On the quasisymmetrical classification of infinitely renormalizable maps

I. Maps with Feigenbaum's topology

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§0 Introduction

We begin by considering the set of infinitely renormalizable unimodal maps on the interval $[-1,1]$. A function f defined on $[-1,1]$ is said to be *unimodal* if it is continuous, increasing on $[-1,0]$, decreasing on $[0,1]$ and symmetric about 0, and if it fixes -1 and maps 1 to -1 . Moreover, it is said to be *renormalizable* if there is an integer $n > 1$ and a subinterval $I \neq [-1,1]$ containing 0 such that $f^{\circ n}(I) \subseteq I$. We will assume that one endpoint q of I is fixed by $f^{\circ n}$ and I is symmetric about 0. If we normalize I to $[-1,1]$ by the linear map α, which maps q to -1, then $\mathcal{R}(f) = \alpha \circ f^{\circ n} \circ \alpha^{-1}$ is unimodal, too. This map $\mathcal{R}$ will be called the *renormalization operator*. To fix our notation, we will assume that n is the minimum such integer and call it the *return time*.

A unimodal map f is *infinitely renormalizable* if every $\mathcal{R}^{\circ k}(f)$ is renormalizable, say with return time n_k. Furthermore, f is of *bounded type* if all the return times are less than a constant integer, otherwise, f is of *unbounded type*. In particular, we call f a Feigenbaum map if all the return times are 2. A well-known example $f = q_{\lambda_\infty}$ of a Feigenbaum map [4, 6, 18] is obtained by period-doubling cascade in the family $\{q_\lambda(z) = -(1+\lambda)z^2 + \lambda\}_{0 \leq \lambda \leq 1}$.

Let $\mathcal{U}$ be the space of unimodal maps $f = h \circ Q_t$ where $Q_t(x) = -|x|^t$ for some $t > 1$ and h is a diffeomorphism (a unimodal map can always be written in this form by some smooth change of coordinate [12]). We may assume either h is a C^3-diffeomorphism with nonpositive Schwarzian derivative [3] or h is a C^{1+Z}-diffeomorphism [17]. However, the smoothness is not important in this paper as long as h satisfies the distortion properties discussed in [17]. To avoid many technical notations, *henceforth, we will assume that h is a C^3-diffeomorphism with nonpositive Schwarzian derivative.* We note that the Schwarzian derivative $S(h)$ of a C^3-diffeomorphism is

$$S(h) = \frac{h'''}{h'} - \frac{3}{2}\Big(\frac{h''}{h'}\Big)^2.$$

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Suppose f and g in $\mathcal{U}$ are both infinitely renormalizable and topologically conjugate. Suppose H from $[-1,1]$ to $[-1,1]$ is the conjugacy between f and g, that is, $f \circ H = H \circ g$. The map H is said to be quasisymmetric [1] if there is a constant $M > 0$ such that

$$M^{-1} \leq \frac{|H(x) - H(z)|}{|H(z) - H(y)|} \leq M$$

whenever x and y are in $[-1,1]$ and $z = (x+y)/2$ is the midpoint between x and y.

CONJECTURE 1 (SULLIVAN). *The homeomorphism H is quasisymmetric.*

We would like to investigate this conjecture in the three cases, the Feigenbaum case, the bounded case and the unbounded case. In this paper we prove this conjecture for the Feigenbaum case as follows. We note that any two Feigenbaum maps are topologically conjugate.

THEOREM 1. *Suppose f and g in $\mathcal{U}$ are two Feigenbaum maps and H is the conjugacy between f and g. Then H is quasisymmetric.*

The idea to prove this theorem is that we construct a sequence of nested partitions on $[-1,1]$ by using all the periodic points of a Feigenbaum map and the preimages of all the periodic points under iterates of this map. We show that this sequence of nested partitions has bounded geometry and bounded nearby geometry properties [11]. The reader may compare the construction of the sequence of nested partitions here with the construction of the sequence of nested partitions in [19] (see also [7]) for a nonrenormalizable quadratic polynomial.

