Dynamics of Certain Smooth One-dimensional Mappings

II. Geometrically finite one-dimensional mappings

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Abstract

We study geometrically finite one-dimensional mappings. These are a subspace of $C^{1+\alpha}$ one-dimensional mappings with finitely many, critically finite critical points. We study some geometric properties of a mapping in this subspace. We prove that this subspace is closed under quasisymmetrical conjugacy. We also prove that if two mappings in this subspace are topologically conjugate, they are then quasisym-

metrically conjugate. We show some examples of geometrically finite one-dimensional mappings.

Contents

§1 Introduction

Quasisymmetrical conjugacy. Two smooth mappings f and g from a one-dimensional manifold M to itself are topologically conjugate if there is a homeomorphism h from M to itself such that $f \circ h = h \circ g$. The homeomorphism h and its inverse are usually not both Lipschitz; if they are, then all the eigenvalues of f and g at the periodic points have to be the same. Between the class of homeomorphisms and the class of Lipschitz homeomorphisms, there is a class of <u>quasisymmetric homeomorphisms</u>. A quasisymmetric homeomorphism distorts symmetrically placed triples by a bounded amount. A celebrated Ahlfors-Beurling extension theorem [A] tells us that any quasisymmetric homeomorphism of the real line can be extended to a quasiconformal homeomorphism of the complex plane. Thus quasisymmetric property of the conjugating homeomorphism gives us a chance to use some methods and theorems in one complex variable functions to study the dynamics of some smooth one-dimensional mappings. M. Jakobson recently considered a C^3-folding mapping with negative Schwarzian derivative and one non-recurrent critical point. He proved that if two such mappings are topologically conjugate, they are then quasisymmetrically conjugate [Ja]. D. Sullivan [S1], M. Herman [H], J. Yoccoz [Y] and G. Swiatek [SW], etc., have some interesting results on this direction for some folding mappings and critical circle mappings.

What we would like to say in this paper. We consider a subspace of piecewise $C^{1+\alpha}$-mappings with finitely many, critically finite critical points from a compact smooth one-dimensional manifold into itself and study some geometric properties of a mapping in this subspace.

Suppose M is an oriented connected compact one-dimensional C^2-Riemannian manifold with Riemannian metric dx^2 and associated length element dx. Suppose $f : M \mapsto M$ is a C^1-mapping. Furthermore, without loss generality, we will assume that f maps the boundary of M (if it is not empty) into itself and the one-sided derivatives of f at all boundary points of M are not zero.

We say $c \in M$ is a critical point of f if the derivative of f at this point is zero. We say a critical point of f is <u>critically finite</u> if its orbit consists of finitely many points.

Suppose $f : M \mapsto M$ is a C^1-mapping with finitely many, critically finite critical points. There is a natural Markov partition of M by f. This Markov partition consists of the intervals of the complement of the critical orbits of f. We call it the first partition η_1 of M by f. For any positive integer n, the n^{th}-partition η_n of M by f consists of all the intervals I' such that the restriction of the $(n-1)^{th}$-iterate of f is a homeomorphism from it to an interval in the first partition of M by f. We use λ_n to denote the maximum of the lengths of the intervals in the n^{th}-partition of M by f. We say the n^{th}-partition of M by f goes to zero exponentially with n if there are constants $K > 0$ and $0 < \mu < 1$ such that $\lambda_n \leq K\mu^n$ for every n.

A geometrically finite one-dimensional mapping is a $C^{1+\alpha}$-mapping $f : M \mapsto M$ for some $0 < \alpha \leq 1$ with finitely many, critically finite, non-periodic power law critical points such that the n^{th}-partition of M by f goes to zero exponentially with n. The reader may see §2 for a definition of a power law critical point of f. We also note that the definition of $C^{1+\alpha}$ for a mapping with power law critical points is given in §2 and is little different from the usual one.

To study a geometrically finite one-dimensional mapping, we introduce two concepts, bounded geometry and bounded nearby geometry, for a sequence $\eta = \{\eta_n\}_{n=1}^{\infty}$ of nested partitions. We say a sequence $\eta = \{\eta_n\}_{n=1}^{\infty}$ of nested partitions has bounded geometry if there is a positive constant K such that for any $J \subset I$ with $J \in \eta_{n+1}$ and $I \in \eta_n$, the ratio of lengths, $|J|/|I|$, is bounded by K from below. We say this sequence has bounded nearby geometry if there is a positive constant K such that for any J and I in η_n with a common endpoint, the ratio of lengths, $|J|/|I|$, is bounded by K from below. The bounded geometry here is an analogue to the Sullivan's definition of bounded geometry for a Cantor set on the line [S2]. One of the main theorems in this paper is the following (see Theorem A and Lemma 2).

MAIN THEOREM. *Suppose $f : M \mapsto M$ is geometrically finite and $\eta = \{\eta_n\}_{n=1}^{\infty}$ is the induced sequence of nested partitions of M by f. Then the sequence $\{\eta_n\}_{n=1}^{\infty}$ of nested partitions has bounded geometry and bounded nearby geometry.*

The proof of this theorem is an application of the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma in [J2].

Following the methods in [MT], we can classify topologically the geometrically finite one-dimensional mappings by their kneading invariants. Moreover, using these properties, bounded geometry and bounded nearby geometry, we can classify these mappings quasisymmetrically as follows.

A homeomorphism $h : M \mapsto M$ is quasisymmetrical if there is a positive constant K such that for any two points x and y in M and $z = (x+y)/2$,

$$K^{-1} \leq \frac{|h(x) - h(z)|}{|h(z) - h(y)|} \leq K.$$

We say two mappings f and g from M to itself are quasisymmetrically conjugate if they are topologically conjugate and the conjugating homeomorphism is quasisymmetrical.

THEOREM B. *Suppose f and g are geometrically finite and topologically conjugate. They are then quasisymmetrically conjugate.*

Geometrically finite one-dimensional mappings are closed under quasisymmetrical conjugacy in the space of $C^{1+\alpha}$-mappings with only power law critical points as follows.

