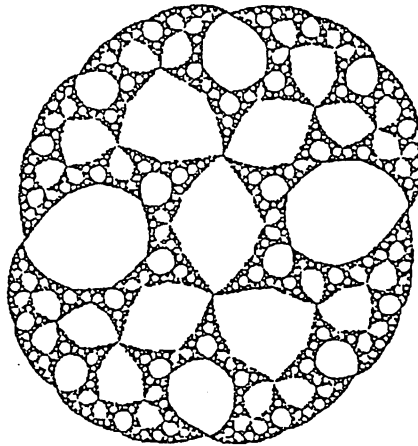


**Pleating Coordinates for
the Maskit Embedding
of the Teichmüller Space
of Punctured Tori**

by L. Keen and C. Series

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Pleating Coordinates for the Maskit Embedding of the Teichmüller Space of Punctured Tori

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1 Introduction

In this paper we introduce a new set of parameters that we call *pleating coordinates* for the Teichmüller space $\mathbf{T}_{1,1}$ of the punctured torus. The coordinate grid is shown in figure 1. These coordinates have this geometric configuration when $\mathbf{T}_{1,1}$ is embedded as a holomorphic family of Kleinian groups $\{G_\mu\}$ depending on a complex parameter μ that varies in a simply connected domain \mathcal{M} in \mathbf{C} . The embedding is made in such a way that the regular set $\Omega(G_\mu)$ has a unique invariant component $\Omega_0(G_\mu)$ and the points in $\mathbf{T}_{1,1}$ are represented by the Riemann surfaces $\Omega_0(G_\mu)/G_\mu$. This embedding is known as the *Maskit embedding* for $\mathbf{T}_{1,1}$. (See section 2 for a more leisurely and detailed explanation of the technical terms and ideas here.)

The advantages of our coordinates are threefold: first, they relate directly to the geometry of the hyperbolic manifold \mathbf{H}^3/G_μ , or more precisely to the component $\partial\mathcal{C}_0$ of the convex hull boundary “facing” $\Omega_0(G_\mu)$ (see section 4.2); second, they reflect exactly the visual patterns one sees in the limit sets; and third, they are directly computable from the generators of G_μ . As is apparent in figure 1, one sees quite explicitly how \mathcal{M} sits inside \mathbf{C} .

The boundary of the convex hull is invariant under G_μ and the coordinates can be read off from the geometry of the punctured torus $\hat{\mathcal{S}}_\mu = \partial\mathcal{C}_0/G_\mu$. The surface $\partial\mathcal{C}_0$ carries a natural hyperbolic metric and is pleated along geodesics that project to a geodesic lamination λ on $\hat{\mathcal{S}}_\mu$. The “vertical” lines in the grid represent lines along which λ remains fixed. We call such a line a *pleating ray*. The set of all possible laminations on a punctured torus is naturally identified with $\hat{\mathbf{R}}$ and all the laminations except the one corresponding to ∞ determine pleating rays. The rays appear in figure 1 in their natural order along \mathbf{R} . For $\lambda \in \mathbf{R}$, the pleating ray \mathcal{P}_λ is asymptotic to the real line $\Re\mu = 2\lambda$ as $\mu \rightarrow \infty$.

When $\lambda \in \mathbf{Q}$ the lamination is a simple closed geodesic $\gamma(\lambda)$ on \mathcal{S}_μ . If $g_\lambda(\mu) \in G_\mu$ is an element representing $\gamma(\lambda)$, the ray \mathcal{P}_λ coincides with a unique branch of the locus $\{\mu \in \mathbf{C}: \text{Tr } g_\lambda(\mu) > 2\}$. These rational rays are dense in \mathcal{M} , and by a recent result of McMullen [11], their endpoints are dense in $\partial\mathcal{M}$.

Along the rational rays, $\Omega_0(G_\mu)$ is a union of overlapping circles that fit together in a manner reflecting the continued fraction expansion of λ . These patterns are visually apparent, at least for values of μ near $\partial\mathcal{M}$, in

pictures of the limit sets of these groups as we see in figure 2. Our interest in these patterns, discovered by David Wright in the course of a computer investigation of $\partial\mathcal{M}$, was the original motivation for the work here.

Each rational ray is naturally parametrized by the length of the pleating lamination. This length, however, does not define a globally continuous parameter: it becomes infinite as we move towards an irrational ray. Therefore, to obtain the “horizontal” lines in figure 1 we have to scale appropriately. To do this, we make a specific choice of transverse measure for the pleating lamination λ and define the *pleating length* of G_μ to be the length of the pleating lamination with respect to this choice. The “horizontal” lines are lines of constant pleating length. Again we find explicit formulae for these lines when λ is rational. By taking appropriate limits, we are able to characterize the irrational pleating rays as the real loci of a family of holomorphic functions.

The paper is organized as follows. In section 2, we set notation and describe the basic theory of the Maskit embedding and its relation to the classical theory of flat tori. In section 3 we discuss simple closed curves on the torus and derive some easy properties of the corresponding trace polynomials.

The combinatorial circle patterns that appear in the limit sets and the rational pleating rays are the topic of section 4. After summarizing the basic facts we need about pleated surfaces and the convex hull boundary of the three manifold \mathbf{H}^3/G_μ , we characterize the groups on the rational pleating rays as those for which the limit set $\Lambda(G_\mu)$ is contained in a particular pattern of overlapping circles.

The first of our main results is proved in section 5: we identify the rational pleating ray with the real locus described above. The proof involves the fact, proved in a more general setting in [6], that the pleating locus of $\partial\mathcal{C}_0(\mu)$ varies continuously with μ .

Section 6 is devoted to real pleating rays. We give the basic facts about measured geodesic laminations and explain the choice of the transverse measure referred to above. To prove the continuity of the pleating length, we use the results, also proved in [6], that the bending measure and the hyperbolic structure of $\partial\mathcal{C}_0(\mu)$ depend continuously on μ . We use this to define a complex length function associated to each lamination whose real locus characterizes the pleating ray.

Finally in section 7 we collect our results to prove the laminations and

their pleating lengths are coordinates for \mathcal{M} (theorem 7.1).

In a future paper, we plan to use the methods developed here to give a complete description of the boundary of \mathcal{M} . In particular, we hope to give proofs of McMullen's theorems [11, 12] for this embedding: that the cusp points are dense in $\partial\mathcal{M}$ and that $\partial\mathcal{M}$ is a Jordan curve.

Although our work here relates to the punctured torus, most of the techniques we have developed apply more generally. In the future we expect to extend our analysis to any surface of finite topological type.

We wish to express our thanks to a number of people. First, to David Wright who introduced us to this problem and has generously allowed us to develop his ideas and to use his computer pictures. Second, to Curt McMullen, who has graciously shared his ideas and work. Third to David Epstein, Michael Handel, Steve Kerckhoff, Paddy Patterson and Bill Thurston who have taught us much background material and patiently discussed the work as it progressed. Finally, we would like to acknowledge the support of the NSF in the US, the SERC in the UK, the Danish Technical University and the IMS at SUNY-Stonybrook.

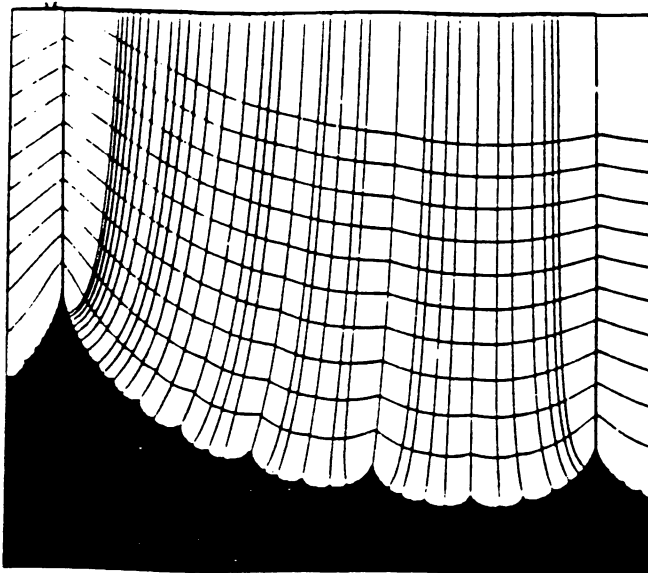


Figure 1: The Maskit Embedding with Pleating Coordinates

2 The Maskit Embedding

2.1 The definition

Let $\mathbf{T}_{1,1}$ be the Teichmüller space of a punctured torus. We want to represent $\mathbf{T}_{1,1}$ as a space of discrete subgroups of $\text{aut}(\hat{\mathbf{C}})$ with a distinguished set of generators having certain special properties. Before describing these properties, we need some notation.

The group $\text{aut}(\hat{\mathbf{C}})$ is the group $PSL(2, \mathbf{C})$. We shall always identify the matrix representing an element of $PSL(2, \mathbf{C})$ with the corresponding linear fractional transformation acting on $\hat{\mathbf{C}}$. A discrete subgroup $G \subset PSL(2, \mathbf{C})$ is called a *Kleinian group*. A Kleinian group is called *marked* if it has a distinguished set of generators. The subset $\Omega = \Omega(G) \subset \hat{\mathbf{C}}$ on which G acts properly discontinuously is called the *regular set* of G , and the limit set $\Lambda = \Lambda(G)$ is its complement.

Denote by $\tilde{\mathcal{M}}$ the space of marked groups characterized by the following conditions:

A group G is in $\tilde{\mathcal{M}}$ if and only if

1. $G = \langle S, T \rangle$ is a free group on two generators and S is parabolic.
2. The connected components of the regular set $\Omega(G)$ are of two kinds:
 - (a) A simply connected G -invariant component Ω_0 for which the orbit space Ω_0/G is topologically conjugate to the punctured torus.
 - (b) Non-invariant components Ω_i , $i \geq 1$, that are conjugate to one another under G and for which each orbit space $\Omega_i/\text{stab}(\Omega_i)$ is conformally the thrice punctured sphere Σ .

A group $G \in \tilde{\mathcal{M}}$ represents the point in $\mathbf{T}_{1,1}$ corresponding to the conformal structure induced on Ω_0/G from the natural complex structure on \mathbf{C} . The torus is marked by curves corresponding to the homotopy classes of the (ordered) pair of generators S and T .

We shall show in the next section that $\tilde{\mathcal{M}} \neq \emptyset$. With this assumption, one obtains a bijective correspondence between conjugacy classes of groups in $\tilde{\mathcal{M}}$ and points in $\mathbf{T}_{1,1}$. The proof of this is as follows.

Let $G \in \tilde{\mathcal{M}}$ and let \mathcal{T} be the corresponding point in $\mathbf{T}_{1,1}$. We may obtain any other point \mathcal{T}' in $\mathbf{T}_{1,1}$ as follows. The conformal structure on \mathcal{T}' relative to that on \mathcal{T} is described by a Beltrami differential on \mathcal{T} that lifts to a G -invariant Beltrami differential ν defined on $\Omega_0(G)$. Extend ν to be zero on $\hat{\mathbf{C}} \setminus \Omega_0(G)$; clearly the extended differential, also denoted by ν , is G -invariant. By the measurable Riemann mapping theorem there is a homeomorphism h^ν of $\hat{\mathbf{C}}$ that conjugates G into a group G' whose regular set $\Omega(G')$ is homeomorphic to $\Omega(G)$; in particular, it is clear that $G' \in \tilde{\mathcal{M}}$ and that $\Omega_0(G')/G'$ represents the torus \mathcal{T}' .

Suppose now that $G, G' \in \tilde{\mathcal{M}}$ represent the same point in $\mathbf{T}_{1,1}$. Since Σ is conformally rigid, there are no deformations of G supported on the components Ω_i , $i \geq 1$. By assumption G and G' induce the same conformal structure on Ω_0 , so since there are no deformations of G supported on $\hat{\mathbf{C}} \setminus \Omega$ by Sullivan's theorem [16], it follows that G and G' must be conjugate in $SL(2, \mathbf{C})$.

The above discussion shows that each point in $\mathbf{T}_{1,1}$ is represented by a unique conjugacy class of groups in $\tilde{\mathcal{M}}$. By choosing an appropriate normalization for these groups, we can represent $\mathbf{T}_{1,1}$ as a one complex dimensional subspace $\mathcal{M} \subset \tilde{\mathcal{M}}$. This representation is known as *the Maskit embedding* of $\mathbf{T}_{1,1}$; details of one specific normalization are described in the next section.

2.2 Normalization

Suppose that $G \in \tilde{\mathcal{M}}$ and consider the subgroup generated by $S, T^{-1}ST = \tilde{S}$ and $K = \tilde{S}^{-1}S$. By assumption, S is parabolic. We claim that the same is true of the commutator K . Clearly, since Ω_0 is simply connected, G may be identified with $\pi_1(\mathcal{T})$. Since the projection onto the orbit space is holomorphic, and since the element corresponding to a loop around the puncture is the commutator of the generators, K must be parabolic.

We define $\mathcal{M}_1 \subset \tilde{\mathcal{M}}$ as the set of groups G_μ for which the fixed points

of S , \tilde{S} and K are normalized to be at ∞ , 0 and -1 respectively. We write elements of $PSL(2, \mathbf{C})$ as matrices so we may assume $\text{Tr } S = 2$. An easy computation shows that

$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

$$\tilde{S} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

and that

$$K = \tilde{S}^{-1}S = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}.$$

It is also easy to compute that the most general element T conjugating S to \tilde{S} is of the form

$$\epsilon \begin{pmatrix} \alpha & i \\ i & 0 \end{pmatrix}$$

for some $\alpha \in \mathbf{C}$. It is standard that the signs of the entries of the generators may be chosen arbitrarily so we take $\epsilon = -1$ and write $\alpha = i\mu$, $\mu \in \mathbf{C}$. We denote

$$-i \begin{pmatrix} \mu & 1 \\ 1 & 0 \end{pmatrix}$$

by T_μ and write G_μ for $G = \langle S, T_\mu \rangle$.

We denote by F the subgroup of G_μ generated by S and \tilde{S} . Notice that F is independent of μ . Clearly F is Fuchsian and stabilizes the upper and lower half planes \mathbf{H} and \mathbf{H}^* .

A fundamental domain for F is

$$D = \{-1 \leq \Re z < 1\} \cup \{|z + 1/2| \geq 1/2\} \cup \{|z - 1/2| > 1/2\}.$$

The orbit spaces of the upper and lower half plane are thrice punctured spheres Σ and Σ^* .

We now show that $\mathcal{M}_1 \neq \emptyset$ by proving that $G_\mu \in \mathcal{M}_1$ whenever $\mu = it$ and $t \in \mathbf{R}$, $t > 2$. This is a special case of the plumbing construction described in section 6.3 of [7].

Let $H_\infty = \{z \in \mathbf{C}: \Im z \geq t\}$ and $H_0 = \{z \in \mathbf{C}: |z - it/2| \leq t/2\}$ be horodisks at ∞ and 0 respectively in \mathbf{H} . It is easy to check that $T_\mu(H_0) = \hat{\mathbf{C}} \setminus H_\infty$ and that the condition $t > 2$ forces $H_0 \cap H_\infty = \emptyset$.

