Scalings in Circle Maps III

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Abstract

Circle maps with a flat spot are studied which are differentiable, even on the boundary of the flat spot. Estimates on the Lebesgue measure and the Hausdorff dimension of the non-wandering set are obtained. Also, a sharp transition is found from degenerate geometry similar to what was found earlier for non-differentiable maps with a flat spot to bounded geometry as in critical maps without a flat spot.

1 Introduction

1.1 Maps with a flat spot

We consider degree one weakly order-preserving circle endomorphisms which are constant on precisely one arc (called the flat spot.) Maps of this kind appear naturally in the study of Cherry flows on the torus (see [1]) as well as "truncations" of smooth non-invertible circle endomorphisms (see [5]). They have been less thoroughly researched than homeomorphisms.

Topologically, one nice thing about maps with a flat spot is that they still have a rotation number. If F is a map with a flat spot, and f is its lifting, the rotation number $\rho(F)$ is the limit

$$\lim_{n \to \infty} \frac{f^n(x)}{n} \ (\mathrm{mod} 1)$$

which turns out to exist for every x and its value is independent of x. The dynamics is most interesting if the rotation number is irrational.

We study first the topology of the non-wandering set, then its geometry. Where the geometry is concerned, we discover a dichotomy. Some of our maps show a "degenerate universality" akin to what was found in a similar case considered by [2] and [10], while others seem to be subject to the "bounded geometry" regime, very much like critical homeomorphisms, i.e. maps which instead of the flat spot have just a critical point.

Before we can explain our results more precisely, it is necessary to define our class and fix some notations.

Almost smooth maps with a flat spot. We consider the class of continuous circle endomorphisms F of degree one for which an arc U exists so that the following properties hold:

1. The image of U is one point.

2. The restriction of F to $\mathbf{S}^1 \setminus \overline{U}$ is a C^2- diffeomorphism onto its image.

3. Consider a lifting to the real line, and let (a, b) be a preimage of U, while the lifting of F itself is denoted with f. On some right-sided neighborhood of b, f can be represented as

$$h_r((x - b)^{p_r})$$

   for $p_r \geq 1$ with h_r which extends as a C^2-diffeomorphism beyond b.

   Analogously, in a left-sided neighborhood of a, f is

$$h_l((a - x)^{p_l}) \ .$$

The ordered pair (p_l, p_r) will be called the *critical exponent of the map*. If $p_l = p_r$ the map will be referred to as *symmetric*.

In the future, we will deal exclusively with maps from this class. Moreover, from now on we restrict our attention to maps with an irrational rotation number.

Basic notations. The critical orbit is of paramount importance in studying any one-dimensional system, thus we will introduce a simplified notation for backward and forward images of U. Instead of $F^i(U)$ we will simply write

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$\underline{i}$. This convention will also apply to more complex expressions. For example, $F^{-3q_n-20}(\underline{0})$ will be abbreviated to $\underline{-3q_n - 20}$. This is certainly different from $F^{-3q_n}(\underline{0}) - 20$ where -20 means an element of the group $\mathbf{T}^1$. In our notation, this difference is marked by 20 not being underlined, i.e. $\underline{-3q_n} - 20$. An underlined complicated expression should be evaluated as a single image of $\underline{0}$. Thus, underlined positive integers are points, and non-positive ones are intervals.

Let q_n denote the closest returns of the rotation number $\rho(F)$ (see [10] for the definition).

Next, we define a sequence of scalings

$$\sigma(n) := \frac{\text{dist}(\underline{0}, \underline{q_n})}{\text{dist}(\underline{0}, \underline{q_{n-2}})} \ .$$

A summary of previous related results. Maps with the critical exponent $(1, 1)$ were studied first. The most complete account can be found in [9]. They turn out to be expanding apart from the flat spot. Therefore, the geometry can be studied relatively easily. One of the results is that the scalings $\sigma(n)$ tend to 0 fast.

Next, critical exponents $(1, \nu)$ or $(\nu, 1)$ were investigated for $\nu > 1$ independently in [2] and [10]. The main result was that $\sigma(n)$ still tend to 0. This was shown to lead to "degenerate universality" of the first return map on $(\underline{q_{n-1}}, \underline{q_n})$. Namely, as n grows, the branches of this map become at least C^1 close to either affine strongly expanding maps, or a composition of $x \to x^\nu$ with such maps.

Finally, we need to be aware of the results for critical maps where U is a point and the singularity is symmetric. The scalings can still be defined by the same formula, but they certainly do not tend to 0 (cite [3] and [7]). Moreover, if the rotation number is golden mean, then they are believed to tend to a universal limit (see [4].) This is an example of bounded geometry, and conjectured "non-degenerate" universality.

In this context, we are ready to present our results.

1.2   Statement of results

We investigate symmetric almost smooth maps with a flat spot with the critical exponent (ν, ν) , $\nu > 1$. First, we get results about the non-

wandering set which are true for any ν. Also, we permanently assume that the rotation number is irrational.

Theorem 1 *For any F with the critical exponent (ν, ν) , $\nu > 1$, the set $\mathbf{S}^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(U)$ has zero Lebesgue measure. Moreover, if the rotation number is of bounded type (i.e. q_n/q_{n-1} are uniformly bounded), the Hausdorff dimension of the non-wandering set is strictly less than 1.*

Corollary.   There are no wandering intervals and any two maps from our class with the same irrational rotation number are topologically conjugate.

Theorem 2 *Again, we assume that the critical exponent is (ν, ν) with $\nu > 1$. Then, we have a dichotomy in the asymptotic behavior of scalings. If $\nu \leq 2$, the scalings $\sigma(n)$ tend to 0. If $\nu > 2$, and the rotation number is of bounded type, then $\liminf_{n \to \infty} \sigma(n) > 0$.*

Comments.   Thus, Theorem 2 shows that a transition occurs from the "degenerate universality" case to the "bounded geometry" case as the exponent passes through 2. This is the first discovery of bounded geometry behavior in maps with a flat spot (which was conjectured in [10].)

