Geometry of quadratic polynomials: moduli, rigidity and local connectivity.

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§1. Introduction.

A key problem in holomorphic dynamics is to classify complex quadratics $z \mapsto z^2 + c$ up to topological conjugacy. The Rigidity Conjecture would assert that any non-hyperbolic polynomial is *topologically rigid*, that is, not topologically conjugate to any other polynomial. This would imply density of hyperbolic polynomials in the complex quadratic family (Compare Fatou [F, p. 73]). A stronger conjecture usually abbreviated as MLC would assert that the Mandelbrot set is locally connected (see [DH1]).

A while ago MLC was proven for quasi-hyperbolic points by Douady and Hubbard, and for boundaries of hyperbolic components by Yoccoz. More recently Yoccoz proved MLC for all at most finitely renormalizable parameter values (see [H], [M2] for the exposition of this work and closely related work of Branner and Hubbard [BH] on rigidity of cubics). One of our goals is to prove MLC for some infinitely renormalizable parameter values. Loosely speaking, we need all renormalizations to have bounded combinatorial rotation number (assumption C1) and sufficiently high combinatorial type (assumption C2) (see §2 for the precise statement of the assumptions).

This result is based on a complex version of a theorem of [L2] which says that the scaling factors characterizing the geometry of a real non-renormalizable quasi-quadratic map decay exponentially. Its complex counterpart proved below (Theorem I) says that the moduli of the principle nest of annuli grow linearly (this result does not need any a priori assumptions). This makes finitely renormalizable maps *geometrically tame* in the sense that the return maps are becoming purely quadratic in small scales. In the infinitely renormalizable case satisfying assumptions (C1) and (C2) Theorem I implies complex a priori bounds (Theorem II) (that is, the bounds from below for the moduli of the fundamental annuli of $R^n f$).

For real quadratic polynomials of bounded combinatorial type the complex a priori bounds were obtained by Sullivan [S]. Our result complements the Sullivan's result in the unbounded case. Moreover, it gives a background for Sullivan's renormalization theory for some bounded type polynomials outside the real line where the problem of a priori bounds was not handled before for any single polynomial.

An important consequence of a priori bounds is absence of invariant measurable line fields on the Julia set (McMullen [McM]) which is equivalent to quasi-conformal (qc) rigidity. To prove stronger topological rigidity we construct a qc conjugacy between any two topologically conjugate polynomials (Theorem III). We do this by means of a pull-back argument, based on the linear growth of moduli and a priori bounds. Actually the argument gives the stronger *combinatorial rigidity* which implies MLC.

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Local connectivity of the Julia set is also a general consequence of a priori bounds (see Hu and Jiang [HJ], [J]), so we have it under assumptions (C1) and (C2). Note that Douady and Hubbard gave an example of an infinitely renormalizable polynomial with non-locally connected Julia set (see Milnor's version of the example in [M2]). In this example the combinatorial rotation numbers of the fixed points of $R^m f$ are highly unbounded, which is ruled out by our first assumption.

We complete the paper with an application to the real quadratic family. Here we can give a precise dichotomy (Theorem IV): on each renormalization level we either observe a big modulus, or essentially bounded geometry. This allows us to combine the above considerations with Sullivan's argument for bounded geometry case, and to obtain a new proof of the rigidity conjecture on the real line (compare McMullen [McM] and Swiatek [Sw]).

This paper is organized as follows. §2 contains a combinatorial framework: Yoccoz puzzle, principle nest, usual and generalized renormalization. Theorems I and II on geometric moduli in the dynamical plane are proved in §3, Theorem III yielding MLC under the above assumptions is proved in §4. The real case is discussed in §5.

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§2. Principle nest and renormalization.

We refer to [D1], [DH2] and [M1] for the background in polynomial-like mappings and tuning (which is called below "quadratic-like renormalization"), and to [H] and [M2] for the introduction to the Branner-Hubbard-Yoccoz puzzle.

Let $f : U' \to U$ be a quadratic-like map with connected Julia set. Let us start with an appropriate combinatorially defined c-symmetric domain ("puzzle-piece") $V^0 \subset U'$ such that (see the construction below). Let us then consider the first return of the critical point back to V^0, and pull V^0 back along the corresponding piece of the critical orbit. In such a way we obtain the critical puzzle-piece $V^1 \subset V^0$ of the first level. If we do the same replacing V^0 by V^1, we obtain the critical puzzle-piece $V^2 \subset V^1$ of the second level. Proceeding in this manner we will construct the *principle nest*

$$V^0 \supset V^1 \supset V^2 \dots$$

of puzzle-pieces. It may happen that on some level the quadratic-like map $f^{\pi(t)} : V^{t+1} \to V^t$ has a connected Julia set (which is equivalent to having non-escaping critical point). Then we say that f is *q-renormalizable*, or that f admits the *quadratic-like renormalization* $Rf = f^{\pi(t)} : V^{t+1} \to V^t$ (usually such a map is just called "renormalizable" but we need to distinguish the quadratic-like renormalization from the generalized renormalization defined below). In this case the puzzle-pieces V^n shrink down to the Julia set $J(Rf)$. Otherwise by the Yoccoz Theorem they shrink down to the critical point.

Let now $V_i^n \subset V^{n-1}$ denote the pull-backs of V^{n-1} corresponding to the first returns of the points $x \in \omega(c) \cap V^{n-1}$ back to V^{n-1}, numbered so that $V^n \equiv V_0^n$. The first return map

$$g_n : \cup V_i^n \to V^{n-1}$$

we call the n-fold (generalized) *renormalization* of f. If f admits a quadratic-like renormalization, then the return time to V^{n-1} is uniformly bounded on $\omega(c)$, and hence the domain of the renormalized maps consists of only finitely many components V_i^n. We select the initial puzzle-piece V^0 in such a way that V^1 is compactly contained in V^0 (see Lemma 0 below). Then the puzzle-pieces V_i^n, $n \geq 2$, are compactly contained in V^{n-1} as well, and hence the g_n are *generalized polynomial-like maps* in the sense of [L1].

Let us call this return to level $n-1$ "central" if $g_n c \in V^n$. If we have several subsequent central returns, we refer to a *cascade* of central returns. In the q-renormalizable case the sequence of puzzle-pieces V^n ends with an infinite cascade of central returns. Let us denote by $\kappa = \kappa(f)$ the number of the levels on which the non-central return occurs. The map f admits a quadratic-like renormalization iff $\kappa < \infty$.

Let us now construct the initial puzzle-piece V^0. Let q/p be the combinatorial rotation number of the dividing fixed point α. This means that there are p disjoint *external rays* Γ_i landing at α which are permuted by the dynamics with rotation number q/p. They cut U' into p initial *Yoccoz puzzle-pieces*. (Warning: we define the external rays in a non-canonical way via a conjugacy to a polynomial. These rays are not necessarily the external rays for the original map f, whose geometry we would not be able to control).

Let Ω^0 be the critical puzzle-piece, that is, the one containing the critical point. Let us pull it back along the critical orbit in the same way as we did above with V-pieces. Then in the beginning we may observe a central cascade

$$\Omega^0 \supset \Omega^1 \supset \dots$$

In what follows we always assume that this first cascade is finite, that is, there is an N, such that $f^p c \in \Omega^{N-1} \setminus \Omega^N$ (this can be viewed as a part of Assumption (C2) that the combinatorial type is sufficiently high).

Let now γ and $\gamma' = -\gamma$ be the periodic and co-periodic points of period p belonging to Ω^0. Let us truncate Ω^N by the external rays landing at these points. The critical puzzle-piece obtained in such a way is the desired V^0 (see Figure 1).

Lemma 0. *The puzzle-piece V^1 is compactly contained in V^0.*

Proof. The argument below is not the shortest possible, but it will later give us important extra information (see Lemma 2). Let W_i, $i = 1, ..., p-1$, be the puzzle-pieces bounded by the external rays landing at α' and the equipotential $f^{-1}U$, numbered in such a way that $f^i W_i \supset U_0$.

Take a point $z \in \Omega^0 \cap J(f)$, push it forward by iterates of $g = f^p$, and find the first moment $r = r(z)$ (if any) such that $g^r z$ lands either at W_i (where $i = i(z)$) or at V^0. In the first case consider the pull-back $X(z)$ of W_i along the orbit of z, in the second case consider the pull-back $Y(z)$ of V^0. (The points which are not covered by the sets X_j and Y_j form an invariant Cantor set in $\Omega^N \setminus V^0$.)

Let us now define a map $G : X_j \cup Y_j \to U^0 \cup V^0$ in the following way: $G = f^i \circ g^r$ on X_j and $G = g^r$ on X_j. Then every X_j is univalently mapped onto Ω^0, while Y_j is univalently mapped onto V^0. Let us push c forward by iterates of G until the first moment l it lands at a set Y_j. Let $Q \ni c$ be the pull-back of Ω^0 under G^l. Then

$$G^l : (Q, V^1) \to (\Omega^0, Y_j). \tag{0}$$

As Y_j is compactly contained in Ω^0, V^1 is compactly contained in Q. Observe finally that $Q \subset V^0$, since Q may not intersect the boundary of V^0. $\square$

If f infinitely q-renormalizable then we can repeat the above construction on the corresponding quadratic-like levels, and consider the full *canonical nest* of puzzle-pieces:

$$V^{0,0} \supset V^{0,1} \supset \ldots V^{0,t(0)+1} \supset V^{1,0} \supset V^{1,1} \supset \ldots \supset V^{1,t(1)+1} \supset V^{2,0} \ldots$$

Here the first index counts the quadratic-like levels, while the second one counts the levels in between. The maps $V^{m,t(m)+1} \to V^{m,t(m)}$ are quadratic-like with non-escaping critical point, while $V^{m+1,0}$ are the critical puzzle-pieces obtained by the above procedure applied to these maps. (The choice of the cutting level $t(m)$ is not canonical). We skip the first index when we work in between two quadratic-like levels.

Let us finish this section with specifying exact conditions under which we will prove MLC. To these end we need several notions. A *limb* of the Mandelbrot M set is the connected components of $M \backslash \{c_0\}$ (which does not contain 0) where c_0 is a bifurcation point on the main cardioid. A limb is specified by specifying a combinatorial rotation number at the dividing fixed point. If we remove from a limb a neighborhood of its root c_0, what is left we call a *truncated* limb. By a *(truncated) secondary limb* we mean the similar object corresponding to the second bifurcation from the main cardioid (see Figure 2).

Two quadratic-like maps are called *hybrid (or internal) equivalent* if they are conjugate by a qc map h with $\bar\partial h = 0$ almost everywhere on the Julia set. By the Douady-Hubbard Straightening Theorem [DH2], any hybrid class with connected Julia set contains a unique quadratic polynomial $z \mapsto z^2 + c$. So such hybrid classes are labeled by points on the Mandelbrot set.

Let $\mathcal{F}$ denote the class of maps admitting infinitely many quadratic-like renormalizations and satisfying the following assumptions:

(C1). First select in the Mandelbrot set a finite number of truncated secondary limbs. We require the hybrid classes of all quadratic-like renormalizations $R^m f$ to be picked from these limbs.

(C2). On the other hand, we also require the *combinatorial type* $\kappa(R^m f)$ to be sufficiently high on all levels (depending on the a priori choice of limbs).

The second condition can be improved by specifying other combinatorial factors producing a big space (see the subsection with Lemma 11 and Lemma 16).

