

**The existence of  $\sigma$ -finite invariant measures,  
Applications to real 1-dimensional dynamics**

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Abstract. A general construction for  $\sigma$ -finite absolutely continuous invariant measure will be presented. It will be shown that the local bounded distortion of the Radon-Nykodym derivatives of  $f_*^n(\lambda)$  will imply the existence of a  $\sigma$ -finite invariant measure for the map  $f$  which is absolutely continuous with respect to  $\lambda$ , a measure on the phase space describing the sets of measure zero.

Furthermore we will discuss sufficient conditions for the existence of  $\sigma$ -finite invariant absolutely continuous measures for real 1-dimensional dynamical systems.

## 1. Introduction

The statistical study of a dynamical system begins with the question whether or not the system has an absolutely continuous invariant measure, finite or  $\sigma$ -finite. In 1947 Halmos gave a characterization of the (bijective) dynamical systems which have a  $\sigma$ -finite absolutely continuous invariant measure, see [Ha]. During this time there was some hope that every dynamical system has  $\sigma$ -finite invariant measures. Unfortunately this turned out not to be true. Ornstein gave an example of a piecewise linear bijective map on the interval not having such a measure ([O]).

Here we will give a characterization of the ergodic conservative dynamical systems on locally compact spaces having  $\sigma$ -finite absolutely continuous invariant measures.

The origin of the characterization presented here can be found in the theory of real 1-dimensional dynamics and the theory of Markov processes. In [H] Harris used limits of ratios of long term transition probabilities to construct infinite stationary states for Markov processes on countable state spaces. In section 2 we will use this idea for constructing  $\sigma$ -finite invariant measures. The construction gives rise to the existence theorem A. The distortion of a measure, used in the theorem, will be defined precisely in section 2.

Furthermore remember that a map  $f : X \rightarrow X$  is ergodic conservative with respect to a measure  $\lambda$  on  $X$  if every set of positive measure is hit by the orbits of  $\lambda$ -almost all points (see [P]).

**THEOREM A.** *Let  $\lambda$  be a Borel probability measure on the  $\sigma$ -compact space  $X$ . The ergodic conservative map  $f : X \rightarrow X$  has a  $\sigma$ -finite invariant measure absolutely continuous with respect to  $\lambda$  if the Radon-Nykodym derivatives of  $f_*^n \lambda$  have locally bounded distortion.*

As in the general construction of invariant probability measures the construction is done by pushing forward some initial measure and then considering limits of these push-forwards. It turns out that the procedure only gives rise to  $\sigma$ -finite absolutely continuous invariant measures if the initial measures are of some special type. In section 3 we will construct the initial measures.

In section 4 we will study the existence of  $\sigma$ -finite invariant measures for real 1-dimensional differentiable dynamics. As we know from [J] there is no general existence theorem for absolutely continuous invariant probability measures: there exist conservative quadratic maps on the interval not having absolutely continuous invariant probability measures. Even the existence question for  $\sigma$ -finite absolutely continuous invariant measures can not be answered in general. Katznelson ([Ka]) constructed diffeomorphisms of the circle not having  $\sigma$ -finite absolutely continuous invariant measures.

Applying the developed theory we can formulate sufficient conditions implying the existence of  $\sigma$ -finite absolutely continuous invariant measures for real 1-dimensional differentiable dynamics.

In [HKe2] Hofbauer and Keller gave an existence theorem for some type of conservative unimodal maps. Now this theorem can be generalized to multimodal and also dissipative maps:

**THEOREM B.** *Let  $f$  be a  $C^3$  map on the interval (or the circle) satisfying*

- 1)  *$f$  has only finitely many critical points, points where the derivative vanishes, and the Schwarzian derivative is everywhere negative except in the critical points;*
- 2) *there exists a dense orbit;*

3) the orbits of the critical points stay in a closed invariant set of Lebesgue measure zero. Then  $f$  has a  $\sigma$ -finite absolutely continuous invariant measure.

In [HKel] quadratic maps are shown to exist having very strange Bowen-Ruelle measures. The same techniques can be used to show that there exists a quadratic map whose critical orbit is in a Cantor set but which doesn't have an absolutely continuous invariant probability measure. This means that in general the invariant measures of Theorem B are really  $\sigma$ -finite. Furthermore we obtain a  $\sigma$ -finite Folklore theorem.

All the results concerns maps whose critical orbits stay in some closed invariant set of Lebesgue measure zero. In the other case, some critical orbits are dense, we state the

**CONJECTURE.** *There exist conservative quadratic maps on the interval not having  $\sigma$ -finite absolutely continuous invariant measures.*

In the appendix we will give a short proof of the Chacon-Ornstein Theorem, the main theorem in  $\sigma$ -finite ergodic theory.

Remember the following notation: if  $g : X \rightarrow X$  is a Borel measurable function and  $\mu$  a Borel measure on  $X$  then  $g_*\mu$  is the measure defined by  $g_*\mu(A) = \mu(g^{-1}(A))$ .

## 2. The construction of $\sigma$ -finite absolutely continuous invariant measures

In this section we are going to construct  $\sigma$ -finite absolutely continuous invariant measures. Let  $\lambda$  be the Borel measure describing the sets of measure zero. A starting point for constructing absolutely continuous invariant measures is considering limits of the Birkhoff sums  $\{\frac{1}{n}\sum_{i=0}^{n-1} f_*^i \lambda\}_{n \geq 0}$ . Indeed, if for all sets  $A$  with  $\lambda(A) > 0$  the measures  $f_*^n \lambda(A)$  stay away from 0 we can construct an absolutely continuous invariant probability measure, simply by taking a converging subsequence of Birkhoff sums. In case our system doesn't have an invariant probability measure, that is there exist closed sets  $A$  with  $f_*^n \lambda(A)$  converging to zero, we have to consider other limits.

In [H] Harris used limits of ratios of long term transition probabilities to construct stationary states for Markov processes. In our construction we will choose some set  $I_0$  of positive

measure and consider limits of the normalized sequence

$$\frac{\sum_{i=0}^{n-1} f_*^i \lambda}{\sum_{i=0}^{n-1} f_*^i \lambda(I_0)}.$$

Our construction is strongly related to the Chacon-Ornstein Theorem or, which is in some sense equivalent to it, the Doeblin-Ratio-Limit Theorem for Markov processes (see resp. [K],[P] and [F]).

