Induced expansion for quadratic polynomials

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Abstract

We prove that non-hyperbolic non-renormalizable quadratic polynomials are expansion inducing. For renormalizable polynomials a counterpart of this statement is that in the case of unbounded combinatorics renormalized mappings become almost quadratic. Technically, this follows from the decay of the box geometry. Specific estimates of the rate of this decay are shown which are sharp in a class of S-unimodal mappings combinatorially related to rotations of bounded type. We use real methods based on cross-ratios and Schwarzian derivative complemented by complex-analytic estimates in terms of conformal moduli.

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1 Introduction

1.1 Overview

In recent years, a rather dramatic progress occurred in the study of real quadratic polynomials, or, more broadly, S-unimodal mappings with a singularity of quadratic type. Examples of progress include better understanding of measure-theoretical attractors and quasisymmetric classification of quadratic polynomials which lead to a proof of the monotonicity conjecture in the real quadratic family.

This progress was partly based on ingenuous new estimates. The main breakthrough, however, was in achieving a better understanding of the rich dynamics of unimodal mappings in conjunction with their geometry, and thus being able to apply appropriate tools in different cases. Most of the progress in this direction seems to be due to the application of the idea of *inducing*. The first striking application of inducing to the study of unimodal maps was in the work [9]. In that work useful geometrical and analytic estimates were obtained only in judiciously chosen special cases. Another notable step was the work of [7]. An attempt was made there to handle all cases, though some patterns emerged as analytically unmanageable. Then, independently, two approaches appeared. One of them was the inducing construction of [10], which is also the underlying approach of the present paper. Here a clear and complete topological model of unimodal dynamics was obtained, with satisfactory estimates in most cases, except in what was called an *infinite box case*. The infinite box case was subsequently solved for mappings with a quadratic singularity based on the phenomenon of *decaying box geometry*. The phenomenon was first noticed in [10], however proved there only in some cases. An estimate called the *starting condition* was provided which, if satisfied, allowed one to prove the decaying geometry in general for S-unimodal maps. It should be noted that about the same time a similar case of decaying geometry was observed independently by [23] and [4] for circle mappings with a flat piece. It is not known whether there is more than an analogy between both cases.

A breakthrough work was [22], see [17] for a description. This was done for non-renormalizable quadratic polynomials, complex as well as real. Inducing was not directly mentioned, but implicitly present in the construction of a Markov partition. It is believed that the approaches of [10] and [22] give

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equivalent sequences of partitions for real polynomials. A great achievement of [22] was in being able to get estimates in the "hard case" which again emerged (and was called *persistently recurrent*). This was done by watching how pieces of the partition nest in one another with certain moduli, and using a computation quite similar to one done in [2] (curiously, this last work was about cubic polynomials.)

In the real situation, analytic methods were refined in the study of one particular topological class of the so-called *Fibonacci polynomial*. This map was proposed in [8] as an interesting example to study. From the point of view of [10], the Fibonacci polynomial shows the simplest example of the infinite box case. In the work of [15] a complex-analytic idea was applied to obtain estimates in the Fibonacci case. Another approach was used in [12] where the same results as in [15] were proved by purely real methods based on negative Schwarzian. Together with arguments of [10] or [16] based on inducing, the non-existence of non-Feigenbaum type Cantor attractors (proved in [14]) implies *induced expansion*.

Finally, there was a work of [20] which addressed the geometry of renormalizable quadratic polynomials. Again, a hard case emerged when the trajectory of the restrictive interval was allowed to be arbitrarily long (unbounded case), and in terms of the construction of [10] needed to be tracked through a long sequence of box returns. In [20], the idea borrowed from [15] was used which consists of introducing an artificial map for which the starting condition holds. Then, a conjugacy with good quasiconformal properties is constructed between this mapping and the given quadratic polynomial which forces the starting condition for the polynomial.

1.2 Statement of results

Introduction In this paper we rely on the inducing approach of [10], and our goal is to verify the starting condition in the box case. Our method is a mix of real and complex arguments and works just as well for renormalizable polynomials of unbounded type as for non-renormalizable ones. The method is direct, that is independent of quasiconformal conjugacies with artificial maps, the tableau computation of [2] or moduli estimates of [22]. On the technical level, our main theorems follow from Theorems A and B stated further in the text. The derivation is by arguments of [20] (fully presented here.)

Main theorems.

Definition of our class of mappings.

Definition 1.1 *We define class $\mathcal{F}_\eta$ to comprise all unimodal mappings of the interval $[0,1]$ into itself normalized so that 0 is a fixed point which satisfy these conditions:*

- *Any $f \in \mathcal{F}$ can be written as $h((x-\frac{1}{2})^2)$ where h is a polynomial defined on a set containing $[0,1/4]$ with range $(-\eta, 1+\eta)$.*

- *The map h has no critical values except on the real line.*

- *The Schwarzian derivative of h is non-positive.*

- *The mapping f has no attracting or indifferent periodic cycles.*

We also define

$$\mathcal{F} := \bigcup_{\eta > 0} \mathcal{F}_\eta .$$

For any mapping $f \in \mathcal{F}$ we define the *fundamental inducing domain* as follows. From the non-existence of attracting or indifferent periodic points it follows that there is a repelling fixed point $q > 1/2$. The fundamental inducing domain is the interval $(1-q, q)$. Almost every orbit passes infinitely many times through the fundamental inducing domain.

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Theorem about non-renormalizable mappings.

Theorem 1 *Let $f \in \mathcal{F}_\eta$ be non-renormalizable. Then on an open, dense and having full measure subset of the fundamental inducing domain one can define a continuous function $t(x)$ with values in positive integers so that $f^{t(x)}$ is an expanding Markov mapping. That is, restricted to a maximal interval on which $t(x)$ is defined an constant, $f^{t(x)}$ is a diffeomorphism onto $(1-q, q)$, expanding, and with distortion (measured as the variation of the logarithm of the jacobian) bounded by depending on η only.*

Theorem 1 has a number of consequences (see [10]). It gives an alternative proof of the non-existence of "exotic" attractors in class $\mathcal{F}$ (already known from [14]). It also gives an approach to constructing invariant measures.

Theorem in the renormalizable case.

Definition 1.2 *Let $f \in \mathcal{F}$. A point x in the domain of f is called* almost parabolic with period m and depth k *provided that:*

- *the derivative of f^m at x is one,*

- *f^m is monotone between x and the critical point,*

- *k consecutive images $f^m(1/2), \ldots, f^{km}(1/2)$ are between x and $1/2$.*

Theorem 2 *Let $f \in \mathcal{F}_\eta$ be renormalizable, and let n be the return time of the maximal restrictive interval into itself. Denote by $k(n)$ the maximum of depths of almost parabolic points with periods less than n. Specify a number $D > 0$. For every given k, a number $N(\eta, D, k)$ exists independent of f so that if $n > N(\eta, D, k)$ and $k(n) \leq k$, then f^n on a neighborhood of $1/2$ is affinely conjugate to a mapping from $\mathcal{F}_D$.*

Theorem 2 "almost complements" the theory of renormalizable mappings developed in [19]. In fact, it says that such a theory at least in some aspects is much simpler for renormalizable mappings of unbounded type. The exclusion of the unbounded case with almost parabolic returns is the only gap. Theorem 2 is a critical step in the proof of monotonicity in the real quadratic family, see [20] (where, by the way, the theorem is stated wrongly without excluding the almost parabolic case.)

Technical theorems. The strongest results of our paper are contained in technical theorems A,B, C, D. They imply theorems 1 and 2. In addition, Theorem C gives exact bounds on the exponential rate of decay of box geometry for S-unimodal rotation-like maps. Theorem D concerns the decay of *complex* box geometry and can serve as an important step in proving the density of hyperbolicity in the real quadratic family, see [20]. The technical theorems are stated for objects that we call *box mappings* and their statement must be postponed until those are defined.

Plan of the work. All three theorems follow from Theorem B which we will formulate later which essentially says that the starting condition holds for any map from $\mathcal{F}$ after a bounded number of box inducing steps. To prove this we will use two complementary methods. The first one is based purely on real-variable considerations. The advantage of a real-variable approach is in robust estimates. The disadvantage is sensitivity to the combinatorial complexity of the problem. Thus, we conduct our estimates only in the case we call "rotation-like" in which estimates are subtle and the combinatorics not harder than for circle rotations. In this way we prove Theorem A for real mappings with rotation-like behavior. Our second method is complex-analytic and based on watching annuli which are mapped by the dynamics and nest inside one another. This method will allow us to cover the full realm of combinatorial possibilities, but estimates are weaker. In particular, they would be too weak in the rotation-like case. More precisely, to prove the starting condition we will need to show that certain ratios are small after a number of box steps. Depending on the dynamics, by each box step the geometrical ratios decrease either by a multiplicative constant uniformly less than 1, or by being raised to a power uniformly greater than 1. The complex approach is capable of accounting for the second type of phenomenon, but, at least at present, misses the first one. It turns out that it is exactly the rotation-like case that exhibits the first, slower, type of decay.

After proving the technical theorems, we will derive Theorems 1 and 2.

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1.3   Box mappings.

Real box mappings.   The method of inducing was applied to the study of unimodal maps first in [9], then in [7]. In [11] and [10] an elaborate approach was developed to study induced maps, that is, transformations defined to be iterations of the original unimodal map restricted to pieces of the domain. We define a more general and abstract notion in this work, namely:

Definition 1.3 *Consider a transformation ϕ defined on an open dense subset of an interval I into I. Call restrictions of ϕ to connected components of the domain* branches. *If each branch is at least three times differentiable and the Schwarzian derivative is non-positive wherever defined, we will call ϕ a* piecewise map.

We will deal with two types of branches, namely monotone and folding. A monotone branch is a diffeomorphism onto its image, while a folding branch arises as a quadratic polynomial pre- and post-composed with diffeomorphisms. Examples include induced maps studied in works cited above.

Next, we want to impose more specific conditions on the images of the branches of a generalized induced map.

Definition 1.4 *A box map on I is a piecewise map on I more combinatorial structure. Namely, there is a finite nesting sequence of intervals, called* boxes *or* real boxes,

$$I = B_0 \supset \cdots \supset B_n$$

indexed by a growing sequence of integers and halves of odd integers. All boxes are assumed symmetric with respect to the critical point. It is further assumed that there is only one central folding branch whose domain is B_n and whose image is contained in some B_k with the boundary of B_n going into the boundary of B_k. It is then said that the rank *of the central branch is k. All other branches are monotone and each maps onto some B_l which determines its rank.*

Furthermore, no branch of positive rank can share a common endpoint with another branch of positive rank or folding, and if the domain of some branch has a non-empty intersection with a box, then it is contained in this box.

This is an extension of the notion of induced map used in [11]. Namely,

Definition 1.5 *A box map is called full if B_1 is the domain of the central branch which is of rank 0, and the highest rank is 1.*

Complex box mappings. We will now give a precise meaning to complex extension of real box mappings. Given a real box B, a corresponding *complex box* must be an open disk symmetrical with respect to the real axis which intersects the real line exactly along B. Given a real box structure, a compatible *complex box structure* will be a sequence of nesting complex boxes corresponding to the real boxes.

Hole structures. Now suppose that a real box mapping ϕ is given whose all branches are real-analytic and pick a compatible complex box structure. A *hole structure* means that for the domain of every branch of ϕ save monotone branches of rank 0 an open disk (hole) is chosen symmetrical with respect to the real line and intersecting the line along the domain. It is further assumed that the branch has an analytic continuation to its hole of the same topological type as the real branch, i.e. either univalent for monotone branches or degree 2 for folding branches. We will not hesitate to talk of monotone or folding branches for complex box mappings. This analytic extension will be called the *complex branch*. Lastly, the image of the complex branch is exactly the complex box corresponding to the rank of the real branch.

Definition 1.6 *Given a real-analytic real box mapping ϕ, the choice of a complex box structure and a hole structure, defines a complex box mapping. Thus, by a complex box mapping we will formally understand the conglomerate of all three compatible structures: a real box map, complex box structure, and hole structure. With a slight abuse of language, we will also extend this name to the union of complex branches of the complex box map.*

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Special type box mappings.

Definition 1.7 *A type I box mapping of rank n is a first of all a box mapping whose central domain is B_n. The central branch has rank $n - 1 \leq n' < n$ corresponding to the element of the box structure directly preceding B_n. All monotone branches of positive rank have rank n. This definition makes equal sense for real and complex box mappings.*

Note that the full map is type I of rank 1, with $n' = 0$. Most of our work will be done in terms of type I box maps.

Definition 1.8 *A type II box mapping of rank n is a box mapping whose central domain is B_n and all branches of positive rank map onto $B_{n'}$ which the element of the box structure directly preceding B_n.*

Separation symbols for complex box mappings.

Definition of the symbols. Now, let φ be any type I complex box mapping. Fix the notations so that the central hole of φ is labeled B_n with $n > 1$. Pick a monotone branch B of rank n. Also, pick another hole C contained in B_{n-1}. We may assign to B its *separation symbol* from C which is simply an ordered quadruple of real non-negative numbers:

$$s(B) := \left(s_1(B), \cdots, s_4(B) \right).$$

A valid separation symbol by definition implies the existence of certain annuli with moduli estimated from below in terms of the components of the symbol. We first assume that there are annuli A_1 and A_2, both selected for the given B even though the dependence on B is not emphasized in the notation. Both annuli are contained in $B_{n'}$. The annulus A_2 surrounds the hole C separating it from the boundary of B_{n-1} as well as the hole B. In addition, A_2 is not allowed to intersect any holes which meet the real on the side of B opposite to C. This, in effect means that A_2 also separates C from all holes strung on the real line "behind" B. Then A_1 separates A_2 from the boundary of B_{n-1}. The number $s_2(B)$ is a lower bound on the modulus of A_2 and $s_1(B)$ is a lower bound on the sum of moduli of A_1 and A_2.

We then proceed to select annuli around B which will give the meaning of the two remaining components of the symbol. First, the annulus A' is

chosen to separate B from the domain of the extension of the branch defined on B. This extension is of rank n'. Then, the existence of A_3 is postulated which surrounds A' separating it from C and the boundary of B_{n-1}. Also, A_3 is not allowed to intersect any holes that meet the real line on the side of C opposite to B. Finally, A_4 separates A_3 from the boundary of B_{n-1}. The component s_3 is then assumed to be a lower bound for the sum of moduli A' and A_3, while s_4 is a lower bound of the sum of all three moduli: A', A_3 and A_4.

In this paper we will always assume C to be the central hole. We will then call the separation symbol of B from C the *critical symbol* of B.

This fully lists the geometrical properties implied by a valid separation symbol.

Normalized symbols. We will now arbitrarily impose certain algebraic relations among various components of a separation symbol. Choose a number β, and $\alpha := \beta/2$, together with λ_1 and λ_2. Assume $\alpha \geq \lambda_1, \lambda_2, \lambda_1 \geq -\frac{\alpha}{2}$, $\lambda_2 \geq -\frac{\alpha}{2}$ and $\lambda_1 + \lambda_2 \geq 0$. If these quantities are connected with a separation symbol $s(B)$ as follows

$$s_1(B) = \alpha + \lambda_1) \, ,$$

$$s_2(B) = \alpha - \lambda_2 \, ,$$

$$s_3(B) = \beta - \lambda_1 \, ,$$

$$s_4(B) = \beta + \lambda_2 \, .$$

we will say that $s(B)$ is normalized with norm β and corrections λ_1 and λ_2.

Separation norm of a box mapping. For a type I complex box mapping ϕ its *separation norm* is defined as the supremum of values of β for which valid normalized critical symbols with norm β exist for all univalent branches.

1.4   Box inducing process

Description of the process. We will now describe a *standard inducing step* which is a procedure that takes a box mapping called ϕ and returns a

type I box mapping called Φ. Φ is induced in terms of ϕ, i.e. all branches of Φ are compositions of branches of ϕ.

Close and non-close returns. If the escape time of the critical point is 1, we say that the mapping shows a non-close return. Otherwise, it is a close return. The box case occurs when the image of the critical point is found in the domain of a monotone branch of positive rank. The remaining two cases when the push-forward image is in an rank 0 domain, or beyond the domain of the induced map (the Misiurewicz case), are well understood by the methods of [10], so we always assume that a box case occurs. The procedure we describe applies equally well to complex and real box mappings. So suppose that a box mapping ϕ. Let the central domain be B_n.

The first filling. Construct ϕ' by replacing ϕ on the central domain (hole) with the identity. This gives a valid box mapping. Next, define ϕ_1 by replacing the central branch ψ of ϕ with $\phi' \circ \psi$. The new central domain is the preimage by ψ of the domain containing the critical value. The new central domain is adjoined to the hole structure as B_{n+1}.

Filling-in. In order to obtain a type I mapping, we "fill in" all monotone branches obtained by the first filling. This process was introduced in [10]. That is, each point in the domain of a monotone branch is mapped by monotone of branches of positive rank until it leaves the union of their domains. Its first image outside of those domains, by definition, is the image of the point under a new map. This definition works except for a Cantor set of points. The mapping obtained in this way has all monotone (univalent) branches of rank $n + 1$.

This construction can be carried out regardless of whether the return was close or not. In the case of a non-close return this completes a *standard inducing step* and the map so obtained is Φ. For close returns, it is convenient to bundle together several steps.

Close returns. In the case of a close return, we proceed as follows. We construct ϕ' in the usual way be replacing the central branch with the identity, and then define ϕ_1 by substituting the central branch ψ with $\phi' \circ \psi$. In other words, the first filling occurs like for a non-close return. The central domain

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is B_{n+1}, but we will more often use the notation B^1. We then look whether the critical value of ϕ_1 (which is still the same as the critical value of ψ) is in B^1. If so, we repeat the same step with ϕ_1 instead of ϕ to obtain ϕ_2 and continue in this way until the critical value is no longer in the central domain B^k. This must happen for some k, or the intersection of all central domains would be a restrictive interval and ϕ would be suitable. The mapping has a bunch of monotone branches all of which have rank n and are compositions of the central branch applied a number of times with branches of ϕ. We then make it a type I mapping ϕ by filling-in of those monotone branches so that they become of rank $n + k$, i.e. all map onto B_k. The central branch remains unchanged of rank $n + k - 1$. Call this new type I mapping $\tilde{\phi}$. Then the last step is to execute the standard inducing step on $\tilde{\phi}$ which, by definition, will give Φ. Finally, we rearrange the box structure of Φ defining B_{n+1} as the central domain of Φ and $B_{n+1/2}$ as B^k.

The reader may check that this is the same map that we would get simply following the standard inducing step described for non-close returns $k + 1$ times. However, our description gives a more direct insight into the origin of branches of Φ.

Immediate preimages. Regardless of how Φ was constructed, its *immediate* or *primary branches* are those univalent branches which are restrictions of the central branch ψ. Note that there may be none or two immediate preimages, depending on whether the real range of the central branch covers the critical point. In the case of a close return the immediate preimages are formed at the last stage of inducing on $\tilde{\phi}$. The well-known "Fibonacci case" is an example when all monotone branches occurring in the constru

Filling-in and hierarchy of branches. Consider a abstract setting in which one has a bunch of univalent branches with common range B' and fills them in to get branches mapping onto some $B \subset B'$. The original branches mapping onto B' will be called *parent branches* of the filling-in process. Clearly, every branch after the filling-in has a dynamical extension with range B'. For two branches, the domains of these respective extensions m

- one branch is mapped onto the central branch, or

- the domains of their dynamical extensions are disjoint.

In the second case we say that the original branches were *independent*. Otherwise, the one mapped onto a monotone domain is called *subordinate* to the other one which was transported onto the central domain.

We then distinguish the set of "maximal" branches subordinate to none. They are mapped by their dynamical extensions directly onto the central domain. Therefore, the domains of extensions of maximal branches are disjoint. They also cover domains of all branches. The extensions of maximal branches exactly the *parent branches* of the filling-in process.

For example, in the non-close return the first filling gives a set of parent branches, two of which may be immediate, which later get filled in. In the close return filling-in is done twice, so we will be more careful in speaking about parent branches i

2 Statement of main results

Rotation-like returns. Let ϕ be a type I map obtained in a standard inducing step. Then its immediate branches are defined as in the previous paragraph.

Definition 2.1 *We say that ϕ exhibits a rotation-like return in the following situation. When the return is not close, it is rotation-like if and only if the critical value lands in an immediate branch of ϕ. When the return is close map the critical value by the central branch until the first exit from the central domain. The return is rotation-like if upon the first exit from the central domain the critical value gets into an immediate branch of ϕ and is mapped by this branch into B^k.*

The above definition is slightly technical however its advantage is that the existence of rotation-like maps follows immediately from the definition.

Proposition 1 *Rotation like maps do exist.*

Proof:
Take a one parameter full family of S-unimodal maps. For each of them do inducing procedure. Then the position of the critical value of the central domain during each step of induction depends continuously on the parameter value. Hence in particular we obtain all rotation-like maps.

$\square$

Now we are in position to give a simple characterization of rotation-like maps which also will justify the name of this class of unimodal maps.

