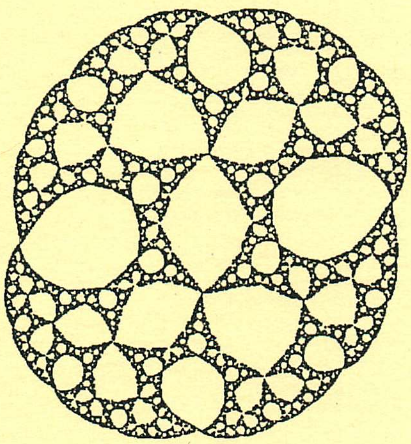
Quasiconformal Mappings, Operators on Hilbert Space, and Local Formulae for Characteristic Classes

by A. Connes, D. Sullivan, and N. Teleman

September 1993



SUNY StonyBrook
Institute for Mathematical Sciences

Preprint #1993/2

Quasiconformal mappings, operators on Hilbert space, and local formulae for characteristic classes

by Alain CONNES, Dennis SULLIVAN and Nicolas TELEMAN

dedicated to Bill Browder
on his 60^{th} birthday

Abstract. We give local formulae for the characteristic classes of a quasiconformal manifold using the subspace of exact forms in the Hilbert space of middle dimensional forms. The method applies to combinatorial manifolds and all topological manifolds except certain ones in dimension four.

By rolling, or better pressing, a sphere $S^{2\ell}$ all around the manifold $M^{2\ell}$, we will construct bounded operators on the space of L^2 middle dimensional forms of $M^{2\ell}$ analogous to the Ahlfors-Beurling operator on the Riemann sphere $\widehat{\mathbf{C}}$

$$\varphi(z,\bar{z})d\bar{z} = \eta \mapsto S\eta = \left(\frac{1}{2\pi i}\int_{\mathbf{C}} \frac{\varphi(\zeta,\bar{\zeta})d\bar{\zeta}\, d\zeta}{(z-\zeta)^2}\right) dz. \tag{1}$$

The kernel of this operator is the biform $\frac{dz\, dz'}{(z-z')^2}$ on $\widehat{\mathbf{C}} \times \widehat{\mathbf{C}}$. Using these operators and their kernels we obtain explicit local cycle representatives of the Hirzebruch-Thom characteristic classes for any quasiconformal manifold. At the end of the introduction we explain how the construction applies to topological manifolds. This answers a question raised by Bill Browder in lectures at Princeton in 1964.

Generalizations of (1) to all even spheres $S^{2\ell}$ with any bounded measurable pointwise $*$ operator on middle forms were constructed for the quasi conformal Yang Mills theory [1] by explicit formulae. For example, here we write

$$S_\mu = (1+\mu)^{-1}\,(\mu+S)\,S(\mu+S)^{-1}\,(1+\mu) \quad , \quad |\mu|<1, \tag{2}$$

for the standard sphere $S^{2\ell}$ can be written out on $R^{2\ell}$ as a quadratic expression in Riesz transforms see [2]. When ℓ is even, these bounded operators could be called ***conformal signature operators***, being essentially the phases of the usual signature operators when the $*$ is smooth (see the proof of theorem 2 part 4). When ℓ is odd they are generalizations of the Ahlfors-Beurling operator to general "curved" measurable conformal structures.

For a general even dimensional quasi conformal manifold provided with a bounded measurable conformal structure or more generally a bounded measurable $*$ on middle forms we do the following:

i) locally copy via charts the structure on $M^{2\ell}$ by structures on $S^{2\ell}$,

ii) get locally defined operators on $M^{2\ell}$ from the operators on $S^{2\ell}$,

iii) collect these together on $M^{2\ell}$ using a partition of unity to construct global operators S on the middle dimensional forms (§1).

Let $I(\ell)$ denote the ideal of compact operators A on Hilbert space satisfying $\mu_n = O(n^{-1/2\ell})$ where μ_n = distance in norm between A and rank n operators. The reader may recall that any degree one smoothing operators in dimension 2ℓ e.g. the Poincaré lemma operator, belongs to the ideal $I(\ell)$. The standard notation for $I(\ell)$ is $\mathcal{L}^{2\ell,\infty}$ (§1).

Theorem 1. *Given quasi conformal $M^{2\ell}$ with a bounded measurable conformal structure or more generally a bounded measurable $*$-structure on the ℓ-forms, the local construction yields an operator S which is determined by $*$ up to the ideal $I(\ell)$. Moreover any such S satisfies i) S agrees* $\operatorname{mod} I(\ell)$ *with the identity on exact ℓ-forms ii) S anticommutes modulo $I(\ell)$ with the involution γ associated to $*$ ($\gamma = *$ if ℓ is even, $\gamma = i*$ if ℓ is odd).*

Let us say an operator S on the Hilbert space $\mathcal{H}$ of middle dimensional forms satisfying i) and ii) of Theorem 1 belongs to the *Hodge class*. It is immediate (§1) that given $*$ any two Hodge class operators differ by a compact operator in $I(\ell)$. Note also that i) and ii) imply iii) $S^2 \equiv I \operatorname{mod} I(\ell)$.

There is a canonical non-local Hodge class operator S_* for the pair $(M,*)$. It is the involution defined up to finite rank by S_* is the identity on exact forms and S_* anticommutes with γ. The projectors associated with S_* are compatible with the usual Hodge decomposition of middle dimensional forms. Examples are the S_μ defined above for $S^{2\ell}$.

An interesting analytical consequence of the formula (2) in the proof of Theorem 1 is the

Corollary 2. *Any* Hodge class operator *S on $\mathcal{H}$ defines a Fredholm module in the precise sense that for the sup norm dense subalgebra of continuous functions on $M^{2\ell}$ satisfying $\int_M |df|^{2\ell} < \infty$, the commutators $[S,f]$ where f denotes the multiplication operator associated to f, belong to the ideal $I(\ell)$ of compact operators.*

The theorem and corollary answers anew the question of Singer [10] about "constructing the operator" cf. [6], [19].

To any Hodge class operator we apply the algebraic procedure of [3] to construct a *refined* Hodge class operator H satisfying the further conditions a) $H^2 = I$ on $\mathcal{H}$ b) $H\gamma + \gamma H$ is trace class. This algebraic process Theorem 2 (§2) preserves locality.

The trace class operator $L = (H\gamma + \gamma H)\, H = H\gamma H + \gamma$ can be used in a simple way to construct representations of the Hirzebruch-Thom characteristic classes or rather their Poincaré dual homology classes. Assume $M^{2\ell}$ is oriented, then the kernel of L, $L(x, x')$ is a biform on $M \times M$ of bigree (ℓ, ℓ). The support of $L(x, x')$ is near the diagonal if we started with a locally constructed Hodge class operator. The trace $L(x)$ on the diagonal is a 2ℓ form, or since M is oriented it is a measure in the Lebesgue measure class.

More generally, consider the cyclic expression $\{\text{trace } L(x_0, x_1)\, L(x_1, x_2) \ldots L(x_{2q}, x_0)\}$ which can be considered as either a top dimensional form or as a measure on $M \times M \times \ldots \times M$ ($2q + 1$ factors), supported near the diagonal.

Theorem 3. *The cyclic expressions* $\{\textit{trace } L(x_0, x_1)\; L(x_1, x_2) \ldots L(x_{2q}, x_0)\}$ *when considered as measures on* M^{2q+1} *near the diagonal define Alexander Spanier cycles. If ℓ is even, these cycles for q even and if ℓ is odd, these cycles for q odd represent the dual Hirzebruch-Thom characteristic homology classes times* $2^{2q+1}(2\pi i)^{-q}\; q!/2q!$. *In particular if ℓ is even* $L(x) =$ *trace* $L(x, x)$ *is a locally constructed measure whose total mass is twice the signature of* $M^{2\ell}$.

The algebraic construction used above to refine Hodge class operator, $S \to H$, and the check that the odd cyclic expressions in the kernel $L(x, y)$ of $L = H\ \gamma\ H + \gamma$ define cycles cover only a page or two. Behind this calculation is the idea that any Hodge duality operator defines a K-homology element because of the corollary, and we know since [4] and [6] what this element and its Chern character should be. We also know from [5] an explicit construction of the Chern character starting from K-theory of an algebra and arriving in the cyclic cohomology of the algebra which for a manifold is related to Alexander Spanier [3]. Concretizing these ideas is the page or two. The connection with Hirzebruch-Thom classes is filled in using the index theorem as in [6], [1] and [3].

The local construction of the operators and the analytical content of the corollary are easy consequences of the formula (2) on spheres relating the (canonical non local) Hodge operator S_μ for any measurable $*$ to the (canonical non local) Hodge operator S for the standard structure on $S^{2\ell}$. This formula also relates to the other discussions.

On the Riemann sphere one knows the remarkable measurable Riemann mapping theorem that any bounded measurable conformal structure is related by a quasiconformal homeomorphism $w = \varphi(z)$ to the standard one. Since S_μ is given by the kernel $\frac{dw\, dw'}{(w-w')^2}$ on $(0,1)$ forms while S is given by the kernel $\frac{dz\, dz'}{(z-z')^2}$ on $(0,1)$ forms our basic formula (2) has a direct relationship with the measurable Riemann mapping theorem. Namely, we can calculate $w = \varphi(z)$ from the kernel of S_μ by expanding out the formula (2).

In dimension 4 the operators S_μ were used in the analytical underpinnings of the Yang-Mills discussion. The formulae (2) show since $|\mu| < 1$ that S_μ determines isomorphisms on $\mathcal{L}^p$-forms for p a little greater than two. Applied to the curvature 2-forms this gives the extra regularity to get past the critical Sobolev exponent for the Yang-Mills connections and gauge transformations. A corollary of this theory [1] was that some closed M^4 have infinitely many distinct quasiconformal structures, and that some topological M^4 have no quasiconformal structure.

Thus our local constructions for characteristic classes based on (2) are higher dimensional relatives of the measurable Riemann mapping in dimension two and the Yang-Mills theory in dimension four.

Outside dimension four there is a proof [7] independent of the theory of [12] that ***stable*** topological manifolds have a quasiconformal structure unique up to isotopy. Here stable refers to the pseudogroup of homeomorphisms of R^n in the connected components of the identity or a reflection [9]. Thus this paper defines local characteristic classes for ***stable*** topological manifolds independent of Novikov's theory [12].

