ON REMOVABLE SETS FOR SOBOLEV SPACES IN THE PLANE

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Let K be a compact subset of $\bar{\mathbf{C}} = \mathbf{R}^2$ and let K^c denote its complement. We say $K \in HR$, K is holomorphically removable, if whenever $F : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is a homeomorphism and F is holomorphic off K, then F is a Möbius transformation. By composing with a Möbius transform, we may assume $F(\infty) = \infty$. The contribution of this paper is to show that a large class of sets are HR. Our motivation for these results is that these sets occur naturally (e.g. as certain Julia sets) in dynamical systems, and the property of being HR plays an important role in the Douady-Hubbard description of their structure. (See [**4**].)

To prove that the sets in question are HR we establish what may be a stronger result. A compact set K is said to be removable for $W^{1,2}$ if every f which is continuous on $\mathbf{R}^2$ and in the Sobolev space $W^{1,2}(K^c)$ (one derivative in L^2 on K^c) is also in $W^{1,2}(\mathbf{R}^2)$. It is a fact that if K is removable for $W^{1,2}$, K is HR. We do not know the answer to the following question:

If K is HR, is K removable for $W^{1,2}$?

To prove the fact we first show that the two dimensional Lebesgue measure of K, $|K|$, is zero. If not let $F_n = \frac{1}{\pi z} * (e^{in(x+y)}\chi_K(z))$. then $\lim_{n\to\infty} \|F_n\|_{L^\infty(\mathbf{R}^2)} = 0$ and F_n is continuous. Since $|\bar{\partial} F_n| = \chi_n$, $\|\bar{\partial} F_n\|_{L^2(\mathbf{R}^2)} = |K|^{1/2}$. On the other hand, $F_n(z) \to 0$ for $z \notin K$, so L^4 bounds on convolution with $\frac{1}{\pi z^2}$ when combined with Hölder's inequality show $\|F_n'\|_{L^2(K^c)} \to 0$. (See [**11**], Chapter 1.) Taking a sum of functions like the F_n, we obtain a globally continuous $F \in W^{1,2}(K^c)$, $F \notin W^{1,2}(\mathbf{R}^2)$. Now using the fact that $|K| = 0$, we deduce $K \in HR$. Take a homeomorphism F with $F'(\infty) = 1$. Then $f(z) = F(z) - z \in W^{1,2}(K^c)$ because integrating $|F'|^2$ gives the area of the image. Now $f \in W^{1,2}(\mathbf{R}^2)$ and $\bar{\partial} f = 0$ except on a set of measure zero implies (Weyl's lemma) f is holomorphic. Therefore $F(z) = z + a$.

We recall some elementary facts concerning HR. If $|K| > 0$, it follows from the "measurable Riemann mapping theorem" (see [**1**]) that there is a nontrivial quasiconformal mapping F which is holomorphic off K. (Thus $K \notin HR$.) If K has Hausdorff dimension less than 1, $\mathrm{Dim}(K) < 1$, the fact that $K \in HR$ follows from the Cauchy integral formula (Painleve's theorem). Similarly, if K is a rectifiable curve, Morera's theorem implies $K \in HR$. Kaufman [**7**] has produced examples of curves where $\mathrm{Dim}(K) = 1$ but $K \notin HR$. The "difficult" case is the one that occurs in conformal dynamics: K is connected and has some "fractal" properties. (The case of "pure" Cantor-type sets is easy; they are HR. By a pure Cantor set, we mean e.g. one arising from a Cantor construction with a constant ratio of dissection, or the Julia set for $z^2 + c$ where c is not in the Mandelbrot set.) We also point out that the case where K is a quasicircle seems to be folklore - again, $K \in HR$. That the property of being HR is related to quasiconformal mappings is seen from the following

Remark. *K is HR if and only if whenever F is a homeomorphism of $\bar{\mathbf{C}}$ which is M quasiconformal on K^c, F is globally quasiconformal (and hence M quasiconformal). (See [**8**], page 200.)*

To prove the remark, first assume that K is HR. By the measurable Riemann mapping theorem there is a globally quasiconformal mapping G such that $G \circ F$ is holomorphic off K. Since $G \circ F$ is a Möbius transformation and $|K| = 0$, F is globally (M) quasiconformal. For the other direction, standard L^p estimates (see [**1**]) show that necessarily $|K| = 0$. If F is a homeomorphism which is analytic off K, F is globally quasiconformal and hence ($|K| = 0$) a Möbius transformation.

For Ω be a domain on the Riemann sphere and let $z_0 \in \Omega$. Then Ω is a John domain (with center z_0) if there is $\varepsilon > 0$ such that for all $z_1 \in \Omega$ there is an arc $\gamma \subset \Omega$ which connects z_0 to z_1 and has the property that

$$d(z) \geq \varepsilon d(z, z_1), \quad z \in \gamma.$$

Here $d(z, z_1)$ is the chordal distance from z to z_1 and $d(z)$ is the chordal distance of z to $\partial\Omega$ We call such an arc γ a John arc. In this paper we will choose coordinates so that $z_0 = \infty$, and this allows us to replace $d(z), d(z, z_1)$ by the corresponding Euclidean distances. The property of being a John domain is preserved under globally quasiconformal mappings. If

Ω is a simply connected John domain, it is easy to show that the arc γ may be taken to be the hyperbolic geodesic from z_0 to ∞. (See [**9**] for an exposition of properties of John domains.) The main result of this paper is

THEOREM 1. *If Ω is a John domain and $K = \partial\Omega$, then K is removable for $W^{1,2}$.*

Notice that the hypothesis demands that $K = \partial\Omega$, but says *nothing* about the other components of $\mathbf{C}\backslash K$. This is because the hypothesis will be seen to force some geometry on those other components. (For example, the interior of a cardioid is a John domain while the exterior is not. The parabolic basin for $z^2 + \frac{1}{4}$ is also a John domain, while the basin for ∞ - the exterior domain - is not.) It is of some philosophical interest to note the similarities between Theorem 1 and the results of [**6**] on extension problems for Sobolev spaces.

Since the John condition is quasiconformally invariant, we obtain directly (see also "Remark")

COROLLARY 1. *If Ω is a John domain and $K = \partial\Omega$, any global homeomorphism which is quasiconformal off K is globally quasiconformal (with the same constant of quasiconformality).*

We say that a polynomial $P(z)$ is subhyperbolic on its Julia set J if there is a metric $\lambda(z)|dz|$ such that $\lambda(z) - \sum_j |z - z_j|^{-\alpha_j}$ is C^∞ for some numbers $\alpha_j < 1$, and $P(z)$ is hyperbolic on J in the metric λ. In other words there are $c, \varepsilon > 0$ such that for all $n \geq 1$,

$$\lambda(z)^{-1}\lambda(P_n(z))|\frac{d}{dz}P_n(z)| \geq c(1+\varepsilon)^n.$$

Here $P_n(z) = P \circ \cdots \circ P(z)$ is the n^{th} iterate of P. (This definition may be a bit restrictive, but it is all we will need for this paper.) The following question is open:

If J is the Julia set for a polynomial, is $J \in HR$?