Numerical findings.   A natural question appears whether bounded geometry, when it occurs, is accompanied by non-trivial universal geometry. More precisely, we have two conjectures:

Conjecture 1 For a map F from our class with the golden mean rotation number, the scalings $\sigma(n)$ tend to a limit.

We found this conjecture supported numerically, albeit only for one map considered. Moreover, the rate of convergence appears to be exponential. The reader is referred to Appendix B for a detailed description of our experiment.

There is a much bolder conjecture:

Conjecture 2 Consider two maps from our class with the same critical exponent larger than 2 and the same irrational (bounded type? noble?) rotation number. Then, the conjugacy between them is differentiable at $\underline{1}$ (the critical value according to our convention.)

This conjecture is motivated by the analogy with the critical case. The same analogy (see [6]) makes us expect that Conjecture 2 would be implied by Conjecture 1 if the convergence in Conjecture 1 is exponential and the limit is independent of F.

Parameter scalings Consider a smooth one parameter family f_t of circle maps in our class with constant critical exponent (ν, ν) for which $d/dt\, f_t > 0$. Assume that f_0 has golden mean rotation number. Denote by I_n the interval of parameters t for which f_t has as rotation number p_n/q_n, the $n - th$ continued fraction approximant to the golden mean. The length $|I_n|$ of the interval I_n tends to zero as n tends to infinity. Define the parameter scaling δ_n as:

$$\delta_n = |I_n|/|I_{n-2}|$$

When $1 \leq \nu \leq 2$ the arguments in [11] yield an asymptotically exact relation between parameter scalings and geometric scalings for f_0:

$$\delta_n = \sigma(n-1)^\nu$$

We conjecture that when $\nu > 2$, the parameter scalings tend to a universal limit only depending on ν. In fact, the same relation between parameter scalings and geometric scalings appears to hold.

1.3   Technical tools

Denote by $(a, b) = (b, a)$ the open shortest arc between a and b regardless of the order of these two points. If the distance from a to b is exactly $1/2$, choose the arc which contains some right neighborhood of 0. The distance between two sets X and Y is defined as

$$\mathrm{dist}(X, Y) = \inf\{\mathrm{dist}(x, y) : x \in X, y \in Y\}\,.$$

We shall write $l(I)$ and $r(I)$ appropriately for the left and the right endpoint of interval I. In particular we set $l = l(U)$ and $r = r(U)$.

The cross-ratio inequality. Suppose we have four points a, b, c, d arranged according to the standard orientation of the circle so that $a < b < c < d$ and $b, c \in (a, d)$. Define their cross-ratio as :

$$\mathbf{Cr}(a, b, c, d) = \frac{\mid (a, b) \mid\mid (c, d) \mid}{\mid (a, c) \mid\mid (b, d) \mid}$$

By the distortion of cross-ratio we mean

$$\mathcal{D}\mathbf{Cr}(a, b, c, d) = \frac{\mathbf{Cr}(f(a), f(b), f(c), f(d))}{\mathbf{Cr}(a, b, c, d)} \ .$$

Let us consider a set of quadruples a_i, b_i, c_i, d_i with the following properties:

1. Each point of the circle belongs to at most k among intervals (a_i, d_i).

2. Intervals b_i, c_i do not intersect U

then

$$\prod_{i=0}^{n} \mathcal{D}\mathbf{Cr}(a_i, b_i, c_i, d_i) \leq C_k$$

and the constant C_k does not depend on the set of triples.

In this paper all sets of triples will be formed by taking iterations of an initial quadruple. Therefore we will only indicate the initial quadruple together with the number of iterations one performs.

This inequality was introduced and proved in [8].

Lemma 1.1 *There is a constant K so that for any two points y, z, if f is a diffeomorphism on (y, z), the following inequality holds:*

$$\frac{\mid f(y, z) \mid}{dist(f(y), f(U))} \leq K \frac{\mid (y, z) \mid}{dist(y, U)}$$

provided $dist(z, U) \leq dist(y, U)$.

Proof:
It is a simple calculation.

$\square$

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The Distortion Lemma. We use the following lemma which can be considered a variant of the "Koebe lemma" which was the basis of estimates in [10].

Let f be a lifting of an almost smooth map with a flat spot, and consider a sequence of intervals I_j with $0 \leq j \leq n$ so that $I_{j+1} = f^{-1}(I_j)$ and $U \cap I_j = \emptyset$ for $0 \leq j < n$. Choose an interval $(a, b) \subset I_0$ and let A be the orientation-preserving affine map from $[0, 1]$ onto I_0. Then, we define the "rescaled" map $\tilde{f} := f^{-n} \circ A$. So, $\tilde{f}$ maps $[0, 1]$ onto I_n.

The nonlinearity of $\tilde{f}$ satisfies the following estimate:

$$\frac{\tilde{f}''}{\tilde{f}'} \leq \frac{K}{\mathbf{Cr}(0, A^{-1}(a), A^{-1}(b), 1)}$$

where K is a uniform continuous function of $\sum_{j=0}^{n-1} |I_j|$ only.

proof The lemma follows directly from the "Uniform Bounded Distortion Lemma" of [7].

2 Estimates valid for any critical exponent

2.1 Geometric bounds

Lemma 2.1 *The sequence $dist(\underline{q_n}, U)$ tends to zero at least exponentially fast.*

Proof:
The orbit of U for $0 \leq i \leq q_{n+1} + q_n - 1$ together with open arcs lying between successive points of the orbit constitute a partition of the circle. Let I be the shortest arc belonging to the set

$$\mathcal{A} := \{(\underline{q_n + i}, \underline{i}) : 0 \leq i \leq q_{n+1}\} \ .$$

Denote the ratio

$$\frac{|\ (\underline{3q_n}, \underline{q_n})\ |}{dist(\underline{q_n}, U)}$$

by $\Gamma(n)$. We will show that $\Gamma(n)$ is bounded away from zero. Lemma 1.1 implies that

$$\frac{|\ (\underline{3q_n + 1}, \underline{q_n + 1})\ |}{|\ (\underline{3q_n + 1}, \underline{1})\ |} \leq K \frac{|\ (\underline{3q_n}, \underline{q_n})\ |}{dist(\underline{3q_n}, U)} \ .$$

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If I coincides with $(\underline{q_n}, \partial U)$ then clearly $\Gamma(n) \geq 1/2$. Otherwise we can iterate i times, mapping the interval $(\underline{q_n + 1}, \underline{1})$ onto I. Note that intervals

$$(\underline{3q_n + 1 + i}, \underline{q_n + 1 + i}) \text{ and } (\underline{1 + i}, \underline{-q_n + q_{n+1} + 1 + i})$$

cover two adjacent intervals to I from the set $\mathcal{A}$.

