematically.

Proof of Proposition 8.1: The previous lemma has still one shortcoming for our purpose: it is still not guaranteed that $\hat{V}_{n,\ell}$ contains the interval $[z_{n+1}, a_{n+1}]$. However, because of the last statement of Lemmma 8.1, there exists a universal integer m and a constant $\gamma_2 > 0$ in the proof of the previous lemma so that $\hat{V}_{n,\ell} \cap \mathbb{R} = \psi_n^m[z_{n+1}, a_{n+1}]$. Now the inverse of ψ_n^m has bounded distortion on $D_*([z_{n+1}, a_{n+1}])$ (it has a univalent extension on a definite neighbourhood of this disc because of the real bounds).

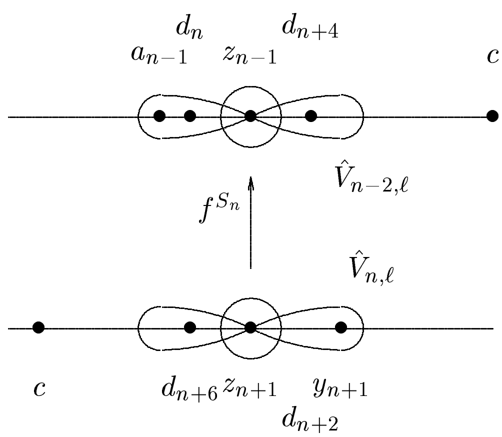

Figure 20: The region $V_{n,\ell}$ mapped into the region $V_{n-2,\ell}$ for large n.

Now $\hat{V}_{n,\ell}$ fits inside $D_*(\hat{V}_{n,\ell} \cap \mathbb{R})$. It follows by Proposition 4.1 that $V_{n,\ell} := \psi_n^{-m}(\hat{V}_{n,\ell})$ satisfies

- ψ_n maps $V_{n,\ell}$ into the interior of $V_{n,\ell}$ for large n;
- there exists $\alpha \in (\pi/2, \pi)$, such that
$$\hat{V}_{n,\ell} \supset D([z_{n+1}, a_{n+1}]; \alpha)$$
provided $n \geq n_0(\ell)$ (in fact, it also contains the other part of the figure eight, but this part is not needed here.)

Let us interpret this information on $\psi_n = h_n \circ f^{S_n}$ for the maps f^{S_n}. Because of the last property we get for each ℓ sufficiently large an integer $n_0(\ell)$ such that f^{S_n} maps $V_{n,\ell}$ into the interior of $V_{n-2,\ell}$ for $n \geq n_0(\ell)$. (Here we use the renormalization result that f^{S_n} converges up to scaling.) Thus we have proved Proposition 8.1. $\square$

9 An induced mapping with Markov properties

In this section we shall use the previous discs to define an induced map which has nice Markov properties. Let D_n be the discs from the previous section and define for $n \geq k_0$,
$$A_n = D_{n+1} \setminus D_{n+2}$$
and let A'_n to be the annulus A_n minus the disc $D^1_{n+1} \subset D_{n+1} \setminus D_{n+2}$. Then
$$f^{S_n - 1} \text{ maps } D^f_{n+1} \text{ diffeomorphically onto } D_{n-1}$$
and
$$f^{S_n - 1} \text{ maps } D^f_{n+2} \text{ diffeomorphically onto } D^1_n$$
So
$$f^{S_n} \text{ maps } A_n \text{ as an } \ell\text{-fold covering onto } D_{n-1} \setminus D^1_n.$$
This last set is equal to
$$(\cup_{i \geq n} A_i) \bigcup A'_{n-1} \bigcup A_{n-2}.$$
Moreover, f^{S_n} maps A'_n again onto $(\cup_{i \geq n} A_i) \cup A'_{n-1} \cup A_{n-2}$ but now the map only $(\ell - 1)$-covers $\cup_{i \geq n} A_i$ (because the missing disc would also have been mapped diffeomorphically onto $\cup_{i \geq n} A_i$) while it is still ℓ-covers the remaining part of the target.

To formalize all this we define X_n to be the disjoint union of A_n and A'_n and $X = \cup X_n$. Define
$$F : \cup X_n \to \cup X_n$$
by
$$F|A_n = f^{S_n} \text{ and } F|A'_n = f^{S_n}.$$

for $n \geq k_0$ and $F|(A_{k_0-1} \cup A_{k_0-2}) = id$. Moreover, let $\mathcal{A}$ be the partition of X into sets A_n and A'_n. Then $F\colon X \to X$ is Markov map with respect to this partition. It sends each element of the partition $\mathcal{A}$ as a covering map onto a union of elements. Now we will iterate F. So define the partition

$$\mathcal{A}_{n+1} = \mathcal{A} \vee F^{-1}(\mathcal{A}) \vee \ldots \vee F^{-n}(\mathcal{A}).$$

Then $\mathcal{A}_1 = \mathcal{A}$. If A is an element from $\mathcal{A}_2$ then $B = F(A) \in \mathcal{A}_1$ and $F\colon A \to B$ is a *covering map*:

- $F\colon A \to B$ is a local homeomorphism;
- there exists k such that for each $y \in B$,

$$\#(F|A)^{-1}(y) = k.$$

Indeed, assume $A \subset A_r$ or $A \subset A'_r$. Then

$$k = \begin{cases} \ell & \text{if } B = A_{r-2}, \\ 0 & \text{if } B = A'_{r-2}, \\ \ell & \text{if } B = A'_{r-1}, \\ 0 & \text{if } B = A_{r-1}, \\ 1 & \text{if } B = A_{r+j}, \\ 0 & \text{if } B = A'_{r+j}, \end{cases}$$

where $j \geq 0$. Of course, there are ℓ components $A \in \mathcal{A}_2$ inside $(F|A_r)^{-1}(A_{r+j})$ whereas there are only $\ell - 1$ components $A \in \mathcal{A}_2$ in $(F|A'_r)^{-1}(A_{r+j})$. However, this will play no role in the future. In fact,

$$F\colon A_r \to (\cup_{i\geq r} A_i) \bigcup A'_{r-1} \bigcup A_{r-2}$$

is also a covering map, while

$$F\colon A'_r \to (\cup_{i\geq r} A_i) \bigcup A'_{r-1} \bigcup A_{r-2}$$

is not. This is because in the latter case, points from $(\cup_{i\geq r} A_i)$ are covered precisely $\ell - 1$ times (due to the missing disc in A'_r) whereas each point from $A'_{r-1} \bigcup A_{r-2}$ is covered ℓ times.

It follows from this that for each component A from an element of $\mathcal{A}_{k+1}$, F^k is also a covering map from A to one element of $\mathcal{A}$. If $F^k(A) = A_r$ or $F^k(A) = A'_r$ then F^{k+1} maps A onto

$$(\cup_{i\geq r} A_i) \bigcup A'_{r-1} \bigcup A_{r-2}.$$

In fact, this map need not be a covering map when $F^k(A) = A'_r$ because – as we pointed out above – the map $F|A'_r$ is not.

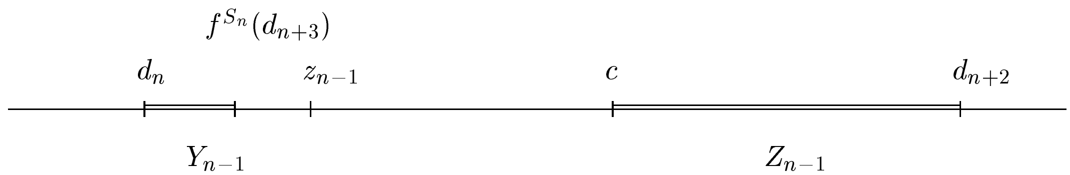

Figure 21: The intervals Y_{n-1} and Z_{n-1}. Note that $Y_{n+1} \subset Z_{n-1}$, see Figure 14. Whether the intervals Y_n and Y_{n+1} lie on the same side of c is determined by the parity of n.

We will have to analyze the distortion of iterates of $F^{k+1}|A$. For this we want to apply the Koebe Lemma and so we would like to see how much one can extend F. Therefore we associate to A_n and A'_n the following slit-regions. Define

$$Z_n = [d_{n+3}, c] \text{ and } Y_n = [d_{n+1}, f^{S_{n+4}}(d_{n+1})] = [d_{n+1}, f^{n+1}(d_{n+4}],$$

(so $Y_n \subset D^1_{n+1} \subset A_n$ and $Z_n \subset \cup_{i \geq n+1} A_i$). Note that

$$f^{S_n}(Z_n) = [d_n, f^{S_{n+3}}(d_n)] = Y_{n-1} \text{ and } f^{S_n}(Y_n) = [d_{n+2}, y_{n+2}] \subset [d_{n+2}, c] = Z_{n-1}.$$

Moreover, let

$$Slit_n = \mathbb{C}_{(d_n, \hat{d}_n)} \setminus Z_n$$

and

$$Slit'_n = \mathbb{C}_{(d_n, \hat{d}_n)} \setminus (Y_n \cup Z_n)$$

and

$$Slit^*_n = \mathbb{C}_{(d_n, \hat{d}_n)} \setminus Y_{n+1}$$

Note that

$$A_n \subset Slit_n \ , \ A'_n \subset Slit'_n$$

and

$$\cup_{i \geq n} A_i \cup A'_{n-1} \cup A_{n-2} \subset Slit^*_{n-2}.$$

Let us study the extensions of $F|A_n$ and $F|A'_n$.

Theorem 9.1 *If $F(A_n) \supset A_m$ then there exists a region E_n with*

- *$E_n \subset Slit_n$;*
- *$F(E_n) = Slit_m$ and*
- *$(F|A_n)^{-1}(A_m) \supset E_n$.*

The same statement holds if we replace A_n by A'_n or A_m by A'_m provided we then replace $Slit_n$ by $Slit'_n$ respectively $Slit_m$ by $Slit'_m$.