First let us remember some general notions. Let  $X$  be a  $\sigma$ -compact topological space, it can be written as a countable union of compact sets, and  $\mathcal{B}(X)$  be the set of Borel measures on  $X$ . The set  $\mathcal{B}_\sigma(X)$  consists of all measures  $\mu \in \mathcal{B}(X)$  for which there exists a collection  $\{X_n | n \in \mathbf{N}\}$  of pairwise disjoint measurable sets in  $X$  such that

$$\begin{aligned} \mu(X_n) \text{ is finite for all } n \in \mathbf{N}; \\ \mu(X - (\bigcup_{n \in \mathbf{N}} X_n)) = 0. \end{aligned}$$

The measures in  $\mathcal{B}_\sigma(X)$  are called  $\sigma$ -finite measures on  $X$ .

For discussing the notion of absolute continuity we fix a measure  $\lambda \in \mathcal{B}_\sigma(X)$  determining the *null sets*, the sets which have to have measure zero. We may assume without restricting generality that  $\lambda$  is a probability measure. Remember a measure  $\mu \in \mathcal{B}_\sigma(X)$  is called absolutely continuous with respect to  $\lambda$ ,  $\lambda \gg \mu$ , iff the sets of  $\lambda$ -measure zero also have  $\mu$ -measure zero. These absolutely continuous measures can be expressed by an integral: for all  $\mu$  with  $\lambda \gg \mu$  there exists an essentially unique integrable non-negative function  $\rho$  such that  $\mu(A) = \int_A \rho d\lambda$  for all measurable sets  $A \subset X$ . This function is called the *Radon-Nykodym derivative* (or just derivative) of  $\mu$  with respect to  $\lambda$ .

The main objects to be studied here will be absolutely continuous measures whose Radon-Nykodym derivatives has locally bounded distortion: we say that the derivative of  $\mu \in \mathcal{B}_\sigma(X)$ ,  $\lambda \gg \mu$ , on  $I \subset X$  with  $\mu(I)$  finite, has *distortion* bounded by  $K$  iff for all measurable sets  $A \subset I$

$$\frac{1}{K} \frac{\lambda(A)}{\lambda(I)} \leq \frac{\mu(A)}{\mu(I)} \leq K \frac{\lambda(A)}{\lambda(I)}.$$

Observe that the constant  $K$  can be taken to be equal to 1 iff  $\mu$  has constant derivative on  $I$ . If you consider a Radon-Nykodym derivative as an object deforming the original measure  $\lambda$  this notion of distortion is related to the concept of distortion for differentiable maps of the interval (see [GuJ], [LB], [MMS] or [Sw]).

In this section we fix a probability measure  $\lambda \in \mathcal{B}_\sigma(X)$  and a measurable map  $f : X \rightarrow X$  on the  $\sigma$ -compact space  $X$ . Assume that  $\lambda$  is quasi-invariant for  $f$ :  $\lambda \gg f_*\lambda$ .

The next step is to give some definitions which enables us to deal with  $\sigma$ -finite absolutely continuous invariant measures, shortly *acim* (if we want to emphasize that some acim is a probability measure we call it *acip*: absolutely continuous invariant probability measure). A  $\lambda$ -partition  $\mathcal{G}$  of  $X$  is a countable collection of pairwise disjoint Borel sets of  $X$ , say  $\mathcal{G} = \{I_n | n \in \mathbf{N}\}$ , such that for all  $n \in \mathbf{N}$

- 1)  $I_n$  is  $\sigma$ -compact;
- 2)  $0 < \lambda(I_n) < \infty$ ;
- 3)  $\lambda(X - (\bigcup_{I \in \mathcal{G}} I)) = 0$ .

If the  $\lambda$ -partition  $\mathcal{G}$  of  $X$  has the additional property

- 4) for all pairs  $I_1, I_2 \in \mathcal{G}$  there exists  $n \geq 0$  such that  $\lambda(f^{-n}(I_1) \cap I_2) > 0$

we will say  $f$  is  $\mathcal{G}$ -irreducible.

The role of the  $\lambda$ -partition is the following: its elements will turn out to be sets of finite measure for the acim of the  $\mathcal{G}$ -irreducible map  $f$ .

In the sequel we fix a  $\lambda$ -partition  $\mathcal{G}$  and assume that  $f : X \rightarrow X$  is  $\mathcal{G}$ -irreducible.

To define the measure spaces  $\mathcal{M}(\mathcal{G}, f)$ ,  $\mathcal{M}_s(\mathcal{G}, f)$  and  $\mathcal{M}_\infty(\mathcal{G}, f)$ , in which the construction of the acims will take place, we need some properties of measures  $\mu \in \mathcal{B}_\sigma(X)$ .

Let  $I \in \mathcal{G}$  and  $K > 0$ :

m1( $I, K$ ):  $\mu(I) \in [\frac{1}{K}, K]$ ;

m2( $I, K$ ): For all  $n \geq 0$  the  $\mu$ -measure of  $f^{-n}(I)$  is finite and positive and the derivatives of the measures  $f_*^n \mu$  on  $I$  with respect to  $\lambda$  has distortions bounded by  $K$ . This property states that measures  $f_*^n \mu$  have locally uniformly bounded distortions.

$$\text{m3}(I): \sup_{n \geq 0} \mu(f^{-n}(I)) < \infty;$$

$$\text{m4}(I): \sum_{n=0}^{\infty} \mu(f^{-n}(I)) = \infty.$$

The measure spaces we need are

$$\mathcal{M}(\mathcal{G}, f) = \{ \mu \in \mathcal{B}_\sigma(X) \mid \text{for all } I \in \mathcal{G} \text{ there is a } K > 0 \text{ with m1}(I, K) \text{ and m2}(I, K) \};$$

$$\mathcal{M}_s(\mathcal{G}, f) = \{ \mu \in \mathcal{M}(\mathcal{G}, f) \mid \text{m3}(I) \text{ holds for all } I \in \mathcal{G} \};$$

$$\mathcal{M}_\infty(\mathcal{G}, f) = \{ \mu \in \mathcal{M}_s(\mathcal{G}, f) \mid \text{m4}(I_0) \text{ holds for some } I_0 \in \mathcal{G} \}.$$

Because we are only considering the fixed  $\lambda$ -partition  $\mathcal{G}$  and the fixed  $\mathcal{G}$ -irreducible map  $f : X \rightarrow X$  we will use the short names  $\mathcal{M}$ ,  $\mathcal{M}_s$  and  $\mathcal{M}_\infty$ . Furthermore observe that  $\mathcal{M}_\infty \subset \mathcal{M}_s \subset \mathcal{M}$  and that  $\mathcal{M}$  only contains measures which are absolutely continuous with respect to  $\lambda$ .

The construction of the acims will be by taking converging subsequences in  $\mathcal{M}$ . Hence we have to describe which kind of convergence we are going to use. A sequence  $\mu_n$  in  $\mathcal{M} = \mathcal{M}(\mathcal{G}, f)$  is said to converge to  $\mu \in \mathcal{B}_\sigma(X)$  iff for all  $I \in \mathcal{G}$  and for every compact  $A \subset I$  we have weak convergence of  $\mu_n|_A \rightarrow \mu|_A$ .

