Forcing of periodic orbits for interval maps and renormalization of piece wise affine maps.

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Abstract.

We prove that for continuous maps on the interval, the existence of a $n-$cycle, implies the existence of $n-1$ points which interwind the original ones and are permuted by the map. We then use this combinatorial result to show that piecewise affine maps (with no zero slope) cannot be infinitely renormalizable.

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1. Introduction

A fascinating feature of real analytic infinitely renormalizable interval maps is that their attracting invariant Cantor sets seem to have a complicated geometry (for the unimodal case, see for instance [S] and references therein). One could hope to avoid this complexity by constructing piecewise affine examples (with no zero slope). This is indeed the case when there are infinitely many intervals of affinity (see e.g. [T]), but we show in section 3 that no example exists with finitely many intervals of affinity. In order to prove this results we had to solve some questions about forcing of permutations which are described in section 2.

We now state our main results. The collection of continuous maps on the interval is denoted by $C^0([0,1])$. A *cycle* of a map $f \in C^0([0,1])$ is a collection of pairwise disjoint closed intervals $\mathcal{I} = \{I_0, I_1, ..., I_q\}$ which are cyclically permuted by f. That means $f(I_i) \subset I_{i+1 \bmod q}$. The cycle is called *trivial* if all intervals are points.

A cycle $\mathcal{I}^j$ refines a cycle $\mathcal{I}^i$ if $\cup I_l^j \subset \cup I_k^i$. In this situation there is a number $a_{i,j} \in \mathbb{N}$ such that every component of $\mathcal{I}^i$ contains $a_{i,j}$ components of $\mathcal{I}^j$. We will always assume $a_{i,j} \geq 2$ and use the notation $\mathcal{I}^i \supset \mathcal{I}^j$ or $\mathcal{I}^j \subset \mathcal{I}^i$.

Let $\mathcal{I}^j$ refine $\mathcal{I}^i$. A connected component G of $\cup I_l^i \setminus \cup I_k^j$ is called a *gap* if $\partial G \subset \cup I_l^j$. The union of gaps is denoted by $\mathcal{G}(\mathcal{I}^i, \mathcal{I}^j)$.

DEFINITION 1.1. *Let $f \in C^0([0,1])$ and $\mathcal{I}^2$ be a cycle which refines the cycle $\mathcal{I}^1$.*
An invariant set P of periodic points of f is called a splitting of the pair of cycles $\mathcal{I}^1 \supset \mathcal{I}^2$ if
- *$P \subset \mathcal{G}(\mathcal{I}^1, \mathcal{I}^2)$,*
- *every gap in $\mathcal{G}(\mathcal{I}^1, \mathcal{I}^2)$ contains exactly one point of P.*

THEOREM A. *Every pair of cycles $\mathcal{I}^1 \supset \mathcal{I}^2$ of a map $f \in C^0([0,1])$ admits a splitting.*

Theorem A is a main ingredient of the proof of Theorem C below. It is a corollary of the following result.

THEOREM B. *If a continuous map f on the interval has a periodic orbit of period n then it permutes $n-1$ points interwinding the periodic orbit.*

Here we say that a set of $n-1$ points on the real line *interwinds* a set of n points if any two consecutive points of any of these sets are separated by a point of the other set. Theorem B in turns follows from a property of markov maps induced by permutations of points in the interval, stated as Theorem 2.1.

A map $f \in C^0([0,1])$ is *infinitely renormalizable* if it has an infinite sequence of refining cycles
$$\mathcal{I}^1 \supset \mathcal{I}^2 \supset \mathcal{I}^3 \supset \ldots$$
The set $\bigcap \cup \mathcal{I}^n$ is called an *infinitely renormalizable invariant set.*
A map $f \in C^0([0,1])$ is called *piecewise affine* if there is a sequence of points $0 = c_{-1} < c_0 < \cdots < c_d < c_{d+1} = 1$, called *corner points*, such that f is affine with non-zero slope on each interval $[c_i, c_{i+1}]$ and f is not affine on any larger interval. The set of corners of f is denoted by C_f. The collection of all piecewise affine maps is denoted by $PL([0,1])$.

