On Invariant Measures for Iterations of Holomorphic Maps

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Let U be an open subset of the Riemann sphere $\hat{\mathbb{C}}$. Consider any holomorphic mapping $f : U \to \hat{\mathbb{C}}$ such that $f(U) \supset U$ and $f : U \to f(U)$ is a proper map, (for a more general situation see [PS]). Consider any $z \in f(U)$. Let $z^1, z^2, ..., z^d$ be some of the f-preimages of z in U where $d \geq 2$. Consider curves $\gamma^i : [0,1] \to \hat{\mathbb{C}}$, $i = 1, ..., d$, also in $f(U)$, joining z with z^i respectively (i.e. $\gamma^i(0) = z, \gamma^i(1) = z^i$).

Let $\Sigma^d := \{1, ..., d\}^{\mathbb{Z}^+}$ denote the one-sided shift space and σ the shift to the left, i.e. $\sigma((\alpha_n)) = (\alpha_{n+1})$. For every sequence $\alpha = (\alpha_n)_{n=0}^{\infty} \in \Sigma^d$ we define $\gamma_0(\alpha) := \gamma^{\alpha_0}$. Suppose that for some $n \geq 0$, for every $0 \leq m \leq n$, and all $\alpha \in \Sigma^d$, the curves $\gamma_m(\alpha)$ are already defined. Suppose that for $1 \leq m \leq n$ we have $f \circ \gamma_m(\alpha) = \gamma_{m-1}(\sigma(\alpha))$, and $\gamma_m(\alpha)(0) = \gamma_{m-1}(\alpha)(1)$.

Define the curves $\gamma_{n+1}(\alpha)$ so that the previous equalities hold (by taking f-preimages of curves already existing; if there are no critical values for iterations of f in $\bigcup_{i=1}^d \gamma^i$ one has a unique choice). For every $\alpha \in \Sigma^n$ and $n \geq 0$ denote $z_n(\alpha) := \gamma_n(\alpha)(1)$.

The graph with the vertices z and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ is called a *geometric coding tree* with the root at z. For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \geq 0$ is called a *geometric branch* and denoted by $b(\alpha)$. The branch $b(\alpha)$ is called *convergent* if the sequence $z_n(\alpha)$ is convergent in cl$U$. We define the *coding map* $z_\infty : \mathcal{D}(z_\infty) \to \mathrm{cl}U$ by $z_\infty(\alpha) := \lim_{n\to\infty} z_n(\alpha)$ on the domain $\mathcal{D}(z_\infty)$ of all such α's for which $b(\alpha)$ is convergent.

There are two basic examples:

1. $f : U \to U$ where U is a simply-connected domain in $\hat{\mathbb{C}}$, deg$f \geq 2$, and the iterates f^n converge to a constant in U, in particular U is an immediate basin of attraction of a sink for f a rational map on $\hat{\mathbb{C}}$.

2. $U = \hat{\mathbb{C}}$, f is a rational mapping.

It is known that except for a "thin "set in Σ^d all branches are convergent (i.e. $\Sigma^d \setminus \mathcal{D}(z_\infty)$ is "thin" and for every $x \in \mathrm{cl}U$, the set $z_\infty^{-1}(x)$ is "thin"). These hold under very mild assumptions about the tree even allowing the existence of critical values in it. Proofs and a discussion of various possibilities of "thiness" can be found in [PS]. In particular one obtains the classical Beurling's Theorem that a holomorphic univalent function R on the unit disc $\mathbb{D}$ has radial limits everywhere except on a set of logarithmic capacity zero, and for every limit point, the set in $\partial\mathbb{D}$ to which radii converge is also of logarithmic capacity 0. One just transports the map $z \mapsto z^2$ to $U := R(\mathbb{D})$, and gets a type 1 situation. There is a 1-to-1 correspondence between the radii and geometric branches.

General Problem. How large is the image: $z_\infty(\mathcal{D}(z_\infty))$?

We shall specify this Problem separately in the basin of attraction case (the situation 1 above) and in the general situation.

To simplify the notation we have restricted ourselves to trees and codings from the full shift space. In the general situation it might be useful to consider also a topological

Markov chain, see [PS].

