A Monotonicity Conjecture for Real Cubic Maps. [1]

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1. Introduction.

This will be an outline of work in progress. We study the conjecture that the topological entropy of a real cubic map depends "monotonely" on its parameters, in the sense that each locus of constant entropy in parameter space is a connected set.

Section 2 sets the stage by describing the parameter triangle $\overline{T}$ for real cubic maps, either of shape $+-+$ or of shape $-+-$, and by describing basic properties of topological entropy. Section 3 describes the monotonicity problem for the topological entropy function, and states the Monotonicity Conjecture. Section 4 describes the family of stunted sawtooth maps, and proves the analogous conjecture for this family. Sections 5 begins to relate these two families by describing the 'bone' structure in the parameter triangle. By definition, a *bone* $B_\pm(\mathbf{o})$ in the triangle $\overline{T}$ is the set of parameter points $\mathbf{v}$ such that a specified critical point (left or right) of the associated bimodal map belongs to a periodic orbit with specified order type $\mathbf{o}$. (Compare [Ma T].) It is conjectured that every bone is a simple connected arc in $\overline{T}$. Although we cannot prove either of these conjectures for cubic maps, we do show that

Generic Hyperbolicity $\Rightarrow$ *Connected Bone Conjecture* $\Rightarrow$ *Monotonicity Conjecture*

(see Theorems 3 and 4 in Sections 7, 8). The paper concludes with a brief Appendix on computation.

This material will be presented in more detail in a later paper [DGMT].

2. The Parameter Triangle $\overline{T}$ and the Topological Entropy Function.

Let f be a cubic map of the unit interval $I = [0,1]$. We will always assume that f maps the boundary of I into itself. To fix our ideas, we consider only those maps which have *shape* $+-+$; that is, f must first increase, then decrease, and then increase. Thus the leading coefficient must be positive, f must have critical points $c_1 < c_2$ in the interior of I, and both boundary points must be fixed by f. (All of the discussion which follows could easily be modified so as to apply also to maps of shape $-+-$; these are dynamically quite different, since the two boundary points must form a period two orbit.) For maps of shape $+-+$, evidently the corresponding *critical values* $v_i = f(c_i)$ must satisfy

$$1 \geq v_1 > v_2 \geq 0 \, . \qquad (1)$$

Lemma 1. *Given any pair $\mathbf{v} = (v_1, v_2)$ satisfying the inequalities (1), there is one and only one cubic map which fixes the boundary of I and has critical values (v_1, v_2).*

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[2] Partially supported by the Miller Institute of the University of California at Berkeley during the preparation of this paper.

Proof Outline. It is easy to check that any real cubic map with distinct real critical points can be written uniquely as

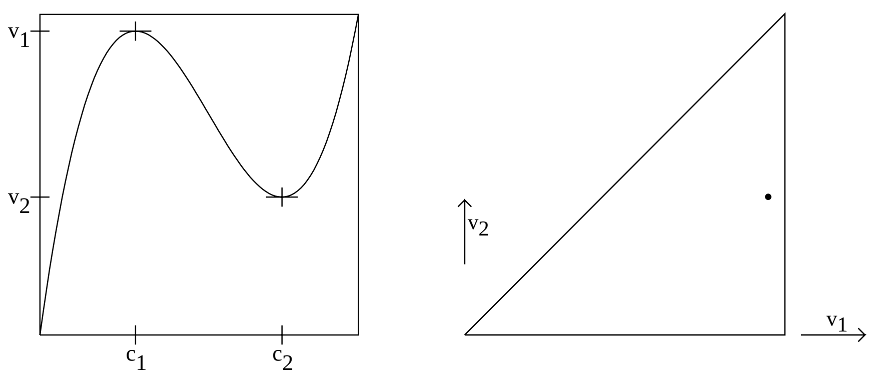
$$f(x) \;=\; c\,F(ax+b)+d$$

with $a>0$, where

$$F(\xi) \;=\; 3\,\xi^2 - 2\,\xi^3$$

is the unique cubic map with fixed critical points 0 and 1. Note that f has critical values $v_1 = d$ and $v_2 = c+d$. We can solve these linear equations for c and d, and then solve the required cubic equations $f(0)=0$ and $f(1)=1$ for b and $a+b$. □

Definitions. This cubic map with *critical value vector* $\mathbf{v}=(v_1,v_2)$ will be denoted by $f=f_{\mathbf{v}}$. It is often convenient to allow the limiting case $v_1=v_2$ also. This corresponds to allowing degenerate cubic maps, for which $c_1=c_2$. The compact set $\overline{T}\subset\mathbf{R}^2$ consisting of all pairs $(v_1\,,\,v_2)$ with $1\ge v_1\ge v_2\ge 0$ will be called the *parameter triangle* for cubic maps of shape $+-+$.



A cubic map $f_{\mathbf{v}}$ *, and the corresponding point* $\mathbf{v}=(v_1,v_2)$ *in parameter space.*

Remarks. The analogue of Lemma 1 for cubic maps of shape $-+-$ can be proved by essentially the same argument. Corresponding statements for higher degree polynomials with distinct real critical points are also true. For example, such a polynomial can be constructed uniquely, from its critical value vector, by constructing the Riemann surface of the corresponding complex polynomial map. A purely real proof may be found in [dM vS, p. 120]. (A somewhat simpler real proof has been given by Douady and Sentenac, unpublished.) For further information, see [DGMT].

How can we measure the dynamic "complexity" of a map $f_{\mathbf{v}}:I\to I$, and how does this complexity vary as the critical value vector $\mathbf{v}$ varies within the parameter triangle $\overline{T}$? One measure of complexity would be the numbers of periodic points of various periods. A particularly useful measure of complexity is provided by the topological entropy h. For our purposes, the *topological entropy* of a piecewise-monotone map can be defined by the formula

$$h(f) \;=\; \lim_{n\to\infty}\frac{\log \ell(f^{\circ n})}{n}\;, \qquad (2)$$

where $\ell(f^{\circ n})$ is the number of *laps* of the n-fold iterate, that is the number of maximal intervals of monotonicity. (Compare [Ro], [M Sz]. For computation of h, see [B K], [BST].) For non-linear polynomial maps, or more generally for piecewise-monotone maps with at most finitely many non-repelling periodic orbits, $h(f)$ can be identified with the number

$$h_{\rm per}(f) \;=\; \limsup_{n\to\infty} \frac{\log \#{\rm fix}(f^{\circ n})}{n} \;, \qquad (3)$$

where $\#$fix is the number of fixed points.[1] (Compare Lemma 7 in §8.) The entropy varies continuously under bimodal C^1-deformation.[2]