Moreover, if $H \in \mathcal{A}_{k+1}$ is contained in A_n then there exists r such that $F^k(H) = A_r$ or $F^k(H) = A'_r$ and there exists a region E such that

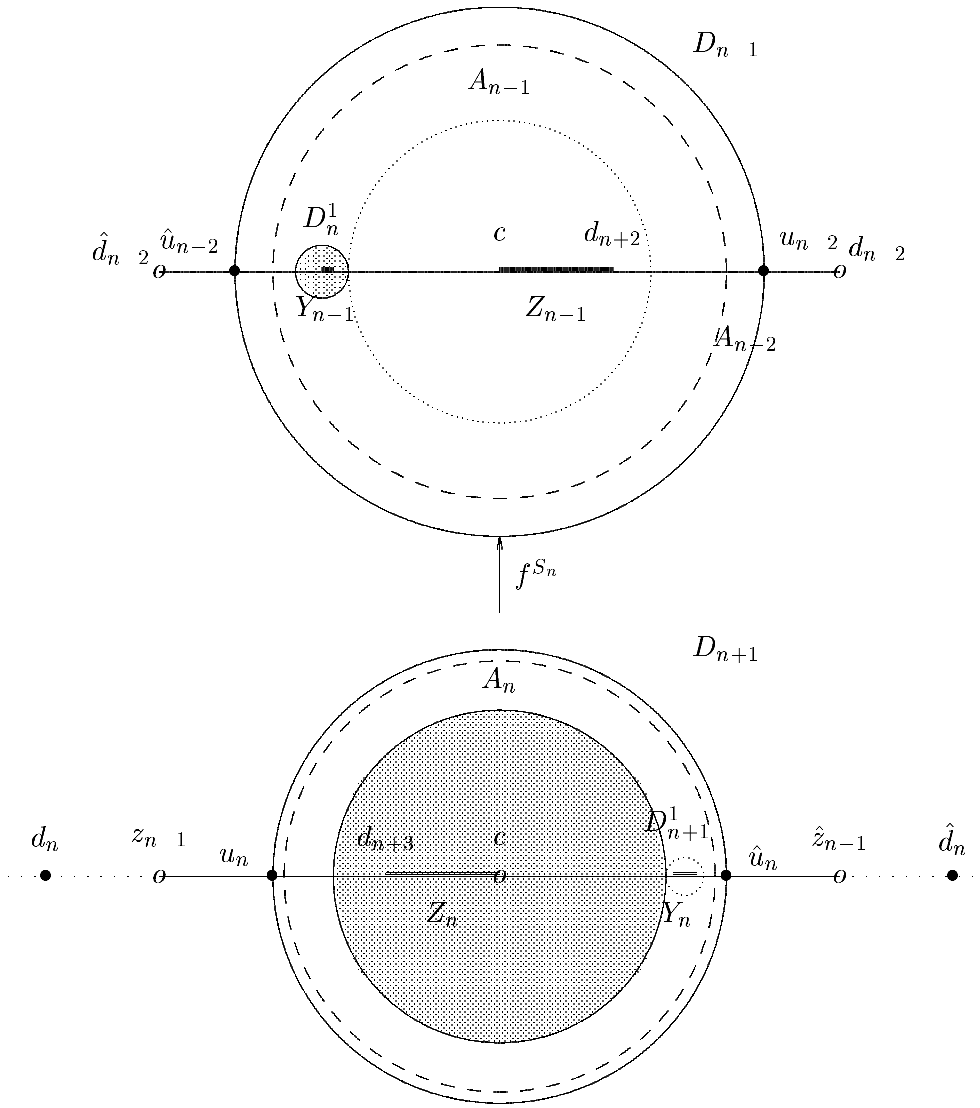

Figure 22: The image under $f^{S_n} = F$ of the annulus A_n on the bottom is an 'asymetric' annulus with a 'small' disc D_n^1 in the annulus $A_{n-1} = D_n \setminus D_{n+1}$ (bounded by a dotted and dashed curve) removed from the disc $D_{n-1} = \cup_{i \geq n-2} A_i$. So $F(A_n) = \cup_{i \geq n} A_i \cup A'_{n-1} \cup A_{n-2}$. The slits $Z_{n-1} = [d_{n+2}, c]$ and $Y_{n-1} = [d_n, f^{S_{n+3}}(d_n)] \subset D_n^1 \subset A_{n-1}$ are drawn with thicker lines.

- *$H \subset E \subset Slit_n$*
- *F^{k+1} is a covering from H onto*

$$\cup_{i \geq r} A_i \cup A'_{r-1} \cup A_{r-2} \subset Slit^*_{r-2}$$

and this map covers E onto $Slit^*_{r-2}$.

- *$F^{k+1}(E) = Slit^*_{r-2}$.*

Proof: f^{S_n-1} maps $[z^f_{n-1}, t^f_n]$ diffeomorphically to $[d_{n-2}, d_{n-4}]$. Therefore, f^{S_n-1} maps some region in

$$\mathbb{C}_{[z^f_{n-1}, t^f_n]}$$

onto

$$\mathbb{C}_{[d_{n-2}, \hat{d}_{n-4}]}.$$

Since $Y^f_n, Z^f_n \subset [z^f_{n-1}, t^f_n]$ and since

$$f^{S_n}(Y_n) \subset Z_{n-1} \text{ and } f^{S_n}(Z_n) = Y_{n-1}$$

it follows that f^{S_n-1} maps some region in

$$\mathbb{C}_{[z^f_{n-1}, t^f_n]} \setminus Y^f_n$$

onto

$$\mathbb{C}_{[d_{n-2}, d_{n-4}]} \setminus Z_{n-1}.$$

Similarly, f^{S_n-1} maps some region in

$$\mathbb{C}_{[z^f_{n-1}, t^f_n]} \setminus (Y^f_n \cup Z^f_n)$$

onto

$$\mathbb{C}_{[d_{n-2}, d_{n-4}]} \setminus (Z_{n-1} \cup Y_{n-1}).$$

It follows that f^{S_n} maps some region in

$$\mathbb{C}_{[z_{n-1}, \hat{z}_{n-1}]} \setminus Y_n$$

as an ℓ-cover onto $Slit_{n-1}$. Because $f^{-1}(f(Z_n))$ consists of ℓ lines through the critical point, this region has a slit in all of these ℓ lines. Similarly, f^{S_n} maps some region in

$$\mathbb{C}_{[z_{n-1}, \hat{z}_{n-1}]} \setminus (Y_n \cup Z_n)$$

onto

$$Slit'_{n-1} = \mathbb{C}_{[d_{n-2}, \hat{d}_{n-2}]} \setminus (Z_{n-1} \cup Y_{n-1}).$$

Since $[d_n, \hat{d}_n] \supset [z_{n-1}, \hat{z}_{n-1}] \supset [u_n, \hat{u}_n]$ the first part of the theorem follows.

The last part of the theorem holds because if $F^k(A) = A_r$ or $F^k(A) = A'_r$ then F^{k+1} covers

$$\cup_{i\geq r} A_i \cup A'_{r-1} \cup A_{r-2} \subset Slit^*_{r-2}.$$

Therefore the last assertion follows immediately by induction from the first part of the theorem. □

Theorem 9.2 *There exists a universal constant $C > 0$ such that if we define Y_n and Z_n as above (so $Y_n \subset D^1_{n+1} \subset A_n$), then*

$$dist\Big((A_n \setminus D^1_{n+1}), Y_n\Big) \geq C|A_n|$$

and

$$dist\,(A_n, Z_n) \geq C|A_n|$$

provided ℓ is sufficiently large and $n \geq n_0(\ell)$ is sufficiently large.

Proof: As we have seen in Theorem 8.1, f^{S_n} maps D^1_{n+1} as a univalent map onto D_n. Moreover, $f^{S_n}(Y_n) = [d_{n+2}, y_{n+2}]$. Now because of the last part of Theorem 8.1 D_n contains a neighbourhood of $[d_{n+2}, y_{n+2}]$ with thickness is comparable to the size of this interval: there exists $C > 0$ such that each point in a neighbourhood of Y_n of the form

$$N_{C|d_{n+2}-y_{n+2}|} := \{x \ ; \ d\,(x, [d_{n+2}, y_{n+2}]) \leq C \cdot |d_{n+2} - y_{n+2}|\}$$

is contained in D_n. In fact, this proves the last assertion of the theorem (but with n replaced by $n-1$). Moreover, the Koebe Lemma this gives that f^{S_n} has uniformly bounded distortion on the subset of D^1_{n+1} (containing Y_n) which is mapped diffeomorphically onto the slightly thinner set

$$N_{(C/2)|d_{n+2}-y_{n+2}|}.$$

Since $f^{S_n}(Y_n) = [d_{n+2}, y_{n+2}]$, it follows that Y_n also has a reasonably thick neighbourhood inside D^1_n. □

Let us now combine the results of this section in the following way. Define A^i_n be the part of A_n which is between the rays l_i and l_{i+1} where l_i is given by $\mathbb{R}^+ \ni t \mapsto te^{2\pi i/\ell} \in \mathbb{C}$. As explained below Theorem 8.1, A^i_n consists of one component and f maps A^i_n diffeomorphically onto $f(A_n)$ (appart from the image of l_i).

Theorem 9.3 *Take $A \in \mathcal{A}_{k+1}$. Then there exists r such that F^k maps A diffeomorphically to $\tilde{A}_r$ where $\tilde{A}_r$ is either A_r or to A'_r. Moreover, let $A^i = F^{-k}(A^i_r)$. Then*

$$F^k \colon A^i \to A^i_r$$

has uniformly bounded distortion.

Proof: The diameter of A^i_r is comparable to the distance of A^i_r to the nearest critical value of F^k. Therefore the result follows from Koebe. □

10 An asymptotic expression for the real induced map

In this section we shall give a very good estimate for the diffeomorphism $f^{S_n-1}|[r_n^f, u_n^f] \to [\hat{u}_{n-2}, u_{n-2}] = \cup_{i\geq -2}(A_{n+i} \cap \mathbb{R})$. Here

$$A_{n+i} \cap \mathbb{R} = [u_{n+i}, \hat{u}_{n+i}] \setminus [u_{n+i+1}, \hat{u}_{n+i+1}].$$

Theorem 10.1 *There exists a constant C and ℓ_0 such that for each $\ell \geq \ell_0$ and each $n \geq n_0(\ell)$, the following estimates for the derivative of the diffeomorphisms $f^{S_n-1}|[r_n^f, u_n^f] \to [\hat{u}_{n-2}, u_{n-2}]$ and $f^{S_{n-2}}|[x_{n-1}, u_{n-1}] \to [\hat{u}_{n-2}, u_{n-2}]$ hold. Let $i \geq 2$ and take $x \in [r_n^f, u_n^f]$ and $y \in [x_{n-1}, u_{n-1}]$ so that $f^{S_n-1}(x) \in A_{n+i}$ and $f^{S_{n-2}}(y) \in A_{n+i}$. Then*

$$|Df^{S_n-1}(x)| \leq C\frac{i^{3/2}}{\ell}\frac{|u_{n-2} - \hat{u}_{n-2}|}{|r_n^f - u_n^f|} \tag{10.1}$$

respectively

$$|Df^{S_{n-2}}(y)| \leq C\frac{i^{3/2}}{\ell}\frac{|u_{n-2} - \hat{u}_{n-2}|}{|x_{n-1} - u_{n-1}|}. \tag{10.2}$$

The remainder of this section shall be occupied with the proof of this theorem.

First we should point out that since $\cup_{i\geq n}A_i$ has a definite amount of Koebe space around $\cup_{i\geq n+\ell}A_i$, it follows that the maps from the previous theorem have uniformly bounded distortion on the piece that maps into $A_{n+\ell}$. Hence we could replace the $i^{3/2}/\ell$ term in (10.1) and (10.2) by $\min(i^{3/2}/\ell, \sqrt{\ell})$.

Secondly, we should emphasize that these estimates are far better than would obtain from a Koebe estimate: using Koebe we would have to replace $i^{3/2}$ by i^2 (for $i \leq \ell$). As we shall show below these 'good' estimates hold because – by the real bounds – the maps $f^{S_{n-2}}$ are far away from Moebius transformations. We shall come back to this issue below. In fact, even though we shall prove that (10.2) holds for $i = \ell$ by a simple cross-ratio argument, to show that (10.2) holds for all $i \geq 2$ we shall use renormalization in a crucial way.

Thirdly, we should note that the map $f^{S_{n-1}-1}$ restricted to $[r_n^f, u_n^f]$ has uniformly bounded distortion because the image of this interval is equal to $[x_{n-1}, u_{n-1}]$ and because this map extends diffeomorphically to an interval with image $[d_{n-5}, c]$. So by the real Koebe Principle, see Proposition 3.1, and the real bounds, see Theorem 3.1, it follows that the distortion of $f^{S_{n-1}-1}|[r_n^f, u_n^f]$ is uniformly bounded (for all $\ell \geq 4$ and all n sufficiently large). So (10.1) follows from (10.2).

