OPTICAL HAMILTONIANS AND SYMPLECTIC TWIST MAPS

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Abstract: This paper concentrates on optical Hamiltonian systems of $T^*\mathbb{T}^n$, i.e. those for which $H_{\boldsymbol{pp}}$ is a positive definite matrix, and their relationship with symplectic twist maps. We present theorems of decomposition by symplectic twist maps and existence of periodic orbits for these systems. The novelty of these results resides in the fact that no explicit asymptotic condition is imposed on the system. We also present a theorem of suspension by Hamiltonian systems for the class of symplectic twist map that emerges in our study. Finally, we extend our results to manifolds of negative curvature.

1. Introduction

In a previous paper [G91b], the author explained how symplectic twist maps could be used to decompose Hamiltonian systems on the cotangent bundle of a compact manifold M^n, thus deriving a discrete variational approach to the search of periodic orbits for such systems. This method can be seen as a generalization of the so called "method of broken geodesics" in differential geometry. A similar method was introduced by Marc Chaperon for Hamiltonian systems [Ch84]. Although our method is very similar to his, it is in fact even more akin to the original method (see e.g. [Mi69].)

In [G91b], we put a boundary condition on the Hamiltonian, however: it had to equal the metric Hamiltonian $H_0(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}\|\boldsymbol{p}\|^2$ on a fixed level set $\{H_0 = K\}$. It could be anything inside $\{H_0 < K\}$, including time dependent. The result (see [G91b,92b]) was then the existence of at least $cl(M)$ (or $sb(M)$ if all nondegenerate) contractible periodic orbits inside $\{H_0 < K\}$ for such systems. Other results for non contractible orbits were obtained

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if M supports a metric with negative curvature, or for $\mathbb{T}^n$ (comparable to Theorem **7.4** in this paper.)

Here we swap the boundary condition (and the compactness of $\{H_0 \leq K\}$) for a convexity condition which gives a Hamiltonian system with a priori no compact invariant set: the systems we study here are *optical*, in the sense that the second derivative in the fiber direction, $H_{\boldsymbol{pp}}$ is positive definite .

The main result of this paper is:

Theorem 5.2 *Let $H(\boldsymbol{q}, \boldsymbol{p}, t) = H_t(\boldsymbol{z})$ be a twice differentiable function on $T^*\mathbb{T}^n \times \mathbb{R}$ satisfying the following:*
(1) $\sup \|\nabla^2 H_t\| < K$
(2) The matrices $H_{\boldsymbol{pp}}(\boldsymbol{z}, t)$ are positive definite and $C < \|H_{\boldsymbol{pp}}\| < C^{-1}$ for some C.

Then the time 1 map of the associated Hamiltonian flow has at least $n+1$ distinct periodic orbits of type $\boldsymbol{m}, d$, for each prime $\boldsymbol{m}, d \in \mathbb{Z}^n \times \mathbb{Z}$, and at least 2^n in the generic case when they are all non degenerate.

(An $\boldsymbol{m}, d$–orbit is one for which the d^{th} iterate of each point of the orbit is a translation by $(\boldsymbol{m}, 0)$ of this point, in the covering space $\mathbb{R}^{2n}$ of $T^*\mathbb{T}^n$.)

Earlier results on the existence and multiplicity of periodic orbits can be found for Hamiltonian systems or symplectic twist maps of $T^*\mathbb{T}^n$ in [BK87], [Che92], [CZ83],[Fe89], [G91a], [J91]. The two first are perturbative results, i.e. for systems close to integrable ones. The four latter works are global in that sense, and do not require the system to be optical, but instead require some asymptotic condition on the first derivative of H (or of the time 1 map F). Only the two last works consider homotopically nontrivial orbits.

Note also that, via the Legendre transformation, Theorem **5.2** applies to Lagrangian systems whose Lagrangian function satisfies the same conditions as H in our theorem (it is not hard to see that these conditions translate under the Legendre transformation.) Hence Theorem **5.2** extends some existing theorems for such systems (see, e.g., [MW89], Theorem **9.3**.)

We start (Sections 2 and 3) with some background on symplectic twist maps. In Section 4, we give the proof of a theorem of existence and multiplicity of periodic orbits for compositions of symplectic twist maps with a convexity condition (Theorem **4.3**.) A proof of such a theorem was given in [KM89]. Unfortunately, the multiplicity part of their proof is wrong. We reproduce here their proof of existence of a minimum, and present a new proof of the multiplicity.

In Section 5, we prove Theorem **5.2**. This results derives from the former theorem on symplectic twist maps, and a decomposition technique. The resulting discrete variational method is interpreted as a method of broken geodesics.

In Section 6 we show that a symplectic twist map with the convexity condition can be suspended by a Hamiltonian (our proof does not force the Hamiltonian to be optical, unfortunately), extending a result of Moser [Mo86] and a remark about it by Bialy and Polterovitch [BP92].

In section 7, we indicate how to extend Theorems **4.3** and **5.2** to the cotangent bundle of a manifold of negative curvature.

It is very likely that, with a little care, these techniques could extend to optical Hamiltonians on the cotangent bundle of any compact manifolds.

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2. Symplectic Twist Maps of $\mathbb{T}^n \times \mathbb{R}^n$

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n–dimensional torus. Its cotangent bundle $T^*\mathbb{T}^n \overset{\pi}{\to} \mathbb{T}^n$ is trivial: $T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$, the cartesian product of n cylinders. We give it the coordinates $(\boldsymbol{q}, \boldsymbol{p})$ in which the symplectic structure is

$$\Omega = d\boldsymbol{q} \wedge d\boldsymbol{p} = \sum_{k=1}^{n} dq_k \wedge dp_k.$$

As in any cotangent bundle, Ω is exact: $\Omega = -d\lambda$, where $\lambda = \boldsymbol{p}d\boldsymbol{q}$.

It is useful to work in the covering space $\mathbb{R}^{2n} = \tilde{\mathbb{T}}^n \times \mathbb{R}^n$ of $T^*\mathbb{T}^n$, with projection $pr : \mathbb{R}^{2n} \to \mathbb{T}^n \times \mathbb{R}^n$.

Of course, pr is an exact symplectic map (see Definition 2.1) , as we have $pr^*\boldsymbol{p}d\boldsymbol{q} - \boldsymbol{p}d\boldsymbol{q} = 0$.