THEOREM C. *If a $C^{1+\alpha}$-mapping $f : M \mapsto M$ for some $0 < \alpha \leq 1$ with only power law critical points is quasisymmetrically conjugate to a geometrically finite one-dimensional mapping, then it is also a* *geometrically finite one-dimensional mapping.*

One example of a geometrically finite one-dimensional mapping is the following (see Section 2 for details).

EXAMPLE 1. *A C^3-mapping $f : M \mapsto M$ with nonpositive Schwarzian derivative and finitely many, critically finite, nonperiodic power law critical points.*

This kind of mappings was systematically studied by M. Misiurewicz in 1979 [Mi] and many other people [Ja], [BL] and [MS], etc.

Let C^{1+bv} stand for C^1 with bounded variation derivative (the definition of C^{1+bv} for a mapping with power law critical points is given in §3.2). We say a periodic point p of a mapping f is expanding if the absolute value of the eigenvalue (some people call an eigenvalue an multiplier) $(f^{\circ n})'(p)$ is greater than one, where n is the period of p. The main theorem in §3 is the following.

THEOREM D. *Suppose $f : M \mapsto M$ is a $C^{1+\alpha}$-, for some $0 < \alpha \leq 1$, and C^{1+bv}-mapping with finitely many, critically finite, nonperiodic power law critical points and only expanding periodic points and suppose $\eta = \{\eta_n\}_{n=1}^{\infty}$ is the induced sequence of nested partitions of M by f.*

Then η has bounded geometry.

The study of this theorem is inspired by the paper [M] where R. Mañe [M] proved that a C^2-endomorphism $f : M \mapsto M$ with only expanding periodic points is actually expanding in a suitable smooth coordinate on M. Theorem D provides another example of a geometrically finite one-dimensional mapping.

EXAMPLE 2. *A $C^{1+\alpha}$- , for some $0 < \alpha \leq 1$, and C^{1+bv}-mapping $f : M \mapsto M$ with finitely many, critically finite, nonperiodic power law critical points and only expanding periodic points.*

In Theorem D and Example 2, the condition that f is a $C^{1+\alpha}$- , for some $0 < \alpha \leq 1$, and C^{1+bv}-mapping can not be weakened to the condition that f is a $C^{1+\alpha}$-mapping for there is a counterexample in [J1]. The construction of the counterexample in [J1] is like the construction of the Denjoy counterexample in circle diffeomorphisms and this example is not topologically conjugate to any geometrically finite one-dimensional mapping.

The condition that a $C^{1+\alpha}$-mapping for some $0 < \alpha \leq 1$ with only power law critical points is quasisymmetrically conjugate to a geometrically finite one-dimensional mapping in Theorem C can not be weakened to the condition that a $C^{1+\alpha}$-mapping with only power law critical points is topologically conjugate to a geometrically finite one-dimensional mapping too for there is an easy counterexample (see Figure 4 in §3.3). This counterexample has a neutral fixed point (namely the absolute value of the eigenvalue of f at this fixed point is one) and suggests a question as follows.

QUESTION 1. *Suppose* $f : M \mapsto M$ *is a* $C^{1+\alpha}$*-mapping for some* $0 < \alpha \leq 1$ *with only power law critical points and only expanding periodic points and is topologically conjugate to a geometrically finite one-dimensional mapping. Is* f *geometrically finite ?*

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§2 Geometrically Finite One-dimensional Mappings

Suppose M is an oriented connected compact one-dimensional C^2-Riemannian manifold with Riemannian metric dx^2 and associated length element dx. Suppose $f : M \mapsto M$ is a continuous mapping. We say an interior point $c \in M$ is a critical point if

(a) f is not differentiable at c, or

(b) f is differentiable at c and the derivative of f at c is zero.

We always assume that f is C^1 at any non-critical point p, namely f is differentiable in a small neighborhood U_p of p and the derivative f' of f in the neighborhood U_p is continuous. We say a critical point c is a power law critical point if

(c) c is an isolated critical point,

(d) for some $\gamma \geq 1$,

$$\lim_{x \mapsto c+} \frac{f'(x)}{|x-c|^{\gamma-1}} \ \ and \ \ \lim_{x \mapsto c-} \frac{f'(x)}{|x-c|^{\gamma-1}}$$

have nonzero limits A and B.

We call the numbers γ and $\tau = A/B$ the exponent and the asymmetry of f at c (see [J2]). We say a critical point c of f is critically finite if the orbit $\{c,\, f(c),\, \cdots\}$ is a finite set.

Although the results in this paper hold for a piecewise C^1-mapping

$f : M \mapsto M$ with both smooth and non-smooth critical points, but we are only interested in a smooth critical point of f. Henceforth we will assume that $f : M \mapsto M$ is a C^1-mapping. Furthermore, without loss generality, we will assume that f maps the boundary of M (if it is not empty) into itself and the one-sided derivatives of f at all boundary points of M are not zero. We note that in the general case, a boundary point of M should count as a critical point anyhow.

We define the term $C^{1+\alpha}$ for a real number $0 < \alpha \leq 1$. Suppose $f : M \mapsto M$ has only power law critical points. We use $CP = \{c_1, \cdots, c_d\}$ to denote the set of critical points of f and use $\Gamma = \{\gamma_1, \cdots, \gamma_d\}$ to denote the corresponding exponents of f. Suppose η_0 is the set of the closures of the intervals of the complement of the set of critical points CP of f in M.

Definition 1. *We say the mapping f is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ if*

(*) *the restrictions of f to the intervals in η_0 are C^1 with α-Hölder continuous derivatives and*

(**) *for every critical point c_i of f, there is a small neighborhood*

U_i of c_i in M such that $r_{-,i}(x) = f'(x)/|x-c|^{\gamma_i-1}$ for $x < c$ in U_i and $r_{+,i}(x) = f'(x)/|x-c|^{\gamma_i-1}$ for $x > c$ in U_i are α-Hölder continuous functions.