It is proven in [**3**] that whenever a polynomial $P(z)$ is subhyperbolic on its Julia set J, then A_∞, the basin of attraction at ∞ for P, is a John domain. Since $J = \partial A_\infty$, we obtain

Corollary 2. *If $P(z)$ is subhyperbolic on its Julia set J, then $J \in HR$.*

The corollary answers a question of A. Douady and J. Hubbard and was the starting point of this investigation. Douady posed the question to the author for the particular (subhyperbolic) case where $P(z) = z^2 + c$ has the (Misiurewicz) property that the origin is preperiodic but not periodic (e.g. $z^2 + i$). This case is not fundamentally different for the general case of subhyperbolic polynomials. An amusing feature of our proof is that the Julia set for a Misiurewicz point (from the family $z^2 + c$) is actually easier to deal with than those arising from the hyperbolic case. (When $K^c = \Omega$, our argument is a bit simpler. The arguments of Sections 5 and 6 are not needed.)

The proof of Theorem 1 starts by proving it in the case where Ω is simply connected on $\bar{\mathbf{C}}$, i.e. K is connected. The general case then follows from

Theorem 2. *If Ω is an (ε) John domain, there is a $(c(\varepsilon))$ John domain Ω' with Ω' simply connected and*

$$\partial\Omega \subset \partial\Omega'.$$

While the proof of Theorem 2 is perhaps not immediately obvious, it turns out to follow from a simple construction with planar graphs.

Section 2 contains background material, and Sections 3-7 are devoted to the proof of Theorem 1. The idea is to redefine F near K so that it is C^∞ near K and so that the Sobolev norm does not change much. Theorem 2 is proven in Section 8.

§2. Background Material

Let $F \in W^{1,2}(K^c)$ be continuous on $\mathbf{C}$. An easy argument with the Dirichlet principle shows that to prove $F \in W^{1,2}(\mathbf{R}^2)$ it is sufficient to treat the case we now assume, where F is harmonic near K. We also assume the reader is familiar with elementary properties of logarithmic capacity, which we denote by $\mathrm{Cap}(\circ)$. See e.g. [**10**] for the first two of the next three lemmata. Let $f : \mathbf{D} \to \mathbf{C}$ be univalent, $f(0) = 0, f'(0) = 1$. Then f has a Fatou extension to $\mathbf{T} = \partial\mathbf{D}$ and this extension is always defined except on a set of capacity (and hence Lebesgue measure) zero. In our applications, all image domains will have locally connected boundaries, and hence f will be continuous on $\bar{\mathbf{D}}$. The following results are due to Beurling. (See e.g. [**10**] for Lemmata 2.1 and 2.2.) The values of c below are various universal constants.

Lemma 2.1. *If $E \subset \mathbf{T}$,*

$$Cap(f(E)) \geq c\ Cap(E)^2.$$

Lemma 2.2. *Let $g_\theta = f(\{re^{i\theta} : 0 \leq r < 1\})$. Then if $\ell(\cdot)$ denotes arclength,*

$$Cap(\{e^{i\theta} : \ell(g_\theta) > \lambda\}) \leq c\lambda^{-1/2}$$

Lemma 2.3. *Suppose H is harmonic and continuous in $\mathbf{D}$, and $(|\nabla H|^2 = |H_x|^2 + |H_y|^2)$,*

$$\iint_{\mathbf{D}} |\nabla H|^2 dxdy = 1.$$

Then

$$Cap(\{e^{i\theta} : |H(e^{i\theta}) - H(0)| \geq \lambda\}) \leq ce^{-\pi\lambda^2}.$$

This last lemma can be found on page 30 of [**2**]. We next require some elementary geometric facts about simply connected John domains. For the next result see [**5**].

Lemma 2.4. *If g is a Poincaré geodesic from ∞ to $z_0 \in \partial\Omega$ where Ω is an (ε) John domain, then g is an arc of a $K(\varepsilon)$ quasicircle.*

Suppose now Ω is a bounded (ε) John domain and suppose the John center z_0 satisfies $d(z_0) = 1$, where

$$d(z) = \text{ distance}(z, \partial\Omega).$$

Then diameter$(\Omega) \sim 1$. Let $f : \mathbf{D} \to \Omega, f(0) = z_0$ be any choice of Riemann mapping, and define for $E \subset \partial\Omega$,

$$\text{Cap}(E, z_0, \Omega) \equiv \text{ Cap}(\{e^{i\theta} : f(e^{i\theta}) \in E\}).$$

LEMMA 2.5. *For any Borel set $E \subset \partial\Omega$,*

$$Cap(E, z_0, \Omega) \sim \; Cap(E).$$

In the last line we mean that $A \sim B$ if there is a constant $M = M(\varepsilon)$ such that

$$M^{-1}A^M \leq B \leq MA^{1/M}.$$

To prove the lemma let $G(z) = G(z, z_0)$ be Green's function for Ω with pole at z_0. Then it follows from the John condition and the Koebe $\frac{1}{4}$ theorem that

$$G(z) \geq cd(z)^\alpha \ , \ \alpha = \alpha(\varepsilon),$$

whenever $|z - z_0| \geq \frac{1}{2}$. Suppose now that $z_j \in E, z_j = f(\zeta_j), j = 1, 2$. Fix a point $\zeta_3 \in \mathbf{D}$ such that

$$(1 - |\zeta_3|) \sim |\zeta_1 - \zeta_3| \sim |\zeta_2 - \zeta_3| \sim |\zeta_1 - \zeta_2|$$

and let $z_3 = f(\zeta_3)$. Then by the John condition

$$|z_1 - z_2| \leq Cd(z_3),$$

while by our last estimate,

$$d(z_3) \leq CG(z_3)^{1/\alpha} \sim C|\zeta_1 - \zeta_2|^{1/\alpha}.$$

In other words,

$$|\zeta_1 - \zeta_2| \geq c|z_1 - z_2|^\alpha,$$

and it follows from the definition of logarithmic capacity that

$$\mathrm{Cap}(E, z_0, \Omega) \geq M^{-1}\mathrm{Cap}(E)^M.$$

The other direction of the lemma follows from Lemma 2.1. ∎

LEMMA 2.6. *Suppose Ω_j are (ε) John domains with centers $z_j, j = 1, 2$, and suppose $d(z_1), d(z_2) \sim 1$. Suppose also that F is harmonic on $\Omega_1 \cup \Omega_2$ and continuous on $\bar{\Omega}_1 \cup \bar{\Omega}_2$. Then if $E \subset \partial\Omega_1 \cap \partial\Omega_2$ satisfies*

$$Cap(E) \geq \delta > 0,$$

there are geodesics $g_j \subset \Omega_j$ from z_j to $\partial\Omega_j$ such that g_1 and g_2 terminate at the same point $\zeta \in \partial\Omega_1 \cap \partial\Omega_2$, and

$$|F(\zeta) - F(z_j)| \leq A(\varepsilon, \delta)\Big(\iint_{\Omega_j} |\nabla F|^2 dxdy\Big)^{1/2}, \ j = 1, 2.$$