The group $\mathbb{Z}^n$ of deck or covering transformations is the set of integer vector translation in $\mathbb{R}^{2n}$ of the form:

$$\tau_{\boldsymbol{m}}.(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{p}), \quad \boldsymbol{m} \in \mathbb{Z}^n.$$

A lift of a map $F : T^*\mathbb{T}^n \to T^*\mathbb{T}^n$ is a map $\tilde{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $pr \circ \tilde{F} = F \circ pr$. Since pr is a local, symplectic diffeomorphism, $\tilde{F}$ is symplectic if and only if F is. On the other hand, $\tilde{F}$ will always be exact symplectic when it is symplectic, which is not the case for F, as the example $(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{q}, \boldsymbol{p} + \boldsymbol{p}_0)$ shows.

We will fix the lift of a map F once and for all, remembering that two lifts only differ by a composition by some $\tau_{\boldsymbol{m}}$.

Definition 2.1 A map F of $T^*\mathbb{T}^n$ is called a **symplectic twist map** if
(1) F is homotopic to Id.
(2) F is **exact symplectic**: $F^*\boldsymbol{p}d\boldsymbol{q} - \boldsymbol{p}d\boldsymbol{q} = dh$ for some $h : T^*\mathbb{T}^n \to \mathbb{R}$.
(3) **(Twist Condition)** If $\tilde{F}(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})$ is a lift of F then the map $\boldsymbol{p} \to \boldsymbol{Q}(\boldsymbol{q}_0, \boldsymbol{p})$ is a diffeomorphism of $\mathbb{R}^n$ for all $\boldsymbol{q}_0$, and thus the map

$$\psi: \ (\boldsymbol{q},\boldsymbol{p}) \to (\boldsymbol{q},\boldsymbol{Q})$$

is a diffeomorphism (change of coordinates) of $\mathbb{R}^{2n}$.

Comments 2.2

1 .The twist condition (3) implies the more familiar looking:

$$\det \ \partial \boldsymbol{Q}/\partial \boldsymbol{p} \neq 0.$$

It also implies that :

$$\tilde{F}^* \boldsymbol{p} d\boldsymbol{q} - \boldsymbol{p} d\boldsymbol{q} = dS(\boldsymbol{q},\boldsymbol{Q}) \tag{2.1}$$

where S is the lift of h written in the $(\boldsymbol{q},\boldsymbol{Q})$ coordinates : $S = \tilde{h} \circ \psi^{-1}$, with $\tilde{h} = h \circ pr$. Equivalently, we can write:

$$\begin{aligned} \boldsymbol{p} &= -\, \partial_1 S(\boldsymbol{q},\boldsymbol{Q}) \\ \boldsymbol{P} &= \partial_2 S(\boldsymbol{q},\boldsymbol{Q}). \end{aligned}$$

$S(\boldsymbol{q},\boldsymbol{Q})$ is called a **generating function** for $\tilde{F}$.

2 . Condition (1) is equivalent to the fact that on *any* lift $\tilde{F}$ of F:

$$\tilde{F} \circ \tau_{\boldsymbol{m}} = \tau_{\boldsymbol{m}} \circ \tilde{F}, \ \text{ i.e. } \quad \tilde{F}(\boldsymbol{q}+\boldsymbol{m},\boldsymbol{p}) = \tilde{F}(\boldsymbol{q},\boldsymbol{p}) + (\boldsymbol{m},0). \tag{2.2}$$

Example 2.3 The family of maps

$$F_s(\boldsymbol{q},\boldsymbol{p}) = \left(\boldsymbol{q} + A\left(\boldsymbol{p} - \nabla V_s(\boldsymbol{q})\right), \boldsymbol{p} - \nabla V_s(\boldsymbol{q})\right)$$

where A is a nondegenerate symmetric matrix, V_s is a C^2 function on $\mathbb{T}^n$ is called the **standard family**. Usually, $V_0 \equiv 0$. The generating function for F_s is given by:

$$S_s(\boldsymbol{q},\boldsymbol{Q}) = S_0(\boldsymbol{q},\boldsymbol{Q}) + V_s(\boldsymbol{q}).$$

Where

$$S_0(\boldsymbol{q},\boldsymbol{Q}) = \frac{1}{2}\langle A^{-1}(\boldsymbol{Q}-\boldsymbol{q}), (\boldsymbol{Q}-\boldsymbol{q})\rangle$$

is the generating function of the **completely integrable map**:

$$F_0 : (\boldsymbol{q},\boldsymbol{p}) \to (\boldsymbol{q}+A\boldsymbol{p},\boldsymbol{p}), \quad A^t = A, \ \det A \neq 0.$$

(The term “completely integrable” comes from the fact that F_0 conserves each torus $\boldsymbol{p} = \boldsymbol{p}_0$, on which it acts as a rigid “translation”.)

This general standard family includes the classical standard family of monotone twist maps of the annulus where $A = 1$ and

$$V_s(q) = \frac{s}{4\pi^2}\cos(2\pi q)$$

and also the Froeschlé family on $\mathbb{T}^2 \times \mathbb{R}^2$ with $A = Id$ and

$$V_s(q_1,q_2) = \frac{1}{(2\pi)^2}\{K_1 cos(2\pi q_1) + K_2 cos(2\pi q_2) + \lambda cos 2\pi(q_1+q_2)\}.$$

In this case the parameter $s = (K_1, K_2, \lambda) \in \mathbb{R}^3$. (See e.g. [KM89].) The standard map can be interpolated by an optical Hamiltonian (see Section 7).

As this paper will suggest, many examples of symplectic twist maps can be derived from Hamiltonian systems, and used to understand these systems.

The following results are also helpful to construct symplectic twist maps. Their proofs can be found in [G93] (see also [H89] for Corollary **2.7**.)