Rotation-like sequences. We will say that a sequence of type I box mappings is rotation-like if the each arises from the previous one by a standard inducing step, and each return, with the exception of the first one for which immediate preimages are not defined, is rotation-like.

Fact 2.1 *For a rotation-like sequence, there is an inductive formula relating consecutive central branches of the inducing procedure*

$$f_{n+1} = f_{n-1} \circ f_n^{a_n},$$

where f_j denotes the central branch of ϕ_j, and a_n is a the smallest i such that the i-th iterate of the central branch of the n-th mapping maps the critical point outside of the central domain. E.g., $a_n = 1$ is equivalent to saying that the n-th map shows a non-close return. We define a_0 by a requirement that $f_2 = f^{a_0} \circ f_1^{a_1}$.

Following the analogy with the circle homeomorphisms we will introduce a concept of a rotation number for our class of maps.

Definition 2.2 *The rotation number $\rho(f)$ of ϕ is equal to*

$$\rho_n = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

which can be written shortly as $[a_0, a_1, \cdots]$ using the formalism of continued fractions.

2.1 Technical theorems

The starting condition.

Definition 2.3 *We say that a type I or type II box mapping of rank n satisfies the* starting condition *with norm δ provided that Let $|B_n|/|B_{n'}| < \delta$ and if D is a monotone domain of ϕ, then also $|D|/dist(D, \partial B_{n'}) < \delta$.*

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Fact 2.2 *Let ϕ be a real box mapping of type I of rank n. Pick $\tau > 1$ and assume that the central branch is τ-extendable. Then, there is a number $\delta(\tau)$ bounded away from 0 for τ in any closed subset of $(1, \infty)$ with the following property. Suppose that ϕ_i with $\phi_0 = \phi$ be a sequence of type I box mappings such that ϕ_{j+1} arises from ϕ_j by a standard inducing step and let ϕ satisfy the starting condition with norm $\delta(\tau)$. Then, in the box construction of [10], the ratios $\delta_{n+i}/\delta_{n+i-1}$ tends to 0 at least exponentially fast with i with rate given by an absolute constant.*

Proof:
This Fact was proved in [10].

$\square$

The box construction of [10] is slightly different from the inducing construction we use. We do not need the details now.

Theorem A about real box mappings. We state the theorem as follows.
Theorem A

Let $\phi = \phi_0$ be a type I real box mapping. Let B_n be its central domain. Suppose that the ratio of lengths $|B_n|/|B_{n'}|$ is $1 - \epsilon$. Next. let ϕ_i be a rotation-like sequence derived from ϕ. Specify a delta > 0. Then, there is a function $K(\epsilon, \delta)$ of ϵ and δ only and independent of ϕ, bounded on any compact compact subinterval of $(0, 1]^2$, with the property that for $i \geq K(\epsilon, \delta)$ the mapping ϕ_i satisfies the starting condition with norm δ.

The proof of Theorem A uses purely real methods. It generalizes the result of [12]. A natural question is whether an analogous result can be demonstrated by real methods for an arbitrary box sequence of induced maps. In principle, that should be possible, but technical difficulties are daunting.

Theorem B about complex box mappings. The technical complexity of the general box case becomes tractable when one works with complex box mappings. Hence the theorem:
Theorem B

Let $\phi = \phi_0$ be a type I complex box mapping, and ϕ_i be a sequence of complex box mappings such that ϕ_{i+1} is derived from ϕ_i in a standard inducing step. Suppose that the separation norm of ϕ is β. Also, specify a $\delta > 0$. Then,

there is a function $K(\beta, \delta)$ depending solely on β and δ and bounded on any closed set contained in $(0, \infty)^2$ with the property that if $i \geq K(\beta, delta)$, then ϕ_i as a real box mapping satisfies the starting condition with norm δ.

Sharp estimates for rotation-like mappings. **Theorem C** *For any S-unimodal rotation-like map there exist positive constants K_1, K_2 and $\kappa_1, \kappa_2 < 1$ depending only on the initial geometry, i.e. the number ϵ of Theorem A, so that*

$$K_1 \kappa_1^{a_1 \cdot \ldots \cdot a_{n-1}} \leq \tau_n \leq K_2 \kappa_2^{a_1 \cdot \ldots \cdot a_{n-1}}.$$

This is an improvement of Theorem A which also gives the lower bound on the rate of decay of box geometry. The proof is by purely real methods.

Growth of conformal moduli. **Theorem D** *Let $\phi = \phi_0$ be a type I complex box mapping of rank n, and ϕ_i form a sequence of complex box mappings derived from ϕ by the box inducing process. Suppose that box ratios on the real line decrease at least exponentially fast, i.e.*

$$\frac{|B_{n+j}|}{|B_{(n+j)'}|} < C^j$$

with $C < 1$. Let β_j denote the separation norm of ϕ_j. Then, there is a number $\overline{C}(C, \beta_0)$ so that if the separation norm of ϕ_0 is at least β_0, then $\beta_j \geq \overline{C} \cdot j$. The constant $\overline{C}$ only depends on its specified parameters.

Theorem D claims a decay of the conformal geometry in a sequence of complex box mappings derived by inducing. This phenomenon seems to be the basis of many recently obtained results, see [20]. In view of Theorem B and Fact 2.2 the assumption of Theorem D regarding the decrease of ratios is automatically satisfied for any complex box mapping with constant C depending only the initial separation

3 Real induction

In this section we prove Theorem A. Suppose that in the situation of Theorem A a rotation-like sequence ϕ_i is given, $i = 0, 1, \cdots$.

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3.1 Rotation-like sequences

Notations. For the real induction, we fix the critical point at 0. The central branch of ϕ_i will be called f_i. Denote the endpoints of the box B_n by z_n^- and z_n^+. We adopt the following convention of ascribing signs to points: '+' written as a superscript indicates the endpoint of B_n which lies closer to the critical value of $f_{[n+1]}$ (where $[\cdot]$ denotes the integral part.) In other words $f_{[n+1]}(0) \in (0, z_n^+)$. Each central branch can be represented as a composition of a diffeomorphism h_n and a quadratic map g. We know that the image of B_n is contained in B_{n-1}. The following Lemma describes extendability of diffeomorphisms h_n.

Generally, we will use (x, y) to denote the interval from x to y, regardless of the ordering of x and y.

Lemma 3.1 *If $n \geq 3$, the diffeomorphism h_n extends on some neighborhood of $g(B_n)$ so that the image of the extension coincides with $(f_{n-2}(0), z_{(n-2)'}^-)$.*

Proof:
By construction, h_n extends to a neighborhood of $g(B_n)$ which is mapped on B_{n-1}. Take arbitrary n. We know that $f_n = f_{n-2} \circ f_{n-1}^{a_n-1}$. Consider two cases.

- If $a_n = 1$ then the range of a monotone extensions of h_n is the same as that of the composition $f_{n-2} \circ h_{n-1}$ which is clearly $(f_{n-2}(0), z_{(n-2)'}^-)$. and next pull it back by the extension of f_n whose image by the inductive hypothesis covers B_{n-2}. The resulting interval is the

- For $a_n > 1$ choose an extension of h_{n-1} so that $f_{n-1}^{a_n-1} \circ h_{n-1}$ maps it diffeomorphically onto $(0, z_{n-2}^+)$. The image of monotone branches of f_{n-2} gives the desired range of the extension.

This proves Lemma 3.1.

$\square$

In the "real part" of this paper we will extend diffeomorphisms h_n every time only over a one side of their domains. We distinguish between two directions of one-sided extendability of h_n. The key observation is that the points $\{f_{n-2}(0), f_n(z_n^+), f_n(0), z_{n-3}^-\}$ are always arranged according either

to the natural or reversed order of the real line. Observe, that f_n extends further "through the head" meaning in the direction of the critical value, where h_n can be extended up to $z^-_{(n-2)'}$, than "through the legs" meaning in the direction of $f_n(\partial B_n)$ where the extension is only up to $f_{n-2}(0)$ which is closer to $B_{n'}$ than $z^+_{(n-2)'}$. The extendability of central branches plays a crucial role in estimates of the distortion. By real Köbe's Lemma the distortion depends only on the relative scale of the images of domains with respect to the images of their extensions. As it happens (and will be proved), these scales will improve during inducing procedure finally forcing *the starting condition*. We will need however some initial extension to start with.

Estimates a priori. Denote by τ_n, n integer, the ratio $|B_n|/|B_{n'}|$. Let τ be the supremum of τ_n with respect to n. We have the following Lemma:

Lemma 3.2 *Under the assumptions of Theorem A, there is a function $K_3(\epsilon)$ bounded on any closed subset of $(0,1]$ with the property that $\tau_n < \tau = 0.37$ for $n \geq K_3(\epsilon)$.*

The proof of this lemma is technical and apart from main ideas of this paper, so we put it in a separate section at the end of the real estimates.

Ratios and cross-ratios. Suppose we have three points a, b, c arranged so that $a \notin [b, c]$. Let us define a few relative scales of the interval (b, c) with respect to (a, c).

Definition 3.1 *The exclusive ratio of the interval (b, c) with respect to (a, c) is given by*

$$\mathbf{R}_e(b, c; a) = \frac{|b - c|}{dist((b, c), a)},$$

whereas their inclusive ratio by

$$\mathbf{R}_i(b, c; a) = \frac{|b - c|}{\max(|b - a|, |c - a|)}.$$

Set $\mathbf{R}(b, c; a)$ to be equal to the geometric mean of the inclusive and exclusive ratios.

$$\mathbf{R}(b, c; a) = \sqrt{\mathbf{R}_i(b, c; a)\mathbf{R}_e(b, c; a)}$$

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Together with ratios we will often use cross-ratios. The types of cross-ratios we use are expanded by homeomorphisms with negative Schwarzian derivative.

Definition 3.2 *Suppose we have a quadruple a, b, c, d ordered so that $a < b < c < d$ or reversely. Define their inclusive cross-ratio as*

$$\mathbf{Cr}_i(a, b, c, d) = \frac{|b - c||d - a|}{|c - a||d - b|},$$

and their exclusive cross-ratio as

$$\mathbf{Cr}_e(a, b, c, d) = \frac{|b - c||d - a|}{|b - a||d - c|}.$$

Finally set

$$\mathbf{Cr}(a, b, c, d) = \sqrt{\mathbf{Cr}_i(a, b, c, d)\mathbf{Cr}_e(a, b, c, d)}$$

Distortion. Suppose that we have an expression A which is defined in terms of distances between points (like ratios and cross-ratios.) Then we consider $f_*(A)$ obtained by replacing given points with their images by f. We will measure the distortion of this transformation by the ratio $f_*(A)/A$. For example, if we set $A = \mathbf{Cr}(a, b, c, d)$ then the distortion by f is equal to

$$\frac{\mathbf{Cr}(f(a), f(b), f(c), f(d))}{\mathbf{Cr}(f(a), f(b), f(c), f(d))}.$$

3.2   Induction parameters

In this subsection we will introduce quantities which will describe geometry of partitions given by our inducing procedure. Next we will compose a quasi-invariant which after a finite number of inducing steps will decrease at least exponentially fast. The real induction parameters formulated here will directly correspond to these in the complex part. The same concerns induction formulae. This suggests that estimates from the complex part of our work can somehow be translated into the corresponding ones in the real line. This would enable one to give a proof by purely real methods. However, the combinatorial complexity of such an approach seems formidable.

Denote by (x_n^-, x_n^+) the domain of the *primary* branch of rank n which contains the critical value of $f_n^{a_n}$. **Set** $v_n = f_n^{a_n}(0)$.

Definition 3.3 *The center of (x_n^-, x_n^+), denoted by c_n, is defined by the condition $\phi_n(c_n) = 0$.*

Following the convention of ascribing superscripts $+$ and $-$ we will define x_n^+ by the condition $v_n \in (x_n^+, c_n)$.

Lemma 3.3 *For rotation-like maps points x_n^- lie closer to zero than x_n^+.*

    Proof:
For rotation-like maps the ranges of central branches always contain the critical point. In particular, it means that the image of B_{n+1} by $f_n^{a_n}$ covers both (0) and the interval (c_n, x_n^-).

$\square$

Parameters of the induction measure sizes of domains of branches as well as and their separation from the critical point and the boundary of the relevant box. The distortion of these quantities will be controlled by bounds on τ and extendability of branches. Here, we provide a full list of parameters.

- $\alpha_n = \mathbf{R}(z_n^+, z_n^-;\ z_{n'}^+),$

- $\gamma_n = \mathbf{R}(x_n^+, x_n^-;\ 0),$

- $\beta_n = \mathbf{Cr}(z_{n'}^?, x_n^+, x_n^-, 0)$

where ? is chosen as $+$ or $-$ so that the points have allowable ordering.

    We will examine how these quantities change after a standard inducing step. Generally, none of these quantities is decreases monotonely in the inducing procedure. Nevertheless, we can choose products of these parameters that show monotone decay. Consider the products $\alpha_n \gamma_n$ and $\alpha_n \beta_n$ (they correspond to the sums $s_1 + s_3$ and $s_2 + s_4$ in the complex induction). We will see soon that that *primary* (immediate) domains (x_n^-, x_n^+) stay always at the definite distance from the boundary of $B_{n'}$. This implies that in the case of rotation-like maps these products are equivalent and it is enough to consider only one of them.

Proposition 2 *Consider the quantity $\alpha_n \gamma_n$ for a rotation-like sequence*

$$\phi_0, \ldots, \phi_n, \ldots \ .$$

If $\tau_n \leq \tau = 0.37$ for every n, then there is an absolute constant $\Lambda < 1$ and a fixed integer N with the property that

$$\alpha_n \gamma_n \leq \Lambda^n \alpha_0 \gamma_0$$

for all $n > N$.

Auxiliary quantities.   Before we pass to the proof of Proposition 2 which will occupy the next three subsections we will introduce three auxiliary induction quantities $\overline{\gamma}_n$, ω_n and Ω_n.

$$\overline{\gamma}_n = \mathbf{R}(v_n, x_n^-; \, 0)$$

$$\omega_n = \frac{|f_n(0)|}{|z_{n'}^+|} \quad \text{and} \quad \Omega_n = \frac{|v_n|}{|z_{n'}^+|}$$

(we recall that $v_n = f_n^{a_n}(0)$.)

In addition, we have already defined τ_n and τ. The estimates in the next two subsections will be quite complicated. It may help the reader to think of central branches as quadratic polynomials, and of monotone branches as affine. In this model estimates are easier and actually give the right idea of the real situation. Then the distortion might be treated as a correction to formulae obtained in the "linear-quadratic" model.

3.3   A non-close return

Throughout this subsection we assume that ϕ_n makes a non-close return. In particular it means that $(n+1)' = n$.

   The distance of a point z to zero is denoted by $|z|$. Observe that $|z_n^+| = |z_n^-|$ and $g(z_n^+) = g(z_n^-)$. Hence, we will often drop superscripts "+" and "−" from the notation of distances if only no confusion can arise. We will start with the following simple observation.

$$\alpha_{n+1}^2 = 4 \cdot \mathbf{R}_e(g(z_{n+1}), g(0); g(z_n)).$$

The image of $\mathbf{R}_e(g(z_{n+1}), g(0); g(z_n))$ by $(h_n)_*$ is equal to

$$\overline{\gamma}_n \frac{\sqrt{|x_n^-||f_n(0)|}}{|x_n^-| + |z_{n'}^-|}.$$

To find the distortion of $(h_n)_*$ on this ratio, we will complete the ratio to the cross-ratio $\mathbf{Cr}_e(g(z_n), g(z_{n+1}), g(0), h_n^{-1}(z_{(n-2)'}^-))$. Since the cross-ratio is expanded, we get

$$\alpha_{n+1}^2 \leq 4 \cdot \overline{\gamma}_n \, \frac{\sqrt{|x_n||f_n(0)|}}{|x_n| + |z_{n'}|} \, \frac{|z_{n'}| + |z_{n-2'}|}{|z_{n-2'}| - |f_n(0)|}. \tag{1}$$

Fact 3.1 *The distortion of $\overline{\gamma}_n$ and γ_n by a quadratic map is at least 2.*

Proof:
This follows directly from the definition of $\overline{\gamma}_n$.

$\square$

We pass to estimating $\overline{\gamma}_{n+1}$. Take the image of $\overline{\gamma}_{n+1}$ by the quadratic map g. Fact 3.1 implies that

$$\overline{\gamma}_n \leq \frac{1}{2} \cdot \mathbf{R}(g(v_{n+1}(0)), g(x_n); \, g(0)).$$

Complete $g_*(\overline{\gamma}_{n+1})$ to the cross-ratio

$$\mathbf{Cr}\{h_n^{-1}(f_{n-2}(0)), g(f_{n+1}(0)), g(x_{(n)}), g(0)\}$$

and then push it forward by h_n. By the property of expanding cross-ratios we have that

$$\overline{\gamma}_{n+1} \leq \frac{1}{2} \cdot \frac{|z_{n+1}| + |f_{n+2}(0)|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \, \frac{|f_{n-2}(0)| + |f_n(0)|}{\sqrt{|f_{n-2}(0)|^2 - |z_{n+1}|^2}}. \tag{2}$$

Comment 1 *Note that the estimate (2) remains true if we replace $f_{n+2}(0)$ by z_{n+1}^+.*

Our next task is to combine estimates on $\overline{\gamma}_{n+1}$ and α_{n+1} and get the best possible upper bound of their product in terms of $\overline{\gamma}_n$ and α_n. To this end we prove

Lemma 3.4 *For arbitrary n the following inequality holds.*

$$\frac{\sqrt{|x_n||f_n(0)|}}{|z_{n'}| + |x_n|} \, \frac{|f_{n+2}(0)| + |z_{n+1}|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \leq \frac{1}{4} \cdot \alpha_n \alpha_{n+1}(1 + \omega_{n+2}) \, \frac{|z_{n-1}|}{|f_n(0)| + |z_{n-1}|}.$$

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Proof:

By the definition of α_n we have that

$$\frac{|z_{n+1}|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \leq \frac{1}{4} \cdot \alpha_{n+1}\alpha_n \frac{\sqrt{|z_{n-1}|^2 - |z_n|^2}}{|z_n|} \sqrt{\frac{|z_n|^2 - |z_{n+1}|^2}{|f_n(0)|^2 - |z_{n+1}|^2}}. \quad (3)$$

The last factor in the inequality (3) is decreasing with respect to $|z_{n+1}|$. Thus, the right-hand side of (3) is bounded by

$$\leq \frac{1}{4} \cdot \alpha_{n+1}\alpha_n \frac{|z_{n-1}|}{|f_n(0)|}.$$

To complete the reasoning we will need the following elementary fact:
For any three positive numbers $0 < x < y < z$ the inequality

$$\frac{\sqrt{xy}}{z + x} < \frac{y}{z + y}$$

holds.
which can be readily proved by calculus. ¿From there,

$$\frac{\sqrt{|x_n||f_n(0)|}}{|z_{n'}| + |x_n|} \leq \frac{|f_n(0)|}{|f_n(0)| + |z_{n'}|}$$

which completes the proof.

$\square$

Comment 2 *Replace $|f_{n+2}(0)|$ by $|z_{n+1}|$ in the estimate of Lemma 3.4. By the same reasoning we obtain*

$$\frac{\sqrt{|x_n||f_n(0)|}}{|z_{n'}| + |x_n|} \frac{|z_{n+1}|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \leq \frac{1}{2} \cdot \alpha_n\alpha_{n-1} \frac{|z_{n-1}|}{|z_{n-1}| + |f_n(0)|}. \quad (4)$$

Multiply the inequalities (1) and (2) and then combine with the inequality in Lemma 3.4. As a result we get the following recursive formula

$$\alpha_{n+1}\overline{\gamma}_{n+1} \leq \lambda_n\alpha_n\overline{\gamma}_n,$$

where λ_n is less than

$$\frac{1}{2} \cdot (1 + \omega_{n+2}) \; \frac{|z_{n'}|}{|f_n(0)| + |z_{n'}|} \; \frac{|f_{n-2}(0)| + |f_n(0)|}{\sqrt{|f_{n-2}(0)|^2 - |z_{n+1}|^2}} \; \frac{|z_{n'}| + |z_{(n-2)'}|}{|z_{(n-2)'}| - |f_n(0)|}. \quad (5)$$

We will bound from above λ_n by maximizing (5) with respect to a location of $f_n(0)$. To this end consider

$$\epsilon_n = \frac{|f_{n-2}(0)| + |f_n(0)|}{|z_{(n-2)'}| - |f_n(0)||z_{n'}| + |f_n(0)|}$$

as a function of $|f_n(0)|$ on the interval $(0, |z_{n'}|)$.