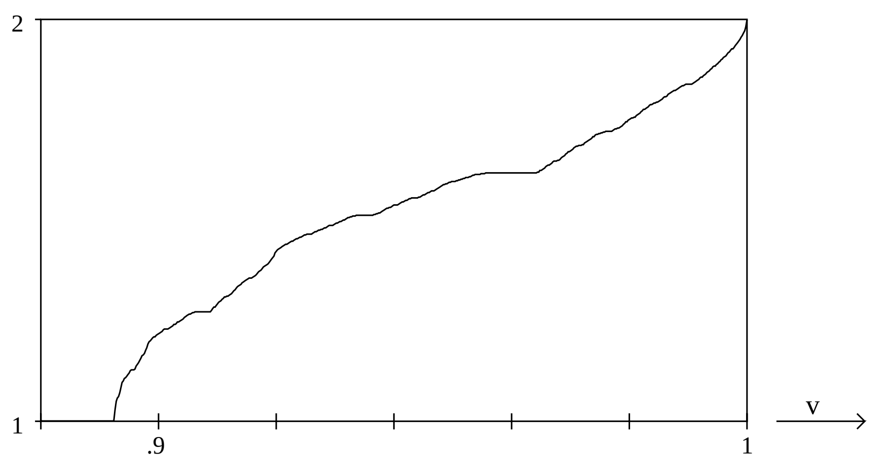
In practice, it is often more convenient to work with the quantity

$$s \;=\; \exp(h) \;=\; \lim_{n\to\infty} \sqrt[n]{\ell(f^{\circ n})} \;,$$

sometimes known as the *growth number* of f. For an m-*modal* map, that is for a map with $m+1$ laps, this number s lies in the closed interval $[1\,,\,m+1]$. In the special case of a piecewise linear map with $|{\rm slope}| = {\rm constant} \ge 1$, the growth number s is precisely equal to this constant $|{\rm slope}|$.

3. The Monotonicity Problem.

In the quadratic case, it is known that the number of period p points for an interval map $x \mapsto 4vx(1-x)$ increases monotonically as the critical value parameter $v \in [0,1]$ increases. (Proofs of this result have been given by Sullivan, Douady and Hubbard, and by Milnor and Thurston. Compare [DH2, n° VI], [M Th], [D], as well as [dM vS].) Hence the entropy h also increases monotonically with v. Compare the picture below.



Graph of $s = \exp(h)$ *as a function of* v *for the family of maps* $x \mapsto 4vx(1-x)$.

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[1] See [M Sz], [M Th], as well as [dMvS, p.268]. It is possible that the equation $h = h_{\rm per}$ is true for a C^r-generic map in any dimension, but no proof is known. (Compare [B, p. 23].)

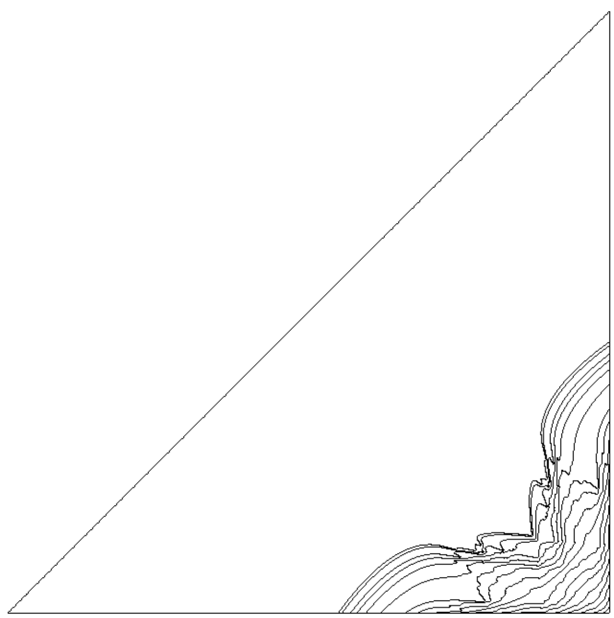
[2] See [M Sz], [M Th], [dM vS]. Conjecturally, entropy remains continuous under C^1-deformation as long as the number of critical points remains bounded; but even in the C^r-case it definitely can jump discontinuously if the number of critical points is unbounded and if $r < \infty$. See [M Sz]. For maps in dimension ≥ 2 or for diffeomorphisms in dimension ≥ 3, the entropy can also drop discontinuously, even in the C^∞ case. See [K], [Mis], [N], [Y] for further information.

Is there some analogous statement for the two-parameter family of cubic maps $f_{\mathbf{v}}$? Can we find curves through an arbitrary point of $\overline{T}$ along which the complexity increases monotonically? (Compare [D G], [DGK], [DGKKY], [DGMT].) Is there some sense in which the topological entropy function

$$\mathbf{v} \mapsto h(f_{\mathbf{v}})$$

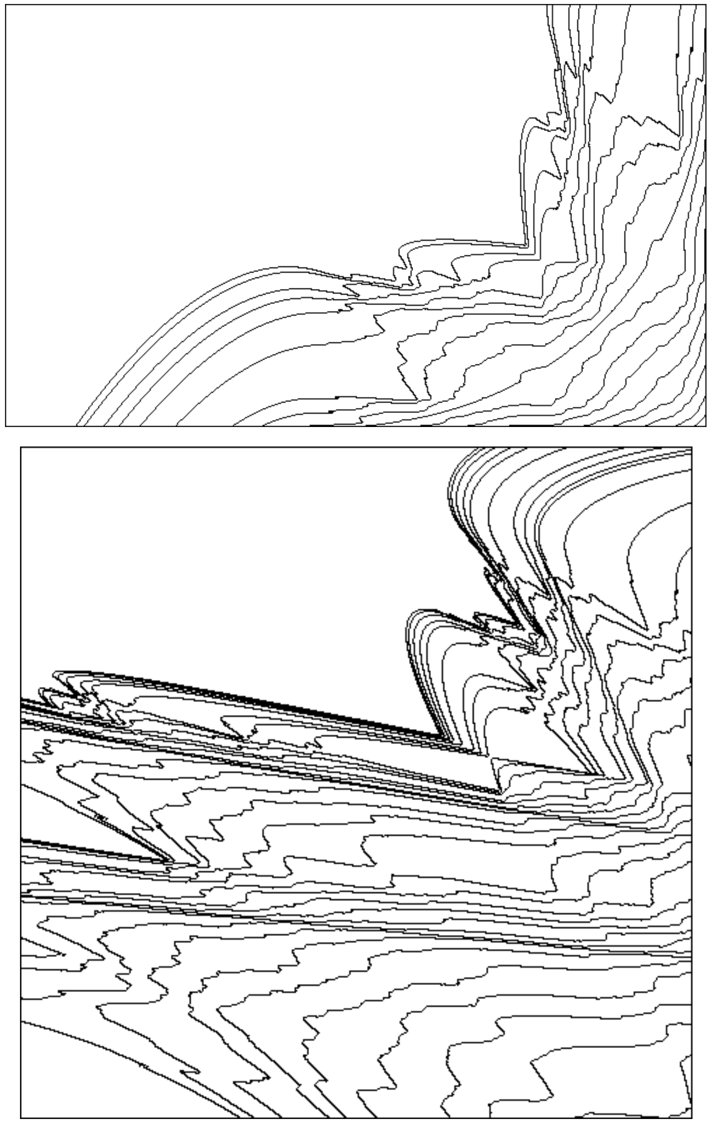
is a "monotone" function on the parameter triangle $\overline{T}$? To formulate this question more precisely, we make the following definition. Fix some constant h_0 in the closed interval $[0, \log 3]$. By the h_0*-isentrope* for the family of cubic maps $f_{\mathbf{v}}$ we will mean the set consisting of all parameter values $\mathbf{v} \in \overline{T}$ for which the topological entropy $h(f_{\mathbf{v}})$ is equal to h_0 . (In the illustrations, it will be convenient to work with $s = \exp(h)$ rather than h .) Since the entropy function is continuous, note that each isentrope is a compact subset of $\overline{T}$.