PROOF: Let

$$E_j = \{z \in \partial\Omega_j : |F(z) - F(z_j)| \geq \lambda\Big(\iint_{\Omega_j} |\nabla F|^2 dxdy\Big)^{1/2}\}.$$

If λ is large enough, Lemmata 2.3 and 2.5 show $\text{Cap}(E_1 \cup E_2) < \delta$. Then $E \backslash (E_1 \cup E_2) \neq \phi$, so we may select ζ from that set. ∎

§3. Quasicircles

We now give a quick outline of our proof for the case where K is a quasicircle. This represents the only idea of the paper. The rest of the sections contain only technical arguments which make the same philosophy work for the general case.

Let Ω_+ and Ω_- denote respectively the unbounded and bounded components of $\bar{\mathbf{C}} \backslash K$. Fix two points $z_\pm \in \Omega_\pm$ satisfying

$$\delta(z_+) \sim \delta(z_-) \sim |z_+ - z_-| \sim \delta,$$

and build domains $\mathcal{D}_\pm \subset \Omega_\pm$ which are bounded by quasicircles and such that $\partial\mathcal{D}_+ \cap \partial\mathcal{D}_-$ is a subarc of K with diameter $\sim \delta$. The points $z_\pm$ are made to be the "centers" of $\mathcal{D}_\pm$. Then by Lemma 2.6 there is A such that

$$|F(z_+) - F(z_-)| \leq A\Big(\iint_{\mathcal{D}_+ \cup \mathcal{D}_-} |\nabla F|^2 dxdy\Big)^{1/2}.$$

Standard smoothing techniques now show there is $\tilde{F} \in W^{1,2}(\mathbf{R}^2)$ such that $\tilde{F} = F$ outside of $K_\delta = \{z : d(z) \leq \delta\}$, $\tilde{F}$ is C^∞ near K, and

$$\iint_{K_\delta} |\nabla \tilde{F}|^2 dxdy \leq c \iint_{K_{c\delta}} |\nabla F|^2 dxdy.$$

Sending δ to zero we see that $F \in W^{1,2}(\mathbf{R}^2)$ and

$$\iint_{\mathbf{R}^2} |\nabla F|^2 dxdy = \iint_{K^c} |\nabla F|^2 dxdy.$$

If F is M quasiconformal on K^c, Lemma 2.2 and an argument similar to the one above show that F is globally quasiconformal. The point of this vague remark is that, *whatever argument we use*, it should show that F being M quasiconformal on K^c implies F is globally quasiconformal. (See the "Remark" in Section 1.)

§4. Some Geometry

In this section we construct certain domains related to a point $x_0 \in K$ and a scale r. Since the John condition is scale invariant, we may assume $x_0 = 0$ and $r = 1$. We will add to K certain curves to obtain a new set $\hat{K}$ so that, in a certain sense, $\mathbf{C} \backslash \hat{K}$ looks like a union of quasidisks of diameter about 1 (near K).

Let $f : \mathbf{D}^* = \{|z| > 1\} \to \Omega$ be univalent with $f(\infty) = \infty$. Since $K = \partial\Omega$ is locally connected, f is continuous up to $\mathbf{T}$. Select angles $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_N = 2\pi$ so that

$$|f(e^{i\theta}) - f(e^{i\theta_j})| \leq 1 \ , \ \theta_j \leq \theta \leq \theta_{j+1},$$

and

$$|f(e^{i\theta_j}) - f(e^{i\theta_{j+1}})| \geq \frac{1}{2}.$$

Now fix $M \geq 1$ and let $r_j < 1$ be the largest value of r so that

$$|f(re^{i\theta_j}) - f(e^{i\theta_j})| = M.$$

Setting $L_j = \{re^{i\theta_j}, r_j \le r < 1\}$ we see that

$$\text{distance}(L_j, L_k) \ge c(1 - r_j)\ ,\ j \ne k, \tag{4.1}$$

for otherwise the John condition would be violated for the corresponding geodesics in Ω.

LEMMA 4.1. $|1 - r_j| \sim |1 - r_{j+1}| \sim$ *distance*(L_j, L_{j+1}).

PROOF: We show that $|\theta_{j+1} - \theta_j| \le C(1 - r_j)$. The proof that $|\theta_{j+1} - \theta_j| \le C(1 - r_{j+1})$ is the same. The lemma will then follow from (4.1). Let $I = \{e^{i\theta} : \theta_j \le \theta \le \theta_j + \pi\}$. By symmetry

$$\omega(\zeta_j, I, \mathbf{D}^*) = \frac{1}{2},$$

where $\zeta_j \equiv r_j e^{i\theta_j}$. Here $\omega(z, E, \mathcal{D})$ denotes the harmonic measure at z of $E \subset \partial\mathcal{D}$ in $\mathcal{D}$. By Beurling's so-called $\frac{1}{2}$ theorem [**10**], if we set $I_j = \{e^{i\theta} : \theta_j \le \theta \le \theta_{j+1}\}$,

$$\omega(\zeta_j, I_j, \mathbf{D}) = \omega(f(\zeta_j), f(I_j), \Omega) \le CM^{-\frac{1}{2}},$$

because diameter $(f(I_j)) \le 1$ and distance $(f(\zeta_j), f(I_j)) \ge M - 1$. Thus

$$\omega(\zeta_j, I \backslash I_j, \mathbf{D}^*) \ge \frac{1}{4}$$

if M is large enough, and the lemma follows from simple estimates on the Poisson kernel. ∎

Let $\mathcal{D}_j$ be the domain bounded by $\mathbf{T}, L_j, L_{j+1}$, and the line segment $[\zeta_j, \zeta_{j+1}]$ and let $\tilde{\zeta}_j = R_j e^{i\varphi_j}$ where $R_j - 1 = \frac{1}{2}\min(r_j - 1, r_{j+1} - 1)$, and $\varphi_j = \frac{1}{2}(\theta_j + \theta_{j+1})$. Then since we are assuming diameter $(K) >> 1$, each $\mathcal{D}_j$ looks like a quadrilateral (in $\mathbf{D}^*$ with one side on $\mathbf{T}$) with bounded geometry.

