

Polynomial Diffeomorphisms of \mathbf{C}^2 . IV: The Measure of Maximal Entropy and Laminar Currents

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§1. Introduction

The simplest holomorphic dynamical systems which display interesting behavior are the polynomial maps of \mathbf{C} . The dynamical study of these maps began with Fatou and Julia in the 1920's and is currently a very active area of research. If we are interested in studying *invertible* holomorphic dynamical systems, then the simplest examples with interesting behavior are probably the polynomial diffeomorphisms of \mathbf{C}^2 . These are maps $f : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ such that the coordinate functions of f and f^{-1} are holomorphic polynomials.

For polynomial maps of \mathbf{C} the algebraic degree of the polynomial is a useful dynamical invariant. In particular the only dynamically interesting maps are those with degree d greater than one. For polynomial diffeomorphisms we can define the algebraic degree to be the maximum of the degrees of the coordinate functions. This is not, however, a conjugacy invariant. Friedland and Milnor [FM] gave an alternative definition of a positive integer $\deg f$ which is more natural from a dynamical point of view. If $\deg f > 1$, then $\deg f$ coincides with the minimal algebraic degree of a diffeomorphism in the conjugacy class of f . As in the case of polynomial maps of \mathbf{C} , the polynomial diffeomorphisms f with $\deg(f) = 1$ are rather uninteresting. We will make the standing assumption that $\deg(f) > 1$.

For a polynomial map of \mathbf{C} the point at infinity is an attractor. Thus the “recurrent” dynamics can take place only on the set K consisting of bounded orbits. A normal families argument shows that there is no expansion on the interior of K so “chaotic” dynamics can occur only on $J = \partial K$. This set is called the Julia set and plays a major role in the study of polynomial maps.

For diffeomorphisms of \mathbf{C}^2 each of the objects K and J has three analogs. Corresponding to the set K in one dimension, we have the sets K^+ (resp. K^-) consisting of the points whose orbits are bounded in forward (resp. backward) time and the set $K := K^+ \cap K^-$ consisting of points with bounded total orbits. Each of these sets is invariant and K is compact. As is in the one dimensional case, recurrence can occur only on the set K . Corresponding to the set J in dimension one, we have the sets $J^\pm := \partial K^\pm$, and the set $J := J^+ \cap J^-$. Each of these sets is invariant and J is compact. A normal families argument shows that there is no “forward” instability in the interior of K^+ and no “backward” instability in the interior of K^- . Thus “chaotic” dynamics, that is recurrent dynamics with instability in both forward and backward time, can occur only on the set J .

The techniques that Fatou and Julia used in one dimension are based on Montel's theory of normal families and do not readily generalize to higher dimensions. A different tool appears in the work of Broliin [Br], who made use of the theory of the logarithmic

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potential. Potential theory associates to any compact subset of the plane a measure which is called the harmonic or equilibrium measure, and the “potential” of this measure which is called the Green function. Brodin showed that for a polynomial map of \mathbf{C} there is an explicit dynamical formula for the Green function. He proceeded to show that the harmonic measure of the Julia set is an invariant measure with interesting dynamical properties. It was later observed that potential theory provides alternate proofs of many of the basic facts of Fatou-Julia theory (see [Si], [T], and [C]).

Potential theory in one variable has a natural extension to several complex variables called pluripotential theory (cf. Klimek [Kl]). In this context the analogs of the Green function corresponding to the sets K^+ and K^- are the functions G^+ and G^- . These functions were studied by J.H. Hubbard from a topological viewpoint (see [H] and [HO]). N. Sibony had the idea of introducing potential theory into the study of these two-dimensional mappings, he introduced the two (1,1) currents $\mu^\pm = (2\pi)^{-1}dd^c G^\pm$ and the measure $\mu = (4\pi)^{-1}(dd^c(G^+ \vee G^-))^2$. Bedford and Sibony established some properties of μ^\pm and μ , the results they obtained are contained in §3 of [BS1]. (See also [Be].) Further results are contained in [BS2–4], [FS]. In the pluri-potential context, μ is the analogue of the equilibrium (or harmonic) measure of the set K (and also of J). Hubbard and Papadopol [HP] have shown that a current like μ^+ also arises naturally from a (non-invertible) holomorphic mapping $f : \mathbf{P}^n \rightarrow \mathbf{P}^n$.

In this paper we combine potential-theoretic methods with tools from ergodic theory, especially Pesin’s theory of non-uniform hyperbolicity. These tools allows us to describe the geometric structure of the currents μ^\pm and to give a geometric description of the relation between μ^\pm and μ . The starting point for these results is a characterization of the measure μ in terms of entropy which we now describe.

We can associate to each invariant probability measure ν its measure-theoretic entropy $h_\nu(f)$. The variational principle states that the supremum of $h_\nu(f)$ taken over the set of all invariant probability measures is the topological entropy, $h_{top}(f)$. A measure ν for which $h_\nu(f) = h_{top}(f)$ is called a *measure of maximal entropy*. For polynomial maps in one dimension the topological entropy is $\log d$ where d is the degree of the polynomial (see [G] and [Lyu1]), and μ is the *unique* measure of maximal entropy (see [Lyu2] and [Ma]). In two complex dimensions the topological entropy is $\log \deg f$ (see [FM] and [S]), and $h_\mu(f) = \log \deg f$ (see [BS4]). In §3 we prove: *The harmonic measure is the unique measure of maximal entropy for a polynomial diffeomorphism of \mathbf{C}^2* (Theorem 3.1).

For polynomial maps of \mathbf{C} , Fatou and Julia used Montel’s theorem to show that expanding periodic points are dense in J . This result can also be proved using potential theory. A key observation in such a potential-theoretic proof is the fact that the support of harmonic measure is the set J . For polynomial diffeomorphisms of \mathbf{C}^2 the situation is not so straightforward. If $J^* \subset \mathbf{C}^2$ denotes the support of μ , then it follows easily that $J^* \subset J$. For polynomial diffeomorphisms which are hyperbolic, we have shown in [BS1] that $J = J^*$. But the question of whether equality holds in general seems to be very difficult.

Periodic saddle points are the analogs of expanding periodic points for two dimensional diffeomorphisms. These are points of period n for which Df^n has one eigenvalue outside and one eigenvalue inside the unit circle. It is relatively easy to show that every saddle

orbit is contained in J . In §9 we prove the more difficult result: *Every saddle orbit is contained in J^** . It was shown in [BS4] that the closure of the saddle orbits contains J^* . Combining these results gives: *J^* is the closure of the set of saddle orbits*. Thus J^* plays a role for polynomial diffeomorphisms of \mathbf{C}^2 analogous to the role played by J for polynomial maps of \mathbf{C} .

Let p be a periodic saddle point. The stable/unstable manifolds of p are defined as

$$W^{s/u}(p) := \{q \in \mathbf{C}^2 : \lim_{n \rightarrow \infty} \text{dist}(f^{\pm n}q, f^{\pm n}p) = 0\}.$$

In §2 we show that for μ almost every point p , the set $W^{s/u}(p)$ is conformally equivalent to \mathbf{C} and is a dense subset of J^\pm . This result was obtained independently by Wu in [W].

For distinct periodic saddle points, p and q , the intersections of $W^s(p)$ and $W^u(q)$ are called heteroclinic intersections. We show in §9 that J^* can be characterized in terms of heteroclinic intersections. *For any pair of periodic saddle points p and q : $J^* = \overline{W^s(p) \cap W^u(q)}$* . It is interesting to contrast this description of J^* with a similar description of J from [BS4]. *For any pair of periodic saddle points: $J = \overline{W^s(p) \cap W^u(q)}$* . The intersections of $W^s(p)$ and $W^u(p)$ other than p itself are called homoclinic intersections. It was observed in [BS4] that the set of periodic saddle points that create homoclinic intersections is dense in J^* . In §9 we prove the more delicate result that *every* periodic saddle point creates homoclinic intersections.

The harmonic measure μ and the currents μ^\pm are related by the analytic equation $\mu = \mu^+ \wedge \mu^-$. This formula does not give much geometric insight into the relation between these objects. The results on periodic saddle points and stable manifolds are consequences of a geometric description of the currents μ^\pm and the way in which these currents “intersect” to give μ . In order to explain the results of this paper about general polynomial diffeomorphisms it is useful to recall results from [BS1] about the special case of uniformly hyperbolic polynomial diffeomorphisms.

A polynomial diffeomorphism f is uniformly hyperbolic if there is a hyperbolic splitting of the tangent bundle over J . Hyperbolicity implies that for every point $p \in J$ the sets $W^{s/u}$ are immersed submanifolds. In the uniformly hyperbolic case, the collection of stable manifolds has the following “laminar” structure. At a point $p \in J$, we may let T^u be a small complex disk transversal to $W^s(p)$. For points $q \in J$ near p , the local stable manifold $W_\epsilon^s(q)$ will intersect T^u in a unique point $a \in T^u$. If we let $A^u \subset T^u$ denote the set of such intersections, then we may parametrize the local stable manifolds by $a \in A^u$, and locally J^+ is topologically equivalent to the product of A^u and a disk. Given two such transversals T_1 and T_2 and corresponding sets $A_j \subset T_j$, $j = 1, 2$, there is a (continuous) *holonomy* map $\chi : A_1 \rightarrow A_2$, defined by following a stable disk from its intersection point $a_1 \in A_1$ to the point $a_2 \in A_2$ where it intersects T_2 . This gives a homeomorphism between the intersections with nearby transversals. In [BS1] we showed that the holonomy map preserves the slice measures $\mu^+|_{T_j}$.

There is a corresponding theory, due to Pesin, of (non-uniform) hyperbolicity with respect to a measure ν . An (ergodic) measure ν is said to be hyperbolic if no Lyapunov exponent is zero. (See §2 for the relevant definitions.) The theory of Pesin for a hyperbolic measure ν implies that for ν almost every point p the sets $W^{s/u}$ are immersed submanifolds. It is shown in [BS4] that the measure μ is ergodic and hyperbolic. In the case of a hyperbolic

measure, we may define a holonomy map on a compact set of positive measure (but not necessarily everywhere). In §4 we show: *The holonomy map preserves the slice measures $\mu^+|_{T_j}$.*

In the uniformly hyperbolic case, we may take a similar transversal $T^s \subset W^s(p)$, and we may parametrize the local unstable manifolds by $A^s \subset T^s$. It follows that a neighborhood in J is homeomorphic to $A^s \times A^u$. In the case of a hyperbolic measure, it is possible to find product sets with positive measure, which we call *Pesin boxes* and denote again as $A^s \times A^u$. The measure μ induces conditional measures on each stable slice. As a byproduct of the characterization of μ as the unique measure of maximal entropy in §3 we show: *The conditional measures on the stable/unstable slices are given by $\mu^{+/-}$.* As a consequence of the holonomy invariance of μ^\pm and the identification with the conditional measures, we obtain in §4 the result: *μ restricted to a Pesin box is a product measure.* This allows us to invoke results of Ornstein and Weiss which imply that μ is Bernoulli. This is the strongest mixing property that a measure can possess.

Now let us pass from the analysis of the slice measures of μ^\pm to the currents themselves. A closed manifold M defines a current of integration, denoted by $[M]$. (See §5 for a general discussion of currents.) In the uniformly hyperbolic case, the laminar structure of $\mathcal{W}^{s/u}$ passes over to a laminar structure for μ^\pm . That is, at a point $p \in J$, we may choose an open set U and a transversal T^u such that for each $a \in A$, the local stable manifold $D^s(a)$ is a closed submanifold of U , and the restriction of μ^+ to U is given by *

$$\mu^+ \llcorner U = \int \lambda^u(a) [D^s(a)], \quad (\ddagger)$$

which is a direct integral of currents of integration with respect to λ^u , which is the measure obtained by restricting μ^+ to T^u . A current of the form (\ddagger) is called *uniformly laminar* if the manifolds $D^s(a)$ are pairwise disjoint. With the family $\mathcal{W}^{s/u}$ given by Pesin theory, there is no uniformity to the size of the manifolds, i.e. we cannot choose U such that for $M \in \mathcal{W}^{s/u}$ every component of $M \cap U$ is closed in U . In §6 we define the more general class of *laminar* currents and show that a laminar current T is given as a countable sum, $T = \sum T_j$, where the T_j 's have disjoint carriers, and T_j is uniformly laminar on some (possibly small) open set U_j . The closed, laminar currents give a natural generalization of the current of integration and seem to be an interesting class in their own right. In §7, it is shown that μ^\pm is laminar. In §8, it is shown that μ^+ contains uniformly laminar "pieces" whose structure is induced by the Pesin boxes. This allows us to show that the wedge product that defines the measure μ is in fact given by an intersection product of stable and unstable manifolds. This yields further structure for the currents μ^\pm .

Much attention has been paid to polynomial diffeomorphisms with real coefficients. In this case the real subspace $\mathbf{R}^2 \subset \mathbf{C}^2$ is invariant and we may let $f_{\mathbf{R}}$ denote the restriction of f to \mathbf{R}^2 . The (real) Hénon map is a much studied example with $\text{deg} = 2$. In contrast to the complex case, where the topological entropy is $\log d$, the topological entropy of

* Throughout this paper we use the following notation for integration. If λ is a measure on A , and if f is an integrable function on A with values in the space of currents, then we write the integral as $\int_{a \in A} \lambda(a) f(a)$.

$f_{\mathbf{R}} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ can be any real number in the interval $[0, \log d]$ (see [FM] and [Mi]). In §10 we give several equivalent criteria for the entropy of $f_{\mathbf{R}}$ to be equal to $\log d$. One of these is that $K \subset \mathbf{R}^2$, that is to say that every complex bounded orbit is actually real. A second criterion is that every periodic point of f is in \mathbf{R}^2 . A third is that: *For any hyperbolic point p , all intersection points $W^s(p) \cap W^u(p)$ lie inside \mathbf{R}^2 .* These results may be used to show that, when topological entropy is maximal, the loss of a single periodic point or homoclinic intersection forces a decrease in the the topological entropy.

This paper is divided into different parts, according to the methods that predominate. In §§2–4 the principal tools are Smooth Ergodic Theory, especially Pesin’s Theory. In §§5–7, the primary tools are the theory of currents and the Ahlfors Covering Theorem. These sections do not use Ergodic Theory. Finally, these methods are combined in §§8–9.

The specific contents are as follows. §2 gives a summary of the part of Smooth Ergodic Theory that we will use. At the end of §2 it is shown that, for μ almost every point p , the stable manifold of p is dense in J^+ and conformally equivalent to \mathbf{C} . In §3 the conditional measures are shown to be induced by the current μ^+ . (This permits estimates on the Hausdorff dimension and Lyapunov exponent at the end of the section.) Then it is shown that μ is the unique measure of maximal entropy. The holonomy map is discussed in §4, and it is shown that the holonomy of the Pesin stable manifolds preserves the restriction measures of μ^+ . Finally, it is shown that μ has a local product structure. In §5 we summarize the main ideas and definitions that we use from the theory of currents. Laminar currents are defined in §6, and the basic structure is developed. In §7 it is shown that μ^{\pm} are laminar currents. In §8 we show that the laminar structure of μ^+ coincides with the structure induced by the Pesin manifolds and the conditional measures. And in §9 we apply the previous work to the study of saddle points. Real Hénon mappings are discussed in §10, and several (equivalent) criteria are given for f to be essentially real. §11 is an appendix which outlines an alternative sequence in which the results of this paper can be obtained. This alternate approach starts with results of Pesin theory and then proceeds to the theory of currents. The main difference is that the use of the methods of entropy theory is delayed until the end.

§2. Preliminaries from Ergodic Theory

2.1. Measurable Partitions and Conditional Measures

The technique of measurable partitions developed by Rokhlin [Ro1] is a powerful tool in measure theory. Somehow it is not widely known beyond ergodic theory. So, we will spend some time to define the main concepts and to establish notation.

Let J be a compact metric space, and let ν be a probability Borel measure on J . A *partition* $\xi = \bigcup \xi_{\alpha}$ of J is a decomposition of J into disjoint, measurable subsets. The element of the partition containing x will be denoted by $\xi(x)$, and will be called the *fiber* through x . Note that all fibers can have zero measure. For example we can consider a partition ε into single points. Two partitions are considered to be equivalent if they coincide on a subset J' of full measure.

Each measurable function ϕ generates a partition whose fibers are level sets of ϕ . Such partitions are called *measurable*. Any countable partition is measurable. An orbit partition of an irrational rotation of the circle (with Lebesgue measure) gives an example

of non-measurable partition. More generally, one can consider an orbit partition of any ergodic transformation; see the discussion below.

The basic property of measurable partitions is for any measure ν there is a *family of conditional measures* $\nu(\cdot|\xi(x))$ on the fibers. This family is uniquely determined by the following properties:

- (i) Each $\nu(\cdot|\xi(x))$ is a probability measure on $\xi(x)$;
- (ii) For any integrable function ϕ , the function

$$\phi_\xi(x) = \int \phi(y) \nu(y|\xi(x))$$

along the fibers is measurable and integrable, and

(iii)

$$\int \phi_\xi(x) \nu(x) = \int \phi \nu.$$

Remark. The above averaging of ϕ over the conditional measures is equivalent to taking of the conditional expectation of ϕ with respect to the σ -algebra generated by ξ .

By “countable” set we will mean “at most countable”. If we have a countable family of measurable partitions ξ_i then we can construct a partition $\bigvee \xi_i$ by intersecting fibers of ξ_i , i.e.

$$(\bigvee \xi_i)(x) = \bigcap \xi_i(x).$$

One can check that this construction leads to a measurable partition.

Finally, let us mention that for an arbitrary (non-measurable) partition η there exists its *measurable envelope*, i.e. the finest measurable partition which is coarser than η .

2.2. Elements of Entropy Theory

The reader can see [Ro2] or [CFS] for the background in entropy theory. Our exposition will be adapted to our goals (in particular, it will not be as general as possible).

Entropy of a countable (mod 0) measurable partition $\xi = \{\xi_i\}$ is defined as

$$H_\nu(\xi) := - \sum \nu(\xi_i) \log \nu(\xi_i) = \int \log \frac{1}{\nu(\xi(x))} \nu(x) \quad (2.1)$$

(it can be infinite). If the partition is not countable then its entropy is infinite by definition.

If we have two measurable partitions ξ and η then we can restrict ξ on the fibers of η , thus we can calculate the entropy of ξ with respect to η in terms of the conditional measures as $H_{\nu(\cdot|\eta(x))}(\xi|\eta(x))$. We then define the *conditional entropy* by averaging this with respect to ν :

$$H_\nu(\xi|\eta) := \int H_{\nu(\cdot|\eta(x))}(\xi|\eta(x)) \nu(x).$$

Let us consider now a homeomorphism $f : J \rightarrow J$ preserving a measure ν . Then it naturally acts on the space of measurable partitions $\xi \mapsto f\xi$, where $(f\xi)(x) = \xi(f^{-1}x)$.

A partition ξ is called *f-invariant* if $f\xi$ is a refinement of ξ . A partition ξ is called a *generator* if

$$\bigvee_{n=-\infty}^{\infty} f^n \xi = \varepsilon.$$

Given a partition ξ , consider the f^{-1} -invariant partition $\xi^u = \bigvee_{n=0}^{\infty} f^n \xi$. Let us call the fibers of this partition ξ -*unstable* fibers. We can define the Jacobian $J^u f$ of f in the “ ξ -unstable direction” as the Radon-Nikodym derivative of f with respect to conditional measures:

$$J^u f(x) = \frac{df^* \nu(\cdot | \xi^u(fx))}{d\nu(\cdot | \xi^u(x))}.$$

Since ν is invariant, $J^u f$ is constant on the fibers of $f^{-1}\xi^u$, and hence

$$J^u f(x) = \frac{1}{p(x)} \tag{2.2}$$

where $p(x) = \nu(f^{-1}\xi^u(fx) | \xi^u(x))$. Now define entropy of f with respect to ξ as

$$h_\nu(f, \xi) = H_\nu(f^{-1}\xi^u | \xi^u) = - \int \log p(x) \nu(x) = \int \log J^u f(x) \nu(x) \tag{2.3}$$

(the middle equality follows from (2.1)). So, from the dynamical point of view entropy of a transformation with respect to a partition is just the logarithm of the geometric average of the Jacobian of f in the ξ -unstable direction.