THEOREM C. *There are no infinitely renormalizable piecewise affine maps.*

Notice that the conclusion of Theorem C fails if C_f is allowed to be countable (see e.g. $[T]$) or if we relax the non-zero slope condition (see e.g. $[BMT]$). Theorem C can be understood as a step toward proving the conjecture in $[GMT]$, that piecewise affine maps on the interval without periodic attractors are eventually expanding. An important ingredient of the proof of Theorem C is the Expansion-Lemma in section 3. It states that there is a reasonable big collection of expanding periodic orbits with exponent away from 0. This Expansion-Lemma should be compared with the Finiteness of Attractors Theorem in [MMS], which states that in smooth maps periodic orbits with sufficiently high period, are expanding with exponent away from 0. Whether this is also true for piecewise affine maps is part of the conjecture in $[GMT]$.

Acknowledgements. After proving Theorem C in the case of period doubling ($a_{i,i+1} \equiv 2$) we checked with some colleagues about its originality. Some time later, Michal Misiurewicz and Karen Brooks reported to us that V.J. Lopez and L. Snoha had recently obtained the same result [LS].

2. Forcing

Let S_n be the collection of permutations of $N_n = \{1, 2, \ldots, n\}$. For every $\pi \in S_n$ we define the $n - 1 \times n - 1$-matrix F_π, with

- $F_\pi(i,j) = 1$, if $(j, j+1) \subset (\pi(i), \pi(i+1))$,
- $F_\pi(i,j) = 0$ otherwise.

A continuous map $f : [1, n] \to [1, n]$ with $f|N_n = \pi$ will in general map the *gaps* $(i, i+1)$ in a very non-monotone way, and one cannot guess the full set of periodic orbits of f by the only knowledge of π. However $f|N_n = \pi$ implies some minimal complexity for the dynamics of f. More specifically, it is known that the subshift of finite type defined by F_π can always be monotonically imbedded into the dynamics of f; we say that $\pi = f|N_n$ *forces* all the dynamics of the subshift defined by F_π. In the above statement, monotonicity refers to the skewed lexicographic order on the sequence space, as used in kneading theory (see e.g. [MT]), and the usual order on $[1, n]$. As usual we identify the matrix F_π with the corresponding subshift.

If $\phi \in S_m$ and F is a $m \times m$-matrix with $0, 1$ entries, such that $F(i, \phi(i)) = 1$, we say that F *contains the permutation* ϕ, and write $\phi << F$.

THEOREM 2.1. *For every $\pi \in S_n$, $n \geq 2$ there exists a $\phi \in S_{n-1}$ with*

$$\phi << F_\pi.$$

REMARK: It is easy to construct examples of subshifts of finite types whose defining matrices have some power with all entries positive but do not contain a permutation.

The proof of this Theorem needs some preparation. We are going to describe a "cutting" procedure on permutations and a related "cutting" procedure on their matrices. Let $n \geq 2$ and $k \leq n - 1$ and let $j_k : N_{n-1} \to N_n \setminus \{k\}$ stand for the order preserving bijection. If $\pi \in S_n$ and $k = \pi^{-1}(n)$ then $\hat{\pi} \in S_{n-1}$ is defined by

$$\hat{\pi}(i) = \pi(j_k(i)),$$

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where $i \in N_{n-1}$.

From the definition of the matrix F_π, the 1's are consecutive in each of its rows. Furthermore if $F_\pi(k, n) = 1$ then $\pi(k) = n$ or $\pi(k+1) = n$. So there are at most two rows with the last entry equal to 1, and such rows have to be consecutive.