Lemma 3.5 *The function ϵ_n achieves a global maximum in 0.*

   Proof:
The sign of the derivative of ϵ_n with respect to $|f_n(0)|$ is the same as the sign of

$$-(|f_{n-2}(0)| - |z_{n'}|)|z_{(n-2)'}| + 2|f_n(0)||f_{n-2}(0)| + |f_n(0)|^2.$$

The smaller root of the above quadratic polynomial is always less than zero. Thus, the function ϵ_n can have only a local minimum in the interval $(0, |z_{n-1}|$. Direct computation shows that if $\tau^2 \leq 1/3$, then $\epsilon_n(0) \geq \epsilon_n(|z_{n-1}|)$

$$\square$$

   Finally, by Lemma 3.5 and the definition of τ, λ_n is less than

$$\frac{1 + \omega_{n+2}}{2} \; \frac{1 + \tau^2}{\sqrt{1 - \tau^6}}. \quad (6)$$

Comment 3 *The same computation based on Comments 1 and 2 yields*

$$\gamma_n \alpha_n \leq \frac{1 + \tau^2}{\sqrt{1 - \tau^6}} \overline{\gamma}_{n-1} \alpha_{n-1}. \quad (7)$$

3.4   A close return.

Throughout this subsection we assume that n-th return is close. In particular it means that $(n+1)' = n + 1/2$. The scheme of the proof is much the same as in the previous case. The only difference is that the reasoning is a bit more way around and requires several repetitions of the estimates similar to these found in the last subsection.

   Let ζ_n be the ratio of $B_{n+1/2}$ to B_n. Put $\overline{\alpha_{n+1}} = \mathbf{R}(z_{n+1}^+, z_{n+1}^-, z_n^+)$. Denote the primary preimage of $B_{n+1/2}$ contained in (x_n^-, x_n^+) by $(x_{n+1/2}^-, x_{n+1/2}^+)$.

Recursion. We will write a recursion for the sequence $\alpha_n \gamma_n$. By definition,

$$\alpha_{n+1} \leq \frac{\overline{\alpha}_{n+1}}{\zeta_n \sqrt{1 - \tau^2}}. \tag{8}$$

For i ranging from 1 to $a_n - 1$ let $x^-_{n+1/2,i}$ stand for $f_n^{-i}(x^-_{n+1/2})$ and $x^+_{n+1/2,i}$ for $f_n^{-i}(x^+_{n+1/2})$. To bound $f_{n*}(\gamma_{n+1}\overline{\alpha}_{n+1})$ from above we will use similar arguments as in the previous section. Push forward γ_{n+1} by the quadratic map g. Then

$$\gamma_{n+1} \leq \frac{1}{2} R(g(x^+_{n+1}), g(x^-_{n+1}); g(0)).$$

Complete $g_*(\gamma_{n+1})$ to the cross-ratio

$$\mathbf{Cr}\{h_n^{-1}(f_{n-2}(0)), g(x^+_{n+1}), g(x^-_{n+1}), g(0)\}$$

and then push it forward by h_n. By the property of expanding cross-ratios we have that

$$\gamma_{n+1} \leq \frac{|z_{n+1}|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \frac{|f_{n-2}(0)| + |f_n(0)|}{\sqrt{|f_{n-2}(0)|^2 - |z_{n+1}|^2}}. \tag{9}$$

In the same way as in the proof of Lemma 3.4 (see the inequality (3)) we obtain

$$\frac{|z_{n+1}|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \leq \frac{1}{4} \cdot \overline{\alpha}_{n+1}\alpha_n \frac{|z_{n'}|}{|f_n(0)|}.$$

By definition of $\overline{\alpha}_{n+1}$,

$$\overline{\alpha}_{n+1} = 4 \cdot \mathbf{R}_e(g(z_{n+1}), g(0); g(z_n)) \leq 4\mathbf{Cr}(g(z_n), g(z_{n+1}), g(0), h_n^{-1}(z^-_{(n-2)'})).$$

Again using the expanding property of cross-ratios we get

$$\overline{\alpha}^2_{n+1} \leq 4 \cdot \frac{|x^-_{n+1/2,a_n-1} - f_n(0)|}{|x^-_{n+1/2}| + |z_{n'}|} \frac{|z_{n'}| + |z_{n-2'}|}{|z_{n-2'}| - |f_n(0)|}.$$

The estimates for γ_{n+1} and $\overline{\alpha}^2_{n+1}$ lead to the following formula

$$\overline{\alpha}_{n+1}\gamma_{n+1} \leq \alpha_n \eta_n \frac{|f_n(0)| - |x^-_{n+1/2,a_n-1}|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2|}},$$

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where η_n is equal to

$$\frac{|z_{n'}|}{|z_{n'}| + |x^-_{n+1/2,a_n-1}|} \frac{|z_{n'}| + |z_{(n-2)'}|}{|z_{(n-2)'}| - |f_n(0)|} \frac{|f_{n-2}(0)| + |f_n(0)|}{\sqrt{|f_{n-2}(0)|^2 - |z_{n+1}|^2}}.$$

Let $0 \le i < a_n$. We shall write $\gamma_{n,i}$ for the ratio

$$\frac{|f_n^{a_n - i}(0)| - |x^-_{n+1/2,i}|}{\sqrt{|f_n^{a_n-i}(0)||x_{n+1/2,i}|}}. \tag{10}$$

Now, compute

$$\frac{|f_n(0) - x^-_{n+1/2,a_n-1}|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \div \mathbf{R}(g \circ f_n(0), g(x^-_{n+1/2,a_n-1}); \ g(z_{n+1})) \le$$

$$\le \frac{|x^-_{n+1/2,a_n-1}|}{|x^-_{n+1/2,a_n-1}| + |f_n(0)|}.$$

Increase this factor to $1/2$ and decrease η_n replacing $|x^{-1}_{n+1/2,a_n-1}|$ with $|f_n(0)|$. As a result we obtain an upper bound of $\overline{\alpha}_{n+1}\gamma_{n+1}$ which can be written in the form

$$\frac{1}{2}\alpha_n \eta'_n \mathbf{R}(g \circ f_n(0), g(x^-_{n+1/2,a_n-1}); \ g(z_{n+1})),$$

where

$$\eta'_n = \frac{|z_{n'}|}{|f_n(0)| + |z_{n'}|} \frac{|f_{n-2}(0)| + |f_n(0)|}{\sqrt{|f_{n-2}(0)|^2 - |z_{n+1}|^2}} \frac{|z_{n'}| + |z_{(n-2)'}|}{|z_{(n-2)'}| - |f_n(0)|}.$$

By Lemma 3.5 we can only worsen estimates letting $|f_n(0)| = 0$ in the expression for η'_n. Therefore,

$$\overline{\alpha}_{n+1}\gamma_{n+1} \le \frac{1}{2}\alpha_n \frac{1 + \tau^2}{\sqrt{1 - \tau^6}} \mathbf{R}(g \circ f_n(0), g(x^-_{n+1/2,a_n-1}), g(z_{n+1})). \tag{11}$$

Complete the last ratio to an appropriate cross-ratio by adjoining a fourth point at $h_n^{-1}(f_{(n-2)}(0))$ and then push it forward by h_n. The points $f_n(z_{n+1})$ and $x^-_{n+1/2,a_n-2})$ lie on the opposite sides of zero. The resulting cross-ratio

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can be only increased if we move the point $f_n(z_{n+1})$ in the direction of zero. Hence, for $a_n > 2$

$$\mathbf{R}(g \circ f_n(0), g(x^-_{n+1/2,a_n-1}), g(z_{n+1})) \leq \frac{\gamma_{n+1/2,a_n-2}}{1-\tau^2} \tag{12}$$

If $a_n = 2$ then put $1 - \tau\Omega_n$ in the place of the denominator of (12) in order to have the correct estimate.

We can use the sequence of estimates starting from (10) again to prove

$$\gamma_{n,i} \leq \frac{1}{2} \cdot \frac{\gamma_{n+1/2,i-1}}{1-\tau^2} \tag{13}$$

provided $i > 1$ and

$$\gamma_{n,1} \leq \frac{1}{2} \cdot \frac{\gamma_{n+1/2,i-1}}{1-\tau\Omega_n} \tag{14}$$

when $i = 1$. ¿From inequalities (11), (12), (13) and (14) we obtain

$$\overline{\alpha}_{n+1}\gamma_{n+1} \leq \overline{\Lambda}_n \alpha_n \gamma_{n+1/2,0} \tag{15}$$

where $\overline{\Lambda}_n$ is bounded from above by

$$\frac{1}{2^{a_n-1}} \frac{1+\tau^2}{\sqrt{(1-\tau^6)(1-\tau^2)^{a_n-2}(1-\tau\Omega_n)}}. \tag{16}$$

In the last step of our reasoning we exploit the fact that $\gamma_{n+1/2,0}$ is substantially less than γ_n. We claim that

Lemma 3.6

$$\gamma_{n+1/2,0} \leq \zeta_n \gamma_n.$$

Proof:
We will actually prove

$$\mathbf{R}(x^+_{n+1/2}, x^-_{n+1/2}; 0) \leq \zeta_n \gamma_n$$

and use

$$\gamma_{n+1/2,0} < \mathbf{R}(x^+_{n+1/2}, x^-_{n+1/2}; 0)$$

which follows directly from the definition of $\gamma(n+1/2,0)$. First observe that $|x^-_{n+1/2} - x^+_{n+1/2}| \leq \zeta_n |x^-_n - x^+_n|$. Indeed, the centers of $B_{n+1/2}$ and B_n coincide

and the hyperbolic length of $B_{n+1/2}$ with respect to B_n is not increased by the pullback by a diffeomorphism with a non-positive Schwarzian. Since the element of the hyperbolic length is the smallest in the middle of an interval we conclude that pullbacks are nested with the ratio at most ζ_{n+1}. Denote by s_1 and s_2 the centers of $(x^-_{n+1/2}, x^+_{n+1/2})$ and (x^-_n, x^+_n). A straightforward calculation shows that if these centers coincide then the Lemma follows. Suppose that s_1 is less than s_2 since if otherwise then we are done. Push x^+_n toward s_1 so far that the centers coincide again. This operation can only increase the ratio of γ_n to $\mathbf{R}(x^+_{n+1/2}, x^-_{n+1/2}; 0)$. The ratio of the lengths of the resulting, concentrical intervals, is again at most ζ_n, which completes the proof.

$\square$

Finally, Lemma 3.6 and inequalities (8) and (15) imply that

$$\alpha_{n+1}\gamma_{n+1} \le \Lambda_n \alpha_n \gamma_n, \tag{17}$$

where Λ_n is less than

$$\frac{1}{2^{a_n-1}} \frac{1 + \tau^2}{\sqrt{(1 - \tau^6)(1 - \tau^2)(1 - \tau^2)^{a_n-2}(1 - \tau\Omega_n)}}.$$

Clearly, Λ_n is the largest for $a_n = 2$ since $\tau^2 < \frac{1}{2}$. For $a_n = 2$ we get

$$\frac{1}{2} \frac{(1 + \tau^2)}{(1 - \tau\Omega_n)\sqrt{(1 - \tau^6)(1 - \tau^2)}} \tag{18}$$

as an upper bound of Λ_n in the case of a close return.

Comment 4 *In this subsection in the unlike for non-close returns we worked with the quantity γ_n instead of $\overline{\gamma}_n$. The estimates become stronger if we decrease the left-hand side of (17) substituting γ_n by $\overline{\gamma}_n$.*

$$\alpha_{n+1}\overline{\gamma}_{n+1} \le \lambda_n \alpha_n \gamma_n, \tag{19}$$

where λ_n is less than

$$\frac{1 + \omega_{n+2}}{4} \frac{(1 + \tau^2)}{(1 - \tau\Omega_n)\sqrt{(1 - \tau^6)(1 - \tau^2)}}. \tag{20}$$

The proof is the same, except that the estimate (9) ought to be replaced by (2).

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3.5 Conclusion of the proof

The sequence ω_n plays a crucial role in the inductive formulae derived in the last two subsections. The decay of geometry depends directly on the separation of this sequence from 1. We will begin by writing a recursion for the sequence Ω_n. Clearly, $\omega_n \leq \Omega_n$. We will consider two cases.

A non-close return. In this case $\Omega_n = \omega_n$.

Lemma 3.7 *Assume that ϕ_n makes a non-close return. Then*

$$\Omega_{n+1}^2 \leq (1 + \tau^2)\frac{\Omega_n + \Omega_{n+2}\tau^2}{1 + \Omega_n}.$$

Proof:

Take the image of Ω_n by the quadratic map g

$$\Omega_{n+1}^2 = \mathbf{R}_i(g(v_{n+1}), g(0); g(z_n))$$

and next push it forward by h_n. As a result we obtain

$$f_{n*}(\Omega_{n+1}) = \frac{|f_n(0)| + |f_{n+2}(0)|}{|f_n(0)| + |z_{n'}|}. \tag{21}$$

To compute the distortion brought in by h_n complete $g_*(\Omega_n)$ to the cross-ratio

$$\mathbf{Cr}_i(g(v_{n+1}), g(x_{n+1}), g(0), h_n^{-1}(z_{(n-2)'}^-)).$$

that fact that the cross-ratio is expanded. We obtain

$$\frac{|z_{n'}| + |z_{(n-2)'}|}{|z_{(n-2)'}| + |f_{n+2}(0)|}.$$

as a correction to (21). Observe that $|f_{n+2}(0)|/|z_{n'}| \leq \Omega_{n+2}\tau^2$ which establishes the claim of Lemma 3.7.

$\square$

A close return. Assume that f_n shows a close return. Then $\Omega_{n+1} = |v_{n+1}|/|z_{n+1/2}|$. By definition

$$\Omega_{n+1}^2 = \mathbf{R}_i(g(v_{n+1}), g(0); g(z_{n+1/2})).$$

Complete the last ratio to an appropriate cross-ratio by adjoining a fourth point at $h_n^{-1}(z_{(n-2)'}^-(0))$ and then push it forward by h_n. By the property of expanding cross-ratios we get

$$\Omega_{n+1}^2 \leq \frac{|f_n(0)| + |f_{n+2}(0)|}{|f_n(0)| + |f_n(z_{n+1/2})|} \frac{|f_n(z_{n+1/2})| + |z_{(n-2)'}|}{|z_{(n-2)'}| + |f_{n+2}(0)|}.$$

Let us denote the ratio $|f_n(z_{n+1/2})|/|z_n|$ by σ. Replace $|f_n(0)|$ by $|f_n(z_{n+1/2})|$ in the inequality above. We obtain a new bound of Ω_{n+1} equal to

$$\frac{1}{2} \cdot (1 + \tau_{n+1} \frac{|z_{n+1/2}|}{|z_n|\sigma}\Omega_{n+2})(1 + \sigma\tau^3) \tag{22}$$

Clearly, $\frac{|z_{n+1/2}|}{|z_n|\sigma} < 1$. We will estimate σ from above under the assumption that the critical value of f_n remains in the central domain for at least two iterates. The point $f_n(z_{n+1/2})$ is $f_n^{-a_n+1}(z_n^-)$. Thus, σ is the greatest when $f_n(z_{n+1/2})$ coincides with a boundary point of B_n. The next inequality is obtained by completing the ratio $g_*(\sigma)$ to an "inclusive" cross-ratio with a fourth point at $h_n^{-1}(z_{(n-2)'})$ and then pushing it forward by h_n.

$$\sigma^2 \leq \frac{|f_n(0)| + |z_n|}{|f_n(0)| + |z_{n'}|} \frac{|z_{(n-2)'}| + |z_{n'}|}{|z_{(n-2)'}| + |z_n|},$$

and finally,

$$\sigma \leq \sqrt{\frac{2\tau(1 + \tau^2)}{1 + \tau}} \tag{23}$$

So, we write (22) as

$$\Omega_{n+1}^2 \leq \frac{1}{2}(1 + \tau\Omega_{n+2})(1 + \sigma\tau^3), \tag{24}$$

where σ bounded by (23).

A bound. We want to find an upper bound of Ω_n. To this end, observe that

- Ω_{n+1} is an increasing function of Ω_n and Ω_{n+2}.

- As long as the value of Ω_n is greater than 0.7, the estimate of Lemma 3.7 gives a lower value of Ω_{n+1} than inequality (24).

The last statement can be easily justified by direct computation. Indeed, the right-hand side of the estimate of Lemma 3.7 is smaller than $(1/2)(1 + \tau^2)(1 + \tau^2 \Omega_{n+2})$ while that of (24) is larger than $(1/2)(1 + \tau \Omega_{n+2})$. We will be done once we show that

$$(1 + \tau^2)(1 + 0.7 \cdot \tau^2) < 1 + 0.7 \cdot \tau,$$

which clearly holds for $\tau < 0.4$.

So we consider the recursion given by assuming equality in (24). The function

$$y \to \sqrt{1 + \frac{\tau y}{2}}$$

has exactly one attracting fixed point for $y \leq 0.823562$ in the positive domain. It follows that if Ω_{n+1} in (24) is greater than this fixed point, then Ω_{n+2} has to be less than Ω_{n+1}. Thus, we set $\Omega = 0.823562$ as an bound of Ω_n and ω_n. We note that this bound is attained in for all values of n sufficiently large depending only on ϵ stipulated by Theorem A. Indeed, by Lemma 3.2 τ_n gets smaller than $\tau = 0.37$ for n sufficiently large in terms of n, and then it is clear that Ω_n decreases at least by a uniform amount for each step of the recursion given by (24) as long as it is greater than Ω.

Final estimates. After these preparations we will prove Proposition 2. We will conduct estimates by splitting the sequence of box mappings into blocks. Each block save perhaps the first will have a mapping with a close return immediately followed by a maximal sequence of consecutive non-close returns. E.g., a single mapping with a close return is a block. The only exception from the above rule of constructing blocks occurs when ϕ_0 makes a non-close return. Then the first block consists of a maximal sequence of box mappings with non-close returns.

Below we list the rules which will give recursive estimates within a given block of box mappings.

1. Suppose ϕ_n exhibits no close return and is the last such mapping in a given block. Then we use formula (7) to estimate

$$\alpha_{n+1}\gamma_{n+1} \leq 1.13837 \cdot \alpha_n \overline{\gamma}_n,$$

If ϕ_n is not last in its block, we use the inequality (6).

$$\alpha_{n+1}\overline{\gamma}_{n+1} \leq 0.56919 \cdot (1 + \omega_{n+2}) \, \alpha_n \overline{\gamma}_n.$$

2. Let n-th box mapping exhibits close return.
   If ϕ_n is not a block in its own right, then we apply formula (18):

$$\alpha_{n+1}\overline{\gamma}_n \leq 0.80402 \cdot \alpha_n \gamma_n.$$

If ϕ_n is a block by itself, then (20) implies that

$$\alpha_{n+1}\gamma_{n+1} \leq 0.881181 \cdot \alpha_n \gamma_n.$$

We will consider two cases:

Blocks with at least two box mappings with non-close returns.
Suppose that a series of at least two box mappings with non-close returns begins at the moment n. We will show that the separation of the critical value $f_n(0)$ from the boundary of the box $B_{n'}$ improves with n growing. Indeed, by Lemma 3.7

$$\Omega_{n+1} \leq (1 + \tau^2)\sqrt{\frac{\Omega}{1 + \Omega}} \leq 0.76403$$

and next

$$\Omega_{n+2} \leq \sqrt{(1 + \tau^2)\frac{0.76403 + \Omega\tau^2}{1.76403}} \leq 0.751714.$$

By monotonicity of this formula with respect to Ω_n, all $\Omega_k \leq 0.751714$ for $k \geq n + 2$. Consequently, for $k \geq n$ we obtain that

$$\alpha_{k+1}\overline{\gamma}_{k+1} \leq 0.9971 \cdot \alpha_k \overline{\gamma}_k.$$

It happens that $0.80402 \cdot 1.13837 < 1$. Thus, if a block of length $k \geq 3$ starting with ϕ_n is taken as whole, then

$$\alpha_{n+k}\gamma_{n+k} \leq (0.9971)^{k/3}\alpha_n\gamma_n .$$

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Shorter blocks. For a block of a single map with a close return, or one close and one non-close, it follows immediately from our rules that the product $\alpha_n \gamma_n$ decreases after passing through a block by a definite constant less than 1.

Conclusion. To see that $\alpha_n \gamma_n$ goes down to 0 at least exponentially fast, first wait N steps for bounds on Ω to be achieved (N is bounded in terms of τ). Then pick a $k > 2N$, and construct the blocks starting from ϕ_N. Cut off the last block at ϕ_k. The uniform exponential estimate follows at once from our considerations of the rate of decay within blocks. So, Proposition 2 follows.

3.6 Decay of box geometry

General picture. In this subsection we will estimate the rate of the decay of box geometry proving eventually that for all S-unimodal rotation-like maps the rate is always at least exponential. This will prove Theorems A and C. In the course of inducing a subtle interaction between α_n and γ_n takes place. Namely, after a long series of non-close returns, γ_n is approximately equal to the second power of α_n. The first close return will violate this simple relation between γ_n and α_n by decreasing α_n stronger than γ_n. If the close return is deep enough (i.e. the critical value needs a lot of iterates to escape from the central domain) then γ_n and α_n can even become comparable. At the moment when we are leaving box maps with close returns, γ_n and αn will quickly regain (exponentially fast with the number of steps of non-close inducing) their square-law relation. However, the product $\alpha_n \gamma_n$ for rotation-like maps of bounded type decreases asymptotically with each step of inducing by a constant uniformly separated from 0 and 1. It means that while switching between patterns of inducing "oscillations" between α_n and γ_n destroy monotone (known from the Fibonacci example) fashion of the decay of boxes. In particular, α_{n+1}/α_n can become arbitrarily large in some cases.