Finally, the entropy of ν with respect to f is defined as

$$h_\nu(f) = \sup_{\xi} h_\nu(f, \xi)$$

where supremum is taken over all measurable partitions ξ . Actually, one can take the supremum over finite partitions only. Moreover, it is enough to evaluate entropy of any generator with finite entropy:

Proposition 2.1. *If ξ is a generator with finite entropy then $h_\nu(f) = h(f, \xi)$.*

In conclusion let us discuss the ergodic decomposition of the transformation f . Let us consider the *orbit partition* O of f (whose fibers are orbits of f). The transformation f is ergodic if the measurable envelope of O is a trivial partition (whose only fiber is the whole space).

In general, let us consider the measurable envelope E of O . The fibers of ε supplied with conditional measures are called *ergodic components* of μ . Note that all ergodic components may have zero measure (consider the identity transformation). However it makes sense to consider the whole space of these components mod 0. Restricting f onto ergodic components and then taking them together we obtain a representation of f as a “direct integral” of ergodic transformations. It follows from (2.3) that

$$h(f) = \int h(f|E(x)) \nu(x). \tag{2.4}$$

This formula gives a method for reducing entropy questions for arbitrary measures to the case of ergodic measures. If f is ergodic then it has a finite generator by the Krieger Theorem [Kr]. So, in the ergodic case we can always compute entropy according to Proposition 2.1.

For ergodic ν let us say that a point x is ν -equidistributed if for any continuous function ϕ

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x) = \int \phi \nu.$$

By the Birkhoff Ergodic Theorem ν almost every point is ν -equidistributed.

2.3. Measures of maximal entropy

For the material of this section we refer to Bowen's book [Bo]. We will not define the topological entropy of f , but a basic property is given by the so-called Variational Principle, which asserts that the topological entropy $h(f)$ is given as

$$h(f) = \sup h_\nu(f) \tag{2.5}$$

where ν runs over all probability Borel measures invariant with respect to f .

A measure μ is called a *measure of maximal entropy* if $h_\mu(f) = h(f)$. This measure does not necessarily exist, but if it does, then by (2.4) all its ergodic components are measures of maximal entropy as well. Hence, existence/uniqueness of a measure of maximal entropy are equivalent to the existence/uniqueness of an *ergodic* measure of maximal entropy.

The problem of uniqueness of the measure of maximal entropy is not handled yet in a general setting. The status of the existence problem is much better:

Newhouse Theorem [Ne]. *If $f : M \rightarrow M$ is a C^∞ diffeomorphism of a compact C^∞ -manifold then f has a measure of maximal entropy.*

2.4. Stable and unstable manifolds

Some basic references for the material in this section are [P1], [FHY], [R2] and [PS]. Let M be a Riemannian C^2 -manifold, $f : M \rightarrow M$ be a C^2 -diffeomorphism, J be an invariant compact subset of M . Let ν be an invariant *ergodic* measure of f supported on J . As usual $T_x M$ denotes the tangent space at x . A measurable function $r(x)$ is called *ϵ slowly varying* if

$$(1 + \epsilon)^{-1} r(x) < r(fx) < (1 + \epsilon) r(x).$$

Oseledec Theorem. *There exist finitely many distinct real numbers χ_i , $i = 1, \dots, s$ called characteristic exponents, an invariant set \mathcal{R} of full measure, and s invariant measurable distributions $E_i(x) \subset T_x M$, $x \in \mathcal{R}$, such that*

- (i) $T_x M = \oplus E_i(x)$;
- (ii) For any nonzero $v \in E_i(x)$,

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|Df^n(x)v\| = \chi_i.$$

(iii) For $i \neq j$ and $\epsilon > 0$ there is an ϵ slowly varying function $s_{ij}(x) > 0$ which is less than the angle between $E_i(x)$ and $E_j(x)$.

The points of the set \mathcal{R} are called *regular*. We can also assume that \mathcal{R} consists of ν -equidistributed points.

Let us state now the Pesin Theorem which says that the above distributions are integrable. Denote by $B(x, r)$ a ball of radius r centered at x , and $B^{s/u}(x, r) = E^{s/u}(x, r) \cap B(x, r)$. Now let us define *stable* and *stable-center* distributions

$$E^s(x) = \bigoplus_{\chi_i < 0} E_i, \quad E^{sc}(x) = \bigoplus_{\chi_i \leq 0} E_i.$$

Similarly one can define the unstable and unstable-center distributions $E^u(x)$ and $E^{uc}(x)$.

Pesin Theorem [P1]. *Let $\dim E^s > 0$. Then for any $\epsilon > 0$ there are ϵ -slowly varying positive functions $C(x) = C_\epsilon(x)$ and $r(x) = r_\epsilon(x)$ on \mathcal{R} , and a family $W_{loc}^s(x)$, $x \in \mathcal{R}$ of smooth manifolds satisfying the following properties:*

- (i) $W_{loc}^s(x)$ is a graph of a function $B^s(x, r(x)) \rightarrow E^{uc}(x)$ tangent to $E^s(x)$ at x ;
- (ii) For any $y \in W_{loc}^s(x)$ and $n = 1, 2, \dots$

$$C(x)^{-1} \exp(-(\lambda + \epsilon)n) \leq \text{dist}(f^n x, f^n y) \leq C(x) \exp(-(\lambda - \epsilon)n);$$

(iii) *The f underflows the manifolds $W_{loc}^s(x)$: $fW_{loc}^s(x) \subset W_{loc}^s(fx)$.*

The manifolds $W_{loc}^s(x)$ are called *local stable manifolds*. For $r \leq r(x)$ let $W_r^s(x)$ be a part of $W_{loc}^s(x)$ lying over $B^s(x, r)$. In order to obtain the Theorem on *local unstable manifolds* $W_{loc}^u(x)$ we just interchange the roles of f and f^{-1} .

Let us indicate one immediate consequence of this result.

Proposition 2.3. *If the measure ν is not supported on a periodic orbit then all characteristic exponents cannot be negative (positive).*

Proof. Otherwise $W_{loc}^s(x) = B(x, r(x))$. Since ν -equidistributed points $x \in \mathcal{R}$ are recurrent, we can find a moment $n > 0$ such that f^n maps $B(x, r)$ into itself uniformly contracting it. It follows that x is periodic, and μ is supported on its orbit. \square

The family of local unstable manifolds does not form a partition. The following statement supplies us with a f^{-1} -invariant measurable partition (called a *Pesin partition*) subordinate to the family of manifolds.

Theorem 2.4 (see [P2], [LS]). *There is a measurable f^{-1} -invariant generator ξ^u whose fibers are open subsets of the local unstable manifolds, and such that*

$$h_\nu(f) = h_\nu(f, \xi^u).$$

Remark. When f has no zero characteristic exponents then the Pesin partition ξ^u is a ξ -unstable partition for some partition ξ with finite entropy [LY]. So, in this case the above entropy formula follows from Proposition 2.1.

Let us now define the *global unstable manifold* $W^u(x)$ at x as the set of points y whose backward orbits are asymptotic to the backward orbit of x . Clearly $fW^u(x) = W^u(fx)$. One can prove that for $x \in \mathcal{R}$

$$W^u(x) = \bigcup f^n W^u(f^{-n}x).$$

This implies the following two consequences:

- (i) The backward orbits $y \in W^u(x)$ are exponentially asymptotic to the orbit of x .
- (ii) The set $W^u(x)$ is an immersed Euclidean space.

The global unstable manifolds form the partition of the measure space (J, ν) which we will call the *global unstable partition*. This partition is in general not measurable.

A partition τ is called *hyperfinite* if there is a sequence of measurable partitions ξ_i such that

$$\xi_1(x) \subset \xi_2(x) \subset \dots, \text{ and } \tau(x) = \bigcup \xi_i(x).$$

Let us call a measure defined up to a scalar factor a *projective measure class*. On a fiber of a hyperfinite partition one can define a conditional projective measure class $\bar{\nu}(\cdot|\tau(x))$ as the class of the measure:

$$\nu(\cdot|\tau(x)) = \lim_{i \rightarrow \infty} \frac{\nu(\cdot|\xi_i(x))}{\nu(\xi_1(x)|\xi_i(x))}.$$

Thus for any other sequence ξ'_n which generates τ , the measure $\nu'(\cdot|\tau(x))$ obtained in this way will be a multiple of the measure above by a constant depending only on x . In fact, for any measurable partition η subordinate to τ , the conditional measures on the fibers of η are just the normalized projective measure classes of τ .

Proposition 2.5. *The global unstable partition is hyperfinite.*

Proof. Take a Pesin partition ξ^u , and represent the global unstable partition as the limit of measurable partitions $f^{-n}\xi^u$. □

It is evident that the preceding discussion may be applied equally well to the stable direction instead of the unstable one.

2.5. Relations between entropy and characteristic exponents

The following inequality was discovered by Margulis in the case of an absolutely continuous measure. It was later generalized by Ruelle [R1]:

Margulis-Ruelle inequality.

$$h_\nu(f) \leq \sum_{\chi_i > 0} \chi_i \dim E_i,$$

and a corresponding inequality holds with the sum of negative characteristic exponents.

Corollary. *If $h_\nu(f) > 0$, and if f has at most two characteristic exponents, then one of these exponents is negative, and another is positive.*

In such a situation we will denote the negative and positive exponents χ^s and χ^u correspondingly.

More recently a number of remarkable relations between entropy, characteristic exponents and Hausdorff dimension have been discovered (see Pesin [P1] and Ledrappier-Young [LY] and the references there.) The *Hausdorff dimension* of a measure ν , written $HD(\nu)$, is defined as the infimum of the Hausdorff dimension of X , for all Borel subsets X with full ν measure. Clearly, the Hausdorff dimension depends on the measure class only.

Lai-Sang Young's Formula [Yg]. *Assume that f has only one characteristic exponent $\chi^s < 0$. Then for ν a.e. x ,*

$$HD(\bar{\nu}(\cdot|W^s(x))) = \frac{h_\nu(f)}{|\chi^s|}.$$

2.6. Complex analytic case

Let M be a Hermitian complex analytic manifold and let f be analytic. Then $E_i(x)$ are complex subspaces in $T_x M$, and all local manifolds are complex analytic.

Assume now that $\dim_{\mathbf{C}} M = 2$, and let ν be any invariant probability measure with two non-zero characteristic exponents of opposite signs, $\chi^s < 0$ and $\chi^u > 0$. (In particular, this will be the case if $h_\nu(f) > 0$, see the Corollary of the Margulis-Ruelle inequality). Hence $\dim_{\mathbf{C}} E^{s/u} = 1$, the global stable/unstable manifolds are regular complex curves. The following statement says that almost all of them are parabolic.

Proposition 2.6. *The stable and unstable manifolds $W^u(x)$ and $W^s(x)$ are conformally equivalent to the complex plane for ν a.e. x .*

Remark. It is possible to prove Proposition 2.6 along the lines of the proof of Theorem 5.4 of [BS1]. That is, for $x \in \mathcal{R}$, $W^u(x)$ contains a sequence of disks $D_1 \subset D_2 \subset \dots$ such that the modulus of $D_{j+1} - D_j$ is bounded below. From this, it follows that $W^u(x)$ is equivalent to \mathbf{C} . To sketch this argument, we note that $W_{r(f^{-k}x)}^u(f^{-k}x)$ is a graph over a disk of radius $r(f^{-k}x) \geq c(1 + \epsilon)^{-k}r(x)$. On the other hand, the derivative of f^{-k} on $W_{r(x)}^u(x)$ is approximately $e^{-n\chi^u}$. For a small disk D_j containing x inside $W^u(x)$, we may choose n sufficiently large that the modulus of the annulus $W_{r(f^{-n}x)}^u(f^{-n}x) - f^{-n}D_j$ is at least 2. Then we may let $D_{j+1} = f^n W_{r(f^{-n}x)}^u(f^{-n}x)$.

Remark. The proof we give below uses a technique that will also be used in §3. Two measurable functions α and β are called *cohomologous* if there is a measurable function ω such that the following cohomology equation

$$\alpha(x) - \beta(x) = \omega(fx) - \omega(x)$$

is satisfied ν -almost everywhere.

Usually, the cohomology equation comes up when we calculate the logarithm of the Jacobian (or norm) of f with respect to two equivalent measures (metrics). The following statement will be useful on several occasions.

Lemma 2.7. *Let α be a measurable function bounded from below. If α is cohomologous to 0 then*

$$\int \alpha \nu = 0.$$

This is trivial if ω is integrable. Otherwise, the proof is based upon the Birkhoff ergodic theorem (see, e.g., [LS, Proposition 2.2].)

Proof of Proposition 2.6. We consider the unstable manifolds $W^u(x)$. Let $F_{W^u(x)}$ denote the Kobayashi metric on $W^u(x)$. This metric depends in a lower semicontinuous manner on x if $W^u(x)$ depends continuously on x . And since we may find compact subsets of J of measure arbitrarily close to 1 on which $W^u(x)$ depends continuously on x , the correspondence $x \mapsto F_{W^u(x)}$ is measurable. $W^s(x)$ is conformally equivalent to either a plane or a disk, depending on whether $F_{W^u(x)}(x, E^u) = 0$ or not. By ergodicity, the type of $W^u(x)$ is the same for almost all x . We assume that it is hyperbolic and derive a contradiction.

Let $\alpha(x) = \log \|Df(x)|_{E^u}\|$, where the norm is taken with respect to the Hermitian metric on M . Similarly, for $x \in \mathcal{R}$ we define the function $\beta(x) = \log |Df(x)|_{E^u}|$, where $|Df(x)|_{E^u}|$ denotes the norm taken with respect to the Kobayashi metric. If we let $\rho(x)$ denote the ratio of the Hermitian to the Kobayashi metrics in the unstable direction, then α and β are cohomologous in the sense that

$$\alpha(x) - \beta(x) = \log \rho(fx) - \log \rho(x).$$

But f is an isomorphism between $W^u(x)$ and $W^u(fx)$ and hence preserves the Kobayashi metric. So $\beta(x) = 0$ almost everywhere, and α is cohomologous to 0. By Lemma 2.7,

$$\chi^u \equiv \int \alpha \nu = 0$$

contradicting the assumption that $\chi^u > 0$. □

Assume now that f is a polynomial automorphism of \mathbf{C}^2 , and J^+ , J^- , $J = J^+ \cap J^-$ etc., be the sets introduced in §1. Consider also the currents μ^+ and μ^- . We can “slice” the current μ^+ with any complex one dimensional variety W (see §5). The result can be interpreted as a measure on W which we denote by $\mu^+|_W$. In particular, we can consider the measure $\mu^+|_{W^u(x)}$ on the unstable leaf containing x . We will call it an *unstable slice* of μ^+ .

Lemma 2.8. *Any ν regular point $x \in \mathcal{R}$ belongs to the support of $\mu^+|_{W^u(x)}$.*

Proof. Let Δ be a disk with $x \in \Delta \subset W^u(x)$. If $\mu^+(\Delta) = 0$ then G^+ is harmonic in Δ . Since the orbit of x is bounded $G^+(x) = 0$ and by the minimum principle G^+ is zero in Δ . Thus $\Delta \subset K^+$. It follows that $f^n(\Delta) \subset K^+$ so $f^n(\Delta)$ is uniformly bounded for all n . By the Schwartz lemma $\|D(f^n|_\Delta)\| \leq C$ but this contradicts the fact that at a regular point the Lyapunov exponent is positive in the unstable direction. □

Proposition 2.9. *For ν a.e. x , $W^u(x)$ is a dense subset of J^- , and $W^s(x)$ is a dense subset of J^+ .*

Proof. For $x \in \mathcal{R}$ it is evident that $W^s(x) \subset K^+$. We show that $W^s(x) \subset J^+$. Let us suppose that $y \in W^s(x) \cap \text{int}K^+$. Since the iterates f^n for $n \geq 1$ form a normal family, it follows that $\|Df_y^n\|$ is bounded. The fact that $y \in W^s(x)$ implies that $d(f^n(x), f^n(y)) \leq Cr^n$ for $r < 1$. This in turn implies that $\|Df_{f^n(x)} - Df_{f^n(y)}\| \leq C'\rho^n$ with some $\rho < 1$. It

follows from [R, Theorem 4.1] that the asymptotic behavior of Df_x^n and Df_y^n is the same. In particular

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df_y^n\| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df_x^n\| = \chi^u$$

We see that $\|Df_y^n\|$ is therefore not bounded. This completes the proof that $W^s(x) \subset J^+$.

We now show that $W^s(x)$ is dense in J^+ . Let D^s denote a disk inside $W^s(x)$ with $x \in D^s$ and such that $\mu^-|_{W^s(x)}(\partial D^s) = 0$. By [BS3], $d^{-n}[f^{-n}D^s]$ converges to $c\mu^+$ as $n \rightarrow \infty$, with

$$c = \mu^-|_{W^s(x)}(D^s).$$

By Lemma 2.8 (in the “stable” setting) $c > 0$.

Now let U be an open set with $U \cap J^+ \neq \emptyset$. Thus $\mu^- \lrcorner U \neq 0$, and so $(f^{-n}D^s) \cap U \neq \emptyset$ for all n greater than some large N . It follows that

$$f^{-n}W^s(x) \cap U = W^s(f^{-n}x) \cap U \neq \emptyset.$$

Let $S_N = \{x \in \mathcal{R} : W^s(f^{-n}x) \cap U \neq \emptyset \text{ for } n \geq N\}$. Clearly, $S_1 \subset S_2 \subset \dots$, and by the previous remark $\bigcup S_N = \mathcal{R}$. Further, $fS_N = S_{N+1}$, and since f is ergodic, each S_N has measure 0 or 1. Thus it follows that S_0 has full measure, which completes the proof. \square

Remark. If p is a (periodic) saddle point, then the average ν of point masses over the orbit of p is a hyperbolic measure. Thus Theorem 1 of [BS3] is a consequence of Proposition 2.9.

§3. The unique measure of maximal entropy

The goal of this section is to prove the following Uniqueness Theorem. Our approach is reminiscent of the proof of the uniqueness theorem for rational endomorphisms of \mathbf{C} given in [EL] and also of Ledrappier’s proof of the “Variational Principle” for absolutely continuous invariant measures [Le]. An important consequence of the proof is that the conditional projective measure class of μ on the unstable foliation is induced by the current μ^+ . At the end of the section we will derive estimates of Hausdorff dimension and characteristic exponents.

Theorem 3.1. *The measure μ is the unique measure of maximal entropy.*

Note that this result gives a characterization μ in terms of topological dynamics which makes no reference to potential theory.

By Theorem 2.2 and the comment preceding it, there is an ergodic measure ν of maximal entropy, $h_\nu(f) = \log d$. We are going to show that $\nu = \mu$ which yields Theorem 3.1. In fact, this gives an alternative proof that μ is a measure of maximal entropy, which was originally proved in [BS4] (yet another approach is outlined in §11).

By the Corollary of the Margulis-Ruelle inequality, ν has two non-zero characteristic exponents of opposite signs, $\chi^s < 0$ and $\chi^u > 0$. So, we can consider the complex one dimensional unstable foliation and the projective measure class $\bar{\nu}(\cdot|W^u(x))$ on its leaves (see Section 2.4). On the other hand, we can consider measures $\mu^+|_{W^u(x)}$ induced by the current μ^+ (see Section 2.6). For an open set $B \subset W^u(x)$, we will use the notation $\mu^+(B) := \mu^+|_{W^u(x)}(B)$.