LEMMA 4.2. $\Omega_j = f(\mathcal{D}_j)$ *is an* (ε') *John domain with John center* $z_j = f(\tilde{\zeta}_j)$.

PROOF: Let $\zeta \in \mathcal{D}_j$ and let $L = [\zeta, \tilde{\zeta}_j]$ be the line segment from ζ to $\tilde{\zeta}_j$. Then $L \subset \mathcal{D}_j$ and if $\zeta' \in L$, distance$(\zeta', \partial\mathcal{D}_j) \ge c|\zeta' - \zeta|$. (This follows from the elementary geometry of $\mathcal{D}_j$.) Now if L' is the geodesic from ζ to ∞ in $\mathbf{D}^*$, $L' = \{R\zeta : R \ge 1\}$, $\rho(\zeta', L') \le C$

for all $\zeta' \in L$, where ρ is the hyperbolic metric on $\mathbf{D}^*$. The lemma now follows from the John property on the arc $f(L')$ and the distortion theorem for f. The details are left to the reader. ∎

Lemma 4.2 is actually a special case of the following fact:

If $f : \mathbf{D} \to \Omega$, $f(0) = z_0$, and Ω is an (ε) John domain with John center z_0, and if $\mathcal{D} \subset \mathbf{D}$ is a (δ) John domain with John center the origin, then $f(\mathcal{D})$ is an $\eta(\varepsilon, \delta)$ John domain with John center z_0.

We leave a proof of this statement as an exercise for the reader.

At this point we remark that $\hat{\Omega}_j$ = interior of $\bar{\Omega}_j$ is a $\delta(\varepsilon)$ quasicircle if $\bar{\mathbf{C}}\backslash K = \Omega$. (This is e.g. the case for the Julia set corresponding to $z^2 + i$.) A most unfortunate complication is that this statement is easily seen to be false if $\bar{\mathbf{C}}\backslash K$ is allowed to have bounded components. This necessitates the technical construction of our next section. The reader interested only in the case where $\Omega = \bar{\mathbf{C}}\backslash K$ may skip to Section 7, noticing that Proposition 6.1 has already been proven for quasicircles.

§5. Some Additional Curves

We now add some additional curves to K. Let $\mathcal{O}_j$ be a bounded component of $\bar{\mathbf{C}}\backslash K$. Then by the definition of the domains Ω_k, each $\partial\Omega_k$ intersects $\partial\mathcal{O}_j$ in either a connected set or the empty set. Let us for the moment reorder the Ω_k so that $\Omega_1, \ldots, \Omega_N$ are exactly those domains such that $\partial\Omega_n \cap \partial\mathcal{O}_j$ consists of more than one point (and hence an arc). Let $\delta > 0$ be a small constant to be fixed later and fix a Riemann mapping $f_j : \mathbf{D} \to \mathcal{O}_j$ so that $I_1, \ldots, I_N$ are intervals with $f_j(I_n) = \partial\Omega_n \cap \partial\mathcal{O}_j$. By selecting $f_j(0)$ to lie very close to $\partial\Omega_1 \cap \partial\mathcal{O}_j$ we may assume $\ell(I_1) \approx 2\pi$. Let T_1 be the tent shaped region bounded by $\mathbf{T}\backslash I_1$ and two straight lines in $\mathbf{D}$ which intersect $\mathbf{T}\backslash I_1$ at angle δ. The T_1 is a "thin sliver". Define $\mathcal{U}_1 = \hat{\mathcal{U}}_1 = \mathbf{D}\backslash T_1$ so that $\partial\mathcal{U}_1 \cap \mathbf{T} = I_1$. For $n \geq 2$ let L_n^1, L_n^2 be the two lines which start at the endpoints of I_n, go into $\mathbf{D}$, and make angle $= \delta$ with $\mathbf{T}\backslash I_n$. Let $J_n = \{(1 - \delta^{-1}\ell(I_n))e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, and let $\hat{\mathcal{U}}_n$ be the domain bounded by the four arcs I_n, L_n^1, L_n^2, J_n. Then $\hat{\mathcal{U}}_n$ almost fills up a rectangle with length (along $\mathbf{T}$) $= \delta^{-2}\ell(I_n)$ and width (in the direction orthogonal to $\mathbf{T}$) $= \delta^{-1}\ell(I_n)$. Then by elementary estimates on the Poisson kernel,

$$\{\omega(z, I_n, \mathbf{D}) \geq c_1\delta\} \subset \hat{\mathcal{U}}_n \subset \{\omega(z, I_n, \mathbf{D}) > c_2\delta\}. \tag{5.1}$$

By reordering we may now assume that

$$\ell(I_2) \geq \ell(I_3) \geq \cdots \geq \ell(I_N).$$

Define $\mathcal{U}_n = \hat{\mathcal{U}}_n \backslash \bigcup_{k=1}^{n} \mathcal{U}_k$ so that $\bigcup_{n=1}^{N} \mathcal{U}_n = \bigcup_{n=1}^{N} \hat{\mathcal{U}}_n$. Recall that a domain $\mathcal{U}$ is called an M Lipschitz domain if there is $z_0 \in \mathcal{U}$ and $R > 0$ such that

$$\partial\mathcal{U} = \{z_0 + Rr(\theta)e^{i\theta} : 0 \leq \theta \leq 2\pi\}$$

where

$$(1+M)^{-1} \leq r(\theta) \leq 1 \quad \text{for all } \theta$$

and

$$|r(\theta) - r(\theta')| \leq M|\theta - \theta'|.$$

LEMMA 5.1. *$\mathcal{U}_n$ is a $M(\delta)$ Lipschitz domain, $1 \leq n \leq N$. Furthermore, if $\zeta \in \mathbf{D} \cap \partial\mathcal{U}_n$, there is $\varphi \in [0, 2\pi]$ such that the line segment $\bar{\mathbf{D}} \cap \{\zeta + re^{i\theta} : r \geq 0\}$ lies in $\bar{\mathcal{U}}_n$ and has endpoint on I_n whenever*

$$|\theta - \varphi| \leq c\delta^2.$$

The proof of the lemma is an exercise in elementary geometry. Now let $\mathcal{O}_j^n = f_j(\mathcal{U}_n)$. If we consider any Ω_k, we have for each $\mathcal{O}_j$, such that $\partial\mathcal{O}_j \cap \partial\Omega_k$ is an arc, obtained a domain $\mathcal{O}_j^k \subset \mathcal{O}_j$ (sometimes $\mathcal{O}_j^k = \mathcal{O}_j$) with the property that $\partial\mathcal{O}_j^k \cap \partial\mathcal{O}_j \subset \partial\Omega_k$. Let $\mathcal{F}_k = \{\mathcal{O}_j^k : \partial\mathcal{O}_j \cap \partial\Omega_k \text{ is an arc}\}$ and let $\tilde{\Omega}_k =$ interior of closure of $\Omega_k \cup \bigcup_{\mathcal{F}_k} \mathcal{O}_j^k$.