For two reasons the space  $\mathcal{M}$  is not compact: the measures are not assumed to be bounded and the underlying space is not compact; mass can disappear to the boundary of the space. Let us try to describe some compact subsets of  $\mathcal{M}$ . A collection  $\mathcal{A} \subset \mathcal{M}$  is called *uniform* iff for all  $I \in \mathcal{G}$  there exists  $K(I) > 0$  such that  $\text{m1}(I, K(I))$  and  $\text{m2}(I, K(I))$  hold for all  $\mu \in \mathcal{A}$ .

The compactness in the weak topology of the set of probability measures on a compact space implies easily

LEMMA 2.1. *Uniform collections in  $\mathcal{M}(\mathcal{G}, f)$  have compact closures in  $\mathcal{M}(\mathcal{G}, f)$ .*

Now we are going to use the above measure spaces to construct the acims. Let  $\mu \in \mathcal{M}$

and define the following measures in  $\mathcal{B}_\sigma(X)$ :

$$S_n\mu = \sum_{i=0}^{n-1} f_*^i\mu;$$

$$Q_n\mu = \frac{S_n\mu}{S_n\mu(I_0)}.$$

for all  $n \geq 0$ .  $I_0$  is a fixed element of  $\mathcal{G}$ . If  $\mu \in \mathcal{M}_\infty$  then we choose  $I_0$  such that  $m4(I_0)$  holds.

The lemma which will assure the existence of limits is

LEMMA 2.2. *Let  $\mu \in \mathcal{M}_s(\mathcal{G}, f)$  then the collection*

$$\mathcal{A}_\mu = \{Q_n\mu | n \geq 0\}$$

*is uniform.*

Before proving this lemma we are going to use lemma 2.1 and 2.2 to define the following limit set in  $\mathcal{M}$ . Let  $\mu \in \mathcal{M}_s$  then

$$\omega(\mu) \subset \mathcal{M}(\mathcal{G}, f)$$

is the set of all limits of the sequence  $\mathcal{A}_\mu$ . We will look for acims in these sets  $\omega(\mu)$ . For the moment we know already that it only contains measures which are equivalent to  $\lambda$ .

The next lemma shows how the measures of backward orbits of two elements in  $\mathcal{G}$  are related. It will be used at several places; it serves for gluing together the information given by the local boundedness of the distortions.

LEMMA 2.3. *Let  $\mu \in \mathcal{M}(\mathcal{G}, f)$ . Then for every pair  $I_1, I_2 \in \mathcal{G}$  there exist  $\epsilon > 0$  and  $n_0 \geq 0$  such that*

$$\mu(f^{-n-n_0}(I_1)) \geq \epsilon\mu(f^{-n}(I_2))$$

*for  $n \geq 0$ .*

PROOF: Because  $f$  is  $\mathcal{G}$ -irreducible there exists  $n_0 \geq 0$  such that  $\lambda(f^{-n_0}(I_1) \cap I_2) > 0$ .

Hence for  $\mu \in \mathcal{M}(\mathcal{G}, f)$  we get for all  $n \geq 0$

$$\begin{aligned} \mu(f^{-n-n_0}(I_1)) &\geq \frac{\mu(f^{-n}(f^{-n_0}(I_1) \cap I_2))}{\mu(f^{-n}(I_2))} \mu(f^{-n}(I_2)) \\ &\geq \frac{1}{K} \frac{\lambda(f^{-n_0}(I_1) \cap I_2)}{\lambda(I_2)} \mu(f^{-n}(I_2)) \\ &= \epsilon \mu(f^{-n}(I_2)), \end{aligned}$$

where  $K > 0$  is such that  $m2(I_2, K)$  holds for  $\mu$ . ■

This lemma is the place where we use the  $\mathcal{G}$ -irreducibility of  $f$ .

LEMMA 2.4. *Let  $\mu \in \mathcal{M}_s(\mathcal{G}, f)$ . For every pair  $I_1, I_2 \in \mathcal{G}$  there exists  $K < \infty$  such that*

$$\frac{1}{K} \leq \frac{S_n \mu(I_1)}{S_n \mu(I_2)} \leq K$$

for all  $n \geq 0$ .

PROOF: Let  $\epsilon, n_0$  be given by lemma 2.3. For  $n \leq n_0$  we have some bound. Let  $n > n_0$ .

Then

$$\begin{aligned} \frac{S_n \mu(I_1)}{S_n \mu(I_2)} &\geq \frac{\sum_{i=0}^{n-n_0-1} \mu(f^{-i-n_0}(I_1))}{\sum_{i=n-n_0}^{n-1} \mu(f^{-i}(I_2)) + \sum_{i=0}^{n-n_0-1} \mu(f^{-i}(I_2))} \\ &\geq \epsilon \frac{\sum_{i=0}^{n-n_0-1} \mu(f^{-i}(I_2))}{n_0 \sup_{i \geq 0} \mu(f^{-i}(I_2)) + \sum_{i=0}^{n-n_0-1} \mu(f^{-i}(I_2))} \\ &\geq \epsilon \frac{\mu(I_2)}{n_0 \sup_{i \geq 0} \mu(f^{-i}(I_2)) + \mu(I_2)} \end{aligned}$$

which is a finite positive number. Remark that we used in the last step that the function  $x \rightarrow \frac{x}{a+x}$  is increasing.

By interchanging the role of  $I_1$  and  $I_2$  we also get an upper bound. ■

PROOF OF LEMMA 2.2: Fix  $I \in \mathcal{G}$ .

m1: Let  $K$  be the number given by lemma 2.4 applied to  $I$  and  $I_0$ . We get directly from lemma 2.4 that  $m1(I, K)$  holds for all  $Q_n \mu$ ,  $n \geq 0$ .

m2: Let  $K > 0$  be such that m2( $I, K$ ) holds for  $\mu$ . Fix  $m \geq 0$  and let  $A \subset I$  be measurable.

Then we get for  $n \geq 0$

$$\begin{aligned} \frac{Q_m \mu(f^{-n}(A))}{Q_m \mu(f^{-n}(I))} &= \frac{S_m \mu(f^{-n}(A))}{S_m \mu(f^{-n}(I))} \\ &= \frac{\sum_{i=n}^{m+n-1} \mu(f^{-i}(A))}{\sum_{i=n}^{m+n-1} \mu(f^{-i}(I))}. \end{aligned}$$

Because m2( $I, K$ ) holds for  $\mu$  we easily get that this last number is in the interval  $[\frac{1}{K} \frac{\lambda(A)}{\lambda(I)}, K \frac{\lambda(A)}{\lambda(I)}]$ . This proves the lemma. ■

The following lemma tells under which conditions on  $\mu, \omega(\mu)$  will contain invariant measures.