We define the term exponential decay. Suppose $f : M \mapsto M$ is a C^1-mapping such that the set of critical orbits $\cup_{n=0}^{\infty} f^{\circ n}(CP)$ is finite. Let η_1 be the set $\{I_1, \cdots, I_n\}$ of the closures of the intervals of the complement of the critical orbits $\cup_{n=0}^{\infty} f^{\circ n}(CP)$ in M. We call it the first partition of M by f. It is a Markov partition, namely f maps every interval in it into and onto the union of some intervals in it. Let $\eta_n = f^{-(n-1)}(\eta_1)$ be the set of all the intervals, to each of which the restriction of the $(n-1)^{th}$-iterate of f is a homeomorphism from this interval to an interval in the first partition η_1. We call it the n^{th}-partition of M by f. We use η to denote the sequence $\{\eta_n\}_{n=1}^{\infty}$ of nested partitions and call it the induced sequence of nested partitions of M by f. Let λ_n be the maximum of lengths of the intervals in the n^{th}-partition η_n. We say the n^{th}-partition η_n tends to zero exponentially with n if there are constants $K > 0$ and $0 < \mu < 1$ such that $\lambda_n \leq K\mu^n$ for all the positive integers n.

§2.1 Geometrically finite.

We now give the definition of a geometrically finite one-dimensional mapping as follows.

Definition 2. *We say a mapping $f : M \mapsto M$ with only power law critical points is geometrically finite if it satisfies the following conditions:*

Smooth condition: f is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$.

Finite condition: the set of critical orbits $\cup_{i=0}^{\infty} f^{\circ}(CP)$ is finite.

No cycle condition: no critical point is a periodic point of f.

Exponential decay condition: the n^{th}-partition η_n tends to zero exponentially with n.

§2.2 Bounded Geometry.

We say a set of finitely many closed subintervals of M with pairwise disjoint interiors is a partition of M if the union of these intervals is M. Suppose $\eta = \{\eta_n\}_{n=1}^{\infty}$ is a sequence of partitions of M. We say it is nested if every interval in η_n is the union of some intervals in η_{n+1} for every $n \geq 1$.

Definition 3. *We say a sequence $\eta = \{\eta_n\}_{n=1}^{\infty}$ of nested partitions has bounded geometry if there is a positive constant K such that for* *any pair $J \subset I$ with $J \in \eta_{n+1}$ and $I \in \eta_n$, the ratio $|J|/|I| \geq K$. We call the biggest such constant BC_f the bounded geometry constant.*

§2.3 From geometrically finite to bounded geometry.

One of the main theorems in this paper is the following:

THEOREM A. *Suppose $f : M \mapsto M$ is geometrically finite and $\eta = \{\eta_n\}_{n=0}^{\infty}$ is the induced sequence of nested partitions of M by f. Then η has bounded geometry.*

Before to prove this theorem, let me state the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma in [J2]. For a geometrically finite one-dimensional mapping, this lemma can be written in the following simple form (see §3.3 in [J2]).

LEMMA 1. (The $C^{1+\alpha}$-Denjoy-Koebe distortion lemma) *Suppose $f : M \mapsto M$ is geometrically finite. There are two positive constants A and B and a positive integer n_0 such that for any inverse branch g_n of $f^{\circ n}$ and any pair x and y in the intersection of one of the intervals in η_{n_0} and the domain of g_n, the distortion $|g_n(x)/g_n(y)|$ of g_n at these two points satisfies*

$$\frac{|g_n(x)|}{|g_n(y)|} \leq \exp\Big(A + \frac{B}{D_{xy}}\Big)$$

where D_{xy} is the distance between $\{x, y\}$ and the post-critical orbits $\cup_{i=1}^{\infty} f^{\circ i}(CP)$.

Proof of Theorem A. Suppose n_0, A and B are the constants in Lemma 1. Suppose $\{c_{i_1}, \cdots, c_{i_k}\}$ is a sequence of critical points of f. We say it is a critical chain of f if there is a sequence $\{l_{i_1}, \cdots, l_{i_{k-1}}\}$ of the integers such that $f^{\circ l_1}(c_{i_1}) = c_{i_2}, \cdots, f^{\circ l_{k-1}}(c_{i_{k-1}}) = c_{i_k}$. We call the integer $l = l_{i_1} + \cdots l_{i_{k-1}}$ the length of this chain. By the no cycle condition, there are only finitely many critical chains. Let N_0 be the maximum of lengths of all the critical chains of f.

We say an interval in η_n is a critical interval if one of its endpoints is a critical point. We may assume that for every critical interval in η_{n_0}, one of its endpoints is not in the critical orbits $\cup_{i=0}^{\infty} f^{\circ i}(CP)$. Let $\mathcal{U}$ be the union of all the critical intervals in η_{n_0} and $K_1 > 0$ be the minimum of ratios, $|J|/|I|$, for $J \subset I$ with $J \in \eta_{n_0+1}$ and $I \in \eta_{n_0}$.

For any $J \subset I$ with $J \in \eta_{n+1}$, $I \in \eta_n$ and $n > n_0$, let $J_i = f^{\circ i}(J)$ and $I_i = f^{\circ i}(I)$ for $i = 0, \cdots, n - n_0$. Then $J_{n-n_0} \in \eta_{n_0+1}$ and $I_{n-n_0} \in \eta_{n_0}$. We consider the intervals $\{I_0, \cdots, I_{n-n_0}\}$ in the two cases. One is that no one of them is in $\mathcal{U}$. The other is that at least one of them is in $\mathcal{U}$.

For the first case, by using the naive distortion lemma (see [J1] or [J2]), there is a constant $K_2 > 0$ (which does not depend on any particular intervals $J \subset I$) such that for any x and y in I,

$$\frac{|f^{\circ(n-n_0)}(x)|}{|f^{\circ(n-n_0)}(y)|} \geq K_2,$$

and moreover,

$$\frac{|J|}{|I|} \geq K_3 = K_2 K_1.$$

For the second case, let $l \leq n - n_0$ be the greatest integer such that $I_l \subset \mathcal{U}$. We note that $I_i \cap \mathcal{U} = \emptyset$ for $i = l+1, \cdots, n-n_0$. By using the naive distortion lemma like that in the first case, we can also show that

$$\frac{|J_{l+1}|}{|I_{l+1}|} \geq K_3 = K_2 K_1.$$

Let $\tilde{I}_i$ be the interval in η_{n_0+l-i} containing I_i for $i = 0, \cdots, l$. Then $\tilde{I}_l$ is an interval in η_{n_0} and is contained in $\mathcal{U}$. Suppose $c_q \in CP$ is an endpoint of $\tilde{I}_l$. The restriction of f to $\tilde{I}_l$ is comparable to the mapping $x :\mapsto |x - c_q|^{\gamma_q} + f(c_q)$. We can find a positive constant K_4 (only depends on K_3) such that