Monotonicity Conjecture for the family of real cubic maps $f_{\mathbf{v}}$. *Every isentrope* $\{\mathbf{v} \in T \ : \ h(f_{\mathbf{v}}) = h_0\}$ *for this family is a connected set.*



Isentropes $s =$ constant *in the parameter triangle for cubic maps of shape* $+-+$ *.*
(Contour interval: $\Delta s = 0.1$ *. Visible isentropes:* $s = 1.1, 1.2, 1.3, \ldots, 2.8$ *.)*

(Compare §6, as well as [M1, p.13].) Evidently this conjecture describes a weak form of monotonicity for this two-parameter family. Such a connected isentrope could be a simple arc with endpoints on the boundary of T, or perhaps could have a more complicated non-locally connected topology although this has not been observed. It is also certainly possible for it to be a compact set with interior points. In the limiting case $h_0 = 0$, the isentrope is the large white region in the picture above, containing the entire upper left hand edge $v_1 = v_2$. (Compare [Ma T].) In the other limiting case $h_0 = \log 3$, it reduces to



Magnified portions of the previous figure. Above: The lower right region $[.5,1]\times[0,.3]$ *, again with contour interval* $\Delta s = 0.1$ *. Below: Detail showing the region* $[.74,.8]\times[.07,.13]$ *near the center of this picture, with* $\Delta s = .02$ *.*

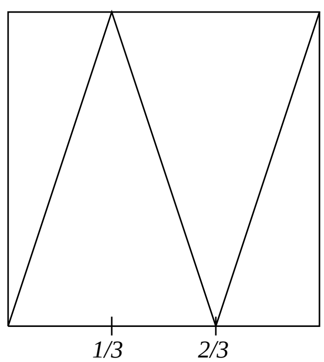
the single corner point $\mathbf{v}=(1,0)$. If the Conjecture is true, then for each $0<h_0<\log 3$ the h_0-isentrope must cut $\overline{T}$ into two connected pieces, one with $h<h_0$ and one with $h>h_0$.

Another interesting consequence would be a "maximum and minimum principle" for entropy: *If the conjecture is true, then the maximum and minimum values for the entropy function on any closed region* $U\subset\overline{T}$ *must occur on the boundary* ∂U *.* In fact every value of entropy which occurs in U must occur already on ∂U . This follows since, by continuity, every value of entropy between 0 and $\log 3$ must occur on $\partial\overline{T}$. Evidently a connected isentrope which contains points both inside and outside U must also intersect ∂U .

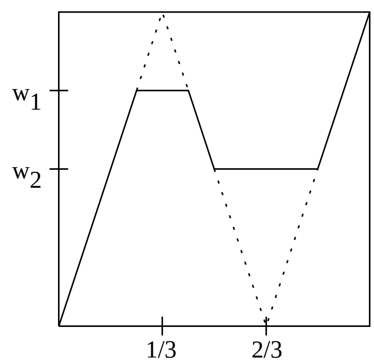
These questions are quite difficult. As a step towards understanding, we will first consider a different family of maps for which they are very much easier.

4. The Stunted Sawtooth Family.

By the *sawtooth map* of shape $+-+$ we mean the unique map $S : I \to I$ which is piecewise linear with slope alternately $+3$, -3 and $+3$. This is a bimodal map for which the topological entropy takes the largest possible value $h = \log 3$. **Definition:** Given any critical value vector $\mathbf{w} = (w_1, w_2)$ satisfying the usual inequalities $1 \geq w_1 > w_2 \geq 0$, we obtain the *stunted sawtooth map* $S_\mathbf{w}$ from S by cutting off the top and bottom at heights w_1 and w_2, as illustrated below. (Compare [Gu].) As with the cubic family, it is often convenient to allow the limiting case $w_1 = w_2$. For the stunted sawtooth family, this means that we may allow the two horizontal plateaus to come together.



The sawtooth map S



A stunted sawtooth map $S_\mathbf{w}$.

For this family it is easy to see that any increase in the parameter w_1, or any increase in $1 - w_2$, can only increase the complexity of the mapping. (Compare [BMT], [Ga].) For example, as we increase w_1 with fixed w_2, no periodic orbit can disappear: If a given periodic orbit misses the left hand plateau, then it remains unchanged as we increase w_1, while if it hits this plateau then it deforms continuously as we increase w_1. Similarly, the topological entropy can never decrease as we increase w_1 or $1 - w_2$. It will be convenient to define a simple partial ordering for the parameter triangle $\overline{T}$ as follows:

Definition : $$\mathbf{w} \ll \mathbf{w}' \iff w_1 \leq w_1' \quad \text{and} \quad 1 - w_2 \leq 1 - w_2' \ .$$

Then it follows from the discussion above that

$$\mathbf{w} \ll \mathbf{w}' \implies h(S_\mathbf{w}) \leq h(S_{\mathbf{w}'}) \ . \tag{4}$$

Remark. Let us temporarily extend the discussion to more general piecewise-monotone maps. By the ***shape*** of an m-modal map we mean an alternating sequence of $m+1$ signs, starting with either $+$ or $-$ according as the map is increasing or decreasing on its initial lap. The above construction for bimodal maps of shape $+-+$ extends easily to m-modal maps for any $m \geq 1$ and for either one of the two possible shapes.

The stunted sawtooth family is very closely related to kneading theory. To make this precise, we will need the following. Again consider an m-modal map with any $m \geq 1$ and with either one of the two possible m-modal shapes.

Definition. By the ***kneading data*** associated with an m-modal map f we will mean its shape, together with the collection of signs

$$\operatorname{sgn}\big(f^{\circ n}(c_i) - c_j\big) \quad \in \quad \{-1, 0, 1\}$$

for $n > 0$ and $1 \leq i, j \leq m$, where the c_i are the critical points of f.

To extend this definition to the case of an m-modal stunted sawtooth map, we simply define the "critical points" to be the center points $\hat{c}_i = i/(m+1)$ of the plateaus, for $1 \leq i \leq m$. With this definition, we can make the following assertion.

