Dynamics of certain non-conformal semigroups

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Abstract

A semigroup generated by two dimensional $C^{1+\alpha}$ contracting maps is considered. We call a such semigroup regular if the maximum K of the conformal dilatations of generators, the maximum l of the norms of the derivatives of generators and the smoothness α of the generators satisfy a compatibility condition $K < 1/l^{\alpha}$. We prove that the shape of the image of the core of a ball under any element of a regular semigroup is good (bounded geometric distortion like the Koebe 1/4-lemma [1]). And we use it to show a lower and a upper bounds of the Hausdorff dimension of the limit set of a regular semigroup. We also consider a semigroup generated by higher dimensional maps.

Contents

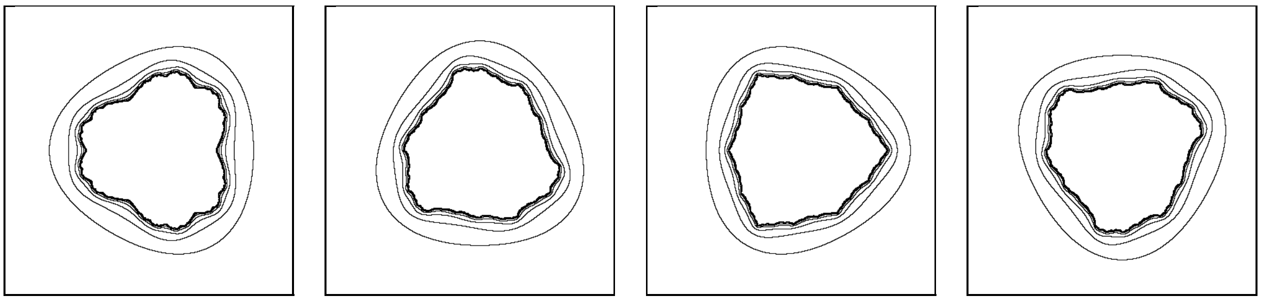
§0 Introduction.

It is a well-known result [11, 13] that the Hausdorff dimension of the Julia set of a complex quadratic polynomial $p(z) = z^2 + c$ is greater than one for a complex number c with small $|c| \neq 0$ (see [3] for a similar result in quasifuchsian groups). Now consider a non-conformal complex map $f(z) = z^2 + b\overline{z} + c$ where b and c are complex parameters (or $f(z) = z^n|z|^{(\gamma-n)} + c$ where $\gamma > 0$ is a real parameter, c is a complex parameter and $n > 0$ is a fixed integer). Let $\lambda = (b, c)$ (or $\lambda = (\gamma - n, c)$ and $|\lambda| = |b| + |c|$ (or $|\lambda| = |\gamma - n| + |c|$). The map $f_0(z) = z^2$ (or $f_0(z) = z^n$) is analytic and expanding on a neighborhood U of $S^1 = \{z \in \mathbf{C}; |z| = 1\}$ which is the maximal invariant set of f_0 in U. By the structural stability theorem [12], for $|\lambda|$ small, there is a set J_λ such that it is the maximal invariant set of f and $f|J_\lambda$ is conjugate to $f_0|S^1$, that is, there is a homeomorphism h from a neighborhood of S^1 onto a neighborhood of J_λ such that $f \circ h = h \circ f_0$. Thus the set J_λ is a Jordan curve. It is easy to see that J_λ is the boundary of the basin $B_\infty = \{z \in \mathbf{C}; |f^{\circ k}(z)| \mapsto \infty \text{ as } k \mapsto +\infty\}$ for $|\lambda|$ small (see Fig. 1 and Fig. 2). We may call J_λ the Julia set of f (ref. [10]).

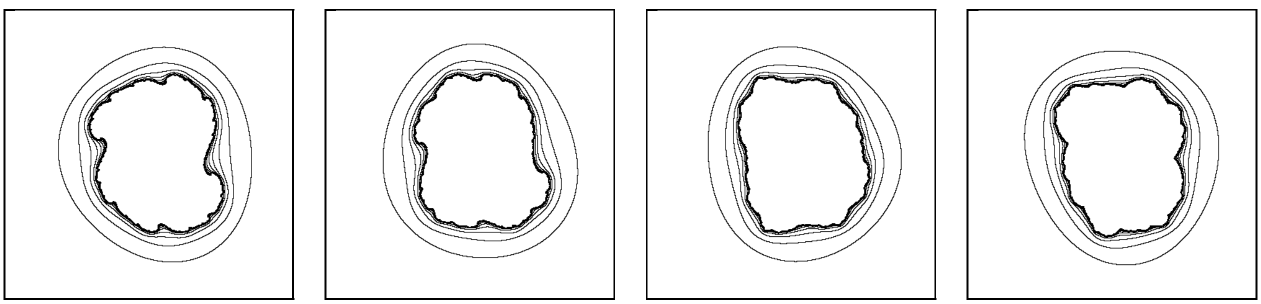
Question 1. *Is the Hausdorff dimension of the Julia set J_λ of $f(z)$ greater than 1 for some small $|b| \neq 0$ and small $|c|$ (or small $|c| \neq 0$ and small $|\gamma - n| \neq 0$) ?*