Now we write the cross-ratio inequality for

$$\{\underline{3q_n + 1}, \underline{q_n + 1}, \underline{1}, \underline{-q_n + q_{n+1} + 1}\}$$

and the number of iterations equal to i. We obtain the following estimate:

$$\frac{\mid (\underline{3q_n + 1}, \underline{q_n + 1}) \mid}{\mid (\underline{3q_n + 1}, \underline{1}) \mid} \frac{\mid (\underline{1}, \underline{-q_n + q_{n+1} + 1}) \mid}{\mid (\underline{q_n + 1}, \underline{-q_n + q_{n+1} + 1}) \mid} \geq 4/C_3 \ .$$

Thus

$$\Gamma(n) \geq 4/C_3 K$$

and $\mathrm{dist}(q_n, U) \leq (1/(1 + \Gamma))\mathrm{dist}(\underline{3q_n}, U)$. The ordering of the orbit of U implies the next inequality

$$\mathrm{dist}(\underline{q_n}, U) \leq (\Gamma/(1 + \Gamma))\mathrm{dist}(\underline{q_{n-4}}, U)$$

which completes the proof.

$\square$

Proposition 1   *1. The sequence $\{\sigma(n)\}$ is bounded away from 1.*

2. The sequence

$$\frac{\mid \underline{-q_{n-1}} \mid}{\mid (\underline{q_n}, \underline{q_{n-2}}) \mid}$$

   is bounded away from zero.

Proof:
Let $\mathcal{U}_n$ be the n-th partition of the circle given by all $q_{n+1} + q_n - 1$ preimages of U, $\mathcal{J}_n = \{\underline{-i} : O \leq i \leq q_{n+1} + q_n - 1\}$, together with the holes between successive preimages of U. It is easy to see that the holes are given by the following formula:

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1. $-q_n$ is on the left side of U. Set

$$\Box_i^n := f^{-i}(r(-q_n), l(U)) \text{ and } \bigcirc_j^n := f^{-j}(r(U), l(-q_{n+1})) .$$

where j ranges from 0 to q_n, and i is between 0 and q_{n+1}.

2. $-q_n$ is on the right side of U. Set

$$\Box_i^n := f^{-i}(r(U), l(-q_n)) \text{ and } \bigcirc_j^n := f^{-j}(r(-q_{n+1}), l(U)) .$$

with i ranging from 0 to q_{n+1} and j from 0 to q_n.

Then

$$\mathcal{U}_n \setminus \mathcal{J}_n = \{\Box_i^n, 0 \leq i < q_{n+1}\} \cup \{\bigcirc_j^n, 0 \leq j < q_n\} .$$

Note that $\bigcirc_j^{n-1} = \Box_j^n$

Take two successive preimages of U which belong to the n-th partition $\mathcal{U}_n$, say $-i$ and $-j$. We may assume that $-i$ lies to the left of $-j$. Take as the initial quadruple the endpoints of the considered preimages of U. We can iterate the quadruple

$$\{l(-i), r(-i), l(-j), r(-j)\}$$

until we hit U . The cross-ratio inequality gives the following estimate:

$$\mathbf{Cr}(l(-i), r(-i), l(-j), r(-j)) \geq$$

$$\geq (\mid U \mid /C_1)\frac{\mid -|i-j| \mid}{\mid -|i-j| \mid + \text{dist}(-|i-j|, U)}$$

where $|i-j|$ is equal to either q_n or q_{n+1}. Thanks to lemma 1.1 we know that the ratio of lengths of intervals adjacent to the plateau can be changed only by a bounded amount.

$$\frac{\mid -|i-j|+1 \mid}{\mid -|i-j|+1 \mid + \text{dist}(-|i-j|+1, 1)} \leq$$

$$\leq K\frac{\mid -|i-j| \mid}{\mid -|i-j| \mid + \text{dist}(-|i-j|, U)} .$$

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Now we form a new quadruple from the endpoints of $-|i-j|+1$ and two additional points: $|i-j|$ and $\underline{1}$. To obtain the next estimate we write the cross-ratio inequality for the quadruple and the number of iterates equal to $|i-j|$. Let us recall that we proved in lemma 2.1 that $\mid (3|i-j|, \underline{|i-j|}) \mid$ was big with comparison to $\mathrm{dist}(\underline{|i-j|}, U)$. Hence

$$\frac{\mid \underline{-|i-j|+1} \mid}{\mid \underline{-|i-j|+1} \mid + \mathrm{dist}(\underline{-|i-j|+1}, \underline{1})}$$

$$\geq \Gamma \mid U \mid /C_3 \ .$$

Combining all above inequalities we get

$$\frac{\mid \underline{-i} \mid}{\mid (l(\underline{-i}), l(\underline{-j})) \mid} \quad \frac{\mid \underline{-j} \mid}{\mid r(\underline{-i}), r(\underline{-j}) \mid} \geq$$

$$\geq \Gamma \mid U \mid /C_3 C_1 \ .$$

To finish the proof note that interval $(\underline{q_{n-2}}, \underline{q_n})$ contains exactly one preimage of U which belong to $\mathcal{U}_{n-2}$, namely $\underline{-q_{n-1}}$.

$\square$

Lemma 2.2 *The lengths of intervals $\square_i^n$ and $\bigcirc_j^n$ tend to zero uniformly exponentially fast with n.*

Proof:
An interval $\square_i^n$ is subdivided into preimages of the flat spot and intervals of the form $\square_j^{n+1}$ and $\bigcirc_k^{n+1}$. We will argue that a certain proportion of measure is lost in the preimages of U. To this end, apply to the cross-ratio inequality to a quadruple given by the endpoints of two neighboring preimages of U in the subdivision. By Proposition 1, this cross-ratio is bounded away from 0.