Proposition 3.2. *If ν is a measure of maximal entropy then for ν almost all x , the conditional projective measure class $\bar{\nu}(\cdot|W^u(x))$ is induced by the current μ^+ .*

Proof of Proposition 3.2. The Jacobian $J_{\mu^+}^u$ of f with respect to the family of unstable slices of μ^+ is equal to $\log d$ since for any $B \subset W^u(x)$ we have

$$\mu^+(fB) = f^*\mu^+(B) = d$$

Let ξ^u be the unstable Pesin partition for ν . By Lemma 2.8 $\rho(x) \equiv \mu^+(\xi^u(x)) > 0$. So, we can normalize the above family of measures in order to get probability measures on the Pesin pieces:

$$\eta(B|\xi^u(x)) = \mu^+(B)/\rho(x).$$

Then the Jacobian J_η^u is multiplicatively cohomologous to the Jacobian $J_{\mu^+}^u$, that is:

$$J_\eta^u(x) = d \frac{\rho(x)}{\rho(fx)}.$$

So, $\log J_\eta^u(x)$ is (additively) cohomologous to $\log d$:

$$\log J_\eta^u(x) - \log d = \log \rho(x) - \log \rho(fx),$$

This formula and the following property of the function $\rho(x)$ imply that $\log J_\eta^u(x)$ is positive.

Claim. $\rho(f(x)) \leq d\rho(x)$.

By the increasing property of ξ^u , we have $(f^{-1}\xi^u)(x) \subset \xi^u(x)$. So

$$\mu^+((f^{-1}\xi^u)(x)) \leq \mu^+(\xi^u(x)).$$

On the other hand $f((f^{-1}\xi^u)(x)) = \xi^u(f(x))$. So

$$\mu^+(\xi^u(f(x))) = d\mu^+((f^{-1}\xi^u)(x)).$$

Thus $\mu^+(\xi^u(f(x))) \leq d\mu^+(\xi^u(x))$ as was to be shown.

Hence $\log J_\eta^u \geq 0$, and Lemma 2.7 yields

$$\int \log J_\eta^u(x) \nu(x) = \log d,$$

or

$$- \int \log q(x) \nu(x) = \log d \tag{3.1}$$

where $q(x)$ is the η -measure of $(f^{-1}\xi)(x)$.

On the other hand, set $p(x) = \nu(f^{-1}\xi^u(fx)|\xi^u(x))$. Then by (2.3) and Theorem 2.4

$$- \int \log p(x) \nu(x) = h_\nu(f, \xi^u) = h_\nu(f) = \log d. \tag{3.2}$$

From (3.1) and (3.2) we conclude

$$\int \log \frac{q(x)}{p(x)} \nu(x) = 0.$$

But

$$\int \frac{q(x)}{p(x)} \nu(x) = \int_X \left(\sum_{\xi^u(y)/f^{-1}\xi^u} \frac{q(z)}{p(z)} p(z) \right) \nu(y) = 1.$$

By concavity of \log , we get $q(x) = p(x)$ almost everywhere. Thus conditional measures coincide on the partition $f^{-1}\xi^u|\xi^u(y)$ for ν almost all y . The same argument applied to f^n shows that they coincide on $f^{-n}\xi^u|\xi^u(y)$. Since

$$\bigvee_{n=0}^{\infty} f^{-n}\xi^u = \epsilon$$

we conclude that $\nu(\cdot|\xi^u(y)) = \eta|\xi^u(y) \equiv \mu^+(\cdot|\xi^u(y))$.

Since $f^n\xi^u$ is also the Pesin partition for any n , the conditional measures of ν and μ^+ coincide on it. Passing to the limit as $n \rightarrow \infty$, we get the required agreement of ν and μ^+ . \square

Proof of Theorem 3.1. By the ergodic theorem ν -almost every point p is equidistributed with respect to ν , that is

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(p)) = \int \phi \nu \quad (3.3)$$

holds for any continuous function ϕ on \mathbf{C}^2 with compact support. Let $\nu_x = \nu(\cdot|\xi^u(x))$ denote the conditional measure on the Pesin piece $\xi^u(x)$. Then for almost every x we have that ν_x -almost every point in $\xi^u(x)$ is equidistributed with respect to ν . By bounded convergence we can average (3.3) over $\xi^u(x)$:

$$\begin{aligned} \int \phi \nu &= \lim_{n \rightarrow \infty} \int \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(p)) \nu_x(p) \\ &= \lim_{n \rightarrow \infty} \int \phi(p) \left(\frac{1}{n} \sum_{i=0}^{n-1} f_*^i(\nu_x) \right)(p) \end{aligned}$$

Since this holds for any continuous function we have:

$$\frac{1}{n} \sum_{i=0}^{n-1} f_*^i(\nu_x) \rightarrow \nu \quad (3.4)$$

in the weak topology of measures.

On the other hand, let $\mu_x^+ = \mu^+(\cdot|\xi^u(x))$ denote the normalized measure $\mu^+|\xi^u(x)$. Then it follows from [BS4] that

$$f_*^i(\mu_x^+) \rightarrow \mu. \quad (3.5)$$

To see this we observe that the proof of Lemma 4.1 in [BS4] is still valid if L is a continuous function and S is a current for which the conclusion of Theorem 1.6 holds. We take $L = G^+$ and S to be the current of integration of the set $\xi^u(x)$. In the notation of [BS4] we have $\Theta_n = \mu^+$ and $\nu_n = d^n \mu_x^+$. The conclusion of Theorem 1.6 holds for S by the third remark following the proof of Theorem 1.6 (cf. example following Corollary 1.7).

Since $\nu_x = \mu_x^+$ by Proposition 3.2, properties (3.4) and (3.5) yield $\nu = \mu$. This completes the proof of the theorem. \square

Corollary 3.3. *The conditional projective measure class $\bar{\mu}(\cdot|W^u(x))$ is induced by the current μ^+ .*

Remark. The Jacobian of f with respect to the conditional measures on the unstable manifolds is thus d . This can be considered as the natural analogue of the balanced property of the Brolin measure.

It is known that for a polynomial endomorphism P of the complex plane the characteristic exponent χ of the measure of maximal entropy (which coincides with harmonic measure of the Julia set $J(P)$) is greater or equal than $\log d$. Moreover, $\chi = \log d$ if and only if $J(P)$ is connected (see [Man], [Pr]). Here we discuss related properties of polynomial automorphisms of \mathbf{C}^2 .

It follows from the Lai Sang Young formula (see §2.5) and Corollary 3.3 that for μ a.e. x ,

$$HD(\mu^-|_{W^u(x)}) = \frac{h_\mu(f)}{|\chi^s|} = \frac{\log d}{|\chi^s|}. \quad (3.6)$$

Let us consider for a moment the dissipative case, i.e. $|a| < 1$, where a is the (constant) Jacobian determinant of f . We have $|\chi^s| = \chi^u - \log |a|$. It was shown in [BS4] that $\chi^u \geq \log d$, so in this case, Young's formula gives

$$HD(\mu^-|_{W^s(x)}) < 1 \quad (3.7)$$

for μ a.e. x . By Corollary 3.3 and the fact that the conditional measures are in the measure class of harmonic measure, we have:

Corollary 3.4. *If f is dissipative, then for μ a.e. x the harmonic measure of $W^s(x) \cap J^-$ inside $W^s(x)$ has Hausdorff dimension strictly less than 1.*

In the following result, we relate the topological property of the connectedness of J to the rate of expansion of f .

Theorem 3.5. *If the map f is hyperbolic, and if J is connected, then $\chi^u = \log d$ and $\chi^s = \log |a| - \log d$.*

Proof. Since f is hyperbolic, J has a local product structure at any point p . That is, there are neighborhoods V^s of p in $W^s(p)$ and V^u in $W^u(p)$ such that $(V^s \cap J^-) \cap (V^u \cap J^+)$ is homeomorphic to a neighborhood of p in J .

We claim that either $W^s(p) \cap J^-$ or $W^u(p) \cap J^+$ has the property: There is a neighborhood U of p such that every connected component of $W^s(p) \cap J^- \cap U$ (resp. $W^u(p) \cap J^+ \cap U$) is noncompact. For otherwise there are compact connected components which are arbitrarily small and arbitrarily close to p inside both $W^s(p) \cap J^-$ and $W^u(p) \cap J^+$. Thus the product neighborhood of p in J contains arbitrarily small compact, connected components. But in this case, J is not connected, which proves the claim.

Thus we may assume that $W^u(p) \cap J^+$ has this property. If $q \in J$ is close to p , then by the local product structure, $W^u(q) \cap J^+$ also has this property. It follows that $U \cap W^u(q) - J^+$ is simply connected. By a Theorem of Makarov [Mak], the harmonic measure of $W^u(q) \cap J^+$ has Hausdorff dimension 1. Now since this holds for a set of q of positive measure, we conclude from the formula of Young, with the stable manifolds replaced by the unstable manifolds, that $\chi^u = \log d$. \square

§4. Product Structure of μ

In this Section we will show that there are sets (“Pesin boxes”) on which μ has a local product structure, and the union of these sets has full μ measure. The main step in doing this is to study the holonomy map along the stable/unstable manifolds and to show that the conditional measures of μ are preserved by the holonomy map.

We consider a family \mathcal{M} of complex manifolds. A complex manifold D is a *transversal* to \mathcal{M} if D intersects each $M \in \mathcal{M}$ in a unique point, and this intersection is transverse. Let D_1 and D_2 be two transversals to \mathcal{M} , and set

$$X_i = \bigcup_{M \in \mathcal{M}} D_i \cap M$$

for $i = 1, 2$. We define the *holonomy map*

$$\chi := \chi(D_1, D_2, \mathcal{M}) : X_1 \rightarrow X_2$$

as $\chi(x_1) = M(x_1) \cap D_2$, where $M(x_1) \in \mathcal{M}$ is the unique manifold containing x_1 .

Throughout this Section we will consider the case where \mathcal{M} is a family of stable (or unstable) disks given by the Pesin theory (see the discussion in §2). If $r > 0$ is sufficiently small, and if $x \in \mathcal{R}$ satisfies $r(x) \geq r$, then each stable (or unstable) disk $W_r^s(x)$ (or $W_r^u(x)$) is given as a graph over the r -ball in the tangent space $E^s(x)$ (or $E^u(x)$). More generally, we will work with complex disks that are graphs over $E_r^u(x)$, i.e. which have the form

$$M = \{(z, \varphi(z)) : z \in E_r^u(x), \varphi(z) \in E_r^s(x)\}.$$

We will say that two such graphs are C^1 *close* if their corresponding graphing functions φ are C^1 close.

For any subset $F \subset \{x \in \mathcal{R} : r(x) \geq r\}$, we write

$$W_{loc}^{s/u}(F) \equiv W_r^{s/u}(F) = \bigcup_{x \in F} W_r^{s/u}(x) \quad \text{and} \quad \mathcal{F}^{s/u} = \{W_r^{s/u}(x) : x \in F\}. \quad (4.1)$$

For $0 < \kappa < 1$ and for $x_0 \in \mathcal{R}$ with $r(x_0) \geq r$, we let $F = \{x \in \mathcal{R} \cap B(x_0, \kappa r) : r(x) \geq r\}$. We may choose κ sufficiently small that if D_1 and D_2 be disks such that D_j is within κr (in the C^1 topology) of $W_r^u(x_j)$ for some $x_j \in F$, then D_j is transversal to the family \mathcal{F}^s for $j = 1, 2$. It follows that the holonomy map $\chi(D_1, D_2, \mathcal{F}^s)$ is defined.

For $m_0 > 0$, $r_0 = r/8 > 0$, and $x \in \mathcal{R}$ such that $r(x) \geq r$, we consider the property

$$\mu^+|_{W_r^u(x)}(W_{r_0}^u(x)) \geq m_0. \quad (4.2)$$

For $C < \infty$ we also consider the properties

$$\text{dist}(f^n(x), f^n(y)) \leq Ce^{-n\lambda} \quad \text{for } n \geq 1 \text{ and } y \in W_r^s(x) \quad (4.3)$$

$$\text{dist}(f^{-n}(x), f^{-n}(y)) \leq Ce^{-n\lambda} \quad \text{for } n \geq 1 \text{ and } y \in W_r^u(x) \quad (4.4)$$

Choosing $m_0 > 0$ sufficiently small and $C < \infty$ sufficiently large, we have

$$\mu(J - Q) < \epsilon \quad (4.5)$$

where

$$Q = \{x \in \mathcal{R} : r(x) \geq r, \text{ and (4.2), (4.3), (4.4) hold}\}.$$

Now let

$$S = \{x \in J : f^n(x) \in Q \text{ for infinitely many } n\}. \quad (4.6)$$

By the Ergodic Theorem, $\mu(S) = 1$. In the sequel, we let Q denote the set $Q \cap S$, which differs from the original Q by a set of measure zero. Now let us fix $x_0 \in Q$ and use the following notation:

$$F = Q \cap B(x_0, \kappa r), \quad (4.7)$$

let $D = W_r^u(x_0)$, and let D' be a transversal which is within C^1 -distance κr of D . The domain of the holonomy map $\chi(D, D', \mathcal{F}^s)$ is given by

$$X = D \cap W_r^s(F)$$

and the range is

$$X' = D' \cap W_r^s(F).$$

We recall that the construction of the Pesin unstable manifolds (as given, for instance, in [PS]) may be carried out by applying the graph transform, starting with disks, called ‘‘trial disks,’’ that are transverse to the stable direction. It is shown that these trial disks, under forward iteration, approach the stable manifolds in a semi-global C^1 sense. Now let us consider a large n such that $f^n x_0 \in Q$. We define $y_0 = D' \cap W_r^s(x_0)$ and view D' as a trial disk for the unstable manifold $D = W_r^u(x_0)$. Let D'_n denote the portion of $f^n D'$ which can be represented as a graph over $E_r^u(f^n x_0)$ and which contains $f^n y_0$. By [PS, Corollary 3.11] D'_n converges to $D_n \equiv W_r^u(f^n x_0)$ in the C^1 topology, and the distance is bounded by $Ce^{-n\lambda}$. Let us define

$$X_n = (f^n X) \cap W_{\kappa r}^u(f^n x_0).$$

Evidently, $f^{-n}X_n \subset X$.

For each $x \in f^n X_n$, the local stable manifold $W_{r(f^n x)}^s(f^n x)$ intersects D'_n transversally because $r(f^n x) \geq (1 + \epsilon)^{-n}r$, and the angle between $W_{r(f^n x)}^s(f^n x)$ and D_n is at least $\theta_0(1 + \epsilon)^{-n}$, whereas D'_n is exponentially close to D_n . Thus D'_n and D_n are transversals to the family $\mathcal{F}_n = \{W_{r(f^n x)}^s(f^n x) : x \in X_n\}$, and so the holonomy

$$\chi_n := \chi(D_n, D'_n, \mathcal{F}_n) \quad (4.8)$$

is defined.

Lemma 4.1. *If n is sufficiently large, and $f^n x \in Q$, then $\chi_n(f^n x) = f^n \chi(x)$ for all $x \in X_n$. Given $r_0 < \kappa r$, n may be taken sufficiently large that for $a \in f^{-n}X_n$*

$$\begin{aligned} \chi_n(X_n \cap B(f^n a, r_0 - 2Ce^{-n\lambda})) &\subset (\chi_n X_n) \cap B(f^n \chi(a), r_0) \subset \\ &\subset \chi_n(X_n \cap B(f^n a, r_0 + 2Ce^{-n\lambda})). \end{aligned}$$

Proof. For $x \in X_n$, let $y = \chi(x)$, and let γ be a path inside $W_r^s(x)$ connecting x to y . Then $f^n \gamma$ lies inside $f^n W_r^s(x)$. Further, by (4.3), $f^n \gamma$ has diameter less than $Ce^{-n\lambda}$, and thus $f^n \gamma \subset W_{r(f^n x)}^s(f^n x)$. Since $f^n \gamma$ connects $f^n x$ to $f^n y$ inside $W_r^s(f^n x)$, it follows that $\chi_n(f^n x) = f^n y$. This proves the first assertion. The required inclusions are now a consequence of (4.3). \square

Remark. Sometimes abusing rigour we will write $\chi \circ f^n = f^n \circ \chi$ and say that f^n commutes with holonomy. Let us use the notation $y_0 = \chi(x_0)$, $x_n = f^n(x_0)$, and $y_n = f^n(y_0)$.

Lemma 4.2. *Let $\{\eta_n\}$ be a sequence of numbers decreasing to zero. Let us pass to a subsequence $n = n_j$ for which $x_n \in Q$, and let D'_n be a sequence of complex disks such that $\text{dist}_{C^1}(D_n, D'_n) \leq \eta_n$. Then there exists ρ with $r_0/2 \leq \rho \leq r_0$ such that*

$$\lim_{n \rightarrow \infty}' [\mu^+|_{D'_n} B(y_n, \rho \pm 2Ce^{-n\lambda}) - \mu^+|_{D_n} B(x_n, \rho \mp 2Ce^{-n\lambda})] = 0,$$

where \lim' means that the limit is taken through a further subsequence.

Proof. Without loss of generality, we may assume that Q is compact, and a subsequence of $\{x_n\}$ converges to $\bar{x} \in Q$. Thus the unstable disks D_n converge in C^1 to $\bar{D} = W_r^u(\bar{x})$. Now choose $r_0/2 \leq \rho \leq r_0$ such that $\mu^+|_{\bar{D}}$ puts no mass on $\partial B(\bar{x}, \rho)$. The Lemma then follows because the measures $\mu^+|_{D'_n}$ converge weakly to $\mu^+|_{\bar{D}}$. \square

Lemma 4.3. *If, in addition to the hypotheses of Lemma 4.2, we require that $x_n \in Q$, then*

$$\lim_{n \rightarrow \infty}' \mu^+|_{D'_n} B(y_n, \rho \pm 2Ce^{-n\lambda}) (\mu^+|_{D_n} B(x_n, \rho \mp 2Ce^{-n\lambda}))^{-1} = 1.$$

Proof. Lemma 4.3 follows from Lemma 4.2 by property (4.2). \square

Lemma 4.4. *Let $F \subset Q$, $D = W_r^u(x_0)$, and D' be as above. With the notation $\nu := \mu^+|_D$, $\nu' = \mu^+|_{D'}$, and $\chi = \chi(D, D', \mathcal{F}^s)$, we have*

$$\chi_*(\nu|_X) = \nu'|_{X'}. \quad (4.9)$$

Proof. It will suffice to show that (4.9) holds for X replaced by $X \cap B(x, \epsilon)$ for some small $\epsilon > 0$. Then we can add over a partition of X to obtain (4.9). We will define two coverings \mathcal{C}^\pm of X and a covering \mathcal{C}' of X' . The coverings will have the property that if $a \in X$, there are elements $C^\pm(a) \in \mathcal{C}^\pm$ and $C'(\chi(a)) \in \mathcal{C}'$ of arbitrarily small size containing a such that

$$\chi C^-(a) \subset C'(\chi(a)) \subset \chi C^+(a). \quad (4.10)$$

For $a \in X$ we may choose n arbitrarily large such that $f^n a \in Q$. We define

$$C_n^\pm(a) = f^{-n}(W_r^u(f^n a) \cap B(f^n a, \rho \pm 2Ce^{-n\lambda}))$$

with ρ as in Lemmas 4.2 and 4.3. In analogy with notation used earlier in this Section, we let D'_n denote the portion of $f^n D'$ which lies as a graph over $E_r^u(f^n a')$, $a' = \chi(a)$, and which contains $f^n a'$. Now we define

$$C'_n(a') = f^{-n}(D_n \cap B(f^n a', \rho)).$$

The inclusions in (4.10) are a consequence of Lemma 4.1. By Lemma 4.3, we have

$$\lim_{n \rightarrow \infty} \frac{\nu(C_n^\pm(a))}{\nu'(C'_n(a'))} = 1. \quad (4.11)$$

By the overflowing property of the unstable disks, $f^{-n} : W_r^u(f^n a) \rightarrow W_r^u(a)$. Since W_r^u is a graph, we may identify it with the disk $\{|\zeta| < r\}$, and thus we may consider $h(\zeta) := f^{-n}(r\zeta)$ as a univalent mapping of the disk $\{|\zeta| < 1\}$ to \mathbf{C} . By the Koebe Distortion Theorem,

$$\left| \frac{f''(\zeta)}{f'(\zeta)} \right| \leq \frac{(1 + r_0/r)^4}{(1 - r_0/r)^5}$$

for $|\zeta| < r_0/r$. The image of the disk $\{|\zeta| < \rho\}$ is a convex set if $(1 + r_0/r)^4(1 - r_0/r)^{-5} \leq (2\rho)^{-1}$. We conclude, then, since $C_n^\pm(a)$ is the image of such a disk, and since $\rho \leq r_0 = r/8$, that $C_n^\pm(a)$ is convex. Similarly, $C'_n(a')$ is convex.