Theorem C expresses what we mean by an exponential decay of box geometry. From Theorem C it follows immediately that there is a whole class of S-unimodal maps with at most exponential decay of box geometry. The dynamics of maps from this class, purely characterized in terms of rotation number, is certainly different from the "Fibonacci pattern".

Corollary 1 *For any S-unimodal rotation-like map with a rotation number of the constant type[1] there exist constants $K > 0$ and $0 < \kappa < 1$ so that*

$$\tau_n > K\kappa^n.$$

The constant κ depends only on the upper bound of the coefficients of the continued fraction representation of the rotation number while the constant K depends solely on the initial geometry of ϕ_0, in particular in uniformly controlled by the parameter ϵ of Theorem A.

¿From now on we will denote positive constants dependent only on the initial geometry by K and call them uniform. Whenever confusion of the type $K < K$ can arise we will distinguish constants K by adding appropriate subscripts.

Proof of Theorem C: We will start the proof with two Lemmas.

Lemma 3.8 *The is a uniform constant K so that*

$$\gamma_n \leq K\alpha_n.$$

Proof:
The proof follows from the observation that the image of γ_n by f_n is comparable with γ_n and $f_{n*}(\gamma_n)$ can exceed α_n by no more than a uniform constant.

$\square$

Lemma 3.9 *The ratio τ_n of two consecutive boxes goes to zero at least exponentially fast.*

Proof:
Suppose that n is so large that $\tau_n < \tau = 0.37$ Let us recall that $\alpha_n \gamma_n$ goes to zero at least exponentially fast. We will actually prove that α_n decreases exponentially which is easily equivalent. Consider two cases:

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[1]Let us recall that a number ρ is of the constant type if and only if all coefficients in its continuous fraction representation are bounded.

- f_n shows a close return.
  We push forward α_n by $f_{n*}^{a_n}$. Using (13) we obtain

$$\alpha_{n+1}^2 \leq K\lambda^{a_n}\gamma_n \ , \tag{25}$$

  where $\lambda < 1$ depends only on τ. Finally, by Lemma 3.8

$$\alpha_{n+1}^4 \leq K\lambda^{2a_n}\alpha_{n-1}\gamma_{n-1}$$

  Proposition 2 concludes the proof.

- f_n shows a close return.
  Similarly as before we get that

$$\alpha_{n+1}^2 \leq K\gamma_n.$$

  Lemma 3.9 likewise follows.

$\square$

Lemma 3.9 together with the inequality 25 give the upper estimate of Theorem C.

To prove the opposite estimate make τ go to 0 arbitrary small in all distortion estimates. So, we can reverse the directions of the inequalities estimating $\alpha_n\gamma_n$ from below. Next, observe that λ_n and Λ_n appearing in the recursive scheme for for $\alpha_n\gamma_n$ are after a finite number of inducing step greater than $(\frac{1}{2} - \epsilon)^{a_n}$. This completes the proof of Theorem C.

Theorem A follows directly from Theorem C and Lemma 3.2.

We finish this section with an important technical statement concerning the starting condition.

Proposition 3 *For any type I real box map ϕ which satisfies*

$$\frac{|B_n|}{|B_{n'}|} \leq \tau$$

and every $\delta > 0$ there is a number $k(\tau, \delta)$ only depending on its stated parameters so that if only a close return occurs and the critical value remains in the domain of the central branch for more than k iterations, then the starting condition is satisfied with norm δ.

Proof:

Define $B^i = f^{-i}(B_n)$ where f is the central branch of ϕ. Also, suppose that ϕ is of rank n. By the non-positive Schwarzian property, in each component of $B_n \setminus B_{n+1/2}$ there is at most one point at which the derivative of f_n is equal to 1. This point is between fixed points of f, or there would be a restrictive interval. Hence, there exists a uniform constant $K_1(\tau)$ so that for all $1 < i < a_n$

$$\frac{|B^i \setminus B^{i-1}|}{|B_n|} \leq \frac{K_1(\tau)}{i}.$$

Push forward α_{n+1} by f_n. The bounded distortion yields

$$\alpha_{n+1} \leq K_2(\tau)\sqrt{\zeta_n}.$$

On the other hand (see (8))

$$\alpha_{n+1} \leq K_3(\tau)\overline{\alpha}_{n+1}/\zeta_n.$$

By the real Köbe Lemma and the definition of $\overline{\alpha}_{n+1}$

$$\overline{\alpha}_{n+1} \leq K_4(\tau)\sqrt{\zeta_n/a_n}.$$

Combining the above inequalities we get finally that

$$\alpha_{n+1} \leq K_5(\tau)/\sqrt{a_n},$$

which completes the proof of the Proposition.

$\square$

4   Initial bounds

Our goal is to prove that 0.37 works as an eventual bound of $\tau_n = |B_n|/|B_{n-1}|$. The estimates will be uniform in the sense of Theorem A. This will prove Lemma 3.2. The reasoning naturally splits into two parts. In the first we will find very initial estimates by solving a certain extremal problem for so called 'auxiliary inducing'. Then using these bounds to control distortion we will refine previous estimates referring to the particular features of both types of inducing.

4.1   Geometrical setup

Notations.   The new domain B_{n+1} is formed as the preimage by f_n of an immediate branch of ϕ_n which is always filled-in so that it maps onto $B_{(n+1)'}$. Let us call this domain (u_n, v_n) and let us say that u_n is closer to the critical point. The branch defined on (u_n, v_n) extends at least onto $B_{n'}$ as the image. The domain of this extension will be called (s_n, t_n), and again say that s_n is closer to the critical point. Then, call $B_{(n+1)'}$ $(-w_n, w_n)$ and say that w_n is on the side of the critical value. Lastly, let $(-w_{n-1}, w_{n-1})$ be $f_n^{-1}((-w_n, w_n))$. The reader may try to get familiar with this notation by trying to see that the ordering of points is

$$-w_{n-1}, -w_n, 0, w_n, s_n, u_n, v_n, t_n, w_{n-1}$$

or perhaps the reverse. These notations are applicable for both close and non-close returns.

Observe that

$$\mathbf{Cr}_i\big(s_n, u_n, v_n, t_{n-1}\big) \leq \mathbf{Cr}_i\big(-w_{n-1}, -w_n, w_n, w_{n-1}\big) \ .$$

Let us mention now that the size of the interval (s_{n-1}, t_{n-1}) with respect to (w_n, w_{n-1}) will play an important role later on when we refine the first estimates obtained by assuming in estimates that (s_{n-1}, t_{n-1}) is equal to (w_n, w_{n-1}).

Formulation of extremal problem.   Let us recall that f_n can be represented as a composition of the quadratic map g and the diffeomorphism h_n.

The extendability properties of f_n are formulated in Lemma 3.1. We will denote preimages of points by h_n by adding primes to the notation.

¿From the previous two paragraphs,

$$\tau_n^2 \leq \frac{|u_n'| - |v_n'|}{|w_n'| + |v_n'|}. \tag{26}$$

Complete the ratio $|w_{n-1}'| - |w_n'|/|w_{n-1}'| + |w{\textrm{\`{}}}_n|$ to the cross-ratio

$$\mathbf{Cr}_e\big(-w_{n-1}', w_n', w_{n-1}', (z_{(n-2)'}^-)'\big) .$$

By the expanding property of cross-ratios,

$$\frac{|(z_{(n-2)'}^-)' - w_n'|}{|(z_{(n-2)'}^-)' + |w{\textrm{\`{}}}_n|} \leq \frac{1 - \alpha}{1 + \alpha} \frac{1 + \nu_n^2}{1 - \nu_n^2},$$

where $\alpha = |w_n'|/|w_{n-1}'|$. Denote the right-hand side of the above inequality by $1/L$. To find preliminary bounds we solve the following extremal problem:

Problem 1 *Suppose that the inclusive cross-ratio of the interval (u_n', v_n') with respect to s_n', t_{n-1}' is equal to to C and the cross-ratio*

$$\mathbf{Cr}_e\big(-s_{n-1}', u_n', v_{n-1}', w_{n-3}'\big) .$$

is equal to $1/L\beta$, $\beta \geq 1$. The interpretation of β is that it accounts for the nesting of (s_n, t_n) in (w_n, w_{n-1}) as well as for possible additional extendability of f_n.

Find a maximum of
$$\frac{|u_n'| - |v_n'|}{|w_n'| + |v_n'|}.$$
depending on the location of points u_n' and v_n'.

Solution:
We assume that the interval s_{n-1}', t_{n-1}' has the unit length. Then $|w_{n-1}'| + |w_n'| > L$. Denote $|s_{n-1}'| - |u_n'|$ by ρ and $|v_n' - w_n'|$ by λ. Then

$$C = \frac{\rho + \lambda - 1}{\rho \lambda}. \tag{27}$$

We want to maximize
$$\frac{\rho + \lambda - 1}{L + \lambda} = C\frac{\rho\lambda}{L + \lambda},$$
which is equivalent to minimizing

$$M = \frac{1}{\rho}\left(1 + \frac{L}{\lambda}\right).$$

In the extremal point the gradients of M and C are parallel. Hence

$$\frac{1 - \lambda}{1 - \rho} = 1 + \frac{\lambda}{L}. \tag{28}$$

Calculate ρ from (27) and substitute into (27). As we solve the resulting quadratic equation for λ^{-1} we get the first coordinate λ_0 of the extremal point.
$$\lambda_0^{-1} = 1 + \sqrt{(1 - C)(1 + L^{-1})}.$$

By algebra we calculate the second coordinate and then the minimum of M

$$L(\sqrt{1 - C} + \sqrt{1 + L^{-1}})^2.$$

Finally, we get

$$\frac{C}{(\sqrt{1 - C} + \sqrt{1 + L^{-1}})^2}.$$

as a solution of our problem. Let $T = 1 - \nu_n^2/1 + \nu_n^2$ and $\gamma = \sqrt{1 - C}$. Then by inequality (26)

$$\tau_{n+1} \leq \frac{\sqrt{1 - \gamma^2}}{\sqrt{T\beta\gamma} + \sqrt{1 + \frac{\beta T}{\gamma}}}.$$

$\square$

Set $T' = 1 - \tau_{n+1}^2/1 + \tau_{n+1}^2$. If $T' > T$ then $\tau_{n+1} < \mu_n = \sqrt{\tau_{n-1}\tau_{n-2}}$. Observe that T' is a growing function of T. We will determine when the difference $T' - T$ is positive. To this aim we bring $T' - T$ to the common denominator and examine the sign of the numerator.

$$T\left((1 - T)\beta(\gamma + \frac{1}{\gamma}) - 2\right) + \gamma_2(1 + T) + 2T\beta(1 - T)\sqrt{(1 - T)}\sqrt{1 + \frac{\gamma}{T}}.$$

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4.2 Computation of initial bounds

Basic procedures.

General description. We begin with the condition $T' - T > 0$ which can be analytically rewritten as

$$T((1-T)\beta(\gamma + \frac{1}{\gamma}) - 2) + \gamma^2(1+T) + 2\beta T(1-T)\sqrt{1 + \frac{\gamma}{\beta T}} > 0 . \quad (29)$$

The first procedure aims to find a possibly large T independent of γ in some range, but depending on β, which will force this estimate to be fulfilled.

The second condition will find T' from the formula

$$T' = \frac{t(\gamma + \frac{1}{\gamma} + 2\sqrt{1 + \frac{\gamma}{t}}) + \gamma^2}{2 - \gamma^2 + t(\frac{1}{\gamma} + \gamma + 2\sqrt{1 + \frac{\gamma}{t}})} \quad (30)$$

where $t = \beta * T$. More precisely, an upper bound will be found depending on t, but not depending on γ varying in some specified range.

We also obtain as a corollary:

Fact 4.1 *Let ϕ be a type I box mapping of rank n, let $|B_n|/|B_{n'}| \leq 1 - \epsilon$ and assume that the central branch of ϕ is ϵ-extendable. If another type I box mapping of rank m is obtained from ϕ in a number of standard inducing steps, then*

$$\frac{|B_m|}{|B_{m'}|} \leq 1 - K(\epsilon)$$

where K is a continuous function of ϵ only, positive when $\epsilon > 0$.

Proof:
¿From Fact 4.1 we see that the ratio will remain bounded away from 1 for two first standard inducing steps. Then, formula 29 implies that it will not deteriorate as long as it is close to 1 (the condition 29 is clearly satisfied for T close to 0.)

$\square$

Implementation of the first procedure. The first procedure will take five parameters. γ_u and γ_l will give an upper and lower bound of the allowed range of γ. For practical reasons, we assume $\gamma_l \leq 0.1$, but one easily checks that the condition of [29] is satisfied for any $\gamma < 0.1$ and $0 \leq T \leq 1$. Another parameter σ gives the step size. We will cover the range $[\gamma_l, \gamma_u]$ with finitely many closed intervals of length σ. We will estimate from below the range of positive values of T which satisfy [29] on each subinterval, and finally take the maximum of all estimates with respect to the subintervals. Another parameter β whose meaning is clear. Lastly, we have ν which must no less than the answer (thus $\nu = 1$ will always work, but the point is to sharpen the estimate by picking ν just about as small as possible.)

To finish the description, we have to explain how the lower estimate is found on a subinterval $[\gamma_1, \gamma_2]$. The left-hand side of [29] is clearly bounded from below by

$$T\left((1-T)\beta(\gamma_2 + \frac{1}{\gamma_2}) - 2\right) + \gamma_1^2(1+T) + 2\beta T(1-T)\sqrt{1 + \frac{\gamma_1}{\beta\nu}} > 0 \ .$$

This gives us a quadratic inequality on T which is solved algebraically to give us the answer.

Implementation of the second procedure. This procedure will take four parameters. As previously, γ_u and γ_l will give the range of γ with respect to which the T' must be minimized. Again, $\gamma < 0.01$ will give an answer greater than 0.9 which is better than we will ever use, so we assume $\gamma_l \geq 0.01$. We use the same procedure of dividing $[\gamma_l, \gamma_u]$ into subintervals, taking a lower bound for T' on each interval, and taking the minimum with respect to all subintervals for a final answer. The parameter σ gives the length of subintervals. The parameter t is βT.

On each subinterval $[\gamma_1, \gamma_2]$, formula [30] bounds T' from below by

$$T' = \frac{t(\gamma_2 + \frac{1}{\gamma_2} + 2\sqrt{1 + \frac{\gamma_1}{t}}) + \gamma_1^2}{2 - \gamma_1^2 + t(\frac{1}{\gamma_1} + \gamma_1 + 2\sqrt{1 + \frac{\gamma_2}{t}})} \ .$$

This is evaluated on each subinterval.

The first estimate on τ.   The first estimate on τ is obtained by calling the first procedure with parameters $\gamma_l = 0.1$, $\gamma_u = 1$, $\beta = 1$, $\mu = 0.7$, $\sigma = 10^{-5}$. The result is more than $T > 0.69901$. Since

$$T = \frac{1 - \tau^2}{1 + \tau^2} \, ,$$

$\tau(1) = 0.4209$ will guarantee that $T' - T > 0$. We claim that from any initial box mapping the value of τ will eventually drop below $\tau(1)$ in a uniformly bounded number of inducing steps. Indeed, from (29) by a compactness argument it is clear that if $0 < \epsilon < T < 0.69901$, the increment $T' - T$ is bounded away from 0. However, ϵ is uniformly bounded away from 0. On the other hand, the formula (30) gives T' as an increasing function of T, thus a decreasing function of τ. Hence, once τ gets below $\tau(1)$ it will stay there. We conclude that eventually, after a number of inducing steps bounded in terms of ϵ only, the box ratios τ become smaller that $\tau(1)$.

Better estimates on the box ratio.   Here we concentrate on a complex box mapping ϕ. We call b_n the central domain of ϕ. We assume that an estimate τ for the box ratio is already satisfied by ϕ as well as two preceding mappings in the inducing sequence. We will assume that $\tau \leq tau^1$. We will try to get better estimates for the next box ratio $|b_{n+1}|/|b_{(n+1)'}|$. The procedure will depend on what ϕ does.

   A close return for ϕ.   In this case, we see that opportunity to significantly improve β in formula (30). Indeed, $\beta = 1$ corresponds to the assumption that either of intervals between B^k and B^{k-1} maps onto B_{k-1}. In fact, however, we know that each of them contains an extended branch mapping of rank n'. By standard estimates, we get that the branch mapping onto B^{k-1} occupies the fraction of either space between B^k and B^{k-1} which is no more than τ. Thus, one can take $\beta = tau^{-1}$. Next, one uses procedure two with

$$t = \beta \frac{1 - \tau^2}{1 + \tau^2}$$

$\sigma = 10^{-5}$ and the full range of γ from 0.01 to 1. For $\tau = \tau(1)$ this gives $\tau(21) = 0.29728$.

A non-close return following a close return. In this case, we will use formula (29) and get an improvement from two sources. First, the lower estimate for γ will be quite large as a result of the box ratio being very small in the case of a close return. Secondly, the extendability factor normally given by

$$\frac{1 - \tau^2}{1 + \tau^2}$$

will also grow, since the ratio of the length of B^k to the length of $B_{(n-2)'}$ will be at most the ratio between lengths of B^1 and B_n times τ^2. This last ratio b is given by

$$b = \sqrt{\frac{2\tau}{1 + \tau}(1 + \tau^2)} \ .$$

Now β is at least

$$\beta \geq \frac{1 - bd^2}{1 + bd^2} \cdot \frac{1 + d^2}{1 - d^2}$$

where $d \leq \tau$. Thus, if we want to get a better estimate on the box ratio, d should be no more than this expected lower estimate. Using $d = 0.37$ and $\tau = \tau(1)$ we get $\beta \geq 1.04694$. Also, $\gamma_l \geq \frac{1 - \tau(21)}{1 + \tau(21)}$. With $\gamma_u = 1$, $\sigma = 10^{-5}$ and $t = 0.749$, procedure one gives $T = 0.7483$ corresponding to $\tau(22) = 0.3795$. The meaning of this result is that once that estimates apriori given by $\tau(1)$ and $\tau(21)$ hold, in this case τ the box ratio will be less than τ as long as $\tau \geq tau(22)$. This would imply that eventually τ gets smaller than τ^{22}, but we have one more case.

A non-close return followed by a non-close return. In this case we also use procedure one. The improvement is obtained from a better γ_l given by

$$\gamma_l = \frac{1 - \tau(1)}{1 + \tau(1)}$$

as well as better β. Here, β can easily be estimated

$$\beta \geq \sqrt{\frac{1 + \tau}{2\tau(1 + \tau^2)}}$$

With $\sigma = 10^{-5}$, $\gamma_u = 1$ and $t = 0.749$ we get $T = 0.7449$ which gives $\tau(23) = 0.38236$.

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Now, the combination of estimates in all cases implies that the box ratio will eventually decrease below the maximum of τ^{21}, τ^{22} and τ^{23} which happens to be τ^{23}. Again, we argue that this will happen in a uniformly bounded number of inducing steps. So, we can take $\tau(2) = \tau(23) = 0.38236$.

A second round of estimates. We now repeat the same sequence of estimates in all three cases using τ^2 as our original bound instead of τ^1. In the case of a close return this gives

$$\tau(31) = 0.27736 \ .$$

In the second situation, we get $b \leq 0.79629$. With this, and $d = 0.369$, we obtain $\beta \geq 1.05793$. Also, $\gamma_l = 0.56572$. With $t = 0.76$ and $\sigma = 10^{-5}$ procedure one yields $\tau(32) = 0.36983$. In the last case, we get $\gamma_l = 0.4468$ and $\beta = 1.25582$. We feed those into procedure one together with $\gamma_u = 1$ and $t = 0.78$ to get $T = 0.75983$ which corresponds to $\tau(33) = 0.36942$.

We see that indeed the box ratio eventually goes below 0.37 which proves Lemma 3.2.

5 Complex induction

5.1 Non-decreasing moduli

Statement of the result. Suppose now that a complex box mapping ϕ is given of rank $n > 1$ which later undergoes k consecutive steps of general inducing in the box case. We denote by ϕ_i the complex box mappings obtained in the process so that $\phi_0 = \phi$ and ϕ_{i+1} arises from ϕ_i in a general inducing step.

Proposition 4 *Choose $0 < j < k$. Suppose that a constant β can chosen independently of B so that normalized critical symbols can be chosen with norm β for all monotone branches B whose domains intersect the real line. Then, for ϕ_{j+1} all normalized critical symbols can be constructed with the same norm β. Assume in addition that $j \geq 3$ and ϕ_j does not show a rotation-like return. If the critical point is in the range of the real central branch, let l_j mean the number of consecutive images of the critical point by the central branch of ϕ_j which remain in the central hole. Otherwise, set*

$l_j = 0$. *Put l equal to the maximum of l_j, $0 \leq j < k$. Then, there is a positive function $K(l)$ such that normalized symbols can be constructed for all univalent branches of ϕ_{j+1} with norm $(1 + K(l))\beta$.*

The first part of Proposition 4 which says that the symbols exist with norm β follows from the proof of Lemma 3.4 of [20]. We will repeat the proof here and refine the argument to show the second part of the statement.