Proposition 2.4 *There is a homeomorphism between the set of lifts $\tilde{F}$ of C^1 symplectic twist maps of $T^*\mathbb{T}^n$ and the set of C^2 real valued functions S on $\mathbb{R}^{2n}$ satisfying the following:*
(a) $S(\boldsymbol{q}+\boldsymbol{m}, \boldsymbol{Q}+\boldsymbol{m}) = S(\boldsymbol{q},\boldsymbol{Q}), \quad \forall \boldsymbol{m} \in \mathbb{Z}^n$
(b) The maps: $\boldsymbol{q} \to \partial_2 S(\boldsymbol{q},\boldsymbol{Q}_0)$ and $\boldsymbol{Q} \to \partial_1 S(\boldsymbol{q}_0,\boldsymbol{Q})$ are diffeomorphisms of $\mathbb{R}^n$ for any $\boldsymbol{Q}_0$ and $\boldsymbol{q}_0$ respectively.
(c) $S(0,0) = 0$.
This correspondence is given by:

$$\tilde{F}(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{Q},\boldsymbol{P}) \Leftrightarrow \begin{cases} \boldsymbol{p} = & -\partial_1 S(\boldsymbol{q},\boldsymbol{Q}) \\ \boldsymbol{P} = & \partial_2 S(\boldsymbol{q},\boldsymbol{Q}). \end{cases} \tag{2.3}$$

Lemma 2.5 *Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be a local diffeomorphism at each point, such that:*

$$\sup_{x\in\mathbb{R}^N} \left\|(Df_x)^{-1}\right\| = K < \infty.$$

Then f is a diffeomorphism of $\mathbb{R}^N$.

Corollary 2.6 *Let $S : \quad \mathbb{R}^{2n} \to \mathbb{R}$ be a C^2 function satisfying:*

$$\begin{aligned} &\det \partial_{12} S \neq 0 \\ &\sup_{(\boldsymbol{q},\boldsymbol{Q})\in\mathbb{R}^{2n}} \left\|(\partial_{12} S(\boldsymbol{q},\boldsymbol{Q}))^{-1}\right\| = K < \infty. \end{aligned} \tag{2.4}$$

Then the maps: $\boldsymbol{q} \to \partial_2 S(\boldsymbol{q},\boldsymbol{Q}_0)$ and $\boldsymbol{Q} \to \partial_1 S(\boldsymbol{q}_0,\boldsymbol{Q})$ are diffeomorphisms of $\mathbb{R}^n$ for any $\boldsymbol{Q}_0$ and $\boldsymbol{q}_0$ respectively, and thus S generates an exact symplectic map of $\mathbb{R}^{2n}$.

Thus , if S satisfies (2.4) , as well as the periodicity condition *(a)* of Proposition **2.4** , it generates a symplectic twist map.

Corollary 2.7 *Let F be an exact symplectic map of $T^*\mathbb{T}^n$, homotopic to Id. Let $\tilde{F}(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{Q},\boldsymbol{P})$ be a lift of F. Suppose that*

$$(2.5) \qquad \sup_{\boldsymbol{z}\in\mathbb{R}^{2n}} \left\| \left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}} \right)_{\boldsymbol{z}}^{-1} \right\| < \infty.$$

Then $\tilde{F}$ is a symplectic twist map.

3. The Variational Setting

As in the classical case of twist map ($n = 1$), the generating function of a symplectic twist map is the key to the variational setting that these maps induce.

Proposition 3.1 (Critical Action Principle) *Let $F_1, \ldots, F_N$ be symplectic twist maps of $T^*\mathbb{T}^n$, and let $\tilde{F}_k$ be a lift of F_k, with generating function S_k. The sequence $\{(\boldsymbol{q}_k, \boldsymbol{p}_k)\}_{k\in\mathbb{Z}}$ is an orbit under the successive $\tilde{F}_k$'s (i.e. $\{(\boldsymbol{q}_{k+1}, \boldsymbol{p}_{k+1}) = \tilde{F}_k(\boldsymbol{q}_k, \boldsymbol{p}_k)\}_{k\in\mathbb{Z}}$, with $\tilde{F}_{k+N} = \tilde{F}_k,\ S_{k+N} = S_k$) if and only if the sequence $\{\boldsymbol{q}_k\}_{k\in\mathbb{Z}}$ in $(\mathbb{R}^n)^{\mathbb{Z}}$ satisfies:*

$$(3.1) \qquad \partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) + \partial_2 S_{k-1}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) = 0, \quad \forall k \in \mathbb{Z}.$$

The correspondence is given by: $\boldsymbol{p}_k = -\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$.

Equation (3.1) can be interpreted formally as:

$$\nabla W(\overline{\boldsymbol{q}}) = 0 \qquad \text{with}$$
$$W(\overline{\boldsymbol{q}}) = \sum_{-\infty}^{\infty} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}).$$

This interpretation makes mathematical sense when one is concerned with periodic orbits of a symplectic twist map F:

Definition 3.2 A point $(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^{2n}$ is called a $\boldsymbol{m}, d$–**point** for the lift $\tilde{F}$ of F if $\tilde{F}^d(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{p})$, where $\boldsymbol{m} \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$.

Let $\tilde{F} = \tilde{F}_N \circ \ldots \circ \tilde{F}_1$. The appropriate space of sequences in which to look for critical points corresponding to $\boldsymbol{m}, d$–points of $\tilde{F}$ is:

$$X^* = \{\overline{\boldsymbol{q}} \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \boldsymbol{q}_{k+dN} = \boldsymbol{q}_k + \boldsymbol{m}\}$$

which is isomorphic to $(\mathbb{R}^n)^{dN}$: the terms $(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN})$ determine a whole sequence in X^*.

To find a sequence satisfying (3.1) in X^* is equivalent to finding $\overline{\boldsymbol{q}} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN})$ which is a critical point for the function:

$$W(\overline{\boldsymbol{q}}) = \sum_{k=1}^{dN} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}),$$

in which we set $\boldsymbol{q}_{dN+1} = \boldsymbol{q}_1 + \boldsymbol{m}$. To see this, write :

$$\begin{aligned} \boldsymbol{p}_k &= -\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}), \\ \boldsymbol{P}_k &= \partial_2 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}). \end{aligned}$$

Then $F_k(\boldsymbol{q}_k, \boldsymbol{p}_k) = (\boldsymbol{Q}_k, \boldsymbol{p}_k)$ and with this notation, the proof of Proposition **3.1** (for $\boldsymbol{m}, d$–points) reduces to the suggestive:

$$\nabla W(\overline{\boldsymbol{q}}) = \sum_{k=1}^{dN} (\boldsymbol{P}_{k-1} - \boldsymbol{p}_k) d\boldsymbol{q}_k.$$

A little more care must be taken in order to let the topology of $\mathbb{T}^n$ play a role. Note that because of the periodicity of S (*(a)* in Proposition **2.4**), W is invariant under the $\mathbb{Z}^n$ action on X^*:

$$\tau_{\boldsymbol{m}}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN}) = (\boldsymbol{q}_1 + \boldsymbol{m}, \ldots, \boldsymbol{q}_{dN} + \boldsymbol{m}).$$

Moreover, if we want our variational approach to count $\boldsymbol{m}, d$–orbits, and not the individual $\boldsymbol{m}, d$–points in each orbit, we should use the fact that W is also invariant under the N–shift map:

$$\sigma\{\boldsymbol{q}_k\} = \{\boldsymbol{q}_{k+N}\}.$$

Let :

$$X = X^*/\sigma, \tau$$

be the quotient of X^* by these two actions. We continue to call W the function induced by W on the quotient X.