The proof of Conjecture 1 for infinitely renormalizable maps of bounded type should not have an essential difference from the proof in this paper. We will do it in a short paper [13]. However, a proof of Conjecture 1 for infinitely renormalizable maps of unbounded type may have an essential difference from the proof in this paper. It is still an interesting research problem.

For complex quadratic-like maps [5] in $\mathcal{U}$ which are infinitely renormalizable of bounded type, Conjecture 1 has been proven by Sullivan [17] using a completely different method. I was also told that recently, Paluba [15] reached another proof of Conjecture 1 for infinitely renormalizable maps of bounded type. Jakobson and Swiatek [16, 9] showed some other interesting results about quasisymmetric classification of unimodal maps.

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also [7]) which inspired the work in this paper. He would also like to thank J. Milnor for reading and correcting this manuscript.

§1 The proof of Theorem 1

We prove Theorem 1 by several lemmas.

Suppose $f = h \circ Q_t$, for some $t > 1$, in $\mathcal{U}$ is a Feigenbaum map. Let $p_0 = -1$ and $p_1 \in (0, 1)$ be the fixed points of f. Inductively, let $p_n \in (-p_{n-1}, 0)$ or $(0, -p_{n-1})$ be the fixed point of $f^{\circ 2^{n-1}}$ for an integer $n > 1$. Let I_n be the interval bounded by p_n and $-p_n$ (see Figure 1) and $c_n = f^{\circ n}(0)$ is the n^{th} critical value of f for $n = 0, 1, \cdots$.

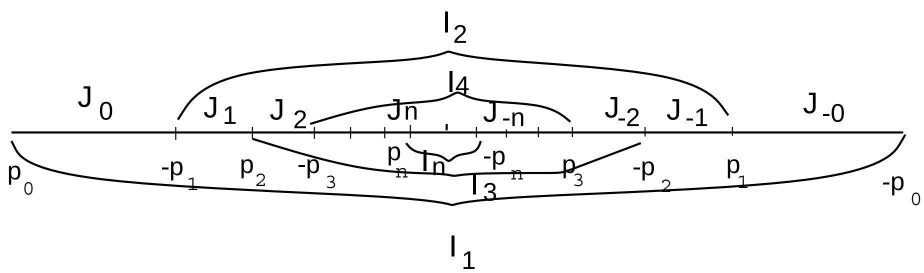

Figure 1

Suppose L_n is the image of I_{n-1} under $f^{\circ 2^{n-1}}$ and T_n is the interval bounded by the points p_{n-1} and p_n. Let M_n be the complement of T_n in L_n (see Figure 2). Then M_n is the interval bounded by p_n and $c_{2^{n-1}}$.

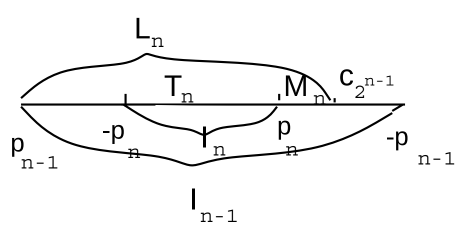

Figure 2

Lemma 1. *There is a constant $C_1 = C_1(f) > 0$ such that $|M_n|/|I_n| \geq C_1$ for all the integers $n \geq 0$.*

Proof. This lemma is actually proved in [17] by using the techniques such as the smallest interval and shuffle permutation on the intervals.

Lemma 2. *There is a constant $C_2 = C_2(f) > 0$ such that $|I_n|/|I_{n-1}| \geq C_2$ for all the integers $n \geq 0$.*

We first prove a more general result as follows. Suppose h is a C^2-diffeomorphism. We use

$$N(h) = \frac{h''}{\big(h'\big)^2}$$

to denote the nonlinearity of h. Let $\mathcal{K} = \mathcal{K}(t, K)$, for fixed numbers $t > 1$ and $K > 0$, be the subspace of maps $f = h \circ Q_t$ in $\mathcal{U}$ such that

$$|\Big(N(h)\Big)(x)| \leq K$$

for all x in $[-1, 0]$. Remember that $Q_t(x) = -|x|^t$.