Using elementary combinations theorems (for instance the packing theorems in [1] page 103 or [10] page 171) the region R consisting of the part of D exterior to H_0 and H_∞ is a fundamental domain for G_μ . Furthermore, it is easy to see that R/G_μ consists of two connected components, a punctured torus $R \cap \mathbf{H}/G_\mu$ and a thrice punctured sphere $R \cap \mathbf{H}^*/G_\mu$. It is also a consequence of Maskit's second combination theorem, [9], that the connected component of $\Omega(G_\mu)$ that contains R is simply connected, and that the remaining components of Ω are all conjugate to \mathbf{H}^* by elements of G_μ .

We now identify \mathcal{M}_1 with the connected component of $\{\mu \in \hat{\mathbf{C}}: G_\mu \in \mathcal{M}_1\}$ containing $\mu = it$ with $t > 2$. The above discussion shows that \mathcal{M} may be identified with $\mathbf{T}_{1,1}$ so we refer to \mathcal{M} as the *Maskit embedding* of $\mathbf{T}_{1,1}$.

2.2.1 Shape of the limit set

In the groups G_μ constructed above, with $\mu = it$, $t > 2$, $T_\mu(\infty) = \mu = it$. Hence the lower half plane \mathbf{H}^* is a non-invariant component of the regular set. The real axis is in the limit set, and therefore the limit set consists of the closure of the translates of the real axis. We see that $\Omega_0(G)$ is carved out of the upper half plane \mathbf{H} by removing the translates by elements of G of $\overline{\mathbf{H}^*}$; more precisely,

$$\Omega_0(G) = \mathbf{H} \setminus \overline{\bigcup_{W \in G, W \neq id} W(\mathbf{H}^*)}.$$

The interiors of the domains $W(\mathbf{H}^*)$, $W \in G$ are the components $\Omega_i(G)$, $i \geq 1$ referred to in 2(b) above, and the stabilizers of these components are the conjugates of the subgroup F .

Now consider any deformation h of G_μ to another group G' in $\tilde{\mathcal{M}}$, and suppose h is normalized so that 0 , -1 , and ∞ are fixed. Then h conjugates F into a group F' generated by parabolics S', \tilde{S}' with parabolic product $K' = \tilde{S}'^{-1}S'$, fixing 0 , ∞ and -1 respectively. It is easy to see that the only such group is F itself, thus $F = F'$, $S = S'$, $\tilde{S} = \tilde{S}'$. Now if $T' = hT_\mu h^{-1}$ then T' conjugates S to S' and hence is of the form $T_{\mu'}$ for some $\mu' \in \mathbf{C}$. Thus we find $G' = G_{\mu'} \in \mathcal{M}$. Therefore set $\Omega_0(G_{\mu'})$ is obtained by carving out images of \mathbf{H}^* exactly as described for the case $\mu = it$ above.

2.3 Recognizing the boundary

The observations in the last section on the shape of Ω_0 give us a useful criterion for recognizing, for $\mu \in \overline{\mathcal{M}}$, whether $\mu \in \text{int } \mathcal{M}$ or $\mu \in \partial \mathcal{M}$. Groups with $\mu \in \partial \mathcal{M}$ are described by a classical result of Bers [2]. First, all the groups in $\partial \mathcal{M}$ are discrete. Second, either G_μ contains, in addition to those containing S and K , another conjugacy class of parabolic elements (accidental parabolics), or G_μ is degenerate. In the first case $\Omega_0(G_\mu)$ degenerates into a countable union of round disks, tangent at the fixed points of the new accidental parabolics; in the second, $\Omega_0(G_\mu)$ completely disappears so that Ω consists entirely of the images of \mathbf{H}^* under G_μ .

Therefore we have,

Proposition 2.1 *If $\mu \in \overline{\mathcal{M}}$ and if there is a connected component of $\Omega(G_\mu)$ that is not a disk, then $\mu \in \text{int } \mathcal{M}$.*

2.3.1 Locating the cusps

A recent result of McMullen, [11], shows that cusps are dense in the boundary of the Bers embedding of the Teichmüller space of any surface of finite topological type. The same methods can be applied to show that in our case, the cusps are dense in $\partial \mathcal{M}$. Thus a very good picture of $\partial \mathcal{M}$ could be obtained if one knew the exact position of the cusps. This was the direction taken by David Wright [19] in his original investigation of $\partial \mathcal{M}$ (made before McMullen's proof of the density conjecture and adding credence to it).

The only elements of G that can become accidentally parabolic are those in conjugacy classes that represent simple closed curves on the torus $\mathcal{S}_\mu = \Omega_0(\mu)/G_\mu$. Therefore if these elements are enumerated systematically, and points in \mathbf{C} are found where their traces are ± 2 in a coherent way, a picture of \mathcal{M} will emerge. The method of enumeration is explained in the sections that follow. For the moment let us simply note that, given an element $g \in G$ representing a simple closed curve γ on the torus, there is *exactly one* point on $\partial \mathcal{M}$ for which $\text{Tr } g = 2$. The existence of such a point is standard and is proved by exhibiting a deformation of the torus that shrinks the length of γ to zero (see for example, [2]). Maskit showed us how to prove uniqueness and this is done in [5]. We do not need to use this result here.

2.4 Relation to the classical theory

Suppose that $\mu \in \mathcal{M}$. Since the orbit space $\mathcal{S}_\mu = \Omega_0(G_\mu)/G_\mu$ is a punctured torus, it admits an intermediate covering space which is the plane \mathbf{C} punctured at a lattice L . The marking on \mathcal{S}_μ by the generators S and T_μ of G_μ determines a set of generators for the lattice L that we denote by \underline{S} and \underline{T} . Without loss of generality we may normalize and write

$$\underline{S}: z \rightarrow z + 1, \quad \underline{T}: z \rightarrow z + \tau,$$

and we may assume that $\Im\tau > 0$. We may therefore identify the unpunctured torus $\underline{\mathcal{S}}_\tau$ whose classical modulus is τ with our punctured torus \mathcal{S}_μ . In this way we obtain a holomorphic homeomorphism $\phi: \mathbf{H} \rightarrow \mathcal{M}$. In particular, \mathcal{M} is a simply connected domain in \mathbf{C} . The parametrization of \mathcal{M} has of course been chosen so that \mathcal{M} looks as much like \mathbf{H} as possible. See figure 1 and section 2.3.1 for more details.

As indicated in section 2.3.1 we need to enumerate the homotopy classes of simple closed curves on the torus. The solution to the same problem on the flat torus is much easier. For each $(p, q) \in \mathbf{Z}^2$, $(p, q) \neq 1$, there is a homology class $\underline{S}^{-p}\underline{T}^q$ that represents a family of parallel closed geodesics on $\underline{\mathcal{S}}$. All closed geodesics on $\underline{\mathcal{S}}$ are simple, and all arise in this way.

The map $\pi_1(\mathcal{S}_\mu) \rightarrow \pi_1(\underline{\mathcal{S}}_\tau)$ induced by the covering $\Omega_0 \rightarrow \mathbf{C}/L$ maps G_μ to its abelianization \mathbf{Z}^2 . Thus $\underline{S}^{-p}\underline{T}^q$ represents only a homology class in $\pi_1(\mathcal{S}_\mu)$. However, we have the following proposition, see [15],

Proposition 2.2 *Given $p, q \in \mathbf{Z}$, $(p, q) \neq 1$, there is a unique conjugacy class in G that represents a homotopy class of simple closed curves in the homology class of $\underline{S}^{-p}\underline{T}^q$. These elements represent all the simple closed curves on \mathcal{S}_μ .*

Remark: In the sequel we will call the above homotopy class of curves the p/q -homotopy class. We denote the geodesic in this class by $\gamma(p/q)$. The details of how to compute these classes explicitly is given in section 3.1. Note that the expression for the group element representing this homotopy class in terms of the generators is independent of μ . This allows us to identify curves on different surfaces \mathcal{S}_μ .

It is also easy to understand the cusps in the flat picture. The boundary of the τ -plane \mathbf{H} is naturally $\hat{\mathbf{R}}$: we think of the point $p/q \in \mathbf{Q}$ as the point

at which the length of the geodesic in the $\underline{S}^{-p}\underline{T}^q$ homology class has shrunk to zero.

By analogy, a p/q -cusp on $\partial\mathcal{M}$ is a boundary point that is reached when the p/q -curve on the punctured torus has length zero; that is, $\gamma(p/q)$ has been pinched to a point and the punctured torus has degenerated to a thrice punctured sphere (with two of its punctures identified). Algebraically, this means that the elements in the p/q -conjugacy class have become parabolic and are the groups are cusps in the Bers sense as explained above.

On the flat torus \underline{S} one can interpolate between the rational (p, q) curves with linear foliations of irrational slope. We can think of a point $\lambda \in \mathbf{R} \setminus \mathbf{Q}$ as a point where the length of such a foliation has shrunk to zero. We again wish to make the analogous construction for the punctured torus \mathcal{S} . In fact, as described in [15], to each irrational foliation of \underline{S} there corresponds a unique compactly supported geometric lamination of \mathcal{S} . We denote by $\gamma(\lambda)$ the lamination corresponding to $\lambda \in \mathbf{R} \setminus \mathbf{Q}$ in this way.

The original motivation of much of what follows was that, just as in the τ -plane the irrational points interpolate between the rational cusps in $\hat{\mathbf{R}} = \partial\mathbf{H}$, so in the μ -plane the rational p/q -cusps in $\partial\mathcal{M}$ should be interpolated by *unique* boundary points corresponding to groups for which the length of the lamination $\gamma(\lambda)$ has shrunk to zero. This is equivalent to Bers' conjecture that the map $\phi: \mathbf{H} \rightarrow \mathcal{M}$ extends to a homeomorphism $\partial\mathbf{H} \rightarrow \partial\mathcal{M}$; or equivalently, that $\partial\mathcal{M}$ is a Jordan curve.

2.5 Rough shape of \mathcal{M}

In this section we include several easy propositions about \mathcal{M} . As remarked above, the parametrization was chosen so that the shape of \mathcal{M} roughly resembles that of \mathbf{H} . As we have already seen, the part of the imaginary axis above $\Im\mu = 2$ lies in \mathcal{M} and one can easily check that $\mu = 2i$ is a cusp for the element T_μ , corresponding to $(p, q) = (0, 1)$.

Proposition 2.3 $\mu \in \mathcal{M}$ if and only if $\mu + 2 \in \mathcal{M}$.

Proof: This follows immediately from the observations that $T_{\mu+2} = ST_\mu$ and that $\langle S, ST_\mu \rangle$ is an appropriately normalized pair of generators. \square

Corollary 2.4 *The lines $\mu = 2n + it$, $t > 2$, $n \in \mathbf{Z}$ are all in \mathcal{M} and the point $\mu = 2n + 2i$ is a cusp for $(p, q) = (-n, 1)$, corresponding to the element $S^{-n}T_\mu$ in G_μ .*

Remark: We shall see shortly that on the Riemann sphere $\hat{\mathbf{C}}$, the point at infinity belongs to $\partial\mathcal{M}$. This point should be thought of as a cusp corresponding to the accidental parabolic S ; that is, $(p, q) = (1, 0) = \lim_{n \rightarrow \pm\infty} (-n, 1)$. This point corresponds naturally to the point at infinity in the τ -plane which is the limiting case for the flat torus as the length of the \underline{S} curve shrinks to zero. There is also an easy symmetry associated to \mathcal{M} .

Proposition 2.5 $\mu \in \mathcal{M}$ if and only if $-\bar{\mu} \in \mathcal{M}$.

Proof: If we write any word of $G_\mu = \langle S, T_\mu \rangle$ as a matrix, the matrix for the corresponding word in $G_{-\bar{\mu}} = \langle S, T_{-\bar{\mu}} \rangle$ is obtained by replacing μ with $-\bar{\mu}$ in each entry. The fixed points of these words are therefore mapped into one another by the map $z \rightarrow -\bar{z}$. This reflection thus maps $\Lambda(G_\mu)$ to $\Lambda(G_{-\bar{\mu}})$; therefore, either both groups are in \mathcal{M} or both are not in \mathcal{M} . \square

Proposition 2.6 *The boundary of \mathcal{M} is contained in the horizontal strip*

$$\{\mu: 0 \leq \Im\mu \leq 2\}.$$

Proof: If $\Im\mu > 2$, we see as we did in section 2.2, that the curves $\gamma = \{z \in \mathbf{C}: \Im z = t\}$, $\tilde{\gamma} = \{z \in \mathbf{C}: |z - it/2| = t/2\}$ satisfy the conditions of the packing theorem. Therefore if $\Im\mu > 2$, $\mu \in \mathcal{M}$. \square

In [19] it is proved that $\Im\mu > 1$ for all μ in \mathcal{M} ; we will not need to use this fact here.

3 Special Words and Trace Identities

3.1 Special words

In this section we give an inductive procedure for constructing an element $W_{p/q} \in G_\mu$ corresponding to the $\gamma(p/q)$ -homotopy class. It will be important for our analysis to be precise about the actual word $W_{p/q}$ and not just its conjugacy class in G_μ . Much of what we do in this section follows David Wright in [19].

We begin by recalling the formation of rationals by Farey sequences. A pair of rationals $(p/q, r/s)$ are called *neighbors* if $ps - rq = \pm 1$. All rationals are obtained in a unique way by repeated application of the process $(p/q, r/s) \mapsto \frac{p+r}{q+s}$ to Farey neighbors. Note that if $p/q < r/s$ and if $(p/q, r/s)$ are neighbors then $p/q < \frac{p+r}{q+s} < r/s$ and both pairs $(p/q, \frac{p+r}{q+s})$, $(\frac{p+r}{q+s}, r/s)$ are again neighbors.

The words $W_{p/q}$ are formed inductively as follows:
if $n \in \mathbf{Z}$, then

$$W_{n/1} = S^{-n}T$$

and if $(p/q, r/s)$ are neighbors with $p/q < r/s$, then

$$W_{\frac{p+r}{q+s}} = W_{r/s}W_{p/q}.$$

It is easy to check inductively that $W_{p/q}$ always has the special form

$$S^{\epsilon_1}T^{\epsilon_2 n_1}S^{\epsilon_1}T^{\epsilon_2 n_2} \dots S^{\epsilon_1}T^{\epsilon_2 n_p}$$

where $\epsilon_i = \pm 1$, $\epsilon_1 = -\text{sgn}(p)$, $\epsilon_2 \sum_{i=1}^p n_i = q$, and $|n_i - n_j| \leq 1$ for $1 \leq i, j \leq p$. There are further restrictions on the patterns of the n_i 's that may occur; a more detailed account (which we do not need here) is in [15].

One can also prove, although again we do not need it here, that if we order the cyclic permutations of $W_{p/q}$ lexicographically by increasing size of the n_i , then $W_{p/q}$ is first in this order.

As $(p/q, r/s)$ run over all Farey neighbors, so the words $(W_{p/q}, W_{r/q})$ run over all possible pairs of generators of G_μ (see [3, 15]). One can also inductively verify the relations $K = [T^{-1}, S^{-1}] = [W_{r/s}^{-1}, W_{p/q}]$ which we will need later.

3.2 Trace identities

It is clear that the trace of any element in G_μ , and in particular the trace of any of the special words $W_{p/q} = W_{p/q}(\mu)$, is a polynomial in μ .