Now let $E \subset X$ be a compact subset, and let $E' = \chi E$. For $\delta > 0$, choose an open set $\mathcal{O} \subset W_r^u(x)$ containing E such that

$$\nu(\mathcal{O}) < \nu(E) + \delta.$$

The coverings \mathcal{C}^\pm and \mathcal{C}' are fine in the sense that any point is contained in an element of arbitrarily small diameter. Since the elements of the cover are convex, we may apply

the Covering Theorem of A.P. Morse [Mo] to conclude that there is a disjointed family $\{C'_j : j = 1, 2, \dots\} \subset \mathcal{C}'$ such that

$$\nu' \left(E' - \bigcup_{j=1}^{\infty} C'_j \right) = 0. \quad (4.12)$$

Each C'_j is of the form $C'_{n_j}(\chi(a_j))$ for some a_j and n_j . The corresponding sets C_j^- in the cover \mathcal{C}^- satisfy $C_j^- \subset \chi^{-1}(C'_j)$ by (4.10) and are thus pairwise disjoint. Since χ^{-1} is continuous, and since the diameters of the C'_j may be taken arbitrarily small, we may assume that $C_j^- \subset \mathcal{O}$. Thus

$$\nu(\mathcal{O}) \geq \sum_{j=1}^{\infty} \nu(C_j^-).$$

Since we may take the diameters arbitrarily small, it follows from (4.11) that

$$\nu(\mathcal{O}) \geq (1 + \delta)^{-1} \sum_{j=1}^{\infty} \nu'(C'_j).$$

By (4.12), then, $\nu(\mathcal{O}) \geq \nu'(E')$. It follows that

$$\nu(E) \geq \nu'(E').$$

Now if we cover E by a disjointed subcover of \mathcal{C}^+ and repeat the previous argument, we conclude that

$$\nu(E) \leq \nu'(E').$$

Thus $\nu(E) = \nu'(E')$, and this completes the proof. \square

Theorem 4.5. *Let $F \subset Q$, and \mathcal{F}^s be as above. Let D_1, D_2 be two transversals, and set $\mu_j := \mu^+|_{D_j}$ for $j = 1, 2$, $X_j = D_j \cap W_r^s(F)$. Then the holonomy $\chi := \chi(D_1, D_2, \mathcal{F}^s)$ satisfies*

$$\chi_*(\mu_1|_{X_1}) = \mu_2|_{X_2}.$$

Proof. We may assume that for each $x_1 \in X_1$, there is a point $z \in Q$ such that $x_1, \chi(z) \in W_{\kappa r}^s(x')$. For otherwise we may apply f^n and use (4.3) and the fact that $f^n \circ \chi = \chi \circ f^n$.

As in Lemma 4.4 we work locally on X_1 , so we may assume that $W_r^u(z)$ is a transversal to \mathcal{F}^s . Let us define $\chi_1 = \chi(D_1, W_r^u(z), \mathcal{F}^s)$ and $\chi_2 = \chi(W_r^u(z), D_2, \mathcal{F}^s)$. By Lemma 4.4, then, $\chi_2 \circ \chi_1 = \chi$ takes $\mu_1|_{X_1}$ to $\mu_2|_{X_2}$. \square

Let $F \subset Q$ denote a compact subset, and let $W_r^s(F), \mathcal{F}^{s/u}$ be as in (4.2). If the diameter δ of F is sufficiently small, then we may assume that F is contained in a δ -ball about the origin, and that every leaf $W_r^s(x)$ (resp. $W_r^s(x)$) is a graph over the horizontal (resp. vertical) coordinate axis. Further, for $x \in F$, $W_r^s(x)$ is transversal to \mathcal{F}^u , and $W_r^u(x)$ is transversal to \mathcal{F}^s . Since the holonomy induces a homeomorphism on transversals, there is a fixed compact set P^u which is homeomorphic to $W_r^u(x) \cap W_r^s(F)$ for all $x \in F$.

Similarly, there is a fixed P^s which is homeomorphic to $W_r^s(x) \cap W_r^u(F)$ for all $x \in F$. We call the set $P := W_r^s(F) \cap W_r^u(F)$ the *Pesin box* generated by F , and we note that P is naturally homeomorphic to $P^s \times P^u$. It is evident that, up to a set of measure zero, \mathcal{R} is a countable union of (not necessarily disjoint) Pesin boxes.

If P is a Pesin box, then the partitions $\xi^{s/u}$ of P , whose elements are $W_r^{s/u}(x) \cap P$ are measurable partitions of P . We let $c := \mu(P)$ so that $\nu := c^{-1}\mu \llcorner P$ is a probability measure. It follows from Theorem 3.1 that the conditional measures of ν are given by

$$\nu(\cdot | \xi^s(x)) = c^s(x)^{-1} \mu^- |_{W_r^s(x)} \llcorner P$$

where $c^s(x)$ is the total mass of $\mu^- |_{W_r^s(x)} \llcorner P$, and a similar expression for $\nu(\cdot | \xi^u(x))$. By Theorem 4.5 we see that $c^s = c^s(x)$ is constant for $x \in F$. In fact:

Theorem 4.6. *If P is a Pesin box as above, then the holonomy maps along $\mathcal{F}^{s/u}$ preserve the conditional measures of $\nu = c^{-1}\mu \llcorner P$.*

To explore the product structure further, we let $E^{s/u} \subset P^{s/u}$ be Borel sets. For $x \in F$, we may define the measures $\lambda^{s/u}$ on $P^{s/u}$ to be the measures induced by the conditional measures $c^{s/u}\nu(\cdot | \xi^{s/u}(x))$ via the homeomorphism between $\xi^{s/u}$ and $P^{s/u}$. By Theorem 4.6, the measures $\lambda^{s/u}$ are independent of the point $x \in F$. By properties (ii) and (iii) of conditional measures, we have

$$\begin{aligned} \nu(E^s \times P^u) &= \int_{x \in \tilde{P}} \nu(E^s | \xi^s(x)) \nu(x) \\ &= \int_{x \in \tilde{P}} (c^s)^{-1} \lambda^s(E^s) = (c^s)^{-1} \lambda^s(E^s). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \nu(E^s \times E^u) &= \int_{x \in P^s \times E^u} \nu(E^s | \xi^s(x)) \nu(x) \\ &= \int_{x \in P^s \times E^u} (c^s)^{-1} \lambda^s(E^s) \\ &= (c^s)^{-1} \lambda^s(E^s) \nu(P^s \times E^u) = (c^s c^u)^{-1} \lambda^s(E^s) \lambda^u(E^u). \end{aligned}$$

Thus we have the following.

Theorem 4.7. *If P is a Pesin box, then there are measures $\lambda^{s/u}$ on $P^{s/u}$ such that $\mu \llcorner P = \lambda^s \otimes \lambda^u$ has the structure of a product measure.*

Corollary 4.8. *The measure μ is Bernoulli.*

Proof. D. Ornstein and B. Weiss discuss invariant measures with nonzero Lyapunov exponents in [OW, p. 86]. Given such a measure which is mixing with respect to f , they remark that it is Bernoulli if it is locally equivalent to a product measure with respect to the stable and unstable manifolds. By Theorem 4.7, then, we conclude that μ is Bernoulli. \square

Since the entropy, $\log d$, depends only on the degree of f , and the entropy is the unique invariant for Bernoulli measures, it follows that any two polynomial automorphisms with the same degree are measurably conjugate with respect to their equilibrium measures.

§5. Uniformly Laminar Currents

Let $\Omega \subset \mathbf{C}^n$ be an open set, and let $\mathcal{D}^{p,q}$ denote the smooth (p, q) -forms $\alpha = \sum \alpha_{IJ} dz^I \wedge d\bar{z}^J$, $|I| = p$, $|J| = q$, with compact support in Ω . The dual space $\mathcal{D}_{p,q}$ of $\mathcal{D}^{p,q}$ is the set of (p, q) -currents or *currents of bidimension* (p, q) . A current of dimension 0 acts on test functions and may thus be considered as a distribution. \mathbf{C}^n itself may be identified with the $2n$ -dimensional current $[\mathbf{C}^n]$, which acts on an (n, n) form φ by integration: $[\mathbf{C}^n](\varphi) = \int \varphi$. If T is a (p_1, q_1) -current, and ψ is a smooth (p_2, q_2) -form, then the *contraction* $T \lrcorner \psi$, defined by

$$(T \lrcorner \psi)(\varphi) = T(\psi \wedge \varphi)$$

is a $(p_1 - p_2, q_1 - q_2)$ -current. The space $\mathcal{A}^{n-p, n-q}$ of smooth $(n - p, n - q)$ forms on \mathbf{C}^n may be identified with a set of currents of bidimension (p, q) via the mapping

$$\mathcal{A}^{n-p, n-q} \ni \psi \mapsto [\mathbf{C}^n] \lrcorner \psi \in \mathcal{D}_{p,q}.$$

The *mass norm* of a current T is given by

$$\mathbf{M}[T] = \sup_{|\varphi| \leq 1} |T(\varphi)|.$$

If T is an $(0, 0)$ current, then the mass norm is finite if and only if T is represented as a distribution by a finite, signed Borel measure ν , and $\mathbf{M}[T]$ is the total variation of ν . A current T is *representable by integration* if χT has finite mass norm for any test function χ on Ω . If T is representable by integration, then there is a Borel measurable function t from Ω to the (p, q) -vectors (the dual of the (p, q) -forms) and a Borel measure ν on Ω such that $T = t\nu$ holds in the sense that

$$T(\varphi) = \int_{x \in \Omega} \langle \varphi(x), t(x) \rangle \nu(x).$$

We will require that $|t|^* = 1$ at ν a.e. point. ($|\cdot|^*$ denotes the norm on (p, q) -vectors which is dual to the norm on (p, q) forms.) In this case t and ν are uniquely determined, and $\nu = |T|$ is the variation measure associated with the current T . We will call $t\nu$ the *polar representation* of T . If T is representable by integration, and if $S \subset \Omega$ is a Borel subset, then we will use the notation

$$T \lrcorner S = t\nu \lrcorner S$$

for contraction, which coincides with restriction in this case.

A (p, p) -current T is *positive* if $T(i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_p \wedge \bar{\alpha}_p) \geq 0$ for all $(1, 0)$ forms $\alpha_j = \sum_k \alpha_j^k dz_k$ with compact support. This definition of positivity is analogous to the positivity of a distribution. And as in the case of distributions, a positive current is representable by integration. Further, if we let $\beta = \sum \frac{i}{2} dz_j \wedge d\bar{z}_j$ denote the standard

Kähler form on \mathbf{C}^n , then for a positive (p, p) current T , the contraction $T \lrcorner \beta^p / p!$ is a Borel measure, and the mass norm $\mathbf{M}[T]$ is just the total variation of this measure.

Let M be a k -dimensional complex manifold of Ω . If either M is locally closed (without boundary) or if M is a smooth submanifold-with-boundary (or more generally, if the area of M is locally finite), then the pairing with test (k, k) -forms given by

$$[M](\varphi) = \int_M \varphi$$

defines $[M]$ as a current of bidimension (k, k) on Ω . We call $[M]$ the *current of integration* associated to M . The mass norm of $[M]$ is the Euclidean $2k$ -dimensional area of M . It is evident that $[M]$ is representable by integration, and

$$[M] = t_M \sigma_M, \tag{5.1}$$

where t_M is the $2k$ -vector of norm 1 defining the tangent space to M (a vector which is uniquely defined, since M is an oriented submanifold of \mathbf{C}^n), and $\sigma_M = \mathcal{H}^{2k} \lrcorner M$ is the Hausdorff $2k$ -dimensional measure restricted to M . The *boundary* ∂T of a current T is defined by

$$\partial T(\varphi) = T(d\varphi).$$

If ∂M is regular, we may apply Stokes' theorem to obtain $\partial[M](\xi) = \int_{\partial M} \xi$. We say that T is *closed* if $\partial T = 0$, and so $[M]$ is closed if M has no boundary.

More generally, if V is a (closed) subvariety of Ω , then the set $Reg(V)$ of regular points (where V is locally a manifold) are a dense open set, and it may be shown that $[V](\varphi) = \int_{Reg(V)} \varphi$ defines a positive, closed current. The device of studying the current of integration $[V]$ has been useful in the study of metric properties of V , such as the area growth. For instance, the fact that $[V]$ is a current at all corresponds to the fact that the area of $[Reg(V)]$ is locally bounded near singular points. And $\partial[V] = 0$ holds because the amount of mass in a neighborhood of the singular set is small.

It is useful to apply similar considerations to the stable and unstable manifolds. However, since $W^s(x)$ (resp. $W^u(x)$) is often dense in \mathcal{W}^s (resp. \mathcal{W}^u) an individual stable manifold does not define a current of integration, since the amount of mass is not locally bounded. Thus we wish to consider the whole stable and unstable laminations as currents, as was suggested by Ruelle and Sullivan [RS] and Sullivan [S].

Let us consider a family of graphs of analytic functions $f_a : \Delta \rightarrow \Delta$, $a \in \Delta$. We assume that the graphs $\Gamma_a = \{(x, f_a(x)) : x \in \Delta\}$ are pairwise disjoint, i.e. if $a_1 \neq a_2$, then $f_{a_1}(x) \neq f_{a_2}(x)$ for all $x \in \Delta$. We denote the set of graphs as $\mathcal{G} = \{\Gamma_a : a \in A\}$. Without loss of generality, we may take the parameter space to be a closed subset of the unit disk, and we may take $a = f(0)$. Further, since the graphs are disjoint, it follows that $a \mapsto f_a$ is continuous.

A current T on Δ^2 is *uniformly laminar* if it has the form

$$T = \int_{a \in A} \lambda(a) [\Gamma_a] \tag{5.2}$$

where λ is a positive measure on A , the parameter space for the set \mathcal{G} of graphs. The action on a (1,1) form φ is given by

$$T(\varphi) = \int_A \lambda(a) \int_{\Gamma_a} \varphi.$$

We say that a current S is *locally uniformly laminar* on an open set Ω if for each $p \in \Omega$ there is a coordinate neighborhood equivalent to Δ^2 on which S is uniformly laminar. The currents of integration $[\Gamma_a]$ are positive, closed currents on Δ^2 , so T , too, is positive and closed.

For a transversal M to the family \mathcal{G} , the set of all intersection points, A_M , could equally well be taken as a parameter space. Further, let M_1 and M_2 be transversals. Then the holonomy map $\chi_{M_1, M_2} : A_{M_1} \rightarrow A_{M_2}$ gives a homeomorphism between parameter spaces. For a point $p \in \mathbf{C}^2$, we let $[p]$ denote the 0-current which puts a unit mass at the point p . For each transversal, the current (measure) $[\Gamma_a \cap M]$ depends continuously on a . We define the *restriction* of T to M by

$$T|_M = \int_A \lambda(a) [\Gamma_a \cap M], \quad (5.3)$$

which is a measure on M . If M_1 and M_2 are transversals, then the restrictions are preserved by the holonomy map $\chi = \chi_{M_1, M_2}$, i.e.

$$\chi_* T|_{M_1} = T|_{M_2}. \quad (5.4)$$

A family of measures $\{T|_M\}$ on transversals induces a *transversal measure* on \mathcal{W}^s if it satisfies (5.4). T may be reconstructed from any transversal (or, equivalently, from any family of transversal measures) as

$$T = \int_{a' \in A_M} T|_M(a') [\Gamma_{a'}]. \quad (5.5)$$

Equations (5.4) and (5.5) are trivial if $T = [\Gamma_a]$ is a current of integration, and the general case is obtained by integrating with respect to λ . Let h be a holomorphic function on Δ^2 such that $M = \{h = 0\}$ and $dh \neq 0$ on M . Then $\log |h|$ is locally integrable on each Γ_a , and

$$dd^c \log |h| [\Gamma_a] = [M \cap \Gamma_a]$$

holds in the sense of currents. Thus

$$\frac{1}{2\pi} T|_M = dd^c (\log |h| T).$$

We may ask, more generally, which positive, closed currents on \mathbf{C}^2 may be represented in the form

$$T = \int_{a \in A} \eta(a) [V_a] \quad (5.6)$$

where $A \ni a \mapsto V_a$ is a measurable family of varieties in \mathbf{C}^2 , and η is a Borel measure on A . This is closely related to the Choquet representation of T as an integral over extremal rays on the cone of positive, closed currents. It is known that an irreducible subvariety $V_a \subset \mathbf{C}^2$ generates an extreme ray (see [D] and [L]). On the other hand, not all extremal rays are of the form $c[V]$. This will also be a consequence of the examples below.

Examples. Let (x, y) denote coordinates on \mathbf{C}^2 , and define

$$u_1 = \log^+ |(x, y)| = \max\{0, \frac{1}{2} \log(|x|^2 + |y|^2)\}$$

$$u_2 = \max\{\log|x|, \log|y|, 0\}.$$

For $\alpha \in \mathbf{C}^2$, we let L_α denote the complex line through 0 and α , and we set $L_\alpha^+ = L_\alpha \cap (\mathbf{C}^2 - \overline{\mathbf{B}}^2)$. Then we may compute

$$T_1 := dd^c u_1 = 2\pi \int_{\alpha \in \mathbf{P}^1} [L_\alpha^+] \sigma(\alpha) + S_1,$$

where σ is normalized spherical measure on \mathbf{P}^1 , and S_1 is supported on $\partial\mathbf{B}^2$. Similarly,

$$\begin{aligned} T_2 := dd^c u_2 &= \int_0^{2\pi} [L_{(1, e^{i\theta})}^+] d\theta + \int_0^{2\pi} [x = e^{i\theta}, |y| < 1] d\theta \\ &+ \int_0^{2\pi} [y = e^{i\theta}, |x| < 1] d\theta + S_2 \end{aligned}$$

where S_2 is supported on the 2-torus $\{|x| = |y| = 1\}$.

It is evident, then, that T_1 is locally uniformly laminar on $\mathbf{C}^2 - \partial\mathbf{B}^2$, and T_2 is locally uniformly laminar on $\mathbf{C}^2 - \{|x| = |y| = 1\}$.

Now if $T = t\lambda$ is any positive current satisfying $T \leq T_1$, then at λ a.e. point $\alpha \in \mathbf{C}^2 - \overline{\mathbf{B}}^2$, the $(1,1)$ vector $t(\alpha)$ must be tangent to L_α . If, in addition, T has the form (5.6), then it follows that for η a.e. a the variety V_a must be contained in L_α for some α . Since V_a is a subvariety, we must have $V_a = L_\alpha$. On the other hand, since $T_1 = 0$ on \mathbf{B}^2 , it follows that $T = 0$ on $\mathbf{C}^2 - \partial\mathbf{B}^2$. But now for η a.e. a , we must have $V_a \subset \partial\mathbf{B}^2$, which is impossible, so $T = 0$. A similar argument shows that if $0 \leq T \leq T_2$ and T has the form (5.6) then $T = 0$.

These examples then show that: *There are extreme rays in the cone of positive, closed currents which are not generated by currents of integration over varieties.* This observation was made by Demailly in [D], using the current T_2 written in a somewhat different form.