§3. Geometric moduli.

Let us summarize the results of the section in two Theorems. We say that a quadratic-like map $f : U' \to U$ has a definite modulus if mod$(U \backslash U') \geq \bar\mu > 0$ (with an a priori selected quantifier $\bar\mu$).

Theorem I. *Let f be a polynomial-like map with a definite modulus whose internal class is selected from a given finite family of truncated secondary limbs. Let $n(k)$ count the levels of non-central returns (preceding the next quadratic-like level). Then the principle moduli $\mu_{n(k)+1} = \mathrm{mod}(V^{n(k)} \backslash V^{n(k)+1})$ grow with k at uniformly linear rate.*

Theorem II. *Let $f \in \mathcal{F}$. Then all its quadratic-like renormalizations $R^n f$ have definite moduli.*

A compact set $K \subset \bar{C}$ is called *removable* if given a neighborhood $U \supset K$, any conformal embedding $\phi : U\backslash K \to \bar{\mathbf{C}}$ allows the conformal continuation across K (see [AB]). A simple condition for removability is the following.

Assume that for any point $z \in K$ there is a nest of disjoint annuli $A_i \subset \bar{\mathbf{C}}\backslash K$ with definite moduli ($\mathrm{mod}(A_i) > \delta > 0$) shrinking to z. Then K is removable.

Removable sets have zero Lebesgue measure. Now Theorem II immediately implies.

Corollary IIa. *Given an $f \in \mathcal{F}$, its critical set $\omega(c)$ is a removable Cantor set.*

By [HJ], [J] the a priori bounds also imply the following (see the argument in §3).

Corollary IIb. *The Julia set $J(f)$ of a map $f \in \mathcal{F}$ is locally connected.*

According to McMullen [McM], an infinitely q-renormalizable quadratic polynomial f is called *robust* if for arbitrary high level m there exists an annulus in $\mathbf{C}\backslash\omega(c)$ with definite modulus which is homotopic rel $\omega(c)$ to a Jordan curve enclosing $J(R^m f)$ but not enclosing any point of $\omega(c)\backslash J(R^m f)$.

Corollary IIc. *Any $f \in \mathcal{F}$ is robust.*

By [McM], robust quadratic polynomials have no invariant measurable line fields on the Julia set. Absence of invariant line fields for a quadratic polynomial $f : z \mapsto z^2 + c_0$ is equivalent to the property that its topological class has empty interior [MSS]. Theorem III below will show that these topological classes are actually single points for $f \in \mathcal{F}$.

Outline for Theorem II. First we show that if a quadratic-like map f satisfying (C1) has a definite modulus then the first annulus of the principle nest also has a definite modulus. However the bound for this modulus is certainly smaller than the a priori bound for f. To compensate this loss, we go through the cascade of generalized renormalizations, and observe (according to Theorem I) a linear growth of the principle moduli. So if we proceed for long enough (assumption (C2)), we will arrive at the next quadratic-like level with a definite modulus controlled by the same quantifier $\bar{\mu}$. Then we start over again.

Most of this section is occupied with the proof of Theorem I.

Initial geometry.

Lemma 1. *If the annulus A has a definite modulus then the starting configuration (U, γ_i) of external rays has a bounded geometry.*

Proof. Indeed, the map f can be conjugate to a polynomial g by a qc map with a bounded dilatation, where g belongs to the finite set of selected limbs. Let g vary within one of these limbs. Then the finite intervals of the external rays vary continuously with g.

Since the truncated limbs don't touch the main cardioid, the absolute value of the multiplier λ of the α-fixed point of g is bounded away from 1. Hence the fundamental annulus around this point has a definite modulus. So the external rays landing at α will meet this annulus on some definite distance from the Julia set. Outside of the annulus they have a bounded geometry by the previous argument. Near the fixed point the geometry is bounded by a local consideration. □

Set $A^n = V^{n-1}\backslash V^n$.

Lemma 2. The annulus $A^1 = V^0 \smallsetminus V^1$ has a definite modulus (depending on the modulus of $U\backslash U'$ only).

Proof. Let us go back to the proof of Lemma 0. Because of Assumption (C1), V^0 is well inside Ω^0. As the puzzle-pieces Y_j are obtained by pulling V^0 back by univalent iterates of g. they are well inside Ω^0 as well. Finally, as G^l in (0) is two-to-one branched covering, V^1 is well inside of Q. □

A priori bounds.

Lemma 3. *Let $i(1), ...i(l) = 0$ be the itinerary of a puzzle-piece V_j^{n+1} through the puzzle-pieces V_i^n by iterates of g_n until the first return back to V^n. Then*

$$\mathrm{mod}(V^n\backslash V_j^{n+1}) \geq \frac{1}{2}\sum_{k=1}^{l} \mathrm{mod}(V^{n-1}\backslash V_{i(k)}^n).$$

Proof. The Grötcsz inequality. □

Let D be a puzzle-piece which we call an "island" (compare below). Let W_i, $i \in I$, be a finite family of disjoint puzzle-pieces containing a critical puzzle-piece W_0. We will freely identify the label set I with the family itself. For $W_i \subset D$ let

$$R_i \equiv R_i(I, D) \subset D\backslash \bigcup_{j\in I} W_j$$

be an annulus of maximal modulus enclosing W_i but not enclosing other puzzle-pieces of the family I. Such an annulus exists by the Montel Theorem. We will briefly call it the *maximal annulus* enclosing W_i in D (rel the family I).

Let us now define *the asymmetric modulus of the group I in D* as

$$\sigma(I|D) = \sum_{i\in I} \epsilon_i \mathrm{mod}(R_i), \tag{1}$$

where the weight ϵ_i is equal to 1 for the critical puzzle-piece and 1/2 for all others (if D is a non-critical island then all weights are actually 1/2). This parameter for a group of two puzzle-pieces was suggested by Jeremy Kahn as a complex analogue of the asymmetric Poincaré length [L2].

Let us now specify $D = V^{n-1}$, and I to be a finite group of at least two puzzle-pieces V_i^n of level n containing the critical one. Then set $\sigma_n(I) \equiv \sigma(I|V^{n-1})$ and

$$\sigma_n = \min_I \sigma_n(I), \tag{2}$$

where I runs over all groups of puzzle-pieces just specified.

Let us use a special notation for the *principle moduli*

$$\mu_n = \operatorname{mod}(V^{n-1} \backslash V^n). \tag{3}$$

The μ_n and σ_n are the principle geometric parameters of the renormalized maps g_n.

Our goal is to show that the asymmetric moduli monotonically and linearly grow with n. Let us fix a level $n \geq N$, denote $V^{n-1} = \Delta,\ V_i = V_i^n$, $g = g_n$, and mark the objects of the next level $n+1$ with prime.

Let I' be a finite family of puzzle-pieces V_i'. Let us organize them in *isles* in the following way. Take two non-symmetric puzzle-pieces V_i' and V_j' and push them forward by iterates of g through the puzzle-pieces V_k of the previous level. Find the first moment t when they are separated by those puzzle-pieces, that is, such that $g^m V_i'$ and $g^m V_j'$ belong to the same piece $V_{k(m)}$ for $m = 0, ..., t-1,$, while $g^t V_i'$ and $g^t V_j'$ land at different pieces. (In other words, the itineraries of V_i' and V_j' coincide until moment $t-1$). Then let us produce an island D by pulling $V_{k(t-1)}$ back by the corresponding inverse branch of g^{t-1}. Let $\phi_D = g^t : D \to \Delta$. This map is either a double covering or a biholomorphic isomorphism depending on whether D is critical or not.

The family $\mathcal{D} = \mathcal{D}(I')$ of isles form a lattice with respect to inclusion. Let *depth* $: \mathcal{D} \to \mathbf{N}$ be the minimal strictly monotone function on this lattice, assigning to the biggest island $V_0 \equiv \Delta'$ depth 0.

Let us now consider the asymmetric moduli $\sigma(I|D)$ as a function on the family $\mathcal{D}$ of isles. This function is clearly monotone:

$$\sigma(I|D) \geq \sigma(I|D_1) \quad \text{if} \quad D \supset D_1, \tag{4}$$

and superadditive:

$$\sigma(I|D) \geq \sigma(I|D_1) + \sigma(I|D_2), \tag{5}$$

provided D_i are disjoint subisles in D.

We call a puzzle-piece $V_j' \subset D$ *pre-critical rel* D if $\phi_D(V_j') = V_0$. If $D = \Delta'$ is the trivial island, we skip "rel". There are at most two pre-critical pieces in any D. If there are actually two of them, then they are non-critical and symmetric with respect to c.

Let D be a deepest island of family $\mathcal{D}(\mathcal{I}')$, and $V_j',\ j \in J$, be the group of puzzle-pieces contained in D, that is $J = I'|D$. Let $i(j)$ is defined for $j \in J$ by the property $\phi_D(V_j') \subset V_{i(j)}$, and $I = \{i(j) : j \in J\}$.

Lemma 4. *Under the circumstances just described the following estimate holds:*

$$\sigma(J|D) \geq \frac{1}{2}\left((|J|-s)\mu + s \operatorname{mod}(R_0) + \sum_{i \in I, i \neq 0} \operatorname{mod}(R_i)\right), \tag{6}$$

where $s = \#\{j : i(j) = 0\}$ *is the number of pre-critical pieces rel* D.

Proof. As D is the deepest island, each puzzle-piece $V_i,\ i \in I,$ contains a single puzzle-piece $\phi_D V_j$ (though there might be two symmetric puzzle-pieces in J with $\phi_D V_j = \phi_D V_k$). Let $R_i \subset \Delta$ denote an annulus of maximal modulus enclosing V_i rel I, and let $T_j \subset D$ be an annulus of maximal modulus enclosing V_j' rel family J. Let δ_{st} denote the Kronecker symbol. Fix a $j \in J$ and let $i = i(j)$. Let us consider now two cases:

(i) Let V'_j be non-critical. Then

$$\mathrm{mod}(T'_j) \geq \mathrm{mod}(R_i) + \delta_{0i}\ \mu. \tag{7}$$

To see that, observe that $\mathrm{mod}\,(V_i \backslash \phi_D V'_j)$ is at least μ, provided $i \neq 0$. Observe also that the pull-back of the topological disc $Q_i = R_i \cup V_i$ to D is univalent. Indeed, if ϕ_D were a double covering then the island D would be critical, and hence would contain the critical puzzle-piece V'_0. It follows that Q_i does not contain the critical value of ϕ_D.

(ii) Let $V'_j = V'_0$ is critical. Then

$$\mathrm{mod}(T'_0) \geq \frac{1}{2}(\mathrm{mod}(R_i) + \delta_{0i}\ \mu). \tag{8}$$

Summing up the estimates (7) and (8) with the weights 1/2 and 1 correspondingly over the family J, we obtain the desired estimate. □

Corollary 5. *For any island D of the family $\mathcal{D}(I')$ the following estimates hold:*

$$\sigma(I'|D) \geq \frac{1}{2}\mu \quad \text{and} \quad \sigma(I'|D) \geq \sigma.$$

Proof. By monotonicity (4), it is enough to check the case of a deepest island D. Let us use the notations of the previous lemma. Observe first that the family $I = \{i(j) : j \in J\}$ contains at least two puzzle pieces. Indeed, the only case when $|I| < |J|$ can happen is when ϕ_D is a double covering, and there are two symmetric puzzle-pieces in the family J. But then this family must also contain the critical piece V'_0, and hence $|I| > 2$.