We obtain these polynomials by inductive use of the trace relations. To do this we invoke the notion of the *Farey level* of a rational p/q . This is best described in terms of the well-known Farey tessellation \mathcal{F} of the upper half plane \mathbf{H} obtained by joining each pair of neighboring rationals by a semi-circular arc. This construction divides \mathbf{H} into triangles with vertices at all the rational points. The *level* of p/q is the number of sides of \mathcal{F} cut by the vertical line $\Re z = p/q$ in \mathbf{H} . Thus, for example, $1/2$ is at level 1. (By convention, $0/1$ and $1/1$ are at level 0.)

Proposition 3.1 *The trace of $W_{p/q}$ is a polynomial of the form:*

$$\mathrm{Tr} W_{p/q} = (-i)^q (\mu^q - 2p\mu^{q-1} + b_{q-2}\mu^{q-2} + \dots + b_0), \quad b_i \in \mathbf{Z}.$$

Proof: The proof proceeds by induction on the level j of p/q .

Since

$$\begin{aligned} W_{0/1} &= T & \mathrm{Tr} T &= -i\mu \\ W_{1/1} &= S^{-1}T & \mathrm{Tr} S^{-1}T &= (-i)(\mu - 2) \end{aligned}$$

the conclusion of the proposition holds for $j = 0$.

Suppose now that $\mathrm{Tr} W_{m/n}$ has the stated form for all rationals m/n of level less than j . Let p/q be a rational of level j . Then $p/q = \frac{r+m}{s+n}$ where $r/s, m/n$ are at level less than j , and $r/s < m/n$ say. By the trace identity

$$\mathrm{Tr} W_{p/q} = \mathrm{Tr} W_{m/n} \mathrm{Tr} W_{r/s} - \mathrm{Tr} W_{m/n} W_{r/s}^{-1}$$

it is easy to check using the induction hypothesis that the first two terms in the first product are $(-1)^{n+s}(\mu^{s+n} - 2(m+n)\mu^{n+s-1})$. We shall show that $\mathrm{Tr} W_{m/n} W_{r/s}^{-1}$ is a polynomial of degree $n-s$ in μ . Since $n-s < n+s-1$ (since $s > 1/2$) the result follows.

The semi-circular arc joining r/s to m/n is a side of exactly two triangles of the tessellation \mathcal{F} . Exactly one of these two triangles has its third vertex k/l outside the interval $[r/s, m/n]$. Clearly $\mathrm{level}(k/l) < \mathrm{level}(p/q) = j$.

There are two cases to consider:

1.

$$k/l < r/s$$

In this case, $r/s = \frac{k+m}{l+n}$, and so $W_{r/s} = W_{m/n}W_{k/l}$. Thus, $W_{m/n}^{-1}W_{r/s} = W_{k/l}$ and so by the induction hypothesis

$$\text{Tr}(W_{m/n}^{-1}W_{r/s}) = \text{Tr}W_{k/l}^{-1} = \text{Tr}W_{k/l}$$

is a polynomial of degree $d = n - s$ as required.

2.

$$k/l > m/n$$

In this case $m/n = \frac{r+k}{s+l}$ and so $W_{m/n} = W_{k/l}W_{r/s}$, hence

$$\text{Tr}W_{m/n} = \text{Tr}(W_{k/l}W_{r/s})$$

is again a polynomial of degree $d = n - s$, as required. This completes the proof. \square

3.3 The special branch and asymptotic behavior

We shall see below that there are interesting consequences when the trace of the word $W_{p/q} \in G_\mu$ is real. We are therefore interested in studying the locus

$$\check{V}_{p/q} = \{\mu \in \mathbf{C} : \Im \text{Tr} W_{p/q}(\mu) = 0, |\Re \text{Tr} W_{p/q}(\mu)| > 2\}.$$

Set $\mu = s + it$ and consider the asymptotics of this locus for large values of t in this strip. Proposition 3.1 implies:

Proposition 3.2 *Along the locus $\check{V}_{p/q}$ in the strip $2[p/q] \leq \Re \mu \leq 2([p/q] + 1)$, as $t \rightarrow \infty$, $|\text{Tr} W_{p/q}| \rightarrow \infty$ and $s \rightarrow 2p/q$.*

Proof: Expand the polynomial in terms of s and t . Since s stays bounded the term $(-i)^{2q}t^q$ dominates the absolute value. Because we are assuming that the polynomial is real, and the dominating term of the imaginary part is $(-i)^{2q-1}(qs - 2p)t^{q-1}$, it is clear that as $t \rightarrow \infty$, $s \rightarrow 2p/q$. \square

Since the polynomial $\text{Tr} W_{p/q}$ is holomorphically conjugate to the function μ^q in a neighborhood of $\infty \in \hat{\mathbf{C}}$, the locus $\check{V}_{p/q}$ has q branches that are

asymptotic to q rays of the form $\exp(2\pi i k/q + \pi i/2)$, $k = 0, \dots, q-1$. Therefore, there is a unique branch of $\tilde{\mathcal{V}}_{p/q}$ in the strip $2[p/q] \leq \Re\mu \leq 2([p/q] + 1)$, for $\Im\mu$ large enough. We call the connected component of this branch of $\tilde{\mathcal{V}}_{p/q}$ the *vertical p/q -branch* and denote it by $\mathcal{V}_{p/q}$.

One of our main goals (see section 5) is to prove that the vertical p/q -branches are non-singular analytic curves that fill \mathcal{M} densely and extend to a foliation of \mathcal{M} ; this is illustrated in figure 1.

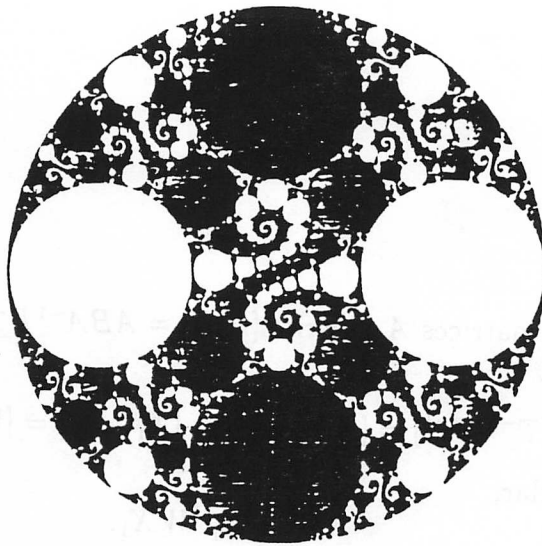


Figure 2: circle chains for $p/q = 1/9$

4 Circle Chains and the Pleating Locus

The computer pictures generated by David Wright in [19] show limit sets for groups he thought should be cusps on $\partial\mathcal{M}$. In these pictures, it appears that the invariant component Ω_0 of the regular set has degenerated into a tree of mutually tangent circles (shaded black in figure 2). These circles are arrayed in a combinatorial pattern that depends on p/q . In trying to prove the existence of such chains for all cusp groups in $\partial\mathcal{M}$, we were led to look at those points $\mu \in \mathcal{M}$ for which $\text{Tr } W_{p/q}(\mu)$ is real and greater than 2. We were then led to investigate the pleated surface forming the convex hull boundary of G_μ facing $\Omega_0(\mu)$ for such μ (see section 4.2). The subject of this section is the relationship between the form of the pleating locus of this pleated surface and the circle chain patterns in the limit set $\Lambda(G_\mu)$.

4.1 Real traces and p/q -circle chains

We begin by investigating the relationship between the condition $\text{Tr } W_{p/q}(\mu) \in \mathbf{R}$ and the existence of circle chains in the limit set of G_μ .

Recall that a Fuchsian group is any discrete subgroup of $\text{aut}(\hat{\mathbf{C}})$ that leaves invariant the interior and exterior of a fixed circle.

Lemma 4.1 *Let $(p, q) \in \mathbf{Z}^2$, $(p, q) = 1$. If $\mu \in \mathcal{M}$ is such that $\text{Tr } W_{p/q}(\mu)$ is real and greater than 2, then the subgroup of G_μ generated by K and W is Fuchsian. Its fixed circle is the circle δ_0 through -1 , (the fixed point of K) and the fixed points of $W_{p/q}(\mu)$. This circle is tangent to \mathbf{R} at -1 .*

Proof: Suppose that $p/q = \frac{r+s}{s+n}$ where $r/s, m/n$ are neighbors and $r/s > m/n$. Write $X_1 = W_{r/s}$, $X_{-1} = W_{m/n}$ and $W = W_{p/q}$, so that $W = X_{-1}X_1$.

For any two matrices A, B write $[A, B] = ABA^{-1}B^{-1}$. It is easy to check inductively that

$$K = [T^{-1}, S^{-1}] = [X_1^{-1}, X_{-1}] = [X_1^{-1}, W] = [W^{-1}, X_{-1}].$$

Thus in particular,

$$KW = X_1^{-1}WX_1.$$

Now by assumption $\text{Tr } W > 2$, and from the above $\text{Tr } KW = \text{Tr } W$. By definition we know $\text{Tr } K = -2$. It now follows from [1] (Theorem 5.2.1) that $\langle W, K \rangle$ is Fuchsian.

The fixed circle δ_0 certainly contains the fixed points of W and the fixed point -1 of K . Since K also fixes the real axis, we see that δ_0 must be tangent to \mathbf{R} at -1 . \square

Definition: A p/q -combinatorial circle chain is a sequence $\{\delta_i\}_{i \in \mathbf{Z}}$ of possibly overlapping circles in \mathbf{C} with the following properties:

1. δ_0 and δ_q are tangent to \mathbf{R} at -1 and 1 respectively.
2. $W_{p/q}(\delta_0) = \delta_0$.
3. $\delta_{r+p} = T(\delta_r)$ for $0 \leq r < q$.
4. $\delta_{r+q} = S(\delta_r)$, for all $r \in \mathbf{Z}$.

Starting from an invariant circle δ_0 as in the lemma above, we can always construct a p/q -combinatorial circle chain as follows:

For $0 \leq r < q$, let $n_r = rp \bmod q$, $0 \leq n_r < q$ and set $n_q = q$. Since $(p, q) = 1$, the sequence n_0, \dots, n_{q-1} is a permutation of $0, \dots, q-1$. Inductively define words E_i , $0 \leq i \leq q$, in $G = \langle S, T \rangle$ by

$$E_0 = id$$

$$E_{n_{r+1}} = TE_{n_r}, \quad 0 \leq p + n_r < q,$$

$$E_{n_{r+1}} = S^{-1}TE_{n_r} \quad p + n_r \geq q.$$

Now define $\delta_i = E_i(\delta_0)$, $1 \leq i < q$. Clearly δ_i is the invariant circle of $E_i F_0 E_i^{-1}$. Further, if $k \in \mathbf{Z}$, $k = rq + s$, $0 \leq s < q$, define

$$\delta_k = S^r \delta_s.$$

To see that $\{\delta_i\}$ is a combinatorial circle chain we need the following lemma. We defer its proof to appendix A.2. A different proof is in [19].

Lemma 4.2 *Suppose that p/q is formed from Farey neighbors $a/b < c/d$ as in section 3.1 so that $W_{p/q} = W_{c/d}W_{a/b}$. Then $E_1 = W_{a/b}$, $E_{-1} = W_{c/d}$ and $E_q = W_{p/q}$.*

Remark: From this lemma we see, in the notation of lemma 4.1, that $E_1 = X_1$ and $E_{-1} = X_{-1}$.

Lemma 4.3 *The circle δ_1 is $X_1(\delta_0)$ and is invariant under $W = W_{p/q}$; hence δ_0 intersects δ_1 in the fixed points of W . Likewise, $\delta_{-1} = X_{-1}(\delta_0)$ and it intersects δ_0 in the fixed points of $X_{-1}WX_{-1}^{-1} = WK$.*

Proof: By the remark above, we have that $E_1 = X_1$, $E_{-1} = X_{-1}$ and $W = E_{-1}E_1$. Since δ_0 is the invariant circle of the Fuchsian subgroup, it contains the fixed points of W , KW and WK . Using the commutator identities, we see that the fixed points of KW are obtained by applying X_1^{-1} to the fixed points of W . Hence the images of the fixed points of KW under X_1 lie on both δ_1 and δ_0 ; therefore, $\delta_1 = X_1(\delta_0)$ intersects δ_0 in the fixed points of W . Similarly, the fixed points of WK are obtained by applying X_{-1} to the fixed points of W and the circles δ_0 and δ_{-1} intersect in the fixed points of WK . \square

Remark: The Fuchsian subgroup $\langle W, K \rangle$ represents a punctured cylinder whose boundary geodesics have equal length. Denote the attracting and repelling fixed points of W and WK by w^\pm and v^\pm respectively. Then the points $-1, w^-, w^+, W(-1), v^+, v^-$ are arranged counterclockwise around δ_0 . (For a more detailed analysis see A.1).

The next definition is a crucial tool in proving our main results.

Definition: A p/q -combinatorial circle chain for a group G_μ is called *proper* if the interiors of adjacent circles intersect and the inside of each circle contains only points of $\Omega(G_\mu)$.

David Wright [19] studied the limiting case of our proper circle chains in which the circles are mutually tangent and form a circle packing. He showed that the existence of such a tangent chain implies that the group G_μ is a cusp group and the point μ is on $\partial\mathcal{M}$. To do this he constructed certain curves to which he could apply one of Maskit's combination theorems. If the interiors of the circles in the chain overlap, we can extend his ideas to construct a fundamental domain for the group G_μ and use it to show that μ is inside \mathcal{M} . This is a stronger result than we need in this paper. Since the construction is involved we omit it and instead, prove the weaker result:

Proposition 4.4 *Suppose that $\mu \in \overline{\mathcal{M}}$ and that G_μ has a proper p/q -circle chain. If $\text{Tr } W_{p/q} > 2$, then $\mu \in \mathcal{M}$.*

Proof: Since $W_{p/q}$ is hyperbolic it has two fixed points so by lemma 4.3 the interiors of the circles δ_0 and δ_1 intersect. Clearly by the discussion preceding proposition 2.1, $\Omega_0 \supset \delta_0 \cup \delta_1$. The result follows from proposition 2.1 itself. \square

Remark: We show in appendix A.3 that it is not possible for neighboring words $W_{p/q}$ and $W_{r/s}$ to have real traces simultaneously. This fact will also follow from the existence of proper circle chains (as in proposition 4.11).

4.2 The convex hull boundary and pleated surfaces

To continue our study, we need to discuss the relation of the existence of circle chains in $\Omega_0(G_\mu)$ to the geometry of the boundary in \mathbf{H}^3 of the convex hull of $\Lambda(G_\mu)$. In this section, we briefly describe the background we need.