An outline. The proof of Proposition 4 has to be split into a number of cases. The major dichotomy is between close returns and others. We remind the reader that the situation is classified as a close return if the critical value is in the central branch. As analytic tools, we will use the behavior of moduli of annuli under complex analytic mappings. Univalent maps transport the annuli without a change of modulus, analytic branched covers of degree 2 will at worst halve them, and for a sequence of nesting annuli their moduli are superadditive (see [13], Ch. I, for proofs, or [2] for an application to complex dynamics.)

We assume that a mapping ϕ_j of rank n is given as in the hypothesis of Proposition 4. We will construct ϕ_{j+1} and show that necessary separation estimates. On the level of notation, quantities related to ϕ_{j+1} will be distinguished by writing a bar above them.

5.2 Non-close returns

Dynamical classification of branches. First, we classify branches of ϕ_{j+1} according to their parent branches (compare the description of the inducing construction for the definition of parent branches.) The main split is between immediate parent branches and non-immediate parent branches. Among domains with immediate parent branches we distinguish maximal branches, or immediate preimages of B_{n+1}, and others. Otherwise, a non-immediate parent branch gets mapped forward by ψ_j. This gives a univalent branch of ϕ_j, denoted with B'. We also have another univalent domain of ϕ_j which contains the critical value. This will be denoted with B. Then, we distinguish three subcases according to whether B and B' are independent, or one is subordinate to the other. The results of our computation can be summarized as follows.

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Lemma 5.1 *Suppose that a sequence of type I complex box mappings ϕ_j is given which satisfies the assumptions of Proposition 4. Let $1 \leq j \leq k$ and assume that ϕ_j shows a non-close return. If β is the separation norm of ϕ_0, then normalized critical symbols can be constructed for branches of*

- *for immediate preimages of B_{n+1} the symbol has norm β with corrections*
$$\overline{\lambda}_1 = \frac{\lambda_2(B)}{2} \, , \, \overline{\lambda}_2 = \frac{\lambda_1(B)}{2} \, .$$

- *for domains whose parent branches are not immediate, and whose domains B' described above are independent with the postcritical domain B, the norm is β with corrections*

$$\overline{\lambda}_1 = \frac{\lambda_2(B)}{2} \, ,$$

$$\overline{\lambda}_2 = \frac{\alpha - \delta}{2}$$

where $\delta = \max(-\lambda_2(B), -\lambda_2(B'))$.

- *in all other situations the norm is at least $(1 + K(l))\beta$ with K a positive function of l only (l is defined in the statement of Proposition 4.)*

We proceed to prove Lemma 5.1.

Immediate preimages. Let B denote the hole which contains the critical branch. In all cases the new central hole B_{n+1} is separated from the boundary of B_n by an annulus of modulus at least $(\beta + \lambda_2(B))/2$. We will first construct the symbols for immediate preimages of B_{n+1}, meaning the preimages by the central branch. Naturally, these preimages exist on the real line exactly if the image of the real central branch covers the central domain. The annulus $\overline{A}_2$ around B_{n+1} will be the preimage by the central branch of the encompassed by A_3 around B with B removed. This region consists at least of the union of A_3 and A'. Then, $\overline{A}_1$ is the preimage of A_4. It follows that we can take

$$\overline{s}_1 = \frac{\beta + \lambda_2(B)}{2} \text{ and}$$

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$$\overline{s}_2 = \frac{\beta - \lambda_1(B)}{2} \ .$$

Of course, since components of the symbol are only lower estimates, we are always allowed to decrease them if needed. The annulus $\overline{A}'$ is naturally given as the preimage of the annulus between B_{n+1} and the boundary of B_n by the central branch, likewise $\overline{A}_3$ is the preimage of A_2, and $\overline{A}_4$ is the preimage of A_1. Since the first two preimages are taken in an univalent fashion, we get

$$\overline{s}_3 = \frac{\beta + \lambda_2(B)}{2} + \alpha - \lambda_2(B) \text{ and}$$

$$\overline{s}_4 = \overline{s}_3 + \frac{\lambda_1(B) + \lambda_2(B)}{2} = \frac{\beta}{2} + \alpha + \frac{\lambda_1(B)}{2} \ .$$

Thus, if we put

$$\overline{\lambda}_1 = \frac{\lambda_2(B)}{2} \ , \ \overline{\lambda}_2 = \frac{\lambda_1(B)}{2}$$

This gives the critical symbol for immediate preimages.

Reduction to maximal branches. A branch $\overline{B}$ can be either maximal, i.e. the preimage of B_{n+1} by its parent branch, or it can be inside another domain of rank n' nested inside the parent domain. We now argue that it is sufficient to do the estimates for maximal branches. In the process we describe, the annulus $\overline{A}'$ is always chosen as the preimage of the annulus between B_{n+1} and $B_{n'}$. If a maximal branch inside the parent branch is replaced with another branch, the annuli $\overline{A}_i$ with $i = 1, 2, 3, 4$ stay the same, as they are all outside of the parent branch. So, we only need to show that $\overline{A}'$ can be chosen larger than for the immediate preimage. But this is clear, since the branch defined on $\overline{B}$ has a univalent extension onto $B_{n'}$, thus an annulus of the desired modulus will always sit inside this extended domain. Then, there is an extra annulus between the extended domain and the complement of the parent branch, whose annulus we can usually bounded away from 0.

Immediate parent branches. Choose a branch $\overline{B}$ of ϕ_1 which is neither critical nor immediate. Assume that the parent domain is an immediate preimage of B_n by the central branch. Since we assumed that $\overline{B}$ was not an immediate preimage of B_{n+1}, we see that while annuli $\overline{A}_i$ for $i = 1, 2, 3, 4$ can be chosen as for immediate preimages, $\overline{A}'$ will be larger by an annulus

between the complement of the extended immediate domain and the rank n extension of $\overline{B}$. Since all branches of ϕ sit inside B_n with separating moduli at least $\frac{1}{2}\alpha$, all parent branches are nested inside B_n with moduli at least $\frac{1}{4}\alpha$. The extension of $\overline{B}$ will be mapped by the extension of the immediate branch inside some parent branch. Thus, the extra contribution to $\overline{A}'$ will be at least $\frac{1}{4}\alpha$. That means $\overline{s}_3$ and $\overline{s}_4$ will both grow by the this amount compared with the estimate for the immediate preimages. Then, one can choose $\beta' = \frac{9}{8}\beta$ and

$$\overline{\lambda}_1 = \frac{\lambda_2(B)}{2} - \frac{\alpha}{8}$$

and

$$\overline{\lambda}_2 = \frac{\lambda_1(B)}{2} + \frac{\alpha}{8} \ .$$

One checks directly that this gives a normalized symbol.

The case of independent B and B'. Let us first consider the independent case. To pick $\overline{A}_2$, we consider the annulus separating B from the boundary of the domain of its rank $(n-1)'$ extension. We claim that its modulus in all cases is estimated from below by $\alpha + \delta$ where δ can be chosen as the maximum of $-\lambda_2(B)$ and $-\lambda_2(B')$. Indeed, if B is carried onto B_n by the extended branch, the estimate is α plus the maximum of $\lambda_1(B)$ and $\lambda_1(B')$ which is at least δ since $\lambda_1 + \lambda_2 \geq 0$ in any normalized symbol. On the other hand, if B is mapped by the extension onto something different from the central hole, the estimate $\beta - \lambda_2 \geq \frac{3}{4}\beta$ applies which is better than $\alpha + \lambda_1 \leq \frac{3}{4}\beta$. To pick A_1, consider the annulus of modulus $\beta + \lambda_2(B)$ separating B from the boundary of B_{n-1}. Pull these annuli back by the central branch to get $\overline{A}_2$ and $\overline{A}_1$ respectively. By the hypothesis of the induction, the estimates are

$$\overline{s}_1 = \frac{\beta + \lambda_2(B)}{2} \text{ and}$$

$$\overline{s}_2 = \frac{\alpha + \delta}{2} \ .$$

As always, $\overline{A}'$ is determined with modulus at least $\overline{s}_1$. The annulus $\overline{A}_3$ will be obtained as the preimage by the central branch of the annulus surrounding the preimage of B' in the domain of the extended branch. This has modulus

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at least $\alpha + \delta$ in all cases as argued above. The annulus $\overline{A}_4$ is the preimage of the annulus surrounding the extension in B_{n-1}. By induction,

$$\overline{s}_3 = \frac{\beta + \lambda_2(B)}{2} + \alpha + \delta \text{ and}$$

$$\overline{s}_4 = \overline{s}_3 + \frac{\beta + \lambda_2(B') - \alpha - \delta}{2} .$$

We put $\overline{\lambda}_1 = \frac{\lambda_2(B)}{2}$ and $\overline{\lambda}_2 = \frac{\alpha - \delta}{2}$. We check that

$$\overline{s}_3 + \overline{\lambda}_1 = \frac{\beta}{2} + \alpha + \lambda_2(B) + \delta \geq \beta - \lambda_2(B) + \lambda_2(B) \geq \beta .$$

In a similar way one verifies that

$$\overline{s}_4 - \overline{\lambda}_2 \geq \beta .$$

Also, the required inequalities between corrections $\overline{\lambda}$ follow directly. In this case, it is not evident how to obtain a symbol with norm greater than β, so we will return later to this case with more careful estimates.

The case of B' subordinate to B. Presently, we will consider the case of B' being subordinate to B. This means that the same univalent branch of rank $n-1$ transforms B onto B_n and B' onto some B''. Consider the annulus separating B_n from B'', and a larger annulus surrounding the previous one in B_{n-1}. Their preimages first by the extended branch and then by the central branch give us $\overline{A}_2$ and $\overline{A}_1$ respectively. Notice that if an annulus surrounds the domain of the extended branch, its preimage can be used to get another layer of $\overline{A}_1$. The estimates are

$$\overline{s}_1 = \frac{\alpha + \delta'}{2}$$

where δ' is the maximum of $\lambda_1(B'')$ and 0. This is allowed, since if $\lambda_1(B'')$ is negative, we can always use the fact that B' is nested in B_{n-1} with modulus at least $\beta + \lambda_2(B') > \alpha$.

$$\overline{s}_2 = \frac{\alpha - \lambda_2(B'')}{2} .$$

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The annulus $\overline{A}'$ is uniquely determined with modulus $\overline{s}_1$, and $\overline{A}_3$ will be the preimage of the annulus separating B'' from B_n. Finally, $\overline{A}_4$ will separate the image of $\overline{A}_3$ from B_{n-1}. Again, the separation between the extended domain and B_{n-1} will give us another layer of $\overline{A}_4$. The estimates are

$$\overline{s}_3 = \frac{\alpha + \lambda_1(B'')}{2} + \beta - \lambda_1(B'') = \beta + \frac{\alpha - \lambda_1(B'')}{2} \text{ and}$$

$$\overline{s}_4 = \overline{s}_3 + \frac{\lambda_1(B'') + \lambda_2(B'')}{2} = \beta + \frac{\alpha + \lambda_2(B'')}{2} .$$

Set

$$\overline{\lambda}_1 = \frac{-\alpha + \delta'}{2} \text{ and}$$

$$\overline{\lambda}_2 = \frac{\alpha + \lambda_2(B'')}{2} .$$

The requirements of a normalized symbol are clearly satisfied. Moreover, as we noted, an estimate of the separation between the extended domain and B_{n-1} will give better $\overline{s}_1$, $\overline{s}_3$ and $\overline{s}_4$. If, say, this extra modulus is at least $K\beta$, $K < 1/2$, then $\overline{s}_1$, $\overline{s}_3$ and $\overline{s}_4$ will grow by at least $\frac{K}{2}\beta$.

Since this situation will recur, we emphasize the following reasoning as a lemma.

Lemma 5.2 *Suppose that a separation symbol (s_1, s_2, s_3, s_4) can be represented by a normalized symbol with norm β and corrections λ_1 and λ_2. If a number $0 < K < 1/2$ exists such that*

$$s_1 \geq (1 + K)\alpha + \lambda_1 ,$$

$$s_3 \geq (1 + \frac{K}{2})\beta - \lambda_1 \quad and$$

$$s_4 \geq (1 + K)\beta + \lambda_2 ,$$

then another normalized symbol can be built for (s_1, s_2, s_3, s_4) with norm $(1 + \frac{K}{4})\beta$.

Proof:
We will distinguish the new normalized symbol by writing primes. Set

$$\lambda_1' = \lambda_1 + K\frac{\alpha}{4} \text{ and}$$

$$\lambda'_2 = \lambda_2 + K\frac{\alpha}{4} \; .$$

Since the corrections were increased the bounds from below on λ'_i and $\lambda'_1 + \lambda'_2$ will remain in force. Also, the bounds from above on will remain since α' will grow by the same amount as λ'_i. Now, the corrections λ'_i were set so that

$$\alpha - \lambda_2 = \alpha' - \lambda'_2 \; ,$$

hence s_2 will be correctly estimated. Finally, we check directly that

$$\beta' - \lambda'_1 = \beta - \lambda_1 + K\frac{\beta}{8} < s_3 \; ,$$

$$\alpha' + \lambda'_1 = \alpha + \lambda_1 + \frac{3}{4}K\alpha < s_1 \;\; \text{and}$$

$$\beta + \lambda'_2 = \beta + \lambda_2 + \frac{3}{8}K\beta < s_4$$

which concludes the proof.

$\square$

Coming back to our situation, we see that a uniform bound on the separation between the extended domain and B_{n-1} will give us a uniform increase of the norm in the case of B' subordinate to B by Lemma 5.2. This is provided by the following lemma:

Lemma 5.3 *Under the hypotheses of Proposition 4, let $0 < j \leq k$. Let B_n be the central hole of ϕ_j. Choose D to be the domain of an extended univalent branch of ϕ_j of rank n'. Then there is a positive bound $K(l)$ such that D is nested inside $B_{n'}$ with modulus at least $K(l)\beta$.*

Proof:
Consider the previous inducing step on ϕ_{j-1}. All extended branches of ϕ_j are nested inside parent branches created after the first filling of ϕ_{j-1}. So, it is enough to prove the bound for the parent branches. If B' is a parent non-immediate branch, consider its push-forward image. By separation bounds, the push-forward image is surrounded inside $B_{(n-2)'}$ by an annulus of modulus at least α which also separates it from the central hole. The bound by $\alpha/2$ for

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the parent branch follows. Extended immediate preimages are nested with annuli

$$2^{-k-1}(\alpha + \lambda_1(B)) \geq 2^{-k-2}\alpha$$

where k is the number of consecutive images of the critical point inside the central hole. This gives a uniform bound in terms of l of Proposition 4.

$\square$

The case of B subordinate to B'. This situation is analogous to the situation of immediate preimages considered at the beginning. Indeed, by mapping B to B_n and composing with the central branch one can get a folding branch of rank $n - 1$ defined on B. We now see that the situation inside the domain of the rank $n - 1$ extension of B and B' is analogous to the case of immediate preimages, except that the folding branch maps onto a larger set B_{n-1}. Like in the previous case, the estimates do not use the separation between the extended domain and B_{n-1}. By Lemma 5.3 this gives a definite improvement in terms of β. Hence $\overline{s}_1$, $\overline{s}_3$ and $\overline{s}_4$ all improve, and we can increase the norm of the symbol by Lemma 5.2.

This concludes the proof of Lemma 5.1.

5.3   Close returns

Topological description.

Notation. We will use the notation $\tilde{\varphi}$ for the type I mapping obtained from ϕ_j in the first stage of the box inducing step. Also, the i-th preimage of B_n by the central branch will be called B^i, thus $B_n = B^0$. We take k to be the smallest i such that the critical value does not belong to B^i. Let B^* mean the parent domain of the branch of $\tilde{\varphi}$ which contains the critical value of ψ_j. B^* is the k-th preimage of some B by the central branch, where B is a univalent domain of ϕ_j.

Classification of branches. According to the definition of the box inducing step, ϕ_{j+1} is obtained from $\tilde{\phi}$ by a box inducing step defined for non-close returns. The first filling of B^k (which becomes $B_{n+1/2}$) gives parent

branches.[2] Those will be of two types: immediate preimages of B^k by the central branch and preimages of branches of rank $n + 1/2$. We will split preimages of B_{n+1} into two classes. First, we consider D-type preimages. Those can be mapped by a composition of immediate branches only until they hit B_{n+1}. All other domains of ϕ_{j+1} will be called δ-type preimages. It follows that any δ-type preimage falls into a non-immediate parent branch before hitting B_{n+1}. A reasoning similar to one conducted in the case of non-close returns shows that separation estimates for δ-type preimages will be the worst for domains which are mapped by immediate branches only until they hit a *maximal* domain inside a non-immediate parent branch. Thus, we only consider this kind of δ-preimages.

The outcome of our computations will be as follows:

Lemma 5.4 *Suppose that a sequence of type I complex box mappings ϕ_j is given which satisfies the assumptions of Proposition 4. Let $0 \le j \le k$ and assume that ϕ_j shows a close return whereby the critical value stays in the central domain for k steps. If β is the separation norm of ϕ_0, then normalized critical symbols*

- *for immediate preimages of B_{n+1} the symbol has norm β with corrections*
$$\overline{\lambda}_1 = \frac{\alpha}{2} - \frac{\alpha - \lambda_2(B)}{2^{k+1}} \ , \ \overline{\lambda}_2 = -\frac{\alpha}{2} + \frac{\alpha + \lambda_1(B)}{2^{k+1}} \ .$$

- *for other D-type preimages the same symbol will work as for immediate preimages, or a better symbol with norm $(1 + K(k))\beta$ can be used, where K is a positive function of β, but tends to 0 with k growing to infinity.*

- *in all other situations the norm is at least $(1 + K(l))\beta$ with K a positive function of l only (l is defined in the statement of Proposition 4.)*

Immediate preimages. We first consider the annulus C surrounding B_k inside B_n. Let us denote its modulus by γ. Clearly,

$$\gamma \ge (\alpha + \lambda_1(B))(1 - 2^{-k}) \ . \tag{31}$$

To construct $\overline{A}_2$, we first consider the annulus surrounding B^* inside B^{k-1} and separating B^* from B^k. This annulus is obtained as the preimage of

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[2]One must avoid confusion between parent branches of $\tilde{\phi}$ and parent branches of ϕ_{j+1}.

$A_3 \cup A'$ which existed for b by a univalent map. Then, inside B^* the preimage of B^k nests with modulus γ. Thus,

$$\overline{s}_2 = \frac{\beta - \lambda_1(B) + \gamma}{2} \ .$$

Then, $\overline{A}_1$ can be obtained as the preimage by the central branch of the annulus which separates B^* from the boundary of B^{k-1} and is the k-th preimage of A_4. This gives

$$\overline{s}_1 = \frac{\beta - \lambda_1(B) + \gamma}{2} + \frac{\lambda_1(B) + \lambda_2(B)}{2^{k+1}} \ .$$

The annulus $\overline{A}'$ is defined in the natural way as the preimage of the annulus surrounding B_{n+1} inside B^k. We have already estimated its modulus by $\overline{s}_1$.

The modulus of $\overline{A}_3$ is easily estimated by

$$\frac{\alpha - \lambda_2(B)}{2^k} \ .$$

The annulus $\overline{A}_4$ will be the preimage of the the skinny annulus encircling B^* in B^{k-1}. Its modulus is bounded from below by

$$\frac{\lambda_1(B) + \lambda_2(B)}{2^{k+1}} \ .$$

This gives

$$\overline{s}_3 = \frac{\alpha - \lambda_2(B)}{2^k} + \frac{\beta - \lambda_1(B) + \gamma}{2} + \frac{\lambda_1(B) + \lambda_2(B)}{2^{k+1}} \ ,$$

$$\overline{s}_4 = \overline{s}_3 + \frac{\lambda_1(B) + \lambda_2(B)}{2^{k+1}} =$$

$$\frac{\beta - \lambda_1(B) + \gamma}{2} + \frac{\alpha + \lambda_1(B)}{2^k} \ .$$

We now observe that γ occurs in all estimates with the positive sign, so we can replace it with the lower estimate [31].

This means the choice of

$$\overline{\lambda}_1 = \frac{\alpha}{2} - \frac{\alpha - \lambda_2(B)}{2^{k+1}} \quad \text{and}$$

$$\overline{\lambda}_2 = -\frac{\alpha}{2} + \frac{\alpha + \lambda_1(B)}{2^{k+1}} \ .$$

¿From this one can see directly that the requirements of a normalized symbol are satisfied.

δ-type preimages. Choose a δ-type preimage $\overline{B}$. It is mapped by a composition of immediate parent branches into a non-immediate parent domain which we call B_p. We argued that it suffices to consider the case when $\overline{B}$ gets mapped onto a m Now, consider the parent branch of $\psi_j(B_p)$ (as a branch of $\tilde{\phi}$) and call it $\tilde{B}$. We distinguish between two cases depending on whether $\tilde{B}$ and B^* are the same or not.