Consider the rows with last entry equal to 1 and assume that row k has the shortest block of 1's among those: that means, if $F_\pi(l, n) = 1$ and $l \neq k$ then there exists $j \leq n$ with $F_\pi(l, j) = 1$ and $F_\pi(k, j) = 0$. Now we define a $(n-1) \times (n-1)$ matrix $\hat{F}_\pi$ by

$$\hat{F}_\pi(i, j) = F_\pi(j_k(i), j).$$

This ˆ-operation on matrices has been defined for matrices induced by permutations. To the contrary of the ˆ-operation on permutations, in general it cannot be squared because the new matrix is maybe not induced by a permutation (this would always be the case for permutations realizable as restrictions of unimodal maps). In particular $F_{\hat{\pi}}$ is in general not equal to $\hat{F}_\pi$. These two matrices are only equal for unimodal permutations. In general we have

PROPOSITION 2.2. *For every* $\pi \in S_n$, $n \geq 2$,

$$F_{\hat{\pi}} \leq \hat{F}_\pi.$$

PROOF: Fix $\pi \in S_n$. For $k \leq n - 1$ let $h_k : N_{n-2} \setminus \{k-1\} \to N_{n-1} \setminus \{k-1, k\}$ be the order preserving bijection.

If $\pi^{-1}(n) = k$ then the $(k-1)^{th}$ and k^{th} row have a 1 in there last entry. Denote these rows by the vectors V and v. Assume that V has a longer block of $1's$. In the case that $k = 1$ or $k = n$ there is only one row whose last entry equals one, resp. the first or the last row. In these cases let V be this row and $v = 0$.

Now the Proposition follows immediately from

CLAIM. *If* $i \neq k - 1$ *then*
$$F_{\hat{\pi}}(i, j) = F_\pi(h_k(i), j),$$
$$\hat{F}_\pi(i, j) = F_\pi(h_k(i), j).$$

For $j \leq n - 2$ *and* $k \neq 1, n$

$$F_{\hat{\pi}}(k-1, j) = V(j) - v(j),$$
$$\hat{F}_\pi(k-1, j) = V(j).$$

PROOF: The matrix $\hat{F}_\pi$ was obtained from F_π by erasing the last column and row v. The result of this operation is expressed in the claim.

Now consider the $\hat{\pi}$ image of the gap $(i, i+1)$. If $i \leq k - 2$ then the $\hat{\pi}$-image is the same as the π-image. Hence the first $k - 2$ rows of $F_{\hat{\pi}}$ equal the first $k - 2$ rows of F_π. If $i \geq k$ then the $\hat{\pi}$-image is the π-image of the gap $(i+1, i+2)$. So the last $(n-2) - (k-1)$ rows of $F_{\hat{\pi}}$ equal the last $(n-1) - k$ rows of F_π. These properties are expressed in

$$F_{\hat{\pi}}(i, j) = F_\pi(h_k(i), j),$$

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whenever $i \neq k-1$. It just remains to consider the image of the $(k-1)^{th}$ gap. The boundary points are $k-1$ and k. Hence the $\hat{\pi}$-image of this gap equals the interval $(\pi(k-1), \pi(k+1))$. It follows easily that if a gap $(j, j+1)$ is covered by the interval $(\pi(k-1), \pi(k+1))$ then $V(j) - v(j) = 1$. $\blacksquare$ (Claim and Proposition 2.2)

PROOF OF THEOREM 2.1: The proof of Theorem 2.1 is by induction. For a permutation $\pi \in P_S$ we have that the only entry of F_π equals 1. Hence F_π contains a permutations. Now suppose that every matrix F_π with $\pi \in S_n$, contains a permutations.

Let $\pi \in S_{n+1}$. Then $\hat{\pi} \in S_n$ and hence $F_{\hat{\pi}}$ contains a permutation $\phi_0 << F_{\hat{\pi}}$. From Proposition 2.2 we get $F_{\hat{\pi}} \leq \hat{F}_\pi$. Hence $\phi_0 << \hat{F}_\pi$.