$$\frac{|J_l|}{|I_l|} \geq K_4.$$

We may assume that both endpoints of $\tilde{I}_l$ are not in the post-critical orbits $\cup_{n=1}^{\infty} f^{\circ n}(CP)$. Otherwise, by the no cycle condition, there is $k \leq N_0$ such that $\tilde{I}_{l-k}$ has this property, one of its endpoint is a critical point of f and both of its endpoints are not in the post-critical orbits $\cup_{n=1}^{\infty} f^{\circ n}(CP)$. Then we can use $\tilde{I}_{l-k}$ to instead of $\tilde{I}_l$ because there is a constant K_5 (only depends on K_4) such that

$$\frac{|J_{l-k}|}{|I_{l-k}|} \geq K_5 \frac{|J_l|}{|I_l|}.$$

Let K_6 be the minimum of lengths of the critical intervals in η_{n_0}. Now using Lemma 1 (the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma), for any x and y in $I_l \subset \tilde{I}_l$,

$$\frac{|f^{\circ(n-l)}(x)|}{|f^{\circ(n-l)}(y)|} \geq K_7 = -\exp\Big(A + \frac{B}{K_6}\Big),$$

and moreover,

$$\frac{|J|}{|I|} \geq C_8 = K_7 K_4.$$

The bounded geometry constant BC_f is greater that the maximum of K_3 and K_8.

§2.4 Quasisymmetrical classification.

Topologically, we can classify the geometrically finite one-dimensional mappings by their kneading invariants just following the methods in

[MT] (see [MT] for a definition of a kneading sequence). By this we means that for two geometrically finite one-dimensional mappings f and g, there is an orientation-preserving homeomorphism $h : M \mapsto M$ such that $f \circ h = h \circ g$ if and only if the kneading invariants of f and g are the same.

A homeomorphism $h : M \mapsto M$ is quasisymmetrical if there is a positive constant K such that for any two points x and y in M and $z = (x + y)/2$,

$$K^{-1} \leq \frac{|h(x) - h(z)|}{|h(z) - h(y)|} \leq K.$$

We call the smallest such constant QC_h the quasisymmetrical constant of h. We say two mappings f and g from M to itself are quasisymmetrically conjugate if they are topologically conjugate and the conjugating homeomorphism is quasisymmetrical. In this subsection, we study the quasisymmetrical property of a conjugating homeomorphism between two geometrically finite one-dimensional mappings.

§2.4.1 Bounded nearby geometry.

The bounded geometry is a nice geometric property of a hierarchical structure of intervals. But it is still not enough to get the quasisymmetrical property of the conjugating mapping. So we introduce another concept, bounded nearby geometry.

DEFINITION 4. *We say a sequence* $\eta = \{\eta_n\}_{n=1}^{\infty}$ *of nested partitions of* M *has bounded nearby geometry if there is a positive constant* K *such that for any pair* J *and* I *in* η_n *with a common endpoint, the ratio* $|J|/|I| \geq K$. *We call the biggest such constant* NC_f *bounded nearby geometry constant.*

LEMMA 2. *Suppose* f *is geometrically finite and* $\eta = \{\eta_n\}_{n=1}^{\infty}$ *is the induced sequence of nested partitions of* M *by* f. *Then* η *has bounded nearby geometry.*

Proof. We use the same notations as that in the proof of Theorem A. Let $n_2 > n_0$ be a positive integer such that if two intervals I and J in η_{n_2} with a common endpoint, then either both of them are in $\mathcal{U}$ or both of them are not in $\mathcal{U}_1$, where $\mathcal{U}_1$ is the union of the critical intervals in η_{n_2}. Let $K_1 > 0$ be the minimum of ratios, $|J|/|I|$, where J and I are intervals in η_{n_2} with a common endpoint.

For any $n > n_2$ and any two intervals J and I in η_n with a common endpoint, let $J_i = f^{\circ i}(J)$ and $I_i = f^{\circ i}(I)$ for $i = 0, \cdots, n - n_2$. We consider the intervals $\{J_i\}_{n=0}^{n-n_2}$ and $\{I_i\}_{n=0}^{n-n_0}$ in the two cases. One is that for some $0 < l \leq n - n_0$, $J_l = I_l$. The other is that J_i and I_i are different (but they have a common endpoint) for every i.

For the first case, let l be the smallest such integer, then the common endpoint of J_{l+1} and I_{l+1} is an extremal critical point of f (this means that it is either maximal or minimal point of f). It is easy to see now that there is a positive constant K_2 such that

$$\frac{|J_{l+1}|}{|I_{l+1}|} \geq K_2.$$

Now we use the arguments like that of the second case in the proof of Theorem A to verify that there is a positive constant K_3 such that

$$\frac{|J|}{|I|} \geq K_4 = K_3 K_2.$$

For the second case, again use the arguments like that of the second case in the proof of Theorem A to demonstrate that there is a positive constant K_5 such that

$$\frac{|J|}{|I|} \geq K_6 = K_5 K_1.$$

The bounded nearby geometry constant NC_f is greater than the maximum of K_4 and K_6.

§2.4.2 Quasisymmetry.

One of the consequences of these properties, bounded geometry and Bounded nearby geometry is the quasisymmetrical classification of geometrically finite one-dimensional mappings as follows.

Theorem B. *Suppose f and g are geometrically finite and topologically conjugate. They are then quasisymmetrically conjugate.*

Proof. Suppose h is the topological conjugacy between f and g and $h \circ f = g \circ h$. Suppose BC_f, NC_f, BC_g and NC_g are the bounded geometry constants and bounded nearby geometry constants of the induced sequences $\{\eta_{n,f}\}_{n=1}^{\infty}$ and $\{\eta_{n,g}\}_{n=1}^{\infty}$ of nested partitions of M by f and g and $\lambda_{n,f}$ and $\lambda_{n,g}$ are the maximum lengthes of the intervals in $\eta_{n,f}$ and $\eta_{n,g}$, respectively.