We can also apply our constructions or those of [7] and [8] to general topological manifolds but this uses Kirby's result on the stable homeomorphism conjecture [9]. The proof in [9] properly contains Novikov's theory needed for his original proof of the topological invariance of rational Pontryagin classes.

Historical Remark. If $g_{\alpha\beta}$ denotes the Jacobian matrices of the overlap homeomorphisms of charts covering a manifold M^n, the curving or non-flatness of M^n is measured by $\theta_{\alpha\beta} = \{g_{\alpha\beta}^{-1}\, d\, g_{\alpha\beta}\}$ which is a Cech 1-cocycle with values in matrices of 1-forms. Thus if the overlap homeomorphisms have Lipschitz derivatives (or even 2^{nd} derivatives in L^n) there is a Chern-Weil type construction of characteristic forms by forming products and traces.

By considering normal bundles to smooth foliations and the Bott vanishing theorem [15] one finds serious obstructions to the possibility of reducing this smoothness requirement and staying in the context of differential forms.

In our context of Lipschitz or quasiconformal manifolds we have exactly one less derivative than required above.

It seems natural then try to interpret $g^{-1}\, dg$ as a distribution or as an Alexander Spanier cochain. This was attempted in 1976 when the possibility of having Lipschitz or quasiconformal coordinates appeared. However, the distribution idea founders because of the impossibility of forming products. This difficulty is removed in Alexander Spanier at the expense of non-commuting products. But then the trace step in the classical Chern-Weil procedure becomes problematical. In other words there is an analytical qua algebraic barrier to copying the pointwise "curvature" route to characteristic classes for quasiconformal or Lipschitz charts.

In this paper these difficulties are surmounted by using trace ideals of operators on Hilbert space [5] and an algebraic addition to the Chern-Weil algorithm coming from cyclic cohomology [5]. The quasiconformal charts provide enough analysis to "quantize the manifold" in the sense of constructing a Hilbert space and a relevant operator replacing curvature.

This "quantized curvature" is then treated algebraically in a manner guided by the formulae of cyclic cohomology. Perhaps the essence of the latter point is that the cocycles in the cyclic context are just those multilinear functionals which when applied to (projector, projector, ...) remain constant when *the* projector is varied by a homotopy.

The reader may recall that exactly this kind of consideration appears classically when showing the Chern-Weil forms are cohomology invariants.

In summary we have treated a problem with one missing derivative in a classical context using the ideas and tools of "non-commutative geometry" [16].

1. Preliminaries on quasiconformal geometry

A quasiconformal (qc) homeomorphism h between two open domains Ω_1, Ω_2 in $\mathbf{R}^n$ is a homeomorphism with the property that relative distances are boundedly distorted, i.e. for each x in Ω_1,

$$(1) \qquad \overline{\lim_{a \to 0}} \frac{\max\{|h(x) - h(y)| \; ; \; |x - y| = r\}}{\min\{|h(x) - h(y)| \; ; \; |x - y| = r\}} = K(x) \leq K < \infty.$$

We also assume the analogous statement for h^{-1}.

Gehring [15] proved that when $n > 1$ a qc-homeomorphism is a.e. differentiable; moreover, the first order partial derivatives of the component functions of h belong to the Banach space $L^{n+\varepsilon}_{\text{loc}}$, where $\varepsilon = \varepsilon(K) > 0$. It follows that h is a Hölder continuous function with exponent ε, h is non-singular with respect to the Lebesgue measure class, and the best K that works in (1) almost everywhere for h also works for h^{-1}.

Let g be an arbitrary Euclidean metric on the tangent space to $\mathbf{R}^n$ at some of its points x_0. Recall that the metric g and all other similar metrics rg, where r is an arbitrary positive real number, define the conformal class $[g]$ of the metric g.

If $[g_0]$ and $[g_1]$ are two conformal structures, the conformal distance between them is by definition

$$(2) \qquad d([g_0],[g_1]) = \log \frac{\max\{|v|_{g_1} \; : \; |v|_{g_0} = 1\}}{\min\{|v|_{g_1} \; : \; |v|_{g_0} = 1\}},$$

From now on we suppose that the dimension $n = 2\ell =$ even, and we choose the standard orientation on $\mathbf{R}^n$.

Let Λ denote the vector space of all differential forms of degree ℓ at x_0. For any Euclidean metric g as above, the Hodge star operator $*_g$ associated to g defines an endomorphism

$$*_g : \Lambda \to \Lambda$$

with

$$*_g^2 = (-1)^\ell.$$

The main property of the operator $*_g$ acting on Λ is that it remains unchanged under dilations of the metric g, i.e. it depends only on the conformal class of g.

We let γ_g be the involution of the complexification $\Lambda_{\mathbf{C}}$ of Λ given by

$$\gamma_g = i^\ell \; *_g \,. \tag{3}$$

We let $\Lambda^{\pm}(g)$ be the ± 1 eigenspaces of γ_g. These subspaces are maximal definite subspaces for the quadratic form

$$\omega \to \omega \wedge \omega.$$

The conformal distance between two conformal classes $[g_0]$ and $[g]$ may be estimated in terms of the relative position of the eigenspaces $\Lambda^{\pm}$. Indeed, there exists a unique linear mapping

$$\mu : \Lambda^-(g_0) \to \Lambda^+(g_0) \tag{4}$$

with the property that the graph of μ is precisely $\Lambda^-(g)$. The operator norm of μ relative to the metric g_0 satisfies

$$|\mu|_{g_0} < 1 \tag{5}$$

and

$$\frac{1}{2} \log \frac{1+|\mu|_{g_0}}{1-|\mu|_{g_0}} \le d([g_0],[g_1]) \le \log \frac{1+|\mu|_{g_0}}{1-|\mu|_{g_0}}. \tag{6}$$

A field $c(x)$ of conformal structures over a domain U in R^n is called a bounded measurable (bm) conformal structure on U if there exists a Riemannian metric g over U, whose components are measurable functions, such that for any x in U, $c(x) = [g(x)]$, and

$$d([g(x)],[e]) \le C < \infty,$$

where e denotes the standard Euclidean metric.

Equivalently, c is a bounded measurable conformal structure iff the corresponding field of endomorphisms μ, relative to e, is a matrix field with measurable entries, and

$$\|\mu\|_\infty \doteq \sup |\mu(x)| < 1.$$

If c is a bounded measurable conformal structure on U, then γ_c is a field of matrices with bounded measurable entries. If c is a bounded measurable conformal structure on V and $h : U \to V$ is a qc-homeomorphism, then h^*c is a bounded measurable conformal structure on U because a qc-homeomorphism induces uniformly quasi-homotheties on the tangent spaces, a.e.

A qc-manifold is a topological manifold equipped with an atlas whose changes of coordinates are qc-homeomorphisms. It possesses a well defined measure class, the Lebesgue measure class, since qc-homeomorphisms are absolutely continuous. The tangent bundle of a qc-manifold is a measurable real vector bundle.

A bounded measurable conformal structure on a qc-manifold is a field of conformal structures on its tangent spaces whose restriction to any qc-chart is bounded measurable. Any paracompact qc-manifold has such structures.

On a compact smooth manifold, a conformal structure is bounded iff the conformal distance (defined point by point) between it and the underlying conformal structure of a smooth Riemannian metric, is a bounded function.

On a compact qc-manifold M the space $L^{n/r}(M, \wedge^r)$ of r-forms with coefficients in $L^{n/r}$, $n = \dim M$, is well defined. Any bounded measurable conformal structure specifies a Banach space norm on $L^{n/r}$ by:

$$(\|\omega\|)^{n/r} = \int_M |\omega|^{n/r}.$$

Given $\omega_1 \in L^{n/r}(M, \wedge^r)$, $\omega_2 \in L^{n/r+1}(M, \wedge^{r+1})$ we write $d\omega_1 = \omega_2$ iff this holds, in the sense of distributions, in any qc local chart. This yields [17] a densely defined closed operator $d : L^{n/r} \to L^{n/r+1}$ which commutes with qc homeomorphisms. We let $\operatorname{Im} d$ be the image of d, it is closed in $L^{n/r+1}$ provided $r \geq 1$ [17] [1].

The underlying topological vector spaces only depend on the quasiconformal structure.

2. Statement of the main result

Let M be a compact oriented quasiconformal manifold of even dimension 2ℓ. Let $[g]$ be a bounded measurable conformal structure on M and γ, $\gamma^2 = 1$, the associated $*$ operator in the Hilbert space $\mathcal{H} = L^2(M, \wedge^\ell)$ of square integrable forms of degree ℓ on M.

Given an open neighborhood U of the diagonal in $M \times M$ and any bounded operator T in $\mathcal{H} = L^2(M, \wedge^\ell)$, we say that Support $(T) \subset U$ iff the following holds for any open $V \subset M$

$$\text{(1)} \qquad \omega \in \mathcal{H}\,,\ \text{Support}\ \omega \subset V \Rightarrow \text{Support}\ (T\omega) \subset U \circ V$$

where $U \circ V = \{x \in M\ ;\ \exists y \in V\ ,\ (x, y) \in U\}$.

For any $p \in [1, \infty[$ the following conditions define two sided ideals of compact operators in Hilbert space. We let for any compact operator T in $\mathcal{H}$, $\mu_n(T)$ be the n-th characteristic value of T, i.e. the n-th eigenvalue of $|T| = (T^*T)^{1/2}$, or the distance in operator norm between T and rank n operators.

(2)
$$\mathcal{L}^p(\mathcal{H}) = \left\{ T \text{ compact} \; ; \; \sum_1^\infty \mu_n(T)^p < \infty \right\}$$

(3)
$$\mathcal{L}^{(p,\infty)}(\mathcal{H}) = \{ T \text{ compact} \; ; \; \mu_n(T) = O(n^{-1/p}) \}.$$

We can now define the following key notion:

Definition 1. *Let U be an open neighborhood refining that of Hodge class operators in the introduction of the diagonal in $M \times M$. A U-local Hodge decomposition is a bounded operator H in $L^2(M, \wedge^\ell T^*_{\mathbb{C}})$ such that:*

α) $H^2 = 1$

β) Support $H \subset U$

γ) $(H-1)/\operatorname{Im} d \in \mathcal{L}^{(2\ell,\infty)}$ ($= I(\ell)$ of the introduction)

δ) $H\gamma + \gamma H \in \mathcal{L}^1$ (= trace class by definition).