LEMMA 5.2. *$\partial\tilde{\Omega}_k$ is an $\eta(\varepsilon,\delta)$ quasicircle.*

PROOF: Let $\gamma_j^k = \mathcal{O}_j \cap \partial\mathcal{O}_j^k$. Then $\partial\tilde{\Omega}_k \subset \partial\Omega_k \cup \bigcup_{\mathcal{F}_k} \gamma_j^k$. We first claim $\tilde{\Omega}_k$ is an $\eta(\varepsilon,\delta)$ John domain. It is only necessary to find for every $z_0 \in \partial\tilde{\Omega}_k$ an arc $\gamma \subset \tilde{\Omega}_k$ which has endpoints z_0 and z_k such that

$$\text{distance}(z, \partial\tilde{\Omega}_k) \geq \eta|z - z_0| \ , \ z \in \gamma.$$

If $z_0 \in \partial\Omega_k$ this is clear by Lemma 4.2. We therefore assume $z_0 \in \gamma_j^k$ for some j. By Lemmata 2.2, 2.3 and 5.1 there are angles $\varphi_{-1} < \varphi_0 < \varphi_1$ such that $|\varphi_\ell - \varphi_m| \sim \delta^2$, $\ell \neq m$, such that

$$\mathbf{D} \cap \{f_j^{-1}(z_0) + re^{i\theta}, r > 0\} \subset \mathcal{U}_j,$$

whenever $\varphi_{-1} \leq \theta \leq \varphi_1$, and such that

$$\ell(\Gamma_m) \equiv \ell(f_j(\{f_j^{-1}(z_0) + re^{i\varphi_m} : r > 0\}) \leq Cd(z_0).$$

Furthermore, Lemma 5.1 allows us to assume that Γ_m is a Jordan arc and if $z \in \Gamma_\ell$ and $|z - z_0| \geq \frac{1}{2}d(z_0)$,

$$\text{distance}(z, \Gamma_m) \geq cd(z_0) \ , \ \ell \neq m. \tag{5.2}$$

Now let δ_m be the endpoint of Γ_m on $\partial\Omega_k$ and let γ_{-1} (resp. γ_1) be the John geodesic from ζ_{-1} (resp. ζ_1) in Ω_k to z_k (the John center of Ω_k). Then the curve $\gamma = \Gamma_{-1} \cup \Gamma_1 \cup \gamma_{-1} \cup \gamma_1$ surrounds ζ_0 and by the John condition on Ω_k,

$$\text{distance}(\zeta_0, \gamma) \geq cd(z_0). \tag{5.3}$$

(Notice here that we are implicitly using the fact that $z \in \mathcal{O}_j^k$ implies $d(z) \leq C$. This in turn follows from (5.1) and either Lemma 2.2 or 2.3.) Notice also that the interior of γ must lie entirely in $\tilde{\Omega}_k$.

Let γ_0 be the John geodesic in Ω_k from ζ_0 to z_k. We claim that the John condition for $\tilde{\Omega}_k$ holds on $\Gamma_0 \cup \gamma_0$. First suppose that $z \in \Gamma_0$ and $|z - z_0| \leq \frac{1}{2}d(z_0)$. Then by the distortion theorem for f_j,

$$\text{distance}(z, \partial\tilde{\Omega}_k) \geq \text{ distance}(z, \gamma) \geq c|z - z_0|.$$

Now by inequality (5.2),

$$d(z, \partial\tilde{\Omega}_k) \geq d(z, \gamma) \geq c|z - z_0|$$

whenever $z \in \Gamma_0$ and $|z - z_0| \geq \frac{1}{2}d(z_0)$. (Here we have used the John property on Ω_k to obtain distance $(z, \gamma_{-1} \cup \gamma_1) \geq c|z - z_0|$.)

We must finally check the John condition on γ_0. If $z \in \gamma_0$ and $|z - \zeta_0| \leq cd(z_0)$, the inequality for distance$(z, \partial\tilde{\Omega}_k)$ follows from (5.3) and the fact that $|z_0 - \zeta_0| \leq Cd(z_0)$. If $z \in \gamma_0$ and $|z - \zeta_0| > cd(z_0)$, the inequality for distance$(z, \partial\tilde{\Omega}_k)$ follows from the John condition distance$(z, \partial\Omega_k) \geq c|z - \zeta_0|$. We have thus established that $\tilde{\Omega}_k$ is a John domain.

We now claim that $G_k = \bar{\mathbf{C}} \backslash \overline{(\tilde{\Omega}_k)}$ is a John domain. We note that by the definition of $\tilde{\Omega}_k$, $\partial\tilde{\Omega}_k = \partial G_k$. Now fix a point $z_0 \in G_k$.

Case A. $z_0 \in \overline{(\tilde{\Omega}_j)}$ for some j. (Then $j \neq k$). First draw the John geodesic in Ω_j from z_0 to z_j. We then draw the geodesic (in the Poincaré metric of Ω) from z_j to ∞. By the construction of the domains Ω_j (Lemma 3.2) this is a John geodesic. The union of these two geodesics provides the arc joining z_0 to ∞.

Case B. $z_0 \in \Omega \backslash \bigcup\limits_j \bar{\Omega}_j$. Let γ be the Poincaré geodesic in Ω from z_0 to ∞. Then by Lemma 3.2, γ is a John geodesic in G_k.

Case C. $z_0 \notin \Omega \cup \bigcup\limits_j \tilde{\Omega}_j$. Then $z_0 \in \mathcal{O}_{j_0}$ for some j_0. Let

$$A = \sup_j \text{ diameter } \Omega_j,$$

so that $A \sim 1$. If $d(z_0) \geq 2A$ there is a half line γ (to ∞ from z_0) which is a John geodesic in G_k. If $d(z_0) < 2A$ there is a hyperbolic geodesic (which is also a John arc in G_k) γ_1

from z_0 to $z_1 \in \mathcal{O}_{j_0,\ell}$ where $\ell \neq k, d(z) \geq 1$ on γ_1, and $\ell(\gamma) \leq C$. (This follows from the definition of the domains $\mathcal{O}_{j,k}$.) By Case A there is a John geodesic γ_2 from z_1 to ∞ in G_k. The curve $\gamma = \gamma_1 \cup \gamma_2$ is the required John arc.