Suppose that $\mu \in \mathcal{M}$ so that G_μ is a discrete group. The convex hull \mathcal{C} of the limit set $\Lambda(G_\mu)$ in \mathbf{H}^3 is the intersection with \mathbf{H}^3 of all closed hyperbolic half spaces of $\mathbf{H}^3 \cup \hat{\mathbf{C}}$ containing $\Lambda(G_\mu)$. The connected components of the boundary, $\partial\mathcal{C}$, correspond bijectively to the connected components of the regular set $\Omega(G_\mu)$. This correspondence is made using the canonical retraction map $r: \mathbf{H}^3 \cup \hat{\mathbf{C}} \rightarrow \mathcal{C}$. If $\xi \in \hat{\mathbf{C}}$, then $r(\xi)$ is the unique point of contact with of the largest horoball based at ξ with interior disjoint from \mathcal{C} . If $\xi \in \Lambda(G_\mu)$ then $r(\xi) = \xi$. We shall be interested entirely in the component corresponding to the invariant component $\Omega_0(G_\mu)$ of the regular set. We denote this component $\partial\mathcal{C}_0(\mu)$. By [4](theorem 1.12.1) the quotient surface $\hat{\mathcal{S}}_\mu = \partial\mathcal{C}_0(\mu)/G_\mu$ is a complete hyperbolic surface. Since $\Omega_0(G_\mu)$ is simply connected so is $\partial\mathcal{C}_0(\mu)$ and since $\Omega_0(G_\mu)/G$ is a punctured torus so is $\hat{\mathcal{S}}_\mu$. Notice that the conformal structure coming from the hyperbolic structure on $\hat{\mathcal{S}}_\mu$ is *not* the same as the conformal structure on $\Omega_0(G_\mu)/G_\mu$. It is proved however, in [6], that the hyperbolic structure on $\hat{\mathcal{S}}_\mu$ varies continuously with μ .

The quotient surface $\hat{\mathcal{S}}_\mu$ is a pleated surface in the sense of Thurston [17]. This means that it is an isometric image of a complete hyperbolic surface X in \mathbf{H}^3/G_μ , under a map $f: X \rightarrow \mathbf{H}^3/G_\mu$ that has the property that every point in X lies in some geodesic arc which maps to a geodesic arc in the image.

The *pleating locus* of \hat{S}_μ is the set of points in \hat{S}_μ that lie in the image of exactly one geodesic arc in X . The pleating locus of \hat{S}_μ , which we denote by $\text{pl}(\mu)$, is always a geodesic lamination. It is non-empty because for $\mu \in \mathcal{M}$ the groups G_μ are not Fuchsian. Further, the lamination $\text{pl}(\mu)$ carries a natural transverse measure, the *bending measure*. Following Thurston, [18], we call a lamination that can support a transverse measure a *projective geodesic lamination*. As discussed in section 2.4, the projective geodesic laminations on a punctured torus are identified with $\hat{\mathbf{R}}$.

See [4, 6] for more details on this material.

The following result which will be important is a special case of one of the main results of [6].

Theorem 4.5 *For $\mu \in \mathcal{M}$, the map $\text{pl}(\mu): \mathcal{M} \rightarrow \hat{\mathbf{R}}$ is continuous.*

4.3 The pleating locus and circle chains

In this section we characterize those groups with a proper p/q -circle chain as those for which the pleating locus of \hat{S}_μ is exactly the geodesic $\gamma(p/q)$ in the $W_{p/q}$ homotopy class.

The following easy lemma is central to our whole analysis.

Lemma 4.6 (Real Trace) *Suppose $\text{pl}(\mu)$ is a simple closed geodesic γ on \hat{S}_μ . Then any lift of γ to \mathbf{H}^3 is an axis of an element $g_\mu \in G_\mu$. For any such element, $\text{Tr } g_\mu$ is real and $|\text{Tr } g_\mu| > 2$.*

Proof: Recall that the boundary of the convex hull is invariant under the group. Recall also that a support plane H of a convex set $X \subset \mathbf{H}^3 \cup \hat{\mathbf{C}}$ is a hyperbolic plane that intersects X , and is such that $X \setminus H$ is entirely contained in one of the two half-spaces determined by H .

Let $\tilde{\gamma}$ be an infinite connected lift of γ in \mathbf{H}^3 . Since γ is a closed simple geodesic, there is some loxodromic element $g_\mu \in G_\mu$ identifying points on $\tilde{\gamma}$, so that $\tilde{\gamma}$ is an axis. This axis is the intersection of two support planes of the convex hull boundary.

Now, if $\text{Tr } g_\mu$ were complex, applying g_μ to the boundary of the convex hull would fix its axis but rotate the support planes that intersect to form the pleating by an amount depending on $\arg \text{Tr } g_\mu$. Since these planes must rotate into themselves, $\text{Tr } g_\mu$ is real and since g_μ is loxodromic, $|\text{Tr } g_\mu| > 2$.

□

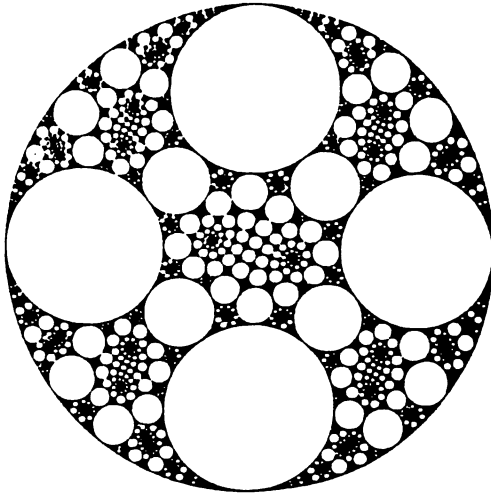


Figure 3: Element with real trace not in the pleating locus

Remark: The converse of this lemma is not necessarily true. Figure 3 shows a group in which there is an element whose trace is real but whose axis does not lie on the convex hull. The circles we see that are not members of the proper circle chain are invariant circles of Fuchsian subgroups that contain these elements.

Corollary 4.7 *If $\text{pl}(\mu)$ equals $\gamma(p/q)$ then G_μ admits a p/q -circle chain.*

We shall now improve this to show that the pleating locus is $\gamma(p/q)$ if and only if G_μ admits a proper p/q -circle chain.

Denote the hemisphere based on the circle δ_0 of the circle chain by H_0 .

Lemma 4.8 *G_μ admits a proper p/q -circle chain if and only if H_0 is a support plane of $\partial\mathcal{C}_0(\mu)$.*

Proof: For the sake of readability, we omit the subscripts μ and p/q in this proof. Suppose first that G admits a proper p/q -circle chain. Since H_0 clearly contains points of $\partial\mathcal{C}_0$, we have only to show that $\partial\mathcal{C}_0$ lies entirely above H_0 . Now suppose that there are points of $\partial\mathcal{C}_0$ lying beneath H_0 . Consider the family of hemispheres centered on the center of δ_0 . Among these is a hemisphere H of minimal radius which intersects $\partial\mathcal{C}_0$. This hemisphere is by definition a support plane for $\partial\mathcal{C}_0$. By construction its radius is less than that of H_0 . By [4][1.6],

$$H \cap \partial\mathcal{C}_0 = \mathcal{C}(H \cap \Lambda)$$

where $\mathcal{C}(X)$ is the convex hull of X in $\mathbb{H}^3 \cup \hat{\mathbb{C}}$. Now since $H \cap \partial\mathcal{C}_0 \neq \emptyset$, we must have $H \cap \Lambda \neq \emptyset$. This implies, however, that there are limit points of G inside δ_0 , contrary to our assumption.

Conversely, suppose that H_0 is a support plane for $\partial\mathcal{C}_0$. This means that all of $\partial\mathcal{C}_0$ lies above H_0 and in particular, that there are no points of Λ beneath H_0 . That is, there are no points of Λ in the interior of the circle δ_0 . Since $\Omega_0(G_\mu)$ is simply connected, the circles overlap and the circle chain is proper. \square

Lemma 4.9 *Suppose that $\text{pl}(\mu)$ equals $\gamma(p/q)$. Then the p/q -combinatorial circle chain for G_μ is proper.*

Proof: Here again, we omit the subscripts μ and p/q . By the assumptions of the lemma, the axis of $W_{p/q}$ in \mathbf{H}^3 and all its conjugates are in $\partial\mathcal{C}_0$. In addition, all the conjugates of this axis by elements of the Fuchsian subgroup $\langle K, W \rangle$ whose invariant circle is δ_0 lie in H_0 . Thus, by convexity, H_0 must be a support plane of $\partial\mathcal{C}_0$ and so by lemma 4.8 the circle chain is proper. \square

Lemma 4.10 *If G_μ has a proper p/q -circle chain, then $\text{pl}(\mu)$ equals $\gamma(p/q)$.*

Proof: By lemma 4.8, H_0 is a support plane for $\partial\mathcal{C}_0$ and similarly, so are H_1, H_{-1} , the hemispheres based on the circles δ_1, δ_{-1} . By lemma 4.3 these two hemispheres intersect H_0 in the axes of W and WK respectively. Thus these axes are bending lines of $\partial\mathcal{C}_0$, and hence $\gamma(p/q)$ is contained in the pleating locus of \hat{S}_μ .

Clearly, the invariant component Ω_0 of the regular set is the region interior to the circle chain and corresponds to $\partial\mathcal{C}_0$. Since Ω_0/G_μ is a punctured torus so is $\hat{S}_\mu = \partial\mathcal{C}/G_\mu$ and since $\gamma(p/q)$ is a maximal lamination on a punctured torus, the result follows. \square

Remark: A more direct way of seeing that \hat{S} is a punctured torus pleated along $\gamma(p/q)$ is as follows. By [4](lemma 1.6.2),

$$H_0 \cap \overline{\mathcal{C}(\Lambda)} = \overline{\mathcal{C}(H_0 \cap \Lambda)},$$

where the closure is taken in $\mathbf{H}^3 \cup \hat{\mathbf{C}}$, and hence $-1 \in H_0 \cap \overline{\mathcal{C}(\Lambda)}$. Let P denote the intersection point on the axis of W with the perpendicular in \mathbf{H}^3 from -1 to this axis. Then the geodesics joining the points

$$\{-1, P, W(P), W(-1), X_{-1}W(P), X_{-1}(P), -1\}$$

in order, bound a polygon Π in $H_0 \cap \overline{\partial\mathcal{C}_0(\Lambda)}$ whose sides are paired by the elements, W, WK and X_{-1} of G . Note that $W(-1) = WK(-1)$ and $X_{-1}W(P) = WKX_{-1}(P)$.

Since $\partial\mathcal{C}_0$ is a complete hyperbolic surface, it is easy to check that the cycle conditions of Poincaré's theorem hold for the vertices of Π . Applying this theorem we see that Π is a fundamental domain for the action of G on $\partial\mathcal{C}_0$. It is also easy to see that the surface obtained by identifying the sides of Π is a punctured torus pleated along $\gamma(p/q)$.

We can summarize the results of this section in the following:

Proposition 4.11 *The group G_μ has a proper p/q -circle chain if and only if $\mu \in \mathcal{M}$ and $\text{pl}(\mu)$ equals $\gamma(p/q)$. In this situation $\text{Tr } W_{p/q}$ is real and $|\text{Tr } W_{p/q}| > 2$.*

5 Pleating Rays for the Rationals

For $p/q \in \mathbf{Q}$, we define the p/q -pleating ray as

$$\mathcal{P}_{p/q} = \{\mu \in \mathcal{M} \mid \text{pl}(\mu) = p/q\}.$$

Clearly, $\mathcal{P}_{p/q} \cap \mathcal{P}_{p'/q'} = \emptyset$ whenever $p/q \neq p'/q'$.

Recall from section 3.3 that the vertical branch $\mathcal{V}_{p/q}$ is the connected component of the locus $\tilde{\mathcal{V}}_{p/q} = \{\mu \in \mathbf{C} : |\text{Tr } W_{p/q}(\mu)| > 2\}$ that is asymptotic to $\Re \mu = 2p/q$ as $\Im \mu \rightarrow \infty$. Clearly, by proposition 4.11 we have $\mathcal{P}_{p/q} \subset \tilde{\mathcal{V}}_{p/q}$. In this section we prove our first main result:

Theorem 5.1 (Rational pleating rays) *For $p/q \in \mathbf{Q}$, the pleating ray $\mathcal{P}_{p/q}$ coincides with the vertical branch $\mathcal{V}_{p/q}$ of $\tilde{\mathcal{V}}_{p/q}$. This branch contains no singularities and $\tilde{\mathcal{V}}_{p/q} \setminus \mathcal{V}_{p/q}$ consists of a single point on $\partial \mathcal{M}$ at which $\text{Tr } W_{p/q} = 2$.*

The boundary point in question is a cusp in the sense of Bers.

5.1 Integral pleating rays

We begin by establishing theorem 5.1 in the special case for which $p/q = n \in \mathbf{Z}$. We refer to the rays $\mathcal{P}_{n/1}$ as *integral pleating rays*.

By definition $\tilde{\mathcal{V}}_{n/1} = \{\mu \in \mathbf{C} : \Im \text{Tr}(S^{-n}T_\mu) = 0 \text{ and } |\Re \text{Tr}(S^{-n}T_\mu)| > 2\}$. By an easy computation we have $S^{-1}T_{\mu+2} = T_\mu$ and so $\tilde{\mathcal{V}}_{n/1} = \{\mu \in \mathbf{C} : \mu = 2n + it, t \in \mathbf{R}, |t| > 2\}$.

As in the discussion in section 2.5, the point $2n + it \in \mathcal{M}$ provided $t > 2$ and clearly $\mathcal{V}_{n/1} = \{\mu \in \mathbf{C} : \mu = 2n + it, t > 2\}$. By proposition 4.11 it is therefore sufficient to establish the existence of a proper $n/1$ circle chain whenever $\mu \in \mathcal{V}_{n/1}$. We carry this out for $n = 0$ and $n = 1$; all other cases are similar.

Proposition 5.2 *Let $\epsilon = 0$ or 1 . Then for $\mu \in \mathcal{V}_{\epsilon/1}$ the group G_μ admits a proper $\epsilon/1$ -circle chain.*

Proof: First note that $W_\epsilon = T$ if $\epsilon = 0$ and $W_\epsilon = S^{-1}T$ if $\epsilon = 1$. From the combinatorics of circle chains in 4.1, we see that we must find a circle δ_0 tangent to \mathbf{R} at -1 and passing through the fixed points of W_ϵ . If $\epsilon = 0$

then δ_0 is invariant under T and the other circles in the chain are given by $\delta_n = S^n(\delta_0)$, $n \in \mathbf{Z}$. If $\epsilon = 1$ then δ_0 is invariant under $S^{-1}T$ and the other circles in the chain are $\delta_1 = T(\delta_0)$, $\delta_{2n} = S^n(\delta_0)$, $\delta_{2n+1} = S^n(\delta_1)$, $n \in \mathbf{Z}$.

Suppose then that $\mu \in \mathcal{V}_{\epsilon/1}$, $\epsilon = 0$ or 1 . In either case we draw the circle δ_0 through the fixed points of W_ϵ and the fixed point of K at -1 . By lemma 4.1 this circle is tangent to \mathbf{R} at -1 and is invariant under W_ϵ . Now draw the other circles δ_i as defined above to form the chain. Notice that the limit set of the Fuchsian subgroup $\langle W_\epsilon, K \rangle$ is a Cantor set in δ_0 . What we have to do is to show that there are no points of the limit set $\Lambda(G_\mu)$ inside the circle δ_0 . To do this, we shall construct a fundamental domain for G_μ inside $\delta_0 \cup \delta_1$ whose images under $\langle W_\epsilon, K \rangle$ and $S^{\pm 1}$ fill out the region inside $\delta_0 \cup \delta_1$.

