Wandering Domains for Holomorphic Maps

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Let f be a rational or entire function. A connected component D of the complement of the Julia set $J(f)$ is called wandering domain if for all $m > n \geq 0$ we have $f^m D \cap f^n D = \emptyset$, where f^m stands for m-th iterate of f. One of the most important theorems in holomorphic dynamics due to D. Sullivan states that rational functions have no wandering domains [11]. We ask for possible generalizations of this theorem. All known proofs of Sullivan's theorem use heavily the fact that the space of quasiconformal deformations of a rational function is finitely dimensional (see e.g. [3]).

Here is one situation where a similar result could be proved. We say that an entire function f belongs to the class S if there is a finite set of points $\{a_1, \ldots, a_q\}$ such that

$$f : \mathbf{C} \backslash f^{-1}\{a_1, \ldots, a_q\} \to \mathbf{C} \backslash \{a_1, \ldots, a_q\}$$

is a covering map. The space of quasiconformal deformations of an entire function of the class S has finite dimension and the following result can be proved by extending the Sullivan's method: entire functions of the class S have no wandering domains [6], [8], [2].

On the other hand it is known that wandering domains D may exist for some entire functions f. The examples with the following properties have been constructed:

1). $f^n D \to \infty$, [1], [2], [5], [9]. The example in [5] has an additional property that the iterates f^n are univalent in D.

2). The orbit $\{f^n D\}$ has infinitely many limit points, including ∞, [5].

Question 1 *Does there exist an entire function f with a wandering domain D such that the orbit $\{f^n D\}$ is bounded?*

Remark that there are entire functions not in the class S, for which the negative answer can be obtained easily. We say that the function f has order less then one half if

$$\log \log^{+} |f(z)| \leq \alpha \log |z|, \quad |z| > r_0$$

for some $\alpha < 1/2$. It follows from a classical theorem by Wiman and Valiron (see, for example, [10]) that such functions have the following property: there exists a sequence $r_k \to \infty$ such that

$$|f(r_k e^{i\theta})| > r_k, \quad 0 \leq \theta \leq 2\pi.$$

It follows that there is an increasing sequence of domains G_k, $\cup G_k = \mathbf{C}$ such that the restrictions of f on G_k are polynomial-like maps [4]. So f has no wandering domains with bounded orbit because polynomial-like maps have no wandering domains.

Now we consider a special type of wandering domains whose orbits tend to a finite point z_0. Let φ be a germ of holomorphic function with the point z_0 fixed. Suppose that $\lambda = \varphi'(z_0) = \exp 2\pi i \alpha$, α irrational. It was proved by Fatou [7] that in this situation $\varphi^n(z)$ cannot tend to z_0 in an *invariant* domain. So we have the following

Question 2 *Is it possible that $\varphi^n(z) \to z_0$ uniformly in some domain D?*

In the case when φ can be analytically continued to an entire function positive answer would imply the existence of wandering domain whose orbit tends to z_0. It would be also interesting to know the answer to the question 2 with other additional assumptions on the germ φ, for example, when φ is a germ of an algebraic function.

Finally remark that the answer to the following question is also unknown

Question 3 *Under the assumptions of Question* 2 *can it happen that there is an orbit tending to z_0?*

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