LEMMA 2.5. *Let  $\mu \in \mathcal{M}_\infty(\mathcal{G}, f)$ . Then  $\omega(\mu)$  contains only invariant measures.*

PROOF: Let  $\nu \in \omega(\mu)$ , say  $\nu = \lim Q_n \mu$  (lim means : the limit of a certain converging subsequence). Because  $\lambda(X - \cup \mathcal{G}) = \nu(X - \cup \mathcal{G}) = 0$  and  $\lambda$  is quasi-invariant for  $f$  we only have to consider  $A \subset I, I \in \mathcal{G}$ . Let  $A \subset I$  be compact. Then

$$\begin{aligned} \nu(f^{-1}(A)) &= \lim \frac{S_n \mu(f^{-1}(A))}{S_n \mu(I_0)} \\ &= \lim \frac{\sum_{i=0}^{n-1} \mu(f^{-i}(A)) - \mu(A) + \mu(f^{-n}(A))}{\sum_{i=0}^{n-1} \mu(f^{-i}(I_0))} \\ &= \nu(A) + \lim \frac{\mu(f^{-n}(A)) - \mu(A)}{S_n \mu(I_0)}. \end{aligned}$$

Now we use that m3( $I$ ) and m4( $I$ ) hold for  $\mu$  and we get

$$\nu(f^{-1}(A)) = \nu(A).$$

The measure is invariant. ■

PROPOSITION 2.6. *Let  $\lambda \in \mathcal{B}_\sigma(X)$  be a Borel measure on the  $\sigma$ -compact space  $X$  and  $f : X \rightarrow X$  a measurable map. The map  $f$  has a  $\lambda$ -equivalent  $\sigma$ -finite invariant measure if it is  $\mathcal{G}$ -irreducible for some  $\lambda$ -partition  $\mathcal{G}$  of  $X$  with*

$$\mathcal{M}_\infty(\mathcal{G}, f) \neq \emptyset.$$

The elements of  $\mathcal{G}$  will be pieces of  $X$  with bounded measure.

PROOF: If  $\mu \in \mathcal{M}_\infty(\mathcal{G}, f)$  then from lemma 2.2, 2.1 we get  $\omega(\mu) \neq \emptyset$ . Furthermore Lemma 2.5 tells us that  $\omega(\mu)$  only contains invariant measures. ■

In fact we also want the reverse statement:  $f$  has an acim iff it is  $\mathcal{G}$ -irreducible for some  $\lambda$ -partition  $\mathcal{G}$  of  $X$  with  $\mathcal{M}_\infty(\mathcal{G}, f) \neq \emptyset$ .

It is not hard to prove that for every map  $f : X \rightarrow X$  which has an acim there exists a  $\lambda$ -partition  $\mathcal{G}$  such that  $\mathcal{M}_\infty(\mathcal{G}, f) \neq \emptyset$ . A problem arises when we want to get it such that  $f$  becomes  $\mathcal{G}$ -irreducible. Probably it is possible to get this property. This technical problem can be illustrated by the question: does the feigenbaum map have an acim?

We can overcome this technical problem by assuming that  $f$  is ergodic and conservative with respect to  $\lambda$ : every set of positive  $\lambda$ -measure will intersect every other set of positive  $\lambda$ -measure after some time. If  $f$  is ergodic and conservative it will be  $\mathcal{G}$ -irreducible for every  $\lambda$ -partition  $\mathcal{G}$ .

So we get the following: an ergodic conservative map  $f$  has an acim iff there exists a  $\lambda$ -partition  $\mathcal{G}$  such that  $\mathcal{M}_\infty(\mathcal{G}, f) \neq \emptyset$ .

Using the following lemma we even can state a stronger existence theorem.

LEMMA 2.7. Let  $\mu \in \mathcal{B}_\sigma(X)$  with  $\lambda \gg \mu$ . If  $f$  is ergodic and conservative with respect to  $\lambda$  then for every set  $A$  with  $\lambda(A) > 0$

$$\sum_{i=0}^{\infty} \mu(f^{-i}(A)) = \infty.$$

PROOF: The ergodicity and conservatively tells us that almost every point in  $A$  will return to  $A$  infinitely many times. Now use the Borel-Cantelli Lemma. ■

COROLLARY 2.8. An ergodic conservative map  $f$  has a  $\sigma$ -finite absolutely continuous invariant measure iff there exists a partition  $\mathcal{G}$  with

$$\mathcal{M}_s(\mathcal{G}, f) \neq \emptyset.$$

The precise formulation of Theorem A in the introduction goes as follows.

THEOREM 2.9. *An ergodic conservative map  $f$  has a  $\sigma$ -finite absolutely continuous measure if there exists a  $\lambda$ -partition  $\mathcal{G}$  with*

$$\lambda \in \mathcal{M}(\mathcal{G}, f).$$

This means: once the derivatives of  $f_*^n \lambda$  have locally bounded distortion the existence of an acim is assured.

### 3. The initial measures

In this section we are going to study a condition which will imply  $\mathcal{M}_\infty(\mathcal{G}, f) \neq \emptyset$ , that is, it will imply the existence of acims.

Fix  $\lambda \in \mathcal{B}_\sigma(X)$  and a  $\lambda$ -partition  $\mathcal{G}$  of  $X$ . We say that the measurable map  $f : X \rightarrow X$  is *finite-to-1 with respect to  $\mathcal{G}$*  iff for all  $I \in \mathcal{G}$   $f^{-1}(I)$  is up to a nullset contained in a finite subcollection of  $\mathcal{G}$ . Furthermore  $PL_\lambda(\mathcal{G})$  is the set of all *distortion free* measures: these measures are equivalent to  $\lambda$  with densities which are constant on the element of  $\mathcal{G}$ .

PROPOSITION 3.1. *Let  $\lambda \in \mathcal{B}_\sigma(X)$  be a Borel measure on  $X$  and  $\mathcal{G}$  a  $\lambda$ -partition such that the measurable map  $f : X \rightarrow X$  is  $\mathcal{G}$ -irreducible.*

*If there exists a  $\lambda$ -partition  $\mathcal{G}_0$  with*

- 1)  $\mathcal{G}$  is a refinement of  $\mathcal{G}_0$ ;
- 2)  $PL_\lambda(\mathcal{G}_0) \subset \mathcal{M}(\mathcal{G}, f)$

*then*

$$\mathcal{M}_\infty(\mathcal{G}, f) \neq \emptyset.$$

PROOF: The condition  $PL_\lambda(\mathcal{G}_0) \subset \mathcal{M}(\mathcal{G}, f)$  is a strong condition. It implies that  $f$  is finite-to-1 with respect to  $\mathcal{G}_0$ : if  $f$  is not finite-to-1 it is easy to find  $I \in \mathcal{G}_0$  and  $\mu \in PL_\lambda(\mathcal{G}_0)$  such that  $\mu(f^{-1}(I)) = \infty$ .