For any $x < y$ in M, let z be the midpoint $(x+y)/2$ of them. Suppose $N > 0$ is the smallest integer such that there is an interval I in η_N and is contained in $[x, y]$ (see Figure 1, 2 and 3).

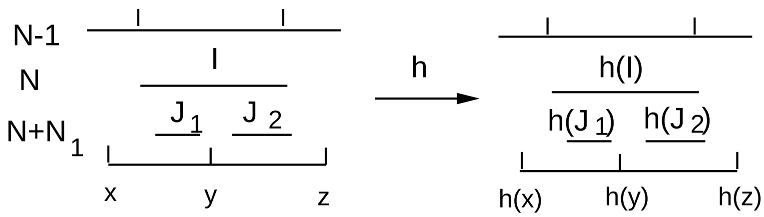

Figure 1

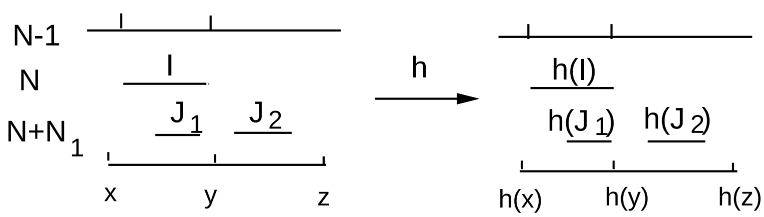

Figure 2

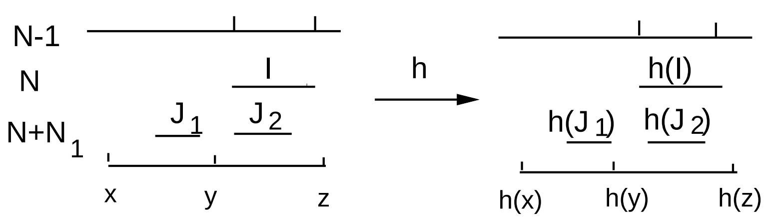

Figure 3

Because f and g are both geometrically finite, the n^{th}-partitions $\eta_{n,f}$ and $\eta_{n,g}$ tend to zero exponentially with n. We can find two constants $L_f = K(BC_f, NC_f) > 0$ and $0 < \mu_f = \mu(BC_f, NC_f) < 1$ such that $\lambda_{n,f} \leq L_f(\mu_f)^n$ for any $n > 0$. Moreover, we can find a positive integer $N_1 = N_1(BC_f, NC_f)$ such that there are intervals J_1 and J_2 in η_{N+N_1} contained in $[x, z]$ and $[z, y]$, respectively. By the bounded geometry and bounded nearby geometry, we can find a constant $K = K(N_1, BC_g, NC_g)$ (see Figure 1, 2 and 3) such that

$$K^{-1} \leq \frac{|h(x) - h(z)|}{|h(z) - h(y)|} \leq K.$$

The quasisymmetric constant QC_h is less than K.

§2.5 Closeness under quasisymmetrical conjugacy.

Another consequence of these properties, bounded geometry and bounded nearby geometry, is that geometrically finite one-dimensional mappings is closed under quasisymmetrical conjugacy in the space of $C^{1+\alpha}$-mappings with only power law critical points as follows.

THEOREM C. *If a $C^{1+\alpha}$-mapping $f : M \mapsto M$ for some $0 < \alpha \leq 1$ with only power law critical points is conjugate to a geometrically finite one-dimensional mapping, then it is also a geometrically finite one-dimensional mapping.*

Proof. The proof of this theorem is the use of the quasisymmetrical property of the conjugating homeomorphism.

§3 Examples Of Geometrically Finite

One-dimensional Mappings

The definition of a geometrically finite one-dimensional mapping is quit abstract. To concrete it, we show some examples. The main theorem in this section is Theorem D.

§3.1 A C^3-mapping with nonpositive Schwarzian derivative.

Suppose $f : M \mapsto M$ is a C^3-mapping. The Schwarzian derivative of f is defined by

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\Big(\frac{f''(x)}{f'(x)}\Big)^2.$$

We say f has nonpositive Schwarzian derivative if $S(f)(x) \leq 0$ for all x in M and has nonnegative Schwarzian derivative if $S(f)(x) \geq 0$ for all x in M. We note that a C^3-diffeomorphism f has nonpositive Schwarzian derivative if and only if the inverse of f has nonnegative Schwarzian derivative. The first example of a geometrically finite one-dimensional mapping is the following:

Example 1. *A C^3-mapping $f : M \mapsto M$ with finitely many, critically finite, nonperiodic power law critical points and nonpositive Schwarzian derivative.*

Suppose I and J are two intervals and g is a C^3-diffeomorphism from I to J. A measure of the nonlinearity of g is the function $n(g) = g''/g'$. If the absolute value of $n(g)$ on I is bounded above by a positive constant C, then the distortion $|g'(x)|/|g'(y)|$ of g at any pair x and y in I is bounded above by $exp(C|x-y|)$. Suppose $d_I(x)$ is the distance from x to the boundary of I.

LEMMA 3 (the C^3-Koebe distortion lemma). *Suppose g has non-negative Schwarzian derivative. Then $|n(g)(x)|$ is bounded above by $2/d_I(x)$ for any x in I.*

Proof. See, for example, [J1] for a proof.

LEMMA 4. *Suppose f is the mapping in Example 1 and $\eta = \{\eta_n\}_{n=1}^{\infty}$ is the induced sequence of nested partitions of M by f. Then η has bounded geometry.*

Proof. The proof is similar to that of Theorem A. Here we use the Lemma 3 (the C^3-Koebe distortion lemma) to replace the role of Lemma 1 (the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma) in the proof of Theorem A.

COROLLARY 1. *Suppose f is the mapping in Example 1 and $\eta = \{\eta_n\}_{n=1}^{\infty}$ is the induced sequence of nested partitions of M by f. Then* *the n^{th}-partition η_n induced by f goes to zero exponentially with n.*

Proof. Suppose l_1 is the number of the intervals in the first partition η_1. Because every critical point of f is not periodic and critically finite and every periodic point of f is expanding, we can find an integer $k > 0$ such that every interval in the first partition η_1 contains at least two but no more than kl_1 intervals in the k^{th}-partition. By using the bounded geometry, we can prove this corollary.

