Thurston's algorithm without critical finiteness

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Thurston's algorithm is a powerful method for passing from a topological branched covering $S^2 \to S^2$ to a rational map having closely related dynamical properties. (See [DH].) The same method can be used to pass from a piecewise monotone map of the interval to a closely related polynomial map of the interval.

Suppose that we start with an orientation preserving branched covering map $f_0 : S^2 \to S^2$. We identify S^2 with the Riemann sphere $\bar{\mathbf{C}} = \mathbf{C} \cup \infty$. In order to anchor this sphere, choose three base points. (For best results, choose dynamically significant base points, for example periodic points of f_0, or critical points, or critical values.)

> **Lemma:** *There is one and only one homeomorphism $h_0 : S^2 \to S^2$ which fixes the three base points, and which has the property that the composition $r_0 = f_0 \circ h_0$ is holomorphic, or in other words is a rational map.*

[Proof: Let σ_0 be the standard conformal structure on the 2-sphere, and let $\sigma = f_0^*(\sigma_0)$ be the pulled back conformal structure, so that f_0 maps (S^2, σ) holomorphically onto (S^2, σ_0). Then h_0 must be the unique conformal isomorphism from (S^2, σ_0) onto (S^2, σ) which fixes the three base points.] Now consider the map $f_1 = h_0^{-1} \circ f_0 \circ h_0$, which is topologically conjugate to f_0. In this way, we obtain a commutative diagram

$$\begin{array}{ccccc} S^2 & & \xrightarrow{f_1} & & S^2 \\ h_0 \downarrow & & r_0 \searrow & & h_0 \downarrow \\ S^2 & & \xrightarrow{f_0} & & S^2 \,. \end{array}$$

Continuing inductively, we produce a sequence of branched coverings f_n, and a sequence of homeomorphisms h_n fixing the base points, so that $f_{n+1} = h_n^{-1} \circ f_n \circ h_n$, and so that each composition $r_n = f_n \circ h_n$ is a rational map. The marvelous property of this construction is that in many cases the homeomorphisms h_n seem to tend uniformly to the identity, so that the successive maps f_n, which are all topologically conjugate to f_0, come closer and closer to the rational maps r_n. In fact the sequence of compositions $\phi_n = (h_0 \circ \cdots \circ h_n)^{-1}$ may converge uniformly to a limit map ϕ, at least on the non-wandering set. In this case, it follows that the rational limit map is topologically semi-conjugate (or perhaps even conjugate) to f_0,

$$r_\infty \circ \phi \;=\; \phi \circ f_0$$

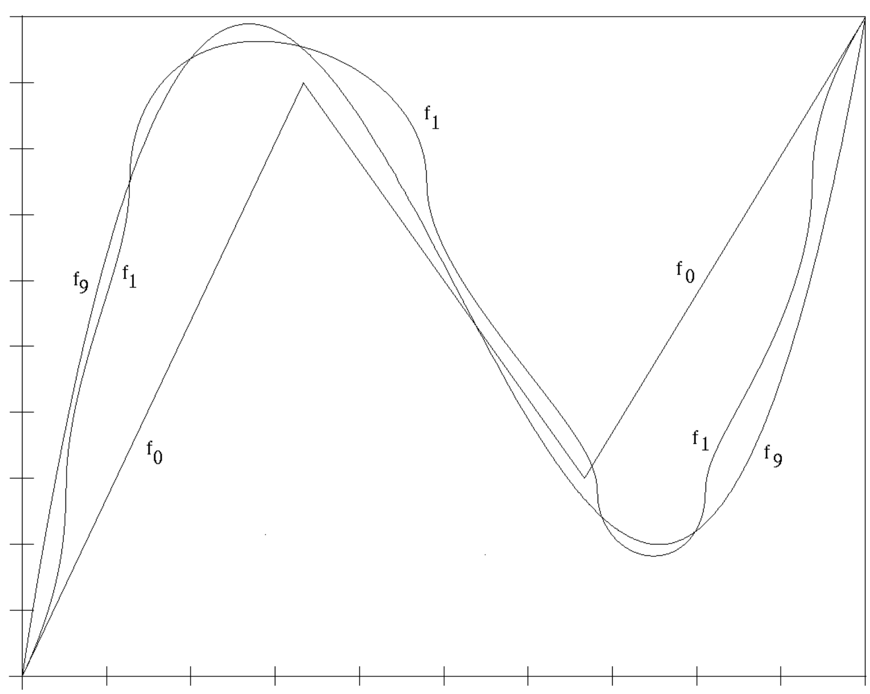
on the non-wandering set.

> **Problem:** *Under what conditions will this sequence of rational maps r_n converge uniformly to a limit map r_∞? Under what conditions, and on what subset of S^2, will the maps ϕ_n converge uniformly to a limit?*

In the post-critically finite case, Thurston defines an obstruction, which vanishes if and only if the restriction of the ϕ_n to the post-critical set converges uniformly to a one-to-one limit function. If this obstruction vanishes, then it follows that the r_n converge.

However, there would be interesting applications where f_0 is not post-critically finite, so that no such criterion is known. A typical example is provided by the problem of "mating". (Compare Bielefeld's discussion, as well as [Ta], [Sh].) Let p and q be monic polynomial maps having the same degree $d \geq 2$. Conjugating p by the diffeomorphism $z \mapsto z/\sqrt{1+|z|^2}$ from $\mathbf{C}$ onto the unit disk D, we obtain a map p^* which extends smoothly over the closed disk $\bar{D}$. Similarly, conjugating q by $z \mapsto \sqrt{1+|z|^2}/z$ we obtain a map q^* which extends smoothly over the complementary disk $\bar{\mathbf{C}} \smallsetminus D$. Now p^* and q^* together yield a C^1-smooth map $f_0 : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$, and we can apply Thurston's method as described above. If this procedure converges to a well behaved limit, then the resulting rational map r_∞ of degree d may be called the "mating" of p and q.

Maps of the interval. The situation here is quite similar. Let f_0 be a piecewise-monotone map of the interval $I = [0,1]$ with d alternately ascending and descending laps, and suppose that f_0 carries the boundary points 0 and 1 to boundary points. Then there is one and only one orientation preserving homeomorphism h_0 of the interval such that the composition $p_0 = f_0 \circ h_0$ is a polynomial map of degree d. Setting $f_1 = h_0^{-1} \circ f_0 \circ h_0$, we can proceed inductively, constructing homeomorphisms h_n, polynomials $p_n = f_n \circ h_n$, and topologically conjugate maps $f_{n+1} = h_n^{-1} \circ f_n \circ h_n$. Again the problem is to decide when and where this procedure converges.



A typical run of Thurston's method, starting with a piece-wise linear map f_0 of the interval. (Horizontal scale exaggerated.) The graphs of f_0, f_1 and f_9 are shown. The latter seems indistinguishable from $f_\infty = p_\infty$.

References.

[DH] A. Douady and J. H. Hubbard, A proof of Thurston's topological characterization of rational functions, preprint, Mittag-Leffler 1984.

[Sh] M. Shishikura, On a theorem of M. Rees for the matings of polynomials, preprint, IHES 1990.

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