As $\mu > \mathrm{mod}(R_0)$, $|J| \geq 2$ and $|I| \geq 2$, the right-hand side in (6) is bounded from below by

$$\frac{1}{2}\left(|J|\,\mathrm{mod}(R_0) + \sum_{i \in I, i \neq 0} \mathrm{mod}(R_i)\right) \geq \sigma(I) > \sigma, \tag{9}$$

□

Let us decompose $g_n : V^n \to V^{n-1}$ as $h_n \circ \Phi$ where Φ is purely quadratic, while h_n is univalent. The *non-linearity* or *distortion* of h_n is defined as

$$\max_{z,\zeta \in V^n} \log \left| \frac{Dh_n(z)}{Dh_n(\zeta)} \right|,$$

and measures how far g_n is from being purely quadratic.

Corollary 6 (a priori bounds). *The asymmetric moduli σ_n grow monotonically and hence stay away from 0 on all levels (until the next quadratic-like level). The basic moduli μ_n stay away from 0 everywhere except for tails of long cascades of central returns. Moreover, the non-critical puzzle-pieces V_i^n are also well inside V^{n-1} except for pre-critical pieces on the levels which immediately follow the long cascades of central returns. The distortion of h_n is uniformly bounded on all levels.*

Proof. On the first non-degenerate level $N+1$ we have a definite principle modulus by Lemma 2. Hence by the previous Corollary we have a definite value of σ on the next level which then begins to grow monotonically. So, it stays definite on all levels until the next quadratic-like one. By Lemma 3, the basic moduli stay definite as well, except for tails of long cascades of central returns. The next statement also follows from Lemma 3.

To check the last statement, it is enough to observe that h_n has a Koebe space spread over V^{n-2}. Hence its distortion is controlled by the principle scaling factor μ_{n-1}. So we are OK outside the tails of central cascades. But observe also that within the central cascade we keep the same return map, just shrinking its domain. □

Linear growth. Our goal is to prove that $\sigma' \geq \sigma + a$ with a definite $a > 0$ at least on every other level except for the tails of central cascades. Corollary 6 shows the reason why these tails play a special role. The growth rate of σ definitely slows down in the tails. So let us assume that the level $n-1$ is not there, so that the principle modulus μ is definitely positive.

Corollary 7. If a deepest island D contains at least three puzzle-pieces V'_j, $j \in J$, then

$$\sigma(J|D) \geq \sigma(I) + \frac{1}{2}\mu.$$

Proof. Let us in (6) split off $(1/2)\mu$ and estimate all other μ's by $\mathrm{mod}\,(R_0)$. This estimates the right-hand side by

$$\frac{1}{2}\mu + \frac{|J|-1}{2}\mathrm{mod}(R_0) + \frac{1}{2}\sum_{i \in I, i \neq 0} \mathrm{mod}(R_i),$$

which immediately yields what is claimed. □

Let us now consider the case when the island D contains only two puzzle puzzle-pieces. In order to treat it, we need some preparation in geometric function theory.

Moduli defect, capacity and eccentricity. Let D be a topological disk, $\Gamma = \partial D$, $a \in D$, and $\psi : (D, a) \to (\mathbf{D}_r, 0)$ be the Riemann map onto a round disk of radius r with $\psi'(a) = 1$. Then $r \equiv r_a(\Gamma)$ is called the *conformal radius* of Γ about a. The *capacity* of Γ rel a is defined as

$$\mathrm{cap}_a(\Gamma) = \log r_a(\Gamma).$$

Lemma 8. *Let $D_0 \supset D_1 \supset K$, where D_i are topological disks and K is a connected compact. Assume that the hyperbolic diameter of K in D_0 and the hyperbolic dist $(K, \partial D_1)$ are both bounded by a Q. Then there is an $\alpha(Q) > 0$ such that*

$$\mathrm{mod}(D_1 \backslash K) \leq \mathrm{mod}(D_0 \backslash K) - \alpha(Q).$$

Proof. Let us take a point $z \in \partial D_1$ whose hyperbolic distance to K is at most Q. Then there is an annulus of a definite modulus contained in D_0 and enclosing both K and z.

Let us uniformize $D_0 \backslash K$ by a round annulus $A_r = \{\zeta : r < |\zeta| < 1\}$, and let $\tilde{z}$ correspond to z under this uniformization. Then $\tilde{z}$ stays a definite Euclidian distance d from the unit circle.

If $R \subset A_r$ is any annulus enclosing the inner boundary of A_r but not enclosing $\tilde{z}$ then by the normality argument $\mathrm{mod}\,(R) < \mathrm{mod}(A_r) - \alpha_r(d)$ with an $\alpha_r(d) > 0$. (Actually, the extremal annulus is just A_r slit along the radius from $\tilde{z}$ to the unit circle).

We have to check that $\alpha_r(d)$ is not vanishing as $r \to 0$. Let us fix an outer boundary Γ of B (the unit circle + the slit in the extremal case). We may certainly assume that the inner boundary coincides with the r-circle. Then the defect $\text{mod}\,(R) - \log(1/r)$ monotonically increases to the $\text{cap}_0(\Gamma)$. By normality this capacity is bounded above by an $-\alpha(d) < 0$, and we are done. □

Let A be a standard cylinder of finite modulus, $K \subset A$. Let us define the mod(K) as the modulus of the smallest concentric sub-cylinder $A' \subset A$ containing K (see Figure 3).

Lemma 9 (Definite Grötcsz Inequality). *Let A_1 and A_2 be homotopically non-trivial disjoint topological annuli in A. Let K be the set of points in their complement which are separated by $A_1 \cup A_2$ from the boundary of A. Then there is a function $\beta(x) > 0$ ($x > 0$) such that*

$$\text{mod}(A) \geq \text{mod}(A_1) + \text{mod}(A_2) + \beta(\text{mod}(K)).$$

Proof. For a given cylinder this follows from the usual Grötcsz Inequality and the normality argument. Let us fix a K, and let $\text{mod}\,(A) \to \infty$. We can assume that A_i are lower and upper components of $A \backslash K$ correspondingly. Then the modulus defect

$$\text{mod}(A) - \text{mod}(A_1) - \text{mod}(A_2)$$

decreases by the usual Grötcsz inequality. At the limit the cylinder becomes the punctured plane, and the modulus defect converges to -($\text{cap}_0(K)+ \text{cap}_\infty(K)$). It follows from the area estimates that this sum of capacities is negative, unless K is a circle centered at the origin. This estimates depends only on $\text{mod}\,(K)$ by normality. □

Let $d_a(\Gamma)$ and $\rho_a(\Gamma)$ be the Euclidian radii of the inscribed and circumscribed circles about Γ centered at a. Then let us define the eccentricity of Γ about a as

$$e_a(\Gamma) = \log \frac{\rho_a(\Gamma)}{d_a(\Gamma)}.$$

By Koebe and Schwarz,

$$e_a(\Gamma) = -(\text{cap}_a(\Gamma) + \text{cap}_\infty)(\Gamma) + O(1),$$

with $O(1) \leq 2 \log 4$.

Lemma 10. *Under the circumstance of Lemma 9 assume also that the annulus $A \subset \mathbf{C} \backslash \{a\}$ is homotopically non-trivially embedded in the punctured plane, and mod $(A_i) \geq \alpha > 0$. If $e_a(K)$ is big then mod $(K|A)$ is big as well.*

Proof. Let us consider the uniformization $\phi : \bar{A} \to A$ of A by a round annulus. If $\text{mod}\,(K|A)$ is bounded then $\bar{K}$ is well inside of $\bar{A}$. Then by the normality argument K must have a bounded eccentricity about a. □

The case of two puzzle-pieces. Let us now go back to the estimates of asymmetric moduli. Suppose we have a deepest island D containing two puzzle-pieces V_j^{n+1}, $j \in J$. Let $\phi \equiv \phi_D$ and let $\phi V_j^{n+1} \subset V_i^n$ with $i = i(j)$. Let us split the argument into several cases.

Case (i). There is a non-critical puzzle-piece V_i^n, $i \in I$, which stays on a bounded Poincaré distance in V^{n-1} (controlled by a given big quantifier Q) from the critical point.

Then by Lemma 8

$$\mu_n \geq \mathrm{mod}(R_0) + \alpha \tag{10}$$

with a definite $\alpha = \alpha(Q) > 0$. But observe that when we passed from Lemma 4 to Corollary 5 we estimated μ by $\mathrm{mod}\,(R_0)$. Using the better estimate (10), we obtain a definite increase of σ.

Case (ii). Let each non-critical puzzle-piece V_i^n, $i \in I$, stay hyperbolically far away from the critical point. Then V_0^n may not belong to any non-trivial island together with some non-critical piece V_i^n, $i \in I$. Indeed, it follows from Corollary 6 that any non-trivial island is well inside of V^{n-1}.

Assume first that both V_i^n are non-critical. Then $\sigma(J|D)$ is estimated by $\sigma_n(\tilde{I})$ where the family $\tilde{I}$ consists of V_i^n and the central puzzle-piece V_0^n. If no two of these puzzle-pieces belong to the same non-trivial island, then by Corollary 7 $\sigma(\tilde{I}) \geq \sigma_{n-1} + a$ with a definite $a > 0$.

Otherwise the puzzle-pieces V_i^n, $i \in I$, belong to an island W. Since W is well inside of V^{n-1}, it stays on the big Poincaré distance from the critical point. Hence $\mathrm{mod}\,(R_0) \approx \mu$ (this sign means the equality up to a small constant controlled by the quantifier Q, while the sign $\succ$ below means the inequality up to a small error), and

$$\sigma(\tilde{I}) \geq \sigma(I|Q) + \mathrm{mod}(R_0) \succ \sigma^{n-1} + \mu.$$

So we have gained some extra growth, and can pass to the next case.

Fibonacci returns. *Let one of the puzzle-pieces V_i^n be critical.* So we have the family I^n of two puzzle-pieces V_0^n and V_1^n. Remember that we also assume that *the hyperbolic distance between these pieces is big*. Hence, V^{n-1} is the only island containing both of them, so that $g_{n-1}V_0^n$ and $g_{n-1}V_1^n$ belong to different puzzle-pieces of level $n-1$. For the same reason we can assume that one of these puzzle-pieces is critical. Denote them by V_0^{n-1} and V_1^{n-1}. Then one of the following two possibilities on level $n-2$ can occur:

1) *Fibinacci return* when $g_{n-1}V_0^n \subset V_1^{n-1}$ and $g_{n-1}V_1^n = V_0^{n-1}$ (see Figure 4);

2) *Central return* when $g_{n-1}V_0^n = V_0^{n-1}$ and $g_{n-1}V_1^n \subset V_1^{n-1}$.