Sullivan conjectured in [S] that a positive, closed current might be written locally in the form (5.6) on a dense, open set. This cannot be the case, however, because of the following examples, which are taken from [BT2]. For a number $r > 0$ let $\chi_r(z) = rz$ denote dilation, and for a point $a \in \mathbf{C}^2$ let $\tau_a(z) = z + a$ denote translation. Let $\{r_j : j = 1, 2, 3, \dots\}$ be dense in \mathbf{R}^+ , and let $\{a_j : j = 1, 2, 3, \dots\}$ be dense in \mathbf{C}^2 . Then the currents

$$\tilde{T}_1 = \sum 2^{-j} \chi_{r_j}^* T_1 \tag{5.7}$$

$$\tilde{T}_2 = \sum 2^{-j} \tau_{a_j}^* T_2 \tag{5.8}$$

are positive and closed, and both have the property of being nowhere locally uniformly laminar. From this it may be shown that neither current can be represented in the form (5.6) on any open set.

We note that the manifolds of \tilde{T}_1 intersect correctly in the sense of §6, although \tilde{T}_1 is not a weakly laminar current, even locally (cf. Proposition 6.2). In fact, if L_j is uniformly laminar on an open set U_j , and if $\sum_{j=1}^{\infty} L_j \leq T_1$, then in fact $\sum_{j=1}^{\infty} L_j \leq T_1 - S_1$ (with a similar property for T_2). Thus

$$\mathbf{M}[\sum L_j] \leq \mathbf{M}[T_1] - \mathbf{M}[S_1],$$

and so T_1 and T_2 cannot be approximated from below by uniformly laminar currents, even in the sense of measure.

Remark. Let us observe that it is possible to define the wedge products $\tilde{T}_j \wedge \tilde{T}_j$ for $j = 1, 2$ (see §8.) We do not know of an example as above with the additional property that $\tilde{T}_j \wedge \tilde{T}_j = 0$.

§6. Laminar Currents

The currents that arise in dynamical systems often derive their structure from the stable and unstable manifolds. The examples in §5 show that the category of positive, closed currents is too general for the dynamical context. Stable (or unstable) manifolds have no self-intersections and are pairwise disjoint, so a representation (5.6) should involve the additional requirement that the varieties V be pairwise disjoint. In fact, the context in which currents have been constructed from dynamical systems has been the uniformly hyperbolic case, and the currents obtained in this case are uniformly hyperbolic. In the case of a hyperbolic measure, this uniformity is lost, and so we turn to the study of laminar currents. The philosophy behind the Sullivan conjecture is substantiated by Proposition 6.2 below, which says: *A laminar current is uniformly laminar outside a set of small measure.*

We say that two manifolds M_1 and M_2 *intersect correctly* if either $M_1 \cap M_2 = \emptyset$ or $M_1 \cap M_2$ is an open subset of M_j for $j = 1, 2$, i.e. they intersect in a set of codimension 0. We consider a measurable set $A \subset \mathbf{C}$ and a measurable function $f : \Delta \times A \rightarrow \mathbf{C}^2$ such that $f(\zeta, a)$ is an analytic injection in ζ for fixed a . We assume that any pair of image disks

$$M_a = \{f(\zeta, a) : \zeta \in \Delta\}$$

intersects correctly. Let λ denote a σ -finite measure on A . If

$$\int_A \lambda(a) \mathbf{M}[M_a \cap U] < \infty \tag{6.1}$$

for all relatively compact open sets $U \subset \mathbf{C}^2$, then

$$T = \int_{a \in A} \lambda(a) [M_a]$$

defines a positive current on \mathbf{C}^2 . A current obtained in this way is called a *weakly laminar current* on \mathbf{C}^2 . The current T is *laminar* if the disks M_a are pairwise disjoint. With suitable modifications, we can also define (weakly) laminar currents on an open set $\Omega \subset \mathbf{C}^2$. Thus

if U is open, then $T \llcorner U$ is again (weakly) laminar. We will say that T is *represented* by the data $(A, \mathcal{M}, \lambda)$. We note that for fixed $a \in A$, the function $\zeta \mapsto f(\zeta, a)$ in the definition of T is far from unique. If we fix $[M_a]$, then we can replace $f(\cdot, a)$ by any holomorphic imbedding $f' : \Delta \rightarrow M_a$ such that $M_a - f'(\Delta)$ has zero area.

The parametrizing function f in the definition is not, strictly speaking, necessary. If we consider $\tilde{M} := \bigcup_{M \in \mathcal{M}} M$ be a total space, then \mathcal{M} is a partition of \tilde{M} , and $A = \tilde{M}/\mathcal{M}$ is the quotient. The essential point is the requirement that this partition be measurable. We say that the families \mathcal{M}_1 and \mathcal{M}_2 *intersect correctly* if all of the component manifolds intersect correctly.

A Borel set E is a *carrier* for T if $T \llcorner E = T$, or equivalently, E carries all the mass of $|T|$. A carrier for a (weakly) laminar current may be taken to be a union of complex disks.

Lemma 6.1. *Let T_j , $j = 1, 2, 3, \dots$ be a sequence of weakly laminar currents with representations $(A_j, \mathcal{M}_j, \lambda_j)$. If the \mathcal{M}_j intersect correctly, and if for every bounded open U*

$$\sum_{j=1}^{\infty} \mathbf{M}[T_j \llcorner U] < \infty, \quad (6.2)$$

then $\sum T_j$ is a weakly laminar current. If the T_j are laminar with pairwise disjoint carriers, and if (6.2) holds, then $\sum T_j$ is laminar.

Proof. We let \mathcal{M} (resp. A) denote the disjoint union of the \mathcal{M}_j (resp. A_j), and we define the measure $\lambda = \sum \lambda_j$ by setting $\lambda \llcorner A_j = \lambda_j$. By (6.2), it follows that (6.1) holds, so $(A, \mathcal{M}, \lambda)$ represents a positive current, which must coincide with $\sum T_j$. \square

Example. Weakly laminar currents are well behaved with respect to taking summations, but for our applications we will need to take the supremum of an increasing family of laminar currents. To understand some of the technical points of the sequel, it may be helpful to note that although T_1 and T_2 are uniformly laminar currents, and $T_1 \leq T_2$, it may happen that the positive current $T_2 - T_1$ is not weakly laminar. Similarly, $T_1 + T_2$ and $\max(T_2, 2T_1)$ may fail to be laminar. For a simple example, consider $T_1 = [M_1] \leq T_2 = [M_2]$, where $M_1 \subset M_2 \subset \mathbf{C}$, but $M_2 \cap \partial M_1$ has positive area.

Let us discuss the polar representation $T = t\nu$ of a laminar current. From (5.1) we have $[M] = t_M \mathcal{H}^2 \llcorner M$. Thus the underlying measure is

$$\nu = \int_{a \in A} \lambda(a) \mathcal{H}^2 \llcorner M_a = |T|, \quad (6.3)$$

and the set $\bigcup_{a \in A} M_a$ carries full measure for ν . By (6.3), $\nu(E) = 0$ holds for a Borel set E if and only if $\text{Area}(M_a \cap E) = 0$ for λ a.e. a . Since the manifolds M_a intersect correctly, it follows that for ν a.e. $x \in \bigcup_{a \in A} M_a$, the 2-vector is $t(x) = t_{M_a}(x)$. Thus t is a simple 2 vector at ν a.e. point. In other words, there are vectors t_1 and t_2 such that $t = t_1 \wedge t_2$. The field of 2-vectors t and ν depend only on T and are independent of the representation used to define them.

We let ξ denote a family of 1-dimensional complex manifolds $\alpha \subset \mathbf{C}^2$ such that each $\alpha \in \xi$ defines a current of integration $[\alpha]$ with finite mass norm. We will say that ξ is a *stratified carrier* for a weakly laminar current T if

- (i) $E := \bigcup_{\alpha \in \xi} \alpha$ is a Borel set.
- (ii) ξ is a measurable partition of E .
- (iii) For λ a.e. $M \in \mathcal{M}$ there is a countable family $\{\alpha_i\} \subset \xi$ such that $M - \bigcup_i \alpha_i$ has zero area.

If T is laminar, then \mathcal{M} is a stratified carrier. It is a consequence of (6.3) that if ξ satisfies (i), (ii), and (iii), then E is a carrier for T . In Corollary 6.7 it will be shown that condition (iii) is in fact independent of the choice of representation $(A, \mathcal{M}, \lambda)$. We note that the main difference between \mathcal{M} and ξ is that the complex manifolds in ξ are disjoint. We will say that two stratified carriers *intersect correctly* if the complex manifolds in the stratifications intersect correctly. We say that a representation $(A, \mathcal{M}, \lambda)$ is *subordinate* to ξ if for λ a.e. $a \in A$ there exists $\alpha \in \xi$ with $M_a \subset \alpha$.

The point of considering a stratified carrier is as follows. Let us suppose that a laminar current T has a representation $(A, \mathcal{M}, \lambda)$ which is subordinate to a stratified carrier ξ . (It will be shown in Lemma 6.8 that any representation may be refined to be subordinate to a given stratified carrier ξ .) For $\alpha \in \xi$ we set $A_\alpha = \{a \in A : M_a \subset \alpha\}$. We may let λ_ξ denote the measure λ restricted to the (coarser) σ -algebra which is generated by ξ . For λ_ξ almost every $\alpha \in \xi$ there is a conditional measure $\lambda(\cdot|\alpha)$ on A_α , as in §2. Let us define a function on α by setting

$$\varphi^\alpha := \int_{a \in A_\alpha} \chi_{M_a} \lambda(a|\alpha), \quad (6.4)$$

where χ_{M_a} denotes the function which is 1 on the set M_a and 0 on $\alpha - M_a$. Since M_a is an open subset of α , and since the conditional measure is positive, φ^α is lower semicontinuous on α . It is immediate that

$$\varphi^\alpha[\alpha] = \int_{a \in A_\alpha} [M_a] \lambda(a|\alpha).$$

It follows from the defining property of the conditional measures that

$$T = \int_{\alpha \in \xi} \varphi^\alpha[\alpha] \lambda_\xi(\alpha). \quad (6.5)$$

This differs from the original representation of T as a direct integral in that the currents involved are not locally closed, but it has the advantage that the supports may be taken to be essentially disjoint.

Proposition 6.2. *Let T be a weakly laminar current. Then for $\epsilon > 0$ and any bounded, open set U , there exist uniformly laminar currents T_j with disjoint supports such that*

$$\mathbf{M} \left[\left(T - \sum T_j \right) \llcorner U \right] < \epsilon. \quad (6.6)$$

If T is a laminar current, then there exist uniformly laminar currents T_1, T_2, \dots with disjoint supports such that $T = \sum T_j$. Further, there is a compact $K \subset U$ such that

$\mathbf{M}[T \llcorner (U - K)] < \epsilon$ and $T \llcorner K$ is the finite sum of uniformly laminar currents with disjoint supports.

Proof. Let T have a representation $(A, \mathcal{M}, \lambda)$. Let \mathcal{Q}_n denote the decomposition of \mathbf{C} into squares of side 2^{-n} and vertices at the points $(j + ik)2^{-n}$ for $j, k \in \mathbf{Z}$. Let $\pi(x, y) = x$. We may assume that the set of $a \in A$ such that $\pi(M_a)$ is a point has λ measure zero. For each $a \in A$, we call a component M' of $M_a \cap \pi^{-1}Q$ *good* if $\pi|_{M'} : M' \rightarrow Q$ is a homeomorphism. We let $\hat{M}_a(Q)$ be the union of all of the good components of $M_a \cap \pi^{-1}Q$, and we set

$$T_Q = \int_{a \in A} \lambda(a) [\hat{M}_a(Q)]. \quad (6.7)$$

It is immediate that

$$\sum_{Q \in \mathcal{Q}_n} T_Q \leq T.$$

Let \mathcal{N} denote the set of every disk which arises as a good component of $M_a \cap \pi^{-1}Q$ for some $a \in A$. Thus there is a measure λ_Q on \mathcal{N} such that

$$T_Q = \int_{\mathcal{N}} \lambda_Q(N) [N].$$

We observe that if $N_1, N_2 \in \mathcal{N}$, then the condition of correct intersection implies that either $N_1 \cap N_2 = \emptyset$, or $N_1 = N_2$. Thus each T_Q is uniformly laminar.

We let

$$T^{(1)} = \sum_{Q \in \mathcal{Q}_1} T_Q,$$

so that $T^{(1)}$ is the sum of uniformly laminar currents with disjoint carriers.

Now we suppose that $T_Q^{(j)}$ have been constructed for $1 \leq j \leq n-1$ and $Q \in \mathcal{Q}_j$. Each $T_Q^{(j)}$ is uniformly laminar, and $T^{(j)} = \sum_Q T_Q^{(j)}$ is laminar. Further, $T^{(1)} + \dots + T^{(n-1)} \leq T$. Since $T - T^{(1)} - \dots - T^{(n-1)}$ is weakly laminar, we may let

$$T_Q^{(n)} := (T - T^{(1)} - \dots - T^{(n-1)})_Q$$

be the uniformly laminar current obtained in the construction (6.7).

We observe that if $U = \{|Re x|, |Re y|, |Im x|, |Im y| < m\}$ for some integer m , then $(T^{(1)} + \dots + T^{(n)}) \llcorner U$ is a finite sum of uniformly laminar currents with disjoint carriers. By the construction above, the mass norm in (6.6) is given by

$$\int \lambda(a) \text{Area} \left(M_a \cap U - \bigcup_{Q \in \mathcal{Q}_n} \hat{M}_a(Q) \right).$$

For fixed $a \in A$, the area decreases to zero as $n \rightarrow \infty$, so this integral tends to zero by monotone convergence.

If $Q \in \mathcal{Q}_n$, then $(T^{(1)} + \dots + T^{(n)}) \llcorner (Q \times \mathbf{C})$ is uniformly laminar. Thus the currents in the family $\{T_j\} := \{(T^{(1)} + \dots + T^{(n)}) \llcorner (Q \times \mathbf{C}) : Q \in \mathcal{Q}_n\}$ are uniformly laminar and have disjoint carriers and satisfy (6.6) for n sufficiently large. If T_j is restricted to a smaller compact inside its carrier, the supports of $\{T_j\}$ will be pairwise disjoint.

Finally, let us observe that if T is laminar, then the carriers of $T_Q^{(j)}$ are already pairwise disjoint, and $T = \sum_{Q,j} T_Q^{(j)}$. By subdividing the support of each $T_Q^{(j)}$ into countably many compact sets, we have the first assertion. The existence of K with the required properties is a property of Radon measures. \square

Remark. It follows that the currents \tilde{T}_1 and \tilde{T}_2 defined in (5.7–8) are not locally weakly laminar on any open set.

Lemma 6.3. *If T_1, \dots, T_k are laminar currents with representations that intersect correctly, then there exists ξ which is a stratified carrier for T_j for $1 \leq j \leq k$.*

Proof. The proof of this Lemma is a repetition of the proof of Proposition 6.2 with \mathcal{M} replaced by $\mathcal{M}_1 \cup \dots \cup \mathcal{M}_k$. To obtain a stratified carrier, we fix n and $Q \in \mathcal{Q}_n$. We use the notation \mathcal{N}_Q^n for the set \mathcal{N} defined above: the union over $a \in A$ of the set of disks which are good components of $M_a \cap \pi^{-1}Q$. We let $\xi_1 = \bigcup_{Q \in \mathcal{Q}_1} \mathcal{N}_Q^1$. We continue inductively, setting $\xi_n = \bigcup_{Q \in \mathcal{Q}_n} \mathcal{N}_Q^n - \xi_{n-1}$. Finally, $\xi = \bigcup \xi_n$ has the desired properties. \square

Given a representation $(A, \mathcal{M}, \lambda)$ of T , we may define a family of germs of complex manifolds as follows: for $x \in \bigcup_{a \in A} M_a$, we let $\hat{M}(x)$ be the germ of x of the manifold $M_{a(x)}$ containing x . The correspondence $x \mapsto \hat{M}(x)$ is thus well defined ν a.e. in terms of the representation. By (5.1) and (6.3), we have

$$T = \int_{a \in A} \lambda(a) t_{M_a} \sigma_{M_a}.$$

Since the M_a overlap correctly, it follows that if $(A', \mathcal{M}', \lambda')$ and $(A'', \mathcal{M}'', \lambda'')$ are two representations, then

$$T_x \hat{M}'(x) = T_x \hat{M}''(x) \tag{6.8}$$

holds for ν a.e. x (so the germs intersect tangentially). We now show that these germs coincide at ν a.e. point.

First we need a lemma.

Lemma 6.4. *Let M_1 and M_2 be complex submanifolds of \mathbf{C}^2 such that $M_1 \cap M_2 = \{p\}$, and $T_p M_1 = T_p M_2$. If M'_1 is sufficiently close to M_1 , but $M'_1 \cap M_1 = \emptyset$, then the intersection $M'_1 \cap M_2$ is nonempty, and nontangential at all intersection points.*

Proof. Let k be the multiplicity of the intersection of M_1 and M_2 at p . By the continuity of the intersection of complex manifolds, the intersection of M'_1 and M_2 (with multiplicity) near p is k . Thus it suffices to show that $M'_1 \cap M_2$ contains k distinct points near p .

Without loss of generality, we may work in a small neighborhood of $p = 0$ and assume that $\{y = f(x) : |x| < 1 + \epsilon\} \subset\subset M_2$ for some holomorphic function $f(x) = x^k + \dots$ and $\{|x| < 1 + \epsilon, y = 0\} \subset\subset M_1$. We may assume that $\{y = f(x) = 0 : |x| < 1 + \epsilon\} = \{0\}$. A manifold M'_1 which is C^1 close to M_1 is of the form $\{y = g(x) : |x| < 1 + \epsilon\} \subset\subset M'_1$. The

hypothesis that $M_1 \cap M'_1 = \emptyset$ implies that $g \neq 0$. By the Harnack inequalities there is a constant C_ϵ such that

$$C_\epsilon^{-1}|g(0)| \leq |g(x)| \leq C_\epsilon|g(0)|$$

for $|x| \leq 1$. This implies that the higher order terms in $g(0)/g(x) = 1 + \dots$ are uniformly small. Since $M'_1 \cap \{y = f(x) : |x| < 1\}$ is given by

$$g(0) = f(x) \frac{g(0)}{g(x)} = x^k + \dots,$$

and the higher order terms are uniformly bounded, this equation has k distinct solutions near $x = 0$ for $g(0)$ sufficiently small. \square

Lemma 6.5. *Let $(A', \mathcal{M}', \lambda')$ and $(A'', \mathcal{M}'', \lambda'')$ be two representations for the weakly laminar current T . Then $\hat{M}'(x) = \hat{M}''(x)$ for ν a.e. x .*

Proof. Let $B = \{x : \hat{M}'(x) \neq \hat{M}''(x)\}$. Removing a set of measure zero, we may assume that (6.8) holds at every point of B . We must show that $\nu(B) = 0$. Otherwise, we may choose ϵ such that $0 < \epsilon < \nu(B)$ and let $T = \sum T'_j$ be the sum of uniformly laminar currents obtained in Lemma 6.2 corresponding to \mathcal{M}' . If $T_j \llcorner B = 0$ for all j , then

$$\mathbf{M} \left[T - \sum T'_j \right] \geq \mathbf{M}[T \llcorner B] = \nu(B)$$

so it follows that $T'_j \llcorner B \neq 0$ for some j . Now the current $T' := T'_j$ is uniformly laminar and has the form

$$T' = \int_{a' \in A'} \lambda'(a') [\Gamma'_{a'}]. \quad (6.9)$$

Let us set

$$B' = \bigcup_{a' \in A'} \Gamma'_{a'} \cap B.$$

Since $T' \llcorner B \neq 0$, we have

$$|T'| \llcorner (B) = \int_{A'} \lambda'(a') \text{Area}(\Gamma'_{a'} \cap B) > 0, \quad (6.10)$$

as in (6.3). It follows that $\text{Area}(\Gamma'_{a'} \cap B) > 0$ for a set of positive λ' measure, so $\nu(B') > 0$. Now we let $T = \sum T''_j$ be as in Lemma 6.2 for $\epsilon < \nu(B')$. As before there exists k such that $T''_k \llcorner B' \neq 0$. Now we set $T'' := T''_k$, and we represent T'' in a form analogous to (6.9). By the analogue of (6.10), we know that there exists a'' such that

$$\text{Area}(\Gamma''_{a''} \cap B') > 0.$$

Now let $b \in \Gamma''_{a''} \cap B'$ be a point of density with respect to area measure. thus there is a sequence $\{b_j\} \subset \Gamma''_{a''} \cap B'$ converging to b . Since $b, b_j \in B'$, there exist $a', a'_j \in A'$ with $b \in \Gamma'_{a'}$ and $b_j \in \Gamma'_{a'_j}$. Let $M_1 = \Gamma'_{a'}$, $M'_1 = \Gamma'_{a'_j}$, and $M_2 = \Gamma''_{a''}$. Since $b \in B$, M_1 and M_2 define different germs of complex manifolds, and we may intersect them with $B(b, \epsilon)$, if necessary, to have $M_1 \cap M_2 = \{b\}$. Since (6.8) holds at b , M_1 , M'_1 , and M_2 satisfy the hypotheses of Lemma 6.3, so we conclude that all intersection points of M'_1 and M_2 are transversal. But $b_j \in M'_1 \cap M_2$, and the intersection at b_j is tangential by (6.8). By this contradiction we conclude that $\nu(B) = 0$. \square

Corollary 6.6. *Let $(\mathcal{M}_1, \lambda_1)$ and $(\mathcal{M}_2, \lambda_2)$ be two representations of a weakly laminar current T . Then for λ_1 a.e. $M_1 \in \mathcal{M}_1$ and λ_2 a.e. $M_2 \in \mathcal{M}_2$, M_1 and M_2 intersect correctly.*

In other words, the set of manifolds \mathcal{M} associated with a weakly laminar current T are unique, up to subdivision or refinement. We say that weakly laminar currents T' and T'' *intersect correctly* if they have representations $(A', \mathcal{M}', \lambda')$ and $(A'', \mathcal{M}'', \lambda'')$ such that the disks of \mathcal{M}' and \mathcal{M}'' intersect correctly. By Corollary 6.6, this condition is independent of the representations \mathcal{M}' and \mathcal{M}'' chosen. Another consequence is the following.