We are going to define a measure  $\mu \in PL_\lambda(\mathcal{G}_0)$  satisfying m3( $I_0$ ) and m4( $I_0$ ), where  $I_0 \in \mathcal{G}$  is fixed. Using  $PL_\lambda(\mathcal{G}_0) \subset \mathcal{M}(\mathcal{G}, f)$  and lemma 2.4 we get  $\mu \in \mathcal{M}_\infty(\mathcal{G}, f)$ .

Let  $\mathcal{G}_0 = \{J_n | n \geq 0\}$  and  $\mathcal{G} = \{I_n | n \geq 0\}$ . We can assume  $I_0 \subset J_0$ . Define  $L_N = \cup_{i=0}^N J_i$  for  $N \geq 0$ . We are going to define  $\mu$  by giving its density  $\delta$  with respect to  $\lambda$

$$\delta = \sum_{N \geq 0} c_N 1_{J_N}.$$

The numbers  $c_N > 0$  will be defined inductively satisfying the following induction hypothesis

$$\sup_{n \geq 0} \mu|L_N(f^{-n}(I_0)) = 1 - \left(\frac{1}{2}\right)^{N+1}.$$

Because  $\lambda(I_0) < \infty$  we can choose  $c_0 > 0$  such that the induction hypothesis holds for  $N = 0$ . Suppose that  $c_0, c_1, \dots, c_N$  are defined satisfying the induction hypothesis. This means that the measure  $\mu|L_N$  is well defined. Now we have to define the value  $c_{N+1}$  of the density on  $J_{N+1}$ : let the map  $c \rightarrow \mu_{N,c} \in PL_\lambda(\mathcal{G}_0)$  with  $c \in [0, \infty)$  be defined as follows

$$\mu_{N,c} = \mu|L_N + c\lambda|J_{N+1}.$$

Using the fact that  $f$  is  $\mathcal{G}$ -irreducible, that is there exists an  $n \geq 0$  such that  $\lambda(f^{-n}(I_0) \cap J_{N+1}) \neq 0$ , it is easy to see that the map  $\phi : [0, \infty) \rightarrow \mathbf{R}$  defined by

$$\phi(c) = \sup_{n \geq 0} \mu_{N,c}(f^{-n}(I_0))$$

tends continuously to infinity for  $c \rightarrow \infty$ . From the definition of  $c_0, c_1, \dots, c_N$  we get  $\phi(0) = 1 - \left(\frac{1}{2}\right)^{N+1}$ . Hence there exists  $c_{N+1} > 0$  such that

$$\sup_{n \geq 0} \mu|L_{N+1}(f^{-n}(I_0)) = \phi(c_{N+1}) = 1 - \left(\frac{1}{2}\right)^{N+2}.$$

We finished the induction step; the measure  $\mu$  is well defined.

PROOF OF M3( $I_0$ ): Suppose there exists  $n \geq 0$  such that  $\mu(f^{-n}(I_0)) > 1$ . Because  $\mathcal{G}_0$  is an exhausting partition of  $X$  there exists  $N \geq 0$  such that

$$\mu|L_N(f^{-n}(I_0)) > 1$$

which contradicts the construction of  $\mu$ .

PROOF OF M4( $I_0$ ): We are going to construct a sequence  $n_k \rightarrow \infty$  such that

$$\mu(f^{-n_k}(I_0)) \geq \frac{1}{2}.$$

This will imply m4( $I_0$ ).

Suppose we have a finite set  $\{n_i | i = 0, 1, \dots, k\}$  such that for all of them

$\mu(f^{-n_i}(I_0)) \geq \frac{1}{2}$ . Let us find another one having this property. Because  $f$  is finite-to-1 with respect to  $\mathcal{G}_0$  there exists  $N \geq 1$  such that

$$\bigcup_{i=0}^k f^{-n_i}(I_0) \subset L_N$$

up to a set of measure zero. Hence  $\mu(f^{-n_i}(I_0)) \leq 1 - (\frac{1}{2})^{N+1}$  for  $i = 0, 1, \dots, k$ . From the definition of  $c_{N+1}$  we easily get a number  $n_{k+1}$  such that

$$1 - (\frac{1}{2})^{N+1} < \mu(f^{n_{k+1}}(I_0))$$

which is obviously not one of the previous ones. ■

Observe that theorem 2.9 gives a much weaker sufficient condition for the existence of acims for conservative maps. The use of proposition 3.1 will be for general maps. Indeed it can be shown that lemma 2.7 gives a characterization for dissipative unimodal maps: A unimodal map is dissipative iff  $\sum_{i=0}^{\infty} \lambda(f^{-i}(A)) < \infty$  for all sets  $A \subset \cup \mathcal{G}$  where  $\mathcal{G}$  is some  $\lambda$ -partition.

#### 4. Applications to 1-dimensional real dynamics

In this section we will discuss the existence of absolutely continuous invariant measures for maps on the interval having negative Schwarzian derivative. The existence of invariant probability measures is strongly related to the expansion along the orbits of the critical points, the points where the derivative vanishes. In [CE] the existence of acips was shown for unimodal maps having exponential growth of the derivative along the critical orbit. In [NS] this result was obtained for a weaker growth of the derivative, for example (non-linear) polynomial growth turns out to be sufficient.

Another type of existence theorems is described in [Mi] and [S]: if the orbits of the critical points are not accumulating at critical points, maps having this property are called *Misiurewicz* maps, then an acip exists. Here we will describe some results continuing in this direction.

In general the orbit closures of the critical points of Misiurewicz maps will lie in some closed invariant set of measure zero which doesn't contain critical points. In the sequel we will allow our maps to exhibit some recurrence, critical orbits accumulate at critical points but we continue imposing the critical orbits to be in some closed invariant set of measure zero. For this type of maps the existence of acims will be shown. A result in this direction was already obtained in [HKe2]: if the critical point of a conservative unimodal map with negative Schwarzian derivative stays in a Cantor set then an acim exists.

The question whether these unimodal maps are always conservative is the main open problem in the theory of interval dynamics. A natural candidate for having an absorbing Cantor set was recently proved to be conservative, see [LM].

The results presented in this section can be formulated shortly as follows: multimodal maps whose critical orbits lie in a closed invariant set have a  $\sigma$ -finite absolutely continuous invariant measure. In particular, unimodal maps having an absorbing Cantor set have an acim.

In the sequel  $X$  denotes the interval  $[0, 1]$  or the circle endowed with the Lebesgue measure  $\lambda$ .