Say, that $\hat{F}_\pi$ was obtained by cutting the kth row of F_π. Then F_π contains the permutation $\phi \in S_{n+1}$ defined by

$$\phi(k) = n$$
$$\phi(i) = \phi_0(j_k^{-1}(i)) \text{ if } i \neq k.$$

$\blacksquare$ (Theorem 2.1)

PROOF OF THEOREM B: Let f be a continuous map with a periodic orbit $\{p_1, p_2, \ldots, p_n\}$ of period n. Assume $p_1 < p_2 < \cdots < p_n$. And let $G_i = (p_i, p_{i+1})$, with $i = 1, 2, \ldots, n-1$. By Theorem 2.1 we know that there exists a permutation $\phi \in S_{n-1}$ and intervals $T_i \subset G_i$ such that $f : T_i \to G_{\phi(i)}$ is onto. The continuity of f assures the existence of points $s_i \in \overline{T_i}$ such that $f(s_i) = s_{\phi(i)}$, with $i = 1, 2, \ldots, n-1$. Clearly the points s_i are periodic with period less than n. In particular $s_i \in int(G_i)$, they actually interwind the original orbit.

$\blacksquare$ (Theorem B)

PROOF OF THEOREM A: To prove Theorem A we may collapse the cycle $\mathcal{I}^2$ into a periodic orbit, say $\mathcal{I}^2 = \{p_1, p_2, \ldots, p_{q_2}\}$. Again, as in the proof of Theorem B, let $G_i = (p_i, p_{i+1})$. Apply Theorem 2.1 to get a permutation $\phi \in S_{q_2-1}$ and the intervals $T_i \subset G_i$ such that $f : T_i \to G_{\phi(i)}$ is onto. Let $P' = \{s_1, s_2, \ldots, s_{q_2-1}\}$ be the corresponding points which interwind $\{p_1, p_2, \ldots, p_{q_2}\}$, that means $s_i \in T_i$ and $f(s_i) = s_{\phi(i)}$.

Unfortunately P' will not be a splitting for the pair $\mathcal{I}^2 \subset \mathcal{I}^1$, it contains also points outside $\mathcal{I}^2$. Let $P = P' \cap G(\mathcal{I}^1, \mathcal{I}^2)$. To show that P is a splitting it suffices to show that P is invariant. Take $s_i \in P$. Then T_i is subset of the cycle $\mathcal{I}^1$. Hence $f(T_i) = G_{\phi(i)}$ is subset of the cycle $\mathcal{I}^1$, in particular $G_{\phi(i)}$ is a gap. So $s_{\phi(i)} \in G(\mathcal{I}^1, \mathcal{I}^2) \cap P' = P$.

$\blacksquare$ (Theorem A)

3. Renormalization

In his section we are going to prove Theorem C. The main reason why piecewise affine maps cannot be infinitely renormalizable, is the fact that there are enough periodic orbits with some definite expansion.

A cycle $\mathcal{I}^2$ is called a *doubling* of a cycle $\mathcal{I}^1$ if $\mathcal{I}^1 \supset \mathcal{I}^2$ and $a_{1,2} = 2$.

EXPANSION-LEMMA 3.1. *Let $f \in PL([0,1))$ having two cycles $\mathcal{I}^1 \supset \mathcal{I}^2$ with splitting P. If*
 - *$\mathcal{I}^2$ is not a doubling of some $\mathcal{I} \subset \mathcal{I}^1$ and*
 - *$C_f \subset \cup \mathcal{I}^2$*

then there exists $x \in P$ with

$$|Df^p(x)| \geq 1 + e^{-11V},$$

where p is the period of x and V the variation of $\log(|Df|)$.