That Example 1 is geometrically finite follows from Lemma 3, 4 and Corollary 1.

§3.2 A C^1-mapping with bounded variation derivative.

We say a function $u : U \mapsto \mathbf{R}^1$ has bounded variation if

$$Var(u) = \sup_{x_1 < \cdots < x_l \in U} \sum_{i=1}^{l-1} |u(x_i) - u(x_{i+1})| < +\infty$$

where U is a subset of M.

We definite the term C^{1+bv}. Suppose $f : M \mapsto M$ is a C^1-mapping with only power law critical points. We use $CP = \{c_1, \cdots, c_d\}$ to denote the set of critical points of f and use $\Gamma = \{\gamma_1, \cdots, \gamma_d\}$ to denote the the corresponding exponents. Suppose η_0 is the set of intervals in the complement of the set CP of critical points of f.

Definition 5. *We say the mapping f is a C^{1+bv}-mapping if*

(i) the restrictions of f to the intervals in η_0 are C^1 with bounded variation derivatives and

(ii) for every critical point c_i of f, there is a small neighborhood U_i of c_i such that the functions $r_{-,i}(x) = f'(x)/|x-c_i|^{\gamma_i - 1}$, $x < c_i$ and $x_i \in U_i$, and $r_{+,i}(x) = f'(x)/|x-c_i|^{\gamma_i-1}$, $x > c_i$ and $x_i \in U_i$, have bounded variations.

The main theorem in this section is the following.

Theorem D. *Suppose $f: M \mapsto M$ is a $C^{1+\alpha}$- , for some $0 < \alpha \leq 1$, and C^{1+bv}-mapping with finitely many, critically finite, nonperiodic power law critical points and only expanding periodic points and suppose $\eta = \{\eta_n\}_{n=1}^{\infty}$ is the induced sequence of nested partitions of M by f. Then η has bounded geometry.*

We prove this theorem by several lemmas. Suppose f is the mapping in Theorem D. We say an interval I is a n-homterval of f if the restriction of the i^{th}-iterate of f to I is a homeomorphism from I to $I_i = f^{\circ i}(I)$ for any $i = 0, 1, \cdots, n$. If, moreover, the intervals $\{I_i\}_{i=0}^{n}$ have pairwise disjoint interiors, then we call it a n-wandering homterval.

LEMMA D1. *There are constants A, $B>0$ such that for any n-wandering homterval I of f and points x and y in I,*

$$\frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(y)|} \leq \exp\Big(A+\frac{B}{D_{x_n y_n,\partial I_n}}\Big)$$

where $x_n = f^{\circ n}(x)$, $y_n = f^{\circ n}(y)$, $I_n = f^{\circ n}(I)$ and $D_{x_n y_n,\partial I_n}$ is the distance between $\{x_n, y_n\}$ and the boundary of I_n.

Proof. The idea of the proof of this lemma is the same as that of the proof of the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma in [J2]. We outline the proof here.

Suppose U_i is the set in Definition 5. We say an interval in η_n is an critical interval if one of its endpoints is a critical point of f. Suppose n_0 is a positive integer such that every critical interval I in η_{n_0} is contained in some U_i and one of its endpoints is not in the critical orbits $\cup_{n=0}^{\infty} f^{\circ n}(CP)$.

The ratio, $f^{\circ n}(x)/f^{\circ n}(y)$, equals the product $\prod_{i=0}^{n-1} f'(x_i)/f'(y_i)$ where $x_i = f^{\circ i}(x)$ and $y_i = f^{\circ i}(y)$. We divide this product into two products,

$$\prod_{x_i, y_i \in \mathcal{U}} \frac{f'(x_i)}{f'(y_i)} \ \ and \ \ \prod_{x_i, y_i \in \mathcal{V}} \frac{f'(x_i)}{f'(y_i)}$$

where $\mathcal{U}$ stands for the union of all the critical intervals in η_{n_0} and $\mathcal{V}$ stands for the union of all the noncritical intervals in η_{n_0}. The second product is bounded by $\exp(Var(f')/\beta)$, where $\beta > 0$ is the minimum of the absolute value of the restriction of the derivative f' to $\mathcal{V}$.

Let $\tilde{r}_{-,i}(x) = |f(x) - f(c_i)|/|x - c_i|^{\gamma_i}$, $x < c_i$ and $x_i \in U_i$, and $\tilde{r}_{+,i}(x) = |f(x) - f(c_i)|/|x - c_i|^{\gamma_i}$, $x > c_i$ and $x_i \in U_i$. Then both of them have bounded variations. We may write the first product into

$$\prod_{x_i, y_i \in \mathcal{U}} \frac{|f'(x_i)|}{|f'(y_i)|} = \prod_{x_i, y_i \in \mathcal{U}} \frac{|r_{a_i b_i}(x_i)|}{|r_{a_i b_i}(y_i)|} \frac{(\tilde{r}_{a_i b_i}(y_i))^{\gamma_i - 1}}{(\tilde{r}_{a_i b_i}(x_i))^{\gamma_i - 1}} \frac{|f(x_i) - f(c_{b_i})|^{m_{b_i}}}{|f(y_i) - f(c_{b_i})|^{m_{b_i}}}$$

where a_i is $+$ or $-$, b_i the integer such that x_i and y_i are in U_{b_i} and $m_{b_i} = 1 - 1/\gamma_{b_i}$. The first two products satisfy that

$$\prod_{x_i, y_i \in \mathcal{U}} \frac{|r_{a_i b_i}(x_i)|}{|r_{a_i b_i}(y_i)|} \frac{(\tilde{r}_{a_i b_i}(y_i))^{\gamma_i - 1}}{(\tilde{r}_{a_i b_i}(x_i))^{\gamma_i - 1}} \leq$$

$$\exp\Big(\sum_{i=1}^{d_1} \Big(Var(r_{i+}) + Var(r_{i-}) + \frac{1}{\gamma_i - 1}\Big(Var(\tilde{r}_{i+}) + Var(\tilde{r}_{i-})\Big)\Big)\Big).$$

To estimate the last product, we write each

$$\frac{f(x_i) - f(c_{b_i})}{f(y_i) - f(c_{b_i})} = 1 + \frac{f(x_i) - f(y_i)}{f(y_i) - f(c_{b_i})}$$

for x_i and y_i in $\mathcal{U}$ and

$$\prod_{x_i, y_i \in \mathcal{U}} \frac{|f(x_i) - f(c_{b_i})|^{m_{b_i}}}{|f(x_i) - f(c_{b_i})|^{m_{b_i}}} \leq \exp\Big(\frac{1}{m_i} \sum_{k=1}^{l} \log\Big(1 + \frac{|f(x_{i_k}) - f(y_i)|}{|f(x_{i_k}) - f(c_{b_{i_k}})|}\Big)\Big)$$

where $i_1 < \cdots < i_l$.