> **Lemma 2.** *To any m-modal map f there is associated a canonical stunted sawtooth map $S_{\mathbf{w}}$ which has exactly the same kneading data.*

The proof can be outlined as follows (details in [DGMT]). Let S be the m-modal sawtooth map with the same shape, with critical points $\hat{c}_i = i/(m+1)$. First consider a point x in the domain of definition of f which is not pre-critical. That is, we assume that the orbit $\{x\,,\, f(x)\,,\, f^{\circ 2}(x)\,,\, \ldots\}$ does not contain any critical point. Then there exists one and only one point $\hat{x} \in [0,1]$ so that the ***itinerary*** of $\hat{x}$ under S is the same as the itinerary of x under f. By definition, this means that

$$\operatorname{sgn}\big(f^{\circ n}(x) - c_j\big) \quad = \quad \operatorname{sgn}\big(S^{\circ n}(\hat{x}) - \hat{c}_j\big) \tag{5}$$

for every $n \geq 0$ and for every critical point c_j of f. In the case of a pre-critical point x, we must weaken this condition slightly by requiring equation (5) only up to the first n for which $f^{\circ n}(x)$ is critical. Then again there is a unique associated $\hat{x}$. Now let $v_1\,,\,\ldots\,,\,v_m$ be the critical values of f. Then the associated points $w_i = \hat{v}_i$ are the critical values for the required stunted sawtooth map $S_{\mathbf{w}}$. □

In particular, for each $n > 0$ we have a matrix equality

$$\big[\operatorname{sgn}\big(f^{\circ n}(c_i) - c_j\big)\big] \quad = \quad \big[\operatorname{sgn}\big(S_{\mathbf{w}}^{\circ n}(\hat{c}_i) - \hat{c}_j\big)\big]\ ,$$

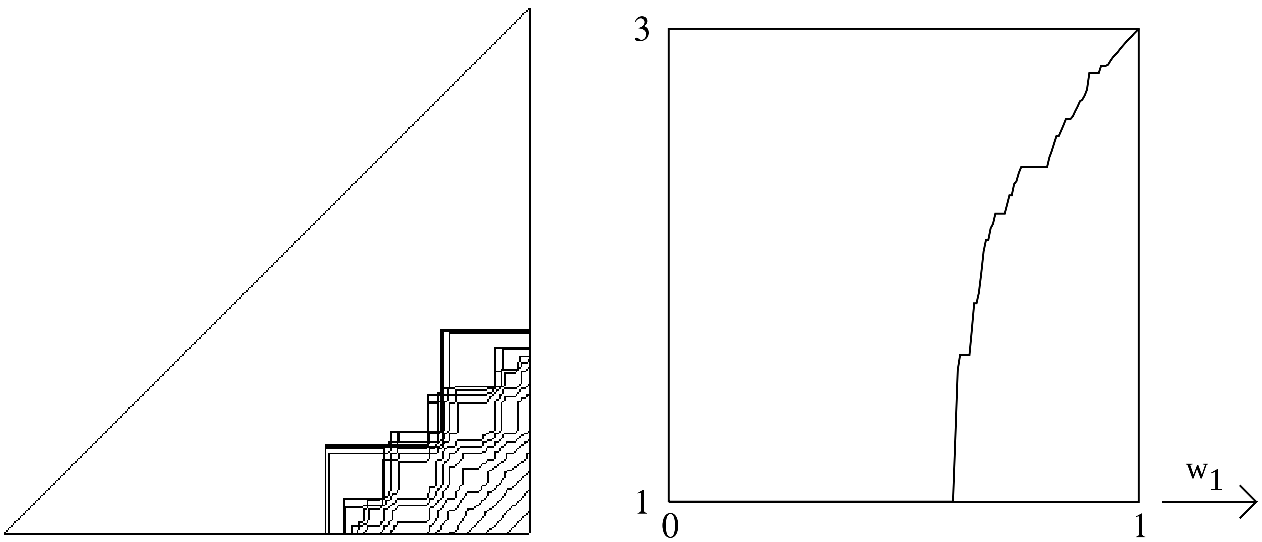
where the c_i are the critical points of f and where the $\hat{c}_i = i/(m+1)$ are the critical points of $S_{\mathbf{w}}$.

Now let us again specialize to $+-+$ bimodal maps. We will show that the Monotonicity Conjecture for the stunted sawtooth family is true:

Theorem 1. *For every constant* $0 \le h_0 \le \log 3$ *, the isentrope*

$$\mathcal{I}(h_0) \;=\; \{\mathbf{w} \in \overline{T} \;:\; h(S_{\mathbf{w}}) = h_0\}$$

is compact and connected.



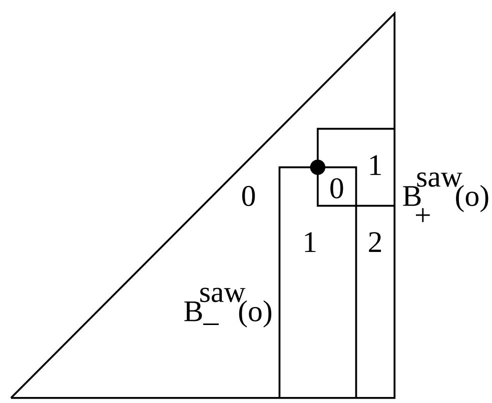
Left: *Isentropes* $s = \text{constant}$ *in the stunted sawtooth parameter triangle, with* $\Delta s = 0.1$ *.*
Right: *Graph of* $s = \exp(h)$ *as a function of* w_1 *along the bottom edge* $w_2 = 0$ *of* $\overline{T}$ *.*

Proof of Theorem 1. Compactness is clear, since the entropy function is continuous. Let $\mathcal{I}^+$ be the union of the isentrope $\mathcal{I}(h_0)$ with those seqments of the edges $w_1 = 1$ or $w_2 = 0$ on which $h \le h_0$. It follows from (4) that for each line $w_1 + w_2 = \text{constant}$ which intersects $\mathcal{I}(h_0)$ the intersection must consist of a point or a closed connected interval. It follows that we can deformation retract the entire triangle $\overline{T}$ onto $\mathcal{I}^+$ by pushing each point towards $\mathcal{I}^+$ along such a line $w_1 + w_2 = \text{constant}$. To check the continuity of this deformation, it is convenient to rotate the parameter triangle $45°$ by taking $w_2 + w_1$ and $w_2 - w_1$ as independent parameters. Then the upper and lower boundaries of the isentrope will be (not necessarily disjoint) Lipschitz curves with $|\text{slope}| \le 1$, and it follows easily that our deformation is continuous. Finally, we can certainly deformation retract $\mathcal{I}^+$ onto $\mathcal{I}(h_0)$. Since $\overline{T}$ is contractable, it follows that $\mathcal{I}(h_0)$ is contractable, and hence connected. □

Remark. This statement for the $+-+$ sawtooth family generalizes naturally to m-modal stunted sawtooth maps for any $m \ge 1$ and for either one of the two possible shapes. However, the proof is somewhat harder in the general case. (See [DGMT].)

5. "Bones" in the Parameter Triangle.

The stunted sawtooth family is well understood, but the corresponding cubic family is poorly understood. In order to relate theses two families, we introduce some terminology from MacKay and Tresser [Ma T]. By a *bone* in the parameter triangle $\overline{T}$ we mean the compact set consisting of all parameter values for which a specified critical point has periodic orbit with specified order type. More precisely, the *left bone* $B_-(\mathbf{o})$ is the set of parameter values for which the left hand critical point is periodic with *order type* $\mathbf{o}$. By definition, this means that the points of the orbit, numbered as $x_1 < \cdots < x_p$, satisfy $x_i \mapsto x_{\mathbf{o}(i)}$ where $\mathbf{o}$ is some given cyclic permutation of $\{1, \ldots, p\}$. The *dual right bone* $B_+(\mathbf{o})$ is the set of parameter values for which the right critical point is periodic with this same order type. We will usually assume that the period p is two or more, and we only allow those order types which can actually occur for a bimodal map of shape $+-+$. These definitions make sense either for the stunted sawtooth family or for the cubic family. (By definition, we take the center points $1/3$ and $2/3$ of the two plateaus as the "critical points" for the stunted sawtooth map.) We will insert the superscript **saw** respectively **cub** in order to distinguish these two cases.