$\tilde{B}$ is distinct from B^*. Let B' be $\psi^k(\tilde{B})$ and B mean the $\psi^k(B^*)$. Let C be the modulus on an annulus separating B from B_n, B' and the complement of B_{n-1}. Analogously, let C' separate B' from B, B_n and the complement of B_{n-1}. The triple alternative between one independent and two subordinate cases tells us that we can always choose $C + C' = \alpha$.

To get the symbol, we use

$$\overline{s}_1 = \frac{\beta - \lambda_1(B) + \gamma}{2}$$

obtained for immediate preimages (we skipped a positive term.) The reader is reminded that γ is a lower estimate on the modulus between B^k and B_n, thus

$$\gamma \geq (\alpha + \lambda_1)(1 - 2^{-k})$$

where λ_1 is the supremum of $\lambda_1(D)$ over all monotone holes of ϕ.

We put

$$\overline{s}_2 = \frac{\gamma + C}{2} \ .$$

The annulus $\overline{A}'$ has modulus $\overline{s}_1$. $\overline{A}_3$ has two layers: one is the preimage of the annulus between B^k and B_n inside the parent branch, another is the preimage by the central branch of an annulus separating the parent branch from B^k inside B_{k-1}. Those give

$$\overline{s}_3 = \frac{\beta - \lambda_1(B)}{2} + \frac{3}{2}\gamma + \frac{\beta - \lambda_1(B')}{2} + \frac{C'}{2}$$

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Finally, we put $\bar{s}_4 = \bar{s}_3$.

The only way k enters the estimates is through γ. Thus, we only consider $k = 1$ which gives the smallest value of γ, namely $\gamma = \frac{\alpha + \lambda_1}{2}$. Then, pick $\beta' = \frac{9}{8}\beta$. Take

$$\bar{\lambda}_1 = -\frac{\alpha}{8} \ .$$

Then

$$\bar{s}_1 \geq \alpha + \frac{\alpha + \lambda_1 - 2\lambda_1(B)}{4} \geq \alpha = \frac{9}{8}\alpha - \bar{\lambda}_1$$

while

$$\bar{s}_3 \geq \beta + 2\gamma - \frac{\lambda_1(B) + \lambda_1(B')}{2} - \frac{\gamma}{2} + \geq \beta + \frac{\alpha}{2} = \frac{5}{4}\beta \geq \beta' - \bar{\lambda}_1 \ .$$

Next, put

$$\bar{\lambda}_2 = \frac{-C - \gamma + 2\alpha}{2} + \frac{1}{8}\alpha \ \ \text{if } \bar{s}_2 < \alpha \text{ and}$$

$$\bar{\lambda}_2 = \frac{\alpha}{8} \ \ \text{otherwise.}$$

Thus guarantees that $\bar{s}_2$ is suitably bounded by $\beta'/2 - \bar{\lambda}_2$. We have already seen that

$$\bar{s}_4 \geq \frac{5}{4}\beta \geq \beta' + \frac{\alpha}{8} \ .$$

So we only need to check the case when $\bar{\lambda}_2$ is given by the first more complicated formula.

$$\bar{s}_4 - \bar{\lambda}_2 = \beta - \frac{\lambda_1(B) + \lambda_1(B')}{2} + 2\gamma + \frac{C + C'}{2} - \frac{9}{8}\alpha =$$

$$\beta - \frac{\alpha}{8} + \frac{C + C'}{2} + \frac{2\lambda_1 - \lambda_1(B) - \lambda_1(B')}{2} \geq \beta + \frac{\alpha}{2} - \frac{\alpha}{8} \geq \beta' \ .$$

Also, other inequalities between $\bar{\lambda}_i$ are easily verified.

$\tilde{B}$ **is equal to** B^*. In this case, we try mapping forward by parent branches of $\tilde{\phi}$ until images of $\psi(B_p)$ and the postcritical branch of $\tilde{\phi}$ land in different domains of ϕ_k. Then, we distinguish between the situation in which both these domains are monotone, or one of them is central. If both are monotone, the same estimates apply as in the previous case. Indeed, the

two distinct branches obtained here can be used as $\tilde{B}$ and B^* in the previous case to obtain the same separation bounds.

We are then left with two subordinate cases. First, assume that the postcritical branch gets mapped onto B^k, and $\psi(B_p)$ lands in another domain of ϕ_k, call it B'_p. Then, we estimate

$$\overline{s}_1 = \frac{\beta}{2}$$

$$\overline{s}_2 = 0 \ .$$

These are both weak estimates that hold in all cases. The annulus $\overline{A}'$ as always will have modulus at least $\overline{s}_1$. The annulus $\overline{A}_3$ will have two layers. The inner one will be of modulus at least γ and is the preimage of the annulus which surrounds the image of $\psi(B_p)$ inside B'_p. The outer one has modulus $\beta - \lambda_1$ and is the preimage of the annulus that separates B'_p from B^k inside B_n. The worst case is when $k = 1$ which gives

$$s_3 \geq \frac{3}{2}\beta \ .$$

As $\overline{s}_4$ we take the preimage of the annulus surrounding $\tilde{B}$ inside B^{k-1} by the central branch. That will add modulus at least $\alpha/2$. So, we can take

$$s_4 = \frac{7}{4}\beta \ .$$

Clearly, a normalized symbol with $\beta' = \frac{9}{8}\beta$, $\lambda_1 = -\frac{\beta}{16}$ and $\lambda_2 = \frac{\beta'}{2}$ will work.

Finally, let $\psi(B_p)$ be mapped on B^k with the postcritical domain going into some B^{p*}. Let l be the least positive integer such that $\psi^l(B^{p*})$ is not in B_n. Certainly, $l \leq k$. Then, $\overline{A}_2$ has two layers. One is the preimage of the annulus surrounding the image of the postcritical branch inside B^{p*}, and that has modulus at least γ. Another is obtained as the preimage of the annulus separating B^{p*} from B^k inside B^{k-1}, that gives modulus at least $\frac{\beta - \lambda_1(B^{t*})}{2}$ where $B^{t*} = \psi^l(B^{p*})$. Thus,

$$\overline{s}_2 = \frac{\beta - \lambda_1(B^{t*}) + \gamma}{2} \ .$$

The annulus $\overline{A}_1$ will have three layers. The inner one will be the preimage of the annulus surrounding $A_3(B^{p*})$ inside B_{l-1}. Its modulus is

$$\frac{\lambda_1(B^{t*}) + \lambda_2(B^{t*})}{2^{l+1}}$$

Next, we have the preimage of an annulus between B^{l-1} and B_n. This is at least $(\alpha + \lambda_1)(1 - 2^{1-l})$. Finally, we have the preimage of an annulus separating B^k from B^* inside B^{k-1}, that is at least $\alpha/2$. All this gives

$$\overline{s}_1 = \frac{1}{4}\beta + (\alpha + \lambda_1)(1 - 2^{1-l}) + \frac{\lambda_1(B^{t*}) + \lambda_2(B^{t*})}{2^{l+1}} + \overline{s}_2 \ .$$

One easily sees that this has the least value for $l = 1$, namely

$$\overline{s}_1 \geq \frac{\beta - \lambda_1(B^{t*})}{2} + \frac{\alpha + \lambda_1}{4} + \frac{\lambda_1(B^{t*}) + \lambda_2(B^{t*})}{4} + \frac{1}{4}\beta \geq$$

$$\geq \frac{3}{4}\beta + \frac{\alpha + \lambda_2(B^{t*})}{4} \geq \frac{13}{16}\beta \ .$$

Thus, we take

$$\overline{s}_1 = \frac{13}{16}\beta$$

$$\overline{s}_2 = \frac{\beta}{2} \ .$$

Then, we trivially have

$$\overline{s}_3 = \overline{s}_1 = \frac{13}{16}\beta$$

and

$$\overline{s}_4 = \overline{s}_3 + \frac{\alpha}{2} = \frac{17}{16}\beta \ .$$

The additional term $\alpha/2$ in $\overline{s}_4$ comes from $\overline{A}_4$ which is taken at least as the preimage by the central branch of the annulus separating B^* from B^k inside B^{k-1}. If $\overline{s}_i$, i not 3 were all decreased by a sixteenth, a normalized symbol with norm β could obviously be set up. Thus, we get a symbol with greater norm by Lemma 5.2.

This ends the analysis of δ-type preimages.

D-type preimages. Choose a non-immediate D-type preimage and call it B'. We go back to the estimates done previously for immediate preimages in the case of a close return. The bound $\overline{s}_1$ certainly remains, moreover, we observe that the annulus $\overline{A}_2$ for immediate preimages was chosen in such a way that it also separates their extensions rank $B_{n+1/2}$ from B_{n+1}, Thus, $\overline{s}_2$ also remains in force for all D-type preimages. For non-immediate D-type preimages, the annulus $\overline{A}'$ can be chosen with modulus equal to $\overline{s}_4$ in the immediate case. Indeed, B' will be mapped onto an immediate preimage by a mapping which extends in a univalent way onto B^k. Then $\overline{A}_3$ is the annulus surrounding the domain of the rank $n + 1/2$ extension of B' inside the parent branch (which is one of the extended immediate preimages.) Its modulus is at least as large as the modulus between an extended immediate preimage of B^k inside B_k, and that is half of the modulus of B^k inside B^{k-1}, or at least

$$\left(\alpha + \lambda_1(B)\right) \cdot 2^{-k} \geq \beta \cdot 2^{-k-2} \ .$$

So, $\overline{s}_3$ will grow by a half of this amount compared to the estimate for immediate preimages. Since $\overline{A}_4$ remains as in the immediate case, $\overline{s}_4$ will also grow by the same correction. For simplicity, denote this correction by $K\beta$ with K uniformly bounded away from 0 in terms of k and no more than $1/2$. Let

$$\beta' = (1 + \frac{K}{4})\beta \ .$$

Now let us mark estimates from the immediate case with bars, and the estimates from the present case with tildes. Then put

$$\tilde{\lambda}_1 = \overline{\lambda}_1 - \frac{K}{8}\beta$$

$$\tilde{\lambda}_2 = \overline{\lambda}_2 + \frac{K}{8}\beta \ .$$

A comparison with the bounds for immediate preimages shows that this gives a legitimate normalized symbol. Thus, for D-type preimages the norm of the symbol can be increased by an amount bounded away from 0 in terms of k.

This concludes the proof of Lemma 5.4.

5.4   Proof of Theorem B.

Statement of results.   The estimates we have done so far show that in all cases the symbols after inducing can be chosen with at least the same norm. This proves the first part of Proposition 4, namely that the norms of critical symbols stay at least β. We will now work to get the improved claim.

Lemma 5.5 *In the situation of Proposition 4, let $2 < j \leq k$. Consider a univalent hole $\overline{B}$ of ϕ_j other than an immediate preimage of the central hole. Then, the critical symbol of $\overline{B}$ can be chosen with norm $\beta' \geq (1 + K(l))\beta$ with $K(l)$ always positive and depending only on l.*

Proof:
Until we say otherwise, our reasoning will be valid for $j = 2$ as well. A large portion of the proof is already in Lemmas 5.1 and 5.4. We must must only return to the "independent" case when ϕ_{j-1} shows no close return while the B' (defined as the image by ψ of the parent branch of $\overline{B}$) and the postcritical branch B are independent.

Let B_n be the central hole of ϕ_{j-1}, B the postcritical branch, and B' the push-forward of the parent branch of $\overline{B}$. The extended branch of B maps the domain of B onto some D, and similarly B' is mapped onto D' by its extension. By Lemma 5.3, we know that the domains of the extended branches are surrounded inside $B_{(n-1)'}$ by an annulus with modulus at least $K_1(l)\beta$. Now we consider a number of cases depending on whether D and D' are central or not.

First, assume that D is not central. Then, the same estimates we got in our previous analysis of the independent case will be valid with B replaced with D. However, $\overline{A}_1$ will contain an extra layer of modulus at least $\frac{\beta}{2}K_1(l)$. Thus, $\overline{s}_1$ will be improved. This will also cause $\overline{s}_3$ and $\overline{s}_4$ to improve. By Lemma 5.2, the norm β can be increased by $\frac{\beta}{8}K_1(l)$.

Knowing that D is central, we have an option of estimating

$$\overline{s}_1 \geq K_1(l)\beta + \frac{\alpha + \lambda_1}{2} \quad \text{and}$$

$$\overline{s}_2 \geq \frac{\alpha + \lambda_1}{2}$$

where λ_1 is the supremum of $\lambda_1(b)$ over all univalent holes of ϕ_{j-1}.

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Now suppose that D' is not central. In this case, use the new estimate for $\overline{s}_1$ and the old estimate $\overline{s}_2$. Also, we take

$$\overline{s}_3 = \overline{s}_1 + \beta + \lambda_2(D') \ ,$$

$$\overline{s}_4 = \overline{s}_3 + K_1(l)\frac{\beta}{2} \ .$$

Then put $\beta' = \beta + \frac{\beta}{4}K_1(l)$ where $K_1(l)$ does not exceed $1/2$. Also, choose

$$\overline{\lambda}_1 = \frac{3K_1(l)\alpha}{4} - \frac{\alpha - \lambda_1}{2} \quad \text{and}$$

$$\overline{\lambda}_2 = \frac{\alpha - \delta}{2} + \frac{K_1(l)}{4}\alpha \ .$$

We easily check that $\overline{\lambda}_1 + \overline{\lambda}_2 > 0$, also it is clear that $\overline{s}_1$ and $\overline{s}_2$ are bounded as desired. Next, check

$$\overline{s}_3 + \overline{\lambda}_1 = \frac{7}{4}K_1(l)\alpha + \lambda_1 + \beta + \lambda_2(D') > \beta'$$

and

$$\overline{s}_4 - \overline{\lambda}_2 = \frac{\alpha + \lambda_1}{2} + \beta + \lambda_2(D')\frac{\alpha - \delta}{2} + \frac{7}{4}K_1(l)\alpha >$$

$$\beta' + \frac{\lambda_1 + \lambda_2(D')}{2} + \frac{\delta + \lambda_2(D')}{2} \ .$$

Since both fractions are non-negative, we are done.

Thus, we are left with the case when both D and D' are central. We first consider the situation of δ very large, i.e.

$$\delta \geq \frac{\alpha}{2} - \frac{1}{4}K_1(l)\alpha \ . \tag{32}$$

As λ_1
$delta$, we use

$$\overline{s}_1 = \frac{\alpha + \delta}{2} + K_1(l)\alpha \ ,$$

$$\overline{s}_2 = \frac{\alpha + \delta}{2} \ ,$$

$$\overline{s}_3 = \overline{s}_1 + \alpha + \delta \ ,$$

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$$\overline{s}_4 = \overline{s}_3 .$$

Then, pick

$$\beta' = (1 + \frac{K_1(l)}{8})\beta ,$$

$$\overline{\lambda}_1 = \frac{\delta - \alpha}{2} + \frac{7}{8}K_1(l)\alpha ,$$

$$\overline{\lambda}_2 = \frac{\alpha - \delta}{2} + \frac{1}{8}K_1(l)\alpha .$$

Provided that $K_1(l) \leq 1/2$ we have no trouble seeing that the inequalities between corrections are satisfied. We need to check

$$\overline{s}_3 + \overline{\lambda}_1 = 2\delta + \alpha + \frac{15}{8}K_1(l)\alpha \geq 2\alpha + \frac{11}{16}\beta > \beta'$$

and

$$\overline{s}_4 - \overline{\lambda}_2 = 2\delta + \alpha + \frac{7}{8}K_1(l)\alpha = 2\alpha + \frac{3}{16}\beta > \beta' .$$

So, we are done in this case and can assume the converse of inequality [32]. We proceed to note that for $j > 1$ there is a positive bound $K_2(l)$ such that

$$\lambda_1(B) + \lambda_2(B) \geq K_2(l)\beta \tag{33}$$

for all univalent holes B of ϕ_j. In the cases already dealt with in the proof of Lemma 5.5 this estimate follows. Indeed, once the norm of the symbol is larger than β, one can always decrease the norm and at the same time increase the corrections λ_i by the same amount getting weaker estimates. If a correction exceeds α as a result, it means that it was close to α to begin with, and thus the sum of the corrections was already positive. So we only need to prove estimate [33] in the remaining case. However, then our original choice of

$$\overline{\lambda}_1 = \frac{\lambda_2(B)}{2} \quad \text{and}$$

$$\overline{\lambda}_2 = \frac{\alpha - \delta}{2}$$

proves correct in view of the converse of inequality [32]. Thus, for $j \geq 3$ we can proceed assuming that estimate [33] holds for all holes of ϕ_{j-1}.

In this situation, we set

$$\overline{s}_1 = \frac{\beta + \lambda_2(B)}{2} \, ,$$

$$\overline{s}_2 = \frac{\alpha + \delta}{2} \, ,$$

$$\overline{s}_3 = \overline{s}_1 + \alpha + \lambda_1 \, ,$$

$$\overline{s}_4 = \overline{s}_3 + \frac{\beta + \lambda_2(B') - \alpha - \lambda_1}{2} \, .$$

Let $\beta' = (1 + \frac{1}{4}) K_2(l)\beta$. Put

$$\overline{\lambda}_1 = \frac{\lambda_2(B)}{2} - \frac{1}{4} K_2(l)\alpha \text{ and}$$

$$\overline{\lambda}_2 = \frac{\alpha - \delta}{2} + \frac{1}{4} K_2(l)\alpha \, .$$

We see that $\overline{\lambda}_1 + \overline{\lambda}_2 > 0$ and the bounds on $\overline{\lambda}_i$ also hold provided that $K_2 < 1/2$ which we can be assumed. Also, $\overline{s}_1$ and $\overline{s}_2$ are correctly estimated. We check

$$\overline{s}_3 + \overline{\lambda}_1 = \beta + \lambda_1 + \lambda_2(B) - \frac{1}{8} K_2(l)\beta \, .$$

By [33], $\lambda_1 + \lambda_2(B) \geq K_2(l)\beta$ for any B, thus we get

$$\overline{s}_3 + \overline{\lambda}_1 \geq \beta + \frac{7}{8} K_2(l)\beta > \beta' \, .$$

Also,

$$\overline{s}_4 - \overline{\lambda}_2 = \beta + \frac{\lambda_2(B) + \lambda_2(B')}{+} \frac{\lambda_1 + \delta}{2} - \frac{1}{8} K_2(l)\beta \geq \beta + \frac{3}{8} K_2(l)\beta > \beta' \, .$$

This concludes the proof of Lemma 5.5.

$\square$

Proof of Proposition 4. The part of Proposition 4 which says that norms of critical symbols do not deteriorate in the inducing process follows from the first round of our estimates. To prove that the norm improves, assume that $j \geq 3$ and the critical value falls into a non-immediate preimage B. By Lemma 5.5 we see two things. First, there is a normalized critical symbol of B with norm at least $(1 + K(l))\beta$. Secondly, for all holes of ϕ_{j+1} other than the immediate ones normalized critical symbols can be constructed with increased norms. Thus, we are left to show that the norm will also increase for critical symbols of the immediate holes of ϕ_{j+1}. This norm is at least equal to the norm of the critical symbol of B for immediate preimages, so we are done.

Theorem B. Suppose that a mapping ϕ satisfies the assumptions of Theorem B. Our first remark is this:

Lemma 5.6 *Under the hypotheses of theorem B, there is a function $K_1(\beta)$ bounded on any closed set in $(0, \infty)$ such that if in the sequence ϕ_i there is a rotation-like sequence*

$$\phi_j, \phi_{j+1}, \cdots, \phi_{j+K_1(\beta)} \,,$$

then $\phi_{j+K_1(\beta)}$ as a real mapping satisfies the starting condition.

Proof:
This follows directly from Theorem A. We only need to check that if B_n is the central domain of ϕ_j, then $|B_n|/|B_{n'}|$ is bounded away from 1 uniformly in terms of β. However, Proposition 4 implies that as complex box mappings, the central hole B_n is nested inside $B_{n'}$ with modulus at least $\beta/2$. The bound we need follows from classical analysis, see [13].

$\square$

By Corollary 5.2, we can assume that numbers l_j are uniformly bounded, or the starting condition is immediately satisfied. Thus, we can regard the parameter l in Proposition 4 as an absolute constant. Now let us fix some j. We observe that either $j + 3 + K_1(\beta)$ satisfies the starting condition, or there is a $j' \geq j + 3$ such that $\phi_{j'}$ does not make a rotation-like return. But then it means that for any j either $\phi_{j+3+K_1}(\beta)$ already satisfies the starting condition, or its separation norm is at least $(1 + K_2(l))\beta$ from Proposition 4.

Next, we infer that for any k, $\phi_{k(j+3+K_1(\beta)}$ either already satisfies the starting condition, or its separation norm is at least $\beta(1 + K_2(l))^k$. But if the separation norm becomes sufficiently large, the starting condition follows. Thus, Theorem B has been demonstrated.

