A Possible Approach to a Complex Renormalisation Problem

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Preliminary Definitions. For a branched covering $f : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$, we define

$$X(f) = \{f^n(c) : c \text{ critical}, \ n > 0\}.$$

Then f is **critically finite** if $\#(X(f))$ is finite. Two critically finite branched coverings f_0, f_1 are **(Thurston) equivalent** if there is a path f_t through critically finite branched coverings connecting them with $X(f_t)$ constant in t.

We are only concerned, here, with orientation-preserving degree two branched coverings for which one critical point is fixed and the other is periodic. By a theorem of Thurston's ([**T**], [**D-H**]), any such branched covering f_0 is equivalent to a unique degree two polynomial f_1 of the form $z \mapsto z^2 + c$ (some $c \in \mathbf{C}$).

Now let f_1, f_2 be two degree two polynomials of the form $z \mapsto z^2 + c_i$ ($i = 1$, 2), with 0 periodic of periods m, n respectively. Then we define **the tuning of** f_1 **about** 0 **by** f_2, written $f_1 \vdash f_2$, as follows. This is simply a branched covering defined up to equivalence. Let D be an open topological disc about 0 such that the discs $f_1^i(D)$ ($0 \le i < m$) are all disjoint, $f_1^m(D) \subset D$ and $f_1 : f_1^i(D) \to f_1^{i+1}(D)$ is a homeomorphism for $1 \le i < m$. Let g be a rescaling of f_2, and V a closed bounded topological disc with $V \subset gV \subset f_1^m(D)$ whose complement is in the attracting basin of ∞ for g. Then we define

$$f_1 \vdash f_2 \begin{array}{l} = f_1 \text{ outside } D, \\ = f_1^{-(m-1)} \circ g \text{ in } V, \end{array}$$

and extend to map the annulus $D \setminus V$ by a two-fold covering to $f_1^m(D) \setminus g(V)$. Then

$$(f_1 \vdash f_2)^m = g \text{ in } V.$$

Thus $f_1 \vdash f_2$ is critically finite with 0 of period $n \cdot m$, and is equivalent to a unique polynomial $z \mapsto z^2 + c$.

For any sequence $\{f_i\}$ of polynomials, we can also define $f_1 \vdash \cdots \vdash f_n$ for all n.

For concreteness, we consider the following renormalisation problem, but different versions are possible.

Let $\{f_i\}$ be any sequence of polynomials of the form $z \mapsto z^2 + c_i$, where the f_i (and c_i) take only finitely many different values, and 0 is of period m_i under f_i. Write g_n for the polynomial $z \mapsto z^2 + c$ equivalent to $f_1 \vdash \cdots \vdash f_n$.
and

$$n_k = \prod_{i \le k} m_i.$$

Problem. Prove geometric properties of $X(g_n)$. Specifically, show that the set

$$\{g_n^{n_k \ell + i}(0) : 0 \le \ell < m_{k+1}\} \tag{1}$$

has uniformly bounded geometry for all $i \le n_k$, $k < n$ and all n.

Of course, this problem (and stronger versions) is not new, has been the focus of much effort, and, in the real case, has been resolved by Sullivan [**S**]. The most obvious method of approach (which was not, in the end, efficacious in the real case) is through analysis of the main technique used to prove Thurston's theorem mentioned above. We now recall this.

Thurston's Pullback Map on Teichmüller space.

To simplify, we stick to orientation-preserving degree two critically finite branched coverings with fixed critical value v_2 and periodic critical value v_1. Let g be one such. Let $X = X(g)$. We let $s : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be given by $s(z) = z^2$. Let $\mathcal{T} = \mathcal{T}(X)$ be the Teichmüller space of the sphere with set of marked points X, so that

$$\mathcal{T} = \{[\varphi] : \varphi \text{ is a homeomorphism of } \overline{\mathbf{C}}\}$$

and $[\varphi]$ denotes the quotient of the isotopy class under isotopies constant on X by left Möbius composition, that is, $[\varphi] = [\sigma \circ \varphi \circ \psi]$ for any Möbius transformation σ and ψ isotopic to the identity rel X. It is convenient to choose representatives φ so that $\varphi(v_1) = 0$, $\varphi(v_2) = \infty$. Then

$$\tau : \mathcal{T} \to \mathcal{T}$$

is defined by

$$\tau([\varphi]) = [s^{-1} \circ \varphi \circ g].$$

(The righthand side makes perfectly good sense as a homeomorphism.)

By Thurston's theorem (in this setting), τ is a contraction with respect to the Teichmüller metric d on $\mathcal{T}$, and has a unique fixed point $[\varphi]$. Then there are ψ isotopic to the identity via an isotopy fixing $X(g)$ (unique given φ) and a Möbius transformation σ such that

$$g = \varphi^{-1} \circ s \circ \sigma \circ \varphi \circ \psi.$$

In particular, g and $s \circ \sigma$ are equivalent, and $X(s \circ \sigma) = \varphi(X(g))$.

The "Obvious" Method of Approach.