We can assume that one of these schemes occur on several previous levels $n-3, n-4, ...$ as well (otherwise we gain an extra growth by the previous considerations). To fix the idea, let us first consider the following particular case

Fibonacci cascade. Assume that on both levels $n-1$ and $n-2$ the Fibonacci returns occur. Let us look more carefully at the estimates of Lemma 4. In the Fibonacci case we just have:

$$\mathrm{mod}(R_1^n) \geq \mathrm{mod}(R_0^{n-1}), \tag{11}$$

$$\mathrm{mod}(R_0^n) \geq \frac{1}{2}\mathrm{mod}(g_{n-1}V_0^n|Q_1^{n-1}), \tag{12}$$

where $Q_i^n = V_i^n \cup R_i^n$. Applying g_{n-2} we see that

$$\mathrm{mod}(Q_1^{n-1}\backslash g_{n-1}V_0^n) \geq \mathrm{mod}(Q_0^{n-3}\backslash V_0^{n-1}). \tag{13}$$

But since V_1^{n-2} is hyperbolically far away from the critical point,

$$\mathrm{mod}(Q_0^{n-3}\backslash V_0^{n-1}) \approx \mathrm{mod}(V_0^{n-3}\backslash V_0^{n-1}). \tag{14}$$

By the Grötcsz Inequality there is an $a \geq 0$ such that

$$\mathrm{mod}(V_0^{n-3}\backslash V_0^{n-1}) = \mu_{n-1} + \mu_{n-2} + a. \tag{15}$$

Clearly

$$\mu_{n-1} \geq \mathrm{mod}(R_0^{n-1}). \tag{16}$$

Furthermore, let P_1^{n-1} be the pull-back of Q_0^{n-2} by g_{n-2}. Since $\partial(P_1^{n-1})$ is hyperbolically far away from V_1^{n-1}, we have:

$$\mu_{n-2} \geq \mathrm{mod}(R_0^{n-2}) = \mathrm{mod}(V_1^{n-1}|P_1^{n-1}) \approx \mathrm{mod}(R_1^{n-1}). \tag{17}$$

Combining estimates (12)-(17) we get

$$\mathrm{mod}(R_0^n) \succ \frac{1}{2}(\mathrm{mod}(R_0^{n-1}) + \mathrm{mod}(R_1^{n-1}) + a). \tag{18}$$

We see from (11) and (18) that the only thing to check that the constant a in (15) is definitely positive. Assume that this is not the case. Set $\Gamma_n = \partial V^n$. Then by the Definite Grötscz Inequality, the mod (Γ_{n-2}) in the annulus $A = V^{n-3}\backslash V^{n-1}$ is very small. Since Γ_{n-2} is well inside of A, we conclude by the Koebe Distortion Theorem that Γ_{n-2} is contained in a narrow neighbourhood of a curve γ with a bounded geometry. Moreover, this curve has a definite eccentricity around the critical point.

On the other hand, the puzzle-piece V_1^{n-1} is hyperbolically far away from the critical point. Hence it must be located Euclidianly very close to Γ_{n-2} (relatively the Euclidian distance to the critical point). Hence the critical value $g_{n-1}c$ is also extremely close to Γ_{n-2}.

By Corollary 6, g_{n-1} is a quadratic map up to a bounded distortion. Hence the curve Γ_{n-1} which is the pull-back of Γ_{n-2} by g_{n-1} must have a huge eccentricity around the critical point. By Lemmas 10 and 9 it will contribute towards the definite extra constant on the $(n+1)$ st level.

Remark. The actual shape of a deep level puzzle-piece for the Fibonacci cascade is shown on Figure 5. There is a good reason why it resembles the filled-in Julia set for $z \mapsto z^2 - 1$ (see [L3]). As the geodesic in V_0^{n-1} joining the puzzle-pieces V_0^n and V_1^n goes through the pinched region, the Poincaré distance between these puzzle-pieces is, in fact, big.

General case. Let us now allow the central returns along with the Fibonacci ones. Suppose we have a cascade of central returns on $N-1$ subsequent levels $V^m \supset ... \supset V^{m+N-2} \equiv V^{n-2}$, preceded by the Fibonacci return on level $m-1$. So $g_{m+1}c \in V_0^{m+N-1}$, while $g_m c \in V_1^m$. By our convention, this cascade is not too long, so that we have a definite space in between any two levels.

Let us now pass from the island $D \subset V^n \equiv V^{m+N}$ all way up the cascade to the level $m-1$, that is, consider the map

$$G = g_m \circ g_{m+1}^{N-1} \circ \phi_D : D \to V^{m-1}. \tag{19}$$

Then $S \equiv G\, V^{m+N+1} \subset V^m$. Now we again should split the analysis depending on where the puzzle-piece V^{m+N+1} lands. Let us start with the most interesting case when it lands at the deepest possible level.

Subcase (a). Let $S = V^{m+N}$. Pulling the annuli R_0^m and R_1^m back by G to D, we get the following estimates:

$$\mathrm{mod}(R_0^{m+N+1}) \succ \frac{1}{2}\mathrm{mod}(V^{m-1}\backslash S) = \frac{1}{2}\mathrm{mod}(V^{m-1}\backslash V^{m+N}), \tag{20}$$

$$\mathrm{mod}(R_1^{m+N+1}) \geq \frac{1}{2^N}(\mathrm{mod}(R_1^m) + \mathrm{mod}(R_0^m)). \tag{21}$$

By the Grötcsz inequality there is an $a \geq 0$ such that

$$\mathrm{mod}(V^{m-1}\backslash V^{m+N}) \geq \mathrm{mod}(A^m) + \mathrm{mod}(V^m\backslash V^{m+N}) + a \geq$$

$$\mathrm{mod}(R_0^m) + \sum_{k=m+N}^{m+1} \mathrm{mod}(A^k) + a \geq \mathrm{mod}(R_0^m) + \sum_{k=0}^{N-1} \frac{1}{2^k}\mathrm{mod}(R_0^{m+1}) + a. \tag{22}$$

Since $\mathrm{mod}(R_0^m) \approx \mathrm{mod}(R_1^{m+1})$, the above estimates imply

$$2\,\sigma(I^{m+N+1}|D) = 2\,R_0^{m+N+1} + R_1^{m+N+1} \succ$$

$$\frac{1}{2^N}\mathrm{mod}(R_1^m) + \frac{1}{2^{N-1}}\mathrm{mod}(R_0^m) + (1 - \frac{1}{2^N})\mathrm{mod}(R_1^{m+1}) + (2 - \frac{1}{2^{N-1}})\mathrm{mod}(R_0^{m+1}) + a \approx$$

$$\approx \frac{1}{2^{N-1}}\sigma(I^m) + (2 - \frac{1}{2^{N-1}})\sigma(I^{m+1}) + a.$$

We see that if the curve Γ^m has a definite modulus in the annulus $V^{m-1}\backslash V^{m+N}$ then we have a definite growth of σ. Otherwise arguing as in the case of the Fibonacci cascade we conclude that the curve Γ^k has a big eccentricity around the puzzle-piece V_1^{k+1}, $k = m, ..., m+N-1$.

Let us now go one central cascade up to the level V^{m-1} (until the Fibonacci level). If this cascade is not too long, then by the above considerations we either have a definite growth of σ within this cascade, or Γ^{m-1} has a big eccentricity about V_1^m. But then Γ^m has a big modulus in $V^{m-1}\backslash V^{m+N}$, and we are done.

Finally, if V^{m-1} is in the tail of a long central cascades then Γ^m has *always* a big eccentricity about the critical point (see the next subsection). If we actually have a central return on level m (so that $N \geq 2$), then Γ^{m+1} has a big eccentricity around c as well. But this curve is for sure well inside $V^m\backslash V^{m+N}$. So we can use it instead of Γ^m to contribute to the definite a in estimate (22).

If a non-central return on level m occurs (that is, $N = 1$), then we don't see a definite growth for σ_{m+N+1} but we gain it one level down.

Subcase (b). Assume now that $S \subset V^m\backslash V^{m+N}$. Let us consider the Markov family of puzzle-pieces W_i^k, $k = m+1, ..., m+N$, the pull-backs of pieces $V_i^{m+1} \equiv W_i^{m+1}$ to the annuli A^k. Let $S \subset W \equiv W_i^k$. Then

$$\mathrm{mod}(W\backslash S) \geq \mathrm{mod}(V^m\backslash V^{m+N}),$$

and we have

$$\mathrm{mod}(V^{m-1}\backslash S) \geq \frac{1}{2}(\mathrm{mod}(A^m) + \mathrm{mod}(W\backslash S) + \mathrm{mod}(V^m\backslash W)) \succ$$

$$\frac{1}{2}(\mathrm{mod}(R_0^m) + \mathrm{mod}(V^m\backslash V^{m+N}) + a$$

where $a > 0$ is definite, unless V^{m-1} is in the tail of a long central cascade. But then argue as in Subcase (a). Theorem A is proved.

Other factors yielding big space. Theorem A ensures that mod (Rf) is sufficiently high if the type is sufficiently high, that is, there are sufficiently many non-central levels. However, there are other combinatorial factors which imply big mod (Rf) as well. For example, if the return time of some V_j^{n+1} back to V^n under iterates of g_n is high, then Lemma 3 implies big space.

Sometimes long central cascades imply big space as well. Let us consider such a cascade

$$V^m \supset ... \supset V^{m+N-1},$$

where $gc \equiv g_{m+1}c \in V^{m+N-1}\backslash V^{m+N}$. The quadratic-like map $g : V^{m+1} \to V^m$ can be viewed as a small perturbation of a quadratic-like map G with a definite modulus and with non-escaping critical point. Let $c \in \partial M$ be the internal class of G.

Lemma 11. *Under the above circumstances let us assume that $z \mapsto z^2 + c$ does not have neither parabolic points nor Siegel disks. If g is sufficiently close to G (depending on c and a priori bounds) then mod (A^{m+N+3}) is big.*

Proof. The above assumptions mean that the Julia set $J(G)$ has empty interior. If g is sufficiently close to G then $\Gamma^{m+N-1} = \partial V^{m+N-1}$ is close in the Hausdorff metric to $J(G)$. Hence Γ^{m+N-1} has a big eccentricity with respect to any point $z \in V^{m+N-1}$.

As g_m are quadratic maps up to bounded distortion, the curves Γ_{m+N}, Γ_{m+N+1} and Γ_{m+N+2} also have big eccentricity with respect to any enclosed point. Moreover, there is a definite space in between these two curves. Hence mod $(V^{m+N}\backslash V^{m+N+2})$ is big. This implies that mod (A^{m+N+3}) is big as well.

Indeed, if central return on level $m + N$ occurs then straightening the quadratic-like map $g_{m+N+1} : V^{m+N+1} \to V^{m+N}$ by a qc map we conclude that

$$\mathrm{mod}(V^{m+N}\backslash V^{m+N+2}) \asymp \mathrm{mod}(A^{m+N+1}).$$

Hence A^{m+N+1} has a big modulus.

So we can assume that non-central returns occur on levels $m + N$ and $m + N + 1$. Let us show that then mod (A^{m+N+3}) is big. Let $\psi^{\circ s}c \in V_j^{m+N+2}$. Then it is easy to see that

$$\mathrm{mod}(A^{m+N+3}) \geq \frac{1}{2}\mathrm{mod}(V^{m+N+1}\backslash\psi^{\circ s}(V^{m+N+3})).$$

Let now t be the return time of V_j^{m+N+2} back to V^{m+N+1} under iterates of g_{m+N+1}. Under this iterate $\psi^{\circ s}(V^{m+N+3})$ is mapped onto V^{m+N+2}, and we conclude that

$$\mathrm{mod}(A^{m+N+3}) \geq \frac{1}{4}\mathrm{mod}(V^{m+N}\backslash V^{m+N+2}),$$

which is big. □

Remark. In the real case we will give a complete description of the combinatorial factors producing big space (see Lemma 16).