Giving H is the same as giving the decomposition of $\mathcal{H} = L^2(M, \wedge^\ell T^*_{\mathbb{C}})$ as the linear sum of the two definition closed subspaces:

(4)
$$\{\xi \in \mathcal{H} \; ; \; H\xi = \pm \xi\}.$$

We shall now explain how to construct for each $q \in \{0, 1, \ldots, \ell\}$ an Alexander Spanier cycle on M from a U-local Hodge decomposition. To define Alexander Spanier homology on a compact space X we consider for each integer d the linear space A_d of totally antisymmetric measures σ on X^{d+1}. Such a measure σ is uniquely determined by the value of $\sigma(\varphi) = \int \varphi d\sigma$ on bounded borel antisymmetric functions φ on X^{d+1}. We let $\delta : A_d \to A_{d-1}$ be the boundary operator given by the equality:

(5)
$$(\delta\sigma)(\varphi) = \sum_0^d (-1)^j \int \varphi(x_0, \ldots, \check{x}_j, \ldots, x_d) \, d\sigma \qquad \forall \varphi.$$

Let U be a neighborhood of the diagonal in $X \times X$. We shall say that $\sigma \in A_d$ is U-local iff:

(6)
$$\text{Support}\,\sigma \subset \{(x_j) \in X^{d+1} \; ; \; (x_i, x_j) \in U \qquad \forall i, j \in \{0, 1, \ldots, d\}\}.$$

One checks that condition (6) is preserved by δ. This defines the complex (A_U, δ) of U-local elements of (A, δ). The Alexander Spanier homology $H_*(X, \mathbb{R})$ is obtained as the projective limit $\varprojlim H^*(A_U, \delta)$, when U runs through all open neighborhoods of the diagonal.

Given a measure space (X, ν) and a measurable hermitian vector bundle Λ on X, the Hilbert-Schmidt operators in $\mathcal{H} = L^2(X, \Lambda)$ are all given [18] by measurable kernels,

$$k(x,y) \in \mathrm{Hom}(\Lambda_y, \Lambda_x) \qquad x, y \in X \tag{4}$$

such that:

$$\int_X \mathrm{trace}(k(x,y)^* \, k(x,y)) \, d\nu(x) \, d\nu(y) < \infty. \tag{5}$$

In particular, for any such kernel k the following expression defines a measure σ on X^{d+1} for any $d \geq 1$:

$$\sigma(\varphi) = \int_{X^{d+1}} \mathrm{trace}(k(x_0, x_1) \, k(x_1, x_2) \ldots k(x_d, x_0)) \, \varphi(x_0, \ldots, x_d) \prod d\nu(x_i) \tag{6}$$

as follows from the inequality:

$$|\sigma(\varphi)| \leq \|k\|_2^{d+1} \, \|\varphi\|_\infty \qquad \forall \varphi \in L^\infty(X^{d+1}, \nu^{d+1}). \tag{7}$$

We shall use the notation:

$$\mathrm{trace}(\wedge^{d+1} k) = \text{ Total antisymmetrisation of } \sigma \tag{8}$$

(where σ is associated to k by (6)).

For $d = 0$ this formula continues to make sense provided the operator in $\mathcal{H} = L^2(X, \Lambda)$ associated to k is of *trace class* [18].

We can now state the main result of this paper.

Theorem 2. *Let M be a compact oriented quasiconformal manifold of even dimension 2ℓ, γ the $\mathbb{Z}/2$ grading of $\mathcal{H} = L^2(M, \wedge^\ell T^*_{\mathbb{C}})$ associated to a measurable bounded conformal structure $[g]$ on M and U a neighborhood of the diagonal in $M \times M$.*

1) There exists a locally constructed U-local Hodge decomposition H.

2) Let H be a U-local Hodge decomposition, $k = H\gamma H + \gamma$ and d an even integer. Then the measure $\sigma = \mathrm{trace}(\wedge^{d+1} k)$ in a U^d local Alexander Spanier cycle of dimension d.

3) The homology class of σ among U^q local cycles, $q = d(6\ell + 2)$ is independent of the choice of H.

4) The homology class of σ is equal to $\lambda_d(L_{2\ell - d} \cap [M])$ where L is the Hirzebruch-Thom L-class and where $\lambda_d = 2^{2m+1} \, (2\pi i)^{-m} \, \frac{m!}{(2m)!}$, $2m = d$.

3. Local construction of a U-local Hodge decomposition

Let M be a quasiconformal manifold and $[g]$ a bounded measurable conformal structure on M. In this section we shall show how to construct local Hodge decompositions H using a covering of M by domains of qc local charts:

$$\rho_\alpha : V_\alpha \to S^{2\ell}.$$

The obtained formula for H will be algebraic in terms of the following ingredients:
1) A partition of unity subordinate to the covering (V_α) of M.
2) The pull back by ρ_α of Hodge decompositions on $S^{2\ell}$ associated to a bounded measurable conformal structure which agrees with $\rho_\alpha[g]$ on $\rho_\alpha(V_\alpha)$.

We shall begin by describing the canonical Hodge decomposition on $S^{2\ell}$ associated to a bounded measurable conformal sructure.

α) *Canonical Hodge decomposition on* $S^{2\ell}$

Let $[g_0]$ be the standard conformal structure on the sphere $S^{2\ell}$, and $[g]$ an arbitrary bounded measurable conformal structure on $S^{2\ell}$. Let γ_0, γ be the corresponding $*$ operations in the vector bundle Λ of middle dimensional forms. If we let $\Lambda_\pm$ be the two eigenspaces of γ_0 we get two subbundles of Λ and a unique measurable bundle homomorphism:

$$\mu_+ : \Lambda_- \to \Lambda_+$$

whose graph at each point $p \in S^{2\ell}$ gives the subspace:

$$\{\omega \in \Lambda_p \ ; \ \gamma\omega = -\omega\}.$$

We endow the vector bundle Λ with the metric associated to the standard conformal structure $[g_0]$. The boundedness of the measurable conformal structure $[g]$ then means that:

$$\|\mu_+\|_\infty = \operatorname*{Sup}_{S^{2\ell}} \ |\mu_+(p)| < 1. \tag{1}$$

Let $\mu = \begin{bmatrix} 0 & \mu_+ \\ \mu_+^* & 0 \end{bmatrix}$ viewed as an endomorphism of the vector bundle Λ. One has:

$$\mu\ \gamma_0 = -\gamma_0\ \mu \ , \ \mu = \mu^* \tag{2}$$

$$\gamma = (1+\mu)\ \gamma_0\ (1+\mu)^{-1}. \tag{3}$$

Indeed $\gamma^2 = 1$ and if $\gamma_0\ \xi = -\xi$ one has $\gamma(1+\mu)\xi = -(1+\mu)\xi$ as desired.

Finally on the vector bundle Λ the metric associated to the conformal structure $[g]$ is given by:

$$(4) \qquad \langle \omega_1, \omega_2 \rangle_g = \langle \omega_1, \gamma_0 \ \gamma \ \omega_2 \rangle_{g_0} = \left\langle \omega_1, \left(\frac{1-\mu}{1+\mu}\right) \omega_2 \right\rangle_{g_0}.$$

Note that since $\|\mu\| < 1$ the operator $(1-\mu)(1+\mu)^{-1}$ is positive. We now consider the Hilbert space $\mathcal{H}_0 = L^2(S^{2\ell}, \Lambda)$ with the inner product given by $[g_0]$. We view all the above endomorphisms of the vector bundle Λ as operator in $\mathcal{H}_0$. The equalities (2) and (3) continue to hold.

The standard Hodge decomposition on $S^{2\ell}$ decomposes $\mathcal{H}_0$ as the direct sum of two orthogonal subspaces, the exact forms and the coexact forms. Let H_0 be the linear operator such that $H_0 \ \omega = \omega$ for any exact form and $H_0 \ \omega = -\omega$ for any coexact form. One has:

$$(5) \qquad H_0 = H_0^* \ , \ H_0^2 = 1$$

$$(6) \qquad H_0 \ \gamma_0 = -\gamma_0 \ H_0.$$

Moreover since H_0 is a standard singular integral operator of order 0 the following subalgebra $A(S^{2\ell})$ of the algebra of continuous functions $C(S^{2\ell})$ contains all smooth functions [2] and is therefore norm dense:

$$(7) \qquad A(S^{2\ell}) = \{f \in C(S^{2\ell}) \ ; \ [H_0, f] \in \mathcal{L}^{(2\ell,\infty)}\}$$

where $f \in C(S^{2\ell})$ is considered as a multiplication operator in $L^2(S^{2\ell}, \Lambda)$ and recall $\mathcal{L}^{(2\ell,\infty)}$ is the two sided ideal of compact operators in $\mathcal{H}_0$ given by the condition:

$$(8) \qquad T \in \mathcal{L}^{(2\ell,\infty)} \Leftrightarrow \mu_n(T) = O(n^{-1/2\ell})$$

where $\mu_n(T)$ is the n-th characteristic value of T.

Let us now consider on the locally convex vector space $\mathcal{H}_0$ the new inner product given by the metric g, using (4) this can be expressed by:

$$(9) \qquad \langle \omega_1, \omega_2 \rangle = \left\langle \omega_1, \frac{1-\mu}{1+\mu} \ \omega_2 \right\rangle_0.$$

The Hodge decomposition on $S^{2\ell}$ relative to the bounded measurable conformal structure $[g]$ is given by:

Proposition 1. a) *The orthogonal complement for the inner product* (9) *of* $\mathrm{Im}\, d = \{\omega \ ; \ H_0\, \omega = \omega\}$ *is equal to* $\gamma(\mathrm{Im}\, d)$.
b) *Let* H *be the linear operator equal to* 1 *on* $\mathrm{Im}\, d$ *and to* -1 *on* $\gamma(\mathrm{Im}\, d)$, *then:*

$$H = (1-\mu)^{-1}\,(H_0 - \mu)\, H_0 (H_0 - \mu)^{-1}\,(1-\mu).$$

Proof. Since $H_0^2 = 1$ and $\|\mu\| < 1$ the operator $H_0 - \mu$ is invertible. Let us consider the operator $T = (1-\mu)^{-1}\,(H_0-\mu)\,H_0(H_0-\mu)^{-1}\,(1-\mu)$. It is conjugate to H_0 so that $T^2 = 1$. Its eigenspaces are obtained from those of H_0 by applying $(1-\mu)^{-1}\,(H_0-\mu)$. Hence $\{\xi \ ; \ T\xi = \xi\} = \mathrm{Im}\, d$. One has $\gamma = (1-\mu)^{-1}\,\gamma_0(1-\mu)$ and hence $\gamma T = -T\gamma$ which shows that $\{\xi \ ; \ T\xi = -\xi\} = \gamma(\mathrm{Im}\, d)$. The orthogonality of $\mathrm{Im}\, d$ with $\gamma_0(\mathrm{Im}\, d)$ for the inner product $\langle\ \rangle_0$ implies the orthogonality of $\mathrm{Im}\, d$ with $\gamma(\mathrm{Im}\, d)$ for the inner product (9), using $\gamma_0\gamma = \frac{1-\mu}{1+\mu}$. Thus we have shown a) and b).