Because each critical point of f is mapped eventually to an expanding periodic point and all the periodic points of f are expanding, there is a positive constant K_1 (by the naive distortion lemma in [J2]) such that

$$\frac{|f(x_{i_1}) - f(y_{i_1})|}{|f(x_{i_1}) - f(c_{b_{i_1}})|} \leq K_1 \frac{|x_n - y_n|}{D_{x_n y_n, \partial I_n}},$$

and

$$\frac{|f(x_{i_k}) - f(y_{i_k})|}{|f(x_{i_k}) - f(c_{b_{i_k}})|} \leq K_1 \frac{|x_{i_{k-1}} - y_{i_{k-1}}|}{|y_{i_{k-1}} - f^{i_k - i_{k-1}}(c_{b_{i_k}})|}$$

for any $0 < k \leq l$. We may assume that $f^{i_k - i_{k-1} - 1}(c_{b_{i_k}})$ is not a critical point of f. Otherwise, $f^{i_k - i_{k-1}}(c_{i_k}) = c_{i_{k-1}}$ and note that there are only finitely many critical chains of f like that in the proof of Theorem A. Let L be the minimum of lengths of the critical intervals in η_{n_0}. Then

$$\frac{|f(x_{i_k}) - f(y_{i_k})|}{|f(x_{i_k}) - f(c_{b_{i_k}})|} \leq K_1 \frac{|x_{i_{k-1}} - y_{i_{k-1}}|}{L},$$

and moreover, there are constants K_2, $K_3 > 0$ such that

$$\prod_{x_i, y_i \in \mathcal{U}} \frac{|f(x_i) - f(c_{b_i})|^{m_{b_i}}}{|f(x_i) - f(c_{b_i})|^{m_{b_i}}} \leq K_2 + \frac{K_3}{D_{x_n y_n, \partial I_n}}.$$

Combining all the estimates together, we get two positive constants A and B.

Lemma D2. *Every ∞-homterval I of f is an ∞-wandering homterval.*

Proof. Suppose there are integers $m > n > 0$ such that I_n and I_m are overlap. Let $k = n - m$, then I_0 and I_k are overlap, and moreover, I_{lk} and $I_{(l+1)k}$ are overlap for any $l > 0$. Let $T = \cup_{l=0}^{\infty} I_{kl}$. It is a connected interval of M and $f^{\circ k} : T \mapsto T$ is a homeomorphism. Then $f^{\circ k}$ has to have a fixed point which is not topologically expanding. This contradiction proves the lemma.

From Lemma D1 and Lemma D2, we have the following lemma.

Lemma D3. *The maximal length of the intervals in η_n tends to zero as n goes to infinity.*

Proof. Suppose there is an $\epsilon_0 > 0$ such that for any positive integer n, there is an interval $I_n \in \eta_n$ with $|I_n| > \epsilon_0$. Because M is a compact manifold, there is a subset $\{n_i\}_{i=1}^{\infty}$ of the integers such that I_{n_i} goes to an interval $\tilde{I}$ as i goes to infinity and the length of $\tilde{I}$ is greater than ϵ_0. There is an interval $I \subset \tilde{I}$ such that $I \subset I_{n_i}$ for large i. The restriction of the i^{th}-iterate of f to I_{n_i} is an embedding for any $i \leq n_i$. Hence I is an ∞-homterval of f, and moreover, it is an ∞-wandering homterval. Suppose I is a maximal such interval. Let $T_n \supset I$ be the maximal n-homterval. Then it is again a n-wandering homterval. Let L_n and R_n be the intervals in the complement of I in T_n. The lengths of L_n and R_n go to zero as n tends to infinity. The boundary of $f^{\circ n}(T_n)$ is contained in the union of the boundary of M and the set of critical values $f(CP)$ of f for T_n is a maximal n-homterval of f. Suppose $\{n_i\}_{i=0}^{\infty}$ is a subsequence of the integers such that the boundary of $f^{\circ n_i}(T_{n_i})$ are the same for all i. By using Lemma D1, one of the lengths of $f^{\circ n_i}(I \cup L_{n_i})$ and $f^{\circ n_i}(R_{n_i} \cup I)$, say $f^{\circ n_i}(I \cup L_{n_i})$, has to go to zero as i tends to infinity. Because every critical point is mapped to a periodic point eventually, the interval $f^{\circ n_i}(I \cup L_{n_i})$ tends to a periodic orbit eventually. This periodic point is not topologically expanding. The contradiction proves the lemma.

Recall that in the proof of Lemma D1, $\mathcal{U}$ stands for the union of all the critical intervals in η_{n_0} and $\mathcal{V}$ stands for the union of all the noncritical intervals in η_{n_0}, where n_0 is a fixed positive integer such that every critical interval is contained in U_i in Definition 5 and one of its endpoints is not in the critical orbits $\cup_{n=0}^{\infty} f^{\circ n}(CP)$.

Lemma D4 and Lemma D5 are two of the key lemmas in the proof of Theorem D.

LEMMA D4. *There is a constant $K > 0$ such that for an interval $I \in \eta_{n+n_0}$, if $I_i = f^{\circ i}(I)$ is in $\mathcal{V}$ for every $1 \leq i \leq n$, then*

$$\frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(y)|} \leq K$$

for any x and y in I.