Let us first define the main analytical tool we need. Let  $g : I \rightarrow J$  be  $C^3$  and mapping the interval  $I \subset X$  to the interval  $J \subset X$ . The Schwarzian derivative  $Sg : I \rightarrow \mathbf{R}$  of  $g$  is defined to be

$$Sg(x) = \frac{D^3g(x)}{Dg(x)} - \frac{3}{2} \left( \frac{D^2g(x)}{Dg(x)} \right)^2.$$

An important and easy to derive property of maps with negative Schwarzian derivative is that the iterates of these maps also have negative Schwarzian derivative.

The Schwarzian derivative enables us to formulate the following distortion result.

**KOEBE-LEMMA.** *For every  $\epsilon > 0$  there exists  $K > 0$  with the following property. Let  $g : I \rightarrow J$  be a diffeomorphism mapping the interval  $I \subset X$  to the interval  $J \subset X$ . Assume that  $Sg(x) < 0$  for all  $x \in I$ .*

If  $M \subset I$  is an interval such that the components of  $I - M$ , denoted by  $L$  and  $R$ , satisfy

$$\frac{\lambda(g(L))}{\lambda(g(M))} \geq \epsilon \text{ and } \frac{\lambda(g(R))}{\lambda(g(M))} \geq \epsilon$$

then

$$\frac{1}{K} \leq \frac{|Dg(x_1)|}{|Dg(x_2)|} \leq K$$

for all  $x_1, x_2 \in M$ .

The proof of this fundamental lemma can be found in different places ([GuJ], [MMS]).

The class  $\mathcal{D}(X)$  of functions which we are going to consider is the class of piecewise diffeomorphic maps on  $X$ . These maps are defined as follows. Let  $f \in \mathcal{D}(X)$  then there exists a  $\lambda$ -partition  $\mathcal{P}$  of  $X$  consisting of open intervals such that for all  $I \in \mathcal{P}$  the restriction  $f|I$  is a diffeomorphism with negative Schwarzian derivative. Furthermore we assume  $f$  to have a dense orbit.

An open interval  $T$  is called a branch of  $f^n$ ,  $f \in \mathcal{D}(X)$ , if  $T$  is a maximal interval on which  $f^n$  is diffeomorphic.

For every map  $f \in \mathcal{D}(X)$  we define the following functions

$$r_n : X \rightarrow \mathbf{R},$$

where  $n \geq 1$  and

$$r_n(y) = \inf\{\epsilon > 0 | B_\epsilon(y) \subset f^i(T_i) \text{ with } T_i \text{ branch of } f^i, i \leq n, \text{ with } y \in f^i(T_i)\}.$$

with  $B_\epsilon(y) = (y - \epsilon, y + \epsilon)$ . Furthermore define  $r : X \rightarrow \mathbf{R}$  to be  $r = \lim r_n$ .

The sufficient condition for the existence of acims will be formulated in terms of the set

$$S = \{y \in X | r(y) > 0\}.$$

**THEOREM 4.1.** *Every conservative ergodic map in  $\mathcal{D}(X)$  having  $\lambda(S) > 0$  exhibits an acim.*

**PROOF:** The set  $S$  is backward invariant. Hence the conservativity of  $f$  implies  $|S| = 1$ .

Let  $S_\rho = r^{-1}(\rho, 1)$  with  $\rho > 0$ .

CLAIM. For every compact set  $I \subset S_\rho$  there exist finitely many pairs  $\{U_i, V_i\}, i = 1, \dots, s$ , of intervals such that

- 1)  $I \subset \cup\{U_i | i = 1, \dots, s\}$ ;
- 2) for all  $i = 1, \dots, s$   $|U_i| = \rho$  and  $U_i \subset V_i$  with both components of  $V_i - U_i$  have length  $\frac{1}{2}\rho$ ;
- 3) if  $T$  is a branch of  $f^n$  with  $f^n(T) \cap (U_i \cap I) \neq \emptyset$  for some  $i \leq s$  then  $V_i \subset f^n(T)$ .

PROOF OF CLAIM: For every  $y \in I$  the interval  $V_y = B_\rho(y) = (y - \rho, y + \rho)$  has the following property: if  $T$  is a branch with  $y \in f^n(T)$  then  $V_y \subset f^n(T)$ .

Consider a branch  $T$  which covers a point  $z \in V_y \cap I, z \in f^n(T)$ . Because  $I \subset S_\rho$  we get immediately  $y \in f^n(T)$ . Hence  $V_y \subset f^n(T)$ . Conclusion: for every  $y \in I$  and every branch  $T$  with  $f^n(T) \cap (V_y \cap I) \neq \emptyset$  we have  $V_y \subset f^n(T)$ .

Let  $U_y = B_{\frac{1}{2}\rho}(y)$ . By using compactness of  $I$  we can cover  $I$  by finitely many intervals of the form  $U_y$ . The corresponding pairs  $\{U_y, V_y\}$  will satisfy the claim.

The claim implies that every compact set  $I \subset S_\rho$  has a natural finite partition in sets  $I_i = I \cap U_i$ . Let us use this partitions for constructing  $\lambda$ -partitions which will allow us to apply Theorem 2.9.

As we saw  $|S| = 1$ . Hence for every set  $K$  with  $|K| > 0$  there exist a compact set  $I \subset K$  of positive Lebesgue measure and a  $\rho > 0$  such that  $I \subset S_\rho$ . This observation easily implies the existence of a  $\lambda$ -partition  $\mathcal{G}_0$  such that for every  $I \in \mathcal{G}_0$   $I \subset S_{\rho_I}$  for some  $\rho_I > 0$ .

Now partition every  $I \in \mathcal{G}_0$  as described above:  $I = \cup_{i=1}^{s_I} I_i$ . Define  $\mathcal{G}$  to be the collection consisting of the sets  $I_i, i = 1, \dots, s_I$  and  $I \in \mathcal{G}_0$ .

For applying Theorem 2.9 we have to bound the distortion of the measures  $f_*^n \lambda | I, I \in \mathcal{G}$ . Fix  $I \in \mathcal{G}$ , say  $I \subset S_\rho$ , and let  $A \subset I$ . The definition of the sets  $I \in \mathcal{G}$  allows us to cover  $f^{-n}(I)$  by branches  $T_1, \dots, T_{k_n}, k_n \in \mathbf{N} \cup \{\infty\}$ , satisfying:

- 1)  $f^n | T_i$  is diffeomorphic;
- 2) both components of  $f^n(T_i) - \{\text{convex-hull}(I)\}$  have length bigger than  $\frac{1}{2}\rho$ .

Now the Koebe-Lemma states the existence of  $K > 0$ , only depending on  $\rho$ , such that

$$\frac{1}{K} \leq \frac{|Df^n(x_1)|}{|Df^n(x_2)|} \leq K$$

for all  $x_1, x_2 \in f^{-n}(I) \cap T_i$ ,  $i = 1, \dots, k_n$ .