Corollary 2. *For any* $f \in L^\infty(S^{2\ell})$ *and any two sided ideal* J *of operators in* $\mathcal{H}_0$ *one has:*

$$[H_0, f] \in J \Leftrightarrow [H, f] \in J.$$

Proof. Since μ commutes with f and $[H, \cdot]$ satisfies Leibniz rule, direct calculation yields

$$[H, f] = -(1+\mu)(H_0-\mu)^{-1}\,[H_0, f](H_0-\mu)^{-1}\,(1-\mu). \tag{10}$$

Corollary 3. *For any two sided ideal* J *of operators in Hilbert space the class of functions* $f \in L^\infty(S^{2\ell})$ *such that* $[H_0, f] \in J$ *is invariant under qc-homeomorphisms.*

Proof. Let φ be a qc-homeomorphism of $S^{2\ell}$ then φ is a.e. differentiable and it defines a bounded operator $U(\varphi)$ in $\mathcal{H}_0 = L^2(S^{2\ell}, \Lambda)$ by the formula:

$$U(\varphi)\,\omega = (\varphi^{-1})^*\,\omega \qquad \forall \omega \in \mathcal{H}_0. \tag{11}$$

This yields a bounded operators in $\mathcal{H}_0$ such that

$$U(\varphi)\, f\, U(\varphi)^{-1} = f \circ \varphi^{-1} \qquad \forall f \in L^\infty(S^{2\ell}) \tag{12}$$

$$U(\varphi)\, H_0\, U(\varphi)^{-1} = H_g \tag{13}$$

where $[g]$ is the mb conformal structure $(\varphi^{-1})^*\,[g_0]$. To prove (13) note first that $U(\varphi)\ \mathrm{Im}\, d = \mathrm{Im}\, d$ (cf. [1]), while $\langle U(\varphi)\,\omega_1, U(\varphi)\,\omega_2\rangle_0 = \langle \omega_1, \omega_2\rangle_g \quad \forall \omega_1, \omega_2 \in \mathcal{H}_0$. Thus $U(\varphi)\, H_0\, U(\varphi)^{-1}$

is equal to 1 on $\mathrm{Im}\, d$ and to -1 on its orthogonal complement for the inner product $\langle\ \rangle_g$. Hence $U(\varphi)\ H_0\ U(\varphi)^{-1} = H_g$. Let then $f \in L^\infty(S^{2\ell})$, if $[H_0, f] \in J$ then $[U(\varphi)\ H_0\ U(\varphi)^{-1}\ ,\ U(\varphi)\ f\ U(\varphi)^{-1}] \in J$ and $[H_g, f\circ\varphi^{-1}] \in J$ so that $[H_0, f\circ\varphi^{-1}] \in J$ by the above corollary.

We shall now see that, modulo the ideal $J = \mathcal{L}^{(2\ell,\infty)}$, the class of the operator H_g is locally determined by the bounded measurable conformal structure $[g]$.

Proposition 4. *Let $U \subset S^{2\ell}$ be an open subset and f_1, f_2 be continuous functions with support in U. Let g_1, g_2 be two bm conformal structures on $S^{2\ell}$ which agree on U. Then $f_1(H_{g_1} - H_{g_2})f_2 \in \mathcal{L}^{(2\ell,\infty)}$.*

Proof. We can replace f_i by smooth functions equal to 1 on the support of the previous ones. Thus we can assume that $f_i \in A(S^{2\ell})$. By proposition 1 both operators H_{g_i} are the sum of a geometrically norm convergent series:

$$H_{g_i} = (1-\mu_i)^{-1}\ (H_0 - \mu_i)\sum_0^\infty(\mu_i\ H_0)^n\ (1-\mu_i) \tag{14}$$

whose terms are monomials of the form

$$a_{1,i}\ H_0\ a_{2,i}\ H_0 \ldots a_{n,i}\ H_0\ a_{n+1,i} = T_{n,i} \tag{15}$$

where $a_{k,i}$ belongs to the commutant of $L^\infty(S^{2\ell})$ in $\mathcal{H}_0$ and where $f_1\ a_{k,1} = f_1\ a_{k,2}$ for all k. It follows using $[H_0, f_1] \in \mathcal{L}^{(2\ell,\infty)}$ that

$$f_1(T_{n,1} - T_{n,2}) \in \mathcal{L}^{(2\ell,\infty)} \tag{16}$$

One thus expresses $f_1(H_{g_1} - H_{g_2})$ as the sum of a series convergent in the Banach space $\mathcal{L}^{(2\ell,\infty)}$, and the conclusion follows.

β) The class of H modulo $\mathcal{L}^{(2\ell,\infty)}$

Let now M be a quasiconformal manifold of dimension 2ℓ.

Let $A(M)$ be the subalgebra of $C(M)$ of functions f such that for any qc local chart: $V \xrightarrow{\rho} S^{2\ell}$ and any $h \in C_c(\rho(V)) \cap A(S^{2\ell})$ one has $h\ f\circ\rho^{-1} \in A(S^{2\ell})$. By corollary 3 one has $\rho^*(A(S^{2\ell}) \cap C_c(\rho(V))) \subset A(M)$ for any qc local chart $V \xrightarrow{\rho} S^{2\ell}$. It follows that $A(M)$ is a norm dense subalgebra of $C(M)$ and that it has partitions of unity subordinate to any finite open covering of M.

Let $[g]$ a bounded measurable conformal structure on M.

Let (V_α) be a finite open cover of M by domains of qc local charts $\rho_\alpha : V_\alpha \to S^{2\ell}$ and g_α a bounded measurable conformal structure on $S^{2\ell}$ which agrees with $\rho_\alpha[g]$ on $\rho_\alpha(V_\alpha)$. For each α let $H_\alpha = H_{g_\alpha}$ be the corresponding Hodge decomposition on $S^{2\ell}$. Let (φ_α) be a partition of unity, $\varphi_\alpha \in A(M)$, support $\varphi_\alpha \subset V_\alpha$ and for each α let $\psi_\alpha \in A(M)$ be equal to 1 in a neighborhood of Support φ_α, with Support $\psi_\alpha \subset V_\alpha$. We let as above Λ be the measurable vector bundle $\Lambda^\ell\, T^*_{\mathbb{C}}$ on M and we consider the following locally constructed operator in $L^2(M, \Lambda)$:

$$S = \Sigma\, \psi_\alpha\, (H_\alpha)^{\rho_\alpha}\, \varphi_\alpha \tag{17}$$

where we used ρ_α to let H_α act in $L^2(M, \Lambda)$.

Proposition 5. *The class of S modulo $\mathcal{L}^{(2\ell,\infty)}$ only depends upon the bm conformal structure $[g]$ on M and one has:*

1) $S\gamma + \gamma S \in \mathcal{L}^{(2\ell,\infty)}$

2) $[S, f] \in \mathcal{L}^{(2\ell,\infty)} \quad \forall f \in A(M)$

3) $S^2 - 1 \in \mathcal{L}^{(2\ell,\infty)}$

4) $(S-1)/\operatorname{Im} d \in \mathcal{L}^{(2\ell,\infty)}$.

Proof. We first need to show that $S - S' \in \mathcal{L}^{(2\ell,\infty)}$ for any two operators S, S' constructed by formula 17. Using the compactness of M it is enough to show that for any $x \in M$ there exists $f \in A(M)$, $f(x) \neq 0$ such that $(S - S')f \in \mathcal{L}^{(2\ell,\infty)}$. Using a local qc chart the proof follows from proposition 4. To check 1) note that any of the operators S given by 17 satisfies $S\gamma + \gamma S = 0$.

The condition 2) follows from Corollary 3.

To check 3) we take a representative $S = \Sigma\, \psi_\alpha\, H_\alpha\, \varphi_\alpha$ constructed from a covering (V_α) by domains of qc charts, such that for any α the open set $\cup V_\beta$, $V_\beta \cap V_\alpha \neq \emptyset$ is the domain of a qc chart. The result then follows from corollary 3 and proposition 4.

Let us check 4). On the sphere $S^{2\ell}$ the operators H_g, g a bounded measurable conformal structure, are equal to 1 on $\operatorname{Im} d$ (proposition 1). Thus, using corollary 3, the operator $S = \Sigma\, \psi_\alpha\, H_\alpha\, \varphi_\alpha$ satisfies $(S-1)/E_\alpha \in \mathcal{L}^{(2\ell,\infty)}$ where E_α is the closure in $L^2(M,\Lambda))$ of $\{d\omega\ ;\ \omega \in L^{2\ell/\ell-1}(V_\alpha, \Lambda)\}$. Thus 4) follows if we show that the map $(\omega_\alpha) \in \oplus E_\alpha \to \Sigma\, \omega_\alpha \in L^2(M, \Lambda)$ is surjective on $\operatorname{Im} d$. Let P_α (resp. $Q_\alpha = \gamma\, P_\alpha\, \gamma$) be the orthogonal projection on E_α (resp. $\gamma\, E_\alpha$). For any α, β the closed subspaces $E_\alpha \subset \operatorname{Im} d$ and $\gamma\, E_\beta \subset \gamma(\operatorname{Im} d)$ are orthogonal. Thus it is enough to show that the following operator is equal to 1+ compact:

$$T = \Sigma\, (P_\alpha + Q_\alpha)\, \varphi_\alpha.$$

(Its range is then closed by the Fredholm theory.)