THE CASE OF THE BASIN OF ATTRACTION

Problem 1.1 If f extends holomorphically to a neighbourhood of cl$U$, is every periodic point in ∂U accessible from U ?

Comment. Accessible means being $\varphi(1)$ for a continuous curve $\varphi : [0,1] \to \mathrm{cl}U$ where $\varphi([0,1)) \subset U$ what is equivalent to being in the radial limit (i.e. $\lim_{r\nearrow 1} R(r\zeta)$ for $\zeta \in \partial\mathbb{D}$, R denoting a univalent map from $\mathbb{D}$ onto U). For g denoting the holomorphic extention of $R^{-1} \circ f \circ R$ to a neighbourhood of cl$\mathbb{D}$ and $\bar{R}$ the radial limit of R wherever it exists, it is known that at every g-periodic $\zeta \in \partial\mathbb{D}$, $\bar{R}$ exists and f at $\bar{R}(\zeta)$ is f-periodic (equivalently we could speak about σ-periodic points in Σ^d and the mapping z_∞, for a tree in U). Are there other periodic points in ∂U ? It seems it does not matter if one assumes here that f is defined only on a neighbourhood of ∂U. This is the case of an RB-domain U (the boundary is repelling on the U side) considered in [PUZ]. Problem 1.1 has a positive answer in the case where f is a polynomial on $\mathbb{C}$ and U is the basin of attraction to ∞, (Douady, Yoccoz, Eremenko, Levin), even if U is not simply-connected, see [EL]. Here the fact $f^{-1}(U) \subset U$ helps.

Problem 1.2. In the situation of Problem 1.1 is every point $x \in \partial U$ of positive Lyapunov exponent (i.e. such that $\liminf_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| > 0$) accessible from U ?

Problem 1.3. In the situation of Problem 1.1 is it true that the topological entropy $\mathrm{h_{top}}(f|_{\partial U}) = \log \deg(f|_U)$?

Comment The $\geq$ inequality is known and easy. The problem is with the opposite one. It would be true if every point $x \in \partial U$ had at most $\deg(f|_U)$ pre-images in ∂U.

A positive answer to problem 1.2 would give a positive answer to 1.3. The reason is that topological entropy is approximated by measure-theoretic entropies for f-invariant measures which having positive entropies would have positive Lyapunov exponents (Ruelle's inequality). Then they would be images under $\bar{R}$ of g-invariant measures on $\partial\mathbb{D}$ which all have entropies upper bounded by $\log d$ (as g is a degree d expanding map on $\partial\mathbb{D}$).

Problem 1.4. Can there be periodic points or points with positive Lyapunov exponents in the boundary of a Siegel disc S ? Is it always true that $\mathrm{h_{top}}(f|_{\partial S}) = 0$?

THE GENERAL CASE

We suppose here only that f extends holomorphically to a neighbourhood of the closure of the limit set Λ of a tree , $\Lambda = z_\infty \mathcal{D}(z_\infty)$. Then Λ is called a quasi-repeller, see [PUZ]. Denote the space of all probability f-invariant ergodic measures on the closure of a quasi-repeller Λ by $M(\Lambda)$. The space of measures in $M(\Lambda)$ which have positive entropy will be denoted by $M^+(\Lambda)$.

Problem 2.1. Is it true that every $m \in M(\Lambda)$ is the image of a measure on the shift space Σ^d through a geometric coding tree with z in a neighbourhood of $\mathrm{cl}\Lambda$. What about measures in $M^+(\Lambda)$? The same questions for f a rational mapping of degree d on $U = \hat{\mathbb{C}}$ and measures on the Julia set $J(f)$.

Comment. It is easy to see at least, due to the topological exactness of f on the Julia set $J(f)$ (for every open V in $J(f)$ there exists $n > 0$ so that $f^n(V) = J(f)$), that for every z except at most two, $z_\infty(\mathcal{D}(z_\infty)$ is dense in $J(f)$. The answer is of course positive in the case f is expanding on Λ because then z_∞ is well defined and continuous on Σ^d, hence Λ is closed.