We will prove some general results (Theorem 1 and Theorem 2) in §1, §2 and §3, which can be used to give the answer (Corollary 3) to this question. We note that the general results themselves are interesting and have other applications [9].

Acknowledgement. I would like to thank Professor Dennis Sullivan for very useful discussions and remarks. The conjecture in Remark 3 is formulated when the author visited the Mathematics Institute at University of Warwick. I would like to thank Professor David Rand for useful conversations.

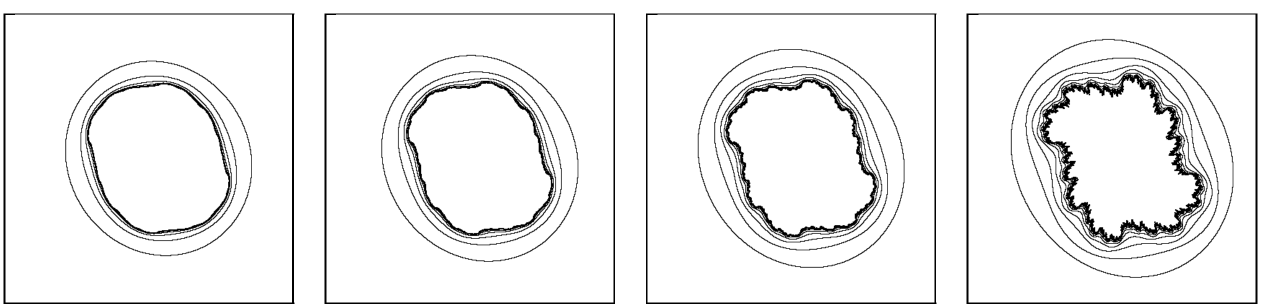


$\lambda = (0.2, 0)$ $\lambda = (0.2i, 0)$ $\lambda = (-0.2, 0)$ $\lambda = (-0.2i, 0)$



$\lambda = (0.1, 0.2 + 0.1i)$ $\lambda = (0.1i, 0.2 + 0.1i)$ $\lambda = (-0.1, 0.2{+}0.1i)$ $\lambda = (-0.1i, 0.2{+}0.1i)$

Fig. 1: Preimages of a circle with large radius under iterates of $f(z) = z^2 + b\overline{z} + c$ and $\lambda = (b, c)$.



$\lambda = (0.5, 0.1 + 0.2i)$ $\lambda = (0.1, 0.1 + 0.2i)$ $\lambda = (-0.2, 0.1{+}0.2i)$ $\lambda = (-0.5, 0.1{+}0.2i)$

Fig. 2: Preimages of a circle with large radius under iterates of $f(z) = z^2|z|^{\gamma-2} + c$ and $\lambda = (\gamma - 2, c)$.

§1 Statements of main results.

Suppose V and U are two bounded and open sets of the complex plane $\mathbf{C}$ with $\overline{V} \subset U$ and f is a C^1-map from U into $\mathbf{C}$. The restriction $f|\overline{V}$ is said to be $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ if