PROOF: Let $I \in \mathcal{I}^1$ and $G \subset I$ a gap, say $\partial G \subset I_l^2 \cup I_r^2$. Now there exists a periodic point $x \in P \cap G$. Say, it has period $p \in \mathbb{N}$.

CLAIM 1. *There exists a gap $G' \subset I$ and $G' \neq G$ and an interval $K \subset G$ with $f^p : K \to G'$ onto.*

PROOF CLAIM 1: Say, $\mathcal{I}^2$ has q_2 intervals. So $p \leq q_2 - 1$. Assume Claim 1 is not true then $f^p(G) \subset I_l^2 \cup G \cup I_r^2$ and $f^p(I_l^2 \cup I_r^2) \subset I_l^2 \cup I_r^2$. Because $p < q_2$, the two intervals I_l^2 and I_r^2 have to be interchanged. In particular $p = \frac{1}{2}q_2$. Let $H = I_l^2 \cup G \cup I_r^2$ then $f^p(H) \subset H$. If we can show that the orbit $\{H, f(H), \ldots, f^{p-1}(H)\}$ is pairwise disjoint, then we would have shown that $\mathcal{I}^2$ is a doubling of some cycle $\mathcal{H} \subset \mathcal{I}^1$. This contradicts the assumptions and Claim 1 would be proved.
So suppose $f^j(H) \cap f^p(H) \neq \emptyset$, for some positive $j \leq p - 1$. Now I_{l+j}^2 and I_{r+j}^2 do not intersect H, since only I_l^2 and I_r^2 intersect H. Assume with no loss of generality $I_l^2 \subset f^j(H)$. This implies $I_{l+p-j}^2 = f^{p-j}(I_l^2) \subset f^p(H) \subset H$. This is impossible because only I_l^2 and I_r^2 are in H. We proved that the orbit of H forms a cycle. $\blacksquare$ (Claim 1)

REAMRK: In Claim 1, G' can be considered to be a gap adjacent to I_l^2 or I_r^2.

Now let $T \subset G$ be the maximal interval containing x such that $(f^p(T) \cap \cup \mathcal{I}^2) \subset I_l^2 \cup I_r^2$. Using Claim 1 we find an interval $K \subset T$ with $f^p : K \to G'$ onto. Because all corner points and their orbits are in $\cup \mathcal{I}^2$, the map $f^p : K \to G'$ is in fact affine and onto. A collection of intervals in $[0,1]$ is said to have *intersection multiplicity w* if every point in $[0,1]$ is contained in at most w intervals of the collection.

CLAIM 2. *The intersection multiplicity of $\{T, f(T), f^2(T), \ldots, f^{p-1}(T)\}$ is at most 11. In particular*
$$Var(\log(|Df^p|T|)) \leq 11V.$$

PROOF OF CLAIM 2: To prove Claim 2 it is enough to show that there are at most 10 values $1 \leq i \leq p - 1$ such that $f^i(T) \cap f^p(T) \neq \emptyset$.
Let I_L^2 be the left neighbor of I_l^2 and I_R^2 be the right neighbor of I_r^2. The interval between I_L^2 and I_R^2 is denoted by S. Clearly $f^p(T) \subset S$. Observe that S contains at most 3 gaps. Because the orbit of x is one of the splitting periodic orbits in P, the orbit of x intersects S in at most 3 points, say in x, $f^a(x)$ and $f^b(x)$. These three intersections can also give rise to an intersection of $f^a(T)$ or $f^b(T)$ with $f^p(T)$.
Consider an intersection $f^i(T) \cap f^p(T) \neq \emptyset$, where $i \leq p - 1$ and $i \notin \{0, a, b\}$. Because $f^i(x) \notin I_L^2 \cup S \cup I_R^2$, the interval I_L^2 (or I_R^2) is contained in $f^i(T)$. Hence $L + p - i \in \{l, r, l + q_2, r + q_2\}$ (or $R + p - i \in \{l, r, l + q_2, r + q_2\}$). This means that there are at most 8 possible values for $i \notin \{0, a, b\}$ giving rise to an intersection.
All together we get at most $2 + 4 + 4 = 10$ intersections $f^i(T) \cap f^p(T) \neq \emptyset$, with $1 \leq i \leq p-1$. The intersection multiplicity is at most 11. $\blacksquare$ (Claim 2)

Because the orbits of the corners are contained in $\cup \mathcal{I}^2$ there exists an interval $D \subset T$ with $x \in D$ and $f^p : D \to G$ affine and onto.