5.5 Proof of Theorem D

Superadditivity of conformal moduli. In the course of our complex induction we have repeatedly encountered nesting annuli. The modulus of their union was always estimated from below by the sum of moduli. However, we will now show that unless the curve separating the two nesting annuli is quite smooth, a definite increment can be added to the modulus of the union. The following is a classical result:

Fact 5.1 *Let A_1 and A_2 be two disjoint open annuli situated so that A_1 separates 0 from A_2 while A_2 separates A_1 from ∞. Assume further that both are contained in the ring $A = \{z : r < |z| < R\}$ for some $0 < r < R$. By C denote the set (annulus) of all points from $A \setminus (A_1 \cup A_2)$ separated from 0 and ∞ by $A_1 \cup A_2$. Then, for every $\delta > 0$ there is a number ϵ with the following property: if*

$$mod\ A_1 + mod\ A_2 \geq mod\ A - \delta \ ,$$

then a ρ exists for which the ring

$$\{z : \rho < |z| < (1 + \epsilon)\rho\}$$

contains C.

Fact 5.1 follows directly from a "Modulsatz" of [21].

Corollary. Our next lemma is a simple corollary to Fact 5.1:

Lemma 5.7 *Let α, w and β be non-intersecting Jordan curves, all separating 0 from ∞. Suppose that w also separates α from β and that w passes through 1 and -1 in the complex plane. Let be the annulus A between α and w and B the annulus bounded by w and β. Suppose further that the moduli of A and B are both at least ζ. Then, there is a number $K \geq 1$ with the property that for each $\delta > 0$ at least one of these possibilities occurs:*

- *the curve w is in the Hausdorff distance less than δ from a $K(\zeta)$-quasicircle passing through -1 and 1, or*

-

$$mod\ A + mod\ B < mod\ (A \cup B) - \epsilon(\delta)$$

where $\epsilon(\delta) > 0$ depends on δ only.

Proof:
Apply the uniformizing map which carries α onto a circle with radius r, while β goes to a circle with radius $R > r$ and w is mapped onto a Jordan curve v in between so that v passes through 1. By Fact 5.1 unless the second part of the alternative holds, v is contained in a neighborhood of the unit circle of width δ. Since the annuli of A and B are at least ζ, for $\delta < 1/2$ the set of all inverses to uniformizing maps is normal on the ring $\{z : 1 < |z| < 1 + \delta\}$. From this, it first follows that the preimage of the unit circle is a uniform quasicircle, and secondly that the preimage of this ring is a narrow neighborhood of this quasicircle with width going to 0 uniformly with δ. So, the first part of the alternative holds.

$\square$

Proof of Theorem D.

Lemma 5.8 *Consider a type I complex box mapping ϕ of rank n. Suppose that ϕ shows a close return which results in a mapping ϕ_1. Let v be the ratio $|B_n|/|B_{n'}|$. Let β be the separation norm of ϕ. There exists a function $V(\beta) > 0$ so that if $v < V(\beta)$ then at least one possibility occurs:*

- *the separation norm of ϕ is at least $-K(\beta) \log v$ where K is a positive function of β alone, or*

- *the separation norm of ϕ_1 at least $\beta + \epsilon$ where $\epsilon > 0$ is an absolute constant.*

Proof:
By Lemma 5.4 we only need to consider D-type preimages, including immediate ones. Also, by the same Lemma the symbol for immediate preimages is also good for all D-type preimages, so it is enough to show that unless

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the first part of the alternative holds, the second possibility must occur for immediate preimages. Let us track again the proof of Lemma 5.4 in the case immediate preimages using the same notations. The annulus $\overline{A}_2$ is obtained as the preimage by the central branch ψ of a certain annulus Γ composed of two nesting annuli. The outer one surrounds the branch B^* inside B^k while the inner one is isomorphic to C and surrounds the postcritical domain of $\tilde{\varphi}$ inside its parent domain B^*. Both annuli can be mapped by ψ^k which is univalent on Γ. Then, B^* gets mapped onto a domain B of the original mapping ϕ. The outer layer of Γ gets mapped onto $A_3(B) \cup A'(B)$. The estimate used in Lemma 5.4 gives the modulus of Γ as the sum of $s_3(B)$ and the modulus of C. However, we can get a positive correction here due to the nesting of the image of C inside $A'(B)$. First, map both by the extension of the branch defined on B, which is univalent. Then A' goes onto the annulus surrounding B_n inside $B_{n'}$. As the moduli of C and A' can both be bounded away from 0 in terms of β, for a suitably chosen β, by Lemma 5.7 the nesting of the image of C inside this annulus gives a definite correction ϵ unless B_n is $\delta(\epsilon)$-close to a quasidisc uniform in terms of β, where $\delta(\epsilon)$ goes to 0 with ϵ. If this correction occurs, it will be factored into $\overline{s}_2$ and consequently into all other components of the critical symbol. So in this case we indeed get the second possibility allowed by Lemma 5.8.

Thus, we can assume that B_n normalized so that its boundary passes through 1 and -1 is δ-close to a $K(\beta)$-quasidisc where $\delta > 0$ can be specified. It remains to show that this last possibility implies the first part of the alternative claimed in Lemma 5.8. The region B_n is the preimage of $B_{n'}$ by a quadratic mapping composed with a univalent transformation with bounded distortion. If B_n is a uniform quasidisc, it contains a round disc centered at 0 of radius comparable to the length of $B_n \cap \mathbf{R}$. This property carries over to $B_{n'}$. On the other hand, again by virtue of B_n being a uniform quasidisc, it is contained in a round disc centered at 0 with radius comparable to the length of the real section of B_n. So, the modulus between the complex boxes B_n and $B_{n'}$ is $\log v + C(\beta)$. Thus, the component s_1 of any separation symbol can be as large as the lemma claims. But then A' always has at least the same modulus, so s_3 and s_4 can be chosen with the same size. The lemma follows.

$\square$

Lemma 5.9 *Consider a type I complex box mapping ϕ of rank $n-1$. Assume that ϕ arises from some ϕ_{-1} in a box inducing step. Suppose that ϕ shows a non-close return which results in a mapping ϕ_1, and ϕ_1 and also shows a non-close return whereby which gives ϕ_2 after a box inducing step. Suppose also that in both cases the critical value falls into an immediate preimage. Let v be the ratio $|B_n|/|B_{n'}|$. Let β denote the separation norm of ϕ. There exists a function $V(\beta) > 0$ so that if $v < V(\beta)$, then at least one possibility occurs:*

- *the separation norm of ϕ_1 is at least $-K(beta)\log v$ where $K > 0$ is an absolute constant, or*

- *the separation symbol for immediate preimages of ϕ_2 at least $\beta + \epsilon$ where $\epsilon > 0$ is an absolute constant.*

Proof:
We ask the reader to return to the proof of Lemma 5.1 in the case of immediate preimages. For ϕ_2, the annulus $\overline{A}_2$ is constructed as the preimage by the central branch ψ_1 of $A'(B) \cup A_3(B)$ where B is the postcritical domain of ϕ_1. In the estimates, the modulus of $A'(B) \cup A_3(B)$ is taken to be equal to the sum of moduli of its components. These nesting components are separated by the boundary of B which is the preimage of the boundary of B_n by the extension of the branch defined of B which is univalent and its distortion on the boundary of B is bounded in terms of β_0. Assume in addition (as will be verified later) that moduli of both $A'(B)$ and $A_3(B)$ are bounded from below uniformly in terms of β. Then, Lemma 5.7 implies that either the nesting involves a positive correction ϵ, or the boundary of B_n after normalization is δ-close to a uniform quasidisc. Then we proceed as in the proof of Lemma 5.8 to prove that the first part of the alternative holds. On the other hand, if the nesting gives a correction, the same correction will appear in all components of the symbol, and the second possibility occur. So we must show that the moduli of $A'(B)$ and $A_3(B)$ are both bounded away from 0. This is clear for $A'(B)$ which is the preimage of the annulus between B_n and $B_{n'}$ of modulus at least $\beta/2$. Next, we have to see how $A_3(B)$ arises from the previous inducing step. Denote the postcritical domain of ϕ_0 by b. Then, $A_3(B)$ is the preimage by ψ_{n-1} of $A_2(b)$. So it is sufficient to show that $s_2(b)$ is a bounded away from 0 proportion of β. Recall that b is an immediate preimage, and the

inspection of separation bounds given by Lemmas 5.8 and 5.9 for immediate preimages shows that $\lambda_1(b) \leq \beta/4$.

$\square$

We will now conclude the proof of Theorem D. Consider a sequence of type I complex box mappings ϕ_i as defined by the hypothesis of Theorem D. Observe that if the first possibility occurs in Lemma 5.8 or Lemma 5.9 for $\phi = \phi_j$, the estimate claimed by Theorem D follows for ϕ_j with a uniform C. Take an index j. First, look for j_0 defined as the largest index i not exceeding j for which the estimate of Theorem D follows with this C. Clearly, we are done if we show that from j_0 to j the moduli grow at a definite linear rate. By Lemma 5.8 we know that each close return causes the modulus to grow by a constant (that is because now only the second possibility can occur.) Thus, we are done if we show that a sequence of, say, five consecutive non-close returns also increases the modulus. Unless the fourth or fifth of those mappings has the critical value fall into an immediate domain, we are done by Lemma 5.8. Otherwise, we also get an increase by Lemma 5.9.

6 From unimodal to box mappings

We will need to delve a little more deeply in the inducing construction of [10]. So we begin by recalling certain crucial constructions.

6.1 The basics of inducing

Extendability. The whole fragment on extendability is a repetition of arguments of [10] put in a somewhat different language. The idea of extendability is to provide a condition which will imply bounded distortion of branches, at least a fair part of all branches, and will automatically reproduce itself by the inducing construction. What is required is the "metric extendability" condition which we state following [11]:

Definition 6.1 *A diffeomorphism g with a non-positive Schwarzian derivative defined on an interval (a,b) is said to be ϵ-extendable if there is a larger interval $(c,d) \supset (a,b)$ and an extension $\tilde{g} \supset g$ such that $\tilde{g}$ is still a diffeomorphism with a non-positive Schwarzian, and*

$$\frac{\tilde{g}(c) - \tilde{g}(a)}{\tilde{g}(d) - \tilde{g}(a)} \frac{\tilde{g}(d) - \tilde{g}(b)}{\tilde{g}(c) - \tilde{g}(b)} > \epsilon \; .$$

We will call the interval (c,d) from Definition 6.1 the *collar* of extendability, and its image by g the *margin* of extendability.

For branches of our box mappings Schwarzian derivative is non-positive by definition. We can therefore use the powerful "real Köbe lemma" (see [6]). This will say that

Fact 6.1 *If a monotone branch or its restriction satisfies ϵ-extendability, the distortion of that branch measured as the maximum logarithm of the ratio of derivatives taken at two points is bounded as uniform function of ϵ.*

A folding branch can also be said to be ϵ-extendable provided that it is the composition of $x \to (x-1/2)^2$ with an ϵ-extendable diffeomorphism. The margin of extendability is equal to the margin determined for this diffeomorphism. The collar is the preimage of the margin by the complete branch.

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Induced box maps. Given a mapping from class $\mathcal{F}$, we will consider its induced box maps, that is real box mappings in sense of Definition 1.4 whose branches are iterations of f.

A real box mapping will be called *suitable* if there is a symmetric neighborhood of the critical point which is mapped by the central branch inside itself. If a suitable box mapping is induced, then this neighborhood must be a restrictive interval of the underlying f.

General inducing process. We proceed to describe the general inducing process introduced in [10]. This will work for any S-unimodal f representable as a power law composed with a bounded distortion diffeomorphism which is either renormalizable or whose critical orbit is recurrent. The main features of this process are as follows:

1. Any map in the sequence is defined except on a set of points whose forward orbits avoid neighborhoods of the critical point.

2. All rank 0 branches show a uniform margin of extendability which is independent of the original map f as well as of the place in the construction. This naturally implies ϵ-extendability with a uniform ϵ.

3. For any rank 0 branch, its collar of extendability is contained in the smallest box containing the domain of this branch, with the obvious exception of the branches adjacent to the boundary of a box whose collars stick out to one side. Also, except for the central branch the collar of extendability does not contain the critical point.

4. If a map constructed in the process is not full, it is still of type I, and any branch of positive rank k extends with the margin $B_{k'}$.

We will now describe the process. This part is essentially a summary of [10]. The reader can refer there for more details.

The beginning. A full induced map exists which satisfies the extendability properties. In most cases, one just takes the first return map of f into its fundamental inducing domain. A problem may occur since the central branch can be arbitrarily short which prevents us from asserting any uniform margin of extendability for the branch next to the critical one. This problem is taken

care of by more inducing as described in the section 2.2 of [11]. The margin of extendability established for this induced map will remain in force throughout the construction.

We now suppose that an induced box mapping mapping ϕ, not necessarily of type I, is given which is not suitable, whose critical value is in the domain of definition of ϕ, and which satisfies our postulates. We will show how to get the next induced mapping. From the procedure, the postulates will be also satisfied. We distinguish a few cases.

The basic case. The basic case occurs if the critical value falls into the domain of a monotone rank 0 branch. In the basic case, the construction of the new induced mapping proceeds as follows. Define ϕ' to be ϕ with the central branch replaced with the identity. Then compose ϕ with ϕ'. This, first of all, gives you the new central branch which is of rank 0. Overall, we now have a box mapping ϕ'_1 of complicated structure. Now consider the extendability of rank 0 branches of ϕ'_1. The only ones of questionable extendability are preimages of rank 0 branches of ϕ' by the central branch of ϕ. This is because the critical value of ϕ may have entered their collars of extendability. However, the new central branch is uniformly extendable (compare the definition of extendability for folding branches.) To regain the extendability of all rank 0 branches of ϕ_1, we apply the process of *boundary refinement* to branches of ϕ' that are images of non-extendable branches of ϕ_1. The process is described in [10]. For us. it suffices to say that the boundary refinement involves composing monotone branches of ϕ' with ϕ' and allows us shrink the collar of extendability of the branch adjacent to the postcritical domain so that it no longer contains the critical value. The easy proof is provided in [10]. This phenomenon is based on the fact that the boundary refinement increases the number of iterations on branches. The boundary of the maximal possible margin of extendability then gets into a vicinity of the repelling fixed point q of f, and as the number of iterations increases, this point will be repelled from q, thus increasing the margin. After the adjustment of ϕ' by boundary refinement followed again by replacing the central branch of ϕ with its composition with ϕ', we get a mapping ϕ_1 with all branches of rank 0 uniformly extendable.

This is followed by a filling-in process. The objective is to obtain a type I mapping. To this end, monotone branches of ranks between 0 and $n + 1$

have to be refined. Prepare ϕ_1' by replacing its central branch with the identity map. Then take every branch of ϕ_1 of rank strictly between 0 and 1 and replace it with the composition of this branch with ϕ_1'. Again, some non-extendable branches of rank 0 may appear which are preimages of rank 0 monotone branches of ϕ_1'. Like in the previous step, one then returns to ϕ_1' to boundary-refine the preimages of those troublesome branches and that takes care of the problem. This gives us ϕ_2. The mapping ϕ_2 still has branches of ranks bigger than 0, though less than $n+1$, but the set occupied by their domains has shrunk. Then, exactly the same filling-in step is performed with ϕ_2 instead of ϕ_1. In the limit of filling-in a type I induced map is regained. Checking the properties of this mapping, we notice that compared with the domain of definition of ϕ, this limit map is not defined on the Cantor set points which forever stay in domains of ranks between 0 and $n+1$. But the orbits of these points forever avoid B_{n+1}. Other postulates are easily satisfied from the construction.

The box case. The box case occurs when the critical value of ϕ is found in a monotone branch of positive rank. We then follow the *standard inducing step* defined earlier on branches of positive rank. The only difference is that monotone branches of rank 0 are adjusted by boundary refinement to ensure their uniform extendability.

Close returns. The remaining case is when the critical value falls into the central domain. This again follows the process described in the *standard inducing step* with the exception that some monotone branches of ϕ'. One first constructs the mapping $\tilde{\phi}$ in the same way it was described in the standard inducing step. The difference may occur is some branches of this map of rank 0 do not have standard extendability. That is helped by going back to ϕ' and boundary-refining some of its monotone branches of rank 0. Then a dichotomy occurs for $\tilde{\phi}$ since it can show either a box or a basic return. We simply follow the appropriate step as described above.

The refined inducing process. In addition to the general inducing process just described we will need yet another inducing procedure, which we call *refined inducing process*. The refined inducing also allows type II mappings in the sense of Definition 1.8 and switches back forth between the two cases.

Let us assume that an induced map ϕ of type I or II and rank n is given. We first consider the case when ϕ is like a map coming from the general inducing process, i.e. it is of type I and satisfies the properties true for maps obtained in the general inducing.

Inducing on a type I map. Let us first assume no close return. The first filling is done in the usual way. A map ϕ' is built by replacing the central branch with the identity, and then the central branch of ϕ is replaced with its composition with ϕ'. If that leads to non-extendable rank 0 branches, we return to ϕ' to apply the boundary-refinement, and then compose the central branch with this modified ϕ'. In what we get all monotone branches of positive rank have rank n, while the new central branch has rank n or 0 depending on whether the basic or box case occurred. In the basic case, we exactly follow the general inducing step (there is no difference between general and refined inducing in the box case.) Otherwise, we stop at this stage. The only thing we do is to adjoin the new central domain to the box structure as B_{n+1}. This gives a type II mapping which differs from the outcome of the general step by a lack of filling-in.

In a close return, the mapping $\tilde{\phi}$ is obtained as in the general step, and then the process just described for non-close returns is used. We observe that this modified step preserves all properties inductively claimed for the general step.

Inducing on a type II mapping. The procedure depends on whether or not the range of the real central branch covers the critical point. If so, we start with a filling-in of all monotone branches of rank n'. This is a general observation that there always is a passage from type II to type I by filling-in of monotone branches of rank n' to obtain only monotone branches of rank n. After that, we follow the refined inducing step for type I maps just described.

If the image of the central branch does not cover the critical point, we follow the general step as in the refined step for type I mappings, and again skip the filling-in stage. In this case, if the rank of ϕ was B_n and, then after this step all branches will be of rank n'. We will thus remove the old B_n from the box structure and replace it with the new central domain. This means that this particular step does not increase the rank.

This completes the description of the refined inducing.

Boxes shrink uniformly.

Decay in terms of return time. We start with a general lemma.

Lemma 6.1 *For a map $f \in \mathcal{F}$, let k denote the maximum depth of almost parabolic points with periods less than the return time of the restrictive interval if f is renormalizable, or ∞ if not. For any $\delta > 0$ there exists an integer $m(\delta, k)$ with the property that if the stopping time on the central branch of a type I induced map in the sequence constructed from f the general inducing process is at least $m(\delta, k)$, then the size of any domain of positive rank, as well as of the central domain, are less than δ. The function $m(\delta, k)$ is independent of f. Conversely, for any integer m there is a number $\delta(m) > 0$ so that if the return time of the central branch is no more than m, the length of the central domain is greater than $\delta(m)$.*

Proof:
We first prove the bound from below on the rate of decay. We begin by noting that $\mathcal{F}$ is a normal family in the $C^{2,1}$ topology. Indeed, all members of this family are in the form $h_f(z - 1/2)^2$. Diffeomorphisms h_f are of negative Schwarzian derivative and uniformly ϵ-extendable by the principle of [19]. It is a well known fact the Schwarzian derivative of an ϵ-extendable iterate of a one-dimensional map with finitely many polynomial-type singularities is bounded from below uniformly in terms of ϵ (see a proof of a very similar estimate in [5].) Thus the normality follows.

Now, proceed by contradiction and consider a limit g of maps from $\mathcal{F}$ which have increasing return times on the central branch while the sizes of positive rank or folding domains remain bounded away from 0. One easily sees that g has a homterval, i.e., an interval on which all iterations of g are monotone. By the general result of [1], g must have a non-repelling, thus neutral cycle. We also notice that g continues not to expand cross-ratios, thus by [18] this neutral orbit is unique and the critical point is in the immediate basin of one point, say p. Now carry out the inducing process for g. The critical point and p will always stay together in the central branch, since branches in the inducing construction are separated either by preimages of the fixed point. Next, it is a property of the construction that for any branch, no intermediate images enter the central domain. This statement is verified by induction. The easiest way to see it is by reviewing the description we

provided asking which branches can change when we change f on its central domain only. By induction, for ϕ the central branch is the only one. Now, observe that when constructing ϕ' to be precomposed with other branches, we always replace the central branch by the identity, which eliminates the dependence except for the new central branch. From this observation it follows that return times on the central branch in the inducing process for g cannot jump the period of p. Thus, after finitely many steps an induced map is obtained which exhibits a close return (which must be low, i.e. the image of the real central branch does not cover the critical point). Now, if we take a map f from the sequence which allegedly contradicts the claim of the lemma which is very close to g in the C^2 topology, the construction is conducted in the same way for f, since the course of the construction only depends on where the critical value falls. The map f will show a low return, but will recover from it after a large number of steps. By Lemma 6.2 the central branch of this map is uniformly ϵ-extendable. Since it takes a long time for the critical value to escape the central domain, and this time can be made arbitrarily large by choosing f close enough to g, we can obtain a map f with an almost parabolic point of arbitrary depth, contradiction.