$$f(w) = f(z) + \Big(D(f)(z)\Big)(w - z) + R(w, z)$$

satisfies $|R(w, z)| \leq L_0|w - z|^{1+\alpha}$ for $z \in \overline{V}$ and $w \in U$ where $L_0 > 0$ is a constant and $D(f)(z)$ is the derivative of f at z. For a $C^{1+\alpha}$ diffeomorphism f from $\overline{V}$ onto $\overline{W}$, we use g to denote its inverse. The map g is said to be contracting if there is a constant $0 < \lambda < 1$ such that $\left|\Big(D(g)(z)\Big)(v)\right| \leq \lambda |v|$ for all z in $\overline{W}$ and all v in $\mathbf{C}$. Suppose V_i and U_i, $i = 0, 1, \ldots, n-1$, are pairs of bounded open sets of $\mathbf{C}$ with $\overline{V}_i \subset U_i$ and f_i are maps from U_i into $\mathbf{C}$ such that the restriction $f_i|\overline{V_i}$ from $\overline{V_i}$ onto $\overline{W_i}$ are $C^{1+\alpha}$ diffeomorphisms for some $0 < \alpha \leq 1$ and the inverses g_i of $f_i|\overline{V_i}$ are contracting. To simplify the notations, we assume that $W = W_i$ for all i and $\cup_{i=0}^{n-1} V_i \subset W$. We will use $\mathcal{G} = \langle g_0, g_1, \ldots, g_{n-1}\rangle$ to denote the semigroup generated by all g_i and use $\Lambda = \cap_{g\in\mathcal{G}}\Big(g(\overline{W})\Big)$ to denote the limit set of $\mathcal{G}$, which is compact, completely invariant (the existence of Λ can be proven by using Hausdorff distance on subsets).

Suppose $z = x + yi$ is a point in $\mathbf{C}$ and $\overline{z} = x - yi$ is the conjugate of z. By the complex analysis [1], we know that for $z \in \overline{W}$ and $w \in \mathbf{C}$ with $|w| = 1$,

$$\Big||(g_i)_z| - |(g_i)_{\overline{z}}|\Big| \leq \left|\Big(D(g_i)(z)\Big)(w)\right| \leq |(g_i)_z| + |(g_i)_{\overline{z}}|.$$

Let

$$l_i(z) = |(g_i)_z| + |(g_i)_{\overline{z}}|, \quad s_i(z) = \Big||(g_i)_z| - |(g_i)_{\overline{z}}|\Big|$$

and $K_i(z) = l_i(z)/s_i(z)$, the conformal dilatation of g_i at z. Let $l = \max\{l_i(z)\} < 1$, $s = \min\{s_i(z)\} > 0$ and $K = \max\{K_i(z)\} < +\infty$ where max and min are over all z in $\overline{W}$ and all $0 \leq i < n$.

Definition 1. *We say $\mathcal{G}$ is regular if $K < 1/l^{\alpha}$.*

Denote by $B(z,r)$ the closed disk of radius r centered at z. One of the main results, which generalizes the Koebe 1/4-lemma [4] in some sense, is the following:

Theorem 1 (geometric distortion). *Suppose* $\mathcal{G} = \langle g_0, g_1, \ldots, g_{n-1}\rangle$ *is regular. There are two functions* $\delta = \delta(\varepsilon) > 0$ *and* $C = C(\varepsilon) \geq 1$ *with* $\delta(\varepsilon) \mapsto 0$ *and* $C(\varepsilon) \mapsto 1$ *as* $\varepsilon \mapsto 0+$ *such that*

$$g\Big(B(z,r)\Big) \supset g(z) + C^{-1} \cdot \Big(D(g)(z)\Big)\Big(B(0,r)\Big) \quad \textit{and}$$

$$g\Big(B(z,r)\Big) \subset g(z) + C \cdot \Big(D(g)(z)\Big)\Big(B(0,r)\Big)$$

for any $0 < r \leq \delta(\varepsilon)$, *any* $g \in \mathcal{G}$ *and any* $z \in \overline{W}$ *(see Fig. 3).*

Let $\angle\Big(g(w) - g(z), \Big(D(g)(z)\Big)(w-z)\Big)$ be the smallest angle between the vectors $g(w) - g(z)$ and $\Big(D(g)(z)\Big)(w-z)$.

Corollary 1 (angle distortion). *Moreover, there is a function* $D(\varepsilon) > 0$ *with* $D(\varepsilon) \mapsto 0$ *as* $\varepsilon \mapsto 0+$ *such that*

$$\left|\log\Big(\angle\Big(g(w) - g(z), \Big(D(g)(z)\Big)(w-z)\Big)\right| \leq D(\varepsilon)$$

for $0 < r \leq \delta(\varepsilon)$, $g \in \mathcal{G}$, $z \in \overline{W}$ *and* $w \in B(z,r)$.