To prove Lemma 3.1, assume that G was chosen the smallest gap in I, $|G| \leq |G'|$. Furthermore $|K| \leq |T| - |D|$. Observe $\frac{|G|}{|D|} = |Df^p(x)|$ and $\frac{|G'|}{|K|} = |Df^p_{|K}|$. Then

$$11V \geq \log(|Df^p|K|) - \log(|Df^p(x)|)$$
$$= \log(\frac{|G'|}{|K|} \times \frac{|D|}{|G|})$$
$$\geq \log \frac{|D|}{|K|}$$
$$\geq \log \frac{|D|}{|T| - |D|}.$$

This implies

$$|Df^p(x)| = \frac{|G|}{|D|} \geq \frac{|T|}{|D|} \geq 1 + e^{-11V}.$$

■ (Lemma 3.1)

PROOF OF THEOREM C: The proof will be given in PL. This is the collection of piecewise affine maps, not defined on the interval, but defined on a finite union of intervals. The proof of Theorem C will be by contradiction. Suppose that $PL([0, 1])$ has an infinitely renormalizable map. Then also PL has an infinitely renormalizable map. Let $f \in PL$ be an infinitely renormalizable map, whose number of corner points is minimal. Denote the cycles by

$$\mathcal{I}^1 \supset \mathcal{I}^2 \supset \mathcal{I}^3 \supset \dots$$

Use the notation $q_n = \#\mathcal{I}^n$. We may assume that this sequence of cycles is complete. This means that if there is some cycle $\mathcal{I}$ with $\mathcal{I}^{n+1} \subset \mathcal{I} \subset \mathcal{I}^n$ then $\#\mathcal{I} = q_n$ or $\#\mathcal{I} = q_{n+1}$. It can be shown that the infinitely renormalizable invariant set is a minimal Cantor set (this follows from the non-existence of wandering intervals for maps in PL. The minimality of the action of f on its infinitely renormalizable Cantor set and the minimality of the number of corner points allow us to assume

CLAIM 1. $C_f \subset int(\cup \mathcal{I}^n)$ for all $n \in \mathbb{N}$. ■ (Claim 1)

Let $V = Var(\log(|Df|))$ and set $R_{n,j} = f^{q_n}|I^n_j$, $n \in \mathbb{N}$ and $j = 0, 1, 2, \dots, q_n - 1$.

CLAIM 2. There exists $K \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $j < q_n$

$$|DR_{n,j}(x)| \leq K,$$

for all $x \in I^n_j$.

PROOF: Observe that $R_{n,j} \in PL(I^n_j)$ and that it has uniform, that is independent of n and j, bounds on the numbers of corner points. Furthermore

$$Var(\log(|DR_{n,j}|)) \leq V.$$

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Now Claim 2 follows easily.   ∎ (Claim 2)

Let P_n be a splitting for the pair $\mathcal{I}^n \supset \mathcal{I}^{n+1}$. Claim 1 allows us to define

$$B_n = \sum_{x \in \partial(\cup \mathcal{I}^n)} \log(|Df(x)|),$$

$$M_n = \sum_{x \in P_n} \log(|Df(x)|).$$

CLAIM 3. *For every $n \in \mathbb{N}$*

$$B_{n+1} = B_n + 2M_n.$$

In particular

$$B_n = B_1 + \sum_{k=1}^{n-1} 2M_n.$$

PROOF: Consider $y \in \partial I_j^{n+1} \subset I_i^n$. If y is also in the boundary of some gap G, then $Df(y) = Df(x)$, where $x \in P_n \cap G$. This equality holds because all corner points are in $\cup \mathcal{I}^{n+1}$. In the other case, y is not in the boundary of some gap then $Df(y) = Df(z)$, where $z \in \partial I_i^n$. Again equality holds because $C_f \subset \cup \mathcal{I}^{n+1}$.