One can show ([BK87], Proposition 1 or [G91a]) that X is the total space of a fiber bundle over $\mathbb{T}^n$, and that the projection map $X^* \to X$ is a covering map. (One makes the change of variables:

$$\boldsymbol{v} = \frac{1}{d} \sum_{1}^{dN} \boldsymbol{q}_k$$

$$\boldsymbol{t}_k = \boldsymbol{q}_k - \boldsymbol{q}_{k-1} - \boldsymbol{m}/d$$

in which $\boldsymbol{v}$ is the base coordinate, $\boldsymbol{t}$ the fiber.) In particular, each critical point of W on X corresponds to an infinite lattice of critical points of W on X^*. Whereas the original variational problem $\nabla W = 0$ on X^* would pick up the (infinitely many) $\boldsymbol{m}, d$*–points* of the lift $\tilde{F}$ of F, when we restrict it to X it exactly gives $\boldsymbol{m}, d$*–orbits* of F.

4. Periodic Orbits and the Convexity Condition

Let $\tilde{F}(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{Q},\boldsymbol{P})$ be the lift of a symplectic twist map of $T^*\mathbb{T}^n$, and $S(\boldsymbol{q},\boldsymbol{Q})$ its generating function. In this section, we impose the:

4.1 Convexity Condition There is a positive a such that:

$$\langle \partial_{12}S(\boldsymbol{q},\boldsymbol{Q}).\boldsymbol{v},\boldsymbol{v}\rangle \leq -a\,\|\boldsymbol{v}\|^2 .$$

uniformly in $(\boldsymbol{q},\boldsymbol{Q})$.

Remark 4.2 . Note that:

$$\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}}(\boldsymbol{q},\boldsymbol{p}) = -\left(\partial_{12}S(\boldsymbol{q},\boldsymbol{Q})\right)^{-1},$$

as can easily be derived by implicit differentiation of $\boldsymbol{p} = -\partial_1 S(\boldsymbol{q},\boldsymbol{Q})$. The convexity condition **4.1** thus translates to:

$$\left\langle \left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}}\right)^{-1} \boldsymbol{v},\boldsymbol{v}\right\rangle \geq a\,\|\boldsymbol{v}\|^2, \quad \forall \boldsymbol{v}\in\mathbb{R}^n. \tag{4.1}$$

uniformly in $(\boldsymbol{q},\boldsymbol{p})$. This means that F has bounded twist. MacKay, Meiss and Stark [MMS89] imposed this condition on their definition of symplectic twist maps, a terminology that we have taken from them.

Theorem 4.3 *Let $F = F_N \circ \ldots \circ F_1$ be a finite composition of symplectic twist maps F_k of $T^*\mathbb{T}^n$ each satisfying the convexity condition. Then, for each prime $(\boldsymbol{m},d)\in\mathbb{Z}^n\times\mathbb{Z}$, F has at least $n+1$ distinct periodic orbits of type $\boldsymbol{m},d$. It has at least 2^n of them when they are all nondegenerate.*

By a prime pair $\boldsymbol{m},d$ we mean that at least one of the components m_k of $\boldsymbol{m}$ is prime with d.

Remark 4.4 . One can show ([G91a]) that an $\boldsymbol{m},d$–point of F is nondegenerate if and only if the sequence $\overline{\boldsymbol{q}}$ of X^* it corresponds to is a nondegenerate critical point for W. The hypothesis that F has only nondegenerate $\boldsymbol{m},d$–points is thus equivalent to the one that W is a Morse function. Furthermore, this is a generic condition on the space of symplectic twist maps [G92a]. Note also that an $\boldsymbol{m},d$ point is also an $k\boldsymbol{m},kd$–point for all $k\in\mathbb{N}$. The reason for restricting ourselves to prime $\boldsymbol{m},d$ is that if we were to look for $k\boldsymbol{m},kd$ orbits, we would also find the prescribed number of them, but with no guarantee that they would be any different from the $\boldsymbol{m},d$ orbits already found.

Proof. The first part of the proof, due to Kook and Meiss, [KM89] consists in proving that the function W is proper, and hence has a minimum.

The following lemma and corollary were proven in [MMS89], and [KM89].

Lemma 4.5 *Let S be the generating function of a symplectic twist map satisfying the convexity condition* **4.1** *. Then there is an α and positive β and γ such that:*

$$S(\boldsymbol{q},\boldsymbol{Q}) \geq \alpha - \beta \|\boldsymbol{q}-\boldsymbol{Q}\| + \gamma \|\boldsymbol{q}-\boldsymbol{Q}\|^2 . \tag{4.2}$$

Proof. We can write:

$$S(\boldsymbol{q},\boldsymbol{Q}) = S(\boldsymbol{q},\boldsymbol{q}) + \int_0^1 \partial_2 S(\boldsymbol{q},\boldsymbol{Q}_s).(\boldsymbol{Q}-\boldsymbol{q})ds,$$

where $\boldsymbol{Q}_s = (1-s)\boldsymbol{q} + s\boldsymbol{Q}$. Applying the same process to $\partial_2 S$, we get:

$$\begin{aligned} S(\boldsymbol{q},\boldsymbol{Q}) &= S(\boldsymbol{q},\boldsymbol{q}) + \int_0^1 \partial_2 S(\boldsymbol{Q}_s,\boldsymbol{Q}_s).(\boldsymbol{Q}-\boldsymbol{q})ds \\ &\quad - \int_0^1 ds \int_0^1 \langle \partial_{12} S(\boldsymbol{Q}_r,\boldsymbol{Q}_s).(\boldsymbol{Q}-\boldsymbol{q}),(\boldsymbol{Q}-\boldsymbol{q})\rangle dr \\ &\geq \alpha - \beta \|\boldsymbol{Q}-\boldsymbol{q}\| + \gamma \|\boldsymbol{Q}-\boldsymbol{q}\|^2 , \end{aligned}$$

where $\alpha = \min_{\mathbb{T}^n} S(\boldsymbol{q},\boldsymbol{q})$, $\beta = \max_{\mathbb{T}^n} \|\partial_2 S(\boldsymbol{q},\boldsymbol{q})\|$ and $\gamma = \frac{a}{2}$. □

Corollary 4.6 *For F as in Theorem* **4.4** *, there is a minimum for W (and hence an $\boldsymbol{m},d$–point for F.)*

Proof. Equation (4.2) as applied to each S_k implies that S_k has a lower bound, thus W does as well. We have to prove that this lower bound is not attained at infinity, i.e., that W is a proper map.