Proof of Corollary IIb. (local connectivity of the Julia sets). I learned the following argument from J. Kahn and C. McMullen. It follows from Theorem II that the renormalized Julia sets $J(R^m f)$ shrink down to the critical point. Let us take an $\epsilon > 0$, and find an m such that $J(R^m f)$ is contained in the ϵ-neighborhood of the critical point.

Let α_m denote the dividing fixed point of the Julia set $J(R^m f)$, and α'_m denote the symmetric point. Let us consider a topological disk bounded by an equipotential level, and cut it by the external rays landing at $\alpha_0, ..., \alpha_{m-1}$ into the puzzle-pieces $P_j^{m,1}$ (as Yoccoz did in the finitely q-renormalizable case). Let us then pull these puzzle-pieces back in the usual way, and use the notation $P^{m,l}(a)$ for the puzzle-piece of level l containing a point a.

Consider the nest $P^{m,1}(c) \supset P^{m,1}(c) \supset ...$ of the critical puzzle-pieces. This nest shrinks down to the Julia set $J(R^m f)$. Hence there is a puzzle-piece $P^{m,l}(c)$ contained in the ϵ-neighborhood of the critical point. As $J(f) \cap P^{m,l}(c)$ is clearly connected, the Julia set $J(f)$ is locally connected at the critical point.

Let us now prove local connectivity at any other point $z \in J(f)$. Consider two cases.

Case (i). Let the orbit of z eventually land at all Julia sets $J(R^m f)$. Take the first moment $k = k(m)$ such that $f^k z \in J(R^m f)$. Let us show that the domain U can be univalently pulled back along the orbit $z, ..., f^k z$. Let $U'_m \equiv V^{m,t(m)}$, $U_m \equiv V^{m,t(m)-1}$, p be the return time of c back to U_m, and

$$\mathbf{Q}_m \equiv \bigcup_{t=1}^{p} f^t U'_m \qquad (23).$$

Let us find the smallest natural number l such that $f^l z \in \mathbf{Q}_m$, and moreover let $f^l z \in f^s U_m$, $1 \leq s \leq p$. Then $f^{l-1} z$ belongs to the domain Ω which is c-symmetric to $f^{s-1} U_m$. As Ω is disjoint from $\mathbf{Q}_m \supset \omega(c)$, there is a single-valued branch $f^{-l} : \Omega \to Z \ni z$. On the other hand, clearly there is a single-valued branch $f^{-(s+1)} : U_m \to \Omega$. Hence there is a single-valued branch $f^{-k} : U_m \to Z$ as it was claimed.

Because of the a priori bounds, the Julia set $J(R^m f)$ is well inside of U_m. Hence there is a puzzle piece $P^{m,l}(c) \supset J(R^m f)$ which is well inside of U_m as well. It follows from the Koebe Theorem that its pull-back $Y \ni z$ has a bounded shape and hence a small diameter (for sufficiently big m). As $Y \cap J(f)$ is connected, we are done.

Case (ii). Assume that the orbit of z never lands at $J(R^m f)$. Then it never lands at the forward orbit J_m of $J(R^m f)$. Hence it accumulates on some point $a \notin J_m$. But the puzzle-pieces $P^{m,l}(a)$ are disjoint from the critical set for sufficiently big l. Pulling them back to z, we again obtain small pieces $Y \ni z$ containing a connected part of the Julia set. □

§4. Pull-Back Argument.

Any quadratic polynomial induces an equivalence relation on the rational points of the circle $\mathbf{T}$ by identifying the external arguments whose external rays land at the same point of the Julia set (see Douady and Hubbard [DH1], [D2] and [H]). Two quadratic-like maps are called *combinatorially equivalent* if they induce the same equivalence relation (the combinatorial classes are clearly bigger than the topological ones). The combinatorial class of quadratic polynomials is obtained by intersecting a nest of parameter puzzle-pieces bounded by appropriate external rays and equipotentials. The definition of combinatorially equivalent quadratic-like maps is straightforward.

Our goal is to prove the following result.

Theorem III. *Let f and $\tilde{f}$ be two quadratic-like maps of class $\mathcal{F}$. If these maps are combinatorially equivalent then they are quasi-conformally conjugate.*

Corollary. *Any quadratic polynomial $f : z \mapsto z^2 + c$ of class $\mathcal{F}$ is combinatorially rigid, so that MLC holds at c.*

Proof of Corollary. The well-known argument: combinatorial classes of quadratic polynomials are closed, while qc classes are either open or single points. So if a combinatorial class coincides with a qc class, both must be single points. MLC follows since the intersection of the Mandelbrot set with puzzle-pieces is connected. □

Strategy. The method we use for proof of Theorem III is called "the pull-back argument". The idea is to start with a qc map respecting some dynamical data, and then pull it back so that it will respect some new data on each step. In the end it becomes (with some luck) a qc conjugacy. This method originated in Sullivan's work, and then was developed in several other works (see [K] and [Sw]). Our way is to pull back through the cascade of generalized renormalizations. The linear growth of moduli gives us enough dilatation control until the next quadratic-like level, while complex a priori bounds allow us to interpolate and pass to the next level.

We will use tilde for marking the corresponding objects. Referring to a qc-map, we always mean that it has a definite dilatation. All puzzle-pieces have a natural boundary marking coming, e.g, from the uniformization of the basin at ∞ (we can always assume that we have started with a polynomial map). Let us call two configurations of puzzle-pieces W_i and $\tilde{W}_i$ qc pseudo-conjugate if there is a qc map between them respecting the boundary marking.

Let $\{V^{m,n}\}$ be the principle nest of critical puzzle-pieces (see §1). We switch from the nest $V^{m,n}$, $n = 0, 1, ..., t(m)$, to the next nest $V^{m+1,n}$, $n = 0, 1, ...,$ when the modulus $A^{m,t(m)+1}$ is bounded from the both sides (not only from below).

By Lemma 1 the starting configurations $\{V_i^{m,0}\}$ and $\{\tilde{V}_i^{m,0}\}$ have bounded geometry, so there is a qc pseudo-conjugacy h_m between them. It is possible to pull it back to the first non-degenerate level, no matter how deep it is (The Initial Construction below). Let us then pull it back through the cascade of generalized renormalizations (the Main Step below). Since the geometric moduli of these maps linearly increase, the positions of their critical values are localized with an exponentially high precision. It follows that the qc dilatation of the pseudo-conjugacy on the next level can jump only by an exponentially small amount. Hence we will arrive at the next quadratic-like level $m+1$ with a qc map H_m with bounded dilatation.

Finally, since the annuli $A^{m,t(m)+1}$ and $\tilde{A}^{m,t(m)+1}$ have a definite moduli, we can qc interpolate in between H_m on their outer boundaries and h_{m+1} on the inner ones (keeping the map in the right homotopy class mod the critical set). This gives us a qc pseudo-conjugacy between the nests of critical puzzle-pieces. Then it can be easily spread around to the whole critical set. Sullivan's pull-back argument completes the construction.

Main Step. Let $g : \cup V_i \to \Delta$ and $\tilde{g} : \cup \tilde{V}_i \to \tilde{\Delta}$ be two generalized polynomial-like maps. The objects on the next renormalization level will be marked with prime. So $g' : \cup V'_j \to \Delta'$ is the generalized renormalization of g, $\Delta' \equiv V_0$. Let μ the principle modulus of g.

Remark. We don't assume that the non-critical puzzle-pieces V_i^n, $i \neq 0$, are well inside Δ, since this is not the case on the levels which immediately follow long cascades of central returns. We even allow the annuli $\Delta \backslash V_i^n$, $i \neq 0$ to be degenerate which actually happens in the beginning.

Let $\lambda(\nu)$ be the maximal hyperbolic distance between the points in the hyperbolic plane enclosed by an annulus of modulus ν. Note that $\lambda(\nu) = O(e^{-\nu})$ as $\nu \to \infty$. Set $\lambda = \lambda(\mu)$.

Let

$$h : (\Delta, V_i) \to (\tilde{\Delta}, \tilde{V}_i) \tag{24}$$

be a K-qc pseudo-conjugacy between the corresponding configurations. Our goal is to pull this map back to the next level. The problem is that h does not respect the positions of the critical values. We assume first that we have a non-central return on this level, that is $c_1 \equiv g(c) \in V_k$ with $i \neq 0$.

Let P_l be the pull-backs of V_0 by the univalent branches of iterates g intersecting the critical set. We can pull h back by these branches to obtain a K-qc pseudo-conjugacy

$$h_1 : (\Delta, \cup P_l) \to (\tilde{\Delta}, \cup \tilde{P}_l).$$

This *localizes* the positions of the critical values in the sense that the hyperbolic distance between $h_1(c_1)$ and $\tilde{c}_1$ in V_k is $O(\lambda)$. Indeed, they belong to the same puzzle-piece $\tilde{P}_l$ whose hyperbolic diameter in V_k is at most λ.

Hence we can find a diffeomorphism $\psi : \tilde{\Delta} \to \tilde{\Delta}$ which is id outside $\tilde{V}_k$, moves $h_1 c$ to $\tilde{c}$, and has a qc dilatation $1 + O(\lambda)$. Then the $K(1 + O(\lambda))$-qc map

$$h_2 = \psi \circ h_1 : (\Delta, V_i) \to (\Delta, V_i)$$

respects the same configurations as h, and also carries c to $\tilde{c}$.

Now we can pull h_1 back to

$$H : (\Delta', U'_i) \to (\tilde{\Delta}', \tilde{U}'_i), \tag{25}$$

where U'_i are g-pull-backs of V_i to Δ'. However U'_i are not the same as V'_j, so we have to do more. What we need is to localize the positions of the critical values $a = g'c$ and $\tilde{a}$ of the next renormalizations. The argument depends on where they are. Let $a_1 = g(a) \in V_j$.

Case (i). Assume V_j is non-critical and different from V_k. Then we can simultaneously move of c_1 and a_1 to the right positions, and then pull the map back to Δ'.

Case (ii). Assume that $V_j = V_k$. If the hyperbolic distance between a_1 and c_1 in V_j is greater than $\lambda(\mu/2)$ then the hyperbolic distance between the corresponding tilde-points is greater than $\lambda(K\mu/2)$. Then we can simultaneously move these points to the right positions by a qc map ψ with dilatation $1+O(\lambda(K\mu/2))$.

Otherwise let us first move c_1 to the right position, and pull the map back to H as in (25). Then a and c stay in V_0 on hyperbolic distance $O(\lambda(K\mu/4))$, and the corresponding tilde-points stay on distance $O(\lambda(K^2\mu/4))$. Hence $H(a)$ and $\tilde{a}$ stay in $\tilde{V}_0$ on hyperbolic distance $\delta = O(\lambda(K^2\mu/4))$. So we can move these points to the right positions by a $K(1+\delta)$-qc map respecting the boundary marking of V_0 (though not respecting the critical points any more).

Case (iii). Let us finally assume that $V_j = V_0$ is critical. Then a belongs to a pre-critical puzzle-piece $V'_s \subset V_0$. Since mod($V_0 \backslash V'_s) \leq \mu/2$, the map H constructed above (see (25)) almost respects the positions of a-points in V_0. So we can make it respect these points keeping ∂V_0 untouched.