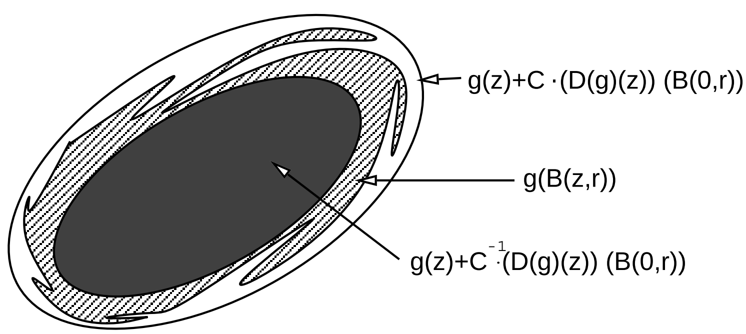


Fig. 3

A regular semigroup $\mathcal{G} = \langle g_0, \ldots, g_{n-1}\rangle$ is said to be Markov for a real number $\delta_0 > 0$ if there are simple connected, pairwise disjoint open sets $\Omega_0, \Omega_1, \ldots, \Omega_{q-1}$ such that

(a) $\max_{0\le l\le q-1} diam(\Omega_l) \le \delta_0$,

(b) $\cup_{l=0}^{q-1}\overline{\Omega_l} \supset \Lambda$, and

(c) $f_i(\overline{\Omega_l \cap \Lambda}) = \left(\cup_{t=1}^{k_l}\overline{\Omega_{i_t}}\right) \cap \Lambda$ for every $0 \le l < q$ and $\Omega_l \subset V_i$ where $f_i = g_i^{-1}$.

Without loss of generality, we may assume $q = n$ and $g_i = (f_i|\Omega_i)^{-1}$ if $\mathcal{G}$ is Markov.

Suppose $\mathcal{G} = \langle g_0, \ldots, g_{n-1}\rangle$ is a regular and Markov semigroup. Let $A = (a_{ij})$ be the $n \times n$ matrix of 0 and 1 such that $a_{ij} = 1$ if $f_i(\Omega_i \cap \Lambda) \supset \Omega_j \cap \Lambda$ and $a_{ij} = 0$ otherwise. A sequence $w_p = i_0 i_1 \cdots i_{p-1}$ of symbols $\{0, 1, \ldots, n-1\}$ is said to be admissible if $a_{i_l i_{l+1}} = 1$ for $l = 0, 1, \ldots, p-1$ (p may be ∞). Let Σ_p be the space of all admissible sequences w_p of length p, $\sigma(i_0 i_1 \cdots) = i_1 \cdots$ be the shift map on Σ_∞ and $\pi(i_0 i_1 \cdots) = \cap_{k=0}^{\infty} g_{i_k}(\overline{W})$ be the projection from Σ_∞ to Λ [2, 11] (note that π is the semi-conjugacy). We call the functions

$$\phi_{up}(w) = \log\left(l_i \circ \pi(w)\right) \quad \text{and} \quad \phi_{lo}(w) = \log\left(s_i \circ \pi(w)\right),$$

for $w = i i_1 \cdots \in \Sigma_\infty$, the upper and lower potential functions of $\mathcal{G}$. They are Hölder [2].

Let P be the pressure function (see, for example, [2, 11]) defined on C^H, the space of Hölder continuous functions on Σ_∞. Then [2]

$$P(\phi) = \lim_{p\mapsto\infty} \frac{1}{p} \log\Big(\sum_{w\in fix(\sigma^{\circ p})} \exp\Big(\sum_{k=0}^{p-1} \phi\big(\sigma^{\circ k}(w)\big)\Big)\Big).$$

For $\phi = \phi_{up}$ or ϕ_{lo}, $P(t\phi)$ is continuous, strictly monotone and convex function on the real line and tends to $-\infty$ and $+\infty$ as t goes to $+\infty$ and $-\infty$. There is a unique $t_{up} > 0$ ($t_{lo} > 0$) such that $P(t_{up}\phi_{up}) = 0$ ($P(t_{lo}\phi_{lo}) = 0$) [3, 11].

Theorem 2. *Suppose $\mathcal{G} = \langle g_0, \ldots, g_{n-1}\rangle$ is a regular and Markov semigroup and $HD(\Lambda)$ is the Hausdorff dimension of the limit set Λ of $\mathcal{G}$. Then $t_{lo} \le HD(\Lambda) \le t_{up}$.*

Suppose $\mathcal{G}_\lambda = \langle g_{0,\lambda}, \ldots, g_{n-1,\lambda}\rangle$ is a family of regular and Markov semigroups such that every $g_{i,\lambda}(z)$ is C^1 on both variables λ and z. Let $HD(\lambda)$ be the Hausdorff dimension of the limit set Λ_λ of $\mathcal{G}_\lambda$.