Dual bones for the stunted sawtooth family. There is a preferred intersection point, called the common "center point" of these bones, such that the two critical points of the associated map belong to a common periodic orbit. It is marked by a heavy dot in the figure. (The complementary regions have been labeled by the number of 'negative' periodic orbits with the given order type $\mathbf{o}$*. Compare §8.)*

Note that two left bones, or two right bones, are disjoint, almost by definition. For the stunted sawtooth family, we have the following simple description:

Lemma 3. *Each non-vacuous bone $B_\pm^{\mathrm{saw}}(\mathbf{o})$ of period $p \geq 2$ is a simple arc with both endpoints on a common edge of the triangle $\overline{T}$, and is made up out of three line segments which are alternately horizontal and vertical. Any pair $B_-^{\mathrm{saw}}(\mathbf{o})$ and $B_+^{\mathrm{saw}}(\mathbf{o}')$ intersect transversally in either 0, 2, or 4 points. Dual bones always intersect in exactly two points, as illustrated.*

(Compare the schematic picture above.) The proof is not difficult. □

Remark. Here it is essential that we exclude the case $p = 1$, which behaves quite differently. (For maps of shape $-+-$ the case $p = 2$ is also different, so one must assume that $p \geq 3$.) In the $+-+$ case, note that a map with a critical fixed point must necessarily have zero entropy.

The corresponding statement for the cubic family is much more difficult. First note the following.

Lemma 4. *Each bone $B^{\rm cub}_{\pm}(\mathbf{o})$ is a smooth 1-dimensional manifold with exactly two boundary points, and these boundary points belong to a common edge of $\overline{T}$ (provided that $p \geq 2$). Any intersection between bones is necessarily transverse.*

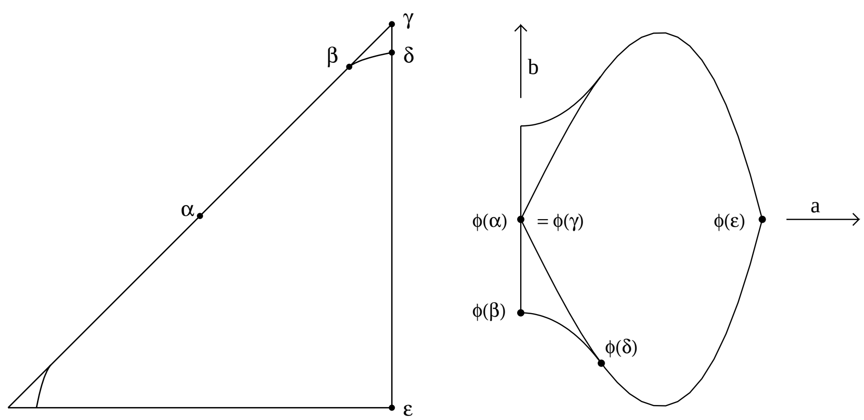
Remark. Evidently the intersection points of bones are precisely those points in parameter space for which *both* critical points are periodic. In general, the two critical points will belong to disjoint periodic orbits. The only exception is for the preferred intersection point of two dual bones. In this exceptional case, the two critical points belong to a common orbit.

Proof of Lemma 4. First consider the corresponding statement for the family of complex maps $z \mapsto z^3 - 3a^2 z + b$, with critical points $\pm a$. It is proved in [M3] that the locus $\mathcal{S}_{\pm}(p)$ of points for which $\pm a$ has period p is a smooth complex curve. Furthermore, for each p and q the curves $\mathcal{S}_+(p)$ and $\mathcal{S}_-(q)$ intersect transversally. In fact, $\mathcal{S}_+(p)$ has transverse intersection with any curve consisting of points for which the other critical point $-a$ is preperiodic. (The proofs make essential use of quasi-conformal surgery. Compare [St], where analogous results for quadratic rational maps are proved by similar methods.)

Restricting to the real (a, b)-plane, we obtain a corresponding statement for real cubic maps: *In the family of real maps*

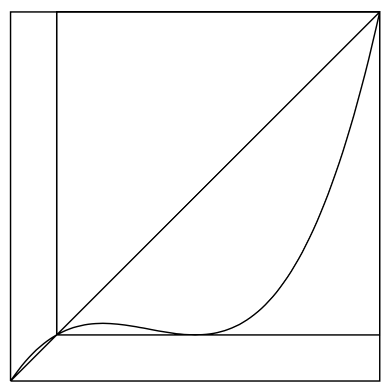
$$x \mapsto x^3 - 3a^2 x + b \;, \tag{6}$$

the locus of pairs (a, b) for which a (or $-a$) is periodic of period p forms a smooth 1-dimensional manifold without boundary.



The parameter triangle $\overline{T}$, and its image $\phi(T)$ in the (a, b)-plane. The two edges $\alpha\beta$ and $\beta\gamma$ fold together in the negative b-axis, and the triangle $\beta\gamma\delta$ folds over so that its image is covered twice.

In order to relate this to the family of cubic maps $f_{\mathbf{v}}$, we note that each $f_{\mathbf{v}}$ is *positively affinely conjugate* to a unique map in the normal form (6) with $a > 0$. That is, there is a unique affine map $L(x) = c\,x + d$ with $c > 0$ so that $L \circ f_{\mathbf{v}} \circ L^{-1}$ has the required form. It follows that there is a well defined smooth map $\phi : T \to \mathbf{R}^2$ which associates to each $\mathbf{v} \in \overline{T}$ the associated pair $\phi(\mathbf{v}) = (a, b)$. Evidently the pre-image of the curve $\mathcal{S}_{\pm}(p)$ under ϕ is the union of all bones $B^{\mathrm{cub}}_{\pm}(\mathbf{o})$ of period p. If ϕ were a diffeomorphism, then it would follow immediately that each bone is a smooth manifold. In fact, the situation is more complicated, since ϕ folds over two of the corners of the triangle $\overline{T}$. Thus ϕ fails to be a local diffeomorphism along two fold curves, which correspond to values $\mathbf{v}$ for which the graph of $f_{\mathbf{v}}$ is tangent to the diagonal at one of its two boundary fixed points. However, for $\mathbf{v}$ along these fold curves or in the folded over regions, both critical orbits converge to one fixed point, so that neither critical point can be periodic of period ≥ 2. It follows easily that each $B^{\mathrm{cub}}_{\pm}(\mathbf{o})$ is indeed a smooth 1-manifold, with boundary points at most on the horizontal or vertical part of the boundary of $\overline{T}$.



A cubic map which carries two distinct intervals bimodally onto themselves. Such a map is positively affinely conjugate to $f_{\mathbf{v}}$ for two distinct critical value vectors $\mathbf{v}$. These two points $\mathbf{v} \in \overline{T}$ will be folded together by the map ϕ. Note that the dynamics of such a map $f_{\mathbf{v}}$ is necessarily rather trivial.

In order to analyze these possible boundary points $B^{\mathrm{cub}}_{\pm} \cap \partial\overline{T}$, we will need to make use of the following basic principle. By definition, a piecewise monotone map is *post-critically finite* if the orbit of every critical point is periodic or eventually periodic.