Hence we shall be interested in the distortion of the map $f^{S_{n-2}}\colon [x_{n-1}, u_{n-1}] \to [\hat{u}_{n-2}, u_{n-2}]$. This map only extends diffeomorphically to an interval whose image

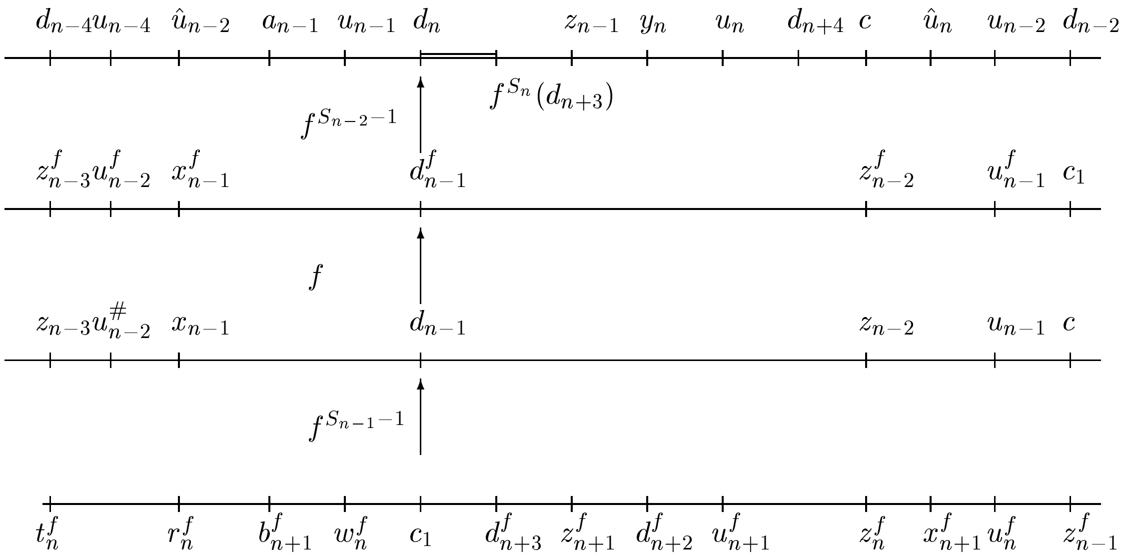

Figure 23: Points and their images under $f^{S_{n-1}-1}$ and f^{S_n-2}. The interval near c_1 shown in the second lowest line should be thought of as ordered in the opposite direction (since c_1 is the mimumum of f). Here $u^{\#}_{n-2}$ stands for either u_{n-2} or $\hat{u}_{n-2}$ depending on whether n is even or odd.

extends on each side a fraction $1/\ell$. So from the real Koebe Principle we can deduce that the distortion of the map is bounded by ℓ^2. In this section we shall improve this bound.

Let $T_{n-1} = [x_{n-1}, u_{n-1}]$, then $f^{S_{n-2}}(T_{n-1}) = [\hat{u}_{n-2}, u_{n-2}]$. We first show that

$$|Df^{S_{n-2}}(u_{n-1})| = \frac{C_n}{\ell} \frac{|f^{S_{n-2}}(T_{n-1})|}{|T_{n-1}|}, \tag{10.3}$$

$$|Df^{S_{n-2}}(x_{n-1})| = \frac{C'_n}{\ell} \frac{|f^{S_{n-2}}(T_{n-1})|}{|T_{n-1}|} \tag{10.4}$$

where $C_n, C'_n > 0$ are uniformly bounded and bounded away from zero. This follows easily from the real Koebe Principle and the real bounds: all the intervals connecting the points $u^{\#}_{n-2}$, x_{n-1}, d_{n-1}, z_{n-2} and u_{n-1} are of the same order (and c is 'far away'). But the interval $[u^{\#}_{n-2}, x_{n-1}]$ is mapped diffeomorphically to $[u_{n-4}, \hat{u}_{n-2}]$ whose size is order $\frac{1}{\ell}$ times $|f^{S_{n-2}}(T)|$. Because there is Koebe space around $[u_{n-4}, \hat{u}_{n-2}]$, formula (10.3) follows. Similarly, (10.4) follows by considering the interval $[f^{S_{n-1}-1}(x^f_{n+1}), u_{n-1}]$ (because the size of this interval is also of the same order as $|T|$ and its image under $f^{S_{n-2}}$ has a size of the order $(1/\ell)|f^{S_{n-2}}(T_{n-1})|$). Since one has again Koebe space around this interval, (10.4) follows.

Since the Koebe space around the interval $f^{S_{n-2}}(T_{n-1})$ is only of order $1/\ell$, we get from the real Koebe Principle that

$$|Df^{S_{n-2}}(x)| \le C\ell^2 |Df^{S_{n-2}}(u_{n-1})| \frac{|f^{S_{n-2}}(T_{n-1})|}{|T_{n-1}|} \le C\ell \frac{|f^{S_{n-2}}(T_{n-1})|}{|T_{n-1}|}.$$

In this section we shall show that this estimate is far from optimal. First we shall give this estimate in a special case using cross-ratios only; this proof does not require renormalization. Since it is not strictly needed in this paper, the reader can skip the next subsection.

10.1 A partial result using cross-ratios

In this subsection we shall show

Proposition 10.1

$$|Df^{S_{n-2}}(z_{n-2})| \leq C\sqrt{\ell}\frac{|f^{S_{n-2}}(T_{n-1})|}{|T_{n-1}|}.$$

In fact, in Proposition 10.3, we shall also obtain estimates for $|Df^{S_{n-2}}(\gamma)|$ when γ is some arbitrary point in T. Those estimates imply Proposition 10.1 but unlike the proof of Proposition 10.1 are based on renormalization results. In fact, we feel that the proof of Proposition 10.1 should have much wider applications: philosophically speaking the proof shows that if a map is not too close to a Moebius transformation then one has improved Koebe estimates!! For this reason we have included Proposition 10.1.

In the proof of Proposition 10.1 we use the following cross-ratio operator: if $J = [\alpha, \beta]$ is an interval and $g\colon J \to \mathbb{R}$ a diffeomorphism, define

$$A(g, J) = \frac{[\frac{|g(J)|}{|J|}]^2}{|Dg(\alpha)||Dg(\beta)|}.$$

Proof of Propositon 10.1: Consider $T' = [x_{n-1}, z_{n-2}]$ and $T'' = [z_{n-2}, u_{n-1}]$. These intervals have one point in common and their union is equal to $T = T_{n-1}$. Because of the next lemma and because $A(g_1 \circ g_2, J) = A(g_1, g_2(J)) \cdot A(g_2, J)$ one has

$$A(f^{S_{n-2}}, T')A(f^{S_{n-2}}, T'') \geq \sqrt{A(f^{S_{n-2}}, T)}.$$

By (10.3) and (10.4), $A(f^{S_{n-2}}, T)$ is of order ℓ^2. Hence

$$A(f^{S_{n-2}}, T')A(f^{S_{n-2}}, T'') \geq C\ell. \tag{10.5}$$

Now $|f^{S_{n-2}}(T_i)|/|T_i|$, $i = 1, 2$ are both of the same order as $|f^{S_{n-2}}(T)|/|T|$ because by the real bounds T_i, T and also $|f^{S_{n-2}}(T_i)|, |f^{S_{n-2}}(T)|$ are of the same order. Using this and (10.3), (10.4),

$$A(f^{S_{n-2}}, T')A(f^{S_{n-2}}, T'')$$

is of the same order as

$$\left[\ell\frac{\frac{|f^{S_{n-2}}(T)|}{|T|}}{|Df^{S_{n-2}}(z_{n-2})|}\right]^2.$$

Using (10.5) this gives that

$$|Df^{S_{n-2}}(z_{n-2})| \leq \sqrt{\ell}\frac{|f^{S_{n-2}}(T)|}{|T|}.$$

□

In the proof of the previous proposition we used the following lemma:

Lemma 10.1 *Assume that $f(z) = z^\ell + c_1$. Then there exists ℓ_0 such that for each $\ell \geq \ell_0$ and each two intervals T', T'' in one component of $\mathbb{R} \setminus \{0\}$ with a unique common endpoint,*

$$A(f,T')A(f,T'') \geq \sqrt{A(f,T' \cup T'')} \tag{10.6}$$

Proof of the lemma: Write $T' = [\alpha, x]$, $T'' = [x, \beta]$ and assume for simplicity that $\alpha < x < \beta$. First we **claim** that the left hand side of (10.6) is minimal if $x = \sqrt{\alpha\beta}$. To prove this, first notice that this expression is invariant if we multiply all the points by the same factor. Hence we may assume that $\alpha = 1$ and write $x = e^{\tau b}$ and $\beta = e^b$ where $\tau \in (0,1)$ and $b > 0$. With this notation the left hand side of (10.6) becomes

$$\frac{(\frac{e^{\ell b\tau}-1}{e^{b\tau}-1})^2(\frac{e^{\ell b}-e^{\ell b\tau}}{e^{b\tau}-e^{b\tau}})^2}{\ell \cdot \ell^2 e^{2(\ell-1)b\tau} \cdot \ell \cdot e^{(\ell-1)b}} = \frac{(\frac{e^{\ell b\tau}-1}{e^{b\tau}-1})^2(\frac{e^{\ell b(1-\tau)}-1}{e^{b(1-\tau)}-1)^2}}{\ell^4 e^{(\ell-1)b}}.$$

Now the denominator of this term,

$$(\frac{e^{\ell b\tau}-1}{e^{b\tau}-1})^2(\frac{e^{\ell b(1-\tau)}-1}{e^{b(1-\tau)}-1})^2$$

is equal to

$$G(\tau) := (1+e^{b\tau}+\ldots+e^{(\ell-1)b\tau})(1+e^{b(1-\tau)}+\ldots+e^{(\ell-1)b(1-\tau)}) = \text{ const } + P(e^{b\tau}) + P(e^{b(1-\tau)})$$

where P is a polynomial with positive constants. Hence

$$G'(\tau) = P'(e^{b\tau}) \cdot e^{b\tau} - P'(e^{b(1-\tau)}) \cdot e^{b(1-\tau)}.$$

Since P has positive coefficients, $\tau \mapsto P'(e^{b\tau}) \cdot e^{b\tau} := G_1(\tau)$ is increasing, $G_1(\tau) = G_1(1-\tau)$ holds only if $\tau = 1/2$. Since $G'(\tau) = 0$ is equivalent to $G_1(\tau) = G_1(1-\tau)$ the first claim holds.