Let us show that for each α the operator $(P_\alpha + Q_\alpha)\, \varphi_\alpha - \varphi_\alpha$ is compact. We can assume that $M = S^{2\ell}$ is the sphere. By proposition 1 a) one has the orthogonal decomposition

$L^2(S^{2\ell},\Lambda) = \operatorname{Im} d \oplus \gamma\ \operatorname{Im} d$, where $d : L^{2\ell/\ell-1}(S^{2\ell},\Lambda^{\ell-1}) \to L^2(S^{2\ell},\Lambda^\ell)$ has closed range. (We assume $\ell \neq 1$.)

Using a compact operator $R : L^2(S^{2\ell},\Lambda) \to L^{2\ell/\ell-1}(S^{2\ell},\Lambda^{\ell-1})$ such that $dR = 1$ on $\operatorname{Im} d$ we thus get a pair of compact operators $R_j : L^2(S^{2\ell},\Lambda) \to L^{2\ell/\ell-1}(S^{2\ell},\Lambda^{\ell-1})$ such that $d\,R_1 + \gamma\ d\,R_2 = 1$. The conclusion then follows using the following equality, with ψ a smooth function with support in V_α and equal to 1 in a neighborhood of Support φ_α:

$$\psi\,\omega = d(\psi\,\omega_1) + \gamma\ d(\psi\,\omega_2) - d\,\psi \wedge \omega_1 - \gamma(d\,\psi \wedge \omega_2) \qquad \omega_i = R_i\,\omega.$$

4. Proof of Theorem 2.

Proof of 1). First choose a neighborhood V of the diagonal such that $V^{2q} \subset U$, $q = 6\ell+2$. Next (proposition 5) let S be an operator such that:

a) Support $(S) \subset V$

b) $S^2 - 1 \in \mathcal{L}^{(2\ell,\infty)}$

c) $S\gamma = -\gamma S$

d) $(S-1)/\operatorname{Im} d \in \mathcal{L}^{(2\ell,\infty)}$.

Let then $\theta = S^2 - 1$. By construction θ commutes with S, it also commutes with γ by c). Let $q(t)$ be the unique polynomial of degree 2ℓ such that:

$$(1) \qquad (1+t)^{-1/2} = q(t) + O(t^{2\ell+1}) \qquad \text{(for } t \text{ small)}.$$

Let $p(t)$ be the polynomial given by:

$$(2) \qquad p(t) = (1+t)\,q(t)^2 - 1.$$

We shall define an operator H by the formula:

$$(3) \qquad H = \gamma\ p(\theta) + \left(1 - \left(\frac{1+\gamma}{2}\right)\ p(\theta)\right)\ q(\theta)\ S.$$

First note that θ is V^{02} local, thus since q (resp. p) has degree 2ℓ (resp. $4\ell+1$) we see that T is U local as required. Next, as $\theta \in \mathcal{L}^{(2\ell,\infty)}$ by b), and as $p(t) = O(t^{2\ell+1})$ we see that $p(\theta) \in \mathcal{L}^1$. We thus have:

$$(4) \qquad H\gamma + \gamma H = 2p(\theta) \in \mathcal{L}^1$$

where we used c) to get the equality.

Since $p(\theta) \in \mathcal{L}^1$ and $\theta \in \mathcal{L}^{(2\ell,\infty)}$ we have $S - H \in \mathcal{L}^{(2\ell,\infty)}$ and hence, using d),

(5) $$(H - id)/\operatorname{Im} d \in \mathcal{L}^{(2\ell,\infty)}.$$

It remains to check that $H^2 = 1$. Note first that the two terms $\gamma\, p(\theta)$, $H - \gamma\, p(\theta)$ anticommute so that H^2 is the sum of their squares:

$$\begin{aligned} H^2 &= (\gamma\, p(\theta))^2 + q(\theta)^2 \left(1 - \frac{1+\gamma}{2}\, p(\theta)\right) \left(1 - \frac{1-\gamma}{2}\, p(\theta)\right) S^2 \\ &= p(\theta)^2 + q(\theta)^2 \,(1 - p(\theta))\,(1+\theta) = 1. \end{aligned}$$

Before we begin the proof of 2) we recall that the cyclic complex (C^*_λ, b) of an algebra $\mathcal{A}$ is given by:

(6) $$\begin{aligned} C^n_\lambda(\mathcal{A}) = \{&\text{multilinear forms } \tau \text{ on } \mathcal{A} \times \ldots \times \mathcal{A} \quad (n+1 \text{ times}) \\ &\text{such that } \tau(a^1, \ldots, a^n, a^0) = (-1)^n\, \tau(a^0, \ldots, a^n) \qquad \forall a^j \in \mathcal{A}\} \end{aligned}$$

(7) $$\begin{aligned} (b\tau)\,(a^0, \ldots, a^{n+1}) &= \sum_0^n (-1)^j\, \tau(a^0, \ldots, a^j\, a^{j+1}, \ldots, a^{n+1}) \\ &+ (-1)^{n+1}\, \tau(a^{n+1}\, a^0, \ldots, a^n) \qquad \forall a^j \in \mathcal{A}. \end{aligned}$$

We note also that if J is an ideal in a larger algebra $\mathcal{A}$ and if $B \subset \mathcal{A}$ is a subalgebra such that $J \cap B = \{0\}$, then the natural extension by 0 on B of the cochains $\tau \in C^*_\lambda(J)$ satisfying:

(8) $$\begin{aligned} \tau(a^0, a^1, \ldots, a^j\, \delta, a^{j+1}, \ldots, a^n) &= \tau(a^0, a^1, \ldots, a^j, \delta\, a^{j+1}, \ldots, a^n) \\ &\forall a^i \in \mathcal{A},\ \forall j,\ \forall \delta \in B \end{aligned}$$

commutes with the coboundary b of $C^*_\lambda(J + B)$.

Proof of 2). It follows from [3] lemma 2. But we shall give the details here. By construction $\sigma =$ trace $(\wedge^{d+1} k)$ is U^{od} local. We need to show that $b\sigma = 0$. We shall first show that, with J the algebra of trace class operators in $\mathcal{H}$, the following formula defines a morphism of complexes: $(A^*, \delta) \xrightarrow{\tau} (C^*_\lambda(J), b)$, where (A^*, δ) is the complex of bounded measurable totally antisymmetric functions (straight cochains) on M.

(9)
$$\tau_\varphi(k^0,\dots,k^n) =$$
$$(-1)^n \int_{M^{n+1}} \text{trace}\ (k^0(x_0,x_1)\ k^1(x_1,x_2)\dots k^n(x_n,x_0)\ \varphi(x_0,\dots,x_n) \prod d\nu(x_i)$$
$$\forall \varphi \in A^n.$$

The coboundary δ in (A^*,δ) is given by:

(10)
$$\delta\varphi = \sum_0^{n+1} (-1)^j\ \varphi_j\ ,\ \varphi_j(x_0,\dots,x_{n+1}) = \varphi(x_0,\dots,\check{x}_j,\dots,x_{n+1}).$$

Next, when one computes τ_{φ_j}, $j \geq 1$, the variable x_j does not occur in φ_j and thus one gets:

(11)
$$\tau_{\varphi_j}(k^0,\dots,k^{n+1}) = -\tau_\varphi(k^0,\dots,k^{j-1}\ k^j,\dots,k^{n+1}).$$

For $j = 0$ it is the variable x_0 which does not occur in φ_0 and using the cyclicity of the trace one gets, using the antisymmetry of φ:

(12)
$$\tau_{\varphi_0}(k_0,\dots,k^{n+1}) = (-1)^{n+1}\ \tau_\varphi(k^{n+1}\ k^0,k^1,\dots,k^n).$$

Thus using 10 we get:

(13)
$$\tau_{\delta\varphi} = b\ \tau_\varphi \qquad \forall \varphi \in A^n.$$

Next note that the compatibility condition (8) is fulfilled by any element of the algebra $B = \{\lambda_0 + \lambda_1\ \gamma\ ;\ \lambda_j \in \mathbf{C}\}$ generated by the operator γ in $\mathcal{H}$. We shall still denote by τ the extension of the above morphism of complexes to $(C_\lambda^*(\mathcal{A}),b)$, $\mathcal{A} = J + B$.

Let us now check that $b\sigma = 0$, i.e. that $\sigma(\delta\varphi) = 0$ for any $\varphi \in A^d$. Using (13) we know that the cyclic cocycle $\tau_{\delta\varphi}$ is a coboundary and thus that it vanishes when evaluated on an idempotent:

(14)
$$\tau_{\delta\varphi}(P,P,\dots,P) = 0 \qquad \forall P\ ,\ P^2 = P\ ,\ P \in \mathcal{A}.$$

Applying this to $P = H\ \frac{1-\gamma}{2}\ H = \frac{1}{2}\ (\frac{1+\gamma}{2}) + \frac{1}{2}\ k$ gives the desired result.

Proof of 3). Let H_0, H_1 be two U-local Hodge decompositions and $Y = H_1 - H_0$. We have:

(15) $$Y/\operatorname{Im} d \in \mathcal{L}^{(2\ell,\infty)}$$

(16) $$Y\gamma + \gamma Y \in \mathcal{L}^1.$$

As $\operatorname{Im} d \oplus \gamma(\operatorname{Im} d)$ is of finite codimension in $\mathcal{H}$ it follows that:

(17) $$Y \in \mathcal{L}^{(2\ell,\infty)}.$$

Let then $H_t = H_0 + tY$. We have:

(18) $$\text{Support } H_t \subset U$$

(19) $$H_t^2 - 1 \in \mathcal{L}^{(2\ell,\infty)}$$

(20) $$(H_t - 1)/\operatorname{Im} d \in \mathcal{L}^{(2\ell,\infty)}$$

(21) $$H_t\, \gamma + \gamma\, H_t \in \mathcal{L}^1.$$

It follows that $S_t = \frac{1}{2}(H_t - \gamma\, H_t\, \gamma)$ anticommutes with γ and satisfies (18) (19) (20). Thus with q and p as in (1) (2) we get a family H_t' of U^n local Hodge decompositions:

(22) $$H_t' = \gamma\, p(\theta_t) + \left(1 - \left(\frac{1+\gamma}{2}\right)\, p(\theta_t)\right)\, q(\theta_t)\, S_t \qquad \theta_t = S_t^2 - 1.$$

For $t = 0$ (or for $t = 1$), the operator $\theta_0 = S_0^2 - 1$ belongs to $\mathcal{L}^1$ we can thus, keeping the relation (2), replace the polynomial q by $1 + \lambda(q-1)$, $\lambda \in [0,1]$ and still get a family of U^n local Hodge decompositions joining H_0' (for $\lambda = 1$) with

(23) $$H_0'' = \gamma\, \theta_0 + \left(1 - \left(\frac{1+\gamma}{2}\right)\, \theta_0\right)\, S_0\,,\ \theta_0 = S_0^2 - 1\,,\ S_0 = \frac{1}{2}\,(H_0 - \gamma\, H_0\, \gamma).$$

Since the obtained path of idempotents (H_t' $(\frac{1-\gamma}{2})$ H_t') in $\mathcal{A}$ are piecewise polynomial it follows that the corresponding straight cycles are U^n homologous. This is, using (13), a restatement of the homotopy invariance of the pairing of K-theory with cyclic cohomology ([5]).