Thurston's Theorem for Real Polynomial Maps. *Given any post-critically finite m-modal map, there exists one and up to positive affine conjugation only one polynomial of degree $m+1$ with the same kneading data.*

Although this statement is well known to experts, it is difficult to find in the literature. The proof makes essential use of complex methods. In fact this statement is an easy corollary of a much more complicated statement for complex polynomials (or for complex rational maps). For further information, the reader is referred to [DH1], [Po] and [dM vS], as well as the discussion in [M Th]. □

Assuming Thurston's Theorem, the proof of Lemma 4 concludes as follows. Consider a map $f_{\mathbf{v}}$ belonging to the intersection $B_-^{\rm cub}(\mathbf{o})\cap\partial\overline{T}$. Thus the left critical point of $f_{\mathbf{v}}$ is periodic, with some period $p\ge 2$. It is easy to check that the two critical points cannot coincide. (Here we make use of the fact that our maps have shape $+-+$. For maps of shape $-+-$ we would rather have to assume that $p\ge 3$ in order to avoid the case $c_1=c_2$.) Since $f_{\mathbf{v}}\in\partial\overline{T}$, it follows that either $v_1=1$ or $v_2=0$. In other words, at least one of the two critical points must map to a boundary fixed point. But c_1 is periodic, so it follows that we must be on the edge $v_2=0$. If we cut the interval I at the points of the period p orbit $\{c_1, v_1, f_{\mathbf{v}}(v_1), \ldots\}$, then since our map must have shape $+-+$ a little work shows that there are exactly two of the $p+1$ complementary intervals where the right hand critical point can be placed. Using Thurston's Theorem, it follows that the intersection $B_-^{\rm cub}(\mathbf{o})\cap\partial\overline{T}$ consists of exactly two points. Further, this intersection is transverse, so these two intersection points must belong to the boundary $\partial B_-^{\rm cub}(\mathbf{o})$. □

6. Three Conjectures.

It follows from Lemma 4 that each bone $B_\pm^{\rm cub}(\mathbf{o})$ consists of a simple arc, possibly together with one or more disjoint simple closed curves, which we may call "bone-loops". Thus a hypothetical *bone-loop* would be a simple closed curve in the parameter triangle $\overline{T}$ consisting entirely of points $\mathbf{v}$ for which one critical point of $f_{\mathbf{v}}$ (say the left one) is periodic. In fact we conjecture that such bone-loops do not exist:

> **Connected Bone Conjecture.** *Every bone* $B_\pm^{\rm cub}(\mathbf{o})$ *for the cubic family is a simple arc.*

Although we cannot prove this conjecture, we will show that it would follow from a well known classical conjecture.

Generic Hyperbolicity. Recall that a polynomial or rational map is said to be *hyperbolic* if the orbit of every complex critical point converges towards an attracting periodic orbit. By the *Generic Hyperbolicity Conjecture* for some given family $\mathcal{F}$ of maps we mean the conjecture that every map $f_0\in\mathcal{F}$ can be approximated arbitrarily closely by a map $f\in\mathcal{F}$ which is hyperbolic. (Compare the discussion in [F, p. 73].) For the family of real quadratic maps, a proof of this statement has been announced by Swiatek [S], and more recently by Lyubich [L]. (See also [Mc2], which proves a weaker but closely related result.) However, we will rather need the statement for cubic maps. In fact, the main result of this note will be to relate these conjectures to the Monotonicity Conjecture, as stated in §3:

Generic Hyperbolicity for Real Cubic Maps

$\Longrightarrow$ *Connected Bone Conjecture*

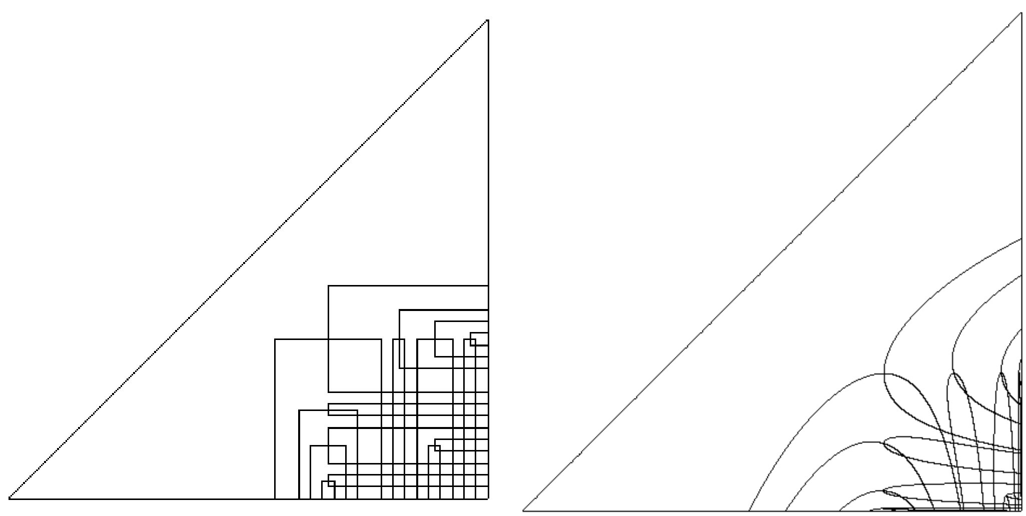
$\Longrightarrow$ *Monotonicity Conjecture.*

The proofs of these implications will be given in §7 and §8 respectively. In order to carry out these proofs, we must first develop a closer relationship between the cubic family and the stunted sawtooth family.

7. Intersections of Bones, and the n-Skeleton.

By the n-*skeleton* $S_n^{\rm saw}$ for the stunted sawtooth family, we will mean the union of all bones $B_\pm^{\rm saw}(\mathbf{o})$ of period $2 \le p \le n$, together with the boundary $\partial\overline{T}$. In the case of the cubic family it will be convenient to modify this definition slightly as follows. Recall that each bone $B_\pm^{\rm cub}(\mathbf{o})$ consists of a simple arc, possibly together with some disjoint simple closed curves. We will use the notation $A_\pm(\mathbf{o})$ for this simple arc. By the n-*skeleton* $S_n^{\rm cub}$ we will mean the union of all of these simple arcs $A_\pm(\mathbf{o})$ with period $2 \le p \le n$, together with $\partial\overline{T}$. For an analysis of the structure of these skeletons, see [R S], [R T], as well as [Ma T]. We will prove the following.

Theorem 2. *For each $n \ge 2$ there exists a homeomorphism η_n from the parameter triangle for stunted sawtooth maps onto the parameter triangle for cubic maps which carries each bone $B_\pm^{\rm saw}(\mathbf{o})$ of period $2 \le p \le n$ onto the corresponding bone-arc $A_\pm^{\rm cub}(\mathbf{o})$, and hence carries the skeleton $S_n^{\rm saw}$ homeomorphically onto $S_n^{\rm cub}$.*



Bones of period $2, 3, 4$ for the stunted sawtooth and cubic families. (As we traverse the bottom edge of $\overline{T}$ from left to right, we meet bones of period $2, 4, 3, 4, 4, 3, 4, 2, 4, 4, 3, 4, 4, 3, 4, 4$ respectively.) In the stunted sawtooth family, the dual bone can be recognized as the unique bone which meets the middle edge of a given bone.