$\square$

2.2   Proof of Theorem 1

The first claim of the Theorem follows directly from Lemma 2.2.

The claim concerning the Hausdorff dimension requires a bit longer argument. Suppose that the rotation number is of bounded type. Take the

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$n - 1$-th partition of the circle S^1. The elements of the next partition subdivide the holes of latter one in the following way:

$$\square_i^{n-1} \subset \bigcup_{j=0}^{a_{n+1}} \square_{i+q_n+jq_{n+1}}^n \cup \bigcirc_i^n \ ,$$

$$\bigcirc_i^{n-1} = \square_i^{n+1} \ .$$

We estimate

$$\sum (\mid \square_i^n \mid^\alpha + \mid \bigcirc_i^n \mid^\alpha$$

where $\sum$ means the sum over all holes of n-th partition. By Proposition 1 follows that there is a constant $\beta < 1$ so that

$$\sum_{j=0}^{a_{n+1}} \mid \square_{i+q_n+jq_{n+1}}^n \mid \leq \beta \mid \square_i^{n-1} \mid$$

holds for all 'long' holes $\square_{i+q_n+jq_{n+1}}^n$ of n-th partition. In particular it means that the holes of n-th partition decrease uniformly and exponentially fast to zero while n tends to infinity. We use concavity of function x^α to obtain that

$$\sum_{j=0}^{a_{n+1}} \mid \square_{i+q_n+jq_{n+1}}^n \mid^\alpha \leq$$

$$\leq \mid a_{n+1} + 1 \mid^{1-\alpha} \beta^\alpha \mid \square_i^{n-1} \mid^\alpha \leq$$

$$\leq \mid \square_i^{n-1} \mid^\alpha$$

if only α is close to 1. Hence the sum over all holes at power α of n-th partition is a decreasing function of n. Consequently, the sum is less than 1. The only remaining point is to prove that for a given ε the holes of n-th partition constitute an ε-cover of Ω if only n is large enought. But this is so since the length of the holes of n-th partition goes to zero uniformly. This completes the proof.

3 Controlled Geometry: recursion on the scalings

3.1 Proof of Theorem 2

The strategy of the proof of the first part of this theorem is to establish recursion relations between scalings (proposition 3.1), similar to what was

done in [10]. A close study of these relations then implies the first part of theorem 2: when $\nu > 2$, these scalings are bounded away from zero.

We will give the derivation of the recursion relation between scalings. Since this derivation is in many respects analogous to what was done in chapter 4 of [10], (in fact the only difference in the proofs is the change of the phrase "essentially linear" to "a priori bounded nonlinearity"), the discussion will be somewhat sketchy. The basic strategy is that closest returns factor as a composition of a power law and a map of a priori bounded distortion. This allows one to control ratio's of lengths of dynamically defined intervals.

Let f be a map satisfying the assumptions of theorem 2: the critical exponent is (ν, ν) and the rotation number is of bounded type. Then proposition 2.1 supplies us with a priori bounds.

In the sequel it is convenient to introduce a symbol ($\approx$) for approximate equality. Let $\{\alpha(n)\}$ and $\{\beta(n)\}$ be two positive sequences. The notation

$$\alpha(n) \approx \beta(n)$$

means that there exists a constant $K \geq 1$ depending only on the a priori bounds and the type of the rotation number so that for all n:

$$\frac{1}{K} \leq \frac{\alpha(n)}{\beta(n)} \leq K$$

Proposition 2.1 (a priori bounds) implies that

$$|\underline{-q_n}| \approx |\underline{q_{n-1}}| \quad (\mathbf{3.1})$$

The interval $[\underline{q_n}, \underline{q_{n-2}}]$ contains the interval $\underline{-q_{n-1}}$ as well as its inverse images: $f^{-iq_n}(\underline{-q_{n-1}})$ ($i = 1,..., a_n - 1$). Each interval $[\underline{i\,q_n}, \underline{(i+1)\,q_n}]$ contains one such inverse image. The distortion lemma (see introduction), the assumption that the singularity is a power law (with power ν), and the assumption that a_n is bounded imply (see also [10], chapter 4]:

$$|(\underline{(i-1)\,q_n}, \underline{i\,q_n})| \approx |\underline{i\,q_n}|; \; for \; i = 2, .., a_n \quad (\mathbf{3.2})$$

This relation immediately implies:

$$|(f(\underline{(i-1)\,q_n}), f(\underline{i\,q_n}))| \approx |f(\underline{i\,q_n})| \quad (\mathbf{3.3})$$

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$$\frac{|((i-1)\,q_n,\ i\,q_n)|}{|f(i\,q_n)|}\,Df(i\,q_n)\ \approx\ \nu\ \ (\mathbf{3.4})$$

This last relation is the analogue of:

$$\frac{x\,.\,\nu\,x^{\nu-1}}{x^\nu}\ =\ \nu$$

Define scalings $\sigma(n,\ i)$ as

$$\sigma(n,\ i)\ =\ \frac{|((i-1)\,q_n,\ i\,q_n)|}{|(i\,q_n,\ (i+1)\,q_n)|},\ \ for\ i=1,..,a_n-1$$

$$\sigma(n,\ a_n)\ =\ \frac{|((a_n-1)\,q_n,\ a_n\,q_n)|}{|(a_n\,q_n,\ q_{n-2})|}$$

Remark: $\sigma(n)$ can not quite be expressed in these scalings. However one has:

$$\sigma(n)\ =\ \frac{|(0,\ q_n)|}{|(0,\ q_{n-2})|}\ \approx\ \frac{|(0,\ q_n)|}{|(a_n\,q_n,\,q_{n-2})|}\ =\ \sigma(n,1)\cdot\ldots\cdot\sigma(n,\ a_n)$$