Let us now specialize to the case $\epsilon = 0$ so $W_\epsilon = T$. It is easy to compute that the points $T(\pm 1)$ lie vertically above ± 1 . By lemma 4.3 or by direct computation the circles δ_0 and δ_1 intersect in the fixed points of T and these lie on the imaginary axis. Let D be the unit disk, $|z| \leq 1$. Using elementary combination theorems (for instance Maskit's first combination theorem [8] or Beardon's packing theorem [1]) the domain between the lines $z = -1$ and $z = 1$ and exterior to the circles D and $T(D)$ forms a fundamental domain for G_μ .

Now divide this domain in two by cutting down the imaginary axis and translate the right hand half to the left by S^{-1} thus obtaining a region contained inside δ_0 . Extend this new region out to the boundary of δ_0 and call this region R .

Clearly R is contained in the regular set $\Omega(G_\mu)$. Further, an easy application of Poincaré's theorem shows that R is a fundamental domain for the subgroup $\langle K, T_\mu \rangle$ that fixes δ_0 . Therefore its translates under this subgroup fill out δ_0 . Thus the inside of δ_0 is completely contained in $\Omega(G_\mu)$ and the proof is complete.

For $\epsilon = 1$ the picture is the same though the elements that pair the sides are different. We now compute that $W_1 = S^{-1}T_\mu$ carries ± 1 to $\pm 1 + it$ so that D is mapped to the upper disk by $S^{-1}T_\mu$ and not by T_μ as before. However, $S^{-1}T_\mu$ and S are still a pair of generators for G_μ so the same region as before is a fundamental domain for G_μ . The same trick as before shows how to fill out the inside of δ_0 using the elements of G_μ ; again note that in the side pairings T_μ must be replaced by $S^{-1}T_\mu$. \square

Remark: Note that the points $\mu = 2i$ and $\mu = 2 + 2i$ are the unique points

at which $\text{Tr } T_\mu$ and $\text{Tr } S^{-1}T_\mu$ are respectively equal to 2. At these points the construction above degenerates to its limit case in which the disks $D, T_\mu(D)$ and the circles δ_0, δ_1 are tangent. The groups at these points are still discrete; in them, the element W_ϵ has degenerated to an accidental parabolic.

5.2 Symmetric tori

Although we don't need it in what follows, it is interesting to note that the punctured tori corresponding to points on the rays $\mathcal{V}_{0/1}$ and $\mathcal{V}_{1/1}$ are exactly the rectangular ones. This implies that $\mathcal{V}_{0/1}$ and $\mathcal{V}_{1/1}$ are respectively the images of the vertical lines $\Re\tau = 0$ and $\Re\tau = 1$ under the Riemann map $\phi: \mathbf{H} \rightarrow \mathcal{M}$.

It is easy to see the symmetry from the construction above. In both cases, the reflection $z \mapsto -\bar{z}$ leaves $\Omega_0(G_\mu)$ invariant, and by computation, conjugates $\langle S, T_\mu \rangle$ to $\langle S^{-1}, T_\mu \rangle$ if $\epsilon = 0$ and $\langle S, S^{-1}T_\mu \rangle$ to $\langle S^{-1}, S^{-1}T_\mu \rangle$ if $\epsilon = 1$. These are exactly the symmetries of the flat tori for which $\Re\tau = 0$ and $\Re\tau = 1$ respectively. It is well known that these are all the tori with these symmetries and since the images of these lines under ϕ are connected analytic curves, all the rectangular tori lie on the rays $\mathcal{V}_{0/1}$ and $\mathcal{V}_{1/1}$ as claimed.

Remark: It is tempting to conjecture that the rational pleating rays are exactly the images of the vertical lines in the τ -plane under the map ϕ . However, our computations indicate that this is *not* the case.

5.3 Rational pleating rays

We now proceed to the proof of theorem 5.1 in the general case.

We begin by establishing that $\mathcal{P}_{p/q}$ is confined to the strip $2[p/q] < \Re\mu < 2([p/q] + 1)$, where $[x]$ denotes the integer part of x .

It is at this point that we need to use the continuity of the pleating locus referred to in section 4.2. It is an easy corollary of theorems 1 and 4 in [6] that the map $\mu \mapsto \text{pl}(\mu)$ is continuous. This result enables us to give an easy proof of the result we need.

Lemma 5.3 *For any $p/q \in \mathbf{Q}$ the pleating ray $\mathcal{P}_{p/q}$ is non-empty. Also, $\mathcal{P}_{p/q} \subset \{\mu \in \mathcal{M} : 2[p/q] < \Re\mu < 2([p/q] + 1)\}$.*

Proof: Let $k > 2$ and consider the horizontal path $\mathbf{R} \rightarrow \mathbf{C}$, $\sigma \mapsto \sigma + ki$. As remarked in section 2.5, this path is completely contained in \mathcal{M} . By the continuity of pl therefore, the map $\sigma \mapsto \text{pl}(\sigma + ik)$ is a continuous map of \mathbf{R} to \mathbf{R} .

We proved in section 5.1 that for $n \in \mathbf{Z}$, $t > 2$, $\text{pl}(2n + it) = n$ and that $\text{pl}^{-1}(n) = \{\mu \in \mathcal{M} : \mu = 2n + it, t > 2\}$.

We also know that for $p/q, p'/q' \in \mathbf{Q}$, we have $\mathcal{P}_{p/q} \cap \mathcal{P}_{p'/q'} = \emptyset$.

Combining these facts gives the result. \square

Proposition 5.4 $\mathcal{P}_{p/q}$ is open and closed in $\tilde{\mathcal{V}}_{p/q}$.

Proof: Let $\mu_0 \in \mathcal{P}_{p/q} \subset \tilde{\mathcal{V}}_{p/q}$ and let K be the connected component of $\tilde{\mathcal{V}}_{p/q} \cap \mathcal{M}$ containing μ_0 . Since \mathcal{M} is open, K is open in $\tilde{\mathcal{V}}_{p/q}$. Thus, since $\tilde{\mathcal{V}}_{p/q}$ is defined by an analytic equation, we can find an arc $\sigma \subset K$ with $\mu_0 \in \text{int}(\sigma)$.

To see that $\mathcal{P}_{p/q}$ is open in $\tilde{\mathcal{V}}_{p/q}$ it suffices to show that σ is entirely contained in $\mathcal{P}_{p/q}$.

Let $\mu_1 \in \sigma$, $\mu_1 \neq \mu_0$, and let f_1 be a quasiconformal homeomorphism of $\hat{\mathbf{C}}$ such that

$$G_1 = G_{\mu_1} = f_1 G_{\mu_0} f_1^{-1}.$$

For $0 \leq t \leq 1$, let f_t be a family of quasiconformal homeomorphisms such that

$$G_t = G_{\mu_t} = f_t G_{\mu_0} f_t^{-1}$$

where μ_t varies along σ between μ_0 and μ_1 .

Note that $\text{Tr } W_{p/q}(\mu_t)$ is real for $t \in [0, 1]$. For each such t , let $\delta_0(t)$ be the circle through the fixed points $\{w^+(\mu_t), w^-(\mu_t)\}$ of $W_{p/q}(\mu_t)$ and the point -1 ; that is, the first circle of the p/q -circle chain for G_t . We claim that the p/q -circle chain is proper for each $t \in [0, 1]$.

The λ -lemma of Mañé, Sad and Sullivan [14] tells us that the restriction of the map

$$f_t: \Lambda(G_{\mu_0}) \rightarrow \Lambda(G_t)$$

is injective. Since it takes fixed points of group elements to corresponding fixed points, the circle $\delta_0(t)$ moves continuously. If F_t is the Fuchsian subgroup fixing $\delta_0(t)$, then by the minimality of $\Lambda(F_t)$,

$$\Lambda(F_t) = \Lambda(G_t) \cap \delta_0(t).$$

Pick a point $\xi_0 \in \Lambda(G_0)$ outside $\delta_0(0)$; in other words, not in $\Lambda(F_0)$. It moves on a continuous path ξ_t ; this path can never cross inside $\delta_0(t)$, for if it did, it would first have to coincide with a boundary point of $\delta_0(t)$ in the limit set of G_t . All such points are in the limit set of F_t , but ξ_0 was chosen not in the limit set of F_0 . This contradicts the injectivity of f_t and proves that the circle chain remains proper. We conclude using proposition 4.11 that $\mathcal{P}_{p/q}$ is open in $\tilde{\mathcal{V}}_{p/q}$.

Next we show that $\mathcal{P}_{p/q}$ is closed in $\tilde{\mathcal{V}}_{p/q}$. Suppose that $\mu_n \rightarrow \mu$, $\mu_n \in \mathcal{P}_{p/q}$, $\text{Tr } W_{p/q}(\mu_n) \rightarrow a$ with $|a| > 2$. Then the family of circles through the fixed points $\{w^+(\mu), w^-(\mu)\}$ of $W_{p/q}(\mu)$ and -1 have a limit. This limit circle contains no limit points in its interior because having interior limit points is an open condition. Moreover, since $|a| > 2$, $w^+ \neq w^-$. The circles of the chain overlap and their union is an invariant component of the set of discontinuity. Therefore the circle chain is proper and by proposition 4.11 again $\mu \in \mathcal{M}$. \square

Corollary 5.5 $\mathcal{P}_{p/q}$ is a union of connected components of $\tilde{\mathcal{V}}_{p/q}$.

It is now fairly easy to complete the proof of theorem 5.1.

Lemma 5.6 No component of $\tilde{\mathcal{V}}_{p/q}$ in $\mathcal{P}_{p/q}$ contains a singularity.

Proof: Suppose $\mathcal{P}_{p/q}$ contained a component of $\tilde{\mathcal{V}}_{p/q}$ which had a singularity at $\mu_0 \in \mathcal{M}$. Since the trace is given by a polynomial function $w = \text{Tr } W_{p/q}$, at a branch point of order k , the local coordinate can be written in the form $\mu - \mu_0 = z^k$. The preimage of a line through w_0 therefore, has $2k$ branches meeting at $z = 0$. Hence starting at a singularity one could move along at least two distinct branches in the direction of increasing trace. Since there are at most finitely many singularities, we can find at least two distinct branches along which trace goes to ∞ . For $\Im \mu \gg 0$ we know that only one branch can lie in the strip $2[p/q] \leq \Re \mu \leq 2([p/q] + 1)$, hence the other must leave this strip while remaining in \mathcal{M} since $\mathcal{P}_{p/q} \subset \mathcal{M}$. Thus it must cross $\mathcal{P}_{[p/q]/1}$ or $\mathcal{P}_{[p/q]+1/1}$. However, this is impossible since we know that the curves $\mathcal{P}_{p/q}$ and $\mathcal{P}_{p'/q'}$ are disjoint for $p/q \neq p'/q'$. \square

Corollary 5.7 The function $\text{Tr} = \text{Tr } W_{p/q}$ has no maximum or minimum on $\mathcal{P}_{p/q}$.

Proof: At a maximum or minimum, $\text{Tr } W_{p/q}(\mu)$ has derivative zero in the direction along $\tilde{\mathcal{V}}_{p/q}$, hence there is a singularity of $\tilde{\mathcal{V}}_{p/q}$ at such a point and this contradicts lemma 5.6. \square

Lemma 5.8 *Any connected component C of $\mathcal{P}_{p/q}$ contains points along which $\text{Tr } W_{p/q} \rightarrow \infty$.*

Proof: Write Tr for $\text{Tr } W_{p/q}$. If the assertion of the lemma were false, the function $\text{Tr}|_C$ restricted to the component C would be bounded. By corollary 5.7, it has no maximum or minimum in C . If $\bar{C} \subset \mathcal{M}$ this is impossible. Otherwise, \bar{C} has at least one point on $\partial\mathcal{M}$ and since $\mathcal{P}_{p/q}$ is closed in $\tilde{\mathcal{V}}_{p/q}$ the only possible value of Tr at these points is 2. But then Tr is constant on C , which is impossible. \square

We put this all together now to obtain:

Proof of Theorem 5.1:

In section 3.3 we established that $\mathcal{V}_{p/q}$ is the unique branch of $\tilde{\mathcal{V}}_{p/q}$ that goes to infinity in the strip $2[p/q] \leq \Re\mu \leq 2([p/q] + 1)$. Hence, taking k sufficiently large in the proof of proposition 5.3, we see that $\mathcal{P}_{p/q} \cap \mathcal{V}_{p/q} \neq \emptyset$. Thus, the only connected component of $\tilde{\mathcal{V}}_{p/q}$ in $\mathcal{P}_{p/q}$ must be $\mathcal{V}_{p/q}$. That $\mathcal{V}_{p/q}$ contains no singularities now follows from lemma 5.6. It follows that $\text{Tr}|_C$ is monotonic. It is clear that $\overline{\mathcal{V}_{p/q}}$ meets $\partial\mathcal{M}$ in a unique point where $\text{Tr } W_{p/q} = 2$.

At this boundary point the element $W_{p/q}$ has become parabolic; its fixed points coincide and the circles of the circle chain all become tangent. By an argument of Maskit [10] the extremal length of curves in the p/q -homotopy class is 0 at this point so it is a cusp both in the classical sense and in the Bers sense. \square

Remark: The above argument does not rule out the possibility that there might be other points $\mu \in \partial\mathcal{M}$ for which $|\text{Tr } W_{p/q}| = 2$. That this does not in fact happen is a special case of the main result of [5].

6 Real Pleating Rays and Pleating Length

In this section we extend theorem 5.1 from simple p/q -curves to laminations $\lambda \in \mathbf{R} \setminus \mathbf{Q}$.

6.1 Real pleating rays

It is apparent from figure 1 that the partial foliation of \mathcal{M} by the rational pleating rays should extend to a foliation by real rays \mathcal{P}_λ , $\lambda \in \mathbf{R}$. Recalling our identification of the set of projective laminations on a punctured torus with $\hat{\mathbf{R}}$ from section 2.4, we set, for $\lambda \in \hat{\mathbf{R}}$,

$$\mathcal{P}_\lambda = \{\mu \in \mathcal{M} : \text{pl}(\mu) = \lambda\}.$$

Clearly, if $\lambda \neq \lambda'$ then $\mathcal{P}_\lambda \cap \mathcal{P}_{\lambda'} = \emptyset$. By the method of lemma 5.3 we find $\mathcal{P}_\lambda \neq \emptyset$. We also note that $\mathcal{P}_\infty = \emptyset$ since $\gamma(\infty)$ is the homotopy class of S and this is fixed as an accidental parabolic in our setup. Therefore, by lemma 4.6, it can never represent the pleating locus of \hat{S}_μ . Since for $\mu \in \mathcal{M}$, the invariant component $\Omega_0(G_\mu)$ is never a circle (see section 2.2), it follows that $\text{pl}(\mu) \neq \emptyset$. Hence, $\mathcal{M} = \bigcup_{\lambda \in \mathbf{R}} \mathcal{P}_\lambda$.