The proof is now completed by first observing that a simply connected domain $\mathcal{D}$ with $\partial\mathcal{D}$ locally connected, $\mathcal{D} =$ Interior$(\bar{\mathcal{D}})$, and $\mathbf{C}\backslash\bar{\mathcal{D}}$ connected is bounded by a Jordan curve, and then invoking the following fact (see [**9**]):

A Jordan domain $\mathcal{D}$ is bounded by a quasicircle if and only if $\mathcal{D}$ and $\bar{\mathbf{C}}\backslash\bar{\mathcal{D}}$ are John domains.

§6. An Estimate on Capacity

We now seek to imitate the proof given in Section 3. What is required is an estimate implying that $|F(z) - F(z_j)|$ is not too large on $\partial\tilde{\Omega}_j$, except for a set of small capacity. While $\tilde{\Omega}_j$ is a quasidisk, $\tilde{\Omega}_j \cap K^c$ is not necessarily connected. This means we cannot simply apply Lemma 2.3. We state our result as a proposition; its proof will be broken into several steps. The result we state is far from optimal, but it is all we need. Let $\tilde{\tilde{\Omega}}_k$ be the domain obtained by adding to $\tilde{\Omega}_k$ the set

$$\bigcup_j \{z \in \mathcal{O}_j : \rho(z, \partial\mathcal{O}_{j,k}) < 1\},$$

where ρ is the hyperbolic metric on $\mathcal{O}_j$. The domains $\tilde{\tilde{\Omega}}_k$ then satisfy

$$\sum \chi_{\tilde{\tilde{\Omega}}_k} \leq C.$$

Proposition 6.1. *Suppose H is continuous on the closure of $\tilde{\tilde{\Omega}}_k$ and harmonic on $\tilde{\tilde{\Omega}}_k\backslash K$. Then if*

$$\iint_{\tilde{\tilde{\Omega}}_k\backslash K} |\nabla H|^2 dxdy = 1,$$

we have the estimate

$$Cap(\{z \in \partial\tilde{\Omega}_k : |H(z) - H(z_k)| > \lambda\}) = o(1)$$

as $\lambda \to \infty$.

PROOF: Let $E_1 = \{z \in \partial\tilde{\Omega}_k \cap \partial\Omega_k : |H(z) - H(z_k)| > \lambda\}$ and let $E_2 = \{z \in \partial\tilde{\Omega}_k \backslash \partial\Omega_k : |H(z) - H(z_k)| > \lambda\}$. Then by Lemmata 2.3 and 2.5, $\text{Cap}(E_1) = o(1)$ as $\lambda \to \infty$, so it is sufficient to show $\text{Cap}(E_2) = o(1)$ as $\lambda \to \infty$.

Step 1. Construction of Some Special Points.

Let $\{z_n\}$ be a collection of points in $\partial\tilde{\Omega}_k \backslash \partial\Omega_k$ satisfying

$$|z_n - z_m| \geq \frac{1}{4} d(z_n) \ , \ \forall n, m$$

and

$$\inf_n |z - z_n| \leq \frac{1}{2} d(z) \ , \ \forall z \in \partial\tilde{\Omega}_k \backslash \partial\Omega_k.$$

We will now form for each z_n a point $z_n^* \in \Omega_k$. For an arbitrary point $z \in \partial\mathcal{O}_{j,k} \backslash \partial\Omega_k$ we let $K_z = \{\zeta \in \partial\Omega_k : |z - \zeta| \leq 2d(z)\}$ so that $\text{Cap}(K_z) \geq cd(z)$. By the John condition and Lemma 2.5, there is a point $z^* \in \Omega_k$ such that $d(z) \sim d(z^*) \sim |z - z^*|$ and there is a set $\tilde{K}_z \subset K_z$ such that

$$\text{Cap}(\tilde{K}_z, z^*, \Omega), \text{Cap}(\tilde{K}_z, z, \mathcal{O}_j) \geq c. \tag{6.1}$$

Denote by f a Riemann mapping from Ω_k to $\mathbf{D}$ with $f(z_k) = 0$, where z_k is the "center" of Ω_k. We can move z^* slightly so that $f(z^*)$ has the form

$$f(z^*) = (1 - 2^{-\ell}) \exp\{im 2^{-\ell} \pi\} \tag{6.2}$$

for some $\ell, m \in \mathbf{N}$. By this method we produce from our collection $\{z_n\}$ a new collection $\{z_n^*\}$. Notice that it is possible that $z_n^* = z_m^*$ even if $n \neq m$, but then $d(z_n) \sim d(z_m) \sim |z_n - z_m|$.

Step 2. Another Geometric Construction.

Let $\{z_n\}$ be the collection of points in Step 1. Let $\{I_n\}$ be a collection of subarcs of $\partial\tilde{\Omega}_k \backslash \partial\Omega_k$ such that $\bigcup_n I_n = \partial\tilde{\Omega}_k \backslash \partial\Omega_k$, $I_n \cap I_m = \varphi$ when $n \neq m$, $\text{diameter}(I_n) \sim d(z_n)$, and $|z - z_n| \le \frac{1}{2} d(z_n)$ for $z \in I_n$. We also define I_n^* to be the arc

$$f^{-1}(\{(1 - 2^{-\ell})\exp\{i(m+t)2^{-\ell}\pi\} : 0 \le t \le 1\}).$$

See (6.2) for notation. Then I_n^* has diameter $\sim d(z_n)$ and if $z \in I_n^*$ the hyperbolic distance from z to z_n^* (in Ω_k) is bounded by C.

With the notation of (6.1) we also denote by J_n the subarc of $\mathbf{T}$

$$J_n = \{e^{i\theta} : m2^{-\ell}\pi < \theta \le (m+1)2^{-\ell}\pi\}$$

and we denote by Q_n the "square"

$$Q_n = \{re^{i\theta} : (1 - 2^{-\ell}) \le r \le 1, e^{i\theta} \in J_n\}.$$

We now use the standard terminology that an arc J_m is maximal in a subcollection $\mathcal{F}$ of $\{J_n\}$ if $J_m \in \mathcal{F}$ and $J_\ell \in \mathcal{F}, \ell \ne m$, implies either $J_\ell \cap J_m = \emptyset$ or $J_\ell \subset J_m$. Notice (by the John condition) that if $J_\ell \subset J_m$,

$$|z_\ell - z_m^*| \le C d(z_m). \tag{6.3}$$

Finally, if $\hat{\mathcal{F}}$ is a subcollection of $\{I_n\}$ and $E = \bigcup_{I_n \in \hat{\mathcal{F}}} I_n$ we denote by E^* the set

$$E^* = \bigcup_{J_n \in \mathcal{F}} I_n^*$$

where

$$\mathcal{F} = \{J_n : I_n \in \hat{\mathcal{F}} \text{ and } J_n \text{ is maximal}\}.$$

Step 3. A Capacitary Estimate.