Corollary 2. *If all g_{i,λ_0} are conformal ($K_{\lambda_0} = 1$), then $HD(\lambda)$ is continuous at λ_0.*

Corollary 3. *Suppose $f(z) = z^2 + b\overline{z} + c$ (or $f(z)= z^n|z|^{(\gamma-n)} + c$) and $\lambda = (b, c)$ (or $\lambda = (\gamma - n, c)$). For each c with small $|c| \neq 0$, there is a $\tau(c) > 0$ such that for every $|b| \leq \tau(c)$ (or $|\gamma - n| \leq \tau(c)$), the Hausdorff dimension $HD(\lambda)$ of the Julia set J_λ of f is bigger than one (see Fig. 4 in §4).*

Remark 1. Biefeleld, Sutherland, Tangerman and Veerman [5] showed recently that for $f(z) = z^2|z|^{(\gamma-2)} + c$ and a small $\gamma - 2 > 0$, there is an $\eta(\gamma) > 0$ such that the Julia set J_λ of $f(z)$ for $|c| < \eta(\gamma)$ is a smooth circle (see Fig. 4 in §4).

§2 Proof of Theorem 1.

By the compactness of $\overline{W}$, there is a function $\delta = \delta(\varepsilon) > 0$ with $\delta(\varepsilon) \mapsto 0$ as $\varepsilon \mapsto 0+$ such that every g_i is defined on $B(z, \delta)$ for z in $\overline{W}$ and $g_i(w) = g_i(z) + \Big(D(g_i)(z)\Big)(w - z) + R_i(w, z)$ satisfies that

$$|R_i(w, z)| \leq \Big(\varepsilon/2\Big) \cdot \Big(\inf_{w\in\overline{W}} ||D(g_i)(z)||\Big) \cdot |w - z|$$

for z and w in $\overline{W}$ with $|w - z| \leq \delta$ and $0 \leq i < n$. This implies that for z in $\overline{W}$ and $0 < r \leq \delta$,

$$g_i\Big(B(z, r)\Big) \supset g_i(z) + (1 + \varepsilon)^{-1} \cdot \Big(D(g_i)(z)\Big)\Big(B(0, r)\Big) \ \textit{ and}$$

$$g_i\Big(B(z, r)\Big) \subset g_i(z) + (1 + \varepsilon) \cdot \Big(D(g_i)(z)\Big)\Big(B(0, r)\Big) \qquad (*).$$

Suppose $L_0 > 0$ and $0 < \beta < \alpha$ are constants such that $|R_i(w, z)| \leq L_0|w - z|^{1+\alpha}$ and $K_i(z) \leq \Big(1/l_i(z)\Big)^\beta$ for $0 \leq i < n$, z and w in $\overline{W}$. Let $\kappa_m = \sum_{i=0}^m l^{(\alpha-\beta)i}$. We take $\delta = \delta(\varepsilon) \leq 1$ so small that

$$\Theta_\varepsilon = \Big(L_0/s\Big)\Big(1 + \varepsilon + \kappa_\infty\Big)^{1+\alpha}\delta^{(\alpha-\beta)} \leq 1$$

and then take

$$C_m(\varepsilon) = 1 + \varepsilon + \delta^\beta \cdot \kappa_m.$$

It is clear that $C_m(\varepsilon) \mapsto 1$ as $\varepsilon \mapsto 0+$.

Claim. For $g = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_m}$ in $\mathcal{G}$,

$$g\Big(B(z,r)\Big) \supset g(z) + C_m^{-1} \cdot \Big(D(g)(z)\Big)\Big(B(0,r)\Big) \ \ \textit{and}$$

$$g\Big(B(z,r)\Big) \subset g(z) + C_m \cdot \Big(D(g)(z)\Big)\Big(B(0,r)\Big).$$

Proof of claim. For $m = 0$, it is the formulae in $(*)$. Suppose the claim holds for $m = 0, 1, \ldots, M-1$ ($M \geq 1$). Then for $g = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_M} = g_{i_0} \circ G$,

$$g\Big(B(z,r)\Big) \supset g_{i_0}\Big(G(z) + C_{M-1}^{-1} \cdot \Big(D(G)(z)\Big)\Big(B(0,r)\Big)\Big) \ \ \textit{and}$$

$$g\Big(B(z,r)\Big) \subset g_{i_0}\Big(G(z) + C_{M-1} \cdot \Big(D(G)(z)\Big)\Big(B(0,r)\Big)\Big).$$