It remains to compare in the same way H_0'' with H_0. One has by hypothesis $S_0 - H_0 \in \mathcal{L}^1$, $\theta_0 \in \mathcal{L}^1$ thus:

$$H_0'' - H_0 \in \mathcal{L}^1. \tag{24}$$

It follows that the idempotents

$$e = H_0 \left(\frac{1-\gamma}{2}\right) H_0 \ , \ e'' = H_0'' \left(\frac{1-\gamma}{2}\right) H_0''$$

satisfy the equality $e'' = W\, e\, W^{-1}$ where $W = H_0''\, H_0 \in 1 + J$ as well as W^{-1}, while W is U^2 local as well as W^{-1}. One then considers the following smooth path of idempotents $e(t) \in M_2(\mathcal{A})$ connecting $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ with $\begin{bmatrix} 0 & 0 \\ 0 & e'' \end{bmatrix}$ and with support in U^3:

$$e(t) = \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} R_t \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} R_t^{-1} \begin{bmatrix} 1 & 0 \\ 0 & W^{-1} \end{bmatrix}$$

where R_t is a rotation matrix.

Using again the homotopy invariance of the pairing of K-theory with cyclic cohomology one gets the desired result.

Proof of 4). We shall first check directly that $\int_M k(x,x) = 2\ \mathrm{Sign}(M)$. Since the Alexander Spanier cocycle given by the constant function $\varphi(x^0) = 1$ is everywhere defined, the proof of 3) shows that $\int_M k(x,x)$ is independent of the choice of the Hodge decomposition H, without any U-locality hypothesis. We can thus choose $H = 2P - 1$ where P is the orthogonal projection on the closed subspace $\mathrm{Im}\, d$. One has $P + \gamma\ P\ \gamma + K = 1$ where K is the harmonic projection ([1]). Thus one gets:

$$H\ \gamma\ H + \gamma = 2\ K\ \gamma$$

$$\mathrm{Trace}\ (H\ \gamma\ H + \gamma) = 2\ \mathrm{Trace}\ (K\gamma) = 2\ \ \mathrm{Sign}(M).$$

To compute the other homology classes let us first assume that M is a smooth manifold. By [3] these classes ω_{2q} represent 2^{2q+1} times the Chern character of the K-homology class of the operator $H = 2P - 1$, with P as above.

Thus it is enough to show that the K-theory class of the symbol of H, $[\sigma(H)] \in K^0(T^*M)$ is the same as the K-theory class of the symbol of the signature operator. The latter is given by the odd endomorphism $s(x,\xi) = e_\xi + i_\xi$, $\xi \in T_x^*(M)$ of the pull back of $\wedge^*\, T_{\mathbb{C}}^*$ (oriented by $\gamma = i^\ell * (-1)^{\frac{p(p-1)}{2}}$) to T^*M. (Here e_ξ, i_ξ are respectively exterior and interior multiplication by ξ.) The symbol $\sigma(H)$ of H is the same as the symbol of $2P - 1$

where P is the orthogonal projection on the image of d. Its restriction to the unit sphere $\{\xi \in T^*V \ , \ \|\xi\| = 1\}$ is thus given by:

$$\sigma(x,\xi) = e_\xi \, i_\xi - i_\xi \, e_\xi \quad \text{acting on} \quad \wedge^\ell \ T^*_{\mathbf{C}}.$$

Let $u(x,\xi) = \frac{1}{\sqrt{2}} \, (1 - e_\xi + i_\xi)$, then for $\|\xi\| = 1$, it is an invertible operator in $\wedge^* \ T^*_{\mathbf{C}}$ with inverse $u^{-1}(x,\xi) = \frac{1}{\sqrt{2}} \, (1 + e_\xi - i_\xi)$. One has

$$(usu^{-1})(x,\xi) = e_\xi \, i_\xi - i_\xi \, e_\xi \quad \text{acting on} \quad \wedge^* \ T^*_{\mathbf{C}}.$$

By construction u commutes with γ. This shows that the K-theory class $[s] - [\sigma]$ is given by the symbol:

$$\rho(x,\xi) = e_\xi \, i_\xi - i_\xi \, e_\xi \quad \text{acting on} \quad \wedge^* \ T^*_{\mathbf{C}} \ominus \wedge^\ell \ T^*_{\mathbf{C}},$$

with the $\mathbf{Z}/2$ grading γ. But using the canonical isomorphism, for $p \neq \ell$,

$$\wedge^p \ T^*_{\mathbf{C}} \simeq (\wedge^p \ T^*_{\mathbf{C}} \oplus \wedge^{2\ell-p} \ T^*_{\mathbf{C}})^{\pm} \ \ ; \ \ \omega \to \tfrac{1}{2}(\omega \pm \gamma \, \omega)$$

one checks that the class of ρ is equal to 0.

This completes the proof the classes we construct agree with Hirzebruch-Thom classes in the smooth case. The extension from the smooth case to the qc case can be done using cobordism as in [6]. One can also use K-theory as in [21].

Remarks. 1) Let M be a compact quasiconformal manifold, one can characterize the subalgebra $A(M)$ of $C(M)$ (section 3) by the following simple criterion

$$f \in A(M) \Leftrightarrow df \in L^{2\ell}(M, T^*_{\mathbf{C}}).$$

A proof of $\Rightarrow$ follows from [25] and the converse from [21] plus an identification of the Besov space there with this Sobolev space. It is sufficient by §3 to work with the standard singular integral operator on the sphere. For a more complete discussion see the Appendix.

2) Let M be a polyhedron of dimension 2ℓ with homological properties to be specified. Let $\mathcal{H} = \oplus \mathcal{H}_\alpha$ be the direct sum of the Hilbert spaces $\mathcal{H}_\alpha$ of square integrable ℓ forms on the 2ℓ dimensional simplices. For each vertex v let E_v be the closed subspace of $\mathcal{H}$ of boundaries of forms with support in the double star of v. Let γ be the $*$ operation on $\mathcal{H}$ given by the canonical flat metric on each 2ℓ-simplex (with equilateral length). Let b_v be the barycentric coordinate assigned to the vertex v. Then let us consider the following operator:

$$\sum_v (P_v - \gamma \, P_v \, \gamma) \, b_v = S$$

where P_v is the orthogonal projection on E_v. It is clear that S is localised, is equal to 1, modulo $\mathcal{L}^{(2\ell,\infty)}$, on the image of d and anticommutes with γ. Now we assume that E_v and $\gamma(E_v)$ are a local decomposition, up to compacts i.e. $(1 - P_v - \gamma \, P_v \, \gamma) / * \oplus h_\alpha$

is compact of forms on the (first star). These are the prescribed homological conditions. Then $S^2 - 1$ is in $\mathcal{L}^{(2\ell,\infty)}$ and one can use the above formulae to define characteristic homology classes. The involved Hilbert space theoretical data are of the same nature as those appearing in transfer matrix theory of statistical mechanics and suggest a purely combinatorial approach to the K-orientation of [4] in the extended context of spaces with singularities.

Appendix. (with Stephen Semmes) An interesting analytical point.

There is an interesting issue concerning operator theory and classical analysis which is related to the topics of this paper and which does not seem to have been treated in the literature. For the sake of clarity we discuss this issue in a restricted setting. Let T be a 0^{th}-order pseudodifferential operator on $\mathbf{R}^d$, $d > 1$, which we also assume to be translation and dilation invariant and nonzero. Thus T could be represented by a Fourier multiplier which is homogeneous of degree 0, and T is a bounded linear operator on $\mathcal{H} = L^2(\mathbf{R}^d)$.

Under what conditions on a function $f(x)$ on $\mathbf{R}^d$ is it true that $[f,T] \in \mathcal{L}^{(d,\infty)}$ $(\mathcal{H})$? (Recall that $\mathcal{L}^{(d,\infty)}(\mathcal{H})$ denotes the space of compact operators A on $\mathcal{H}$ such that $\mu_n(A) = O(n^{-\frac{1}{d}})$, where $\mu_n(A)$ is the n^{th} eigenvalue of $(A^*A)^{\frac{1}{2}}$.) This type of question has been studied extensively (see [26], for instance), but this particular case involves a critical index and has some special features. It follows from [27] that

$$[f,T] \in \mathcal{L}^{(d,\infty)} \Leftrightarrow f \in Osc^{d,\infty}(\mathbf{R}^d) \quad \text{when } d > 1, \tag{1}$$

where $Osc^{d,\infty}(\mathbf{R}^d)$ is a variant of a Besov space whose definition will be reviewed soon. For many purposes it would be preferable to work with a Sobolev space instead of $Osc^{d,\infty}$. It was observed in (Theorem 2.2 on p 228 in) [28] that

$$W^{1,d}(\mathbf{R}^d) \subseteq Osc^{d,\infty}(\mathbf{R}^d) \quad \text{when} \quad d > 1, \tag{2}$$

where $W^{1,d}(\mathbf{R}^d)$ is the Sobolev space of locally integrable functions on $\mathbf{R}^d$ whose distributional first derivatives all lie in $L^d(\mathbf{R}^d)$. In fact we have the following

Theorem. *When $d > 1$, $W^{1,d}(\mathbf{R}^d) = Osc^{d,\infty}(\mathbf{R}^d)$, and so $[f,T] \in \mathcal{L}^{(d,\infty)}(\mathcal{H})$ if and only if $f \in W^{1,d}(\mathbf{R}^d)$.*

It is very important here that the dimension d is the same as the exponent in the function spaces; otherwise this theorem would not work. This theorem is rather surprising from the perspective of classical analysis, because Sobolev spaces normally coincide with Besov-type spaces only when the exponent is 2. Indeed, the second half of the theorem had been conjectured by Jaak Peetre.