Let E and E^* be sets as in the previous paragraph.

Lemma 6.2. *$Cap(E^*) \ge c\, Cap(E)$.*

Proof: Let μ be a probability measure on E satisfying

$$\int \log \frac{1}{|z - \zeta|} d\mu(\zeta) \le \gamma\ ,\ \ z \in \mathbf{C}.$$

We relabel the intervals $J_n \in \mathcal{F}$ so that $\mathcal{F} = \{J_1, J_2, \dots\}$ and $d(z_1) \geq d(z_2) \geq \cdots \geq d(z_n) \geq d(z_{n+1} \geq \cdots$. Define

$$E_n = \{z \in E : |z - z_n| \leq Cd(z_n) \text{ and } |z - z_m| > Cd(z_m), m < n\},$$

so that by (6.3),

$$E = \bigcup_n E_n.$$

Notice that the sets E_n are pairwise disjoint.

Now define a probability measure μ^* by setting μ^* to have uniform distribution on the center half $\frac{1}{2}I_n^*$ of I_n^*, $\mu^*(I_n^* \backslash \frac{1}{2}I_n^*) = 0$, and

$$\mu^*(I_n^*) = \mu(E_n).$$

By the construction of I_n^* and $\frac{1}{2}I_n^*$,

$$\operatorname{dist}(\frac{1}{2}I_n^*, \frac{1}{2}I_m^*) \geq cd(z_n) \ , \ \forall n \neq m.$$

Let $z \in E$ and $z' \in \frac{1}{2}I_n$ where n satisfies $z^* \in I_n$. Then

$$\begin{aligned}
\int \log \frac{1}{|z' - \zeta|} d\mu^*(\zeta) &= \int_{\{|z'-\zeta| \leq Ad(z)\}} + \int_{\{|z'-\zeta| > Ad(z)\}} \\
&\leq c + \int_{\{|z'-\zeta| \leq Ad(z)\}} \log \frac{1}{|z - \zeta|} d\mu(\zeta) \\
&+ c + \int_{\{|z'-\zeta| > Ad(z)\}} \log \frac{1}{|z' - \zeta|} d\mu(\zeta) \\
&\leq 2c + \gamma,
\end{aligned}$$

and Lemma 6.2 is established.

Step 4. Proof of the Proposition.

By Lemmata 2.3 and 2.5 and by estimate (6.1),

$$|H(z) - H(z^*)| \leq 1.$$

Now let $\mathcal{D} = \Omega_k \backslash \bigcup_{\mathcal{F}} \hat{Q}_n$, where the Q_n are as defined in Step 2 and $\hat{Q}_n = f^{-1}(Q_n)$. Then by Lemma 4.1, $\mathcal{D}$ is an (ε') John domain. We define our collection $\mathcal{F}$ to be $\{I_n : \exists z \in I_n, |H(z) - H(z_k)| \geq \lambda\}$. Then if $I_n \in \mathcal{F}, |H(\zeta) - H(z_k)| \geq \lambda - c$ for all $\zeta \in I_n$. (This is why we slightly enlarges $\tilde{\Omega}_k$ to $\tilde{\tilde{\Omega}}_k$.) By our previous estimate,

$$|H(z) - H(z_k)| \geq \lambda - 2c \text{ on } I_n^*,$$

for any $I_n \in \mathcal{F}$. Setting as before

$$E = \bigcup_{\mathcal{F}} I_n$$

$$\text{and} \qquad E^* = \bigcup I_m^*,$$

we have $\mathrm{Cap}(E^*) \geq c\ \mathrm{Cap}(E)$. Now since

$$\iint_{\mathcal{D}} |\nabla H|^2 dxdy \leq 1,$$

it follows from Lemmata 2.3 and 2.5 that

$$\mathrm{Cap}(E^*) = o(1) \text{ as } \lambda \to \infty.$$

This completes the proof of Proposition 6.1.

§7. Proof of Theorem 1

Let $\tilde{\Omega}_j$ and $\tilde{\Omega}_k$ be two domains satisfying

$$\mathrm{distance}(\tilde{\Omega}_j, \tilde{\Omega}_k) \leq 1.$$

It is an exercise to find domains $\tilde{\Omega}_{j_1}, \ldots, \tilde{\Omega}_{j_N}$ where $j_1 = j, j_N = k$, and $\partial\tilde{\Omega}_{j_m} \cap \partial\tilde{\Omega}_{j_{m+1}}$ is an arc of diameter $\geq c$, and $N \leq C$. (Use the fact that Ω is a John domain and each $\tilde{\Omega}_j$ is an η quasicircle, i.e. Lemma 5.2.) Then by Proposition 6.1,

$$\begin{aligned}
|F(z_j) - F(z_k)| &\leq \sum_{m=1}^{N-1} |F(z_{j_m}) - F(z_{j_{m+1}})| \\
&\leq C \sum_{m=1}^{N} \left(\iint_{\tilde{\tilde{\Omega}}_{j_m} \backslash K} |\nabla F|^2 dxdy \right)^{1/2} \\
&\leq C' \left(\iint_{\{z \in K^c : |z - z_j| \leq C\}} |\nabla F|^2 dxdy \right)^{1/2}
\end{aligned}$$

We also notice by (6.1) that if $z \in \tilde{\Omega}_k$ and $d(z) \geq 1$,

$$|F(z) - F(z_k)| \leq C \left(\iint_{\tilde{\tilde{\Omega}}_k \setminus K} |\nabla F|^2 dxdy \right)^{1/2}.$$

Putting our last two estimates together we see there is $\tilde{F} \in C^\infty(\mathbf{R}^2)$ such that $\tilde{F}(z) = F(z)$ when $d(z) \geq 1$ and

$$\iint_{\{z \in K^c : |d(z)| \leq 1\}} |\nabla \tilde{F}|^2 dxdy \leq C \iint_{\{z \in K^c : |d(z)| \leq C\}} |\nabla F|^2 dxdy.$$

Here we are using the fact that, by the construction of the $\tilde{\Omega}_k$, $\{z \in K^c : d(z) \leq 1\} \subset \bigcup\limits_k \tilde{\Omega}_k$. Since the John condition is dilation invariant, we may now build a sequence $\tilde{F}_n \in C^\infty(\mathbf{R}^2)$ with $\tilde{F}_n(z) = F(z)$ when $d(z) \geq \frac{1}{n}$ and

$$\iint_{\{z \in K^c : d(z) \leq \frac{1}{n}\}} |\nabla \tilde{F}_n|^2 dxdy \leq C \iint_{\{z \in K^c : d(z) \leq \frac{c}{n}\}} |\nabla F|^2 dxdy.$$