The set $\{(\boldsymbol{q},\boldsymbol{Q}) \in (\mathbb{R}^n \times \mathbb{R}^n)/\mathbb{Z}^n \mid S(\boldsymbol{q},\boldsymbol{Q}) \leq C\}$ is compact since (4.2) implies that $S \leq C$ corresponds to bounded $\|\boldsymbol{q}-\boldsymbol{Q}\|$. Likewise the set

$$\mathcal{S} = \{\overline{\boldsymbol{q}} \in X \mid W(\overline{\boldsymbol{q}}) \leq C\}$$

is compact. Hence W must have a minimum in the interior of $\mathcal{S}$, for C big enough. This point is a critical point. □

Remark 4.7 . We have thus found at least one $\boldsymbol{m},d$–orbit corresponding to a minimum of W. The reader should be aware that, unlike the 1 degree of freedom case, this does not imply that the orbit is a minimum in the sense of Aubry (see [H89].)

We now turn to the proof of existence of at least $n+1$ distinct orbits of type $\boldsymbol{m},d$, and 2^n when they are all nondegenerate.

Remember that X is a bundle over $\mathbb{T}^n$. Let $\Sigma \cong \mathbb{T}^n$ be its zero section. Let $K = \sup_\Sigma W(\overline{\boldsymbol{q}})$. Trivially, we have:

$$\Sigma \subset W^K \stackrel{\text{def}}{=} \{\overline{\boldsymbol{q}} \in X \mid W \leq K\}$$

(since W is proper, for almost every K, W^K is a compact manifold with boundary, by Sard's Theorem.) From this we get the commutative diagram:

$$\begin{array}{ccc} H_*(\Sigma) & \xrightarrow{k_*} & H_*(X) \\ i_* \searrow & & \nearrow j_* \\ & H_*(W^K) & \end{array} \tag{4.3}$$

where i, j, k are all inclusion maps. But $k_* = Id$ since Σ and X have the same homotopy type. Hence i_* must be injective.

If all the $\boldsymbol{m}, d$–points are nondegenerate, W is a Morse function (a generic situation) and by [Mi69], §3, W^K has the homotopy type of a finite CW complex, with one cell of dimension k for each critical point of index k in W^K. In particular, we have the following Morse inequalities:

$$\#\{\text{critical points of index } k\} \geq b_k$$

where b_k is the kth Betti number of W^K, $b_k > \binom{n}{k}$ in our case since $H_*(\mathbb{T}^n) \hookrightarrow H_*(W^K)$. Hence there are at least 2^n critical points in this nondegenerate case.

If W is not a Morse function, rewrite the diagram (4.3) , but in Cohomology, reversing the arrows. Since $k^* = Id$, j^* must be injective this time. We know that the cup length $cl(X) = cl(\mathbb{T}^n) = n + 1$. This exactly means that there are n cohomology classes $\alpha_1, \ldots, \alpha_n$ in $H^1(X)$ such that $\alpha_1 \cup \ldots \cup \alpha_n \neq 0$. Since j^* is injective, $j^*\alpha_1 \cup \ldots \cup j^*\alpha_n \neq 0$ and thus $cl(W^K) \geq n + 1$. W^K being compact, and invariant under the gradient flow, Lusternik-Schnirelman theory implies that W has at least $n+1$ critical points in W^K (The proof of Theorème 1 in CH.2 §19 of [DNF87], which is for compact manifolds without boundaries can easily be adapted to this case.) $\square$

5. Periodic Orbits for Optical Hamiltonian Systems

Assumption 5.1 $H(\boldsymbol{q}, \boldsymbol{p}, t) = H_t(\boldsymbol{z})$ is a twice differentiable function on $T^*\mathbb{T}^n \times \mathbb{R}$ (or $T^*\tilde{M} \times \mathbb{R}$, where $\tilde{M} = \mathbb{R}^n$) and satisfies the following:
(1) $\sup \left\|\nabla^2 H_t\right\| < K$
(2) The matrices $H_{\boldsymbol{pp}}(\boldsymbol{z}, t)$ are positive definite and $C < \|H_{\boldsymbol{pp}}\| < C^{-1}$.

Theorem 5.2 *Let $H(\boldsymbol{q}, \boldsymbol{p}, t)$ be a Hamiltonian function on $T^*\mathbb{T}^n \times \mathbb{R}$ satisfying Assumption* **5.1** *. Then the time 1 map h^1 of the associated Hamiltonian* *flow has at least $n+1$ distinct periodic orbits of type $\boldsymbol{m}, d$, for each prime $\boldsymbol{m}, d$, and 2^n in the generic case when they are all non degenerate.*

Proof. we can decompose the time 1 map:

$$h^1 = h^{\frac{N}{N}} \circ (h^{\frac{N-1}{N}})^{-1} \circ \ldots \circ h^{\frac{k}{N}} \circ (h^{\frac{k-1}{N}})^{-1} \circ \ldots \circ h^{\frac{1}{N}} \circ Id.$$

and each of the maps $h^{\frac{k}{N}} \circ (h^{\frac{k-1}{N}})^{-1}$ is the time $\frac{1}{N}$ of the (extended) flow, starting at time $\frac{k-1}{N}$, or in other words, the time $1/N$ of the Hamiltonian $K_t = H_{t+\frac{k-1}{N}}$. Proposition **5.4** shows that, for N big enough, such maps are symplectic twist and satisfy the convexity condition **4.1** . The result follows from Theorem **4.3**. □

Remark 5.3 . Remember that Hamiltonian maps on cotangent bundles are exact symplectic. More precisely, the time t map h^t of a Hamiltonian system on T^*M satisfies:

$$(5.1) \qquad (h^t)^*\boldsymbol{p}d\boldsymbol{q} - \boldsymbol{p}d\boldsymbol{q} = dS_t \quad \text{where} \quad S_t(\boldsymbol{q}, \boldsymbol{p}) = \int_{(\boldsymbol{q},\boldsymbol{p})}^{h^t(\boldsymbol{q},\boldsymbol{p})} \boldsymbol{p}d\boldsymbol{q} - Hds,$$

and the path of integration is the trajectory $(h^s(\boldsymbol{q}, \boldsymbol{p}), s)$ of the (extended) flow. Obviously h^t is isotopic to Id. The twist condition is what remains to be checked – it is clearly not always satisfied. The following proposition shows that it is, for small t, under Assumption **5.1** .