Corollary 6.7. *If ξ is a stratified carrier which satisfies condition (iii) for \mathcal{M}_1 , then (iii) holds for any other representation \mathcal{M}_2 .*

Let $(A, \mathcal{M}, \lambda)$ be a representation of a weakly laminar current, and let ξ be a stratified carrier. We will show how to subdivide the elements of \mathcal{M} so that the representation is subordinate to ξ . We set $\tilde{A} = A \times \xi$, and we define a measure $\tilde{\lambda}$ on \tilde{A} by setting $\tilde{\lambda}(E \times \{\alpha\}) = \lambda(E)$ for any measurable $E \subset A$ and any $\alpha \in \xi$. In other words, $\tilde{\lambda} = \lambda \times \mathcal{H}^0$ is the product measure obtained from λ and the counting measure \mathcal{H}^0 on ξ . We define $\tilde{\mathcal{M}}$ by setting $\tilde{M}_{(a,\alpha)} = M_a \cap \alpha$ for any $a \in A$ and $\alpha \in \xi$.

Lemma 6.8. *If $(A, \mathcal{M}, \lambda)$ is a representation of T , and if ξ is a stratified carrier of T , then $(\tilde{A}, \tilde{\mathcal{M}}, \tilde{\lambda})$ is a representation of T which is subordinate to ξ .*

Proof. By definition of \tilde{A} , $\tilde{\mathcal{M}}$, and $\tilde{\lambda}$, we have

$$\begin{aligned} \int_{\tilde{a} \in \tilde{A}} [\tilde{M}_{\tilde{a}}] \tilde{\lambda}(\tilde{a}) &= \int_{(a,\alpha) \in A \times \xi} [M_a \cap \alpha] \tilde{\lambda}(a, \alpha) \\ &= \int_{a \in A} \lambda(a) \int_{\alpha \in \xi} [M_a \cap \alpha] \mathcal{H}^0(\alpha) \\ &= \int_{a \in A} \lambda(a) \sum_{\alpha \in \xi} [M_a \cap \alpha] \\ &= \int_{a \in A} \lambda(a) [M_a] = T \end{aligned}$$

where the second line follows by the Fubini Theorem, and the fourth line is by (iii) of the definition of stratified carrier. \square

Since we may subdivide any representation $(A, \mathcal{M}, \lambda)$ to be subordinate to a given stratified carrier ξ , it follows that T may be given as a direct integral over the elements of ξ , as in (6.4) and (6.5). This yields the following:

Lemma 6.9. *If T is weakly laminar, and if ξ is any stratified carrier, then T may be represented in terms of ξ as follows: For $\alpha \in \xi$, there exists a lower semicontinuous function $\varphi^\alpha \geq 0$ on α such that $\xi \ni \alpha \mapsto \varphi^\alpha$ is measurable, and*

$$T = \int_{\alpha \in \xi} \lambda_\xi(\alpha) \varphi^\alpha[\alpha].$$

T is laminar if and only if φ^α is locally constant a.e. on $\{\varphi^\alpha > 0\}$.

The *maximum*, written $\max(T_1, \dots, T_n)$, of the currents T_1, \dots, T_n (if it exists) is characterized by the properties: $T_j \leq \max(T_1, \dots, T_n)$ for $j = 1, \dots, n$, and if S is any current satisfying $T_j \leq S$, $j = 1, \dots, n$, then $\max(T_1, \dots, T_n) \leq S$.

Lemma 6.10. *Let T_1, \dots, T_n be weakly laminar currents which intersect correctly. Then $\max(T_1, \dots, T_n)$ exists as a positive current and is weakly laminar.*

Proof. By Lemmas 6.3 and 6.8, we may assume that the representations of T_j are subordinate to some carrier ξ . By Lemma 6.9

$$T_j = \int_{\alpha \in \xi} \varphi_j^\alpha[\alpha] \lambda_\xi^j(\alpha)$$

with the measurable family of lower semicontinuous functions φ_j^α on α being given by (6.4). Let us define $\lambda := \lambda_\xi^1 + \dots + \lambda_\xi^n$, and let h_j be a measurable function such that $\lambda_\xi^j = h_j \lambda$. It follows

$$\int_{\alpha \in \xi} \max(h_1 \varphi_1^\alpha, \dots, h_n \varphi_n^\alpha)[\alpha] \lambda(\alpha),$$

defines a laminar current which has the properties of $\max(T_1, \dots, T_n)$. \square

Lemma 6.11. *Let T_1, \dots, T_n be uniformly laminar currents which intersect correctly. Suppose that for any $M_i \in \mathcal{M}_i$ and $M_j \in \mathcal{M}_j$, $M_i \cap \partial M_j$ has zero area in M_i . Then $\max(T_1, \dots, T_n)$ exists as a positive current and is laminar.*

Proof. The existence of $\max(T_1, \dots, T_n)$ follows from Lemma 6.10. Since the relative boundaries have zero area, this current is laminar by Lemma 6.9. \square

Lemma 6.12. *Let $T_1 \leq T_2 \leq \dots$ be an increasing sequence of weakly laminar currents whose mass is locally bounded. Suppose that there exists ξ which is a stratified carrier for all T_n . Then $\sup_n T_n$ exists as a positive current and is weakly laminar.*

Proof. Each current T_n may be written as

$$T_n = \int \varphi_n^\alpha[\alpha] \lambda_\xi^n(\alpha).$$

There exists a sequence of functions $g_n > 0$ on ξ such that $m = \sum g_n \lambda_\xi^n$ is a probability measure. Clearly $\lambda_\xi^n \ll m$ for each n , so there exist measurable functions h_n such that $\lambda_\xi^n = h_n m$. Further, since the currents T_n are increasing, the functions $\varphi_n^\alpha h_n$ are increasing in n for fixed α . Thus the function

$$\tilde{\varphi}^\alpha := \lim_{n \rightarrow \infty} \varphi_n^\alpha h_n$$

is finite for m a.e. α (since the T_n have locally bounded mass) and is thus lower semicontinuous. We conclude, then, that

$$T := \int_{\alpha \in \xi} \tilde{\varphi}^\alpha[\alpha] m(\alpha)$$

is a geometric current, which clearly has the property of $\sup_n T_n$. \square

Remark. Some of the properties of weakly laminar currents may be summarized as follows. Let T be weakly laminar, and let $\mathcal{S}(T)$ denote the set of weakly laminar currents $0 \leq S \leq T$. Then the subset of $\mathcal{S}(T)$ consisting of finite sums of uniformly laminar currents with disjoint supports is dense in the local mass norm (Proposition 6.2). If $0 \leq \psi \leq 1$ is lower semicontinuous, then $\psi\mathcal{S}(T) \subset \mathcal{S}(T)$ (Lemma 6.9). Finally, $\mathcal{S}(T)$ is convex and closed under countable maxima (Lemmas 6.3 and 6.12).

§7. μ^+ is a Laminar Current

In this section, we show that μ^+ is laminar.* Let \tilde{D} denote a 1-dimensional complex submanifold of \mathbf{C}^2 , and let $D \subset \tilde{D}$ be a relatively compact domain with smooth boundary. Let us suppose that

$$\mu^-|_{\tilde{D}}(D) = c > 0 \text{ and } \mu^-|_{\tilde{D}}(\tilde{D} - D) = 0.$$

It follows by [BS3], then, that

$$\lim_{n \rightarrow \infty} d^{-n} f^{*n}[D] = c\mu^+. \quad (7.1)$$

Further, by general properties of the filtration (see [BS1, §2]), we may choose $R < \infty$ such that for all $n < \infty$

$$f^{-n}\tilde{D} \subset \{|y| < R\} \cup \{|y| < |x|\}. \quad (7.2)$$

Let $Q \subset \mathbf{C}$ be a connected open set. For each n , we consider the connected components M of $(f^{-n}D) \cap (Q \times \mathbf{C})$ and the preimage components D' in the domain $D \cap f^n(Q \times \mathbf{C})$. If a component D' of D is relatively compact in D , we say that D' is an *island*; otherwise, it is a *tongue*. Let $\pi : \mathbf{C}^2 \rightarrow \mathbf{C}$ be the projection $\pi(x, y) = x$. If D' is an island, we say that it is a *good island* if the projection $\pi \circ f^{-n}$ is univalent on D' .

We let $\mathcal{G}_n(Q)$ denote the set of components M of $(f^{-n}D) \cap Q \times \mathbf{C}$ which are graphs over Q . This corresponds to the set of good islands, and each good island may be identified with the graph of analytic function $\varphi : Q \rightarrow \mathbf{C}$. If we fix a point $x_Q \in Q$, then each element of $\mathcal{G}_n(Q)$ is uniquely determined by the value $\varphi(x_Q)$, i.e. the intersection $M \cap (\{x_Q\} \times \mathbf{C})$.

By (7.2), the union $\bigcup_n \mathcal{G}_n(Q)$ is a normal family, and we let $\mathcal{G}(Q)$ consist of all graphs $\{y = \varphi(x) : x \in Q\}$ which are obtained as limits of sequences $\varphi_n \in \mathcal{G}_n(Q)$. Since f is a diffeomorphism, the components of $\mathcal{G}_n(Q)$ are disjoint. It follows from the Hurwitz Theorem, then, that any two different graphs in $\mathcal{G}(Q)$ are in fact disjoint. We let $A_Q \subset \mathbf{C}$ denote the closed set of points $\{\varphi(x_Q) : \varphi \in \mathcal{G}(Q)\}$. For $a \in A_Q$ we let $M_j(a)$ denote the element of $\mathcal{G}(Q)$ passing through (x_Q, a) .

For each n we define a measure $\lambda_n^+ = d^{-n} \sum \delta_p$, where the summation is taken over all $p \in \{x_Q\} \times \mathbf{C}$ which are parameters of elements $\varphi \in \mathcal{G}_n(Q)$. For each Q , we choose a subsequence $\{n_k\}$ such that the limit $\lim_{k \rightarrow \infty} \lambda_{n_k}^+$ exists. We let λ_Q^+ denote this limit, and it follows that λ_Q^+ supported on A_Q . Now we define

$$\mu_Q^+ = c^{-1} \int_{a \in A_Q} \lambda_Q^+(a) [M(a)]. \quad (7.3)$$

* We wish to thank Cliff Earle for telling us about the Ahlfors Covering Theorem, which is the principal tool in the proof.

It is evident that

$$d^{-n}[f^{-n}D] \llcorner (Q \times \mathbf{C}) \geq d^{-n} \sum_{M \in \mathcal{G}_n(Q)} [M]. \quad (7.4)$$

Thus, passing to the limit through the subsequence $\{n_k\}$, we have

$$c\mu^+ \llcorner (Q \times \mathbf{C}) \geq c\mu_Q^+.$$

Let us use the following notation. For $k \geq 0$ we let \mathcal{Q}_k denote a dyadic subdivision of the complex plane \mathbf{C} into open squares with vertices of the form $r2^{-k} + is2^{-k}$ with r and s both odd. Let $\mathcal{Q}_k^{(2)}$, $\mathcal{Q}_k^{(3)}$, and $\mathcal{Q}_k^{(4)}$ denote the three different translates of $\mathcal{Q}_k^{(1)}$, so that $\mathcal{Q}_k = \bigcup_{\sigma=1}^4 \mathcal{Q}_k^{(\sigma)}$.

As before, we construct families of graphs $\mathcal{G}_k(Q)$ for $Q \in \mathcal{Q}_k^{(\sigma)}$, for each $\sigma = 1, 2, 3, 4$. If we write

$$\mu_k^+ = \sum_{Q \in \mathcal{Q}_k} \mu_Q^+,$$

then it is evident from (7.4) and (7.3) that

$$\mu_1^+ \leq \mu_2^+ \leq \dots \leq \mu_k^+ \leq \dots \leq \mu^+. \quad (7.5)$$

Now suppose that $j > k$, $Q \in \mathcal{Q}$, $Q' \in \mathcal{Q}_k$, and $Q \subset Q'$. If $\mathcal{G}(Q')|_Q$ denotes the restriction of the disks to Q , then it is evident that $\mathcal{G}(Q')|_Q \subset \mathcal{G}(Q)$. Similarly, making the natural identification via the holonomy for the transversal measures, it follows that $\lambda_{Q'}^+ \leq \lambda_Q^+$. Thus if we set

$$\tilde{\lambda}_Q^+ := \lambda_Q^+ - \max\{\lambda_{Q'}^+ : Q' \supset Q, Q' \neq Q\},$$

then $\tilde{\lambda}_Q^+$ is a positive measure. For each Q , then, we set

$$\tilde{\eta}_Q^+ = \int_{a \in A_Q} \tilde{\lambda}_Q^+(a) [M_Q(a)]$$

and

$$\tilde{\eta}_j^+ = \sum_{Q \in \mathcal{Q}} \tilde{\eta}_Q^+.$$

Thus by Lemma 6.1 we have shown:

Lemma 7.1. *The currents $\tilde{\eta}_j^+$ are uniformly laminar over the squares of \mathcal{Q}_j and have disjoint carriers, and*

$$\lim_{k \rightarrow \infty} \mu_k^+ = \sum_{j=1}^{\infty} \eta_j^+.$$

Further, this limit is a laminar current.

In Theorem 7.4 we will show that this limit is equal to μ^+ .

Let $Q_1, \dots, Q_q \subset \mathbf{C}$ be simply connected, open sets such that $\overline{Q_i} \cap \overline{A_j} = \emptyset$ for $i \neq j$. We let $Q := Q_1 \cup \dots \cup Q_q$, and $I_0 := \text{Area}(Q)$. We set $D_{(n)} := D \cap f^n(Q \times \mathbf{C})$, and we consider the map

$$g_n := \pi \circ f^{-n} : D_{(n)} \rightarrow \mathbf{C}.$$

We let $I_{(n)}$ denote the area (with multiplicity) of $g_n(D_{(n)})$. The *mean sheeting number* of the map g_n is $S_{(n)} := I_{(n)}/I_0$. The length of the *relative boundary* is defined by

$$L_{(n)} := \text{Length}(g_n(\partial D) \cap Q).$$

Fixing the number n of iterates, we write $N(Q_j)$ for the number of good islands over Q_j , i.e. this is just the cardinality of the set $\mathcal{G}_n(Q_j)$. We will use the following celebrated result of Ahlfors (see Nevanlinna [N, Chapt. XIII], or Hayman [Ha]).

Ahlfors' Covering Theorem. *There is a constant h depending only on Q such that the mappings g_n , $n = 1, 2, \dots$ satisfy*

$$\sum_{j=1}^q N(Q_j) \geq (q-2)S_{(n)} - hL_{(n)}.$$

We will use this inequality to estimate the amount of mass in $\sum \mu_{Q_j}^+$. By [BS1] we have

$$\lim_{n \rightarrow \infty} d^{-n} I_{(n)} = c \text{Area}(Q), \text{ or } \lim_{n \rightarrow \infty} d^{-n} S_{(n)} = c, \quad (7.6)$$

with c as in (7.1). Further, by [BS3], there is a constant $C < \infty$ such that

$$L_{(n)}^2 \leq C d^n. \quad (7.7)$$

We note that for a current T , the mass norm of $T \llcorner \frac{i}{2} dx \wedge d\bar{x}$ on the set $B \times \mathbf{C}$ is the same as $\mathbf{M}[\pi_* T \llcorner B]$. Each $M_j(a) \in \mathcal{G}_n(Q_j)$ is the graph of an analytic function on Q_j . Thus the mass norm is

$$\mathbf{M} \left[[M_j(a)] \llcorner \frac{i}{2} dx \wedge d\bar{x} \right] = \text{Area}(Q_j).$$

Lemma 7.2. *If $\text{Area}(Q_1) = \dots = \text{Area}(Q_q)$, then*

$$\mathbf{M} \left[\sum_{j=1}^q \mu_{Q_j}^+ \llcorner \frac{i}{2} dx \wedge d\bar{x} \right] \geq \frac{q-2}{q} \text{Area}(Q).$$

Proof. By the definition (7.3), it follows that the mass norm of $\mu_{Q_j}^+$ is

$$\mathbf{M}[\mu_{Q_j}^+ \llcorner \frac{i}{2} dx \wedge d\bar{x}] = c^{-1} \mathbf{M}[\lambda_j^+] \text{Area}(Q_j). \quad (7.8)$$

In order to estimate $\mathbf{M}[\lambda_j^+]$, we count the number of components M that appear in the right hand side of (7.4). This is the same as the number of good islands over Q_j . Thus we have

$$\begin{aligned} \mathbf{M} \left[\sum_j d^{-n} \sum_{M \in \mathcal{G}_n(Q_j)} [M] \llcorner \frac{i}{2} dx \wedge d\bar{x} \right] &\geq d^{-n} \sum_{j=1}^q \#\mathcal{G}_n(Q_j) \text{Area}(Q_j) \\ &\geq d^{-n} \frac{\text{Area}(Q)}{q} \sum_{j=1}^q N(Q_j) \\ &\geq \frac{d^{-n}}{q} \text{Area}(Q) ((q-2)S_{(n)} - hL_{(n)}), \end{aligned}$$

where the middle inequality follows from the identity $\text{Area}(Q_j) = q^{-1} \text{Area}(Q)$, and the last inequality follows from the Ahlfors Covering Theorem. Applying (7.7), we have

$$\mathbf{M} \left[\sum_j d^{-n} \sum_{M \in \mathcal{G}_n(Q_j)} [M] \llcorner \frac{i}{2} dx \wedge d\bar{x} \right] \geq \frac{q-2}{q} \text{Area}(Q) (d^{-n} S_{(n)} - O(d^{-\frac{n}{2}})).$$

Letting $n \rightarrow \infty$, we see from (7.6) that the right hand side tends to $c(q-2)\text{Area}(Q)/q$ as the left hand side tends to $\sum_j \mathbf{M}[\lambda_j^+] \text{Area}(Q_j)$. Combined with (7.8), this yields Lemma 7.2. \square

Lemma 7.3. *Let $B \subset \mathbf{C}$ denote the unit square. Then*

$$\mathbf{M}[\pi_*(\mu^+ - \mu_k^+) \llcorner B] \leq 8 \cdot 4^{-k}.$$

Proof. We note that $\mathbf{M}[\pi_*\mu^+ \llcorner B] = \text{Area}(B)$ for any open set. And since $\mu^+ \geq \mu_k^+$,

$$\mathbf{M}[\pi_*(\mu^+ - \mu_k^+) \llcorner B] = \mathbf{M}[\pi_*\mu^+ \llcorner B] - \mathbf{M}[\pi_*\mu_k^+ \llcorner B].$$

Thus the Lemma follows by setting $q = 4^{k-1}$ and adding the estimate of Lemma 7.2 over the four partitions $\mathcal{Q}_k^{(\sigma)}$. \square

Theorem 7.4. $\lim_{k \rightarrow \infty} \mu_k^+ = \mu^+$, and μ^+ is a laminar current.

