# Problems on local connectivity.[1]

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If the Julia set $J(f)$ of a quadratic polynomial is connected, then Yoccoz has proved[2] that $J(f)$ is locally connected, unless either:

(1) $f$ has an irrationally indifferent periodic point, or

(2) $f$ is infinitely renormalizable.

**Cremer Points.** To illustrate case (1), consider the polynomial

$$P_\alpha(z) \;=\; z^2 + e^{2\pi i\alpha} z$$

with a fixed point of multiplier $\lambda = e^{2\pi i\alpha}$ at the origin. Take $\alpha$ to be real and irrational. For generic choice of $\alpha$ (in the sense of Baire category), Cremer showed that there is no local linearizing coordinate near the origin. We will say briefly that the origin is a *Cremer point*, or that $P_\alpha$ is a *Cremer polynomial*. According to Sullivan and Douady, the existence of such a Cremer point implies that the Julia set is not locally connected. More explicitly, let $t(\alpha)$ be the angle of the unique external ray which lands at the corresponding point of the Mandelbrot set. For generic choice of $\alpha$, Douady has shown that the corresponding ray in the dynamic plane does not land, but rather has an entire continuum of limit points in the Julia set. (Compare [Sø].) Furthermore, the $t(\alpha)/2$ ray in the dynamic plane accumulates both at the fixed point 0 and its pre-image $-\lambda$.

**Problem 1.** Is there an arc joining 0 to $-\lambda$, in the Julia set of such a Cremer polynomial?

**Problem 2.** Give a plausible topological model for the Julia set of a Cremer polynomial.

**Problem 3.** Make a good computer picture of the Julia set of some Cremer polynomial.

**Problem 4.** Can there be any external rays landing at a Cremer point?

**Problem 5.** Can the critical point of a Cremer polynomial be accessible from $\mathbf{C}\smallsetminus J$?

**Problem 6.** If we remove the fixed point from the Julia set of a Cremer polynomial, how many connected components are there in the resulting set $J(P_\alpha)\smallsetminus\{0\}$, ie., is the number of components countably infinite?

**Problem 7.** The Julia set for a generic Cremer polynomial has Hausdorff dimension two. Is this true for an arbitrary Cremer polynomial? Do Cremer Julia sets have measure zero? (Compare [Sh], [L1], [L2].)

In the quadratic polynomial case, Yoccoz has shown that every neighborhood of a Cremer point contains infinitely many periodic orbits. On the other hand, Perez-Marco [P-M1] has described non-linearizable local holomorphic maps for which this is not true.

**Problem 8.** For a Cremer point of an arbitrary rational map, does every neighborhood contain infinitely many periodic orbits?

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[1] Based on questions by a number of participants in the 1989 Stony Brook Conference.

[2] Compare [Hu].

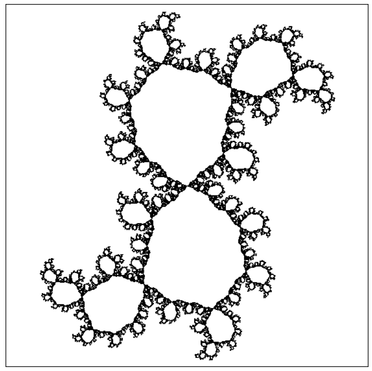

*Figure 1. Julia set of $P_\alpha$ where $\alpha = .78705954039469$ has been randomly chosen.*

**Siegel Disks.** (Compare Carleson's discussion.) If $\alpha$ satisfies a Diophantine condition (in particular, for Lebesgue almost every $\alpha$), Siegel showed that there is a local linearizing coordinate for the polynomial $P_\alpha(z) = z^2 + e^{2\pi i\alpha}z$ in some neighborhood of the origin. Briefly we say that the origin is the center of a *Siegel disk* $\Delta$, or that $P_\alpha$ is a *Siegel polynomial*. Yoccoz has given a precise characterization of which irrational angles yield Siegel polynomials and which yield Cremer polynomials. ([Y], [P-M2].)

Herman, making use of ideas of Ghys, showed that there exists a value $\alpha_0$ so that $P_{\alpha_0}$ has a Siegel disk whose boundary $\partial\Delta$ does not contain the critical point. It follows that the Julia set $J(P_{\alpha_0})$ is not locally connected. On the other hand if $\alpha$ satisfies a Diophantine condition, then Herman showed that $\partial\Delta$ does contain the critical point.

**Problem 9.** Give any example of a Siegel polynomial whose Julia set is provably locally connected. Is $J(P_\alpha)$ locally connected for Lebesgue almost every choice of $\alpha$? (Compare Figure 1.) What can be said about the Hausdorff dimension of $J(P_\alpha)$?