The previous claim implies that it suffices to consider the situation that $\alpha = 1$, $x = e^a$ and $\beta = e^{2a}$. So we need to prove that

$$A(f,T')A(f,T'') = \frac{(\frac{e^{\ell a}-1}{e^a-1})^2(\frac{e^{\ell 2a}-e^{\ell a}}{e^{2a}-e^a})^2}{\ell \cdot \ell^2 e^{2(\ell-1)a} \cdot \ell \cdot e^{(\ell-1)2a}} = \frac{(\frac{e^{\ell a}-1}{e^a-1})^4}{\ell^4 e^{2(\ell-1)a}}$$

is greater or equal than

$$\sqrt{A(f,T)} = \frac{\frac{e^{2\ell a}-1}{e^{2a}-1}}{\ell e^{(\ell-1)a}}.$$

Hence, writing

$$g_1(t) = (e^t - 1)^3 \text{ and } g_2(t) = (e^t + 1)e^t t^3/2,$$

and using $e^{2t} - 1 = (a^t - 1)(e^t + 1)$ the required estimate (10.6) is equivalent to

$$g_1(\ell a)g_2(a) \geq g_2(\ell a)g_1(a). \tag{10.7}$$

To show that there exists ℓ_0 such that this inequality holds for each $a > 0$ and each $\ell \geq \ell_0$ we proceed as follows. First notice that

$$g_1(t) - g_2(t) = (e^t - 1)^3 - (e^t + 1)e^t t^3/2 = (e^{3t} - 3e^{2t} + 3e^t - 1) - (e^{2t} + e^t)t^3/2. \tag{10.8}$$

The coeffient corresponding to the t^n-th term in the Taylor expansion of (10.8) is equal to

$$\frac{1}{n!}[(3^n - 3\,2^n + 3) - \frac{n(n-1)(n-2)}{2}(2^{n-3} + 1)].$$

So these coefficients are zero for $n = 2, 3, 4, 5, 6$ and strictly positive for $n \geq 7$. In fact, the coefficient corresponding to t^7 is equal to $1/240$ and so we get

$$g_1(t) - g_2(t) \geq \frac{1}{240}t^7$$

for $t \geq 0$. Hence

$$\frac{g_1(t)}{g_2(t)} - 1 \geq \frac{\frac{1}{240}t^7}{\frac{(e^t+1)e^t}{2}t^3} \geq \frac{1}{120}\frac{t^4}{e^{2t} + e^t}. \tag{10.9}$$

Moreover, there exists t_0 such that

$$\frac{g_1(t)}{g_2(t)} - 1 = \frac{(e^t - 1)^3 - t^3 e^{2t}/2 - t^3 e^t/2}{\frac{(e^t+1)e^t}{2}t^3} \geq e^{(1/2)t}t^4 \tag{10.10}$$

for all $t \geq t_0$. Combining (10.9) and (10.10) we get that

$$\frac{g_1(t)}{g_2(t)} - 1 \geq \text{ const } \cdot e^{(1/2)t}t^4 \tag{10.11}$$

for all $t \geq 0$ where const is a positive constant. Similarly one gets that

$$\frac{g_1(t)}{g_2(t)} - 1 \leq \text{ const' } \cdot e^{2t}t^4 \tag{10.12}$$

for all $t \geq 0$ where const' is a finite constant. Applying (10.11) and (10.12) it follows that

$$\frac{g_1(\ell a)}{g_2(\ell a)}\frac{g_2(a)}{g_1(a)} \geq \frac{1 + \text{ const } \cdot (\ell a)^4 e^{(1/2)\ell a}}{1 + \text{ const' } \cdot a^4 e^{2a}} \geq (1 + \text{ const'' } a^4) \geq 1$$

for each $a \geq 0$ provided ℓ is sufficiently large. This concludes the proof of (10.7) and the proof of the lemma. $\square$

We would also like to remark that (10.6) and (10.7) also hold for the quadratic case $\ell = 2$. Indeed, in this case (10.7) is equivalent to showing that for all $a \geq 0$,

$$e^{4a} + 6e^{2a} + 1 \geq 4e^{3a} + 4e^{a}.$$

So it suffices to show that the coefficients of the power series of the left hand side dominates those of the right hand side. This means that we have to show that

$$4^n + 6\,2^n \geq 4\,3^n + 4$$

for each $n \geq 1$. This is readily checked. Presumably, this lemma holds for all $\ell \geq 2$.

Moreover, the same ideas also work for C^2 maps because in this case there exists a universal constant such that

$$A(f,T')A(f,T'') \geq \sqrt{A(f,T' \cup T'')}(1 - C|T' \cup T''|).$$

So if we have that $\sum_i |f^i(T' \cup T'')|$ is bounded then one can proceed as before. In particular, such a bound holds for the Fibonacci map (see Section 2 of [BKNS]) Proposition 10.1 also holds for C^2 Fibonacci maps with a critical point of order ℓ.

10.2 An asymptotic expression for $f^{S_{n-2}}$ on $[x_{n-1}, u_{n-1}]$

Now we will give a more precise version of the last proposition. In fact, we will obtain an asymptotic expression for large ℓ of the limit of the sequence of diffeomorphism

$$f^{S_{n-2}}: T_{n-1} \to [u_{n-2}, \hat{u}_{n-2}],$$

$n \in 2\mathcal{N}$ where as before $T_{n-1} = [x_{n-1}, u_{n-1}] \ni z_{n-2}$. Note that this situation is quite remarkable: as ℓ increases the non-linearity increases and the amount of Koebe space decreases. Evenso, we are able to determine the limit function (it is far from linear). So let

$$H_{n-1}: T_{n-1} \to \mathbb{R}$$

be the orientation preserving affine map with

$$H_{n-1}(z_{n-2}) = 0$$

and such that

$$|H_{n-1}(u_{n-1})| = 1.$$

Moreover, let $L_n: [-1,1] \to [u_{n-2}, \hat{u}_{n-2}]$ be the linear orientation preserving bijection.

Theorem 10.2 *There exists $K(\ell) > 0$ which is universally bounded and bounded away from zero such that if we write*

$$\Gamma(x) = \frac{x\sqrt{K(\ell)\ell + 1}}{\sqrt{K(\ell)\ell x^2 + 1}}$$

then for $n > \ell^{3/2}$, $n \in 2\mathbb{N}$ and for each $x \in [x_{n-1}, u_{n-1}]$,

$$L_n \circ \Gamma \circ H_{n-1}(x) \cdot (1 - o(1/\ell)) \leq f^{S_{n-2}}(x) \leq L_n \circ \Gamma \circ H_{n-1}(x) \cdot (1 + o(1/\ell))$$

and

$$D\left(L_n \circ \Gamma \circ H_{n-1}\right)(x) \cdot (1 - o(\ell)) \leq Df^{S_{n-2}}(x) \leq D\left(L_n \circ \Gamma \circ H_{n-1}\right)(x) \cdot (1 + o(\ell)).$$

Here $o(t)$ stands for a function which tends to zero as t tends to zero. For $n \in 2\mathbb{N} + 1$ there exists a similar constant $K(\ell)$.

Remark 10.1 *We should note that Γ can be written as a composition of the square map $x \mapsto x^2$, the Moebius map*

$$M_\ell \colon t \mapsto \frac{t(K(\ell)\ell + 1)}{K\ell t + 1}$$

and the root map $x \mapsto \sqrt{x}$. This Moebius map M_ℓ send the interval $[0, 1]$ onto itself but pushing points extremely far to the right when ℓ is large. The good bounds for the distortion of $f^{S_{n-2}}$ come from the fact that it is close to M_ℓ up to a conjugation with the square map. Presumably, using Ecalle cylinders or so, one can also give good estimates on the domain in the complex plane for which this asymptotic expression holds.

The idea of this result is related to the so-called Ecalle-cylinders used in [Sh1]. Before proving this theorem let us show how Theorem 10.1 follows from it.

Proof of Theorem 10.1: Note that $|u_{n-2}| = |\hat{u}_{n-2}|$ and the symmetry of the map Γ in the previous theorem implies that $1 - o(1/\ell) \leq |H_{n-1}(x_{n-1})| \leq 1 + o(1/\ell)$ for large n. In other words,

$$H_{n-1}[x_{n-1}, u_{n-1}] \to [-1 \pm o(1/\ell), 1 \pm o(1/\ell)].$$

We need to determine the preimages of the intervals $A_{n+i} \cap \mathbb{R}$ under $f^{S_{n-2}} \colon T_{n-1} = [x_{n-1}, u_{n-1}] \to [u_{n-2}, \hat{u}_{n-2}]$ and then determine the size of the derivative of this map in these intervals. First note that the intervals $A_{n+j} \cap \mathbb{R}$ have a size of the order $1/\ell$ of the size of $[u_{n-2}, \hat{u}_{n-2}]$ when $-2 \leq j \leq \ell$. So $H_{n-1}^{-1}(A_{n+j})$ is of the form

$$\pm[(1 - C(j+3)/\ell), (1 - C(j+2)/\ell)].$$

for $-2 \leq j \leq \ell$. The inverse of Γ is

$$\Gamma^{-1}(y) = \frac{y}{\sqrt{K\ell(1 - y^2) + 1}}.$$

This implies that the preimage of $\pm(1 - C(j+2)/\ell)$ is of the form $\frac{C_0}{\sqrt{j+3}}$, $-2 \le j \le \ell$. So if y is in the annulus A_{n+j} then its preimage $\tilde{x}$ under $H_{n-1} \circ \Gamma$ is in an interval of the form $[\frac{C_0}{\sqrt{j+4}}, \frac{C_0}{\sqrt{j+3}}]$. Since

$$D\Gamma(\tilde{x}) = \frac{\sqrt{K\ell + 1}}{(K\ell\tilde{x}^2 + 1)^{3/2}} \le C\frac{\sqrt{\ell}}{(K\ell\frac{1}{j} + 1)^{3/2}} \le C\frac{\sqrt{\ell}}{(\ell + j)^{3/2}} j^{3/2} \le C\frac{j^{3/2}}{\ell}.$$

This gives that if x is the preimage of y under $f^{S_{n-2}}\colon [x_{n-1}, u_{n-1}] \to [u_{n-2}, \hat{u}_{n-2}]$ then x is also in scaled down interval of this form and the previous estimate gives

$$|Df^{S_{n-2}}(x)| \le C(1 + o(1/\ell))\frac{j^{3/2}}{\ell}\frac{|u_{n-2} - \hat{u}_{n-2}|}{|x_{n-1} - u_{n-1}|}.$$

This is the required estimate for $j < \ell$. If $y \in A_{n+j}$ with $j \ge \ell$, then for its preimage x one has $|x| \le \frac{C}{\sqrt{\ell}}$ and therefore we get also

$$D\Gamma(x) \le C\sqrt{\ell}.$$

$\square$

Now we will start with the proof of Theorem 10.2. Notice that $f^{S_{n-2}}$ can be written as

$$f^{S_{i_0}} \circ f^{S_{i_0+1}} \circ f^{S_{i_0+3}} \circ \ldots \circ f^{S_{n-7}} \circ f^{S_{n-5}} \circ f^{S_{n-3}}$$

where $i_0 < n$ and $n - i_0$ is even.