For any w in $B(0,r)$, we know that

$$g_{i_0}\Big(G(z) + C_{M-1}^j \cdot \Big(D(G)(z)\Big)(w)\Big) = g(z) + C_{M-1}^j \cdot \Big(D(g)(z)\Big)(w) + R$$

where $R = R_{i_0}\Big(C_{M-1}^j \cdot \Big(D(G)(z)\Big)(w), z\Big)$ and $j = 1$ or -1, and

$$|R| \leq L_0 C_{M-1}^{1+\alpha} ||D(G)(z)||^{1+\alpha} |w|^{1+\alpha}.$$

But for $z_0 = z$ and $z_i = g_{M-i} \circ \cdots \circ g_{i_M}(z)$, $i = 1, 2, \ldots, M$,

$$||D(G)(z)|| = \prod_{1 \leq k \leq M} ||D(g_{i_k})(z_{M-k})|| \leq \prod_{1 \leq k \leq M} l_{i_k}(z_{M-k}).$$

Hence, by $K_i(z) \leq \Big(1/l_i(z)\Big)^\beta$ for all i, we have that

$$||D(G)(z)||^{1+\alpha} \leq \Big(\prod_{1 \leq k \leq M} s_{i_k}(z_{M-k})\Big) l^{(\alpha-\beta)M}.$$

Let $B_M = \Big(L_0/s\Big) C_{M-1}^{1+\alpha} \delta^\alpha l^{(\alpha-\beta)M}$, then

$$|R| \leq B_M \Big(\prod_{0 \leq k \leq M} s_{i_k}(z_{M-k})\Big)|w|.$$

Since $B_M \leq \Theta_\varepsilon \delta^\beta l^{(\alpha-\beta)M} \leq \delta^\beta l^{(\alpha-\beta)M}$, we get that $C_{M-1} + B_M \leq C_M$. Now we can conclude from the estimates that $g(w) - g(z)$ is in $C_M \cdot \Big(D(g)(z)\Big)\Big(B(0,r)\Big)$ and if $|w| = r$, $g(w) - g(z)$ is outside of $C_M^{-1} \cdot \Big(D(g)(z)\Big)\Big(B(0,r)\Big)$. The proof of the claim is completed.

Take $C = C_\infty(\varepsilon)$. Then δ and C are the functions we want. This completes the proof of Theorem 1.

The proof of Corollary 1 is similar.

§3 Proof of Theorem 2.

According to Theorem 1, each $g_{w_p}(\overline{W})$ contains a translation of the ellipse $C^{-1} \cdot \Big(D(g_{w_p})(z)\Big)\Big(B(0,1)\Big)$ and is contained in a translation of the ellipse $C \cdot \Big(D(g_{w_p})(z)\Big)\Big(B(0,1)\Big)$ where C is independent of w_p and z. For every $w_p = i_0 i_1 \cdots i_{p-1}$ in Σ_p, let $g_{w_p} = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{p-1}}$. Since all g_i are contracting, there is a constant $0 < \lambda_0 < 1$ such that $diam\Big(g_{w_p}(\overline{W})\Big) \leq \lambda_0^p$ for all $w_p \in \Sigma_p$. Thus $\{g_{w_p}(\overline{W}); w_p \in \Sigma_p\}$ is a cover of Λ for every p and $\tau_p = \max\{diam(g_{w_p}(\overline{W})); w_p \in \Sigma_p\}$ tends to zero as p tends to ∞. Use Theorem 1 again, the Hausdorff dimension [6] of Λ is a unique number $t_0 > 0$ satisfying

$$\lim_{p \mapsto \infty} \sum_{w_p \in \Sigma_p} \Big(diam\Big(g_{w_p}(\overline{W})\Big)\Big)^t = \infty \quad for \quad t < t_0 \quad and$$

$$\lim_{p \mapsto \infty} \sum_{w_p \in \Sigma_p} \Big(diam\Big(g_{w_p}(\overline{W})\Big)\Big)^t = 0 \quad for \quad t > t_0.$$

Let $l_{w_p}(z)$ and $s_{w_p}(z)$ be the lengths of longest and shortest axes of the ellipse $\Big(D(g_{w_p})\Big)\Big(B(0,1)\Big)$. Then we have that

$$C^{-1} \cdot s_{w_p}(z) \leq diam\Big(g_{w_p}(\overline{W})\Big) \leq C \cdot l_{w_p}(z).$$