**Problem 10.** Can a Siegel disk have a boundary which is not a Jordan curve?

**Problem 11.** Does any rational function have a Siegel disk with a periodic point in its boundary? Such an example would be extremely pathological. (In the polynomial case, Poirier has pointed out that at least there cannot be a Cremer point in the boundary of a Siegel disk. See [GM].)

**Infinitely Renormalizable Polynomials.** A quadratic polynomial $f_c(z) = z^2 + c$ is *renormalizable* if there exists an integer $p \geq 2$ and a neighborhood $U$ of the critical point zero so that the orbit of zero under $f^{\circ p}$ remains in this neighborhood forever, and so that the map $f^{\circ p}$ restricted to $U$ is polynomial-like of degree $2$. (Thus the closure $\bar{U}$ must contain no other critical points of $f^{\circ p}$, and must be contained in the interior of $f^{\circ p}(U)$.) Let $M$ be the Mandelbrot set, and let $H \subset M$ be any hyperbolic component of period $p \geq 2$. Douady and Hubbard [DH2] show that $H$ is contained in a small copy of $M$. This small copy is the image of a homeomorphic embedding of $M$ into itself, which I will denote by $c \mapsto H * c$. The elements of these various small copies $H * M \subset M$ (possibly with the root point $H * \frac{1}{4}$ removed) are precisely the renormalizable elements of $M$.

Now consider an infinite sequence of hyperbolic components $H_1, H_2, \ldots \subset M$. If the $H_i$ converge to the root point $1/4$ sufficiently rapidly, then Douady and Hubbard (unpublished) show that the intersection $\bigcap_k H_1 * \cdots * H_k * M$ consists of a single point $c_\infty$ such that the corresponding Julia set $J(f_{c_\infty})$ is not locally connected.

**Problem 12.** Suppose that $f_{c_\infty}$ is infinitely renormalizable of bounded type. For example, suppose that $c_\infty \in \bigcap_k H_1 * \cdots * H_k * M$, where the $H_i$ are all equal. Does it then follow that $J(f_{c_\infty})$ is locally connected? As the simplest special case, if we take $H_1 = H_2 = \cdots$ to be the period two component centered at $-1$, then $f_{c_\infty}$ will be the quadratic *Feigenbaum map*. Is the Julia set for the Feigenbaum map locally connected?

**Problem 13.** More generally, if $c$ is real (belonging to the intersection $M \cap \mathbf{R} = [-2\,,\ 1/4]$), does it follow that the Julia set $J(f_c)$ is locally connected?

**The Mandelbrot Set.** Here the most basic remaining question is the following.

**Problem 14.** Does every infinite intersection of the form $\bigcap_k H_1 * \cdots H_k * M$ reduce to a single point? Equivalently, is the set of infinitely renormalizable points totally disconnected? Does this set have measure zero? Does it in fact have small Hausdorff dimension?

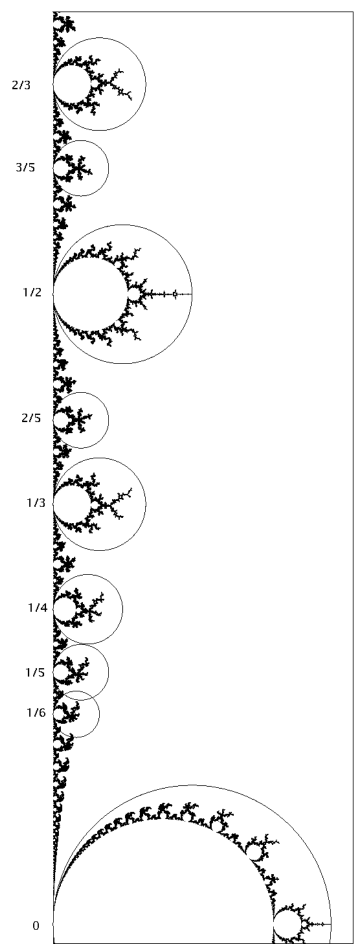


*Figure 2. Picture of the* $\log\lambda$*-plane, showing the Yoccoz disks of radius* $\log(2)/q$. (*Heights in units of* $2\pi$.)

**Problem 15.** For each rational number $0 < p/q < 1$ let $M(p/q)$ be the limb of the Mandelbrot set with interior angle $p/q$. Is the diameter of $M(p/q)$ less than $k/q^2$ for some constant $k$ independent of $p$ and $q$? If not, is it at least less than $k\log(q)/q^2$? (It is actually more natural to work in the $\log\lambda$ plane, where $f(z) = z^2 + \lambda z$. The Yoccoz inequality asserts that the corresponding limb in this $\log\lambda$ plane is contained in a disk of radius $\log(2)/q$. Compare [P], and see Figure 2.)

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