After all, we have constructed a $(1+O(\lambda^d))$-qc map

$$h_2 : (\Delta, V_i, a) \to (\tilde{\Delta}, \tilde{V}_i, \tilde{a}).$$

Let us now start over again, and pull this map back to get a qc map

$$h_3 : (\Delta, \cup P_l) \to (\tilde{\Delta}, \cup \tilde{P}_l).$$

Unlike h_1 above this map also respects the forward orbits of the critical points (that is, the appropriate pre-images of a) until their returns to the central puzzle-pieces. Hence we can pull h_3 back to the critical puzzle-piece Δ'. Since V'_j are the pull-backs of P_l, this map respects the boundary marking of V'_j, and we are done (in the non-central case).

Cascades of central returns. Let $V^m \supset V^{m+1} \supset ... \supset V^{m+N}$ be a cascade of central returns, that is the critical value $g_{m+1}c$ belongs to $V_k,\ k = m+1, ..., m+N-1$, but escapes V^{m+N}. We assume that the following configurations are qc pseudo-conjugate:

$$h : (V^m, V^{m+1}) \to (\tilde{V}^m, \tilde{V}^{m+1}).$$

Set $g = g_{m+1},\ \mu = \mathrm{mod}(V^m \backslash V^{m+1})$.

Let us take non-central puzzle-pieces $V_i^{m+1} \subset A^{m+1} = V^m \backslash V^{m+1}$ and pull them back to the annuli $A^{m+2}, ..., A^{m+N}$. We obtain a Markov family of puzzle-pieces W_i^{m+k}. Let us induce on this Markov scheme *the first landing map*

$$\phi : \cup P_l^{m+k} \to V^{m+N}.$$

Then

$$\mathrm{mod}(W_i^{m+k} \backslash P_l^{m+k}) \leq \mu,$$

so that the dynamically defined points are well localized by this partition.

Now we can proceed along the lines of the Main Construction just using the following substitution: W_i^{m+k} play the role of V_i, V^{m+N} plays the role of $\Delta' \equiv V'$. So we pull h back to

$$h_1 : (\Delta, \cup P_l, V^{m+N}) \to (\tilde{\Delta}, \tilde{\cup} P_l, \tilde{V}^{m+N}),$$

correct this map to make it respect the g-critical values and then pull it back to V^{m+N} as in (25):

$$H : (V^{m+N}, U_i) \to (\tilde{V}^{m+N}, \tilde{U}_i),$$

where U_i are the pull-backs of W_i^{m+N} and V^{m+N}. Take now the first return b of the critical point back to V^{m+N}, and look at Cases (i), (ii), (iii) of the Main Step. The first two cases go in the same way as above. However, the last case is different since the pre-critical puzzle-pieces U_1 and U_2 are not necessarily well inside of V^{m+N}.

To take care of this problem let us first consider the first landing map

$$\psi : \cup Y_j \to V^{m+N+1}$$

from $U_1 \cup U_2$ to V^{m+N+1}, and pull H back to the domain of ψ. Since the components Y_j of this domain are well inside U_s (namely $\mathrm{mod}\,(U_s \backslash Y_j) \geq \mathrm{mod}(A^{m+N+1}) \geq \mu$), this gives us an appropriate localization of the b-points.

Initial Construction. In the beginning we have a cascade $\Omega^0 \supset ... \supset \Omega^N$ of central returns with degenerate annuli. So we may not directly apply the above argument. Below we use the notations of Lemma 0. We start with a qc pseudo-conjugacy respecting the dynamics on the external rays through α, α', γ, γ' and the equipotentials of $\partial\Omega^1$.

Step 1. Let us now construct a qc map $Q \to \tilde{Q}$. Let $a = f^s c$ be the last point of $\mathrm{orb}\,(c)$ landing at a W_j before the return back to V^0. If the points $a = g^N c$ and $\tilde{a}$ are well inside W_j and $\tilde{W}_j$ correspondingly, then we can take a qc map $(W_j, a) \to (\tilde{W}_j, \tilde{a})$ and pull it back to Q-pieces.

Otherwise let us cut Ω^N by the external rays landing at γ', and take the component E attached to the fixed point α. Then the branch of g^{-1} fixing α univalently maps E into itself. So $F = E \backslash g^{-1}E$ is a combinatorially well-defined fundamental domain for g near the fixed point α. Hence if $w \equiv f^i(a) \in E$ (combinatorially close to α) then there is the first moment $l \geq 0$ depending only on combinatorics such that $g^l w \in F$.

Let us also consider the fundamental domain

$$F_* = F \cup g^{-1}F \cup g^{-2}F$$

for the third iterate of g. If the external class of f belongs to the given set of truncated limbs of order two then the configuration $(F_*, g^l a)$ has a bounded geometry. Hence we can start with a qc map respecting these configurations and the dynamical pairing on the ∂F_*.

Let us now pull this qc map back to $g^{-3}E$, $g^{-6}E, ...$ If $l - 1 = 3m$ then the m-fold pull-back will carry the point w to $\tilde{w}$. Then we can pull this map back to Q-pieces by the appropriate iterate of f. If $l - 1$ is not a multiple of 3 then we can replace it by $l - 1 - n$ which is a multiple of 3 (where $n = 1$ or 2) and correspondingly replace F by $g^{-n}F$.

Step 2. Let us now take a point $z \in \Omega^0$ and push it forward by iterates of g until it lands either at $\cup W_j \cup (\Omega^0 \backslash \Omega^1)$ or at Q. If it happens, then we can pull the pseudo-conjugacy to an appropriate piece containing z. The qc maps constructed in such a way agree on the common boundaries of the puzzle-pieces. The set of points where this map is not defined is an expanding Cantor repellor. Hence it is qc removable, and the map automatically allows a qc continuation across it. This provides us with a qc map

$$(\Omega^0, V^0, Q) \to (\tilde{\Omega}^0, \tilde{V}^0, \tilde{Q})$$

.

Step 3. Let us now localize the first return b of the critical point back to V^0. To this end let us push b forward until the first moment t it returns back to Q. Let $u = f^t b$. The procedure depends on whether u and c stay on bounded hyperbolic distance in Q (in terms of a given quantifier R) or not (compare Case (ii) of the Main Construction). In the former case the position of u is already well localized inside of Q. In the latter case we can on Step 1 simultaneously localize positions of $a = g^N c$ and $g^N u$ in W_j and pull them back to Q. Hence we can change the qc pseudo-conjugacy inside of Q so that it respects u-points. Pulling this back as on Step 2, we construct a qc pseudo-conjugacy respecting b-points.

Step 4. Let us consider the full first return map $G : \cup Z_j \to V^0$. Its domain covers the whole piece V^0 except for a removable Cantor set $K \subset V^0$. We can now construct a qs pseudo-conjugacy

$$(V^0, Z_j) \to (\tilde{V}^0, \tilde{Z}_j)$$

by the simple pull back and removing K. Since the puzzle-pieces V_i^1 are among Z_i, we are done.

By Lemma 2, the principle modulus is definite on this level. So we can proceed further by applying the Main Step.

Qc conjugacy on the critical sets. Let us show now that there is a qc map conjugating f and $\tilde{f}$ on their critical sets. Let $t = (m, n) \in T$ runs over the indices of the principle nest of puzzle-pieces. Clearly the lexicographic order on T corresponds to inclusion of the puzzle-pieces. Let $Q_0^t \equiv V^t$, and Q_l^t be all pull-backs of Q_0^t corresponding to the first landing of the orbits of z, $z \in \omega(c)$, at Q_0^t. Then

$$\omega(c) = \bigcap_t \bigcup_l Q_l^t.$$

Let us consider the multiply connected domains

$$P_l^t = Q_l^t \backslash \bigcup_k Q_k^\tau,$$

where $\tau \in T$ immediately follows t in the lexicographic order. The boundaries of P_l^t are naturally marked.

By the *Teicmüller distance* between two marked domains (of the same qc type) we mean the $\log K$ where K is the qc dilatation of the best qc homeomorphism between the domains respecting the marking.

Lemma 11. *The domains P_l^t and $\tilde{P}_l^t$ stay on bounded Teichmüller distance.*

Proof. We have proved that the pairs $(V^t, \cup_j V_j^\tau)$ and $(\tilde{V}^t, \cup_j \tilde{V}_j^\tau)$ stay a bounded Teichmüller distance. Pulling the corresponding qc equivalence back by the univalent branches of g_τ we obtain that P_0^τ and $\tilde{P}_0^\tau$ also stay a bounded Teichmüller distance. Pulling this back by the univalent branches of f we obtain the claim for all l. □

Gluing now together the multiply connected domains under consideration, we construct a homeomorphism $h : V^0 \to \tilde{V}^0$ which is qc on $V^0 \backslash \omega(c)$ and conjugates f and $\tilde{f}$ on their critical sets. Since the critical sets are removable, we are done.

Homotopy. Let $\psi_0 : (U, U') \to (\tilde{U}, \tilde{U}')$ be a homeomorphism conjugating f and $\tilde{f}$. We will show now that the qc map h conjugating f and $\tilde{f}$ on their critical sets can be constructed in such a way that it is homotopic to ψ_0 rel the critical sets.

As in the proof of Corollary 2, let $U_m \equiv V^{m,t(m)}$, $U'_m \equiv V^{m,t(m)+1}$, $U''_m \equiv V^{m,t(m)+2}$, $G_m : U'_m \to U_m$ be the corresponding quadratic-like renormalization of f, and let $\mathbf{Q}_m$ be defined as in (23). These sets nest down to $\omega(c)$.

A selection of the straightenings of the quadratic-like maps G_m and $\tilde{G}_m$ provides us with a choice of conjugacies

$$\psi_m : (U'_m, U''_m) \to (\tilde{U}'_m, \tilde{U}''_m).$$

Let us continue ψ_m to the annuli $U_m \backslash U'_m$ in such a way that $\psi_m \simeq \psi_{m-1}$ (are homotopic) in the annulus $U_m \backslash J(G_m)$ rel the boundary. Then let us spread ψ_m around to the whole set $\mathbf{Q}_m$. Outside $\mathbf{Q}_m$ set $\psi_m = \psi_{m-1}$. Clearly $\psi_m \simeq \psi_{m-1}$ mod J_m where J_m is the orbit of $J(G_m)$.

Let us define a homeomorphism $\psi : U \to \tilde{U}$ as the pointwise $\lim \psi_m$. This homeomorphism is homotopic ψ_0 rel the critical sets. Let us now construct a qc map h homotopic to ψ rel the critical sets. First of all, the above selection of the straightenings should be uniformly qc which is possible because of the a priori bounds (Theorem B). Then let us assume by induction that we have already constructed a map $h_{m-1} \simeq \psi$ mod $\mathbf{Q}_m$ which is qc outside $\mathbf{Q}_m$.

Let us cut U_m by the external rays through the points $\alpha, \alpha', \gamma, \gamma'$ into puzzle-pieces S_i, and go through the above pull-back construction. In the beginning we change ψ_m on the S_i to make it qc. As the pieces S_i are simply-connected, this change can be done via homotopy rel the boundary. Then this homotopy can be pulled back to the deeper puzzle-pieces according to the Starting Construction. This provides us with a homotopy rel the boundary

$$(V^0, \cup V^1_i) \to (\tilde{V}^0, \tilde{V}^1_0).$$

Then this homotopy can be pulled back through the cascade of Main Steps, and spread around to the whole critical set (as in the previous subsection). This gives us a qc map

$$h_m : U' \backslash \mathbf{Q}_{m+1} \to \tilde{U}' \backslash \tilde{\mathbf{Q}}_{m+1}$$

homotopic to ψ_m rel the boundary.