Problem 2.2 For which $m \in M^+(\Lambda)$ for every "reasonable" function $\varphi : \Lambda \to \mathbb{R} \cup \pm\infty$ (for example Hölder, into $\mathbb{R}$ or allowing isolated values $-\infty$ with $\exp\varphi$ nonflat there, as $\log|g|$, g holomorphic) do the probability laws like Almost Sure Invariance Principle, Law of Iterated Logarithm, or Central Limit Theorem hold for the sequence of sums $S_n(\varphi) = \sum_{j=0}^{n-1} t_j$ of the random variables $t_j := \varphi \circ f^j - \int \varphi dm$ provided $\sigma^2(\varphi) = \lim \frac{1}{n} \int S_n(f)^2 dm > 0$?

Comment. If the measure is a z_∞-image of a measure on Σ^d with a Hölder continuous Jacobian (a Gibbs measure for a Hölder continuous function) then the probability laws hold, see [PUZ]. The positive answer in Problem 2.1 would be very helpful in solving Problem 2.2.

The class of measures for which Problem 2.2 has not been solved, but does not seem out of reach, are equilibrium states for Hölder continuous functions, say on the Julia set in the case f is rational. In this case the transfer (Ruelle-Perron-Frobenius) operator is already understood to some extent [DU], [P]. A proof seems to depend on finding an appropriate space of functions on which the maximal eigenvalue has modulus strictly larger than supremum over the rest of the spectrum (by the analogy to the expanding case, [Bowen]).

Actually these equilibrium states are z_∞-images of measures on Σ^d. The Jacobians of these equilibrium states have modulus of continuity bounded by $\mathrm{Const}(m)(\log(1/t))^{-m}$ for any $m > 0$ (I don't know if it is Hölder). The Jacobian of the pull-back of the equilibrium measure to Σ^d is not wild. This gives a chance to prove that mixing in Σ^d is polynomially fast.

Problem 2.3 Is it true for every $m \in M^+(\Lambda)$ that m is absolutely continuous with respect to H_κ (where H_κ is the Hausdorff measure in dimension $\kappa = \mathrm{HD}(m)$) iff $HD(m) = HD(\mathrm{cl}\Lambda)$?

Comment. In such a generality I would expect a negative answer. One should probably restrict the family of measures under consideration and/or impose additional assumptions on the mapping f.

If f is expanding on Λ then the answer is positive for all measures in $M^+(\Lambda)$ with Hölder continuous Jacobian. This is basically Bowen's theorem.

In the discussion here we assume that on every set E on which f is 1-to-1 the measure $(f|_E)^{-1}(m)$ is equivalent to m, and we write $\mathrm{Jac}_m f(z) = \frac{d(f|_E)^{-1}(m)}{dm}(z)$.

When the Jacobian exists in this sense we can replace the absolute continuity hypothesis $m \ll H_\kappa$ or the alternative singularity hypotesis $m \perp H_\kappa$ with another pair of alternative hypotheses.

Problem 2.4. In what class of measures in $M^+(\Lambda)$ does the property: the family $S_n(\log \mathrm{Jac}_m(f) - \kappa \log |f'|)$ is not uniformly bounded in $L^2(m)$, imply $m \perp H_\kappa$ and $HD(m) < HD(\mathrm{cl}\Lambda)$.

Comment. The answer is positive for f expanding and Jacobian Hölder continuous.

It is positive also if $m = z_\infty(\mu)$ for any Gibbs measure μ for a Hölder continuous function on Σ^d. The singularity $\perp$ follows then from the positive answer to Problem 2.2 in this special case, see [PUZ]. From the probability laws one can deduce a stronger singularity, for example with respect to the measure $H_\Phi(\kappa, c)$ which is the Hausdorff measure for the function

$$\Phi(\kappa, c)(t) = t^\kappa \exp c \sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}}$$

for all

$$c < \sqrt{2\sigma^2(\log \mathrm{Jac}_\mu(s) - \kappa \log |f'| \circ z_\infty) / \int \log |f'| dm}.$$

The inequality $HD(m) < HD(\mathrm{cl}\Lambda)$ follows from [Z1].

Problem 2.5. In what class of measures in $M^+(\Lambda)$ s does the property: the family $S_n(\log \mathrm{Jac}_m(f) - \kappa \log |f'|)$ is uniformly bounded in $L^2(m)$, imply $m \ll H_\kappa$?