We now show that the various scalings are related, through suitable derivatives of iterates at the critical value. An application of the chain rule will finally yield an interesting recursion relation. These recursion relations were first discovered in section 4 of [10], under the additional assumption that scalings tended to zero. Denote by $\{\,D(n)\,\}$ the sequence of derivatives of iterates at the critical value:

$$D(n)\ =\ Df^n(1)$$

Of particular interest are those derivatives for closest returns. We now present the relations of interest. As remarked before, their proofs are essentially the same as in [10] if one replaces the phrase "essentially linear" to "a priori bounded nonlinearity". As in lemma 4.8 [10] we have:

$$If\ a_n\ >\ 1\ \ D(q_n)\ \approx\ \frac{\nu}{\sigma(n,\ 1)}\ \ (\mathbf{3.5a}):$$

$$For\ i\ =\ 2,..,a_n-1\ \ \sigma(n,\ i)\ \approx\ \sigma(n,i-1)^\nu\ \ (\mathbf{3.5b})$$

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The last relation implies that $\sigma(n, i)$ can be expressed in terms of $\sigma(n, 1)$:

$$\sigma(n, i) \approx \sigma(n, 1)^{\nu^{i-1}} \quad (\mathbf{3.6})$$

As in Theorem 4.6 [10] we have that:

$$if\ a_n\ =\ 1\ \ D(q_n) \approx \frac{\nu^{a_{n-1}}\,\nu}{\sigma(n,\ 1)} \quad (\mathbf{3.7})$$

$$if\ a_n\ >\ 1\ D(q_n) \approx \frac{\nu^{a_{n-1}}\,\nu}{\sigma(n,\ a_n)}\ \Pi_{i=1}^{a_n-1}\,\sigma(n,i)^{\nu-1} \quad (\mathbf{3.8})$$

Equations **3.5 a,b and 3.6,8** imply that when $a_n\ >\ 1$ (but bounded by the type of the rotation number)

$$\sigma(n,\ a_n) \approx \nu^{a_{n-1}}\,\sigma(n,\ 1)^{\nu^{a_n-1}} \quad (\mathbf{3.9})$$

The previous relations imply that every $\sigma(n,\ i)$ can be expressed in terms of $\sigma(n,\ 1)$. Consequently, $D(q_n)$ can be expressed in terms of $\sigma(n,\ 1)$. The chain rule will finally yield a recursion relation between scalings at various levels. As in proposition 4.5 [10] we have that:

$$D(q_{n+1}) \approx D(q_n)^{a_n}\ \Pi_{i=2}^{a_n}\,\frac{Df(\underline{iq_n})}{Df(\underline{q_n})}\,D(q_{n-1})\,\frac{Df(\underline{q_{n+1}})}{Df(\underline{q_{n-1}})}$$

Expressing this relation in terms of $\sigma(n+1,\ 1)$, $\sigma(n,1)$ and $\sigma(n-1,1)$ one obtains the following simple recursion relation.

Proposition 3.1:

$$\sigma(n+1,\ 1)^{\nu^{a_{n+1}}} \approx \nu^p\,\sigma(n,\ 1)^{\frac{1-\nu^{a_n}}{1-\nu}}\,\sigma(n-1,\ 1)$$

The power p only depends on the values of $a_n,\ a_{n-1}$.

Remark: 1. The quantity $\sigma(n,\ 1)^{\nu^{a_n}}$ has a geometric interpretation as:

$$\sigma(n,\ 1)^{\nu^{a_n}}\ \approx\ \frac{|(\underline{1},\ \underline{1\ +\ a_n\ q_n})|}{|(\underline{1},\ \underline{1\ +\ i\ q_{n-2}})|}$$

2. We have that

$$\sigma(n) \approx \sigma(n,1)^{\frac{1-\nu^{a_n}}{1-\nu}}$$

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Proof of the first part of Theorem 2 : If $\nu > 2$ then $\liminf \sigma(n) > 0$

Proof:

By the second part of the last remark, it suffices to show that $\liminf \sigma(n, 1) > 0$. Define the quantity

$$s(n) = -\nu^{a_n} \, ln(\sigma(n, 1))$$

Proposition 3.1 implies that we have the recursion inequality:

$$\left| s(n+1) - \frac{1 - \nu^{-a_n}}{\nu - 1} s(n) - \nu^{-a_{n-1}} s(n-1) \right| \leq bound$$

Here the quantity *bound* only depends on the apriori bounds, the power ν and the type of the rotation number. It now suffices to show that the sequence $\{ s(n) \}$ is bounded.

Define the sequence of vectors $\{ \zeta(n) \}$ as:

$$\zeta(n) = \begin{pmatrix} s(n) \\ s(n-1) \end{pmatrix}$$

and the sequence of matrices $\{ B(n) \}$ as:

$$B(n) = \begin{pmatrix} \frac{1 - \nu^{-a_n}}{\nu - 1} & \nu^{-a_{n-1}} \\ 1 & 0 \end{pmatrix}$$

Then the recursion inequality implies that

$$\| \zeta(n+1) - B(n) \zeta(n) \| \leq bound$$

Here $\|.\|$ denotes the Euclidean distance on the plane.

We study long compositions of these matrices in appendix A. Since $\nu > 2$, lemma A.2 in the appendix implies the existence of an integer N so that for any n, the composition

$$B(n + N) \circ \, ... \, \circ B(n)$$

uniformly contracts the Euclidean metric by a factor less than .8.