One of the crucial points is that

$$l_{w_p}(z) \leq l_{i_0}(z_{p-1}) \cdots l_{i_p}(z_0) \quad and \quad s_{w_p}(z) \geq s_{i_0}(z_{p-1}) \cdots s_{i_p}(z_0)$$

where $z_k = g_{i_{p-k}} \circ \cdots \circ g_{i_{p-1}}(z)$. Because of these two inequalities, we can conclude our proof by Gibbs theory (see, for example, [2, 11, 14]) as follows: for any $t>0$,

$$\Big(diam\Big(g_{w_p}(W)\Big)\Big)^t \leq C_1 \cdot \exp\Big(\sum_{k=0}^{p-1} t\phi_{up}(w^k)\Big) \ \ and$$

$$\Big(diam\Big(g_{w_p}(W)\Big)\Big)^t \geq C_1^{-1} \cdot \exp\Big(\sum_{k=0}^{p-1} t\phi_{lo}(w^k)\Big)$$

where $\pi(w^k) = z_k$ and C_1 is a constant. Suppose $\mu_{t_{up}\phi_{up}}$ and $\mu_{t_{lo}\phi_{lo}}$ are the Gibbs measures of $t_{up}\phi_{up}$ and $t_{lo}\phi_{lo}$ on $\Big(\Sigma_\infty, \sigma\Big)$. Because $P(t_{up}\phi_{up}) = 0$ and $P(t_{lo}\phi_{lo}) = 0$, there is a constant $d>0$ such that

$$\mu_{t_{up}\phi_{up}}(\Lambda_{w_p}) \in [d^{-1}, d] \exp\Big(\sum_{k=0}^{p-1} t_{up}\phi_{up}\Big(\sigma^{\circ k}(w_0)\Big)\Big) \ \ \text{and}$$

$$\mu_{t_{lo}\phi_{lo}}(\Lambda_{w_p}) \in [d^{-1}, d] \exp\Big(\sum_{k=0}^{p-1} t_{lo}\phi_{lo}\Big(\sigma^{\circ k}(w_0)\Big)\Big)$$

where $w_0 \in \Lambda_{w_p} = \{w \in \Sigma; w = w_p \cdots\}$. Hence there is a constant $C_2 > 0$ such that

$$\Big(diam\Big(g_{w_p}(W)\Big)\Big)^{t_{up}} \leq C_2 \cdot \mu_{t_{up}\phi_{up}}(\Lambda_{w_p}) \ \ \text{and}$$

$$\Big(diam\Big(g_{w_p}(W)\Big)\Big)^{t_{lo}} \geq C_2^{-1} \cdot \mu_{t_{lo}\phi_{lo}}(\Lambda_{w_p}).$$

Moreover,

$$\sum_{w_p \in \Sigma_p} \Big(diam\Big(g_{w_p}(W)\Big)\Big)^{t_{up}} \leq C_2 \cdot \sum_{w_p \in \Sigma_p} \mu_{t_{up}\phi_{up}}(\Lambda_{w_p}) = C_2 \ \ and$$

$$\sum_{w_p \in \Sigma_p} \Big(diam\Big(g_{w_p}(W)\Big)\Big)^{t_{lo}} \geq C_2^{-1} \cdot \sum_{w_p \in \Sigma_p} \mu_{t_{lo}\phi_{lo}}(\Lambda_{w_p}) = C_2^{-1}.$$

This implies that $t_{lo} \leq HD(\Lambda) \leq t_{up}$. The proof is completed.

Proof of Corollary 2. For $\phi = \phi_{lo,\lambda}$ (or $\phi_{up,\lambda}$), the inverse of $P(t\phi)$ is continuous on P and λ. This implies that $t_{lo,\lambda}$ (or $t_{up,\lambda}$) tends to t_{lo,λ_0} (or t_{up,λ_0}) as λ goes to λ_0. But, $t_{lo,\lambda_0} = t_{up,\lambda_0} = HD(\lambda_0)$ because all g_{i,λ_0} are conformal. This completes the proof.

Proof of Corollary 3. Let $\lambda = (b, c)$ (or $\lambda = (\gamma - n, c)$) and $|\lambda| = |b| + |c|$ (or $|\lambda| = |\gamma - n| + |c|$). There is a neighborhood W of $S^1 = \{z \in \mathbf{C}; |z| = 1\}$ so that f is expanding on $\overline{W}$ for small $|\lambda|$. Let $g_{0,\lambda}, \ldots, g_{n-1,\lambda}$ be the inverse branches of $f|\overline{W}$. Then $\mathcal{G}_\lambda$, the semigroup generated by $g_{0,\lambda}, \ldots, g_{n-1,\lambda}$, is regular and Markov for λ with small $|\lambda|$. Now the proof follows from Corollary 2 because for each $\lambda = (0, c)$ with small $|c| \neq 0$, all $g_{i,\lambda}$ are conformal and the Hausdorff dimension $HD(\lambda)$ of J_λ is greater than one.