If f is a complex analytic function defined in a domain $U \subset \mathbf{C}$, we define the *real locus* of f to be the set $f^{-1}(\mathbf{R})$ in U . We have:

Lemma 6.1 *The pleating ray \mathcal{P}_λ is contained in the real locus of an analytic function defined on \mathcal{M} .*

Proof: Suppose that $\mu_0 \in \mathcal{P}_\lambda$. Let C be the cusp of the punctured torus \hat{S}_{μ_0} . Since all the leaves of $\text{pl}(\mu_0)$ lie in a compact part of \hat{S}_{μ_0} , a neighborhood of C is contained in a flat part of \hat{S}_{μ_0} . By area considerations (see [17](p.9.32)), this flat piece must be a punctured bigon B . Choose a geodesic from the cusp C to one end of B and cut along it. Lifting this cut region to \mathbf{H}^3 , we obtain an ideal 4-gon as a flat piece of $\partial\mathcal{C}_0$. Call the four vertices of this 4-gon v_1, v_2, v_3, v_4 and denote their cross ratio by ρ . Since these points are on the boundary of the support plane containing the lifted flat piece they are concyclic and ρ is real.

Now let μ vary in a neighborhood of μ_0 . By the λ -lemma, ([14] and see also [6]), the points $v_i = v_i(\mu)$ depend analytically on μ and they remain distinct for $\mu \in \mathcal{M}$. It follows that $\rho = \rho(\mu)$ also depends analytically on

μ . The points $v_i(\mu)$ are all on the boundary of the same support plane of $\partial\mathcal{C}_\mu$ only when $\rho(\mu)$ is real. Hence, \mathcal{P}_λ is contained in the real locus of ρ as required. \square

Corollary 6.2 *The union of rational rays $\bigcup_{p/q \in \mathbf{Q}} \mathcal{R}_{p/q}$ is dense in \mathcal{M} .*

Proof: Suppose $\mu_0 \in \mathcal{M}$ and let $\lambda = \text{pl}(\mu_0)$, $\lambda \notin \mathbf{Q}$. By lemma 6.1 we can find a path $\alpha: [0, 1] \rightarrow \mathcal{M}$ with $\alpha(0) = \mu_0$ and such that the image of α is locally transverse to \mathcal{P}_λ . By the continuity of the map $\mu \mapsto \text{pl}(\mu)$, $\text{pl} \circ \alpha$ is a continuous map to \mathbf{R} , and by the transversality it is non-constant. Thus we can find points arbitrarily close to μ_0 with $\text{pl}(\mu) \in \mathbf{Q}$. \square

There is a more refined result that will be useful later:

Lemma 6.3 *Suppose that $\mu_0 \in \mathcal{P}_\lambda$ and that $p_n/q_n \in \mathbf{Q}$, $p_n/q_n \rightarrow \lambda$ as $n \rightarrow \infty$. Then there exists $\mu_n \in \mathcal{P}_{p_n/q_n}$ with $\mu_n \rightarrow \mu_0$.*

Proof: Without loss of generality, we may assume that the sequence p_n/q_n is increasing with limit λ . Pick $\mu_1 \in \mathcal{P}_{p_1/q_1}$. Join μ_1 to μ_0 by a path $\sigma: [0, 1] \rightarrow \mathcal{M}$ transversal to \mathcal{P}_λ at μ_0 so that $\sigma \cap \mathcal{P}_\lambda = \{\mu_0\}$.

By the continuity of pl , and since $p_1/q_1 < p_n/q_n < \lambda$, σ must intersect all the pleating rays \mathcal{P}_{p_n/q_n} at points $\mu_n = \sigma(t_n)$. We may clearly assume $\dots < t_n < t_{n+1} < \dots$ so that $\{\mu_n\}$ has a limit $\sigma(t_\infty) \in \sigma$.

Again by the continuity of pl , $\text{pl}(\sigma(t_\infty)) = \lim_{n \rightarrow \infty} p_n/q_n = \lambda$ so that $\sigma(t_\infty) \in \mathcal{P}_\lambda$. Since by construction $\sigma \cap \mathcal{P}_\lambda = \{\mu_0\}$, we have $\sigma(t_\infty) = \mu_0$ which proves $\mu_n \rightarrow \mu_0$. \square

6.1.1 Normalized traces

We should like to attach to each \mathcal{P}_λ an extended trace function $\text{Tr}_\lambda(\mu)$ which would extend the trace polynomials $\text{Tr } W_{p/q}(\mu)$ from simple closed geodesics on \hat{S}_μ to laminations on \hat{S}_μ .

Now the family $\{\text{Tr } W_{p/q}(\mu)\}_{p/q \in \mathbf{Q}}$ is not normal on \mathcal{M} : as $p_n/q_n \rightarrow \lambda \in \mathbf{R} \setminus \mathbf{Q}$, the degree q_n of $\text{Tr } W_{p_n/q_n}$ goes to infinity and the traces are unbounded. In order to produce a more tractable family we first convert the traces to complex length and then scale appropriately.

Recall that the *complex translation length* $L(g)$ of a loxodromic element $g \in SL(2, \mathbf{C})$ is $2 \text{arccosh}(\text{Tr } g)/2$. The motivation for this definition is that $\Re L(g)$ is the hyperbolic translation length of g along its axis in \mathbf{H}^3 and

$\arg L(g)$ is the angle through which a point off the axis is moved about the axis.

Since $|\operatorname{Tr} W_{p/q}(\mu)| \neq 2$ for $\mu \in \mathcal{M}$, and since $\operatorname{Tr} W_{p/q}(\mu)$ is real on the connected set $\mathcal{P}_{p/q}$, we can pick a branch of the complex length of $W_{p/q}$ which is analytic on the (simply connected) set \mathcal{M} and which is real on $\mathcal{P}_{p/q}$.

We prove in appendix A.4 that the coefficients of the polynomial $\operatorname{Tr} W_{p/q}$ are bounded by 8^q . Thus the family $\{L_{p/q}(\mu) = 1/q \operatorname{arccosh} \operatorname{Tr} W_{p/q}(\mu)\}$ is uniformly bounded on compact subsets of \mathcal{M} . We shall take appropriate limit functions of this normal family as $p/q \rightarrow \lambda$, $\lambda \in \mathbf{R} \setminus \mathbf{Q}$ as the normalized length functions.

In order to prove the uniqueness of these limit functions we shall give an alternative characterization of the function $L_{p/q}(\mu)$ on $\mathcal{P}_{p/q}$. This involves introducing several concepts from Thurston's theory of measured laminations; we do this in the next section.

6.2 Pleating measure and pleating length

A *transverse measure* ν on a geodesic lamination L on a hyperbolic surface X of finite area is an assignment of a regular countably additive measure to every interval transversal to L in such a way that these measures are preserved by any isotopy mapping one transversal to another and preserving the leaves of the lamination. We call the pair, (L, ν) a *measured lamination*. By abuse of terminology we usually refer to ν as a measured lamination and write $|\nu|$ for the underlying point set L .

In particular, if γ is a simple closed geodesic on X then we denote by δ_γ the measured lamination whose leaves consist of the geodesic γ and whose measure is an atomic unit mass on γ .

We denote by $\mathcal{ML}(X)$ the space of measured laminations on X . The weak topology on measures gives a natural topology on $\mathcal{ML}(X)$: a sequence $\nu_n \in \mathcal{ML}(X)$ converges to $\nu \in \mathcal{ML}(X)$ if $\int_I f d\nu_n$ converges to $\int_I f d\nu$ for any open interval I transversal to all the $|\nu_n|$ and $|\nu|$ and for any continuous function f of compact support on X .

For $\nu \in \mathcal{ML}(X)$, the *lamination length* of ν , $\ell(\nu)$, is the total mass of the measure on X that is locally the product of the measure ν on transversals to $|\nu|$ and hyperbolic distance along the leaves of $|\nu|$. Note that if γ is a simple closed geodesic then $\ell(\delta_\gamma)$ is exactly the hyperbolic length in the usual sense. It follows easily from the definition of the topology on $\mathcal{ML}(X)$

that $\ell: \mathcal{ML}(X) \rightarrow \mathbf{R}^+$ is continuous. (See [6] for a careful discussion of how one deals with cusps on X .)

Similarly if $\nu \in \mathcal{ML}(X)$ and if γ is a simple closed geodesic on X , the intersection number $i(\gamma, \nu)$ is the minimal measure given by ν to a curve isotopic to γ . In particular, if $\nu = \delta_{\gamma'}$ for some simple geodesic γ' , then $i(\gamma, \delta_{\gamma'})$ is just the intersection number in the usual sense.

Continuity of the map $i_\gamma: \mathcal{ML}(X) \rightarrow \mathbf{R}$, $i_\gamma(\nu) = i(\gamma, \nu)$ also follows easily from the definitions. For brevity we write $\delta_{p/q}$ for $\delta_{\gamma(p/q)}$.

Lemma 6.4 *With the terminology above, if $\mu \in \mathcal{P}_{p/q}$, then*

$$L_{p/q}(\mu) = \ell_\mu(\delta_{p/q})/i(\gamma(\infty), \delta_{p/q})$$

where ℓ_μ denotes the lamination length on the pleated surface \hat{S}_μ .

Proof: Observe that, because $\gamma(p/q)$ is the pleating locus of \hat{S}_μ , the geodesic length of $\gamma(p/q)$ on the hyperbolic surface \hat{S}_μ and its length in the hyperbolic three manifold \mathbf{H}^3/G_μ coincide. Since $\text{Tr } W_{p/q}$ is real for $\mu \in \mathcal{P}_{p/q}$, we have $\text{arccosh } \text{Tr } W_{p/q}(\mu) = \ell_\mu(\delta_{p/q})$. Thus to prove the lemma we need only see that $i(\gamma(\infty), \delta_{p/q}) = q$. Intersection number however, only depends on the topology and not the conformal structure of a surface so we can read this off from the flat picture. Alternatively, one can observe that any curve in $\Omega_0(G_\mu)$ joining $z_0 \in \text{int } \delta_0$ to $S(z_0) \in \text{int } \delta_q$ must run through the circles $\delta_0, \dots, \delta_q$ of the p/q -circle chain in $\Omega_0(G_\mu)$, and hence must intersect q conjugates of the axis of $W_{p/q}$. \square

We now want to find an expression for $L_{p/q}$ that is independent of p/q . To do this, we make use of the bending measure of the pleating locus $\text{pl}(\mu)$. Recall ([4, 6]), that the bending measure $\beta(\mu)$ of $\text{pl}(\mu)$ is a natural transverse measure that measures the total angle through which support planes of the convex hull are bent when moving along a transversal to $\text{pl}(\mu)$. In the case where $\text{pl}(\mu) = p/q$, so that G_μ has a proper p/q -circle chain, $\beta(\mu)$ is just the measure $\theta \delta_{p/q}$, where θ is the angle between adjacent support planes in $\partial \mathcal{C}_0(\mu)$; in other words, θ is the angle between successive circles in the circle chain. Clearly then, we have:

$$L_{p/q}(\mu) = \ell_\mu(\delta_{p/q})/i(\gamma(\infty), \delta_{p/q}) = \ell(\beta(\mu))/i(\gamma(\infty), \beta(\mu))$$

for $\mu \in \mathcal{P}_{p/q}$.

Proposition 6.5 *The map $\psi: \mathcal{M} \rightarrow \mathcal{ML}(\hat{S})$ given by*

$$\mu \mapsto \beta(\mu)/i(\gamma(\infty), \beta(\mu))$$

is continuous and $\psi|_{\mathcal{P}_\lambda}$ is constant for each $\lambda \in \mathbf{R}$.

Proof: The continuity of β is a special case of theorem 4 of [6]. The continuity of $i_{\gamma(\infty)}$ was mentioned above.

It follows from the discussion above that for $\mu \in \mathcal{P}_{p/q}$,

$$\beta(\mu)/i(\gamma(\infty), \beta(\mu)) = \theta \delta_{p/q} / q\theta = \delta_{p/q} / q.$$

Thus we have only to show that $\psi|_{\mathcal{P}_\lambda}$ is constant for each $\lambda \in \mathbf{R} \setminus \mathbf{Q}$. Pick $\mu_0 \in \mathcal{P}_\lambda$. By lemma 6.3, if p_n/q_n is any sequence with $p_n/q_n \rightarrow \lambda$ we can find $\mu_n \in \mathcal{P}_{p_n/q_n}$ such that $\mu_n \rightarrow \mu_0$. By the continuity of pl it follows that $p_n/q_n = \text{pl}(\mu_n) \rightarrow \text{pl}(\mu_0) = \lambda$. Then $\psi(\mu_n) \rightarrow \psi(\mu_0)$, and since $\psi(\mu_n)$ is independent of $\mu_n \in \mathcal{P}_{p_n/q_n}$, the limit is independent of $\mu_0 \in \mathcal{P}_\lambda$. \square

We define the *pleating measure* $\pi_\lambda \in \mathcal{ML}(\hat{S})$ to be the value of ψ on \mathcal{P}_λ . The above shows:

Proposition 6.6 *The map $\mathbf{R} \rightarrow \mathcal{ML}(\hat{S})$ given by $\lambda \mapsto \pi_\lambda$ is continuous.*

We define the *pleating length* of G_μ , $\mu \in \mathcal{M}$ to be $PL(\mu) = \ell_\mu(\pi_{\text{pl}(\mu)})$, where, as usual ℓ_μ denotes the lamination length on the pleated surface \hat{S}_μ .

The following proposition is an immediate consequence of the discussion above.

Proposition 6.7 *The function $PL: \mathcal{M} \rightarrow \mathbf{R}$ is continuous and $PL(\mu) = L_{p/q}(\mu)$ for $\mu \in \mathcal{P}_{p/q}$.*

6.3 Normalized complex length

We shall use the characterization of the normalized length functions given in proposition 6.7 to prove the uniqueness of the limit functions of the normal family $\{L_{p/q}(\mu)\}_{p/q \in \mathbf{Q}}$.

Lemma 6.8 *Suppose that $p_n/q_n \rightarrow \lambda \in \mathbf{R}$ and that $\mu_0 \in \mathcal{P}_\lambda$. Then*

$$\Re L_{p_n/q_n}(\mu_0) \rightarrow PL(\mu_0).$$

Proof: For the sake of readability we omit the dependence on μ_0 where it will cause no confusion. Write π_n for π_{p_n/q_n} , the pleating measure of the lamination p_n/q_n . Following [17](p 8.10.3), we see in this situation, that a generic leaf l of $|\pi_\lambda|$ may be approximated arbitrarily closely by leaves of $|\pi_n|$. For, let α be a local transversal to $|\pi_\lambda|$ and think of π_λ as a measure on the space X of all unit tangent vectors based on α . Let $x \in X$ represent the generic leaf l , so that $x \in |\pi_\lambda|$. Let U be a neighborhood of $x \in X$. Then for large n , π_n must assign positive mass to U and hence $|\pi_n|$ contains leaves l_n close to l .

Since by assumption $|\pi_n|$ is a simple closed curve, the leaf l_n has a lift \tilde{l}_n to \mathbf{H}^3 that is invariant under some conjugate g_n of W_{p_n/q_n} . The endpoints \tilde{l}_n^\pm of \tilde{l}_n in Λ are the fixed points g_n^\pm of g_n . Since l is in the pleating locus of \hat{S} , any lift \tilde{l} to \mathbf{H}^3 is geodesic and has endpoints \tilde{l}^\pm in Λ . Also, since l_n converges to l in \hat{S} , we can choose lifts so that \tilde{l}_n converges to \tilde{l} in \mathbf{H}^3 and hence so that g_n^\pm converge to l^\pm . This says that the geodesics $Ax(g_n)$ also converge to \tilde{l} in \mathbf{H}^3 . Therefore, $Ax(g_n)$ is close to \tilde{l}_n and hence to $\partial\mathcal{C}_0$.