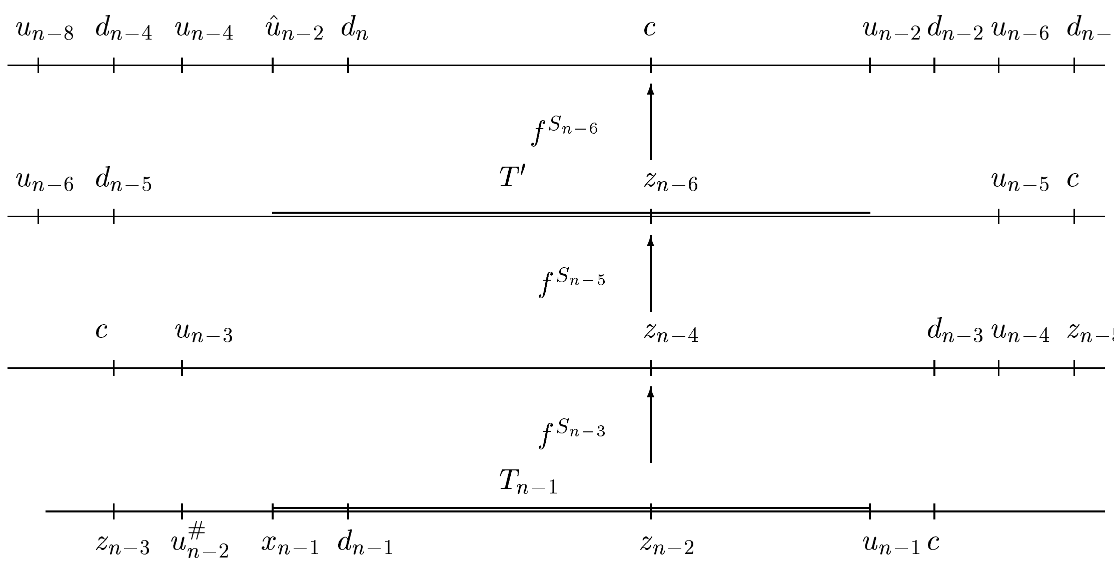

Figure 24: The map $f^{S_{n-2}}|T$ can be factored as shown (or as a longer composition).

Our aim is to give an asymptotic expression for this composition by comparing it with the solution of some particular differential equation which has an almost neutral attracting singularity. To do this we preceed similarly as in Lemma 8.1 and Lemma 8.2. Note that f^{S_i} maps z_{i+1} to z_{i-1} and $[u_{i+2}, x_{i+2}] \ni z_{i+1}$ into $[u_i, x_i] \ni z_{i-1}$. So let $H_i: [u_i, x_i] \to \mathbb{R}$ be the orientation preserving affine map with

$$H_i(z_{i-1}) = 0$$

so that

$$|H(u_i)| = 1.$$

By the real bouds, $|H_i(x_i)|$ is also uniformly bounded and bounded away from zero. Let

$$\Psi_i = H_i \circ f^{S_i} \circ H_{i+2}^{-1}: \mathbb{R} \to \mathbb{R}.$$

As in the proof of Lemma 8.2,

$$\Psi_i(z) = (-1 + \alpha_i'/\ell)z + \beta_i' z^2 + \gamma_i' z^3 + O(z^4)$$

where the coefficients $\alpha_i', \beta_i', \gamma_i'$ do depend on ℓ but for each fixed ℓ converge to constants as $i \to \infty$ with either $i \in 2\mathbb{N}$ or $i \in 2\mathbb{N}+1$. In fact, $\liminf \alpha_i'$ is uniformly bounded away from 0. Hence, in the same way as in Lemma 8.2,

$$\Theta_i := \Psi_{i-2} \circ \Psi_i(z) = (1 - \frac{\alpha_i}{\ell})z + \frac{\beta_i}{\ell} z^2 - \gamma_i z^3 + O(z^4)$$

where α_i, γ_i converge to positive constants (again as $i \in 2\mathbb{N}$ or $i \in 2\mathbb{N}+1$). The expression $O(z^4)$ stands for a function which in norm is dominated by $C|z|^4$; that this last bound holds is explained below (8.6). These limits are uniformly bounded and bounded away for all large ℓ and $\alpha_i, \beta_i, \gamma_i$ converge to constants which are uniformly bounded in norm.

Because of the convergence of the sequence of renormalizations, as $i \in 2\mathbb{N}$, tends to infinity, $[H_i(u_i), H_i(x_i)]$ tends to an interval M (having 1 as an endpoint and containing 0 in its interior) and – up to scaling – $f^{S_{i-2}} \circ f^{S_i}$ also converges (provided i runs through either the even or the odd integers). In particular, $\Psi_{i-2} \circ \Psi_i$ tends to some fixed map $\Theta: M \to M$ of the form

$$\Theta = (1 - \frac{\alpha_{0,\ell}}{\ell})z + \frac{\beta_{0,\ell}}{\ell} z^2 - \gamma_{0,\ell} z^3 + O(z^4)$$

on a neighbourhood of $M \subset \mathbb{C}$ where the coefficients $\alpha_{0,\ell}, \beta_{0,\ell}, \gamma_{0,\gamma}$. In fact, this limit depends on whether i runs through $i \in 2\mathbb{N}$ or $i \in 2\mathbb{N}+1$, but to avoid needless repetition we will not mention this anymore in the remainder of this section. Of course Θ presumably depends on ℓ. So there exists i_0 (depending on ℓ) so that for all $i \geq i_0$ the maps

$$f^{S_{i-2}} \circ f^{S_i}: [x_{i+1}, u_{i+1}] \to [x_{i-3}, u_{i-3}]$$

are – up to scaling – in a given neighbourhood of the limiting map Θ.

Choose $\hat{\imath} \in \{i_0, i_0+1, i_0+2, i_0+3\}$ so that $n - \hat{\imath} \in 4\mathbb{N}$. Then

$$f^{S_{n-2}} = f^{S_{\hat{\imath}-2}} \circ f^{S_{\hat{\imath}-1}} \circ f^{S_{\hat{\imath}+1}} \circ \ldots \circ f^{S_{n-7}} \circ f^{S_{n-5}} \circ f^{S_{n-3}}$$

is a composition of a large number of maps of the form $f^{S_{i-2}} \circ f^{S_i}$ with $i \geq i_0$ and the map $f^{S_{i_0}}$. The image of $f^{S_{n-2}}$ of the interval $T_{n-1} = [x_{n-1}, u_{n-1}]$ is equal to $[\hat{u}_{n-2}, u_{n-2}]$. Let

$$T' = f^{S_{\hat{\imath}-1}} \circ \ldots \circ f^{S_{n-5}} \circ f^{S_{n-3}}(T_{n-1})$$

and let H be the maximal interval containing T' on which $f^{S_{\hat{\imath}-2}}$ is a diffeomorphism. The endpoints of $f^{S_{\hat{\imath}-2}}(H)$ consist of points of the form d_j with $j < 2i_0 - 2$ and so the diffeomorphic image contains a $k(n)$-scaled neighbourhood of $[\hat{u}_{n-2}, u_{n-2}]$ where $k(n) \to \infty$ as $n \to \infty$. It follows that $f^{S_{\hat{\imath}-2}}|T'$ has uniformly bounded distortion and that this distortion even disappears as n tends to infinity:

$$1 - o(1/n) \leq \frac{|Df^{S_{\hat{\imath}-2}}(x)|}{|Df^{S_{\hat{\imath}-2}}(y)|} \leq 1 + o(1/n)$$

for each $x, y \in T'$.

So it suffices to describe the limit of the sequence of maps

$$f^{S_{\hat{\imath}-1}} \circ \ldots \circ f^{S_{n-5}} \circ f^{S_{n-3}}$$

on T_{n-1}. So remember that

$$f^{S_{i-2}} \circ f^{S_i} = H_{i-2}^{-1} \circ \Psi_{i-2} \circ \Psi_i \circ H_{i+2} = H_{i-2}^{-1} \circ \Theta_i \circ H_{i+2}.$$

Hence

$$f^{S_{\hat{\imath}-1}} \circ \ldots \circ f^{S_{n-5}} \circ f^{S_{n-3}} = H_{\hat{\imath}+1}^{-1} \circ \Theta_{\hat{\imath}+1} \circ \ldots \circ \Theta_{n-7} \circ \Theta_{n-3} \circ H_{n-1} \tag{10.13}$$

and each of these maps is up to scaling near Ψ_i because $i \geq i_0$.

Before continuing with our proof, let us describe the idea. By our identification of z_i with the origin, the composition $f^{S_{i-2}} \circ f^{S_i}$ is identified with a map Θ_i having an almost neutral fixed point. So we shall analyse a long composition of such maps which are all close to a given map. Now an orientation preserving one-dimensional map can be essentially imbedded in a flow. So we shall be able to get good estimates for a high iterate of the map, by integrating a certain vector field explicitly. Since the maps Θ_i are not all identical, we shall only be able to use this comparison up to a certain number of iterates. We shall choose $n = \ell^{3/2}$ iterates, because it will turn out that for the remaining iterates the relevant restriction has a distortion which disappears as n and ℓ tend to infinity. Now comes the miracle: the limit of the first $n = \ell^{3/2}$ iterates is a function which only depends on the coefficients $\alpha_{0,\ell}$ and $\gamma_{0,\ell}$ and not on the remainder of the Taylor series of Θ.

To explain this and to analyze such a composition of maps we make a digression to solutions of a particular differential equation.

Lemma 10.2 *Consider the following differential equation*

$$x'(t) = -(\alpha/\ell)x(t) - \gamma \cdot [x(t)]^3, \tag{10.14}$$

where $\alpha, \gamma > 0$ are positive constants and let $\phi_t(x)$ be its flow. Then

$$\phi_t(x) = \frac{x \exp(-\alpha t/\ell)}{\sqrt{(\gamma\ell/\alpha)\left[1 - \exp(-2\alpha t/\ell)\right] x^2 + 1}}. \tag{10.15}$$

In particular, the Taylor expansion of $x \mapsto \phi_t(x)$ at $x = 0$ is $x(1 - \frac{\alpha t}{\ell})(1 - \gamma' t x^2 + \ldots)$ where $\gamma' = \gamma + o(1/\ell)$. Moreover, there exist universal constants C such that

$$\sum_{i=0}^{\infty} |\phi_i(x)| \leq C\sqrt{\ell},\ \sum_{i=0}^{\infty} |\phi_i(x)|^2 \leq C\log(\ell) \text{ and } \sum_{i=0}^{\infty} |\phi_i(x)|^3 \leq C\,, \tag{10.16}$$

provided $|x| \leq 1$ (or $|x|$ is universally bounded).

Remark 10.2 *Essentially the reason for the miracle mentioned above is that for given $x, y \in [1/2, 1]$*

$$\lim_{t\to\infty} \frac{\phi_t(x)}{\phi_t(y)} = \frac{x}{\sqrt{\gamma\ell x^2/\alpha + 1}} \frac{\sqrt{\gamma\ell y^2/\alpha + 1}}{y} = 1 \pm o(1/\ell).$$

So an error in the initial condition becomes less and less important as $t, \ell \to \infty$. We should also remark that

$$\phi_t(x) = \pm\sqrt{M_t(x^2)}$$

where $M_t\colon \mathcal{R}^+ \to \mathcal{R}^+$ is a Moebius transformation which becomes increasingly degenerate as $t \to \infty$.

Proof: The general solution of (10.14) is of the form

$$\frac{1}{(x(t))^2} = \frac{-\gamma\ell}{\alpha} + \exp(2\alpha t/\ell)\cdot c_0 \tag{10.17}$$

(where c_0 is an integration constant). Indeed, differentiation of (10.17) gives:

$$-\frac{2x'}{x^3} = \frac{2\alpha}{\ell}\exp(2\alpha t/\ell)\cdot c_0 = \frac{2\alpha}{\ell}\left[\frac{\gamma}{\alpha}\ell + \frac{1}{x^2}\right].$$

This last expression is (10.14) rewritten. It follows that the integration constant is equal to $c_0 = \frac{\gamma\ell}{\alpha} + \frac{1}{[x(0)]^2}$, which gives the required expression.