We can choose h_n equivalent to g_n so that the sets of (1), with h_n replacing g_n, have uniformly bounded geometry for $i \leq n_k$, $k < n$, and all n. Then let $\mathcal{T}_n = \mathcal{T}(X(h_n))$, and $\tau_n : \mathcal{T}_n \to \mathcal{T}_n$ be the associated pullback. It suffices (!) to prove convergence, as $m \to \infty$, and uniform in n, of the sequences $\{\tau_n^m(\text{identity})\}$. Of course, for fixed n, the convergence would be with respect to the Teichmüller metric d_n on $\mathcal{T}_n$. This seems to be impossible to implement. An alternative is suggested below. One virtue - and probably the only one - of this alternative is that it has not yet been tried (so far as I know). Before making this precise, we need to clarify some properties of the Teichmüller metric.

The Teichmüller metric and its Derivative.

Let $\mathcal{T} = \mathcal{T}(X)$ (for any finite set $X \subset \overline{\mathbf{C}}$) and let d denote the Teichmüller metric. Let $[\varphi]$, $[\psi] \in \mathcal{T}$. Assume without loss of generality that $\infty \in \varphi(X)$, $\psi(X)$. Then there is a unique quasiconformal homeomorphism $\chi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ with the following properties.

1. $\chi(\varphi(X)) = \psi(X)$ and $[\chi \circ \varphi] = [\psi]$.

2. There is a rational function q with at most simple poles in $\mathbf{C}$, all occurring at points of $\varphi(X)$, and at least three more poles than zeros in $\mathbf{C}$, such that the directions of maximal stretch and contraction of χ are tangent to the vector fields

$i\sqrt{\overline{q}}$, $\sqrt{\overline{q}}$ respectively, and the dilatation (ratio of infinitesimal stretch to contraction) is constant.

3. The images under χ_* of these vector fields are of the form $i\sqrt{\overline{p}}$, $\sqrt{\overline{p}}$, for a rational function p with at most simple poles in $\mathbf{C}$, all occurring at points of $\psi(X)$.

The function q is then also unique, up to a positive scalar multiple, and becomes unique if we normalise so that

$$\int \mid q \mid \frac{d\bar{z} \wedge dz}{2i} = 1.$$

Similarly, we normalise p. (Of course, q represents a quadratic differential $q(z)dz^2$, but it is convenient to keep the representing rational function in the foreground.)

Let $h = (h(x)) \in \mathbf{C}^X$ be small, taking $h(x) = 0$ if $\varphi(x) = \infty$. Then by abuse of notation, we write $\varphi + h$ for a homeomorphism near φ with $(\varphi + h)(x) = \varphi(x) + h(x)$. Then the following holds, where q, p are detemined by $[\varphi]$, $[\psi]$ as above **[R]**.

$$d([\varphi + h], [\psi + k]) = d([\varphi], [\psi]) + 2\pi \,\mathrm{Re}\left(\sum_{x \in X} (\mathrm{Res}\,(q, \varphi(x))h(x) - \mathrm{Res}\,(p, \psi(x))k(x))\right)$$
$$+ o(h) + o(k).$$

Now we consider the case $X = X(g)$ and $y = \tau x$. As before we consider only specific g and take $s(z) = z^2$ (as before). The **pushforward** $s_* q$ of a rational function (or quadratic differential) q is defined by

$$s_* q(z) = \sum_{s(w) = z} \frac{q(w)}{s'(w)^2}$$

if $s'(w) \neq 0$ for $s(w) = z$. If q has only simple poles in $\mathbf{C}$, and at least 3 more poles than zeros in $\mathbf{C}$, then $s_* q$ extends to a rational function on $\mathbf{C}$ with the same properties. Then if q, p are determined by $[\varphi]$, $\tau([\varphi])$, taking $\varphi(v_1) = 0$, $\varphi(v_2) = \infty$ as above,

$$d([\varphi + h], \tau([\varphi + h]) = d([\varphi], \tau([\varphi])) + 2\pi \,\mathrm{Re}\left(\sum_{x \in X} (\mathrm{Res}\,(q, \varphi(x)) - \mathrm{Res}\,(s_* p, \varphi(x)))h(x)\right)$$
$$+ o(h).$$

The Suggested Alternative Approach to the Problem. Take $\mathcal{T} = \mathcal{T}(X(g))$, $\tau : \mathcal{T} \to \mathcal{T}$. Let

$$F([\varphi]) = d([\varphi], \tau([\varphi])).$$

Then the derivative formula for F above theoretically enables us to construct flows for which F decreases along orbits. It can be shown that the only critical point of F occurs where $F = 0$. So if we can find a compact subset B of $\mathcal{T}$ with smooth boundary and a vector field v pointing inward on ∂B with $DF(v) < 0$, then the (unique) zero of F must be inside B. Now put a subscript n on everything. Conceivably we can find $B_n \subset \mathcal{T}_n$ and vector field v_n pointing inward on ∂B_n such that if $A \subset X(g_n)$ is any of the sets in (1) and $[\varphi] \in \partial B_n$ then $\varphi(A)$ has bounded geometry (uniformly in A, n) and $DF_n(v_n) < 0$?

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