Orthogonal projection of $Ax(g_n)$ onto $\partial\mathcal{C}_0$ produces a curve with the same endpoints as \tilde{l}_n and by the above, close to it in \mathbf{H}^3 . Therefore, it follows from the definition of the length function ℓ that $\Re L(g_n)(\mu_0)$ and $\ell_{\mu_0}(\delta_{l_n})$ are close. Thus, by linearity, $\Re L(\pi_n) \rightarrow \ell_{\mu_0}(\pi_\lambda)$. \square

We are finally able to extend the normalized length functions as we require. Let $\mathcal{O}(\mathcal{M})$ denote the space of analytic functions from \mathcal{M} to \mathbf{C} with the topology of uniform convergence on compact subsets.

Theorem 6.9 (Normalized complex length) *The family of functions $L_{p/q}: \mathcal{M} \rightarrow \mathbf{C}$ extends to a family of complex analytic functions $L_\lambda: \mathcal{M} \rightarrow \mathbf{C}$, $\lambda \in \mathbf{R}$ in such a way that $L_\lambda(\mu) = PL(\mu)$ for $\mu \in \mathcal{P}_\lambda$, and the map $\mathbf{R} \rightarrow \mathcal{O}(\mathcal{M})$ given by $\lambda \mapsto L_\lambda$ is continuous.*

Proof: We need to show that for any sequence p_n/q_n , $p_n/q_n \rightarrow \lambda$, the sequence of functions $f_n = L_{p_n/q_n}(\mu)$ converges uniformly on compact subsets of \mathcal{M} to a limit function which is independent of the sequence p_n/q_n and has the asserted value on \mathcal{P}_λ .

We saw in section 6.1 that the family $\{L_{p/q}(\mu)\}$ is normal.

Pick a convergent subsequence of the sequence (f_n) , and by abuse of notation write $f_n \rightarrow f \in \mathcal{O}(\mathcal{M})$. Let $\mu_0 \in \mathcal{P}_\lambda$. By lemma 6.8, $\Re f_n(\mu_0) \rightarrow PL(\mu_0)$. Thus $\Re f|_{\mathcal{P}_\lambda} = PL|_{\mathcal{P}_\lambda}$. It follows from the method of lemma 5.3

that \mathcal{P}_λ is uncountable. Hence we see that the value of $\Re f$ on \mathcal{P}_λ completely determines f and therefore f is independent of the sequence p_n/q_n .

Finally we need to establish that $f|_{\mathcal{P}_\lambda}$ is real-valued. Pick $\mu_\infty \in \mathcal{P}_\lambda$. By lemma 6.3 we can find $\mu_n \in \mathcal{P}_{p_n/q_n}$ with $\mu_n \rightarrow \mu_\infty$. Since (f_n) converges uniformly to f on a neighborhood of μ_∞ and since $f_n(\mu_n) \in \mathbf{R}$, we have that $f_n(\mu_n) \rightarrow f(\mu_\infty)$ and so $f(\mu_\infty) \in \mathbf{R}$ as claimed. \square

We call the function L_λ the *complex pleating length* of the lamination λ .

7 Pleating coordinates

In this section we complete the proof of our main result.

Theorem 7.1 *The map*

$$\Pi: \mathcal{M} \rightarrow \mathbf{R} \times \mathbf{R}^+$$

defined by $\Pi(\mu) = (\text{pl}(\mu), PL(\mu))$ is a homeomorphism onto its image.

There is one remaining point that we need to establish to complete the proof of this theorem.

Theorem 7.2 *The real pleating ray \mathcal{P}_λ is a connected component of the real locus of the complex pleating length L_λ in \mathcal{M} . This component contains no singularities and is asymptotic to $\Re\mu = 2\lambda$ as $\Im\mu \rightarrow \infty$.*

The connectivity and non-singularity can be proved without much difficulty using techniques that have already been introduced. That $PL(\mu) \rightarrow \infty$ as $\Im\mu \rightarrow \infty$ in \mathcal{P}_λ follows directly by looking at the trace polynomials.

7.1 Connectivity of \mathcal{P}_λ

We shall prove theorem 7.2 by a method analogous to the construction of the real numbers by Dedekind cuts.

Lemma 7.3 *$\mathcal{P}_{p/q}$ separates \mathcal{M} into two connected pieces.*

Proof: Let

$$\mathbf{A}_{p/q} = \{\mu \in \mathcal{M}: \text{pl}(\mu) < p/q\}$$

and

$$\mathbf{B}_{p/q} = \{\mu \in \mathcal{M}: \text{pl}(\mu) > p/q\}.$$

These sets are obviously both open and closed in $\mathcal{M} \setminus \mathcal{P}_{p/q}$. We claim that both are connected. Suppose $\mu_1, \mu_2 \in \mathbf{A}_{p/q}$. Since $\mathbf{A}_{p/q}$ is open, by corollary 6.2, we can find points μ'_1, μ'_2 in $\mathbf{A}_{p/q}$ near μ_1, μ_2 respectively, on some pair of rational rays $\mathcal{V}_{m/n}, \mathcal{V}_{r/s}$. Now the rays $\mathcal{V}_{m/n}, \mathcal{V}_{r/s}$ can be connected by a path α in $\mathbf{A}_{p/q}$ in the region $\Im\mu > 2$; thus the points μ'_1, μ'_2 are connected by the path $\mathcal{V}_{m/n} \cup \alpha \cup \mathcal{V}_{r/s}$. \square

Proof of theorem 7.2

We keep the notation of the lemma above. We claim that if $\lambda > p/q$, then $\mathcal{P}_\lambda \subset \mathbf{B}_{p/q}$. If $\lambda = r/s \in \mathbf{Q}$ then clearly by lemma 3.2, $\mathcal{P}_{r/s} \subset \mathbf{B}_{p/q}$ for sufficiently large values of $\Im\mu$. Since $\mathcal{P}_{r/s} \cap \mathcal{P}_{p/q} = \emptyset$ the claim is proved. Next suppose that $\lambda \in \mathbf{R}$, $\lambda > p/q$ with $\mathcal{P}_\lambda \cap \mathbf{A}_{p/q} \neq \emptyset$. Pick $x \in \mathcal{P}_\lambda \cap \mathbf{A}_{p/q}$ and let U be a neighborhood of x in $\mathbf{A}_{p/q}$. As in the proof of corollary 6.3 there is there is some $r/s > p/q$ with $\mathcal{P}_{r/s} \cap U \neq \emptyset$. But then $\mathcal{P}_{r/s} \cap \mathbf{A}_{p/q} \neq \emptyset$ which we ruled out above.

Now for $\lambda \in \mathbf{R}$, let

$$\mathbf{A}_\lambda = \cup_{p/q < \lambda} \mathbf{A}_{p/q}$$

and

$$\mathbf{B}_\lambda = \cup_{p/q > \lambda} \mathbf{B}_{p/q}.$$

We claim that $\mathbf{A}_\lambda \cap \mathbf{B}_\lambda = \emptyset$ and

$$\mathcal{P}_\lambda = \mathcal{M} \setminus (\mathbf{A}_\lambda \cup \mathbf{B}_\lambda).$$

The first part is clear from the definitions. It is also clear that

$$\mathcal{P}_\lambda \cap \mathbf{A}_\lambda = \mathcal{P}_\lambda \cap \mathbf{B}_\lambda = \emptyset,$$

while $\mathcal{P}_{\lambda'} \subset \mathbf{A}_\lambda$ for $\lambda' < \lambda$ and $\mathcal{P}_{\lambda'} \subset \mathbf{B}_\lambda$ for $\lambda' > \lambda$. Since every point of \mathcal{M} lies in \mathcal{P}_ξ for some $\xi \in \mathbf{R}$, the claim follows.

Thus \mathcal{P}_λ separates \mathcal{M} into exactly two connected components \mathbf{A}_λ and \mathbf{B}_λ . Since \mathcal{M} is simply connected, and since no branch of \mathcal{P}_λ can end in \mathcal{M} , \mathcal{P}_λ is connected [13]. Now \mathcal{P}_λ is part of the real locus of an analytic function and if it contained a singularity at least three branches would meet there. However, again because no branch of \mathcal{P}_λ can end in \mathcal{M} , these branches would separate \mathcal{M} into more than two connected components which is a contradiction. The statement about the asymptotic behavior of \mathcal{P}_λ should now be clear. \square

7.2 Proof of theorem 7.1

We can now complete the proof that we have defined coordinates.

Injectivity of Π is an immediate consequence of the non-singularity of the \mathcal{P}_λ as in theorem 7.3. Continuity of Π has already been established. It is sufficient therefore to prove that Π is open.

Let $U \subset \mathcal{M}$ be an open disk, choose $(\lambda, c) \in \Pi(U)$ and write $\mu(\lambda; c)$ for $\Pi^{-1}(\lambda, c)$. Since U is open we can find an $\epsilon > 0$ such that $(\lambda, c \pm \epsilon/2) \in \Pi(U)$. Set $c_1 = c + \epsilon/2$ and $c_2 = c - \epsilon/2$. Draw arcs $\sigma_i \subset U$ transversal to \mathcal{P}_λ at the points $\mu(\lambda; c_i)$, $i = 1, 2$.

Using the continuity of $PL \circ \sigma_i$ we may choose both σ_1 and σ_2 to have endpoints on the rays \mathcal{P}_{λ_1} and \mathcal{P}_{λ_2} respectively, with $\lambda_1 < \lambda < \lambda_2$ and short enough that $|PL(\mu) - c_1| < \epsilon/8$ on σ_1 and $|PL(\mu) - c_2| < \epsilon/8$ on σ_2 .

Now consider the region $W \subset \mathbf{C}$ bounded by the lines \mathcal{P}_{λ_i} and σ_j , $i, j = 1, 2$. Since by construction $\partial W \subset U$ and since U is simply connected, we must have $W \subset U$. We claim that $\Pi(W)$ covers the compact neighborhood $[\lambda_1, \lambda_2] \times [c - \epsilon/4, c + \epsilon/4]$ of $(\lambda; c)$. This will complete the proof.

Suppose that $t \in [\lambda_1, \lambda_2]$. By the continuity of $pl \circ \sigma_i$, we see that $\mathcal{P}_t \cap \sigma_i \neq \emptyset$, $i = 1, 2$. Now $PL(\mu) > c + 3\epsilon/8$ for $\mu \in \sigma_1$ and $PL(\mu) < c - 3\epsilon/8$ for $\mu \in \sigma_2$. Using the fact that $\mathcal{P}_t \cap \mathcal{P}_{\lambda_i} = \emptyset$ for $\lambda_1 < t < \lambda_2$ and the monotonicity of PL on \mathcal{P}_t we see that $\Pi^{-1}(t, \xi) \in W$ for $\xi \in [c - \epsilon/4, c + \epsilon/4]$ and we are done. \square

A Appendix

A.1 Right circular cylinders

Proposition A.1 *Let $F = \langle K, W \rangle \subset PSL(2, \mathbb{C})$ where*

1. K is parabolic; that is $\text{Tr } K = -2$
2. W is hyperbolic; $\text{Tr } W = 2t > 2$
3. WK is hyperbolic and $\text{Tr } WK = \text{Tr } W = 2t$

Then F is Fuchsian with an invariant disk Δ . The open Riemann surface $\Sigma = \Delta/F$ is a right circular cylinder; that is, it is a sphere with two holes and one puncture which admits two reflections j_1, j_2 . These have a common fixed point on Σ that we call the elliptic point. They lift to reflections of Δ which commute with F .

Proof: It is proved in [1] that for a two generator group, if the traces of the generators and the product of the generators are all real, the group is Fuchsian. We may assume without loss of generality that the fixed point of K is -1 and that the fixed points of W are at $e^{i\theta}$ and $e^{-i\theta}$. The disk Δ is then the unit circle.

With this normalization the matrices K and W are of the form

$$K = \begin{pmatrix} -1 + 2ti & -2t \\ -2t & -1 - 2ti \end{pmatrix}$$

and

$$W = \begin{pmatrix} t - i & ti \\ -ti & t + i \end{pmatrix}$$

We also have

$$WK = \begin{pmatrix} t + i & ti \\ -ti & t - i \end{pmatrix}$$

Let $\alpha = \arcsin \frac{t^2-1}{t^2+1}$. Then $W(e^{-i\alpha}) = e^{i\alpha}$, and $WK(-e^{i\alpha}) = -e^{-i\alpha}$. Draw the figure in Δ bounded by hyperbolic geodesics joining $-i$ to $e^{-i\alpha}$, $e^{i\alpha}$ to i , i to $-e^{-i\alpha}$ and $-e^{i\alpha}$ to $-i$. (See figure 4.)

This geodesic polygon is clearly a fundamental polygon for the group F ; the sides on the left are identified by WK and those on the right are identified

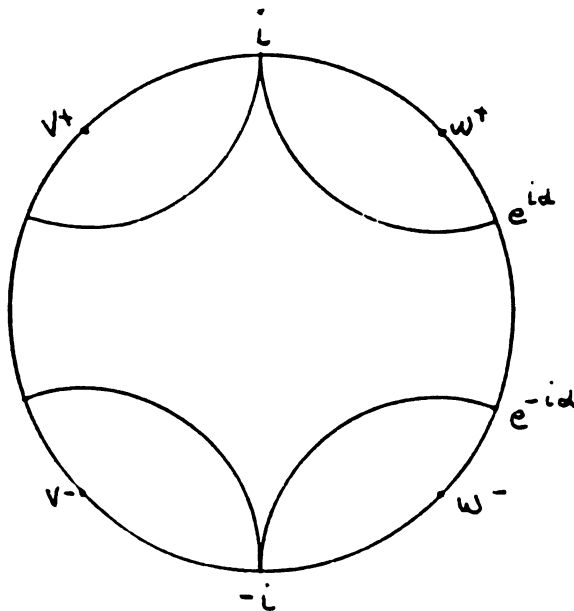


Figure 4: The right circular cylinder

by W . Reflection in the real and imaginary axes maps the polygon to itself so it projects to the required reflections. The elliptic point is the projection of the origin. \square

Remark: For the above normalization, the real axis is the common perpendicular to the axes of W and WK . The cylinder has two infinite hyperbolic ends extending out beyond the projections of these axes. If we truncate these ends symmetrically, we can glue them together. We can measure a "twist" in the gluing by the distance apart of the endpoints of the projection of the real axis.

For a point on a real ray $\mathcal{R}_{p/q}$, the group G contains many Fuchsian subgroups. As in section 4 let F_0 be the one fixing the circle δ_0 . Then the hypotheses of the proposition are satisfied and the quotient δ_0/F_0 is a right circular cylinder. The element X_{-1} gives a gluing and we obtain the punctured torus with a twist.

A.2 Combinatorics of circle chains

We prove lemma 4.2 from section 4.1. Without loss of generality we restrict our attention to the interval $0 < p/q < 1$.