Therefore the sequence of lengths $\{ \|\zeta(n)\| \}$ is bounded. Consequently the sequence $\{ s(n) \}$ is bounded and the sequences $\{ \sigma(n, 1) \}$ and $\{ \sigma(n) \}$ are bounded away from zero.
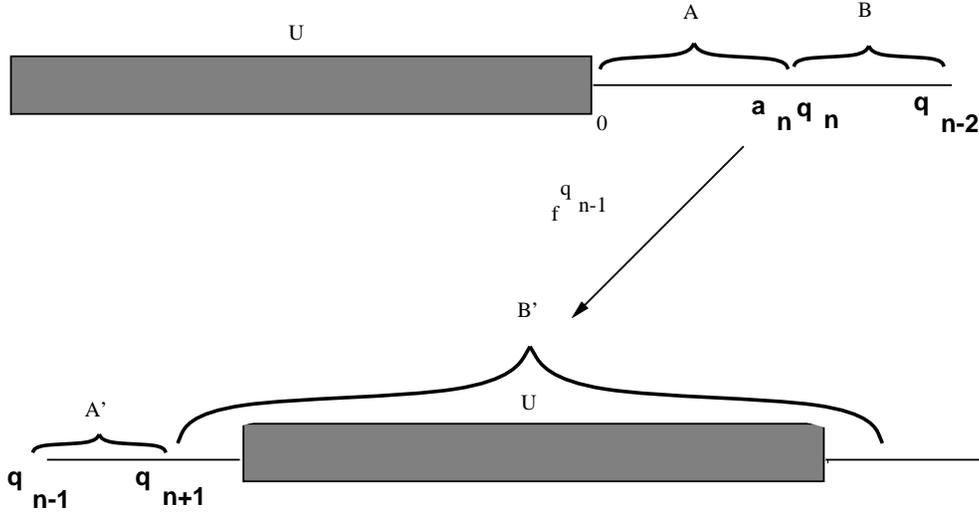
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Figure 1:

$\square$

Proof of the second part of Theorem 2: *If $\nu \leq 2$ then $lim_{n\to\infty}\, \sigma(n) = 0$*

The main idea is that when the power ν is close to 1, the map is actually not very non-linear. Consider the configuration of intervals described in figure 3.1. The intervals are: $A = [\underline{0}, \underline{a_n\, q_n}]$ and $B = [\underline{a_n\, q_n}, \underline{q_{n-2}}]$ Apply q_{n-1} iterates to $A \cup B$. Then A maps to A' and B maps to B'. Note that B' contains U (is asymptotically equal to it) and is therefore large. In particular the ratio of lengths $\frac{|A'|}{|B'|}$ is very small. Therefore, if the q_{n-1}^{th} iterate of f on $A \cup B$ is not very non-linear, one should expect that the initial ratio $\frac{|A|}{|B|}$ is also small. Consequently, the scalings tend to zero. The details for this argument are found in the proof of proposition 3.2 below.

An important observation is that the intervals $\{\, f^i(\partial U, \underline{q_{n-2}})\}_{i=1,..,q_{n-1}}$ do not intersect.

We will need the following lemma.

Lemma 3.1: *Let a,b and z be positive reals:*

$$0 \; < \; a \; < \; b \; < \; z$$

Let S be the map $S : x \rightarrow x^\nu$. Then:

$$\frac{|S(b) \; - \; S(a)|}{|S(z) \; - \; S(b)|} \; \leq \; (\frac{a}{b})^{\nu-1} \frac{|b - a|}{|z - b|}$$

Proof:
Consider the quotient r(z) of ratio's:

$$r(z) \; = \; \frac{b^\nu \; - \; a^\nu}{b \; - \; a} \frac{z \; - \; b}{z^\nu \; - \; b^\nu}$$

Fix a and b and take the supremum over z:

$$sup_{z \in (a, \infty)} \, r(z) \; = \; lim_{z \downarrow b} \, r(z) \; =$$

$$\frac{b^\nu \; - \; a^\nu}{b \; - \; a} \frac{1}{\nu \, b^{\nu-1}} \; \leq \; (\frac{a}{b})^{\nu-1}$$

$$\square$$

Proposition 3.2: For $\nu \leq 2$,

$$lim_{n \rightarrow \infty} \, \sigma(n, \, a_n) \, \sigma(n + 1, \, a_{n+1}) \; = \; 0$$

Proof:
We will find an upper bound for the following quantity, measuring the non-linearity:

$$R_n \; = \; ln \frac{|f^{q_{n-1}}(B)|/|f(B)|}{|f^{q_{n-1}}(A)|/|f(A)|}$$

Decompose the complement of the flat spot U in three overlapping parts (see figure 2):

An interval M $= (x_r, \, x_l)$ in which f has bounded non-linearity.

An interval $right \; = \; (\partial U, \, x_r \, + \, \epsilon)$ to the right of U where f is the composition of $x \rightarrow |x|^\nu$ and a diffeomorphism.

An interval $left \; = \; (x_l \, - \, \epsilon, \, \partial U)$ to the left of U on which f on which f is the composition of the $x \rightarrow -|x|^\nu$ and a diffeomorphism.
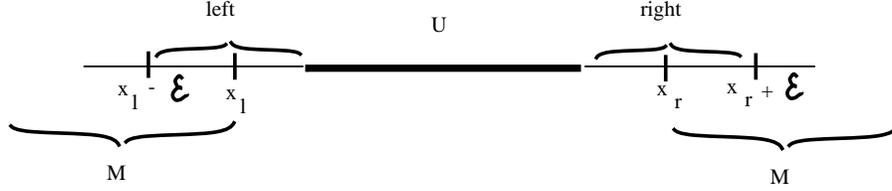
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Figure 2:

We again remark that the interval $f(A \cup B)$ and its $q_{n-1} - 2$ images under f are disjoint and do not land in the interval $[\underline{q_{n-1}}, \underline{q_{n-2}}]$ containing the flat spot U.

Denoting the ith forward image by a subscript i, we then have for $i \in \{1, .., q_{n-1} - 1\}$:

$$R_n = \sum_{A_i \cup B_i \subset M} ln \frac{|f(B_i)|/|B_i|}{|f(A_i)|/|A_i|} +$$

$$\sum_{A_i \cup B_i \subset right} ln \frac{|f(B_i)|/|B_i|}{|f(A_i)|/|A_i|} +$$

$$\sum_{A_i \cup B_i \subset left} ln \frac{|f(B_i)|/|B_i|}{|f(A_i)|/|A_i|} =: \sum_M + \sum_{right} + \sum_{left}$$

In order to avoid over-counting, any couple of intervals which is strictly contained in M, is included in the first sum. We now estimate each of the three contributions separately.

For the intervals that land in M, there are points $\zeta_i \in B_i$ and $\eta_i \in A_i$ such that $(nf(x) = \frac{D^2 f(x)}{Df(x)})$:

$$|\sum_M| = |\sum ln \frac{Df(\zeta_i)}{Df(\eta_i)}| \le$$

$$\sum |\int_{\eta_i}^{\zeta_i} nf(x)\,dx| \le \int_M |nf(x)|\,dx$$

which is bounded by say C_M.