Before that, let us remark that the method of proof that we are using in this section is analogous to the so called *method of broken geodesics* [Mi69]: by (5.1) , the function W that we appeal to above in our use of Theorem **4.3** can be interpreted as:

$$W(\overline{\boldsymbol{q}}) = \sum_k \int_{\gamma_k} \boldsymbol{p}d\boldsymbol{q} - Hds$$

where γ_k is the orbit of h^t starting from $(\boldsymbol{q}_k, \boldsymbol{p}_k)$ at time $\frac{k}{N}$, and ending at $(\boldsymbol{q}_{k+1}, \boldsymbol{P}_k)$ at time $\frac{k+1}{N}$. The broken curve whose pieces are the γ_k projects, via the diffeomorphisms ψ_k (see definition **2.1**) to a continuous, but only piecewise differentiable curve of $\mathbb{T}^n$. In the case where H is the Hamiltonian corresponding to a metric, this curve is a piecewise, or broken geodesic and ψ_k is the exponential map. Proposition **3.1** can then be interpreted as saying that, among broken geodesics, the smooth ones are exactly the ones that are critical for W (See [G93] for more details.)

The following applies without change to Hamiltonians in cotangent bundles of Riemannian manifolds of negative curvature. It is, however, the point at which our method breaks for the cotangent of arbitrary manifolds: symplectic twist maps cannot be defined on *all* of T^*S^2, for instance.

Proposition 5.4 *Let h^ϵ be the time ϵ of a Hamiltonian flow for a Hamiltonian function satisfying Assumption* **5.1** *. Then, for all sufficiently small ϵ, h^ϵ is a symplectic twist map of $T^*\mathbb{T}^n$. Moreover, h^ϵ satisfies the convexity condition* **4.1** *.*

***Proof*.**
We can work in the covering space $\mathbb{R}^{2n}$ of $T^*\mathbb{T}^n$, to which the flow lifts. The differential of h^t at a point $\boldsymbol{z} = (\boldsymbol{q}, \boldsymbol{p})$ is solution of the linear (variation) equation:

$$\dot{U}(t) = J\nabla^2 H(h^t(\boldsymbol{z}))U(t), \quad U(0) = Id, \quad J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} \tag{5.2}$$

We first need a lemma that tells us that $U(\epsilon)$ is not too far from Id:

Lemma 5.5 *Consider the linear equation:*

$$\dot{U}(t) = A(t)U(t), \quad U(t_0) = U_0$$

where $\|A(t)\| < K, \forall t$. Then :

$$\|U(t) - U_0\| < K\,\|U_0\|\,|t - t_0|e^{K|t-t_0|}.$$

***Proof*.** Let $V(t) = U(t) - U(t_0)$, so that $V(t_0) = 0$. We have:

$$\begin{aligned} \dot{V}(t) &= A(t)\,(U(t) - U_0) + A(t)U_0 \\ &= A(t)V(t) + A(t)U_0 \end{aligned}$$

and hence:

$$\|V(t)\| = \|V(t) - V(0)\| \leq \int_{t_0}^{t} K\,\|V(s)\|\,ds + |t - t_0|K\,\|U_0\|$$

For all $|t - t_0| \leq \epsilon$, we can apply Gronwall's inequality to get:

$$\|V(t)\| \leq \epsilon K\,\|U_0\|\,e^{K|t-t_0|}$$

and we get the result by setting $\epsilon = |t - t_0|$. □

We now finish the proof of Proposition **5.4** . By Lemma **5.5** we can write:

$$U(\epsilon) - Id = \int_0^\epsilon J\nabla^2 H(h^s(\boldsymbol{z})).(Id + O_1(s))ds$$

where $\|O_1(s)\| < 2Ks$, for $s \leq \epsilon$ small enough.

Let $(\boldsymbol{q}(t), \boldsymbol{p}(t)) = h^t(\boldsymbol{q}, \boldsymbol{p}) = h^t(\boldsymbol{z})$. The matrix $\boldsymbol{b}_\epsilon(\boldsymbol{z}) = \partial\boldsymbol{q}(\epsilon)/\partial\boldsymbol{p}$, is the upper right $n \times n$ matrix of $U(\epsilon)$. It is given by:

$$\text{(5.3)} \qquad \boldsymbol{b}_\epsilon(\boldsymbol{z}) = \int_0^\epsilon H_{\boldsymbol{pp}}(h^s(\boldsymbol{z}))ds + \int_0^\epsilon O_2(s)ds$$

where $\left|\int_0^\epsilon O_2(s)ds\right| < K\epsilon^2$. From this, and the fact that

$$C\left\|\boldsymbol{v}\right\|^2 < \langle H_{\boldsymbol{pp}}(\boldsymbol{z})\boldsymbol{v}, \boldsymbol{v}\rangle < C^{-1}\left\|\boldsymbol{v}\right\|^2,$$

we deduce that:

$$\text{(5.4)} \qquad (\epsilon C - K\epsilon^2)\left\|\boldsymbol{v}\right\|^2 < \langle \boldsymbol{b}_\epsilon(\boldsymbol{z})\boldsymbol{v}, \boldsymbol{v}\rangle < (\epsilon C^{-1} + K\epsilon^2)\left\|\boldsymbol{v}\right\|^2$$

so that in particular $\boldsymbol{b}_\epsilon(\boldsymbol{z})$ is nondegenerate for small enough ϵ. The set of nonsingular matrices $\{\boldsymbol{b}_\epsilon(\boldsymbol{z})\}_{\boldsymbol{z}\in\mathbb{R}^{2n}}$ is included in a compact set and thus:

$$\text{(5.5)} \qquad \sup_{\boldsymbol{z}\in\mathbb{R}^{2n}} \left\|\boldsymbol{b}_\epsilon^{-1}(\boldsymbol{z})\right\| < K',$$

for some positive K'. We can now apply Corollary **2.7** to show that h^ϵ is a symplectic twist map with a generating function S defined on all of $\mathbb{R}^{2n}$.