Since

$$\frac{1}{\ell(1 - \exp(-2\alpha t/\ell))} \leq \max(C/t, C/\ell),$$

the last three inequalities of this lemma can be derived from (10.15):

$$\sum_{i=0}^{\infty} |\phi_i(x)| \leq C + C\sum_{i=1}^{\ell}\frac{1}{\sqrt{i}} + C\frac{1}{\sqrt{\ell}}\sum_{i=\ell+1}^{\infty}\exp(-ai/\ell) \leq C\sqrt{\ell} + C\frac{1}{\sqrt{\ell}}\ell.$$

$$\sum_{i=0}^{\infty} |\phi_i(x)|^2 \le C + C\sum_{i=1}^{\ell} \frac{1}{i} + C\frac{1}{\ell}\sum_{i=\ell+1}^{\infty} \exp(-ai/\ell) \le C\log(\ell) + C\frac{1}{\ell}\ell.$$

$$\sum_{i=0}^{\infty} |\phi_i(x)|^3 \le C + C\sum_{i=1}^{\ell} \frac{1}{i^{3/2}} + C\frac{1}{\ell^{3/2}}\sum_{i=\ell+1}^{\infty} \exp(-ai/\ell) \le C.$$

□

Let us write $\theta_i = \Theta_{n-2i-1}$. Then

$$\Theta_{\hat{\imath}+1} \circ \ldots \circ \Theta_{n-7} \circ \Theta_{n-3} = \theta_m \circ \ldots \theta_1 \tag{10.18}$$

where $m = (n - \hat{\imath} - 4)/4$. This brings us to the following abstract situation, where are before ϕ_t is the solution of the previous differential equation 10.15:

Theorem 10.3 *Consider a sequence of analytic maps $\theta_{i,\ell}\colon \mathbb{R} \to \mathbb{R}$ for $i = 0, 1, \ldots$ such that*

$$\theta_{i,\ell}(x) = \left(1 - \frac{\alpha_{i,\ell}}{\ell}\right) x + \frac{\beta_{i,\ell}}{\ell} x^2 - \gamma_{i,\ell}\, x^3 + O(|x|^4).$$

Assume that

- *$\alpha_{0,\ell}$, $\gamma_{0,\ell}$ are positive, uniformly bounded away from zero and bounded from above and that $\beta_{0,\ell}$ uniformly bounded in norm;*
- *$\theta_{i,\ell}$ is a diffeomorphism from $(-1,1)$ into itself with $|\theta_{i,\ell}(x)| < |x|$ and such that for each given $\epsilon > 0$ there exists n (which does not depend on i and ℓ) such that $\theta^n_{i,\ell}(-1,1) \subset (-\epsilon, \epsilon)$;*
- *for each $|\alpha_{i,\ell} - \alpha_{0,\ell}| < C/\ell$, $|\beta_{i,\ell} - \beta_{0,\ell}| < C/\ell$, $|\gamma_{i,\ell} - \gamma_{0,\ell}| < C/\ell$ and $\theta_{i,\ell}$ is in a compact set of maps.*

Then writing $F_m = \theta_m \circ \ldots \circ \theta_1$ one has for each $m \ge m(\ell) = \ell^{3/2}$ and each $x \in (-1,1)$,

$$1 - o(1/\ell) \le \frac{F_m(x)}{\phi_m(x)} \le 1 + o(1/\ell) \tag{10.19}$$

and

$$1 - o(1/\ell) \le \frac{DF_m(x)}{D\phi_m(x)} \le 1 + o(1/\ell), \tag{10.20}$$

where $o(s)$ is some universal function which tends to zero as $s \to 0$ and where $t \mapsto \phi_t$ is the flow of the differential equation from the previous lemma, with $\alpha = \alpha_{0,\ell}$ and $\gamma = \gamma_{0,\ell}$.

Because of (10.13) and (10.18), Theorem 10.2 follows from this theorem. So it remains to prove Theorem 10.3. For this we need the following two lemmas.

Lemma 10.3 *The assertion of the previous proposition holds for $m = \ell^{3/2}$ in which case one also has for $|x| < 1$,*

$$|F_m(x)| < \frac{1}{\ell}. \tag{10.21}$$

Proof: In order to be definite choose $x > 0$. The case that $x < 0$ goes similarly.

Step 1. First we claim that if

$$0 < x < \frac{1}{\sqrt{j}} \tag{10.22}$$

and $i \in \mathbb{N}$ then there exists $t' < 1 < t$ with

$$|t-1|, |t'-1| \le C \max\left(\frac{1}{\sqrt{j}}, \frac{1}{\ell}\right) \tag{10.23}$$

such that

$$\phi_t(x) \le \theta_i(x) \le \phi_{t'}(x). \tag{10.24}$$

Here as before C is a universal constant (not depending on j, i, ℓ). This can be seen as follows: if $O(x)$ is a bounded function then

$$x^4 \cdot O(x) \le |t'-1| \cdot |x|^3, \beta |x|^2 \le |t'-1| \cdot |x| \text{ and } \frac{C_0}{\ell} \le |t'-1| \tag{10.25}$$

provided x is as in (10.22) and $t' < 1$ so that equality holds in (10.23). Therefore, taking $t' < 1$ in this way, we get from (10.25) and from the second assumption of this lemma that

$$\theta_{i,\ell}(x) = \left(1 - \frac{\alpha_{i,\ell}}{\ell}\right) x + \frac{\beta_{i,\ell}}{\ell} x^2 - \gamma_{i,\ell} x^3 + x^4 \cdot O(x)$$

is bounded from above by

$$x(1 - \frac{\alpha t'}{\ell})(1 - \gamma' t' x^2 + \ldots) = \frac{x \exp(-\alpha t'/\ell)}{\sqrt{(\gamma \ell/\alpha)\left[1 - \exp(-2\alpha t'/\ell)\right] x^2 + 1}} = \phi_t(x)$$

where $\gamma' = \gamma + o(1/\ell)$ and where we take $\alpha = \alpha_{0,\ell}$, $\gamma = \gamma_{0,\ell}$. As usual, $O(x)$ is some bounded function of x.

Step 2. By assumption $\theta_i(x) \in (0, x)$ and there exists a universal number j such that $\theta_j \circ \ldots \circ \theta_1(x)$ is inside a given neighbourhood of 0. So it follows from the previous step that for i sufficiently large:

$$F_i(x) = \theta_i \circ \ldots \circ \theta_1(x) \le \phi_{i/2}(x).$$

Using the explicit formula for $\phi_t(x)$ one sees that $F_i(x) \le \phi_{i/2}(x) \le C\frac{1}{\sqrt{i}}$.

Step 3. Using Step 2, $F_i(x) \le C\frac{1}{\sqrt{i}}$ and therefore there exists $t_i' < 1 < t_i$ as in Step 1 such that

$$\phi_{t_i} \circ F_i(x) \le \theta_{i+1} \circ F_i(x) = F_{i+1}(x) \le \phi_{t_{i+1}} \circ F_i(x).$$

By induction we get that

$$\phi_{T_i}(x) \le F_i(x) \le \phi_{T_i'}(x) \tag{10.26}$$

with $T_i' < i < T_i$ such that $T_i = t_i + \ldots + t_1$, $T_i' = t_i' + \ldots + t_1'$ and $|t_i' - t_i|$ as above. Hence

$$|T_i - T_i'| \le \sum_{j=0}^{i} |t_i - t_i'| \le C \left[\sum_{j=0}^{i} \left(\frac{1}{\sqrt{j}} + \frac{1}{\ell} \right) \right].$$

For $i \le \ell^{3/2}$ this gives

$$|T_i' - T_i| \le C[\sqrt{i} + i/\ell] \le C[\ell^{3/4} + \ell^{1/2}].$$

Since

$$\phi_t(x) = \frac{x \exp(-\alpha t/\ell)}{\sqrt{(\gamma\ell/\alpha)\,[1 - \exp(-2\alpha t/\ell)]\, x^2 + 1}}$$

it follows that if $i \le \ell^{3/2}$ for such $T = T_i, T' = T_i'$,

$$\frac{\phi_{T'}(x)}{\phi_T(x)} \le \exp(\alpha|T - T'|/\ell) \frac{\sqrt{(\gamma\ell/\alpha)\,[1 - \exp(-2\alpha T/\ell)]\, x^2 + 1}}{\sqrt{(\gamma\ell/\alpha)\,[1 - \exp(-2\alpha T'/\ell)]\, x^2 + 1}}.$$

Because

$$\sqrt{(\gamma\ell/\alpha)\,[1 - \exp(-2\alpha T'/\ell)]\, x^2 + 1} \ge C_0 \sqrt{\min(|T'|, \ell)}$$

this implies that

$$\frac{\phi_{T'}(x)}{\phi_T(x)} \le \left(1 + C\frac{|T - T'|}{\ell} \right) \left(1 + C\ell \frac{[\exp(-2\alpha T/\ell) - \exp(-2\alpha T'/\ell)]}{\sqrt{\min(|T'|, \ell)}} \right).$$

For $i \le \ell^{3/2}$, $T' < i < T$ with $|T' - T| \le \sqrt{i}$, the last factor is at most

$$\le 1 + C\frac{|T - T'|}{\sqrt{(|T'|, \ell)}} \le 1 + \max\left(\frac{1}{\sqrt{i}}, \frac{1}{\sqrt{\ell}} \right).$$

From this it follows that

$$\frac{\phi_{T'}(x)}{\phi_T(x)} \le 1 + C\left[\frac{\sqrt{i}}{\ell} + \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{\ell}} \right] \tag{10.27}$$

for $i \le \ell^{3/2}$ and the the upper bound in (10.19). A lower bound is shown in the same way. Therefore, using (10.26), one gets

$$|\frac{F_i(x)}{\phi_i(x)} - 1| \le C\left[\frac{\sqrt{i}}{\ell} + \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{\ell}} \right] \tag{10.28}$$

which gives the first inequality (10.19) claimed in the theorem. Note that the right hand side is at most $C\ell^{-(1/4)}$ when $i \le \ell^{3/2}$.

Step 4. Now we shall prove the second inequality (10.20) claimed in the theorem. For this note

$$\frac{D\theta_{i+1}(F_i(x))}{D\phi_1(\phi_i(x))} \le (1 + C|D\theta_{i+1}(F_i(x)) - D\phi_1(F_i(x))| + C|D\phi_1(F_i(x)) - D\phi_1(\phi_i(x))|)\,. \tag{10.29}$$

This expression is at most

$$\left(1 + C/\ell^2 + (C/\ell)|F_i(x)| + C\left[|F_i(x)|^2 - |\phi_i(x)|^2\right] + C|F_i(x)|^3\right).$$

By the last part of the lemma,

$$\sum |F_i(x)| \le \sum |\phi_i(x)| \le \sqrt{\ell}.$$

Moreover, by (10.28)

$$\left[|F_i(x)|^2 - |\phi_i(x)|^2\right] \le C \cdot \left[\frac{|F_i(x)|}{|\phi_i(x)} - 1|\right] \cdot |\phi_i(x)|^2$$

whne $i \le \ell^{3/2}$ which gives that

$$\sum_0^{\ell^{3/2}} \left[|F_i(x)|^2 - |\phi_i(x)|^2\right] \le C \cdot \sum_0^{\ell^{3/2}} \left[\frac{\sqrt{i}}{\ell} + \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{\ell}}\right] \left(\frac{1}{\sqrt{i}}\right)^2 \le C\frac{\sqrt{\ell^{3/2}}}{\ell} \le C\ell^{-1/4}.$$

Combining this proves (10.20). $\square$

The remaining iterates have a non-linearity which vanishes as $\ell \to \infty$:

Lemma 10.4 *Let $\theta_{i,\ell}$ be as in the previous lemma. Assume that $|y| < 1/\ell$. Then for any $m, k \in \mathbb{N}$ one has, writing $F_{m,k} = \theta_{m+k} \circ \ldots \circ \theta_{m+1}$,*

$$1 - o(1/\ell) \le \frac{|D\hat{F}_{m,k}(y)|}{|D\hat{F}_{m,k}(0)|} \le 1 + o(1/\ell).$$

This means that the distortion of $F_{m,k}$ on $[-1/\ell, 1/\ell]$ is small for large ℓ.