Proof. If we show that the limit holds, then μ^+ is laminar by Lemma 7.1. By (7.5), it suffices to show that

$$\lim_{k \rightarrow \infty} \mathbf{M}[\mu_k^+ \llcorner \pi^{-1}B_0] = \mathbf{M}[\mu^+ \llcorner \pi^{-1}B_0]$$

for any open $B_0 \subset \mathbf{C}$. Without loss of generality, we may choose B_0 to be relatively compact in the unit square B .

For $\alpha \in \mathbf{C}$, we define the projection $\pi'(x, y) = x - \alpha y$. Let us choose $\alpha \neq 0$ sufficiently small that

$$(\pi^{-1}B_0) \cap \text{spt}\mu^+ \subset \pi'^{-1}B.$$

Following the procedure for constructing the current μ_k^+ , except that the projection π' is used in place of π , we may construct a current $\mu_k'^+$. Thus we use the function $g_n := \pi' \circ f^{-n}$, and $\mathcal{G}'_n(Q_j)$ consists of manifolds which are graphs with respect to the coordinates $x' = x - \alpha y$ and $y' = y$. Corresponding to Lemma 6.2, we have

$$\mathbf{M}[\pi'_*(\mu^+ - \mu_k'^+)\lrcorner B] \leq 8 \cdot 4^{-k}.$$

By Lemma 6.6 there is a geometric current T_k such that $\mu_k^+, \mu_k'^+ \leq T_k \leq \mu^+$. Thus we have

$$\mathbf{M}[(\mu^+ - T_k)\lrcorner \chi_{\pi^{-1}(B_0)} \frac{i}{2} d(x - \alpha y) \wedge \overline{d(x - \alpha y)}] \leq 8 \cdot 4^{-k}.$$

Now we use the values $\alpha = 0$ and $\alpha = \pm a \in \mathbf{R}$ and the identity

$$d(x - ay) \wedge \overline{d(x - ay)} + d(x + ay) \wedge \overline{d(x + ay)} - 2dx \wedge d\bar{x} = 2dy \wedge d\bar{y}$$

to obtain

$$|a|^2 \mathbf{M}[(\mu^+ - T_k)\lrcorner \chi_{\pi^{-1}(B_0)} \frac{i}{2} dy \wedge d\bar{y}] \leq 16 \cdot 4^{-k}.$$

Thus

$$\mathbf{M}[(\mu^+ - T_k)\lrcorner \chi_{\pi^{-1}(B_0)} \beta] \leq 8(1 + 2|a|^{-2})4^{-k}$$

where $\beta = \frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y})$. Since $\mu^+ - T_k$ is positive, this gives

$$\mathbf{M}[(\mu^+ - T_k)\lrcorner \chi_{\pi^{-1}(B_0)}] \leq 8(1 + 2|a|^{-2})4^{-k}.$$

Thus $\lim_{k \rightarrow \infty} T_k = \mu^+$.

Now let us recall that T_k is obtained by taking μ_k^+ and adding all of the currents of integration that appear in $\mu_k'^+$, after removing the sets where a manifold $M \in \mathcal{G}(Q_k)$ overlaps a manifold $M' \in \mathcal{G}'(Q'_k)$. But let us consider such a manifold $M' \in \mathcal{G}'(Q'_k)$. As we increase k to a larger index, say K , we subdivide it into the pieces $\pi^{-1}(Q) \cap M'$ for $Q \in \mathcal{Q}_K^{(\sigma)}$. For any point $P \in M'$, except at the (finite) set where π is branched, there is a square $Q \in \mathcal{Q}_K^{(\sigma)}$ for some large K such that a component of $\pi^{-1}\overline{Q} \cap M'$ contains P , and this component belongs to $\mathcal{G}(Q)$. Thus it follows from monotone convergence that $\lim_{j \rightarrow \infty} \mu_j^+ \geq T_k$. Thus $\lim_{k \rightarrow \infty} \mu_k^+ = \mu^+$. \square

Let us denote the total space of the graphs in $\mathcal{G}(G)$ as $\mathcal{E}(G) = \bigcup_{\Gamma \in \mathcal{G}(G)} \Gamma$. We may write μ^+ in the polar form $\mu^+ = t|\mu^+|$, where $|\mu^+|$ is the total variation measure, and for $|\mu^+|$ a.e. point p , $t(p)$ is the unit 2-vector tangent to $M(p) \in \mathcal{M}$. Thus we may define

$$m^+ := \mu^+ \lrcorner \left(\frac{i}{2} dx \wedge d\bar{x} \right) = \left\langle \frac{i}{2} dx \wedge d\bar{x}, t(p) \right\rangle |\mu^+|.$$

We note, further, that the integral of $\langle \frac{i}{2} dx \wedge d\bar{x}, t(p) \rangle$ over a complex manifold M is just the area (with multiplicity) of the projection of M to the x -axis. Since μ^+ is laminar, and since $\langle \frac{i}{2} dx \wedge d\bar{x}, t(p) \rangle$ does not vanish identically on any stable manifold, it follows that m^+ and $|\mu^+|$ define the same measure class.

Theorem 7.5. *Let $G_1, G_2,$ and G_3 be Jordan domains in \mathbf{C} with disjoint closures. Then for some j ,*

$$m^+(\mathcal{E}(G_j)) \geq \frac{1}{9} \text{Area}(G_j).$$

Proof. Let us recall the current $\mu_{Q_j}^+$, constructed above. The total variation measure associated with this current satisfies $|\mu_{G_j}^+| \leq |\mu^+|$ (with $G_j = Q_j$). It follows that

$$m^+(\mathcal{E}(G_j)) = \int_{\mathcal{E}(G_j)} \langle t_\Gamma(p), \frac{i}{2} dx \wedge d\bar{x} \rangle |\mu^+|(p) \geq \mathbf{M}[\mu_{Q_j}^+ \llcorner \frac{i}{2} dx \wedge d\bar{x}].$$

Without loss of generality, we may enlarge G_j to a larger Jordan domain G'_j , so that the three domains have the same area. If we set $q = 3$ in Lemma 7.2, then we have

$$\sum_{j=1}^3 m^+(\mathcal{E}(G_j)) \geq \frac{1}{3} \text{Area}(G'),$$

where $\text{Area}(G')$ is the area of any of the G'_j . It follows, then, that for some j ,

$$m^+(\mathcal{E}(G'_j)) \geq \frac{1}{9} \text{Area}(G'_j).$$

Finally, since each Γ' is a graph over the (larger) domain G'_j , this inequality remains after we shrink to the domain G_j . \square

Theorem 7.6 (Three Islands). *Let $G_1, G_2,$ and G_3 be Jordan domains with disjoint closures. Then for some j , the total space of $\mathcal{G}(G_j)$ has positive $|\mu^+|$ measure.*

Remark. In §8, Corollary 8.8, it will be shown that (almost every) manifold making up the laminar structure of μ^+ is in fact an open subset of one of the stable manifolds $W^s(p)$, $p \in \mathcal{R}$ given by the Pesin theory. The utility of this theorem is that it gives the existence of stable manifolds that are graphs over arbitrarily large sets.

A more general formulation is as follows. Let h be any polynomial, and let $\mathcal{G}(G, h)$ denote the set of all components M of manifolds obtained in the construction of μ^+ such that $h|_M : M \rightarrow h(M)$ is a conformal equivalence. Thus, with our previous notation, we have $\mathcal{G}(G) = \mathcal{G}(G, \pi_x)$. Thus we have:

Corollary 7.7. *Let $G_1, G_2,$ and G_3 be Jordan domains with disjoint closures, and let h be any polynomial. Then for some j , the total space of $\mathcal{G}(G_j, h)$ has positive $|\mu^+|$ measure.*

§8. Geometric intersection of μ^+ and μ^-

By Sections 6 and 7 we know that $|\mu^\pm|$ almost every point lies inside a uniformly laminar current which makes up part of μ^\pm . In this chapter we will obtain a uniformly laminar structure for the currents μ^+ and μ^- near any regular point for μ . This is possible due to a hyperbolic structure given by Pesin boxes. Given a Pesin box P , we can identify it with $P^s \times P^u$ via an appropriate homeomorphism (see §4). Then by Theorem 4.7, μ also has

a product structure on this box, i.e. $\mu \llcorner P$ is taken via this homeomorphism to $\lambda^s \otimes \lambda^u$, where the measures λ^s and λ^u are induced by the currents μ^- and μ^+ correspondingly. Let us fix an ‘‘origin’’ $o \in P$. For any $a \in P^u, b \in P^s$, denote by $\Gamma^s(a)$ a piece of $W_{loc}^s(a, b)$ which is projected onto the disk $B^s(o, r)$ parallel to $E^u(o, r)$ (it does not depend on b). Similarly we can define a family of disks $\Gamma^u(b)$. Now let us consider the following sets supplied with a uniformly laminar structure:

$$\Gamma^s = \bigcup_{a \in P^u} \Gamma^s(a), \quad \Gamma^u = \bigcup_{a \in P^s} \Gamma^u(a).$$

If a Pesin box P_j is labeled by j , we will use the same label for the corresponding sets $\Gamma_j^s(a)$ etc. We let $\{P_j, j = 1, 2, \dots\}$ be a family of Pesin boxes such that $\bigcup P_j$ has full measure, and we set

$$\eta_j^+ := \int_{b \in P_j^u} \lambda_j^u(b) [\Gamma_j^s(b)]$$

$$\eta_j^- := \int_{a \in P_j^s} \lambda_j^s(a) [\Gamma_j^u(a)],$$

which are uniformly laminar currents. Without loss of generality, we may assume that these currents satisfy the hypotheses of Lemma 6.11. Thus the currents

$$\eta_{[n]}^\pm = \max(\eta_1^\pm, \dots, \eta_n^\pm) \quad \text{and} \quad \eta^\pm = \lim_{n \rightarrow \infty} \eta_{[n]}^\pm$$

exist and are laminar. By the holonomy invariance obtained in §4, it follows that η_j^\pm is well defined independently of the transversal used in the definition.

Lemma 8.1. *The sets $\Gamma_j^{s/u}$ satisfy $\eta_j^+ = \mu^+ \llcorner \Gamma_j^s$ and $\eta_j^- = \mu^- \llcorner \Gamma_j^u$. Thus $\eta_j^\pm \leq \mu^\pm$.*

Proof. Let M be any transversal to the lamination Γ_j^s . Since the measure λ_j^u is induced by the current μ^+ ,

$$\eta^+|_M \leq \mu^+|_M. \quad (8.1)$$

Hence $\eta^+ \leq \mu^+$.

Let $\eta^+ = \tau|\eta^+|$ and $\mu^+ = t|\mu^+|$ be the polar representations. Then $|\eta^+| \leq |\mu^+|$. Since μ^+ is a laminar current, t is a simple 2 vector $|\mu^+|$ a.e. Thus $\tau = t$ $|\eta^+|$ -a.e., and the Lemma follows. \square

Lemma 8.2. *If T is a closed current, $0 \leq T \leq \mu^+$, then locally there is a continuous function u with $dd^c u = T$.*

Proof. Since T is closed, there is locally an integrable function u such that $dd^c u = T$. If $\beta = \frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y})$, then $\Delta u \llcorner \beta = dd^c u$. It follows that $0 \leq \Delta u \leq \Delta G^+$. Let ν denote the positive measure $\Delta G^+ - \Delta u$ on some open set \mathcal{O} , and let $s = -c_4|x|^{-2} * \nu$ denote the convolution with ν , with c_4 chosen so that of $-c_4|x|^{-2}$ is the fundamental solution of Δ on \mathbf{R}^4 . Thus s is subharmonic, and the difference between G^+ and $u + s$ is harmonic on $\mathcal{O} \subset \mathbf{R}^4$. A subharmonic function v on \mathcal{O} satisfies $\liminf_{q \rightarrow q_0} v(q) \leq \limsup_{q \rightarrow q_0} v(q) = v(q_0)$ for all $q_0 \in \mathcal{O}$. Since s and u both satisfy this inequality, and since $u + s$ is continuous at q_0 , it follows that s and u are continuous at q_0 . \square

We will define two different ways of taking the product of two currents. First, we consider a continuous, psh function u and a positive, closed (1,1) current T . We define the (2,2) current $T \wedge dd^c u$ by its action on a test function φ :

$$(T \wedge dd^c u)(\varphi) = T(u dd^c \varphi).$$

(This is essentially just integrating the dd^c by parts since T is closed.) It is evident from the right hand side of the defining equation that if u_j converges uniformly to u , then $T \wedge dd^c u_j$ converges to $T \wedge dd^c u$. We refer the reader to [BT1] for further discussion of the \wedge operation.

If L_1 and L_2 are uniformly laminar currents on Δ^2 , then it is also natural to define

$$L_1 \dot{\wedge} L_2 = \int \lambda_1(a_1) \int \lambda_2(a_2) [\Gamma_{a_1} \cap \Gamma_{a_2}]$$

with $[\Gamma_{a_1} \cap \Gamma_{a_2}]$ defined as the 0-current which puts unit mass on each point of $\Gamma_{a_1} \cap \Gamma_{a_2}$, with the exception that $[\Gamma_{a_1} \cap \Gamma_{a_2}] = 0$ if $\Gamma_{a_1} = \Gamma_{a_2}$. This is analogous to the integrated version of (5.3), except that $\Gamma_{a_1} \cap \Gamma_{a_2}$ is not necessarily transversal or finite.

Lemma 8.3. *Let L and L' be uniformly laminar currents on Δ^2 such that there is a continuous, psh function u with $dd^c u = L$. Then*

$$L \wedge L' = L \dot{\wedge} L'.$$

Proof. Without loss of generality, we may assume that L and L' are represented in the form (5.2), and

$$u = \frac{1}{2\pi} \int \lambda(a) \log |y - \varphi_a(x)|.$$

It will suffice to work over the relatively compact set $\{|x| < 1 - \epsilon\}$. Let us fix $\Gamma' = \Gamma_{a'}$. Choosing a parameter $\zeta = x$ for points $(x, y) = (\zeta, \varphi_a(\zeta)) \in \Gamma'$, we have

$$\log |y - \varphi_a(x)| = \sum_{j=1}^{N_a} \log |\zeta - p_j(z)| + h_a(\zeta),$$

where h_a is harmonic. Since h_a is harmonic on $\{|x| < 1\}$, it is bounded on $\{|x| < 1 - \epsilon\}$. Let us define

$$A_R = \{a \in A : \|h_a\|_{L^\infty(|x| < 1 - \epsilon)} \leq R, N_a \leq R\}.$$

If we set

$$u_R(x, y) = \frac{1}{2\pi} \int_{a \in A_R} \lambda(a) \log |y - \varphi_a(x)|,$$

then, as in Lemma 8.2, u_R is continuous. Further, since the A_R increase to A as $R \rightarrow \infty$, u_R converges uniformly to u . Thus

$$\begin{aligned} (dd^c u_R) \wedge [\Gamma'] &= (dd^c \int \lambda(a) \log |y - \varphi_a(x)|) \wedge [\Gamma'] \\ &= \left(\int \lambda(a) [\Gamma_a] \right) \wedge [\Gamma'] \\ &= \int \lambda(a) [\Gamma_a \cap \Gamma'] = dd^c u_R \dot{\wedge} [\Gamma'], \end{aligned}$$

where the next to last equality follows from the Fubini theorem, since the multiplicity of the intersection is uniformly bounded for $a \in A_R$. Letting $R \rightarrow \infty$, we have

$$L \wedge [\Gamma_{a'}] = L \dot{\wedge} [\Gamma_{a'}]. \quad (8.2)$$

Finally, we integrate (8.2) with respect to $\lambda'(a')$. The right hand side yields $L \dot{\wedge} L'$ by Fubini's Theorem. The left hand side, applied to a smooth test function χ is

$$\begin{aligned} \int \lambda'(a')(L \wedge [\Gamma_{a'}])\chi &= \int \lambda'(a')[\Gamma_{a'}]u dd^c \chi = \\ &= L'(u dd^c \chi) = (L' \wedge dd^c u)(\chi), \end{aligned}$$

which completes the proof. \square

Lemma 8.4. *We have $\eta_j^+ \wedge \eta_j^- = \mu \llcorner P_j$, and thus $\mu^- \wedge \eta_{[k]}^+ \geq \mu \llcorner \bigcup_{j=1}^k P_j$.*

Proof. By Lemmas 8.2 and 8.3, we have $\eta_j^+ \wedge \eta_j^- = \eta_j^+ \dot{\wedge} \eta_j^-$. By the product structure of Theorem 4.7, we have that under the homeomorphism between P_j and $P_j^s \times P_j^u$, $\eta_j^+ \dot{\wedge} \eta_j^-$ is taken to $\lambda_j^s \otimes \lambda_j^u$, which in turn is equivalent to $\mu \llcorner P_j$. Similarly, since $\mu^- \geq \eta_j^-$ and $\eta_{[k]}^+ \geq \eta_j^+$, we have

$$\mu^- \wedge \eta_{[k]}^+ \geq \eta_j^+ \wedge \eta_j^- = \eta_j^+ \dot{\wedge} \eta_j^- = \mu \llcorner P_j.$$

Since this holds for all j , the Lemma follows. \square

Lemma 8.5. $\lim_{n \rightarrow \infty} d^{-n} f^{*n} \eta^+ = \mu^+$.

Proof. Let φ be a test form. We will show that

$$\int \varphi \mu^+ = \lim_{n \rightarrow \infty} \int \varphi d^{-n} f^{*n} \eta^+. \quad (8.3)$$

Without loss of generality, we may assume that $\varphi \geq 0$. Let $\epsilon > 0$ be given. By Lemma 8.4, we may choose k large enough that the total mass of $\mu^- \wedge \eta_{[k]}^+$ is greater than $1 - \epsilon$. By Lemma 6.1, we may write $\eta_{[k]}^+$ as a sum of uniformly laminar currents $\sum L_j$ with disjoint carriers. We may take finitely many terms from this summation and choose test functions $0 \leq \chi_j \leq 1$ such that $\sum \chi_j L_j \leq \eta_{[k]}^+$, and the total mass of $\mu^- \wedge \sum \chi_j L_j$ is $c > 1 - \epsilon$.

Now by [BS3], we have $\lim_{n \rightarrow \infty} d^{-n} f^{*n} (\sum \chi_j L_j) = c \mu^+$. Since $\varphi \geq 0$, we have

$$\begin{aligned} \int \varphi \mu^+ &\geq \int \varphi d^{-n} f^{*n} \eta^+ \geq \\ &\geq \int \varphi d^{-n} f^{*n} \left(\sum \chi_j L_j \right) \geq (1 - \epsilon) \int \varphi \mu^+ \end{aligned}$$

for n sufficiently large. Since ϵ may be made arbitrarily small, we have (8.2). \square

Let us assume further that for λ^s a.e. a , the measure induced by μ^- on the corresponding stable manifold puts no mass on $\partial \Gamma_j^s(a)$. Then we have the following:

Lemma 8.6. *Let \tilde{M} be a 1-dimensional submanifold of \mathbf{C}^2 , and let M be a relatively compact submanifold such that $\mu^+|_{\tilde{M}}(\partial M) = 0$. Then*

$$\lim_{n \rightarrow \infty} (d^{-n} f^{*n} \eta_j^+) \lrcorner [M] = c \mu^+ \lrcorner [M],$$

where $c := \eta_j^+|_{\tilde{M}}[M]$.