Comment. Again the answer is positive for f expanding and Jacobian Hölder continuous.

If $m = z_\infty(\mu)$ then the boundness of the family $S_n(\varphi)$ where $\varphi := \log \mathrm{Jac}_\mu(f) - \kappa \log |f'| \circ z_\infty$ occurs precisely when $\sigma^2(\log \mathrm{Jac}_\mu(f) - \kappa \log |f'| \circ z_\infty) = 0$ assuming the series $\sum_{n=1}^\infty n \int |\varphi \cdot (\varphi \circ s^n)| d\mu$ is convergent. This is equivalent to the existence of a function u in $L^2(\mu)$ so that $\varphi = u \circ s - u$. Then we say that we can solve the cohomology equation for φ. Then we can also solve the cohomology equation for $\log \mathrm{Jac}_m(f) - \kappa \log |f'|$ on Λ. The naive way to compare m with H_κ is to prove that the sequence $S_n(\log \mathrm{Jac}_m(f) - \kappa \log |f'|)(z)$ is bounded at almost every $z \in \Lambda$. In the expanding case this allows comparison of the m-measure and the radius to the κ power of little discs, so the naive method happens to be successful. In the general case we do not have even pointwise boundness, because the function u is only in $L^2(\mu)$.

The problem has the positive answer in the following special cases:

1. In the RB-domain case, where m is equivalent to a harmonic measure on the boundary of a simply-connected domain U, see [PUZ] and [Z2]. Then $m = \bar{R}(\mu)$ where μ

is equivalent to the Lebesgue measure on $\partial \mathbb{D}$. $\log|R'|$ happens to be within a bounded distance from any harmonic extension of u to a neighbourhood of $\partial \mathbb{D}$, in particular radial limits for $\log|R'|$ exist a.e.. In [Z2] it is proved in fact that all this implies that ∂U is analytic, giving the answer to Problem 2.3 in this case.

2. In the case where f is a rational map on $\hat{\mathbb{C}}$ and m is a measure with maximal entropy (in which case Jacobian$\equiv$ deg$f$). Then again a careful look at u proves that f is either $z \mapsto z^n$ or is a Tchebysheff polynomial (in respective holomorphic coordinates on $\hat{\mathbb{C}}$) or else $J(f) = \hat{\mathbb{C}}$ and f has a parabolic orbifold, see [Z1].

In the general case it seems hopeful to treat any harmonic extension of u as a logarithm of a derivative of a "Riemann mapping". In the case $m = z_\infty(\mu)$ one can average u over cylinders in Σ^d extending u to the vertices $z_n(\alpha)$ of the tree.

The mapping z_∞ can be vieved as a dynamical version of a Riemann maping. We can formulate the following problem:

Problem 2.6. Which theorems about the boundary behaviour of Riemann maps hold for geometric coding trees?

Comment. Beurling Theorems hold, see the discussion in Section 1.

One has a natural dictionary:

For R:	For z_∞:
prime end	a geometric branch
impression	$I(\alpha) = \cap_{n=0}^{\infty} z_\infty\{\beta : \beta_i = \alpha_i, i = 0, ..., n\}$
the set of principal points	the limit set for the vertices $z_n(\alpha)$ of $b(\alpha)$.

Problem 2.7. Is it true that $\sup_{m\in M^+(\Lambda)} \mathrm{HD}(m) = \mathrm{HD}(\mathrm{cl}\Lambda)$? Does $\sup_{m\in M(\Lambda)}$ help?

Comment. Of course a negative answer to this Problem for some Λ and positive to Problem 2.3 would mean that $m \perp H_{\mathrm{HD}(m)}$ for all m.

Problem 2.7 has positive answer in the expanding and subexpanding cases where sup is attained, it is so even for a positive measure set of rational mappings on $\hat{\mathbb{C}}$ for which absolutely continuous invariant measures exist (with respect to the Lebesgue), see [R]. The problem has also a positive answer for rational mappings with neutral points but without critical points in the Julia set. But then it may happen that supremum is not attained, see [ADU] and [L].

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