Outline Proof. By a *vertex* of the skeleton $S_n^{\rm cub}$ or $S_n^{\rm saw}$ we will mean either an endpoint of a bone or a point of intersection between two bones (necessarily one left bone and one right bone). Evidently each such vertex corresponds to a post-critically finite map. Every such map is uniquely determined by its kneading data. In fact, any kneading data which can occur for an arbitrary post-critically finite $+-+$ bimodal map must actually occur for one and only one map in this family. In the cubic case, this follows easily from Thurston's Theorem as stated in §5, while in the stunted sawtooth case it follows from an elementary argument as in the proof of Lemma 2. Thus there is a natural one-to-one mapping from the vertices of the skeleton $S_n^{\rm saw}$ onto the vertices of $S_n^{\rm cub}$.

Remark: Here it is essential that we only consider bones of period ≥ 2 , so that none of these vertices lie on the upper left edge $v_1 = v_2$ of the parameter triangle where things behave somewhat strangely. (Compare the proof of Lemma 6 below.)

Next we must check that these vertices occur in the same order as we traverse some bone $B^{\rm saw}_{\pm}(\mathbf{o})$ or as we traverse the corresponding bone-arc $A_{\pm}(\mathbf{o})$ from one endpoint to the other. By the ***center point*** of a bone we will mean the unique point for which both critical points belong to a common orbit. (This is one of the two intersection points with the dual bone, and is evidently also the center point for the dual bone.) We will see that the center point of any bone $B^{\rm cub}_{\pm}(\mathbf{o})$ necessarily belongs to the component $A_{\pm}(\mathbf{o})$. Now, as we traverse any $B^{\rm saw}_{\pm}(\mathbf{o})$ or $A_{\pm}(\mathbf{o})$ from one end to the other, we claim that the entropy decreases monotonically until we reach the center point, and then increases monotonically until we reach the other end. In other words, if we divide each $B^{\rm saw}_{\pm}(\mathbf{o})$ or $A_{\pm}(\mathbf{o})$ into two ***half-bones*** by cutting at this center point, then along each half-bone the entropy changes monotonically. In fact, the kneading sequence for the non-periodic critical point changes monotonically along each half-bone. For the stunted sawtooth family, this can be proved by a direct argument. For the cubic family, it can be proved by an argument which is completely analogous to the proof in [M Th] of monotonicity for the quadratic family. Note first that the kneading sequence for the periodic critical point is fixed as we traverse the half-bone (although it is different from one half-bone to the other). The kneading sequence for the remaining critical point varies continuously with the cubic map, except for discontinuities of a very special form at those maps for which this remaining critical point eventually maps to one or the other critical point. Using these facts, we obtain an "intermediate value theorem" for admissible kneading sequences as we traverse the half-bone. In particular, any post-critically finite kneading data which can occur for any $+-+$ bimodal map must occur somewhere along the appropriate half-bone. It now follows that the kneading sequence must change monotonically. For otherwise some post-critically finite kneading sequence would have to occur twice, which is impossible by Thurston's Theorem. The rest of the proof is reasonably straightforward. □

As an immediate corollary to this argument we obtain the following.

Lemma 5. *A bone-loop in the cubic parameter triangle cannot contain any post-critically finite point. In particular, it cannot intersect any other bone.*

Proof. We have shown in the proof of Theorem 2 that all possible kneading types of post-critically finite points in a bone $B^{\rm cub}_{\pm}(\mathbf{o})$ can be found somewhere along the bone-arc $A_{\pm}(\mathbf{o})$. There cannot be any other post-critically finite points by Thurston's Theorem. □

Lemma 6. *The region enclosed by a bone-loop in $\overline{T}$ cannot contain any hyperbolic maps.*

Proof. By a ***hyperbolic component*** H in the cubic parameter triangle $\overline{T}$ we mean a connected component in the open set consisting of all $\mathbf{v} \in \overline{T}$ such that $f_{\mathbf{v}}$ is hyperbolic. It follows immediately from [M2] that every hyperbolic component which lies completely in the interior of $\overline{T}$ is an open topological 2-cell which contains a unique post-critically finite point called its ***center***. (Compare [Mc1].) Furthermore, every bone which intersects such a hyperbolic component H must intersect it in a simple arc passing through the center point of H. (Compare [M3]. There may be just one bone passing through the component, or there may be two which intersect transversally at the center point.)

Note: Here we exclude the three exceptional hyperbolic components which meet the boundary of $\overline{T}$. These are "centered" at the mid-point and end-points of the edge $v_1 = v_2$. Because of the folding mentioned in the proof of Lemma 4, these three distinct center points actually correspond to just one affine conjugacy class of maps, which is represented for example by $x \mapsto x^3$.

Suppose now that some hyperbolic component H intersects the region enclosed by a bone-loop L. Then H certainly cannot be one of the three exceptional hyperbolic components which meet the boundary of $\overline{T}$. If H intersects the loop itself, then its center point must belong to L, which is impossible by Lemma 5. On the other hand, if H is completely enclosed by L then we can choose a bone-arc $A_\pm(\mathbf{o})$ which passes through the center point of H. By the Jordan Curve Theorem, $A_\pm(\mathbf{o})$ must intersect L, which again contradicts Lemma 5. □

Theorem 3. *The Generic Hyperbolicity Conjecture for real cubic maps implies the Connected Bone Conjecture.*

Proof. This statement, which was promised in §6, clearly follows as an immediate corollary to Lemma 6. □

Remark. There is real hope that something much less than the full Generic Hyperbolicity Conjecture for cubic maps might be enough to prove the Connected Bone Conjecture. In particular, it would be enough to prove generic hyperbolicity along each bone. Thus we need only study very special one-parameter families of cubic maps. The known techniques for dealing with quadratic maps might well suffice to deal with this special case.

8. Bones, Negative Orbits, and the Monotonicity Conjecture.

In order to relate bones and entropy, let us first recall the following result from [M Th]. A fixed point $f(x_0) = x_0$ for a piecewise monotone map will be called either ***positive***, ***negative***, or ***critical*** according as the map f is monotone increasing or monotone decreasing throughout a neighborhood of x_0, or has a turning point at x_0. **Definition:** Let $N(f)$ be the number of critical fixed points of f plus twice the number of negative fixed points. This number is closely related to the total number of fixed points $\#\mathrm{fix}(f)$, but is easier to work with since it is more robust. Note that $N(f)$ is always finite, in fact $N(f) \le \ell(f)+1$ where $\ell(f)$ is the number of laps. It is easy to check that the sequence of numbers $N(f^{\circ k})$ is completely determined by the kneading data for f, as described in §4. We will need the following estimate, which is similar to formula (3) of §3 but true in much greater generality.