Finally, we prove the bound from above on the rate of decay of boxes. This follows immediately by induction. By construction, each central branch defined on B_n is a composition of the quadratic polynomial with a diffeomorphism mapping on B_{n-1}. In the case of a close return, this should be applied to mappings ϕ_i described in the standard inducing step. But the derivative of this diffeomorphism is bounded from above in terms of the return time. So, by each step the box shrinks only by a bounded factor in terms of the return time, and the number of inducing steps is certainly no more than the return time.

$\square$

Decay in terms of the rank.

Lemma 6.2 *If ϕ is a type I or II induced box mapping of rank n derived from some $f \in \mathcal{F}$, then*
$$\frac{|B_n|}{|B_{n'}|} \leq 1 - \epsilon$$
where ϵ is a n absolute constant independent of f.

Proof:
If ϕ is full, the ratio is indeed bounded away from 1 since a fixed proportion of B_0 is occupied by the domains of two branches with return time 2. In a sequence of box mappings this ratio remains bounded away from 1 by Fact 4.1.

$\square$

For type I mappings, we know that monotone branches of rank $r > 0$ are extendable to $B_{r'}$, and the central branch is extendable to $B_{(r-1)'}$, also in type II mappings. Note that it implies ϵ-extendability with ϵ an absolute constant.

6.2 Main proposition

Proposition 5 *For every $\delta > 0$, there are a fixed integer N and $\beta_0 > 0$, both independent of the dynamics, for which the following holds. Given a mapping from $\mathcal{F}$ a mixture of general and refined inducing gives an induced map ϕ which satisfies at least one of these conditions:*

- *ϕ is of type I, suitable and of rank less than N,*

- *ϕ is of type I, satisfies the starting condition with norm δ, its central branch is τ-extendable, and $\delta \leq \delta(\tau)$ in the sense of the hypothesis of Fact 2.2,*

- *ϕ has a structure of a complex box mapping with separation norm greater than β_0 and the rank less than N.*

If the depths of almost parabolic points with periods less than the return time of the restrictive interval (taken as ∞ in the non-renormalizable case) are bounded by k, the bound N on the rank can be replaced by a bound the return time on the central branch by a function $N(k)$.

Taking care of the basic case. Let $\delta > 0$ be given small enough so that if for a full map induced from some $f \in \mathcal{F}$ the central domain and all domains of positive rank are shorter that δ, then this map satisfies the assumption of Fact 2.2. This is certainly possible as all domains of positive rank are separated from the boundary of B_0 by the branches of return time 2 adjacent

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to the boundary of B_0 and the central branch is uniformly extendable (the τ is uniformly bounded away from 0 in Fact 2.2.)

Then, for any f we construct an induced map $\varphi(\delta)$ according to the following procedure. We follow the general inducing process until *one step before* the central domain and all domains of positive rank become smaller than δ. By Lemma 6.1, the stopping time on the central branch of $\varphi(\delta)$ is uniformly bounded in terms of δ, hence by an absolute constant. We can be prevented from being able to construct $\phi(\delta)$ by earlier hitting a suitable map. That, however, means that the return time of the restrictive interval is uniformly bounded in terms of δ, so are done as far the proof of Proposition 5 is concerned.

We take $\varphi(\delta)$ as the beginning point for further inducing. Note that if in the general or refined inducing a full map is ever derived from $\varphi(\delta)$, we are already done with the proof of Proposition 5, since the starting condition holds. Thus, we can assume that general and refined inducing on $\varphi(\delta)$ encounters exclusively box cases.

A way to control stopping times.   As another step in the proof of Proposition 5, we offer a convenient way of controlling stopping times on the central branch in terms of the rank. This will let us derive the estimates claimed in terms of return times in the absence of almost parabolic points of great depth. Let us label the induced mappings which follow $\varphi(\delta)$ in the general inducing procedure as $\varphi_0 := \varphi(\delta)$, φ_1 is the next one, and so on.

Lemma 6.3 *Under the hypothesis and in notations of Proposition 5, there exists a sequence of integer constants $C(j,k)$, independent of f, such that if the stopping time of the central branch of φ_j exceeds $C(j,k)$, then the starting condition holds.*

Proof:
We proceed by induction with respect to j, starting from $j = 0$ where the lemma is obvious. Since the stopping time on the central branch is bounded by $C(j,k)$, the size of the central domain is bounded away from 0. Otherwise, the derivative on the central branch would follow to be uniformly small. But, by Lemma 6.1 if the stopping time on the central branch of φ_{j+1} becomes large, the size of its central domain will become small as a uniform function depending also on k. Observe, however, that the smallness of the central

domain of φ_{j+1} relative to the central domain of φ_j determines box ratio in the starting condition for φ_{j+1}. Also, the extendability of the central branch is uniformly bounded in terms of j by Lemma 6.2. The lemma follows.

$\square$

The meaning of Lemma 6.3 is that in the absence of almost parabolic point of great depth, the requirement regarding the stopping time of the central branch being bounded in a certain way can be replaced by a simpler condition that the number of box steps leading to the suitable map from $\varphi(\delta)$ is appropriately bounded.

6.3   Finding a hole structure

We consider the sequence (φ_k) of consecutive box mappings obtained from $\varphi_0 := \varphi(\delta)$ in the course of the refined inducing. The objective of this section is to show that for some k which is bounded independently of everything else in the construction, a uniformly bounded hole structure exists which extends φ_k as a complex box mapping in the sense of Definition 1.6.

The case of multiple type II maps.   We will prove the following lemma:

Lemma 6.4 *Consider some φ_m of rank n. There is a function $k(n)$ such that if the mappings $\varphi_{m+1}, \ldots, \varphi_{m+k(n)}$ are all of type II, then $\varphi_{m+k(n)}$ has a hole structure which makes it a complex box mapping. The separation norm of the hole structure is bounded away from 0 depending solely on n.*

Proof:
If a sequence of type II mappings occurs, that means that the image of the central branch consistently fails to cover the critical point. The rank of all branches is fixed and equal to n. The central domain shrinks at least exponentially fast with steps of the construction at a uniform rate by Lemma 6.2. Thus, the ratio of the length of the central domain of φ_{m+k} to the length of B_n is bounded by a function of k which also depends on n, and for a fixed n goes to 0 as k goes to infinity.

This means that we will be done if we show that a small enough value of this ratio ensures the existence of a bounded hole structure. We choose two symmetrical circular arcs which intersect the line in the endpoints of B_n at

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angles $\pi/4$ to be the boundary of the box.(α will be chosen in a moment, right now assume $\alpha < \pi/2$). We take the preimages of the box by the monotone branches of rank n. They are contained in similar circular sectors circumscribed on their domains by Poincaré metric considerations given in [19]. Now, the range of the central branch is not too short compared to the length of B_n since it at least covers one branch adjacent to the boundary of the box (otherwise we would be in the basic case). We claim that the domain of that external branch constitutes a proportion of the box bounded depending on the return time of the central branch. Indeed, one first notices that the return time of the external branch is less than the return time of the central branch. This follows inductively from the construction. But the range of the external domain is always the whole fundamental inducing domain, so the domain of this branch cannot be too short. On the other hand, the size of the box is uniformly bounded away from 0 by Lemma 6.1. To obtain the preimage of the box by the complex continuation of the central branch, we write the central branch as $h(z - 1/2)^2$. The preimage by h is easy to handle, since it will be contained in a similar circular sector circumscribed on the real preimage. Since the distortion of h is again bounded in terms of n, the real range of $(x - 1/2)^2$ on the central branch will cover a proportion of the entire $h^{-1}(B_n)$ which is bounded away from 0 uniformly in terms of n. Thus, the preimage of the complex box by the central branch will be contained in a star-shaped region which meets the real line at the angle of $\pi/4$ and is contained in a rectangle built on the central domain of modulus bounded in terms of n.

It follows that this preimage will be contained below in the complex box if the middle domain is sufficiently small. As far as the separation norm is concerned, elementary geometrical considerations show that it is bounded away from 0 in terms of n.

$\square$

Formation of type I mappings. We consider then same sequence φ_k and we now analyze the cases when the image of the central branch covers the critical point. Those are exactly the situations which lead to type I maps in the refined inducing.

A tool for constructing complex box mappings. The reader is warned that the notations used in this technical fragment are "local" and should not be confused with symbols having fixed meaning in the rest of the paper. The construction of complex box mappings (choice of a bounded hole structure) in the remaining cases will be based on the technical work of [15]. We begin with a lemma which appears there without proof.

Lemma 6.5 *Consider a quadratic polynomial ψ normalized so that $\psi(1) = \psi(-1) = -1$, $\psi'(0) = 0$ and $\psi(0) = a \in (-1,1)$. The claim is that if $a < \frac{1}{2}$, then $\psi^{-1}(D(0,1))$ is strictly convex.*

Proof:
This is an elementary, but somewhat complicated computation. We will use an analytic approach by proving that the image of the tangent line to $\partial \psi^{-1}(D(0,1))$ at any point is locally strictly outside of $D(0,1)$ except for the point of tangency. We represent points in $D(0,1)$ in polar coordinates (r, ϕ) centered at a so that $\phi(1) = 0$, while for points in preimage we will use similar polar coordinates r', ϕ'. By school geometry we find that the boundary of $D(0,1)$ is given by

$$(r + a \cos \phi)^2 + a^2 \sin^2 \phi = 1 .$$

By symmetry, we restrict our considerations to $\phi \in [0, \pi]$. We can then change the parameter to $t = -a \cos \phi$, which allows us to express r as a function of t for boundary points, namely

$$r(t) = \sqrt{1 - a^2 + t^2} + t . \tag{34}$$

Now consider the tangent line at the preimage of $(r(t), t)$ by ψ. By conformality, it is perpendicular to the radius joining to 0, so it can be represented as the set of points (r', ϕ')

$$r' = \frac{\sqrt{r(t)}}{\cos^2(\theta/2)} , \quad \phi' = \frac{\pi}{2} - \frac{\phi}{2} + \frac{\theta}{2}$$

where θ ranges from $-\pi$ to π. The image of this line is given by $(\hat{r}(\theta), \phi - \theta)$ where

$$\hat{r}(\theta) = \frac{2r(t)}{1 + \cos \theta} .$$

Let us introduce a new variable $t(\theta) := -a \cos(\phi - \theta)$ so that $t = t(0)$. Our task is to prove that

$$r(t(\theta)) < \hat{r}(\theta) \tag{35}$$

for values of θ is some punctured neighborhood of 0. This will be achieved by comparing the second derivatives with respect to θ at $\theta = 0$. By the formula [34]

$$r(t(\theta)) = \sqrt{1 - a^2 + t(\theta)^2} + t(\theta) \ .$$

The second derivative at $\theta = 0$ is

$$\frac{1 - a^2}{(\sqrt{1 - a^2 + t(\theta)^2})^3}(a^2 - t^2) - t - \frac{t^2}{\sqrt{1 - a^2 + t(\theta)^2}} =$$

$$= -\sqrt{1 - a^2 + t(\theta)^2} - t + \frac{1 - a^2}{(\sqrt{1 - a^2 + t(\theta)^2})^3} \ .$$

The second derivative of the right-hand side of the desirable inequality [35] is more easily computed as

$$\frac{\sqrt{1 - a^2 + t(\theta)^2} + t}{2} \ .$$

Thus, the proof of the estimate [35], as well as the entire lemma, requires showing that

$$\frac{3}{2}(\sqrt{1 - a^2 + t(\theta)^2} + t) - \frac{1 - a^2}{(\sqrt{1 - a^2 + t(\theta)^2})^3} > 0$$

for $|a| < 1/2$ and $|t| \le |a|$. For a fixed a, the value of this expression increases with t. So, we only check $t = -a$ which reduces to

$$\frac{3}{2}(1 - a) - (1 - a^2) > 0$$

which indeed is positive except when $a \in [1/2, 1]$.

$\square$

The main lemma. Now we make preparations to prove another lemma, which is essentially Lemma 8.2 of [15]. Consider three nested intervals $I_1 \subset I_0 \subset I_{-1}$ with the common midpoint at $1/2$. Suppose that a map ψ is defined on I_1 which has the form $h(x - 1/2)^2$ where h is a polynomial diffeomorphism onto I_{-1} with non-positive Schwarzian derivative. We can think ψ as the central branch of a generalized box mapping. We denote

$$\alpha := \frac{|I_1|}{|I_0|}$$

. Next, if $0 < \theta \leq \pi/2$ we define $D(\theta)$ to be the union of two regions symmetrical with respect to the real axis. The upper region is defined as the intersection of the upper half plane with the disk centered in the lower $\Re = 1/2$ axis so that its boundary crosses the real line at the endpoints of I_0 making angles θ with the line. So, $D(\pi/2)$ is the disk having I_0 as diameter.

Lemma 6.6 *In notations introduced above, if the following conditions are satisfied:*

- *ψ maps the boundary of I_{-1} into the boundary of I_0,*

- *the image of the central branch contains the critical point,*

- *the critical value inside I_0, but not inside I_1,*

- *the distance from the critical value to the boundary of I_0 is no more than the (Hausdorff) distance between I_{-1} and I_0,*

then $\psi^{-1}(D(\theta))$ is contained in $D(\pi/2)$ and the vertical strip based on I_1. Furthermore, for every $\alpha < 1$ there is a choice of $0 < \theta(\alpha) < \pi/2$ so that

$$\psi^{-1}(D(\theta(\alpha))) \subset D(\theta(\alpha))$$

with a modulus at least $K(\alpha)$, and $\psi^{-1}(D(\theta(\alpha)))$ is contained in the intersection of two convex angles with vertices at the endpoints of I_1 both with measures less than $\pi - K(\alpha)$. Here, $K(\alpha)$ is a continuous positive function.

Proof:
By symmetry, we can assume that the critical value, denoted here by c, is on the left of $1/2$. Then t denotes the right endpoint of I_0, and t' is the

other endpoint of I_0. Furthermore, x means the right endpoint of B_{n-1}. By assumption, h extends to the range (t', x). To get the information about the preimages of points t, t', c, x one considers their cross-ratio

$$C = \frac{(x-t)(c-t')}{(x-c)(t-t')} \geq \frac{1+\alpha}{4}$$

where we used the assumption about the position of the critical value relative I_0 and I_{-1}. The cross ratio will not be decreased by h^{-1}. In addition, one knows that h^{-1} will map the disk of diameter I_0 inside the disk of diameter $h^{-1}(B_n)$ by the Poincaré metric argument of [19]. As a consequence of the non-contracting property of the cross-ratio, we get

$$\frac{h^{-1}(c) - h^{-1}(t')}{h^{-1}(t) - h^{-1}(t')} < \frac{1+\alpha}{4} \; . \tag{36}$$

When we pull back the disk based on $h^{-1}(I_0)$, we will get a figure which intersects the real axis along I_1. Notice that by the estimate [36] and Lemma 6.5, the preimage will be convex, thus necessarily contained in the vertical strip based on I_1. Its height in the imaginary direction is

$$\frac{|I_1|}{2} \sqrt{\frac{h^{-1}(t) - h^{-1}(c)}{h^{-1}(c) - h^{-1}(t')}} < \frac{|I_1|}{2} \sqrt{\frac{3-\alpha}{1+\alpha}} \; , \tag{37}$$

where we used the estimate [36] in the last inequality. Clearly,

$$\psi^{-1}(D(\pi/2))$$

is contained in the disk of this radius centered at $1/2$. To prove that

$$\psi^{-1}(D(\pi/2)) \subset D(\pi/2) \; ,$$

in view of the relation [37] we need

$$\alpha \sqrt{\frac{3-\alpha}{1+\alpha}} < 1 \tag{38}$$

By calculus one readily checks that this indeed is the case when $\alpha < 1$. To prove the uniformity statements, we first observe that

$$\psi^{-1}(D(\theta)) \subset \psi^{-1}(D(\pi/2))$$

for every $\theta < pi/2$. Since [38] is a sharp inequality, for every $\alpha < 1$ there is some range of values of θ below $\pi/2$ for which $\psi^{-1}(D(\theta)) \subset D(\theta)$ with some space in between. We only need to check the existence of the angular sectors. For the intersection of $\psi^{-1}(D(\theta))$ with a narrow strip around the real axis, such sectors will exist, since the boundary intersects the real line at angles θ and is uniformly smooth. Outside of this narrow strip, even $\psi^{-1}(D(\pi/2))$ is contained in some angular sector by its strict convexity.

$\square$

The assumption of extendability to the next larger box is always satisfied in our construction.

The case when there is no close return. We now return to our construction and usual notations. We consider a map φ_k, type II and of rank n, whose central branch covers the critical point, but without a close return. Then:

Lemma 6.7 *Either the Hausdorff distance from B_n to B_{n-1} exceeds the Hausdorff distance from B_{n-1} to B_{n-2}, or φ_k has a hole structure uniformly bounded in terms of n.*

Proof:
Suppose the condition on the Hausdorff distances fails. We choose the box around B_n and the hole around B_{n+1} by Lemma 6.6. Observe that the quantity α which plays a role in that Lemma is bounded away from 1 by Lemma 6.2. The box is then pulled back by these monotone branches and its preimages are inside similar figures built on the domains of branches by the usual Poincaré metric argument of [19]. For those monotone branches, the desired bounds follow immediately.

$\square$

The case when the close return occurs. Finally, we have to deal with the case when the image of the central branch of φ_k, which as always can be assumed of type II and some rank n, covers the critical point, but also makes it a close return.

Lemma 6.8 *In the situation described above at least one the three statements holds:*

- *the starting condition holds as claimed by Proposition 5,*

- *a hole structure uniformly bounded in terms of k exists for an induced map φ' obtained from φ_k and from which φ_{k+1} can be extracted by more inducing,*

- *the Hausdorff distance from B_{n+1} to B_n exceeds the Hausdorff distance from B_n to B_{n-2}.*

Proof:

Suppose that the condition on the Hausdorff distances does not hold. We look into the description of the inducing process in the close case, and seek an opportunity to apply Lemma 6.6. We build a sequence of temporary boxes which are preimages of B_n by the central branch until for one of them, say B_{n+m}, the critical value escapes. First, we remark by Lemma 6.3 that m is uniformly bounded in terms of k, or the starting condition holds and we have nothing more to prove. To construct φ' we just fill-in all branches of positive rank so that they map onto B_{n+m-1}. The map φ' is of type II and no longer shows a close return, and indeed φ_{k+1} can be obtained from φ' by a step of the refined inducing construction. The hole structure for φ' is obtained by repeating the argument of Lemma 6.7 with the additional information that the central branch is extendable to a margin equal to the whole B_{n-2}.

$\square$

Proof of Proposition 5. This is just a summary of the work done in this section. We claim that we have proved that either a map with a box structure can be obtained from $\varphi(\delta)$ in a uniformly bounded number of steps of the refined inducing process, or the Proposition 5 holds anyway. Since Lemmas 6.4, 6.7 and 6.8 all provide uniform bounds for the hole structures in terms of k or the rank which is bounded in terms of k, it follows that the hole structure is bounded or the starting condition holds anyway. It also follows that by Lemma 6.3 that if the inducing fails within this bounded number of steps because of a suitable map being reached, then the stopping time on the central branch of the suitable map is bounded, hence Proposition 5 again follows.

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So, we need to prove that claim. If the claim fails, then by Lemma 6.4 the situations in which the image of the central branch covers the critical point have to occur with definite frequency. That is, we can pick a function $m(k)$ independent of other elements of the construction which goes to infinity with k such that among $\varphi_1, \ldots, \varphi_k$ the situation in which the critical point is covered by the image of the central branch occurs at least $m(k)$ times. But each time that happens, we are able to conclude by Lemmas 6.7 and 6.8 that the Hausdorff distance between more deeply nested boxes is more than between shallower boxes. Initially, for φ_0 whose rank was n, the B_n distance between and B_{n-1} was a fixed proportion of the diameter of B_n So only a bounded number of boxes can be nested inside B_{n-1} with fixed space between any two of them (or at least between every other pair in the situation of Lemma 6.8.) So we have a bound on the value of $m(k)$, thus on k. This proof of the claim is a generalization of the reasoning used in [15].

The claim concludes the proof of Proposition 5.

6.4   Proofs of main theorems

Proof of Theorem 1.   By [10] an S-unimodal non-renormalizable mapping without attracting or indifferent periodic points is expansion-inducing provided that box ratios shrink to 0. Thus, Theorem 1 follows directly.

Proof of Theorem 2.   We will prove this theorem by showing that if the return time on the restrictive interval is sufficiently large, it follows that for the suitable map obtained in the inducing construction of [10] which is of type I and rank n the ratio $|B_{n'}|/|B_n|$. Namely, given a D in Theorem 2, this ratio should be at least D. Next, specify δ in Proposition 5 equal to that D. Of the three outcomes of Proposition 5 the integer N will give us the minimum return time in Theorem 2. In other cases the starting condition either holds, in which case we are done, or we get a hole structure with a uniform separation norm. Then we use Theorem B. We learn that after a bounded number of box steps, which by Lemma 6.3 means either a starting condition again or a uniformly bounded return time on the central branch, we get the starting condition with norm δ. So, in all cases either the starting condition holds with norm δ and can only improve, or the return time on the central branch is uniformly bounded. Theorem 2 follows.

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