Proof. If $\{I_i\}_{i=0}^{n-1}$ have pairwise disjoint interiors, then

$$|(f^{\circ n})'(x)| \geq \exp\Big(\frac{Var(f')}{\beta}\Big)\frac{|I_n|}{|I|}$$

for any $x \in I$ where $I_n = f^{\circ n}(I) \in \eta_{n_0}$ and $\beta > 0$ is the minimum of the absolute value of $f'|\mathcal{V}$. By using this fact and Lemma D3, we can find a constant $\nu > 1$ such that for a periodic point p of f, if $p_i = f^{\circ i}(p)$ is in $\mathcal{V}$ for every $i \geq 0$, then the eigenvalue $|(f^{\circ k})'(p)| \geq \nu$ where k is the period of p.

By the naive distortion lemma (see [J1] or [J2]), we have that

$$\frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(y)|} \leq \exp\Big(\frac{K_1}{\beta}\sum_{i=0}^{n-1}|I_i|^{\alpha}\Big)$$

for any x and y in I where K_1 is a positive constant and c is the minimum of the absolute value of $f'|\mathcal{V}$.

Suppose $I_0, \cdots, I_{k-1}$ have pairwise disjoint interiors and $I_k \subset I_0$. There is a periodic point p of period k in I_0. Again by using the naive distortion lemma, there is a constant $K_2 > 0$ such that

$$|I_{lk+i}| \leq \frac{K_2}{\nu^l}|I_i|$$

for all $l > 0$ and $0 \leq i < k$ where $K_2 > 0$ is a constant. Last two inequalities imply Lemma D4.

We say a critical point of f is pure if it is not in the post-critical orbits $\cup_{n=1}^{\infty} f^{\circ n}(CP)$. We say an interval I is a pure critical interval in η_{n_0} if one of its endpoint is pure critical point. Remember that the other endpoint of I is not in the critical orbits $\cup_{n=0}^{\infty} f^{\circ n}(CP)$.

LEMMA D5. *There is a constant $K > 0$ such that for an interval $I \in \eta_{n+n_0}$, if $I_n = f^{\circ n}(I)$ is in a pure critical interval in η_{n_0}, then*

$$\frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(y)|} \leq K$$

for any x and y in I.

Proof. By the similar arguments to the proof of Lemma D1 and that I_n is far to the post-critical orbit $\cup_{n=1}^{\infty} f^{\circ n}(CP)$, we can find a positive constant K_1 such that if $\{I_i\}_{i=0}^{\infty}$ have pairwise disjoint interiors, then

$$|(f^{\circ n})'(x)| \geq \exp\Big(K_1\Big)\frac{|I_n|}{|I|}$$

for any $x \in I$. Using this fact and Lemma D3, we can find a constant $\nu > 1$ such that for any periodic point p in a pure critical interval in η_{n_0}, the eigenvalue $|(f^{\circ k})'(p)| \geq \nu$ where k is the period of p.

By the version of the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma in [J2], there is a constant $K_2 > 0$ such that

$$\frac{|(f^{\circ n})'(x)|}{|(f^{\circ n})'(x)|} \leq \exp\Big(K_2\sum_{i=0}^{n-1}|I_i|^{\alpha}\Big)$$

for any x and y in I where K_2 is a positive constant.

Suppose $I_0, \cdots, I_{k-1}$ have pairwise disjoint interiors and $I_k \subset I_0$. There is a periodic point p of period k in I_0. Again by Lemma 1 and the naive distortion lemma (see [J1] or [J2]), there is a constant $K_3 > 0$ such that

$$|I_{lk+i}| \leq \frac{K_3}{\nu^l}|I_i|$$

for all $l > 0$ and $0 \leq i < k$. The last two inequalities imply Lemma D5.

Proof of Theorem D. The proof of Theorem D is now similar to the proof of Theorem A. Here we use Lemma D4 to replace the role of the naive distortion lemma and use Lemma D5 to replace the role of Lemma 1 (the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma).

Corollary D1. *The maximum λ_n of lengths of the intervals in η_n tends to zero exponentially with n.*

Theorem D and Corollary D1 provide another example of a geometrically finite one-dimensional mapping.

Example 2. *A $C^{1+\alpha}$-, for some $0 < \alpha \leq 1$, and C^{1+bv}-mapping $f : M \mapsto M$ with finitely many, critically finite, nonperiodic power law critical points and only expanding periodic points.*

In Theorem D and in Example 2, the condition that f is a $C^{1+\alpha}$-, for some $0 < \alpha \leq 1$, and C^{1+bv}-mapping can not be weakened to the condition that f is a $C^{1+\alpha}$-mapping for there is a counterexample in [J1]. The construction of the counterexample in [J1] is like the construction of the Denjoy counterexample in circle diffeomorphisms and this example is not topologically conjugate to any geometrically finite one-dimensional mapping.

§3.3 A question on $C^{1+\alpha}$-mappings with expanding periodic points.

In Theorem C, the conditions that a $C^{1+\alpha}$-mapping for some $0 < \alpha \leq 1$ with only power law critical points is quasisymmetrically conjugate to a geometrically finite one-dimensional mapping can not be weakened to the condition that a $C^{1+\alpha}$-mapping for some $0 < \alpha \leq 1$ with only power law critical point is topologically conjugate to a geometrically finite one-mapping for there is an easy counterexample $f : [-1, 1] \mapsto [-1, 1]$ with the neutral fixed point -1, namely $f'(-1) = 1$ (see Figure 4).

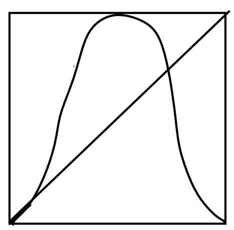

Figure 4

The graph in Figure 4 suggests a question as follows.

QUESTION 1. *Suppose $f : M \mapsto M$ is a $C^{1+\alpha}$-mapping for some $0 < \alpha \leq 1$ with only power law critical points and only expanding periodic points and is topologically conjugate to a geometrically finite one-dimensional mapping. Is f geometrically finite ?*

The answer of this question may be negative. But we do not have a concrete counterexample yet. The reader may refer to the construction of the counterexample in [J1] and Lemma D4 and Lemma D5 in this paper.

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