Since $|K| = 0$, it follows that $F \in W^{1,2}(\mathbf{R}^2)$ and

$$\iint_{\mathbf{R}^2} |\nabla F|^2 dxdy = \iint_{K^c} |\nabla F|^2 dxdy.$$

§8. Proof of Theorem 2

Let Ω be an (ε) John domain with compact boundary K of diameter one, let $\{Q_j\}$ denote the Whitney decomposition of Ω into dyadic squares [**11**], and let z_j be the center of Q_j. Let $A = A(\varepsilon)$ be a large constant and define $\mathcal{F}_n = \{z_j : A^{-n} \leq d(z_j) \leq A\}$. It is an exercise with the John condition to construct a connected graph G_0 such that every edge in G_0 is of the form $[z_j, z_k]$ where $\partial Q_j \cap \partial Q_k \neq \phi$, and where the vertices V_0 of G_0 satisfy

$$\mathcal{F}_0 \subset V_0 \subset \{z_j : 1 \leq d(z_j) \leq A^2\}.$$

We also build G_0 so that

(8.1) If $d(z_j) \geq 1$ and $z_j \notin V_0$ then Q_j is in the unbounded component of $\mathbf{C} \setminus \bigcup\limits_{z_k \in \mathcal{F}_0} Q_k$.

It is now an exercise (with induction) to construct connected graphs G_n with the following properties:

(8.2) Every edge in G_n is of the form $[z_j, z_k]$ for some $z_j, z_k \in \mathcal{F}_{n+1} \cup V_0$ where $\partial Q_j \cap \partial Q_k \neq \emptyset$.

(8.3) Every $z_j \in \mathcal{F}_n$ is in V_n, the vertices of G_n.

(8.4) If $\rho(z_j, z_k)$ is the graph distance on G_n,

$$\inf_{z_j \in \mathcal{F}_n} \rho(z_j, z_k) \leq C \ , \ \ z_k \in G_n.$$

(8.5) $V_n \subset V_{n+1}$

Notice that we have chosen G_0 to be connected. Let $z_0 \in V_0$ be an extreme point of the (planar set) convex hull (G_0). We may assume by induction that each G_n is actually a *directed* graph in the following sense. Each edge $[z_j, z_k]$ is directed in the sense that (perhaps switching j and k)

$$\rho(z_j, z_0) = \rho(z_k, z_0) + 1. \tag{8.6}$$

Such an edge is an outgoing edge from z_j. It is not hard to see that we may choose the G_n so that

(8.7) Each $z_j \neq z_0$ has exactly one outgoing edge.

LEMMA 8.1. *The graph G_n is simply connected, i.e. it contains no loops.*

PROOF: Suppose to the contrary that there is a loop in G_n. Let z_j be a vertex in the loop maximizing $\rho(z_j, z_0)$. Then z_j has two outgoing edges (by (8.6)) and this contradicts (8.7). ∎

Let $G = \lim_n G_n$ be the limiting graph, so that G is simply connected. It is clear that $K \cup G$ is connected. Notice by (8.3) that

(1) (8.7) For every $z_j \in G$ there is an arc $\gamma \subset G$ from z_j to z_0 which satisfies the ε' John condition in Ω.

In other words, G is a John graph.

For a Whitney square Q_j with $z_j \in G$ let $\{\mathcal{L}_k^j\}$ denote all the edges of G with one endpoint being z_j. Define

$$I_k^j = \{z \in \partial Q_j : \ \text{distance}(z, \mathcal{L}_k^j) < \delta \ \text{diam}(Q_j)\},$$

where δ is a small constant, and put

$$S_j = \partial Q_j \backslash \bigcup_k I_k^j \ , \ j \neq 0.$$

For the special point $z_0 \in G$ we select a Whitney square Q_ℓ such that $z_\ell \notin G$, $\partial Q_0 \cap \partial Q_\ell \neq \phi$, and we put

$$S_0 = \partial Q_0 \backslash (I_\ell^0 \cup \bigcup_k I_k^0),$$
$$\hat{\Omega} = \Omega \backslash \bigcup_{z_j \in G} S_j.$$

Lemma 8.2. *$\hat{\Omega}$ is simply connected.*

Proof: Let $\hat{\Omega}_+ = \cup\{\hat{\Omega} \cap Q_j : z_j \notin G\}$, $\hat{\Omega}_- = \cup\{\hat{\Omega} \cap Q_j : z_j \in G\}$ so that

$$\hat{\Omega} = \hat{\Omega}_+ \cup \hat{\Omega}_- \cup I_\ell^0.$$

By condition (8.1), $\hat{\Omega}_+$ is simply connected (in $\bar{\mathbf{C}}$), so it is only necessary to check that $\bar{\Omega}_-$ is simply connected.

We first verify that $\hat{\Omega}_-$ is connected. Let $z \in Q_j \cap \hat{\Omega}_-$ and let γ be an arc in G connecting z_j to z_0. Then $\gamma' = [z, z_j] \cup \gamma$ is an arc in $\hat{\Omega}_-$ which connects z to z_0.

Now suppose that γ is a loop in $\hat{\Omega}_-$ that is not homologous to zero. It is then an elementary exercise to homotopy γ to γ', a loop in G that is not homologous to zero. This contradicts Lemma 8.1. ∎

It is clear from the construction of $\hat{\Omega}$ that $\partial\Omega \subset \partial\hat{\Omega}$. To verify that $\hat{\Omega}$ is a John domain we must look at two cases.

Case 1. $z \in \hat{\Omega}_+ \cap Q_j$. There is arc γ from z_j to some $z_k \notin G$ such that length$(\gamma) \leq C$, $d(z) \geq 1$ on γ, and z_k is not in the convex hull of $\partial\hat{\Omega}$. By selecting a suitable ray R from z_k to ∞ we then see that

$$[z, z_j] \cup \gamma \cup R$$

is the required John arc.

Case 2. $z \in \hat{\Omega}_- \cup I_\ell^0$. Let γ be a John arc from z_ℓ (the center of the special Whitney square Q_ℓ adjacent to z_0) to ∞. Then if $z \in Q_j$ and $\gamma' \subset G$ is the John arc from z_j to z_0 guaranteed by condition (8.7), we see that

$$[z, z_j] \cup \gamma' \cup [z_0, z_\ell] \cup \gamma$$

is the required John arc.

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