Observe that all boundary points of $\cup \mathcal{I}^n$ are counted once and all splitting periodic points twice.   ∎ (Claim 3)

CLAIM 4. *For every periodic point x, with period p, $\log(|Df^p(x)| > 0$. In particular*

$$M_n > 0,$$

for all $n \in \mathbb{N}$.

PROOF: It is easy to see that a periodic attractor attracts a corner point. Because $q_n \geq 2^{n-1}q_1 \to \infty$ every periodic orbit falls eventually outside $\cup \mathcal{I}^n$. An attracting periodic orbit would take a corner point with it, contradicting the minimality of corner points of f. The non-existence of neutral periodic orbits follows by a similar argument.   ∎ (Claim 4)

By Claims 3 and 4 we know that the sequence B_n increases.

CLAIM 5.

$$\lim_{n \to \infty} B_n = B < \infty.$$

PROOF: The corners of f are denoted by $C_f = \{c_1, c_2, \ldots, c_d\}$. Take $x \in I_0^n$. Furthermore let $c_j \in I_{k_j}^n$ and $v_j = \log(|Df_+(c_j)|) - \log(|Df_-(c_j)|)$, $j = 1, 2, \ldots, d$ (the $+$ and $-$ means right and left derivatives). From [MMS] we know that f does not have wandering intervals. This implies that the length of the intervals in $\mathcal{I}^n$ tends to zero for $n \to \infty$. So (in PL) we may assume that every interval in $\mathcal{I}^n$ contains at most one corner point.

It is an easy exercise to compare $\log(|Df^{q_n}(x)|)$ with B_n. There exist numbers $\sigma_{n,j}(x) \in \{-1, 1\}$ such that

$$2 \log(|Df^{q_n}(x)|) = B_n + \sum_{j=1}^{d} \sigma_{n,j}(x)v_j,$$

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where $\sigma_{n,j}(x) = 1$ if and only if $f^{k_j}(x) \in I_{k_j}^2$ is on the right of c_j (remember c_j is the unique corner point in $I_{k_j}^2$).

From Claim 2 we get a bound on $|Df^{q_n}(x)|$. The finite sum in the right hand side of the above equation is clearly uniformly bounded. Hence we get a uniform bound on B_n.

<div align="right">■ (Claim 5)</div>

As a consequence we get

$$\lim_{n \to \infty} M_n = 0.$$

But now we can apply the Expansion-Lemma: eventually $\mathcal{I}^{n+1}$ is a doubling of $\mathcal{I}^n$, for all $n \geq n_0$. This means that the splitting P_n becomes a single periodic orbit hitting every component of the cycle $\mathcal{I}^n$ exactly once.

As above we get the existence of numbers $\sigma_{n,j} \in \{-1, 1\}$ such that

$$2M_n = B_n + \sum_{j=1}^{d} \sigma_j^n v_j.$$

Because $M_n \to 0$ and $B_n \to B$ and the fact that the sum in the above equality takes only finitely many values we get eventually

$$\sum_{j=1}^{d} \sigma_j^n v_j = -B.$$

Combining this with $B_{n+1} = B_n + 2M_n$ we get

$$B_{n+1} = 2B_n - B.$$

For the sequence B_n to be bounded we need $B_n \equiv B$. So $M_n \equiv 0$, contradicting Claim 4.

<div align="right">■ (Theorem C)</div>

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