We begin by being more precise about the construction of the special words $W_{p/q}$ in section 3.1. Associated to any $p/q \in (0, 1)$ we have a sequence

$$0/1 = a_0/b_0 \leq a_1/b_1 \leq \dots \leq a_k/b_k < c_k/d_k \leq \dots \leq c_0/d_0 = 1/1;$$

such that for $0 \leq i \leq k$:

1. $(a_i/b_i, c_i/d_i)$ are Farey neighbors,

2. Either

- (a) $a_i/b_i = a_{i+1}/b_{i+1}$, $c_{i+1}/d_{i+1} < c_i/d_i$ and $c_{i+1}/d_{i+1} = (a_i + c_i)/(b_i + d_i)$ or
- (b) $a_i/b_i < a_{i+1}/b_{i+1}$, $c_{i+1}/d_{i+1} = c_i/d_i$ and $a_{i+1}/b_{i+1} = (a_i + c_i)/(b_i + d_i)$,

and p/q equals either a_k/b_k or c_k/d_k .

We set $W(0/1) = T$, $W(1/1) = S^{-1}T$ and define $W(c_i/d_i)W(a_i/b_i)$ to be $W(c_{i+1}/d_{i+1})$ in case (a) and $W(a_{i+1}/b_{i+1})$ in case (b).

Now let $(\delta_i)_{i \in \mathbf{Z}}$ be a p/q -combinatorial circle chain as defined in section 4.1. With the notation of section 4.1, we have to show that $E_1 = W(a_{k-1}/b_{k-1})$, $E_{-1} = W(c_{k-1}/d_{k-1})$ and $E_q = W(p/q)$.

To do this we give an alternative construction of the words $W(a_i/b_i)$, $W(c_i/d_i)$ based on the pattern of "closest returns to 1" for the orbit of 1 on S^1 under rotation by $e^{2\pi i p/q}$.

We shall identify $(\delta_i)_{i \in \mathbf{Z}}$ with \mathbf{Z} so that, for example, $S^{-1}(\delta_0) = \delta_{-q} = -q$. We define a map $\tau: \mathbf{Z} \rightarrow \mathbf{Z}$ by:

$$\begin{aligned} \tau(\delta_i) &= T(\delta_i) = \delta_{i+p} \text{ if } i < p \\ \tau(\delta_i) &= S^{-1}T(\delta_i) = \delta_{i+p-q} \text{ if } i \geq p. \end{aligned}$$

We inductively define sequences of integers $L^i < R^i$ and $n_i > 0$ as follows:

$$L^0 = -q, \quad R^0 = p, \quad n_0 = 1.$$

Given $L^i < R^i$, then n_{i+1} is the least positive integer r with

$$L^i < \tau^r(0) < R^i.$$

If $\tau^{n_{i+1}}(0) < 0$ we set $L^{i+1} = \tau^{n_{i+1}}(0)$ and $R^{i+1} = R^i$ and if $\tau^{n_{i+1}}(0) > 0$ we set $L^{i+1} = L^i$ and $R^{i+1} = \tau^{n_{i+1}}(0)$. The sequence terminates at the l th stage when $\tau^r(0) = 0$ and in this case we arbitrarily choose either to set $0 = L^l$ or $0 = R^l$. Note that since p and q are relatively prime we necessarily have that $L^{l-1} = -1$ and $R^{l-1} = 1$.

It is also clear that for $0 \leq r < n_i$, $\tau^r(0) = \tau^s(0)$ if and only if $r = s$. Let Γ be the free semigroup generated by S^{-1} and T . To each circle δ_r we can associate the unique word in Γ determined by the sequence of elements of Γ

that form the product τ^r . Denote by $W(L^i)$ and $W(R^i)$ the words associated to L^i and R^i in this way; in particular, this gives $L^0 = S^{-1}$ and $R^0 = T$.

We shall prove that $l = k$ and that $W(L^i) = W(c_i/d_i)$ and $W(R^i) = W(a_i/b_i)$. From this we can conclude that $E_{-1} = W(L^{l-1}) = W(c_{l-1}/d_{l-1})$, $E_1 = W(R^{l-1}) = W(a_{l-1}/b_{l-1})$ and $E_q = W_{p/q}$ as required.

To help keep track of the induction we define one further sequence. Let $\sigma: \mathbf{Q}^+ \rightarrow \mathbf{Q}$ be the map that “subtracts 1 from the first entry of the continued fraction expansion of ξ ”; more precisely, let

$$\begin{aligned}\sigma(\xi) &= \xi - 1 & \text{if } \xi > 1 \\ \sigma(\xi) &= 1/\xi - 1 & \text{if } 1/2 \leq \xi \leq 1 \\ \sigma(\xi) &= \frac{1}{1/\xi - 1} & \text{if } 0 < \xi \leq 1/2.\end{aligned}$$

Let $p_0/q_0 = p/q$ and let $p_i/q_i = \sigma^i(p/q)$. This sequence terminates at $i = m$ with $p_{m-1}/q_{m-1} = 1$, $p_m/q_m = 0$.

We are now in a position to state our inductive claims.

Inductive Claims

For $0 \leq i \leq \min(k, l, m)$:

- (a)_i If $L^i = L^{i+1}$ then $|L^i| = |R^i - R^{i+1}|$
and $p_i/q_i = |L^i - R^{i+1}|/|R^{i+1} - R^i|$.
- If $R^i = R^{i+1}$ then $|R^i| = |L^i - L^{i+1}|$
and $p_i/q_i = |L^i - L^{i+1}|/|L^{i+1} - R^i|$.
- (b)_i $W(L^i) = W(a_i/b_i)$ and $W(R^i) = W(c_i/d_i)$.

Proof of claims

Both (a)₀ and (b)₀ are easily verified. In the step from i to $i+1$ there are various cases. We shall assume $L^{i-1} = L^i$, $R^i < R^{i-1}$ and $|L^{i-1} - R^i|/|R^i - R^{i-1}| > 1$. All the other cases are similar and are left to the untiring reader.

Let $t = R^{i-1} - R^i$. The crucial point to verify is that $\tau^{n_{i+1}}(0) = R^i - t$. We claim first that R^i is the first point $\tau^s(R^{i-1})$, $s \geq 0$, in the forward orbit of R^{i-1} to fall in the interval (L^{i-1}, R^{i-1}) .

Now by construction either $L^{i-1} = \tau^{n_{i-1}}(0)$ or $R^{i-1} = \tau^{n_{i-1}}(0)$. In the latter case, to reach R^i we have to iterate τ on R^{i-1} until we next land in the interval between L^{i-1} and R^{i-1} ; and the claim is clear.

If $L^{i-1} = \tau^{n_{i-1}}(0)$, then the point R^i is the next orbit point to land between L^{i-1} and R^{i-1} ; in other words, R^i is a forward image of L^{i-1} . But

in this case, R^{i-1} must have preceded L^{i-1} in the τ -orbit and so again, R^i has the required form.

Now notice that from the definition of τ :

- (a) $\tau(j+k) \equiv \tau(j) + k \pmod{q}$ for $j, k \in \mathbf{Z}$ and
- (b) if $i \in [p-q, p)$ then $\tau(i) \in [p-q, p)$.

Thus for any $m > 0$, $\tau^m(R^{i-1}) \equiv \tau^m(R^i) + t \pmod{q}$ and since $\tau^m(R^{i-1}), \tau^m(R^i) \in [p-q, p)$, we find

$$\tau^m(R^{i-1}) = \tau^m(R^i) + t, \quad \forall m \in \mathbf{N}. \quad (1)$$

In particular, since $R^i - L^i > R^{i-1} - R^i$, we have

$$\tau^s(R^i) = R^i - t \in (L^i, R^i).$$

Suppose that $\tau^k(R^i) \in (L^i, R^i)$ for some $k < s$. Then by equation 1, we find $\tau^k(R^{i-1}) = \tau^k(R^i) + t$ and hence $\tau^k(R^{i-1}) \in (L^i, R^i)$. This contradicts the fact that $\tau^s(R^{i-1})$ is the first point in the forward orbit of R^{i-1} to fall in (L^{i-1}, R^{i-1}) . From this we deduce that $\tau^s(R^i)$ is the first point in the forward orbit of R^i to land in (L^i, R^i) , in other words, that $\tau^{n_{i+1}}(0) = \tau^s(R^i) = R^i - t$ as required.

It is now easy to check claim (a)_{i+1}.

To check (b)_{i+1} note that by the induction hypothesis (b)_i:

$$W(c_{i-1}/d_{i-1})(0) = W(L^{i-1}(0)) = L^{i-1} = L^i = W(L^i)(0) = W(c_i/d_i)(0)$$

and hence $c_{i-1}/d_{i-1} = c_i/d_i$. Thus $W(a_i/b_i) = W(c_i/d_i)W(a_{i-1}/b_{i-1})$ and

$$R^i = W(a_i/b_i)(0) = W(c_i/d_i)W(a_{i-1}/b_{i-1})(0) = W(c_i/d_i)(R^{i-1}).$$

We established above that $\tau^s(R^{i-1}) = R^i$ and $\tau^s(R^i) = R^{i+1}$ (or L^{i+1} if $\tau^s(R^i) < 0$). Thus we can write $g(R^i) = R^{i+1}$ and $g'(R^{i-1}) = R^i$ for some $g, g' \in \Gamma$. We claim that in fact $g = g'$.

By the definition of τ this can only fail if for some k , $0 < k \leq s$, we have $\tau^k(R^i) < 0 < \tau^k(R^{i-1})$. By equation 1, $\tau^k(R^{i-1}) - \tau^k(R^i) = t$ so $\tau^k(R^{i-1}) < t$. But $\tau^k(R^{i-1}) > R^{i-1} > t$; thus $g = g'$.

We conclude that,

$$R^{i+1} = W(c_i/d_i)R^i = W(c_i/d_i)W(a_i/b_i)(0)$$

and so $W(R^{i+1}) = W(c_i/d_i)W(a_i/b_i)$ as required.

It remains only to establish that $k = l = m$. This follows from the observations

$$i < k \Leftrightarrow a_i/b_i < p/q < c_i/d_i \Leftrightarrow W(a_i/b_i) \neq W(p/q) \neq W(c_i/d_i) \Leftrightarrow L^i \neq 0 \neq R^i \Leftrightarrow i < l$$

and

$$L^i \neq 0 \neq R^i \Leftrightarrow p_i/q_i \neq 0 \Leftrightarrow i < m.$$

□

A.3 No two real traces

In this section we show that the loci corresponding to two nearest neighbors cannot intersect.

Lemma A.2 (Two real traces imply quasi-fuchsian) *Let*

$$F = \langle A, B \rangle \subset \text{PSL}(2, \mathbf{C})$$

be a free group such that $\text{Tr}[A, B] = -2$. Then if $\text{Tr} A$ and $\text{Tr} B$ are both real, F is a quasi-Fuchsian group.

Proof: Since F is a free group, the elements A and B must both be hyperbolic; that is $|\text{Tr} A| > 2, |\text{Tr} B| > 2$. If we consider F as a group of isometries of \mathbf{H}^3 , each generator leaves invariant the geodesic in \mathbf{H}^3 that joins its fixed points; this geodesic is called its axis. There is a unique common perpendicular P to these axes. (If the axes intersect take P as the perpendicular to the plane spanned by the axes through the intersection point.)

Assume the group is normalized so that the endpoints of P are at 0 and ∞ and so that the attracting fixed point of A is at +1. Let $2x = \text{Tr} A$, and $2y = \text{Tr} B$. The normalization implies that A is of the form:

$$\begin{pmatrix} x & \sqrt{x^2 - 1} \\ \sqrt{x^2 - 1} & x \end{pmatrix}$$

and B is of the form:

$$\begin{pmatrix} y & (\sqrt{y^2 - 1})re^{i\theta} \\ (\sqrt{y^2 - 1})/re^{i\theta} & y \end{pmatrix}$$

Set $\text{Tr } AB = 2z$. Then it is easy to compute that

$$\text{Tr } AB = 2z = 2xy + \sqrt{(x^2 - 1)(y^2 - 1)}(re^{i\theta} + 1/re^{i\theta}).$$

Using the trace identities we have the following equation for the trace of the commutator.

$$\text{Tr}[A, B] = 4x^2 + 4y^2 + 4z^2 - 8xyz - 2 = -2$$

Solving this equation for $2z$ and setting the solution equal to the right side of the above we have,

$$\sqrt{(x^2 - 1)(y^2 - 1)}(re^{i\theta} + 1/re^{i\theta}) = \pm \sqrt{(x^2 - 1)(y^2 - 1) - 1}$$

The right hand side is either real or pure imaginary (depending on whether $(x^2 - 1)(y^2 - 1) - 1$ is positive or negative). The left hand side is real iff $\theta = 0$ and is pure imaginary iff $\theta = \pi/2$.

If $\theta = 0$, F is actually Fuchsian. Either its limit set is the full unit circle and its two orbit spaces are punctured tori or its limit set is a nowhere dense set in \mathbf{R} and the regular set has only one component. In this case the orbit space is a surface of genus 2. If $\theta = \pi/2$, the isometric circles of A, A^{-1}, B, B^{-1} are mutually tangent at fixed points of the commutators.

The common exterior is clearly a fundamental domain for the group; both components of the fundamental domain are punctured tori under the boundary identification and so the group is quasi-Fuchsian. \square

A.4 Estimating the coefficients of the trace polynomials

We are indebted to Miller Maley for showing us how to prove this estimate.

Lemma A.3 *If $A^k = (a_{ij}^k)$ are 2×2 matrices such that $|a_{ij}^k| \leq 2$, then*

$$C = (c_{ij}) = \prod_{k=1}^n A^k$$

satisfies $|c_{ij}| \leq 2^{2n-1}$.

Proof: We proceed by induction on n . The lemma is obviously true for $n = 1$. Assume it is true for all $n \leq N$. Write

$$\prod_{k=1}^{N+1} A^k = (d_{ij}).$$

Then

$$d_{ij} = c_{i1}a_{1j}^{N+1} + c_{i2}a_{2j}^{N+1}$$

and hence

$$|d_{ij}| \leq 2(|c_{i1}| + |c_{i2}|) \leq 4 \cdot 2^{2N-1} = 2^{2N+1}.$$

□

Lemma A.4 Given $0 \leq p/q \leq 1, p/q \in \mathbf{Q}$, define

$$P(\mu) = \text{Tr } W_{p/q}(\mu) = b_0\mu^q + b_1\mu^{q-1} + \dots + b_q.$$

Then $\max_{i=0,\dots,q} |b_i| \leq 8^q$.

Proof: On the circle $|\mu| = 1$, the entries a_{ij} of the matrices S and T satisfy $|a_{ij}| \leq 2$. Therefore $W_{p/q} = (d_{ij})$ is the product of $n = p + q$ matrices all of whose entries satisfy $|a_{ij}| \leq 2$. By the lemma above, $|d_{ij}(\mu)| \leq 2^{2n-1}$ so

$$|P(\mu)| = |d_{11}(\mu) + d_{22}(\mu)| \leq 4^{p+q} \leq 8^q.$$

Now think of $P(e^{i\theta})$ as a Fourier series on the circle $|\mu| = 1$. Recall the Parseval Identity

$$1/2\pi \int_{|\mu|=1} |P|^2 = \sum_{i=0}^q |b_k|^2.$$

Since we have just shown that $|P(e^{i\theta})| \leq 8^q$, we conclude $\sum_{i=0}^q |b_k|^2 \leq 8^{2q}$. □

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