In *right* f is a composition of the power law map $x \to x^\nu$ and a diffeomorphism. Therefore we may assume that $nf(x) > 0$ and equals $\frac{\nu - 1}{x} + O(x)$.

Since the intervals avoid the interval $(\partial U, \underline{q_{n-2}})$, an estimate similar to the one above yields:

$$\sum_{right} \leq \int_{\underline{q_{n-2}}}^{x_r + \epsilon} n f(x)\, dx \leq$$

$$\int_{\underline{q_{n-2}}}^{x_r + \epsilon} \frac{\nu - 1}{x} + O(x)\, dx \leq ln\left(|\underline{q_{n-2}}|^{1-\nu}\right) + C_{right}$$

In $left$, we may assume that the nonlinearity n(f) is negative. This implies that if $(A_i \cup B_i) \subset left$, the ratio of lengths decreases when f is applied. For n very large, there are on the order of $\frac{n}{2}$ times when $(A_i \cup B_i) \subset left$, for which moreover $(-q_{n-k}) \subset B_i$ for some $k < n$. An application of lemma 3.3, (reverse the orientation) yields an estimate of the amount of decrease of the ratio. Since for the indices i under consideration $(-q_{n-k}) \subset B_i$ for some $k < n$, we obtain that the ratio is uniformly decreased. Namely, there exists $\delta \in (0, 1)$ such that:

$$\sum_{left} < \frac{n}{2} ln\, \delta < 0$$

Putting the three estimates together, we obtain that:

$$\frac{|f^{q_{n-1}}(B)|/|f(B)|}{|f^{q_{n-1}}(A)|/|f(A)|} = e^{R_n} \leq \delta^{\frac{n}{2}} e^{C_M + C_{right}} |\underline{q_{n-2}}|^{1-\nu}$$

But

$$\frac{|f^{q_{n-1}}(B)|/|f(B)|}{|f^{q_{n-1}}(A)|/|f(A)|} = \frac{|U|/(|\underline{q_{n-2}}|^\nu - |\underline{a_n\, q_n}|^\nu)}{|\underline{q_{n-1}} - q_{n+1}|/|\underline{a_n\, q_n}|^\nu} \approx$$

$$\frac{|\underline{a_n\, q_n}|^\nu}{(|\underline{q_{n-2}}|^\nu - |\underline{a_n\, q_n}|^\nu)} \frac{|U|}{|\underline{q_{n-1}}|}$$

Since $\frac{|\underline{a_n\, q_n}|}{|\underline{q_{n-2}}|} \approx \sigma(n, a_n)$, the previous implies that there exists a constant K so that

$$\sigma(n, a_n)^\nu \leq K\, \delta^{\frac{n}{2}} |\underline{q_{n-2}}|^{1-\nu} |\underline{q_{n-1}}|$$

Multiplying this inequality by the analogous inequality for $\sigma(n-1, a_{n-1})$ yields:

$$\sigma(n-1, a_{n-1})^\nu \sigma(n, a_n)^\nu \leq$$

$$\frac{K^2}{\delta} \delta^n |\underline{q_{n-2}}|^{2-\nu} |\underline{q_{n-3}}|^{1-\nu} |\underline{q_{n-1}}| < \frac{K^2}{\delta} \delta^n |\underline{q_{n-2}}|^{2-\nu} |\underline{q_{n-3}}|^{2-\nu}$$

This goes to zero whenever $\nu \leq 2$.

$\square$

The proof of the second claim of Theorem 2 is now nearly finished. **Proof:** Fix $\epsilon > 0$. We want to show that when n is large enough, $\sigma(n, 1)$ is less than ϵ. Proposition 3.2 implies that we can choose n large enough so that at least one of the scalings $\sigma(n, a_n)$ and $\sigma(n - 1, a_{n-1})$ is much smaller than ϵ. By choosing n still larger, we can arrange that also one of the scalings $\sigma(n, 1)$ and $\sigma(n - 1, 1)$ is much smaller than ϵ (using equation **(3.6)**). We need to show that $\sigma(n, 1)$ is smaller than ϵ. By the previous we only have to consider the case when we only know that $\sigma(n - 1, 1)$ is very small. Then however, the recursion relation in Proposition 3.1 (applied to n-1) shows that also then $\sigma(n, 1)$ is small. This finishes the proof of the second claim of Theorem 2.

$\square$

We remark that as the scalings tend to zero, the recursion relations in Proposition 3.1 converge to recursion equations. This case was studied in [10].

Appendix A

Fix $\nu > 1$. Let $\{\,b(n)\,\}$ be a sequence of positive numbers which are bounded from above by $\frac{1}{\nu}$. Define the sequence of matrices $\{\,B_\nu(n)\,\}$ as:

$$B_\nu(n) \;=\; \begin{pmatrix} \frac{1-b(n)}{\nu-1} & b(n-1) \\ 1 & 0 \end{pmatrix}$$

Lemma A.1: *Assume that $\nu \geq 2$. Then the sequence $\{\,B_\nu^{\circ\,n}\,\}$ defined as:*

$$B_\nu^{\circ\,n} \;=\; B_\nu(n) \circ \cdots \circ B_\nu(1)$$

is relatively compact.

Proof:

Each $B_\nu^{\circ\,n}$ is non-negative and we have that $B_2^{\circ\,n} - B_\nu^{\circ\,n}$ is non-negative also. It therefore suffices to consider the case when $\nu = 2$.

$B_2^{\circ\,n}$ can be written in the form:

$$B_2^{\circ\,n} \;=\; \begin{pmatrix} \alpha(n) & \beta(n) \\ \alpha(n-1) & \beta(n-1) \end{pmatrix}$$

One proves by induction that:

$$\alpha(n) \;=\; 1 \,-\, b(n) \,+\, b(n)\,b(n-1)\cdots \,+\, (-1)^n\, b(n).b(n-1)\cdots b(1)$$

$$\beta(n) \;=\; b(0)\,(1 \,-\, b(n) \,+\, b(n)\,b(n-1)\cdots + (-1)^{n-1}\, b(n).b(n-1)\cdots b(2))$$

Therefore $\alpha(n) \leq 1 + \frac{1}{2}\cdots + \frac{1}{2^n} \leq 2$ and $\beta(n) \leq 1$.