§4 Higher dimensional regular semigroups and some remarks.

Suppose $\mathbf{E^m}$ is the m-dimensional Euclidean space, $V_i \subset U_i$, $i = 0, \ldots, n-1$, are pairs of open sets of $\mathbf{E}^m$ with $\overline{V_i} \subset U_i$ and f_i from $\overline{V}_i$ onto $\overline{W}_i$ are $C^{1+\alpha}$ diffeomorphisms such that the inverses g_i of $f_i|\overline{V}_i$ are contracting. Let $\mathcal{G}_m = \langle g_0, g_1, \ldots, g_{n-1}\rangle$ be the semigroup generated by all g_i. Then l and K for $\mathcal{G}_m$ can be defined similarly. Again $\mathcal{G}_m$ is said to be regular if $K < 1/l^\alpha$. Let $B(x, r)$ be the closed ball of radius r centered at x of $\mathbf{E}^m$. The higher dimensional version of Theorem 1 is the following:

Theorem 3 (geometric distortion). *Suppose $\mathcal{G}_m = \langle g_0, g_1, \ldots, g_{n-1}\rangle$ is regular. There are two functions $\delta = \delta(\varepsilon) > 0$ and $C = C(\varepsilon) \geq 1$ with $\delta(\varepsilon) \mapsto 0$ and $C(\varepsilon) \mapsto 1$ as $\varepsilon \mapsto 0+$ such that*

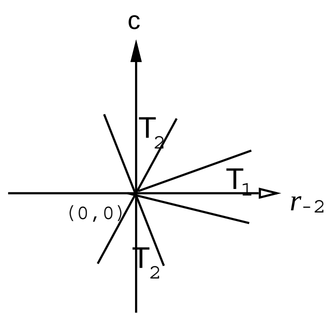
$$g\Big(B(x, r)\Big) \supset g(x) + C^{-1} \cdot \Big(D(g)(x)\Big)\Big(B(0, r)\Big) \ \textit{and}$$

$$g\Big(B(x, r)\Big) \subset g(x) + C \cdot \Big(D(g)(x)\Big)\Big(B(0, r)\Big)$$

for any $0 < r \leq \delta(\varepsilon)$, any $g \in \mathcal{G}_m$ and any $x \in \overline{W}$.

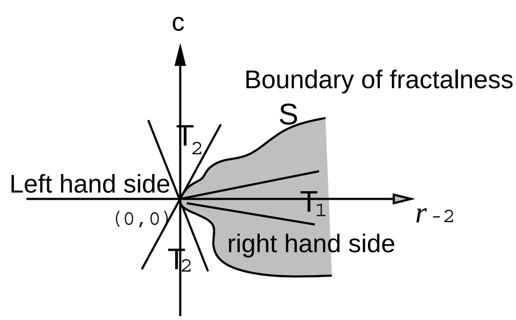
Remark 2. Similarly, we have the higher dimensional versions of Corollary 1 and Theorem 2. We learned recently that Gu [7] showed another upper bound (in higher dimensional case) which is similar to that in Theorem 2.

Remark 3. Suppose $f_\lambda(z) = z^2|z|^{(\gamma-2)} + c$ where $\lambda = (\gamma - 2, c)$. From Corollary 3 and Remark 1, there is an interesting picture on the parameter space λ (three dimensional space) near the point $(0,0)$: there are small sectors T_1 and T_2 (see Fig. 4) such that for λ in T_1, J_λ is a smooth circle and for λ in T_2, J_λ is a fractal circle with Hausdorff dimension > 1. From computer pictures of J_λ for small $|\lambda|$, we conjecture that there is a topological surface S passing $(0,0)$ in a small ball centered at $(0,0)$ such that in the right hand side of S, J_λ is a smooth circle and in the left hand side of S (but not on the $(\gamma-2)$-axis), J_λ is a fractal circle with Hausdorff dimension >1 (see Fig. 5). We may call S the *boundary of fractalness*. If S exists, what can be said about its shape ?



Proven picture

Fig. 4



Conjectured picture

Fig. 5

Remark 4. Sullivan [14] has considered quasiconformal deformations of analytic and expanding systems and Gibbs measures. Moreover, he also studied (uniform) quasiconformality in geodesic flows of negatively curved manifolds. One wonders if Theorem 1 can be used to extend some results [14] to non-conformal expanding systems (or hyperbolic systems) with the compatibility condition $K < 1/l^\alpha$ and to geodesic flows of negatively curved manifolds with pinched condition.

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