Likewise, from (5.3) , and the fact that $\langle H_{\boldsymbol{pp}}^{-1}(\boldsymbol{z})\boldsymbol{v}, \boldsymbol{v}\rangle > C\|\boldsymbol{v}\|^2$, one easily derives that h^ϵ satisfies the convexity condition **4.1**. □

6. Suspension of Symplectic Twist Maps by Hamiltonian Flows

In [Mo86], Moser showed how to suspend a monotone twist map of the compact annulus into a time 1 map of a (time dependent) optical Hamiltonian system. Furthermore, he was careful to construct the Hamiltonian in such a way that its flow leaves invariant the compact annulus (when the map does) and also such that it is time periodic.

As announced by Bialy and Polterovitch [BP92], Moser's method can be adapted to suspend a symplectic twist map whose generating functions S is such that $\partial_{12}S(\boldsymbol{q}, \boldsymbol{Q})$ is a positive definite *symmetric* matrix satisfying the convexity condition **4.1**. In particular their result shows that the Standard Map is the time 1 of an optical Hamiltonian flow, periodic in time.

Here we present a suspension theorem for higher dimensional symplectic twist maps, without the assumption that $\partial_{12}S$ is symmetric. Our result is modest in that we do not obtain a convexity condition on the Hamiltonian, or show that the Hamiltonian we construct can be made time periodic. Our method is different from Moser's.

Theorem 6.1 *Let $F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})$ be a symplectic twist map of $T^*\mathbb{T}^n$ which satisfies the convexity condition* **4.1**. *Then F is the time 1 map of a (time dependent) Hamiltonian H.*

Proof. Let $S(\boldsymbol{q},\boldsymbol{Q})$ be the generating function of F. Condition **4.1** can be rewritten:

$$\inf_{(\boldsymbol{q},\boldsymbol{Q})\in\mathbb{R}^{2n}} \langle -\partial_{12}S(\boldsymbol{q},\boldsymbol{Q})\boldsymbol{v},\boldsymbol{v}\rangle > a\left\|\boldsymbol{v}\right\|^2, \quad a>0, \forall \boldsymbol{v}\neq 0\in\mathbb{R}^n. \tag{6.1}$$

The following lemma, whose proof is left to the reader shows that this inequality implies (2.4). Hence whenever we have a function on $\mathbb{R}^{2n}$ which is suitably periodic and satisfies (6.1) , it is the generating function for some symplectic twist map.

Lemma 6.2 *Let $\{A_x\}_{x\in\Lambda}$ be a family of $n\times n$ real matrices satisfying:*

$$\sup_{x\in\Lambda}\langle A_x\boldsymbol{v},\boldsymbol{v}\rangle > a\|\boldsymbol{v}\|^2, \quad \forall\boldsymbol{v}\neq 0\in\mathbb{R}^n.$$

Then :

$$\sup_{x\in\Lambda}\left\|A_x^{-1}\right\| < a^{-1}.$$

We construct a differentiable family S_t of generating functions, with $S_1 = S$, and then show how to make a Hamiltonian vector field out of it, whose time 1 map is F. Let

$$S_t(\boldsymbol{q},\boldsymbol{Q}) = \begin{cases} \frac{1}{2}af(t)\|\boldsymbol{Q}-\boldsymbol{q}\|^2 & \text{for } 0<t\leq\frac{1}{2} \\ \frac{1}{2}af(t)\|\boldsymbol{Q}-\boldsymbol{q}\|^2 + (1-f(t))S(\boldsymbol{q},\boldsymbol{Q}) & \text{for } \frac{1}{2}\leq t\leq 1. \end{cases}$$

where f is a smooth positive functions, $f(1) = f'(1/2) = 0,\ f(1/2) = 1$, $\lim_{t\to 0^+} f(t) = +\infty$. We will ask also that $1/f(t)$, which can be continued to $1/f(0) = 0$ be differentiable at 0. The choice of f has been made so that S_t is differentiable with respect to t, for $t\in(0,1]$. Furthermore, it is easy to verify that:

$$\sup_{(\boldsymbol{q},\boldsymbol{Q})\in\mathbb{R}^{2n}} \langle -\partial_{12}S_t(\boldsymbol{q},\boldsymbol{Q})\boldsymbol{v},\boldsymbol{v}\rangle > a\left\|\boldsymbol{v}\right\|^2, \quad a>0, \forall\boldsymbol{v}\neq 0\in\mathbb{R}^n, t\in(0,1].$$

Hence S_t generates a smooth family $F_t,\ t\in(0,1]$ of symplectic twist maps, and in fact $F_t(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{q}+(af(t))^{-1}\boldsymbol{p},\boldsymbol{p}), \quad t\leq 1/2)$, so that $\lim_{t\to 0^+} F_t = Id$, in any topology that one desires (on compact sets.) Let us write

$$\tilde{S}_t(\boldsymbol{q},\boldsymbol{p}) = S_t\circ\psi_t(\boldsymbol{q},\boldsymbol{p}),$$

where ψ_t is the change of coordinates given by the fact that F_t is twist. It is not hard to verify that $\psi_t(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{q},\boldsymbol{q}-(af(t))^{-1}\boldsymbol{p}), \quad t\leq 1/2$. so that:

$$\tilde{S}_t(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2}(af(t))^{-2}\|\boldsymbol{p}\|^2$$

In particular, by our assumption on $1/f(t)$, $\tilde{S}_t$ can be differentiably continued for all $t \in [0,1]$, with $S_0 \equiv 0$. Hence, in the $\boldsymbol{q}, \boldsymbol{p}$ coordinates, we can write:

$$F_t^* \boldsymbol{p} d\boldsymbol{q} - \boldsymbol{p} d\boldsymbol{q} = d\tilde{S}_t, \quad t \in [0,1].$$

A familly of maps that satisfies this with $\tilde{S}_t$ differentiable in $(\boldsymbol{q}, \boldsymbol{p}, t)$ is called an *exact symplectic isotopy.* The proof of the theorem derives from the standard:

Lemma 6.3 *Let g_t be an exact symplectic isotopy of $T^*\mathbb{T}^n$ (or T^*M, in general.) Then g_t is a Hamiltonian isotopy.*