Now

$$\begin{aligned} \frac{\lambda(f^{-n}(A))}{\lambda(f^{-n}(I))} &= \frac{\sum_{i=1}^{k_n} \lambda(f^{-n}(A) \cap T_i)}{\lambda(f^{-n}(I))} \\ &= \frac{\sum_{i=1}^{k_n} \frac{\lambda(f^{-n}(A) \cap T_i)}{\lambda(f^{-n}(I) \cap T_i)} \lambda(f^{-n}(I) \cap T_i)}{\lambda(f^{-n}(I))} \\ &= \frac{\sum_{i=1}^{k_n} \frac{\frac{\lambda(I)}{\lambda(f^{-n}(I) \cap T_i)} \frac{\lambda(A)}{\lambda(I)} \lambda(f^{-n}(I) \cap T_i)}{\frac{\lambda(A)}{\lambda(f^{-n}(A) \cap T_i)}}}{\lambda(f^{-n}(I))}. \end{aligned}$$

Using the mean value theorem and the distortion result above we get

$$\frac{1}{K} \frac{\lambda(A)}{\lambda(I)} \leq \frac{\lambda(f^{-n}(A))}{\lambda(f^{-n}(I))} \leq K \frac{\lambda(A)}{\lambda(I)},$$

the measures  $f_*^n \lambda|I$  have distortion bounded by  $K$ . We proved  $\lambda \in \mathcal{M}(\mathcal{G}, f)$ . Hence Theorem 2.9 states the existence of an acim. ■

A map  $f \in \mathcal{D}(X)$  is called a *Markov map* if there exists a  $\lambda$ -partition  $\mathcal{P}$  consisting of intervals such that for every  $I \in \mathcal{P}$  the image  $f(I)$  is a union (up to a set of measure zero) of elements of  $\mathcal{P}$ . The map  $f$  is said to satisfy the Markov property with respect to  $\mathcal{P}$ . Obviously these Markov maps have  $r(y) > 0$  for all  $y \in \cup \mathcal{P}$ .

**COROLLARY 4.2.** *Every conservative ergodic Markov map has an acim.*

This statement has to be compared with a theorem of Harris (see [H]) stating the existence of infinite stationary states for certain Markov processes on countable many state spaces. In fact examples are known of Markov process not having a stationary state (see [D]). These examples also serve for showing that we cannot omit the conservativity. On the other hand we can weaken this condition by imposing a topological condition. By doing so we kill the metrical subtleties and get a general existence theorem which is valid as well in the conservative as in the dissipative case.

**$\sigma$ -FOLKLORE THEOREM.** *Every finite-to-1 Markov map has an acim.*

**PROOF:** Suppose  $f$  satisfies the Markov property with respect to  $\mathcal{G}_0$ . Let  $\mathcal{G}$  be a  $\lambda$ -partition refining  $\mathcal{G}_0$  and consisting of intervals. These intervals  $I \in \mathcal{G}$  are chosen

in such a way that both components of  $T - I$  have length bigger than  $\lambda(I)$ , where  $T \in \mathcal{G}_0$  with  $I \subset T$ .

Once we proved

$$PL_\lambda(\mathcal{G}_0) \subset \mathcal{M}(\mathcal{G}, f)$$

Theorem 3.1 assures the existence of an acim.

Fix  $I \in \mathcal{G}$  with  $I \subset T \in \mathcal{G}_0$ . Furthermore let  $A \subset I$ . Because  $f$  is a finite-to-1 Markov map the set  $f^{-n}(I)$ ,  $n \geq 0$ , can be covered by finitely many intervals  $T_1, \dots, T_{k_n}$  satisfying

- 1)  $f^n|_{T_i}$  is diffeomorphic;
- 2)  $f^n(T_i) = T$ ;
- 3) there exists  $T'_i \in \mathcal{G}_0$  with  $T_i \subset T'_i$ .

for all  $i = 1, \dots, k_n$ .

Take  $\mu \in PL_\lambda(\mathcal{G}_0)$ . To prove that  $\mu \in \mathcal{M}(\mathcal{G}, f)$  first we have to show that  $\mu(f^{-n}(I)) < \infty, n \geq 0$ . However this is a direct consequence of  $f$  being finite-to-1. Secondly we have to study the local distortion of the measures  $f_*^n \mu$ .

Again the Koebe-Lemma gives a constant  $K > 0$  such that

$$\frac{1}{K} \leq \frac{|Df^n(x_1)|}{|Df^n(x_2)|} \leq K$$

for all  $x_1, x_2 \in f^{-n}(I) \cap T_i, i = 1, \dots, k_n$ .

As before

$$\begin{aligned} \frac{\mu(f^{-n}(A))}{\mu(f^{-n}(I))} &= \frac{\sum_{i=1}^{k_n} \frac{\frac{\lambda(I)}{\mu(f^{-n}(I) \cap T_i)}}{\frac{\lambda(A)}{\mu(f^{-n}(A) \cap T_i)}} \frac{\lambda(A)}{\lambda(I)} \mu(f^{-n}(I) \cap T_i)}{\mu(f^{-n}(I))} \\ &= \frac{\sum_{i=1}^{k_n} \frac{\frac{\lambda(I)}{\lambda(f^{-n}(I) \cap T_i)}}{\frac{\lambda(A)}{\lambda(f^{-n}(A) \cap T_i)}} \frac{\lambda(A)}{\lambda(I)} \mu(f^{-n}(I) \cap T_i)}{\mu(f^{-n}(I))}. \end{aligned}$$

In the last step we used the fact that all measures in  $PL_\lambda(\mathcal{G})$  have constant densities on the elements  $T \in \mathcal{G}_0$  and property 3) above. Using the mean value theorem and the distortion result above we get

$$\frac{1}{K} \frac{\lambda(A)}{\lambda(I)} \leq \frac{\mu(f^{-n}(A))}{\mu(f^{-n}(I))} \leq K \frac{\lambda(A)}{\lambda(I)}.$$

We proved  $\mu \in \mathcal{M}(\mathcal{G}, f)$ . ■

The usual Folklore theorem states that every Markov map having derivative bigger and bounded away from 1 has an absolutely continuous invariant probability measure.

Another possible  $\sigma$ -Folklore theorem could be formulated by considering Markov maps whose branches are all mapped onto  $X$ .

As the main consequence of the  $\sigma$ -Folklore Theorem we get Theorem B of the introduction.