Proof: One has

$$|D\theta_i(y) - D\theta_i(0)| \le \frac{\beta}{\ell}|y| + \gamma|y|^2 + O(|y|^3).$$

Since $|y| < 1/\ell$, therefore

$$|D\theta_i(y) - D\theta_i(0)| \le \frac{C}{\ell}|y|$$

and

$$|D\theta_i(z)| \le (1 - C_0/\ell)$$

for each $z \in (0, y)$. It follows that

$$\left| \frac{|D\hat{F}_{m,k}(y)|}{|D\hat{F}_{m,k}(0)|} - 1 \right| \le \frac{C}{\ell} \sum_{i=0}^{k} |F_{m,i}(y)| \le \frac{C}{\ell} \sum_{i=0}^{m} \frac{1}{\ell}(1 - C_0/\ell)^i \le \frac{C}{\ell}.$$

□

Proof of Theorem 10.3: Take as before $m(\ell) = \ell^{3/2}$ and write for $m \ge m(\ell)$, $F_m = F_{m,m(\ell)} \circ F_{m(\ell)}$. Because of Lemma 10.3, one has that the first map $F_{m(\ell)}$ can be compared very well with $D\phi_{m(\ell)}$. Since this lemma also asserts that $F_{m(\ell)}(-1, 1) \subset (-1/\ell, 1/\ell)$, the last map $F_{m,m(\ell)}$ can be compared better and better (as ℓ tends to infinity) with a linear map because of Lemma 10.4. Combined, this gives the required estimates. □

11 The proof of the Main Theorem

In this section we shall complete the proof of the Main Theorem. First we should remark that the filled Julia set of f is nowhere dense. This simply follows from the fact that the critical point is recurrent, $c \in \omega(c)$, and therefore f has no periodic attractors or neutral periodic points, see for example [Blan], [Ly0] or [Mil]. Actually, this also implies that the Julia set of f is connected and that $\cup f^{-k}(c)$ is dense in the Julia set. Rather than showing that the Julia set of f has positive Lebesgue measure we shall prove the following stronger theorem. (In fact, this theorem is equivalent to the main theorem because of [Ly0], [Ly2].)

Theorem 11.1 *For all x from a set of positive Lebesgue measure*

$$\omega(x) \subset \omega(c).$$

Let A_k, A'_k, F, $\mathcal{A}_n$ be defined as in the previous section. In this section we shall show that F and $\mathcal{A}$ satisfy the assumptions of Theorem 7.1.

Theorem 11.2 *For all sufficiently large ℓ holds: The set D of all points x for which the trajectory $(F^k x)_{k>0}$ visits A_n at most finitely often, has positive Lebesgue measure.*

Let us first show that this result implies our Theorem 11.1.

Proof of Theorem 11.1. Because $f^{-k}(c)$ is dense in the Julia set, the length of a maximal of interval of monotonicity of $f^{S_n}|\mathbb{R}$ is at most $\delta(n)$ where $\delta(n) \to 0$ as $n \to \infty$. Consider a point $x \in X$ for which $(F^k x)_{k>0}$ visits each annulus A_n at most finitely often, and denote by $t_1 < t_2 < t_3 < \dots$ the sequence of times for which $F^k x = f^{t_k} x$. We have to show that $\lim_{t\to\infty} \mathrm{dist}(f^t x, \omega_f(c)) = 0$. Along the subsequence t_k this holds as $\lim_{k\to\infty} f^{t_k}x = \lim_{k\to\infty} F^k x = c \in \omega_f(c)$. Consider now $t_k < t < t_{k+1}$ and suppose that $F^k x \in A_n \subset D_{n+1}$. Write $i := t - t_k$ and note that $0 < i < S_k$. Now $f^{S_n - 1}$ is a diffeomorphism from $f(D_{n+1})$ to $D_{n-1} \subset D_*(u_{n-2}, \hat{u}_{n-2})$ and using the Lemma of Schwarz $f^i(A_n) \subset f^i(D_{n+1})$ is contained inside a disc $D_*(h_i, j_i)$ where $(h_i, j_i) \subset \mathbb{R}$ is an interval of monotonicity of f^{S_n-1}. So the diameter of this disc is at most $\delta(n)$ and since $c_{S_{n+1}} \in A_n$ it follows that

$$\operatorname{dist}(f^i x, \omega(c)) \le \operatorname{diam}(f^i(A_n)) \le \delta(n) \to 0 \text{ as } n \to \infty.$$

$\square$

Proof of Theorem 11.2: Let k_0 be a large integer as before, for $k \ge k_0$ let X_k be the disjoint union of A_k and A'_k and finally let $X = \cup_{k\ge k_0} X_k$. So X can be considered as the disjoint union of the disc $\cup A_k$ and the disc with holes $\cup A'_k$. Define $\mathcal{X}$ to be the partition of X in elements X_k and let m the Lebesgue measure on X (i.e., the Lebesgue measure on $\cup A_k$ and on $\cup A'_k$). Let us show that we can apply Theorem 7.1 with these choices. So take $n_0 \ge k_0 + 2$, take $A \in \mathcal{X}_{k+1}$ and assume that $n := \phi(F^k(A)) \ge n_0$. Then F^k either maps A onto $\tilde{A}_n$ where $\tilde{A}_n$ is the annulus A_n or the annulus A'_n with a hole.

Because of Theorem 9.1, the map F^k extends in a univalent way to a map which maps respectively onto the slit $Slit_n$ or onto the slit $Slit'_n$. Because of Theorem 9.2 and because of the Koebe Lemma this implies that $F^k\colon A \to \tilde{A}_n$ has uniformly bounded distortion.

Now we will check that for ℓ sufficiently large the first condition

$$\int_A (\Delta\varphi - 1) \circ F^k \, dm \ge 0 \tag{11.1}$$

from the random walk result, Theorem 7.1, is satisfied, where

$$\Delta\varphi := \varphi \circ F - \varphi$$

and $\varphi(x) = i$ if $x \in A_i$. In other words, we need to show that

$$\frac{1}{m(A)} \int_A \Delta\varphi \circ F^k \, dm \ge 1. \tag{11.2}$$

Let A^i be the part of A which F^k sends to $\tilde{A}^i_n$. Note that this is the part of A_n which is between two infinite rays l_i and l_{i+1}; also note that this piece is connected since $A_n \cap \mathbb{R}$ is equal to $[u_n, \hat{u}_n] \setminus [u_{n+1}, \hat{u}_{n+1}]$ and since A_n is rotational symmetry. From Theorem 9.3 one has that the diffeomorphism $F^k\colon A^i \to A^i_n$ has uniformly bounded distortion. Since $f(A^i_n) = f(A_n)$ this implies that (11.2) follows from

$$\frac{1}{m(\tilde{A}_n)} \int_{\tilde{A}_n} \Delta\varphi \, dm \ge \gamma(\ell). \tag{11.3}$$

where γ stands for some function such that $\gamma(\ell) \to \infty$ as $\ell \to \infty$. Now let $\hat{A}_n = A_n \setminus f^{-1}(D^{1,f}_{n+1})$. So $\hat{A}_n$ is the annulus A_n with *all* the ℓ symmetrics of D^1_{n+1} removed (instead of only one such disc D^1_{n+1} removed as is the case with A'_n). Using this notation,

$$\begin{aligned}\int_{A_n} \Delta\varphi \, dm &= \int_{\hat{A}_n} \Delta\varphi \, dm + \ell \int_{D^1_{n+1}} \Delta\varphi \, dm \\ &\geq -2m(A_n) + \ell \int_{D^1_{n+1}} \Delta\varphi \, dm\end{aligned}$$

and

$$\begin{aligned}\int_{A'_n} \Delta\varphi \, dm &= \int_{\hat{A}_n} \Delta\varphi \, dm + (\ell-1) \int_{D^1_{n+1}} \Delta\varphi \, dm \\ &\geq -2m(A_n) + (\ell-1) \int_{D^1_{n+1}} \Delta\varphi \, dm\end{aligned}$$

where we have used $\Delta\varphi \geq -2$. From this it follows that it is enough to prove that (11.3) holds for $\tilde{A}_n = A_n$ and therefore it suffices to show that

$$\frac{1}{m(A_n)} \int_{A_n} (\varphi \circ F - n) \, dm = \frac{1}{m(A_n)} \int_{A_n} \Delta\varphi \, dm \geq \gamma(\ell) \tag{11.4}$$

Note that

$$F|A_n = f^{S_n} = f^{S_{n-2}} \circ f^{S_{n-1}}$$

and that $f|A_n$ has uniformly bounded distortion since A_n is between two discs centered at 0 of radius $|u_n|$ respectively $|u_n|(1 - C/\ell)$. This implies that it is enough to show that

$$\frac{1}{m(A^f_n)} \int_{A^f_n} (\varphi \circ f^{S_n - 1} - n) \, dm \geq \gamma(\ell).$$

(Here γ is a function with the same properties as before.)