We should now continue this map to the annulus $R_m = U_m \backslash U'_m$. To this end observe that ψ_{m-1} has a bounded twist in this annulus since it can be deformed rel the boundary to a qc map (by the above pull-back argument). Hence ψ_{m-1} has a bounded twist in the annulus $U_m \backslash J(G_m)$ as well, since this homotopy can be pulled back to this annulus (and by a hyperbolic argument will automatically be trivial on the Julia set). Consequently the continuation of ψ_m (and hence h_m) to R_m (such that $\psi_m \simeq \psi_{m-1}$ in $U_m \backslash J(G_m)$ mod the boundary) has a bounded twist as well. Hence this continuation can be realized quasi-conformally.

Finally we can spread the homotopy from U_m around the $\mathbf{Q}_m$.

Sullivan's pull-back argument. Remember that $\psi_0 : (U, U') \to (\tilde{U}, \tilde{U}')$ is a conjugacy between f and $\tilde{f}$, and h is a K-qc map homotopic to ψ_0 rel the critical sets. Sullivan's Pull-back argument allows us to reconstruct h into a qc conjugacy.

Let U^n be the preimages of U under the iterates of f. We can always assume that $h|U^n = \psi$. Since $h(c_1) = \tilde{c}_1$, we can lift h to a K-qc map $h_1 : U^1 = \tilde{U}^1$ homotopic to ψ rel the critical set and ∂U^1. Hence $h_1 = h$ on these sets, and we can continue h_1 to $U \backslash U^1$ as h. This map conjugates f and $\tilde{f}$ on the critical sets and also on $U^1 \backslash U^2$.

Let us now replace h with h_1 and repeat the procedure. In such a way we construct a sequence of K-qc maps h_n conjugating f and $\tilde{f}$ on the critical sets and on $U^1 \backslash U^{n+1}$. Passing to a limit we obtain a desired qc conjugacy.

§5. Real case.

In this section we will prove the following dichotomy: real maps of Epstein class (see below) either have a big complex space on the next quadratic-like level, or essentially bounded real geometry ("essentially" loosely means "up to saddle-node cascades"). The main ingredient is to create a generalized polynomial-like map with a definite modulus on an essentially bounded level. By Theorem I this implies big space, provided the type is sufficiently high. From this dichotomy we derive the real rigidity theorem.

Preliminaries. Let $\phi(z) = (z - c)^2$ denote the purely quadratic map. Let $I' \subset I$ be two nested intervals. A map $f : I' \to I$ is called *quasi-quadratic* if it is S-unimodal and has quadratic-like critical point c.

Let us also consider a more general class $\mathcal{A}$ of maps $g : \cup T_i \to T$ defined on a finite union of disjoint intervals T_i compactly contained in an interval T. Moreover, $g|T_i$ is a diffeomorphism onto T for $i \neq 0$, while $g|T_0$ is unimodal with $g(\partial T_0) \subset \partial T$. We also assume that the critical point $c \in T_0$ is quadratic-like, and that $Sg < 0$. Maps of class $\mathcal{A}$ are real counterparts of generalized polynomial-like maps.

Let $g \in \mathcal{A}$, and $g|T_0 = h \circ \phi$ where and h is a diffeomorphism of an appropriate interval $K \supset \phi(T_0)$ onto T. This map belongs to the so-called *Epstein class* $\mathcal{E}$ (see [S] and [L2]) if the inverse branches $f^{-1} : T \to T_i$ for $i \neq 0$ and $h^{-1} : T \to K$ allow analytic extension to the slit complex plane $\mathbf{C} \backslash T$.

Let $I^0 = [\alpha, \alpha']$ be the interval between the dividing fixed point α and the symmetric one. Let $\mathcal{M}$ denote the full Markov family of pull-backs of the interval I^0. Given a critical interval $J \in \mathcal{M}$ (that is, $J \ni c$), we can define a (generalized) renormalization $R_J f$ on J as the first retun map to J restricted to the components of its domain meeting the critical set. If f admits a unimodal renormalization, then there are only finitely many such components, so that we have a map of class $\mathcal{A}$. Moreover, if f is a map of Epstein class or a polynomial-like map, the renormalization $R_J f$ inherits the corresponding property.

Let $I^0 \supset I^1 \supset \ldots \supset I^{t+1}$ be the real principal nest of intervals until the next quadratic-like level (that is, I^{n+1} is the pull-back of I^n corresponding to the first return of the critical point. Let us use the same notation $g_n : \cup I^n_j \to I^{n-1}$ for the real generalized renormalizations on the intervals I^n.

Our first goal is to fill-in the gap in between the notions of bounded combinatorial type in the sense of period and in the sense of the number of central cascades. To this end we need to analyse in more detail cascades of central returns.

The return on level $n-1$ is called *high* or *low* if $g_n I^n \supset I^n$ or $g_n I^n \cap I^n = \emptyset$ correspondingly. Let us classify the cascades

$$I^m \supset \ldots \supset I^{m+N}, \quad g_{m+1} c \in I^{m+N-1} \backslash I^{m+N} \tag{26}$$

of central returns as *Ulam-Neumann* or *saddle-node* according as the return on the level $m+N-1$ is high or low. There is a fundamental difference between these two types of cascades. Let us call the levels $m+1,...,m+N-1$ of a saddle-node cascade *neglectable*, and all other levels *essential*. Let $m=e(l)$ counts the essential levels.

Let $K_j^{m+i} \subset I^{m+i-1} \setminus I^{m+i}$ denote the pull-back of I_j^{m+1} under $g_{m+1}^{\circ(i-1)}$, $i=1,...,N$, $j \neq 0$. Clearly K_j^{m+i+1} are mapped by g_{m+1} onto K^{m+i}, $i=1,...,N-1$, while $K_j^{m+1} \equiv I_j^{m+1}$ are mapped onto the whole I^m. So we have a Markov scheme associated with any central cascade.

Take now a point $x \in \omega(c) \cap (I^m \setminus I^{m+1})$ on an essential level $m=e(l)$. Let us push it forward by iterates of $g=g_{m+1}$ through the above Markov scheme until it lands at the next essential level I^{m+N}, $m+N=e(l+1)$. Let $o(x)$ ("the order of x") denote the number of times it passes through $I^m \setminus I^{m+1}$ before landing at I^{m+N} (e.g., $o(x)=1$ if $gx \in I^{m+N}$). Let $gx \in I^{m+i}$. Then set $d(x) \min\{i, N-i\}$ ("the depth of the first iterate").

Let us now introduce the scaling factors

$$\lambda_n \equiv \lambda_n(f) = \frac{|I^n|}{|I^{n-1}|}.$$

According to [L2], these scaling factors exponentially decay with the number of central cascades. Moreover, this rate is uniform when the scaling factors become small enough.

Let us call the geometry of f *essentially bounded* (until the next quadratic-like level) if the scaling factors $\lambda_n = |I^n|/|I^{n+1}|$ stay away from 0, while the configurations $(I^{n-1} \setminus I^n, I_k^n)$ have bounded geometry (that is, all intervals I_j^n, $j \neq 0$, and all components of $I^{n-1} \setminus \cup I_k^n$ ("gaps") are commensurable). Remark that we allow the scaling factors λ_n to be close to 1.

Complex bounds. Sullivan's Sector Lemma provides us with complex bounds in the case when f is infinitely q-renormalizable of bounded type. In the non-q-renormalizable case the complex bounds were obtained in [LM] and [L2]. We will complement these results with the following theorem.

Let us pick a class $\mathcal{U}_{\tau,\bar{\mu}}$ of real quadratic-like maps f of the same q-renormalizable type τ, and such that $\mathrm{mod}\,(f) \geq \bar{\mu}$, where $\bar{\mu} > 0$ is an a priori chosen small quantifier.

Theorem D. *One of the following two possibilities occurs for all $f \in \mathcal{U}_{\tau,\mu}$ simultaneously: either $\mathrm{mod}\,(Rf) \geq \bar{\mu} > 0$, or the real geometry of f is essentially bounded (until the next quadratic-like level).*

In the following two lemmas we analyse the geometry of long central cascades. Let us call a quasi-quadratic map saddle-node or Ulam-Neumann if it is topologically conjugate to $z \mapsto z^2 + 1/4$ or $z \mapsto z^2 - 2$ correspondingly.

Lemma 12. *Let us consider an Ulam-Neumann cascade as (26) with commensurable I^m and I^{m+1}. Then there is a bounded l such that the generalized renormalization g_{m+l} allows a polynomial-like extension to the complex plane with a definite modulus. Moreover, the principle modulus μ_{m+N+1} is big, provided the cascade is long.*

Proof. It is easy to see by compactness argument that if the cascade is long enough then the map $g_{m+1} : I^{m+1} \to I^m$ (with the domain rescaled to the unit size) is C^1-close to an Ulam-Neumann map. It follows that I^{m+N} occupies a definite part of I^m, and, moreover, $|I^{m+k} \backslash I^{m+N}|$ decrease with k at a uniformly exponential rate. Hence there is a bounded l such that $I^{m+l} \backslash I^{m+N}$ is ϵ-tiny as compared with I^{m+N}.

Take now the Euclidian disk $D = D(I^{m+l})$ and pull it back by the inverse branches of g_{m+l+1} (as in the previous lemma). As $g_{m+l+l} = h \circ \phi$ where h is a diffeomorphism with a bounded distortion, the central pull-back will be an ellipse based upon the interval I^{m+l+1} whose imaginary axis is $O(\sqrt{\epsilon}|I^{m+l+1}|)$. It follows that this ellipse is well inside of D.

The last statement follows from Lemma 11. □

Lemma 13. *All saddle-node patterns (26) of the same length with commensurable I^m and I^{m+1} are qs equivalent.*

Proof. Let $g : I' \to [0, 1]$ be a unimodal map of Epstein class (and perhaps escaping critical point): $g \in \mathcal{E}_u$. By definition, $g = h \circ \phi$ with a diffeomorphism h whose inverse allows the analytic extension to $\mathbf{C} \backslash [0, 1]$. Let us supply this space with with the Montel topology on h^{-1}.

The set of $g \in \mathcal{E}_u$ with bounded geometry on the real line is compact. Hence given a long saddle-node cascade (26), the map G obtained from $g_{m+1} : I^{m+1} \to I^m$ by rescaling I^m to the unit size must be close to a saddle-node quadratic-like map. Hence we can reduce G to a form $z \mapsto z + \epsilon + \psi(z) > 0$ where $\psi(z)$ is uniformly comparable with z^2, and (as we will see in a moment) ϵ is determined (up to a bounded error) by the length of the cascade.

Take a big $a > 0$. When $|z| < a\sqrt{\epsilon}$, the step $G(z) - z$ is of order ϵ. Otherwise $\psi(z)$ dominates over ϵ, and in the chart $\zeta = 1/z$ the step is of order 1. It follows that the qs class of the cascade is determined by ϵ, which in turn is related to the length of the cascade by $N \asymp 1/\sqrt{\epsilon}$. □

Given two intervals $L \subset S$, let $P(L|S)$ denote the Poincaré length of L in S. Given an interval I, let $D(I)$ denote the Euclidian disk based upon I as a diameter.