$\square$

Lemma A.2: *When $\nu > 2$, there exists an integer N only depending on the bound $\frac{1}{\nu}$ for each $b(n)$ so that when $n \geq N$, each $B_\nu^{\circ\,n}$ contracts the Euclidean metric on the plane by a factor smaller than .8.*

Proof:

Each $B_\nu^{\circ\,n}$ can be expressed in the form:

$$B_\nu^{\circ\,n} \;=\; \begin{pmatrix} \alpha(n,\,\nu) & \beta(n,\,\nu) \\ \alpha(n-1,\,\nu) & \beta(n-1,\nu) \end{pmatrix}$$

Here $\alpha(n, \nu)$ and $\beta(n, \nu)$ are polynomials of degree n in the variable $\frac{1}{\nu-1}$:

$$\alpha(n, \nu) = \sum \alpha_i(n) \frac{1}{\nu - 1}^i$$

$$\beta(n, \nu) = \sum \beta_i(n) \frac{1}{\nu - 1}^i$$

The coefficients $\alpha_i(n)$ and $\beta_i(n)$ only depend on the sequence b(n). We have (Lemma A.1) the estimate:

$$\alpha(n, \nu) \leq \alpha(n, 2) \leq 2$$

$$\beta(n, \nu) \leq \beta(n, 2) \leq 1$$

Fix N_1 so that

$$\frac{1}{\nu - 1}^{N_1} \leq \frac{1}{10}$$

One proves by induction that as n tends to infinity, the finitely many coefficients

$$\{ \alpha_0(n) \cdots \alpha_{N_1}(n),\ \beta_0(n) \cdots \beta_{N_1}(n)\}$$

tend to zero exponentially fast. Consequently, there exists N so that when n is bigger than N, each of these coefficients are all smaller than $\frac{.2}{N_1}$.

Therefore: for $n \geq N$

$$\alpha(n, \nu) \leq \frac{1}{10} \sum_{i=N_1+1}^{n} \alpha_i(n) + N_1 \frac{.2}{N_1} \leq .4$$

and $\beta(n, \nu) \leq .4$. Consequently all the entries in $B_\nu^{\circ\,n}$ are less than or equal to .4. Therefore the Euclidean metric is contracted by a factor less than .8.

$\square$

Appendix B

Description of the procedure. A numerical experiment was performed in order to check Conjecture 1 of the introduction. To this end, a family of almost smooth maps with a flat spot was considered given by the formula

$$x \to (\frac{x-1}{b})^3 (1 - 3\frac{x+b-1}{b} + 6(\frac{x+b-1}{b})^2 - 10(\frac{x+b-1}{b})^3 + (x-1)^3)$$

$$+t \pmod 1 .$$

These are symmetric maps with the critical exponent $(3,3)$. The parameter b controls the length of the flat spot, while t must be adjusted to get the desired rotation number.

In our experiment, b was chosen to be 0.5, which corresponds to the flat spot of the same length. By binary search, a value t_{Au} was found which approximated the parameter value corresponding to the golden mean rotation number $\frac{\sqrt{5}-1}{2}$. Next, the forward orbit of the flat spot was studied and the results are given in the table below.

It should finally be noted that the experiment presents serious numerical difficulties as nearest returns to the critical value tend to 0 very quickly so that the double precision is insufficient when one wants to see more than 15 nearest returns. This problem was avoided, at a considerable expense of computing time, by the use of an experimental package which allows for floating-point calculations to be carried out with arbitrarily prescribed precision.

Results. Below the results are presented. The column y_i is defined by $y_i := \mathrm{dist}(\underline{0}, \underline{q_i})$. The μ_i is given by $\mu := \frac{\sigma(i+2)-\sigma(i+1)}{\sigma(i+1)-\sigma(i)}$.

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n	y_n	$\sigma(n)$	μ_n
10	$3.010 \cdot 10^{-3}$	.2637	.5869
11	$1.544 \cdot 10^{-3}$	.2450	1.683
12	$.7044 \cdot 10^{-3}$	.2340	.4527
13	$.3328 \cdot 10^{-3}$	.2156	1.775
14	$.1460 \cdot 10^{-3}$	.2072	.5079
15	$64.04 \cdot 10^{-6}$	.1924	1.285
16	$26.99 \cdot 10^{-6}$	.1849	.6396
17	$11.22 \cdot 10^{-6}$	.1752	.9773
18	$4.562 \cdot 10^{-6}$	.1690	.7485
19	$1.829 \cdot 10^{-6}$	.1630	.8634
20	$.7229 \cdot 10^{-6}$	.1585	.8015
21	$.2826 \cdot 10^{-6}$	.1546	.8307
22	$.1095 \cdot 10^{-6}$	.1514	.8191
23	$42.07 \cdot 10^{-9}$	.1488	.8243
24	$16.06 \cdot 10^{-9}$	.1467	.8241
25	$6.097 \cdot 10^{-9}$	.1449	.8172
26	$2.305 \cdot 10^{-9}$	.1435	.9982
27	$1.070 \cdot 10^{-9}$	.1423	-2.54
28	$.8677 \cdot 10^{-9}$	.1411	-25.9
29	$.3252 \cdot 10^{-9}$	.1441	-10.9

Interpretation. The most interesting is the third column which shows the scalings. They seem to decrease monotonically. The last column attempts to measure the exponential rate at which the differences between consecutive scalings change. Here, the last three numbers are obviously out of line which, however, is explained by the fact that t_{Au} is just an approximation of the parameter value which generates the golden mean dynamics. Other than that, the numbers from the last column seems to be firmly below 1, which indicates geometric convergence. If 0.82 is accepted as the limit rate, this projects to the scalings limit of about 0.137 which consistent with rough theoretical estimates of [10].

Thus, we conclude that Conjecture 1 has a numerical confirmation.

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