Since $(\varphi \circ f^{S_n-1} - n) \geq -2$, since $(\varphi \circ f^{S_n-1} - n) \geq 0$ on $D^{1,f}_{n+1}$ and since the area of $D^{1,f}_{n+1} \subset A^f_n$ occupies a definite proportion of the area of A^f_n it suffices to show that

$$\frac{1}{m(D^{1,f}_{n+1})} \sum_{i \geq 0} \; i \cdot m(\{x \in D^{1,f}_{n+1} \; ; \; \varphi(\circ f^{S_n-1} x) = n+i\}) \tag{11.5}$$

$$= \frac{1}{m(D^{1,f}_{n+1})} \int_{D^{1,f}_{n+1}} (\varphi \circ f^{S_n-1} - n) \, dm \geq \gamma(\ell). \tag{11.6}$$

For $i \geq 1$, write as before

$$A^i_{n+i} = A_{n+i} \cap \{z \in \mathbb{C} \; ; \; |\arg(z)| < i/\ell\}.$$

For i small this is a very small piece of the annulus A_{n+i} but for $i \approx \ell$, this piece occupies a definite proportion of A_{n+i}. Let B^+_i be the preimage of this set under the map $f^{S_n-1}\colon D^{1,f}_{n+1} \to D_{n+1} = \cup_{i \geq 0} A_{n+i}$:

$$B^+_i = \{x \in D^{1,f}_{n+1} \; ; \; f^{S_n-1}(x) \in A^i_{n+i}\}.$$

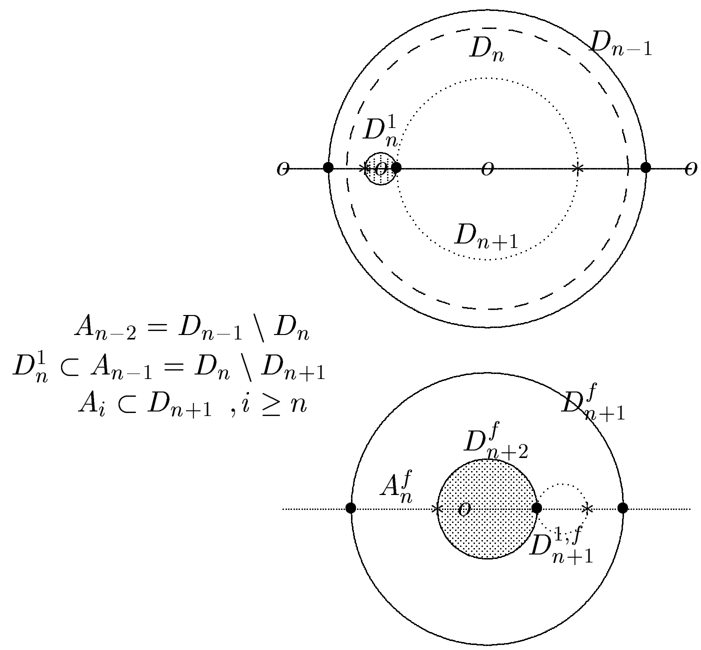

Figure 25: The annulus A^f_n and its image under the map $f^{S_n - 1}$. Moreover, $B^i_i \subset D^{1,f}_{n+1}$ is the preimage of the region $A^i_{n+i} \subset A_{n+i}$.

Then (11.6) is implied by

$$\frac{1}{m(D^{1,f}_{n+1})} \sum_{i \geq 0} \ i \cdot m(B_i^+) \geq \gamma(\ell) \tag{11.7}$$

Because of the real bounds and the shape estimates on the annuli A_j, the diameter of A^i_{n+i} is of the same order as its distance to the nearest critical value of f^{S_n-1}. Hence the distortion of the restriction of $f^{S_n-1}\colon D^{1,f}_{n+1} \to D_{n+1}$ to the diffeomorphism

$$f^{S_n-1}\colon B_i^+ \to A^i_{n+i}$$

is uniformly bounded (for all $i \geq 1$ and all large ℓ and n). From Theorem 10.1 we have very good estimates for this map (on the real line) and combined with the bounded distortion on B_i^+ this gives a uniform constant $C > 0$ such that

$$|Df^{S_n-1}(x)| \leq C \frac{i^{3/2}}{\ell} \frac{|u_{n-2} - \hat{u}_{n-2}|}{|r^f_n - u^f_n|} \tag{11.8}$$

for each $x \in B_i^+$. Since f^{S_n-1} maps B_i^+ diffeomorphically onto A^i_{n+i} and areas are distorted with the *square* of the Jacobian of the map, (11.8) implies

$$\frac{m(A^i_{n+i})}{m(B_i^+)} \leq \frac{i^3}{\ell^2} \left[\frac{|u_{n-2} - \hat{u}_{n-2}|}{|r^f_n - u^f_n|} \right]^2 . \tag{11.9}$$

Here we have used (11.8) and that the distortion of the size of areas is measured by the square of the Jacobian of the map. By the corollary to Theorem 8.1, the 'height' of $D^{1,f}_{n+1}$ is comparable to its 'width' and also comparable to $|r^f_n - u^f_n|$, and in particular,

$$\frac{m(D_{n+1})}{m(D^{1,f}_{n+1})} \geq K \left[\frac{|u_{n-2} - \hat{u}_{n-2}|}{|r^f_n - u^f_n|}\right]^2 \tag{11.10}$$

for some uniform constant K. Combining (11.9) and (11.10), we get

$$\frac{m(B^+_i)}{m(D^{1,f}_{n+1})} \geq C \cdot \frac{\ell^2}{i^3} \cdot \frac{m(A^i_{n+i})}{m(D_{n+1})}. \tag{11.11}$$

Since for $i = 1, 2, \ldots, \ell$ the area of $m(A^i_{n+i})$ is of the order i/ℓ^2 times the area of $m(D_{n+1})$, the last inequality yields

$$\frac{m(B^+_i)}{m(D^{1,f}_{n+1})} \geq C \cdot \frac{1}{i^2}, \text{ for } i = 1, 2, \ldots, \ell. \tag{11.12}$$

(For $i = \ell, \ell + 1, \ldots$, the area of $m(A^i_{n+i})$ is of the order $e^{-i/\ell}/\ell$ times the area of of $m(D_{n+1})$ and so (11.11) also gives

$$\frac{m(B^+_i)}{m(D^{1,f}_{n+1})} \geq C \cdot \frac{\ell}{i^3} \cdot e^{-i/\ell} \text{ for } i = \ell + 1, \ell + 2, \ldots .) \tag{11.13}$$

Hence, because of (11.12), the left hand side of (11.7) can be bounded from below by

$$\sum_{i=0}^{\ell} \frac{1}{i} \geq \text{const} \cdot \log(\ell). \tag{11.14}$$

(The contribution to the expected drift due to the 'tail' terms corresponding to $i \geq \ell+1$, see (11.13), is

$$\sum_{i=\ell+1}^{\infty} \frac{\ell}{i^2} \cdot e^{-i/\ell}$$

which is uniformly bounded and so does not give any essential improvement on our previous bound.) This concludes the proof of the first assumption of Theorem 7.1.

Next we check the second condition from Theorem 7.1. This means that we have to find $M < \infty$ such that

$$\frac{1}{m(A)} \int_A (\Delta\varphi)^2 \circ F^k \, dm \leq M$$

for any $A \in \mathcal{A}_{k+1}$ with $n = \phi(F^k A)$ large enough. We should emphasize that M does *not* need to be uniform in ℓ. Since $F^k \colon A \to A_n$ has as before bounded distortion, it is enough to prove that there exists M such that

$$\frac{1}{m(A_n)} \int_{A_n} (\Delta\varphi)^2 \, dm \leq M$$

for each n sufficiently large. Since the distortion of F on A_n is bounded by ℓ^2 (which is bounded), the last hand side of the previous expression is at most

$$\ell^2 \frac{1}{m(D_{n-1})} \int_{D_{n-1}} (\varphi - n)^2 \, dm \;=\; \frac{1}{m(D_{n-1})} \sum_{i=-2}^{\infty} i^2 \cdot m(A_{n+i}).$$

Since

$$\frac{m(A_{n+i})}{m(D_{n-1})} \leq C_1 \cdot \frac{1}{\ell} \cdot e^{-C \cdot i/\ell},$$

the last infinite sum is, up to a multiplicative constant, bounded from above by

$$\ell^2 \sum_{i=-2}^{\infty} i^2 \cdot \frac{1}{\ell} \cdot e^{-C \cdot i/\ell} \leq \text{Const} \cdot \ell^5$$

which proves the second condition of Theorem 7.1 and concludes the proof of the Main Theorem. □

To conclude this section, we would like to make some comments on the difficulties of giving a computer supported numerical 'estimate' for the smallest value of ℓ for which the statement of the theorem holds. So take $A \in X_{k+1}$ with $\phi(A) = n$ and let $\{W_i\}_{i \geq -2}$ be the partition of A defined by the amount of drift:

$$W_i = \{x \in A \; ; \; \Delta\phi(x) = i\}.$$

First note that

$$\frac{m(W_{-1})}{m(A)} \text{ and } \frac{m(W_{-2})}{m(A)}$$

are uniformly bounded away from zero. So the chance to go 'down' is uniformly bounded away from zero. Moveover, consider the expected drift

$$\frac{1}{m(A)} \sum_{i \geq -2} i \cdot \mu(W_i) = \frac{1}{m(A)} \int_A \Delta\phi \, dm. \tag{11.15}$$

Suppose we would estimate this term by choosing a finite, say k', number of states neighbouring a given state and estimate numerically the finite sum

$$\frac{\sum_{i \geq -2}^{k'} i \cdot \mu(W_i')}{m(A)} \tag{11.16}$$

where $W_i' = W_i$ for $i < k'$ and $W_{k'}' = \cup_{i \geq k'} W_i$. By (11.12) we see that

$$\frac{m(W_i)}{m(A)} \geq \frac{C}{i^2}$$

(in fact, one can show that the left hand side is really of this form) and so we only can expect (11.16) to be equal to

$$\sum_{i=0}^{k'} \frac{1}{i}$$

(at least provided $k' < \ell$) and so the estimate we would obtain in this way does not get better for increasing ℓ. This means that the only way to get a good esimate for (11.15) is to take k' very large!

Similarly, one has to take k very large before one has that

$$\frac{|\{x \in A\ ;\ \phi(F^k(x)) - \phi(x) \geq 1\}|}{|A|}$$

gets close to one. This means that one has to iterate the induced map a very large number of times, before 'observing' the drift.

Let us finally also make a comparison with the real one-dimensional paper [BKNS], where we did not need to take the square in (11.9). This means that – if we had known the estimates of Section 10 of this paper already in that paper – we would have obtained (up to a multiplicative constant which is universally bounded away from zero), the following lower bound for the expression (11.12) in the *real one-dimensional* case. Firstly, then

$$\frac{m(W_i)}{m(A)} \sim \frac{1}{\ell} \cdot \frac{\ell}{i^{3/2}} = \frac{1}{i^{3/2}}, \text{ for } i = 1, 2, \ldots, \ell,$$

where $1/\ell$ corresponds to the size of the one-dimensional annulus A_{n+i} relative to the size of A_n and $\ell/i^{3/2}$ corresponds to the distortion of measure (when $i < \ell$). Similarly,

$$\frac{m(W_i)}{m(A)} \sim \frac{e^{-i/\ell}}{\ell} \cdot \frac{\ell}{i^{3/2}} = \frac{e^{-i/\ell}}{i^{3/2}}, \text{ for } i = \ell+1, \ell+2, \ldots.$$

Hence the expected drift in the real case is

$$\sum_{i=0}^{\ell} i \cdot \frac{1}{i^{3/2}} + \sum_{i=\ell+1}^{\infty} i \cdot \frac{e^{-i/\ell}}{i^{3/2}} \approx C_1\sqrt{\ell} + C_2\sqrt{\ell} \tag{11.17}$$

So the drift grows in the real case much faster with ℓ then in the complex case! Moreover, note that the contribution due to the 'tail' (i.e., the term of the form $\sum_{\ell+1}^{\infty}$ in (11.17) which can be derived immediately from Proposition 10.1) already cause a large drift when ℓ is large. Instead, of the above estimates we used in [BKNS] a simple Koebe estimates which implies that the measure distorts by at most ℓ/i^2 and so we were merely able to get the weaker bound

$$\sum_{i=0}^{\ell} i \cdot \frac{\ell}{i^2}\frac{1}{\ell} = \sum_{i=0}^{\ell} \frac{1}{i} \approx \log(\ell).$$

This – not so sharp estimate – is sufficient in the one-dimensional case. In our 'two-dimensional' case it is not enough because then the i^2 term in (11.12) would have to be replaced by i^3 and the term i would become i^2 in (11.14). This would lead to a series with uniformly bounded sum. Hence the expected drift would be of constant magnititude. Whether it would be positive or not would depend on the numerical value of some constants.

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