Lemma 14. *Assume that $\lambda_n < \epsilon$ with a sufficiently small $\epsilon > 0$. Then there is an interval $T \in \mathcal{M}$ containing I^{n+2} such that the renormalization $R_T f$ allows a (generalized) polynomial-like extension to the complex plane with the principle modulus $\mu \to \infty$ as $\epsilon \to 0$.*

Proof. First of all we can assume that all intervals I^n_j are well inside I^{n-1} (otherwise pass to the next level). The following construction of a polynomial-like map is combinatorially the same as in [L], Lemma 5.3. For the reader's convenience we briefly repeat it.

Let $g \equiv g_n : \cup I^n_j \to I^{n-1}$. Let us inductively define the *cut-off orbit* of I^n_0 as

$$g^l_{cut}(I^n_0) = g\,(g^{l-1}_{cut}(I^n_0) \cap I^n_j),$$

provided $I_j \ni g^{l-1}c,\ j \neq 0$. We stop at the first moment when $g^l_{cut} I^n_0 \cap I^n_0 \neq \emptyset$. Let us define $T \equiv T_0 \ni c$ as the pull-back of I^{n-1} by g^l, and set $G|T_0 = g^l$. Clearly $I^n \supset T_0 \supset I^{n+1}$, and $G(\partial T_0) \supset \partial I^{n-1}$.

Let now $z \in (\omega(c) \cap I^{n-1}) \setminus T_0$. If $z \in I^{n-1} \setminus I^n$ then let $T(z)$ be the interval $I^n_j \equiv I^n(z)$ containing z, and $G|T(z) = g$. If $z \in I^n \setminus T_0$ then let us push z forward by iterates of g until it is separated from the corresponding iterates of c by the intervals I^n_j. Let it happen at moment s, and $g^s c \in I^n_k$. It follows from the choice of l that $s \le l$ and $k \ne 0$. Let us now define $T(z)$ as the pull-back of I^n_k by g^s, and $G|T(z) = g^{s+1}$. It is easy to see that $G : T(z) \to I^{n-1}$ is a diffeomorphism.

So we have constructed a map $G : \cup T_i \to I^{n-1}$ of class $\mathcal{A}$. Let us now take the Euclidian disk $D = D(I^{n-1})$ and pull it back by the inverse branches of G. This provides us with a set of domains D_i based upon the intervals T_i. Moreover, by a little hyperbolic argument (see e.g., Lemma 8.1 of [LM]) $D_i \subset D(T_i)$.

Let us now estimate the shape of D_0. To this end let us consider the following decomposition:

$$G|T_0 = (g|I^n_i) \circ (h|K) \circ \phi|T_0.$$

Here $g^{l-1}c \in I^n_i$, and $g^{l-1}|T_0 = h \circ \phi$, where h is a diffeomorphism of an appropriate interval K onto I^n_i with a Koebe space spreading over I^{n-1}. As I^n_i is well inside of I^{n-1}, $h|K$ has a bounded distortion. Moreover, $g|I^n_i$ is quasi-symmetric (as a composition of the quadratic map and a diffeomorphism of bounded distortion). Hence $G|T_0 = (H|K) \circ \phi|T_0$ with a quasi-symmetric diffeomorphism H. Furthermore, as $G(T_0) \cap I^n_0 \ne \emptyset$,

$$|G(T_0)| \ge \frac{1-\epsilon}{2} |I^{n-1}|.$$

Pulling this back by the qs map H, we conclude that

$$|\phi(T_0)| \ge \delta |K|$$

with $\delta = \delta(\epsilon)$. Let Q be the pull-back of D by H. Then $Q \subset D(K)$. Pulling this back by the quadratic map ϕ, we conclude that D_0 has a bounded shape. As it is based upon a ϵ-tiny interval I^n_0, it is well inside D. Moreover, the annulus $D \setminus D_0$ is getting big as $\epsilon \to 0$.

It follows that $R_T G$ satisfies the desired properties. Finally, it is easily seen from the construction that the first return map to T under f coincides with the first return map under f, so that $R_T f = R_T G$. □

Now we are ready to state the key lemma.

Lemma 15. *There is an interval $T \in \mathcal{M}$ such that the renormalization $R_T f$ allows a polynomial-like continuation to the complex plane with a definite principle modulus μ. Moreover, T lies on an essentially bounded level: $T \supset I^{e(l)}$.*

Proof. Take a small $\epsilon > 0$ and $\delta > 0$, and select the first moment l for which

$$\lambda_l > (1-\delta)\lambda_{l-1}. \tag{27}$$

For such a level [L2,§5] provides us with a polynomial-like map $G : \cup D_i \to D(I^l)$ with a definite modulus and such that the number of central cascades preceding $T_0 = D_0 \cap \mathbf{R}$ is bounded. Moreover, only the last of these cascades may be of Ulam-Neumann type. If this cascade is of bounded length then T_0 lies on an essentially bounded level. Otherwise Lemma 12 provides us with a desired polynomial-like map.

On the other hand, if (27) fails to happen on the first $s = \log\epsilon/log(1-\delta)+1$ levels then we come up with an ϵ-small scaling factor, and can apply Lemma 14. □

Given a q-renormalizable map f, let $\tau(f)$ denote the maximum of the type $\kappa(f)$, the lengths of the Ulam-Neumann cascades, the orders $o(x)$ and the depths $d(x)$ for all $x \in \omega(c)$.

Lemma 16. *Take a $\bar\mu > 0$. Let f be a q-renormalizable unimodal map of Epstein class of a bounded distortion D. If $\tau(f)$ is sufficiently high (depending on D and $\bar\mu$ only), then the renormalization Rf is polynomial-like with mod $(Rf) > \bar\mu$.*

Proof. Assume that (i) occurs. Then by Lemma 15 on an essentially bounded level T we can create a generalized polynomial-like map $R_T f$ with a definite modulus ($> \bar\nu > 0$). Then by Theorem A the moduli of further renormalizations of $R_T f$ will grow at a linear rate with the number of central cascades. Hence the quadratic-like renormalization Rf will have a $\bar\mu$-big modulus, provided there are sufficiently many central cascades.

If (ii) occurs then by Lemma 12 in the end of the Ulam-Neumann cascade we observe a generalized polynomial-like map with a big modulus. Then by Corollary 6 the modulus of the quadratic-like renormalization Rf will be big as well.

Assume further that there is an $x \in \omega(c) \cap (I^m \backslash I^{m+1})$ of a high order $o(x)$, where $I^m \supset \dots \supset I^{m+N}$ is a central cascade as (26) (it may be $N = 1$). Let us consider the above Markov scheme involving the intervals K_j^{m+i}. Let $J \ni x$ denote the pull-back of I^{m+N} corresponding to the first landing of the orb(x) at I^{m+N}.

As the intervals K_j^{m+1} are well inside of $I^m \backslash I^{m+1}$, and orb(x) passes many times through these intervals before the first landing at I^{m+N}, the Poincaré length $P(J|(I^m \backslash I^{m+1}))$ is big. Pulling this interval back to the critical point we will find a level with a small scaling factor. Applying Lemma 14 we get the claim.

Let us finally assume that there is an $x \in \omega(c) \cap (I^m \backslash I^{m+1})$ with high $d(x)$. Then $g_{m+1}x \in I^{m+i} \backslash I^{m+i+1}$ with $d(x) \le i \le N - d(x)$. Then by Lemma 13 $I^{m+i} \backslash I^{m+i+1}$ is tiny in I^m. It follows that the interval $J \ni x$ introduced two paragraphs up is tiny in $I^m \backslash I^{m+1}$. Now we can complete the argument as above. □

Remark. Now a little extra work shows that if $\tau(R^m f)$ is sufficiently high on all levels, then MLC holds at $c \in \mathbf{R}$.

Lemma 17. *If $\tau(f)$ is bounded, then the geometry of f is essentially bounded (until the next quadratic-like level).*

Proof. Assume that the geometry is bounded on level $n-1$, and let us see what happens on the next level. Given an $x \in \omega(c) \cap (I^{n-1} \backslash I^n)$, let $J(x)$ denote the pull-back of I^n corresponding to the first landing of orb(x) at I^n. As the landing time under iterates of g_n is bounded, $J(x)$ is commensurable with I^{n-1}.

To create the intervals I_j^{n+1}, we should pull all intervals $J(x)$ back by $g_n : I^n \to I^{n-1}$. As g_n is a quasi-quadratic map, all non-central intervals I_j^{n+1} and the gaps in between are commensurable with I^n.

The only possible problem is that the central interval I^{n+1} may be tiny in I^n. This may happen only if the critical value $g_n c \in J(x)$ is very close to the $\partial J(x)$. Let l be such that $f^l J(x) = I^n$. Since $f^l : J(x) \to I^n$ is qs, $g_{n+1}c = g_n^{\circ(l+1)}$ turns out to be very close to ∂I^n ("very low return"). But $g_{n+1}c$ belongs to some non-central interval I_j^{n+1} whose Poincaré length in I^n is definite (as we have shown above). This is a contradiction.

So when we pass from one level to the next, the geometric bounds change gradually (provided the conditions of Lemma 16 don't hold). But the same is true when when we pass from level $m = e(l)$ to level $m+N = e(l+1)$ of a saddle-node cascade (26). Indeed, assume that the geometry on level I^m is bounded. Then the geometry of all configurations $(I^{m+i-1} \backslash I^{m+i}, K_j^{m+i})$, $i = 1, ..., m+N$, are bounded as well. Let us define the intervals $J(x)$, $x \in \omega(c) \cap (I^{m+N-1} \backslash I^{m+N})$, as the pull-backs of I^{m+N} corresponding to the first landing of orb (x) at I^{m+N} . Then it follows from boundedness of $o(x)$ and $d(x)$ that the configurations of intervals $J(x)$ has a bounded geometry in I^{m+N-1} . Now we can pull these intervals back to the next level $m+N$, and argue that the geometry is still bounded in the same way as above. □

Now Theorem D follows from the last two lemmas.

Quasi-symmetric conjugacy. We will show below that any two real quadratic-like maps with the same combinatorics are qs conjugate, which implies the real rigidity conjecture (compare [Sw]). To construct the conjugacy, we bounce in between Sullivan's argument for bounded geometry case, and the pull-back argument of §4.

Let us take two maps f and $\tilde{f}$ of Epstein class with a bounded distortion on the real line. Let us consider the alternatives of Theorem D. In the latter case the real geometry is essentially bounded before the next quadratic-like level. This allows us to construct a qc pseudo-conjugacy between the configurations of the Euclidian disks based upon the intervals I_j^n . The construction is the same as in the bounded geometry case (see [MS], Ch. IV, Theorem 3.1), except that Lemma 12 takes care of long saddle-node cascades.

If the first alternative of Theorem D occurs, then by Lemma 15 on some essentially bounded level we can create polynomial-like maps with definite moduli. By Lemma 17 the geometry is essentially bounded until that level, and we can apply the previous argument. On that level we can switch to the pull-back argument of §4. (To begin the argument, use the initial construction of [L3], §5.)

When we arrive at the next quadratic-like level, then we proceed as follows. In the first case we have arrived with a qc pseudo-conjugacy between configurations of Euclidian disks. Then just apply the previous construction to Rf (here we need real a priori bounds for infinitely q-renormalizable maps [G], [BL], [S]). In the second case we have arrived with configurations of topological disks. Then interpolate the qc pseudo-conjugacy as in §4, and conformally map the range of Rf to a slit domain. This gives us a map of Epstein class with a definite distortion on the real line, and we can repeat the construction.

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