If we set $G^s = \cup_j \Gamma_j^s$, then by Lemma 8.1 we have

$$d^{-n} f^{*n} \eta^+ = \mu^+ \lrcorner f^{-n}(G^s).$$

Since $G^s \subset \cup_{x \in \mathcal{R}} W^s(x)$, where \mathcal{R} is the set of all regular points (see §2), it follows from Lemma 8.5 that we have:

Corollary 8.7. $\cup_{x \in \mathcal{R}} W^s(x)$ is a carrier for $|\mu^+|$.

By Corollary 6.6, we have:

Corollary 8.8. *If $(A, \mathcal{M}, \lambda)$ is a representation of μ^+ , then λ almost every $M \in \mathcal{M}$ is an open subset of a stable manifold $W^s(x)$, $x \in \mathcal{R}$.*

Here we give a slightly different formulation of holonomy invariance. This is more general than the one given in §4 because it applies to all stable manifolds. Let $\mathcal{M} = \{M_\alpha : \alpha \in A\}$ be a family of stable manifolds. Let $\mathcal{D} = \{D_t : 0 \leq t \leq 1\}$ be a continuous family of manifolds such that each D_t is a transversal to \mathcal{M} . We define $X_j = D_j \cap \cup_\alpha M_\alpha$, $j = 1, 2$. The holonomy map $\chi : X_0 \rightarrow X_1$ is defined by at a point $x \in X_0$ by following the intersection point with D_t from $t = 0$ to $t = 1$.

Theorem 8.9 (Holonomy Invariance). *The holonomy map preserves the slices of μ^+ , i.e.*

$$\chi_*(\mu^+|_{D_0} \lrcorner X_0) = \mu^+|_{D_1} \lrcorner X_1.$$

Proof. If the family \mathcal{M} consists of leaves of $G^s = \cup_j \Gamma_j^s$, then holonomy is preserved. In general, we consider the compact sets $\gamma_\alpha = \{D_t \cap M_\alpha : 0 \leq t \leq 1\}$. For each α the curve γ_α is contained in a stable manifold, so there is an n such that $f^n \gamma_\alpha$ is contained in one of the leaves of G^s . Thus for $\epsilon > 0$ there exists an n such that $\{x \in X_0 : f^n \gamma_\alpha \not\subset G^s\}$ has measure less than ϵ . Since the holonomy is preserved on the complement of this set, we see that the Lemma holds. \square

§9. Saddle Points and the Support of μ

Questions about saddle points have motivated much of the preceding work on currents, and we are grateful to J.H. Hubbard for several discussions on this subject. In this section we show that there is a homoclinic/heteroclinic intersection between any pair of stable and unstable manifolds. The general idea is that if D^s is a stable disk through a saddle point p , then the normalized pullbacks $d^{-n}[f^{-n}D^s]$ converge to a nonzero multiple of μ^+ . By the results of §8, it follows that the product $d^{-n}[f^{-n}D^s] \wedge \mu^-$ converges to a nonzero multiple of the measure $\mu = \mu^+ \wedge \mu^-$. Since this is also equal to the intersection wedge product, it follows that $d^{-n}[f^{-n}D^s] \lrcorner \mu^-$ must be nonzero for some n . This produces intersections between stable and unstable manifolds.

Lemma 9.1. *Let P be a Pesin box, and Γ^u be the corresponding lamination defined in §8. If p is a saddle point, then $W^s(p)$ must intersect λ^s almost every disk of Γ^u , and the tangential intersections are an isolated subset of $W^s(p)$.*

Proof. Let $D \subset W^s(p)$ be a relatively compact open set. If D contains p , then $\mu^-|_D \neq 0$, so $d^{-n}[f^{-n}D] \rightarrow c\mu^+$ for some $c > 0$ as $n \rightarrow \infty$. Let us suppose that there is a subset $E \subset P^u$ such that the corresponding unstable disks are disjoint from $W^s(p)$. Let us define

$$\nu_E^- = \int_{a \in E} \lambda^s(a) [\Gamma^u(a)].$$

By Lemma 8.2, it follows that $\nu_E^- \wedge d^{-n}[f^{-n}D] \rightarrow c\nu_E^- \wedge \mu^+$. By Lemma 8.3, the left hand side must be zero. But $\mu^+ \geq \nu^+ \equiv \mu^+ \llcorner \Gamma^s$, and

$$\mathbf{M}[\nu_E^- \wedge \nu^+] = \lambda^s(E)\lambda^u(P^s) \neq 0.$$

Thus we must have $\lambda^s(E) = 0$, which completes the proof of the first part. The tangential intersections are isolated by Lemma 6.3. \square

Theorem 9.2. *If p is a saddle point of f , then $p \in J^*$.*

Proof. Let P be a Pesin box. By Lemma 9.1, there is a compact subset $K^u \subset W^u(p)$ and a subset \mathcal{G}_0^s of the leaves of Γ^s such that (i) $\mu^u(\mathcal{G}_0^s) > 0$, (ii) for each $a^u \in K^u$ there is a leaf $\Gamma^s(x)$ of Γ^s such that $\{a^u\} = K^u \cap \Gamma^s(x)$, and (iii) the angle of intersection of $\Gamma^s(a^u)$ and $W^u(p)$ is greater than $\theta_0 > 0$. Similarly, we may find subsets $K^s \subset W^s(p)$ and the family of leaves \mathcal{G}_0^u with analogous properties.

Let us choose a coordinate system in a neighborhood U of p so that $p = 0$, $U = \{|x| < 1, |y| < 1\}$, $\{y = 0\} \cap U$ is the component of $W^s(p) \cap U$ containing 0, and $\{x = 0\} \cap U$ is the component of $W^u(p) \cap U$ containing 0. By the Lambda Lemma (see e.g., [PdM]), we may take n sufficiently large that $f^{-n}K^u \subset \{x = 0, |y| < \epsilon\}$, and for each $a^u \in K^u$ the portion of $f^{-n}W^s(a^u) \cap U$ passing through $f^{-n}a^u$ (denoted $W_1^s(f^{-n}a^u)$), is uniformly C^1 close to $\{y = 0, |x| < 1\}$. Similarly, for each $a^s \in K^s$, the portion of $f^nW^u(a^s) \cap U$ passing through f^na^s (denoted $W_1^u(f^na^s)$), is C^1 close to $\{x = 0, |y| < 1\}$.

Thus we may choose n large enough that every pair of manifolds $W_1^s(f^{-n}a^u)$ and $W_1^u(f^na^s)$ have nonempty intersection. Finally, since $\lambda^u(K^u) > 0$, it follows that

$$\lambda_n^u := f_*^{-n}(\lambda^u \llcorner K^u) = \mu^+|_{\{x=0, |y|<1\} \cap f^{-n}K^u} \neq 0.$$

Thus

$$\nu_n^+ := \int_{a \in f^{-n}K^u} \lambda_n^u(a) [W_1^s(a)] \neq 0.$$

With an analogous definition for ν_n^- , we have $\mu \llcorner U \geq \nu_n^+ \wedge \nu_n^- \neq 0$. Thus p is in the support of μ . \square

Combining Theorem 9.2 with the density of saddle points proved in [BS4] gives the following characterization of J^* .

Corollary 9.3. J^* is the closure of the set of saddle points.

Corollary 9.4. Any hyperbolic measure has support contained in J^* .

Proof. According to Katok [K, Theorem 8], periodic saddle points are dense in the support of any hyperbolic measure. Thus the Corollary follows from Theorem 9.2. \square

Corollary 9.5. If p is a saddle point for f , then every transverse intersection of $W^s(p)$ and $W^u(p)$ is in J^* .

Proof. By the Birkhoff-Smale theorem every transverse homoclinic intersection is the limit of saddle points. So the corollary follows from Theorem 9.2 and the fact that J^* is closed. \square

Theorem 9.6. If p and q are saddle points for f , then the set of transverse intersections of $W^s(p)$ and $W^u(q)$ is dense in J^* .

Proof. Let U be an open set with $U \cap J^* \neq \emptyset$. Then there exists a Pesin box $P \subset U$. Every pair of points $x_1, x_2 \in P$ have the property that $W_r^s(x_1)$ intersects $W_r^u(x_2)$ in a unique point in P and the intersection is transverse. Further, there exists $\epsilon > 0$ such that for any smooth manifolds M^s and M^u such that $\text{dist}_{C^1}(M^s, W_r^s(x)) < \epsilon$ and $\text{dist}_{C^1}(M^u, W_r^u(x)) < \epsilon$, then M^s intersects M^u in a unique point in U and the intersection is transverse. By Lemma 9.1, there exist $x_1, x_2 \in P$ and $y_1 \in W^s(p) \cap W_r^u(x_1)$ and $y_2 \in W^u(q) \cap W_r^s(x_2)$. For infinitely many values $n_j \rightarrow \infty$ we have $f^{-n_j} x_1 \in P$. By the Lambda Lemma, there is a portion D_j^s of $f^{-n_j} W^s(p)$ which contains $f^{-n_j} y_1$ and lies as a graph over $W_r^s(f^{-n_j} x_1)$. Further, D_j^s approaches $W_r^s(f^{-n_j} x_1)$ in C^1 . Similarly, $f^{m_k} y_2 \in P$ for infinitely many $m_k \rightarrow \infty$, and there is a portion $D_k^u \subset f^{m_k} W^u(q) = W^u(q)$ which approaches $W_r^u(f^{m_k} x_2)$ in C^1 . For j, k large, this C^1 distance is less than ϵ , and it follows that D_j^s and D_k^u have a transverse intersection. Since $W^s(p) \cap W^u(q) \cap U \supset D_j^s \cap D_k^u \cap U \neq \emptyset$, it follows that the set of transverse intersections of $W^s(p)$ and $W^u(q)$ is dense in J^* . \square

A saddle point p is said to generate a homoclinic intersection if $W^s(p)$ and $W^u(q)$ intersect in points other than p .

Corollary 9.7. Every saddle point generates a homoclinic intersection.

Proposition 9.8. If p is a saddle point for f , then every point in the intersection of $W^s(p)$ and $W^u(q)$ is a limit of transverse intersections of $W^s(p)$ and $W^u(q)$.

Proof. Let $x \in W^s(p) \cap W^u(q)$. Choose a coordinate system at p as in the proof of Theorem 9.2. Replacing x by $f^n(x)$ we may assume that $x \in \{y = 0\} \cap U$. Now by Theorem 9.4 $W^u(p)$ has a transverse intersection with $W^s(p)$. The Lambda Lemma implies that there are components of $W^u(p) \cap U$ arbitrarily close to $\{y = 0\} \cap U$. Lemma 6.4 then completes the proof. \square

Theorem 9.9. If p is a saddle point for f , then every intersection of $W^s(p)$ and $W^u(q)$ is in J^* .

Proof. This follows from Theorem 9.6 and the previous proposition. \square

§10. Applications to Real Henon mappings

Consider a polynomial diffeomorphism f with real coefficients. f leaves invariant the real subspace \mathbf{R}^2 . In this section we will denote $f : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ by $f_{\mathbf{C}}$ and $f|_{\mathbf{R}^2}$ by $f_{\mathbf{R}}$. Recall that if $d > 1$, then d can be defined as the minimal degree of any map conjugate to f . This number can be computed from $f_{\mathbf{R}}$ without making reference to $f_{\mathbf{C}}$. In [FM] it is shown that $h_{top}(f_{\mathbf{R}}) \leq \log d$. In this section we investigate maps for which equality holds.

Hyperbolic “ d -fold” horseshoes (see [FM] and [HO]) are examples of maps of maximal entropy but these are not the only examples. The set of horseshoes is open in parameter space and continuity of the entropy function ([Mi]) shows the set of maps of entropy $\log d$ is closed. Thus the set of parameters of maps with entropy $\log d$ contains the closure of the set of horseshoe parameters. It would be interesting to know whether it contains other maps as well.

Theorem 10.1. *The following are equivalent:*

- (1) $h_{top}(f_{\mathbf{R}}) = \log d$
- (2) $\mu(\mathbf{R}^2) > 0$
- (3) $J^* \subset \mathbf{R}^2$
- (4) $K \subset \mathbf{R}^2$
- (5) *Every periodic point of $f_{\mathbf{C}}$ is contained in \mathbf{R}^2*
- (6) *If p is a periodic saddle point then $W^s(p, f_{\mathbf{C}}) \cap W^u(p, f_{\mathbf{C}})$ is contained in \mathbf{R}^2 .*

Any of these conditions implies:

- (7) $J^* = J = K$.

Proof. The result of Newhouse, (Theorem 2.2), shows that $f_{\mathbf{R}}$ possesses a probability measure ν of maximal entropy. If (1) holds then the entropy of ν is $\log d$ so $\nu = \mu$ by the uniqueness result, Theorem 3.1. Thus (1) implies $\mu(\mathbf{R}^2) = 1$.

Assume that (2) holds. Since μ is ergodic and \mathbf{R}^2 is an invariant set of positive measure we have $\mu(\mathbf{R}^2) = 1$. Since \mathbf{R}^2 is a closed set of full measure the support of μ is contained in \mathbf{R}^2 . But the support of μ is J^* so (2) implies (3).

We will show that if J^* is real then K is real. Recall that J^* is the Shilov boundary of K which is the minimal closed set $S \subset K$ with the property that for any polynomial P the maximum value of $|P|$ on K is equal to the maximum value of $|P|$ on S . It is a general fact that if the Shilov boundary of a set is real then the set is real. We recall the proof. Assume that K is not real. Say $p = (z_1, z_2)$ is in K but not in \mathbf{R}^2 . Either z_1 or z_2 is not real. For definiteness assume that $z_1 \notin \mathbf{R}$. Let J_1 be the projection of J^* onto the first coordinate. Runge’s theorem assures the existence of a complex polynomial $P_1(z)$ so that $|P_1(J_1)| < 1/10$ and $|P_1(z_1)| > 1$. Thus the polynomial $P(z_1, z_2) = P_1(z_1)$ takes its maximum value outside of J^* contradicting our assumption. Thus (3) implies (4).

If (4) holds then μ is supported in \mathbf{R}^2 so $f_{\mathbf{R}}$ has a measure of entropy $\log d$ so (4) implies (1). This demonstrates the equivalence of conditions (1) through (4).

We prove the equivalence of (5). Since every periodic point is in K , (4) implies (5). Since periodic points are dense in J^* , ([BS3] Theorem 3.4), we have (5) implies (3).

We prove the equivalence of (6). Since every point in $W^s(p, f_{\mathbf{C}}) \cap W^u(p, f_{\mathbf{C}})$ has a bounded orbit this set is contained in K , thus (4) implies (6). Since $W^s(p, f_{\mathbf{C}}) \cap W^u(p, f_{\mathbf{C}})$ is dense in J^* , (Corollary 9.3), we have (6) implies (3).

To show that these conditions imply (7) we argue as follows. By (4) $K \subset \mathbf{R}^2$. The Stone-Weierstrass theorem implies that any continuous function on $K \subset \mathbf{R}^2$ can be approximated by a polynomial function. This implies that the Shilov boundary of K is all of K . Thus $J^* = K$. Since $J^* \subset J \subset K$ we have $J^* = J = K$. \square

The following result gives some consequences of the equivalent conditions described in Theorem 4.1. Note that all these results can be *stated* in terms of $f_{\mathbf{R}}$ without reference to $f_{\mathbf{C}}$ or \mathbf{C}^2 . Nevertheless our *proofs* of these results require complex techniques.

Theorem 10.2. *Let $f_{\mathbf{R}}$ be a polynomial diffeomorphism of \mathbf{R}^2 with entropy $\log d$ then:*

- (1) $f_{\mathbf{R}}$ has a unique measure of maximal entropy
- (2) $f_{\mathbf{R}}|_K$ is topologically mixing
- (3) $f_{\mathbf{R}}$ has no sinks
- (4) Periodic points are dense in the set of bounded orbits
- (5) For any periodic saddle point $W^s(f_{\mathbf{R}}, p) \cap W^u(f_{\mathbf{R}}, p)$ is dense in the set of bounded orbits.

Proof. Since $f_{\mathbf{C}}$ has a unique measure of maximal entropy it follows that $f_{\mathbf{R}}$ has a unique measure of maximal entropy when $h(f_{\mathbf{R}}) = h(f_{\mathbf{C}})$.

We prove (2). By [BS3] f is mixing for the measure μ . It follows that f is topologically mixing on the support of μ which is J^* . By assertion (7) of Theorem 4.1 $J^* = K$.

To prove (3) we note that a sink orbit is in K but not in J (a sink is in the interior of K^+ but $J = \partial K^+ \cap \partial K^-$). So (1) implies that f has no sinks. In the volume preserving case the same argument shows that f has no linearizable elliptic points.

Assertion (4) follows from assertion (7) of Theorem 10.1 because periodic points are dense in J^* and the set of bounded orbits is K .

Assertion (5) follows from (1) because homoclinic intersections are dense in J^* (Corollary 9.3) and $J^* = K$ from assertion (7) of Theorem 4.1. This completes the proof of the theorem. \square

Remark. Let us mention the real quadratic mapping $h : \mathbf{R}^2 \mapsto \mathbf{R}^2$ given by $(x, y) \rightarrow (1 - ax^2 + y, bx)$ with $a = 1.4$ and $b = .3$, which was considered in detail by Hénon. There are eight solutions of $\{(x, y) \in \mathbf{C}^2 : h_{\mathbf{C}}^3(x, y) = (x, y)\}$, two of which are real fixed points, and the other six lie in two cycles of period 3. Numerical computation suggests that the 3-cycles consist of nonreal points. Paul Pedersen gave a mathematical proof that this is indeed the case ([P]). It follows from Theorem 10.1 that h has entropy strictly less than $\log 2$. And for any saddle p , $W^s(p) \cap W^u(p)$ contains points outside \mathbf{R}^2 .

§11. Appendix: Concluding Remarks

Some remarks on the logical inter-relationships between the various sections of this paper are in order. This paper was organized so that the first methods used were Pesin theory and entropy; and the first main results obtained were the identification of the conditional measures and the uniqueness of the measure of maximal entropy. The logical progression we have adopted was not the only one possible. What follows is an outline of a different order in which arguments from this paper could be presented. In this scenario, the use of entropy comes only at the end. And this organization leads to new proofs both that the entropy of μ is $\log d$ and that the topological entropy of f is $\log d$.

Step 1. Let P be a Pesin box, Γ^s be the corresponding stable lamination as defined in §8. First prove the holonomy invariance of measures induced by μ^+ along this lamination (Lemma 4.4). Thus μ^+ induces a transversal measure on Γ^s .

Step 2. From Lemma 4.4 we deduce that $\mu^+ \llcorner \Gamma^s$ is a uniformly laminar current (Lemma 8.1). Next we prove Lemmas 8.2 and 8.3, and it follows that

$$\mu \llcorner P = (\mu^+ \llcorner \Gamma^s) \dot{\wedge} (\mu^- \llcorner \Gamma^u).$$

Thus $\mu \llcorner P$ has a product structure.

Step 3. The product structure of μ on sets P of positive measure, and especially the fact that the wedge product is equal to the intersection $\dot{\wedge}$, allows us to prove the results of §9 concerning saddle points.

Step 4. The product structure of μ on the sets P also implies that the conditional measures of μ on the unstable leaves are induced by μ^+ .

Step 5. Up to this point, these arguments have not involved entropy. But now the facts obtained in the previous steps may be used to calculate the entropy of μ via the formula:

$$h_\mu(f) = \int \log J_\mu^u \mu = \log d.$$

Step 6. The argument in §3 shows that $h_\nu(f) < \log d$ for any invariant measure $\nu \neq \mu$. Hence, μ is the unique measure of maximal entropy. By the Variational Principle, this also gives us an alternative proof of the Friedland-Milnor-Smillie formula for the topological entropy: $h(f) = h_\mu(f) = \log d$.

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