Lemma 7. *The topological entropy of an arbitrary piecewise monotone map is given by the formula*

$$h(f) \;=\; \limsup_{k\to\infty} \frac{\log N(f^{\circ k})}{k} \;.$$

Proof. This follows easily from [M Th], or from [Pr] or [B R]. □

Lemma 8. *Let* $\mathbf{v}$ *and* $\mathbf{v}'$ *be two points in the cubic parameter triangle such that the associated maps have entropy* $h(f_{\mathbf{v}}) \neq h(f_{\mathbf{v}'})$ *. Then any path from* $\mathbf{v}$ *to* $\mathbf{v}'$ *in the parameter triangle* $\overline{T}$ *must cross infinitely many bones.*

For according to Lemma 7 the difference $|N(f_{\mathbf{v}}^{\circ k}) - N(f_{\mathbf{v}'}^{\circ k})|$ must be unbounded as $k \to \infty$. But clearly, as we deform $\mathbf{v}$ along some path in $\overline{T}$, the number $N(f_{\mathbf{v}}^{\circ k})$, which measures the number of decreasing laps of $f_{\mathbf{v}}^{\circ k}$ whose graph crosses the diagonal, will remain constant except as we pass through a map which has a critical periodic orbit of period dividing k . In other words, $N(f^{\circ k})$ remains constant unless we pass through a bone of period dividing k . (Here bones of period 1 must also be allowed, but cause no difficulty.) Further details are easily supplied. □

Completely analogous arguments apply to the stunted sawtooth family, provided that we define the concept of a "negative" fixed point of $S_{\mathbf{w}}^{\circ k}$ in an appropriate formal manner. **Definition:** A fixed point $S_{\mathbf{w}} : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_k = x_0$ of $S_{\mathbf{w}}^{\circ k}$ is *critical* if the orbit passes through one of the critical points $1/3$ or $2/3$, and otherwise is *positive* or *negative* according as the number of x_i in the interval $(1/3\,, 2/3)$ is even or odd. Using this definition, the analogues of Lemmas 7 and 8 are easily verified.

Associated with the skeleton $S_n^{\mathrm{saw}} \subset \overline{T}$ is a *topological cell structure* on $\overline{T}$. That is, we can partition $\overline{T}$ into subsets, each of which is homeomorphic either to a point, an open interval, or a 2-dimensional open unit disk. Furthermore, these subsets fit together nicely so that the closure of each one is topologically a point, a closed interval, or a closed 2-disk. By definition, the open 2-cells in this cell structure are the connected components of the complement $\overline{T} \smallsetminus S_n^{\mathrm{saw}}$, the 0-cells are the vertices as described in §7, and the open 1-cells are the connected components of $S_n^{\mathrm{saw}} \smallsetminus \{\text{vertices}\}$. The resulting cell complex will be denoted by $\overline{T}_n^{\mathrm{saw}}$. There is a completely analogous cell complex $\overline{T}_n^{\mathrm{cub}}$ for the cubic family. These two cell complexes are homeomorphic by Theorem 2, in a homeomorphism which takes each vertex to a vertex of the same entropy and each edge to an edge with the same interval of entropies.

Lemma 9. *For each* $\epsilon > 0$ *there exists an integer* n *so that, for any two points* $\mathbf{w}$ *and* $\mathbf{w}'$ *belonging to a common closed cell of the complex* $\overline{T}_n^{\mathrm{saw}}$ *, we have*

$$|h(S_{\mathbf{w}'}) - h(S_{\mathbf{w}})| \;<\; \epsilon\ .$$

If the Connected Bone Conjecture is true, then there is an analogous statement for the complex $\overline{T}_n^{\mathrm{cub}}$ *and the family of cubic maps* $f_{\mathbf{v}}$ *.*

Proof. Otherwise we could find $\epsilon > 0$ so that for each n there existed points $\mathbf{w}_n$ and $\mathbf{w}'_n$ in a common cell of $\overline{T}_n^{\mathrm{saw}}$ with $|h(S_{\mathbf{w}'_n}) - h(S_{\mathbf{w}_n})| \geq \epsilon$. After passing to infinite subsequences, we could assume that both sequences converge, say $\mathbf{w}_k \to \mathbf{w}$ and $\mathbf{w}'_k \to \mathbf{w}'$, and furthermore that all of the $\mathbf{w}_k$ and $\mathbf{w}'_k$ belong to a common cell of $\overline{T}_n^{\mathrm{saw}}$ whenever $k \geq n$. Thus the limit points $\mathbf{w}$ and $\mathbf{w}'$ would belong to a common closed cell of $\overline{T}_n^{\mathrm{saw}}$ for every n , but by continuity the associated entropies would differ by at least ϵ . This is impossible by Lemma 8. The proof for the cubic family is similar. □

Lemma 10. *The entropy function* $\mathbf{w} \mapsto h(S_{\mathbf{w}})$ *for the stunted sawtooth family, restricted to any closed cell of the cell complex* $\overline{T}_n^{\,\mathrm{saw}}$, *takes its maximum and minimum values on the boundary (and in fact on the set of vertices). If the Connected Bone Conjecture is true, then the analogous statement holds for the entropy function* $\mathbf{v} \mapsto h(f_{\mathbf{v}})$ *for cubic maps, restricted to any closed cell of the cell complex* $\overline{T}_n^{\,\mathrm{cub}}$.

Proof. For the stunted sawtooth family, this follows easily from Theorem 1, together with the fact that the entropy function is monotone along each edge. (Compare the proof of Theorem 2.) To prove the analogous statement for the cubic family, suppose for example that for some point $\mathbf{v}_0$ of a closed cell C, the value $h(f_{\mathbf{v}_0})$ were strictly larger than the maximum value $h_{\max}$ on the boundary of C. Let $2\epsilon = h(f_{\mathbf{v}_0}) - h_{\max}$. According to Lemma 9 we can choose $m > n$ so that h varies by less than ϵ on each cell of $\overline{T}_m^{\,\mathrm{cub}}$. Let $C' \subset C$ be a cell of $\overline{T}_m^{\,\mathrm{cub}}$ which contains this point $\mathbf{v}_0$, and let $\mathbf{v}'$ be any vertex of C'. Then it follows that $h(f_{\mathbf{v}'}) > h_{\max}$. Since the homeomorphism η_m of Theorem 2 carries vertices to vertices with the same entropy, this would yield a vertex in the complex $\overline{T}_m^{\,\mathrm{saw}}$ satisfying a corresponding inequality. But this is impossible, since Lemma 10 is known to be true for the stunted sawtooth family. □

Theorem 4. *If the Connected Bone Conjecture is true, then every isentrope* $\{\mathbf{v} \in T : h(f_{\mathbf{v}}) = h_0\}$ *for the cubic family is connected. Thus, the Connected Bone Conjecture implies the Monotonicity Conjecture.*

Proof. For each n, the union of all closed cells of $\overline{T}_n^{\,\mathrm{saw}}$ which touch the h_0-isentrope forms a compact set, which is connected by Theorem 1. The corresponding union of cells in $\overline{T}_n^{\,\mathrm{cub}}$ forms a compact connected set by Theorem 2, and contains the corresponding isentrope by Lemma 10. The intersection of these sets, as $n \to \infty$, will be precisely equal to the required isentrope by Lemma 9. Since an intersection of compact connected sets is compact and connected, the conclusion follows. □

Appendix on Computation.

As a supplement to this paper, the computer programs which were used to make figures are available, in documented form, via ftp. They include subroutines to compute the topological entropy of an arbitrary unimodal or bimodal map. (The latter is based on Block and Keesling [B K].) For further information, send email to 'IMS@math.sunysb.edu'.

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