Proof. Let g_t be an exact symplectic isotopy:

$$g_t^* \boldsymbol{p} d\boldsymbol{q} - \boldsymbol{p} d\boldsymbol{q} = dS_t$$

for some S_t differentiable in all of $(\boldsymbol{q}, \boldsymbol{p}, t)$. We claim that the (time dependent) vector field:

$$X_t(\boldsymbol{z}) = \frac{dg_t}{dt}(g_t^{-1}(\boldsymbol{z}))$$

whose time t is g_t, is Hamiltonian. To see this, we compute:

$$\frac{d}{dt}(d\tilde{S}_t) = \frac{d}{dt} g_t^* \boldsymbol{p} d\boldsymbol{q} = g_t^* L_{X_t} \boldsymbol{p} d\boldsymbol{q} = g_t^* \left(i_{X_t} d(\boldsymbol{p} d\boldsymbol{q}) - d(i_{X_t} \boldsymbol{p} d\boldsymbol{q}) \right),$$

from which we get

$$i_{X_t} d\boldsymbol{q} \wedge d\boldsymbol{p} = dH_t$$

with

$$H_t = \left((g_t^{-1})^* \frac{dS_t}{dt} - i_{X_t} \boldsymbol{p} d\boldsymbol{q} \right),$$

which exactly means that X_t is Hamiltonian.

□

7.2 Cotangent Bundle of Manifolds with Negative Curvature

We indicate in this section how some of the previous results can be obtained in the cotangent bundle T^*M of a compact manifold M which supports a metric of negative curvature. Such a manifold is always covered by $\mathbb{R}^n$ (As before we denote by $pr : \tilde{M}(= \mathbb{R}^n) \to M$ the covering map.) The definition of symplectic twist map carries through verbatim for the cotangent bundle of such manifolds, as well as Propositions 2.4 and 3.1, Corollaries 2.6 and 2.7. The action by translations of $\pi(\mathbb{T}^n) = \mathbb{Z}^n$ on $\mathbb{R}^{2n}$ is replaced by the more general action of $\pi_1(M)$, the deck transformation group of $T^*\tilde{M}$. Note also that the convexity condition **4.1** still makes sense in this more general context. For more details, see [G91b], [G93].

The first resistance we encounter to an extension of our results to such manifolds is Definition **3.2** of $\boldsymbol{m}, d$–orbits. The clue to define such an orbit in this new context is Remark **5.4**: we saw there that, in the case where the map F considered is Hamiltonian and decomposed into symplectic twist maps, an $\boldsymbol{m}, d$–sequence gives rise to a closed, piecewise smooth curve in $\mathbb{T}^n$ (a "broken geodesic"). The integer vector $\boldsymbol{m}$ classifies these broken geodesics up to homotopy *with or without fixed base points.* This is because the group $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$ is abelian.

In general manifolds, two loops through a base point that represent different elements in $\pi_1(M)$ might be homotopic if we allow the homotopy to move the base point: we say then that the curves are *free homotopic.* Free homotopy classes are in one to one correspondence with the conjugacy classes in $\pi_1(M)$.

Coming back to our broken geodesics, the natural classification for periodic orbits of a Hamiltonian system is that of free homotopy class: each of these classes represent a connected component in the loop space. This motivates:

Definition 7.3 Let $\boldsymbol{m}$ be a representative of a free homotopy class of loops in M. A sequence $\{\boldsymbol{q}_k\}$ of points in $\tilde{M} = \mathbb{R}^n$ is called a $\boldsymbol{m}, d$–**sequence** if, for all $k \in \mathbb{Z}$, $pr(\boldsymbol{q}_k) = pr(\boldsymbol{q}_{k+d})$ and (any) curve $\tilde{\gamma}$ of $\tilde{M}$ that joins $\boldsymbol{q}_k$ and $\boldsymbol{q}_{k+d}$, projects to a closed curve of M in the free homotopy class $\boldsymbol{m}$, independent of k. The orbit $\{(\boldsymbol{q}_k, \boldsymbol{p}_k)\}$ of a map of T^*M is an $\boldsymbol{m}, d$–**periodic orbit** if the sequence $\{\boldsymbol{q}_k\}$ is an $\boldsymbol{m}, d$–sequence.

We can now state:

Theorem 7.4 *Let M be a compact Riemannian manifold with negative curvature. Let $F = F_N \circ \ldots \circ F_1$ be a finite composition of symplectic twist maps F_k of T^*M satisfying the convexity condition* **4.1**. *Then, for each free homotopy class $\boldsymbol{m}$ and period d, F has at least 2 periodic orbits of type $\boldsymbol{m}, d$. If $\boldsymbol{m} = 0$, the class of contractible loops, then there are at least $cl(M)$ orbits of type $\boldsymbol{m}, d$, and $sb(M)$ if they are all nondegenerate.*

Proof. It is shown in [G91b], Lemma **7.2.2**, that the set X of $\boldsymbol{m}, dN$ sequences (modulo $\pi_1(M)$ and shift; X is denoted $O_{m,d}/\sigma$ in that paper), has a deformation retraction onto the set, that we call Σ, formed by the unique geodesic of class $\boldsymbol{m}$ (remember that M has negative curvature) that is, X has the homotopy type of $\mathbb{S}^1$. The proof of Theorem **4.3** can now be repeated, keeping in mind that $sb(\mathbb{S}^1) = cl(\mathbb{S}^1) = 2$.

When $\boldsymbol{m}$ is the trivial class, the set X retracts on the set of constant loops, naturally embedded in it ([G91b], Lemma **6.2**). This set, that we call

Σ again is homeomorphic to M. A simple adaptation of Lemma **7.2.2** in [G91b] shows that Σ is in fact a deformation retract of X and hence once again, we can repeat the proof of Theorem **4.3**. $\square$

Assumption **5.1** and Proposition **5.4** apply without a change to our new context and hence we have:

Theorem 7.5 *Let M be a compact Riemannian manifold with negative curvature. Let $H(\boldsymbol{q}, \boldsymbol{p}, t)$ be a Hamiltonian function on T^*M satisfying Assumption* **5.1** *. Then the time 1 map of the associated Hamiltonian flow can be decomposed into a product of symplectic twist maps. It has at least* 2 *periodic orbits of type $\boldsymbol{m}, d$, for each $\boldsymbol{m}, d$. When $\boldsymbol{m}$ is the trivial class, there are at least $cl(M)$ orbits of type $\boldsymbol{m}, d$, and $sb(M)$ if they are all nondegenerate (i.e. generically.)*

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