**COROLLARY 4.3.** *Let  $f$  be a  $C^3$  map on the interval (or circle) satisfying*

- 1)  *$f$  has only finitely many critical points and the Schwarzian derivative is everywhere negative except in the critical points;*
- 2) *there exists a dense orbit;*
- 3) *the orbits of the critical points stay in a closed invariant set of Lebesgue measure zero.*

*Then  $f$  has a  $\sigma$ -finite absolutely continuous invariant measure.*

**PROOF:** Let  $\Lambda$  be a closed invariant set which contains the critical orbits. Assume it has Lebesgue measure zero. Furthermore consider the  $\lambda$ -partition  $\mathcal{P}$  consisting of the gaps of this set (the connected components of its complement). The map we are considering has only finite critical points. Hence it is finite-to-1. In other words the map is a finite-to-1 Markov map. ■

In the unimodal case the last corollary can be stated simpler. In the unimodal case the measure of the Cantor set containing the critical orbit always has Lebesgue measure zero ([M]). So, unimodal maps which are only finitely renormalizable and having their critical orbit in an invariant Cantor set have acims.

### **Appendix: The Chacon-Ornstein Theorem**

In this section we will give a short proof of the main theorem in conservative  $\sigma$ -finite ergodic theory. It is based on the classical Birkhoff Ergodic Theorem. The proof can be summarized by: a map has an acim iff the return map on some set has an acip.

**THEOREM A.1.** *Let  $f : X \rightarrow X$  be ergodic and conservative with respect to  $\mu \in \mathcal{B}_\sigma(X)$  which is a  $\sigma$ -finite invariant measure for  $f$ . For every pair of Riemann integrable functions*

$\phi, \psi : X \rightarrow \mathbf{R}$  the equality

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=0}^{n-1} \phi(f^i(x))}{\sum_{i=0}^{n-1} \psi(f^i(x))} = \frac{\int \phi d\mu}{\int \psi d\mu}$$

holds for  $\mu$ -almost every point  $x \in X$ .

PROOF: Let  $B \subset X$  having  $\mu(B) < \infty$ . Using the fact that  $f$  is ergodic and conservative we can write the space as a stack:  $X$  is up to a set of measure zero equal to a countable union of pairwise disjoint sets  $B_k, k \geq 0$ , where

$$B_k = \{x \in X \mid x, f(x), \dots, f^{k-1}(x) \notin B \text{ and } f^k(x) \in B\}.$$

The key for the theorem is

CLAIM: Let  $A \subset B_k$  for some  $k \geq 0$ . Then for  $\mu$ -almost every point  $x \in X$

$$\lim_{n \rightarrow \infty} \frac{\#_n(A)}{\#_n(B)} = \frac{\mu(A)}{\mu(B)}.$$

Here  $\#_n(U) = \#\{i = 0, 1, \dots, n-1 \mid f^i(x) \in U\}$ .

PROOF OF CLAIM: The return map on  $B$  is denoted by  $R : B \rightarrow B$ . It has the following properties

- 1)  $R$  is ergodic;
- 2) for all  $x \in B$  the set  $\{f(x), \dots, R(x)\}$  contains at most one point of  $A$ ;
- 3) the measure  $\frac{\mu}{\mu(B)}$  is an acip for  $R$ .

Let

$$X_A = \{x \in B \mid \{f(x), \dots, R(x)\} \cap A \neq \emptyset\}.$$

Then

- 4)  $\mu(X_A) = \mu(A)$ .

The statements 1) and 2) are obvious. Let us prove 3) and 4). Consider a set  $A \subset B_k$  and define inductively the sets  $A_l$  with  $l \geq k$ :

$$\begin{aligned} A_k &= A; \\ A_{l+1} &= f^{-1}(A_l) \cap B_{l+1}. \end{aligned}$$

Define  $R_{l+1} = f^{-1}(A_l) \cap B$  for  $l \geq k$ . Using induction we get

$$\mu(A) = \sum_{l=k+1}^n \mu(R_l) + \mu(A_n)$$

for all  $n \geq k$ . Applying this to  $A = B$  and using the fact that almost every point in  $B$  returns to  $B$  we get  $\mu(B_k) \rightarrow 0$  for  $k \rightarrow \infty$ .

Consider  $A \subset B_k$  and observe  $X_A = \cup_{l=k+1}^{\infty} R_l$ . Using  $\mu(A_n) \leq \mu(B_n) \rightarrow 0$  we get

$$\mu(X_A) = \mu\left(\bigcup_{l=1}^{\infty} R_l\right) = \sum_{l=1}^{\infty} \mu(R_l) = \mu(A).$$

We proved 4). Furthermore observe  $R^{-1}(A) = X_A$  for every  $A \subset B$ . We proved 3).

The proof of the claim is based on the Birkhoff Ergodic theorem. Consider a point  $x$  whose orbit behaves according to the invariant measure of  $R$ . Let  $y = f^{i_0}(x)$  be the first time when  $f^i(x) \in B$ . Furthermore partition the orbit according to the returns to  $B$ . Then we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\#_n(A)}{\#_n(B)} &= \lim_{n \rightarrow \infty} \frac{\#\{i = 0, 1, \dots, \#_n(B) - 1 \mid R^i(y) \in X_A\}}{\#_n(B)} \\ &= \frac{\mu(X_A)}{\mu(B)} = \frac{\mu(A)}{\mu(B)}. \end{aligned}$$

Observe that up to time  $i_0$  we hit  $A$  at most  $i_0$  times. Furthermore the part of the orbit from  $R^{\#_n(B)}(x)$  to  $R^{\#_n(B)+1}(x)$  hit  $A$  at most once. The conservativity and ergodicity implies  $\#_n(B) \rightarrow \infty$ . Hence the initial and final part of the orbit are not influencing the limit. We proved the claim.

For general sets  $A \subset X$  we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\#_n(A)}{\#_n(B)} &= \liminf_{n \rightarrow \infty} \sum_{k \geq 0} \frac{\#_n(A \cap B_k)}{\#_n(B)} \\ &\geq \sum_{k \geq 0} \frac{\mu(A \cap B_k)}{\mu(B)} = \frac{\mu(A)}{\mu(B)}. \end{aligned}$$

Using the symmetry in  $A$  and  $B$  we get

$$\frac{\mu(A)}{\mu(B)} \geq \frac{1}{\limsup_{n \rightarrow \infty} \frac{\#_n(B)}{\#_n(A)}} = \liminf_{n \rightarrow \infty} \frac{\#_n(A)}{\#_n(B)} \geq \frac{\mu(A)}{\mu(B)}.$$

This implies, again using the symmetry, the equality

$$\lim_{n \rightarrow \infty} \frac{\#_n(A)}{\#_n(B)} = \frac{\mu(A)}{\mu(B)}.$$

The Chacon-Ornstein Theorem obviously follows for linear combinations of indicator functions. The general statement is a direct consequence of the definition of Riemann integrability. ■

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