We need only show that the inclusion opposite to (2) holds. Our original proof of this was obtained by understanding the Dixmier trace of $|[f,T]|^d$, which in fact reduces to $\int_{\mathbb{R}^d} |\nabla f(x)|^d \, dx$ for certain T. We shall sketch a more direct proof below.

Let us recall the definition of $Osc^{d,\infty}(\mathbf{R}^d)$. This space is a little bit nonstandard; at this critical index, the standard Besov space is distinct from this one.

Given $(x,t) \in \mathbf{R}^d \times \mathbf{R}_+ = \mathbf{R}_+^{d+1}$, let $B(x,t)$ denote the ball with center x and radius t. Let $m_{x,t}(g)$ denote the average of the locally-integrable function $g(y)$ on $\mathbf{R}^d$. For such a function we let $\Theta(x,t)$ denote its average oscillation on $B(x,t)$, which is given by

$$\Theta(x,t) = m_{x,t}(|g - m_{x,t}(g)|).$$

We shall define $Osc^{d,\infty}(\mathbf{R}^d)$ in terms of a global measurement of these localized oscillation quantities.

Let $\{(x_j,t_j)\}_j$ denote a reasonably thick hyperbolic lattice in $\mathbf{R}_+^{d+1}$: we require that every point in $\mathbf{R}_+^{d+1}$ be no further than 10^{-2} from some (x_j,t_j) in the hyperbolic metric and that no pair of the (x_j,t_j)'s are closer to each other than 10^{-3}. Thus the numbers $\Theta(x_j,t_j)$ measure the average oscillations of g at all possible locations and scales.

Let θ_n, $n = 1,2,3,\ldots$, denote the n^{th} largest value of $\Theta(x_j,t_j)$. In other words, we reorder the $\Theta(x_j,t_j)$'s in decreasing size. Then

$$g \in Osc^{d,\infty}(\mathbf{R}^d) \Leftrightarrow \theta_n = O(n^{-\frac{1}{d}}).$$

This definition does not depend on the particular choice of the lattice (x_j,t_j). (The point of (1), incidentally, is that the θ_n's for f can be related to the $\mu_n([f,T])$'s.)

The main step in the proof of the theorem is to show that if g is smooth then

$$\text{(3)} \qquad \left(\int_{\mathbb{R}^d} |\nabla g(x)|^d \, dx\right)^{\frac{1}{d}} \leq C \limsup_{n\to\infty} n^{\frac{1}{d}} \theta_n$$

for some constant C which does not depend on g. Once we know this, then we know that the $Osc^{d,\infty}(\mathbf{R}^d)$ norm controls the $W^{1,d}(\mathbf{R}^d)$ norm for smooth functions, and the proof of the theorem can be finished with a standard approximation argument (which we omit). (The main point is that if you convolve $g \in Osc^{d,\infty}(\mathbf{R}^d)$ with a function in $L^1(\mathbf{R}^d)$ with bounded norm, then you get a function in $Osc^{d,\infty}(\mathbf{R}^d)$ with bounded norm.) The proof of (3) that follows is pretty sketchy, but it would not be too difficult to give a detailed argument.

Let us first try to understand the right side of (3) better. For each $\lambda > 0$ let $N(\lambda)$ be the number of j's such that $\Theta(x_j,t_j) > \lambda$. Thus $\theta_{N(\lambda)} > \lambda$, and so

$$\text{(4)} \qquad \limsup_{n\to\infty} n^{\frac{1}{d}} \theta_n \geq \limsup_{\lambda\to 0} N(\lambda)^{\frac{1}{d}} \lambda.$$

Now let us try to understand the left side of (3). For the time being we shall work on some fixed large cube K. Let $\{Q_\ell\}$ be a partition of K into tiny cubes of sidelength s. If s is small enough, then ∇g will be almost constant on each Q_ℓ, because g is smooth. Let G_ℓ denote the approximate value of $|\nabla g|$ on Q_ℓ.

Set $\widehat{Q}_\ell = Q_\ell \times (0,s) \subseteq \mathbf{R}_t^{d+1}$. If $(x,t) \in \widehat{Q}_\ell$, then $\Theta(x,t)$ is approximately tG_ℓ. For each ℓ let $N_\ell(\lambda)$ denote the number of j's such that $(x_j,t_j) \in \widehat{Q}_\ell$ and $\Theta(x_j,t_j) > \lambda$. If $sG_\ell < \lambda$ then we should have that $N_\ell(\lambda)$ is approximately 0, while if $sG_\ell > \lambda$ then $N_\ell(\lambda)$ should be approximately the same as the number of j's such that $(x_j,t_j) \in \widehat{Q}_\ell$ and $t_j > \lambda\, G_\ell^{-1}$. Simple considerations of hyperbolic geometry imply that $N_\ell(\lambda)$ is roughly proportional to $\left(\frac{sG_\ell}{\lambda}\right)^d$ in this case. This is also about the same as what we got in the first case.

Since $N(\lambda) \geq \sum_\ell N_\ell(\lambda)$, we conclude that $N(\lambda)$ should dominate $\lambda^{-d} \sum_\ell (sG_\ell)^d$, modulo controllable errors. Form (4) we get that

$$\limsup_{n\to\infty} n^{\frac{1}{d}}\, \theta_n \quad \text{is roughly larger than} \quad \left(\sum_\ell (sG_\ell)^d\right)^{\frac{1}{d}}.$$

On the other hand, $\sum_\ell (sG_\ell)^d$ is just a Riemann sum for $\int_K |\nabla g(x)|^d\, dx$. In the limit we get that

$$\left(\int_K |\nabla g(x)|^d\, dx\right)^{\frac{1}{d}} \leq C \limsup_{n\to\infty} n^{\frac{1}{d}}\, \theta_n,$$

where C does not depend on g or K. This implies (3), and proves the theorem.

Notice, incidentally, that the proof of (3) works also when $d = 1$. However, the approximation argument that gives $Osc^{d,\infty}(\mathbf{R}^d) \subseteq W^{1,d}(\mathbf{R}^d)$ when $d > 1$ gives only $Osc^{1,\infty}(\mathbf{R}) \subseteq BV(\mathbf{R})$, where $BV(\mathbf{R})$ denotes the space of functions on $\mathbf{R}$ of bounded variation (i.e., whose distributional derivatives are finite measures). The reciprocal inclusion is false, because jump discontinuities are bad for $Osc^{1,\infty}(\mathbf{R})$.

References

[1] S. Donaldson and D. Sullivan, Quasiconformal Four Manifolds, *Acta Math.* Vol. 163 (1989), 181-252.

[2] T. Iwaniec and G. Martin, Quasiregular Mappings in even dimensions, Mittag-Leffler Inst., Report N^{0}19. 1989-1990, Acta Math. (to appear).

[3] A. Connes and H. Moscovici, Cyclic Cohomology, the Novikov Conjecture and Hyperbolic Groups, *Topology*, Vol. 29, 3 (1990), 345-388.

[4] D. Sullivan, Geometric Periodicity and the Invariants of Manifolds, Manifolds, Amsterdam Conf. 1970, Lecture Notes, Springer-Verlag 197 (1970), 44-75.

[5] A. Connes, Non Commutative Differential Geometry, Publ. I.H.E.S. 62 (1985), 41-144.

[6] N. Teleman, The Index Theorem on Topological Manifolds, *Acta Mathematica*, Vol. 153 (1984), 117-152.

[7] D. Sullivan, Hyperbolic Geometry and Homeomorphisms, *Geometric Topology Proceed.*, Georgia Topology Conf., Athens, Georgia (1977), 543-555.

[8] D. Sullivan and N. Teleman, An Analytical Proof of Novikov's Theorem on the Topological Invariance of the Rational Pontrjagin Classes, Publ. Math. I.H.E.S., Vol. 58 (1983), 291-295.

[9] R.C. Kirby, Stable Homeomorphisms and the Annulus Conjecture, *Annals of Math.* 89 (1969), 575-582.

[10] I.M. Singer, Future Extensions of Index Theory and Elliptic Operators, in Prospects of Mathematics, *Annals of Math. Studies* 70, Princeton (1971), 171-185.

[11] R.D. Edwards and R.C. Kirby, Deformations of Spaces of Imbeddings, *Annals of Math.* 93 (1971), 63-88.

[12] S.P. Novikov, Topological Invariance of Rational Pontrjagin Classes, Doklady A.N.S.S.S.R., Vol. 163 (1965), 921-923.

[13] R.C. Kirby and L. Siebenmann, Foundational Essays on Topological Manifolds, Smoothings and Triangulations, *Annals of Math. Studies*, Vol. 88 (1977), Princeton.

[14] F.W. Gehring, The L^p-integrability of the Partial Derivatives of a Quasiconformal Mapping, *Acta Mathematica*, 130 (1973), 265-277.

[15] R. Bott, Vanishing Theorem for Characteristic Class of Foliations.

[16] A. Connes, Non Commutative Geometry, book to appear in Academic Press (1993).

[17] D. Sullivan, Quasiconformal Homeomorphism in Dynamics, Topology and Geometry, ICM Berkeley (1986).

[18] B. Simon, Trace Ideals and Applications, *London Math. Soc. Lecture Notes* 35 , Cambridge Univ. Press (1979).

[19] M. Hilsum, The signature operator of Lipschitz manifolds and unbounded Kasparov bimodules, *Lecture Notes Math.* 1132 (1985), 254-288.

[20] M. Hilsum, Fonctorialité en K-théorie bivariante pour les variétés Lipschitziennes, *K-theory* 3 (1989), 401-440.

[22] M.F. Atiyah, Global theory of elliptic operators, *Proc. Internat. Conf. on functional analysis and related topics*, Tokyo, Univ. of Tokyo Press (1970), 21-29.

[23] M. Freedman and F. Quinn, book: Topology of four Dimensional Manifolds.

[24] Morita, Counterexamples to Bott Vanishing in Lower Smoothness.

[25] A. Connes, The action functional in non-commutative geometry, *Commun. Math. Phys.* **117** (1988), 673-683.

[26] S. Janson and T. Wolff, Schatten classes and commutators of singular integral operators, *Ark. Mat.* **20** (1982), 301-310.

[27] R. Rochberg and S. Semmes, Nearly weakly orthonormal sequences, singular value estimates, and Calderon-Zygmund operators, *J. Functional Analysis*, **86** (1989), 237-306.

[28] R. Rochberg and S. Semmes, End point results for estimates of singular values of singular integral operators, in *Operator Theory: Advances and Applications*, vol. 35 (1988), Birkäuser.

[29] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

StonyBrook
Institute for Mathematical Sciences

SUNY, Stony Brook, New York 11794-3651
telephone: (516) 632-7318 email: IMS@math.sunysb.edu

PREPRINTS IN THIS SERIES:

1990/1. B. BIELEFELD (editor), Conformal Dynamics Problem List.

/2.*A. M. BLOKH & M. LYUBICH, Measurable Dynamics of S-Unimodal Maps of the Interval. *Ann. Sci. Éc. Norm. Sup.* **24** (1991) 545–573.

/3.*J. J. P. VEERMAN & F. M. TANGERMAN, On Aubry Mather Sets. *Phys. D.* **46** (1990) 149–162.

/4. A. E. EREMENKO & M. LYUBICH, Dynamical Properties of Some Classes of Entire Functions.

/5. J. MILNOR, Dynamics in One Complex Variable: Introductory Lectures.

/6.*J. MILNOR, Remarks on Iterated Cubic Maps. *Experimental Math.* **1** (1992) 5–24.

/7.*J. J. P. VEERMAN & F. M. TANGERMAN, Intersection Properties of Invariant Manifolds in Certain Twist Maps. *Commun. Math. Phys.* **139** (1991), 245–265.

/8.*J. J. P. VEERMAN & F. M. TANGERMAN, Scalings in Circle Maps (I). *Commun. Math. Phys.* **134** (1990), 89–107.

/9. L. CHEN, Shadowing Property for Nondegenerate Zero Entropy Piecewise Monotone Maps.

/10.*G. ŚWIĄTEK, One-Dimensional Maps and Poincaré Metric. *Nonlinearity* **4** (1992), 81–108.

/11.*J. J. P. VEERMAN & F. M. TANGERMAN, Scalings in Circle Maps (II). *Commun. Math. Phys.* **141** (1991), 279–291.

/12.*P. M. BLEHER & M. LYUBICH, The Julia Sets and Complex Singularities in Hierarchical Ising Models. *Commun. Math. Phys.* **141** (1991) 453–474.

/13. J. J. P. VEERMAN & F. M. TANGERMAN, A Remark on Herman's Theorem for Circle Diffeomorphisms.

/14. L. R. GOLDBERG, Fixed Points of Polynomial Maps, Part I: Rotation Subsets of the Circle, and
*L. R. GOLDBERG & J. MILNOR, Part II: Fixed Point Portraits. *Ann. Sci. Éc. Norm. Sup.* **26** (1993), 51–98.

/15. G. P. PATERNAIN & R. J. SPATZIER, New Examples of Manifolds with Completely Integrable Geodesic Flows.

/16. L. KEEN & C. SERIES, Continuity of Convex Hull Boundaries.

1991/1. Y. JIANG, Dynamics of Certain Smooth One-Dimensional Mappings: I. The $C^{1+\alpha}$-Denjoy-Koebe Distortion Lemma, II. Geometrically Finite One-Dimensional Mappings.

/2. L. KEEN & C. SERIES, Pleating Coordinates for the Maskit Embedding of the Teichmüller Space of Punctured Tori.

/3.*B. BIELEFELD, Y. FISHER, & J. HUBBARD, The Classification of Critically Preperiodic Polynomials as Dynamical Systems. *J. Amer. Math. Soc.* **5** (1992), 721–762.

/4. M. REES, A Partial Description of the Parameter Space of Rational Maps of Degree Two: Part 2.

/5. M. KIM & S. SUTHERLAND, Polynomial Root-Finding Algorithms and Branched Covers.

/6.*Y. JIANG, T. MORITA, & D. SULLIVAN, Expanding Direction of the Period Doubling Operator. *Commun. Math. Phys.* **144** (1992), 509–520.

Continued on other side.

*Starred papers have already appeared in the journal cited.

PREPRINTS IN THIS SERIES: **Continued from other side.**

1991/7. M. SHISHIKURA, The Hausdorff Dimension of the Boundary of the Mandelbrot Set and Julia Sets.

/8. J. GRACZYK & G. ŚWIĄTEK, Critical Circle Maps Near Bifurcation.

/9. E. CAWLEY, The Teichmüller Space of an Anosov Diffeomorphism of T^2.

/10. M. LYUBICH, On the Lebesgue Measure of the Julia Set of a Quadratic Polynomial.

/11. M. LYUBICH, Ergodic Theory for Smooth One-Dimensional Dynamical Systems.

/12. Y. JIANG, Dynamics of Certain Smooth One-Dimensional Mappings: III. Scaling Function Geometry, IV. Asymptotic Geometry of Cantor Sets.

/13. A. M. BLOKH, Periods Implying Almost All Periods, Trees with Snowflakes, and Zero Entropy Maps.

/14. A. M. BLOKH, The "Spectral" Decomposition for One-Dimensional Maps.

/15.*M. LYUBICH & J. MILNOR, The Fibonacci Unimodal Map. *J. Amer. Math. Soc.* 6 (1993), 425–457.

/16. M. JAKOBSON & G. ŚWIĄTEK, Quasisymmetric Conjugacies Between Unimodal Maps.

/17.*M. LYUBICH & A. VOLBERG, A Comparison of Harmonic and Balanced Measures on Cantor Repellors. *Approximation by Solutions of Partial Differential Equations*, NATO ASI series, v. 365, 127–140.

/18. B. BIELEFELD, S. SUTHERLAND, F. TANGERMAN, & J. J. P. VEERMAN, Dynamics of Certain Non-Conformal Degree Two Maps of the Plane.

/19. Y. JIANG, On the Quasisymmetrical Classification of Infinitely Renormalizable Maps: I. Maps with Feigenbaum's Topology, II. Remarks on Maps with a Bounded Type Topology.

/20. A. POIRIER, On the Realization of Fixed Point Portraits.

/21. C. GOLÉ, Periodic Orbits for Hamiltonian Systems in Cotangent Bundles.

/22. P. JONES, On Removable Sets for Sobolev Spaces in the Plane.

/23. L. KEEN, B. MASKIT, & C. SERIES, Geometric Finiteness and Uniqueness for Kleinian Groups with Circle Packing Limit Sets.

1992/1. M. MARTENS, The Existence of σ-finite Invariant Measures, Applications to Real One-Dimensional Dynamics.

/2. J. GRACZYK, G. ŚWIĄTEK, F. M. TANGERMAN, & J. J. P. VEERMAN, Scalings in Circle Maps III.

/3. J. MILNOR, Hyperbolic Components in Spaces of Polynomial Maps (with an appendix by A. POIRIER).

/4. E. CAWLEY, The Teichmüller Space of the Standard Action of $SL(2, Z)$ on T^2 is Trivial.

/5.*Y. JIANG, Dynamics of Certain Non-Conformal Semigroups. *Complex Variables* **22** (1993), 27–34.

/6. F. TANGERMAN & F. PRZYTYCKI, Cantor Sets in the Line: Scaling Functions and the Smoothness of the Shift Map.

/7. B. BIELEFELD & M. LYUBICH (editors), Problems in Holomorphic Dynamics.

/8.*E. BEDFORD, M. LYUBICH, & J. SMILLIE, Polynomial Diffeomorphisms of C^2. IV: The Measure of Maximal Entropy and Laminar Currents. *Inv. Math.* **112** (1993), 77–125.

/9. J. GRACZYK & G. ŚWIĄTEK, Singular Measures in Circle Dynamics.

Continued on outside cover.

*Starred papers have already appeared in the journal cited.

StonyBrook
Institute for Mathematical Sciences

SUNY, Stony Brook, New York 11794-3651
telephone: **(516) 632-7318** email: **IMS@math.sunysb.edu**

PREPRINTS IN THIS SERIES: **Continued from inside cover.**

1992/10. G. Świątek, Hyperbolicity is Dense in the Real Quadratic Family.

/11. J. Milnor, Local Connectivity of Julia Sets: Expository Lectures.

/12. A. Poirier, Hubbard Forests.

/13. P. Boyland, Weak Disks of Denjoy Minimal Sets.

/14. J. Milnor, Remarks on Quadratic Rational Maps.

/15. G. Golé, Optical Hamiltonians and Symplectic Twist Maps.

/16. C. Liverani & M. Wojtkowski, Ergodicity in Hamitonian Systems.

/17. M. Martens, Distortion Results and Invariant Cantor Sets of Unimodal Maps.

/18. M. Lyubich, Combinatorics, Geometry and Attractors of Quasi-quadratic Maps.

1993/1. E. Bedford, M. Lyubich, & J. Smillie, Distribution of Periodic Points of Polynomial Diffeomorphisms of C^2.

/2. A. Connes, D. Sullivan, & N. Teleman, Quasiconformal Mappings, Operators on Hilbert Space, and Local Formulae for Characteristic Classes.

/3. F. Przytycki, Accessability of Typical Points for Invariant Measures of Positive Lyapunov Exponents for Iterations of Holomorphic Maps.

/4. F. Przytycki & A. Zdunik, Density of Periodic Sources in the Boundary of a Basin of Attraction for Iteration of Holomorphic Maps, Geometric Coding Trees Technique.

/5. A. Poirier, On Post Critically Finite Polynomials. Part One: Critical Portraits.

/6. F. Lalonde & D. McDuff, The Geometry of Symplectic Energy.

/7. A. Poirier, On Post Critically Finite Polynomials. Part Two: Hubbard Trees.

/8. J. Graczyk & G. Świątek, Induced Expansion for Quadratic Polynomials.

Several of the above preprints are available via email or anonymous ftp from math.sunysb.edu. Send an email message containing the single line **send info** to **preprints@math.sunysb.edu** for further details.

Publication of this preprint series is made possible in part by grants from the Paul and Gabriella Rosenbaum Foundation and the National Science Foundation