Lemma 3. *There is a constant $C_3 = C_3(t, K) > 0$ such that $f(0) = c_1(f) \geq C_3$ for all Feigenbaum maps f in $\mathcal{K}$.*

Proof. Suppose $f = h \circ Q_t$ is a map in $\mathcal{K}$. Since h is a C^3-diffeomorphism with nonpositive Schwarzian derivative, we can compare h with some Möbius transformations $m(x) = (ax + b)/(cx + d)$. We note that all the Möbius transformation have zero Schwarzian derivatives.

Let m be the Möbius transformation satisfying

(a) $m(-1) = f(-1) = -1$ and $m(0) = f(0)$, and

(b) $\Big(N(m)\Big)(-1) = K$.

Then

$$m(x) = \frac{(f(0) + 1)(1 + \frac{K}{2})x + f(0)}{\frac{K}{2}(f(0) + 1)x + 1}.$$

Since $m(-1) = h(-1)$, $m(0) = h(0)$ and $\Big(N(m)\Big)(-1) \geq \Big(N(h)\Big)(-1)$, we have that $h(x) \geq m(x)$ for all x in $[-1, 0]$.

Suppose, in the moment, $w = f(0) > 0$ in m is a variable. Consider the equation

$$m\Big(-\big(m(0)\big)^t\Big) = \frac{(w + 1)(1 + \frac{K}{2})(-w^t) + w}{\frac{K}{2}(w + 1)(-w^t) + 1} = 0.$$

Let $C_3 = C_3(t, K) > 0$ be the solution of this equation for w.

For any Feigenbaum map $f = h \circ Q_t$ in $\mathcal{K}$, we know, from the property of renormalization and $m(x) \leq h(x)$ for all x in $[-1, 0]$, that $m\Big(-\big(m(0)\big)^t\Big) < 0$. It implies that $f(0) = c_1(f) > C_3$. This proves Lemma 3.

Suppose $f = h \circ Q_t$, for some $t > 1$, in $\mathcal{U}$ is a Feigenbaum map and $f_n = h_n \circ Q_t$ is the n^{th} renormalization $\mathcal{R}^{\circ n}(f)$ of f for an integer $n \geq 0$. We note that the graph of f_n is the rescale of the graph of the restriction of $f^{\circ 2^n}$ to I_n.

LEMMA 4. *There is a constant $C_4 = C_4(f) > 0$ such that*

$$|\Big(N(h_n)\Big)(x)| \leq C_4$$

for all x in $[-1, 0]$ and $n \geq 0$.

Proof. It is the a prior bound proved in [17]

Proof of Lemma 2. It is now a direct corollary of Lemma 1, Lemma 3 and Lemma 4 for $K = C_4$.

The set of the nested intervals $\{I_0, I_1, \cdots, I_n, \cdots\}$ gives a partition of $[-1, 1]$ as follows. Let J_0 and J_{-0} be two connected components of $I_0 \setminus I_1$. Inductively, let J_n and J_{-n} be two connected components of $I_n \setminus I_{n+1}$. Then the set $\eta_0 = \{J_0, J_{-0}, J_1, J_{-1}, \cdots\}$ forms a partition of I_0 (see Figure 1).

Now we are going to define a Markov map F induced by f. Let F be a function of $[-1, 1]$ defined as

$$F(x) = \begin{cases} f(x), & x \in J_0 \cup J_{-0}, \\ f^{\circ 2}(x), & x \in J_1 \cup J_{-1}, \\ \vdots & \\ f^{\circ 2^n}(x), & x \in J_n \cup J_{-n}, \\ \vdots & . \end{cases}$$

It is clear that F is a Markov map in the sense that the image of every J_k under F is the union of some intervals in the partition η_0 (see Figure 3).

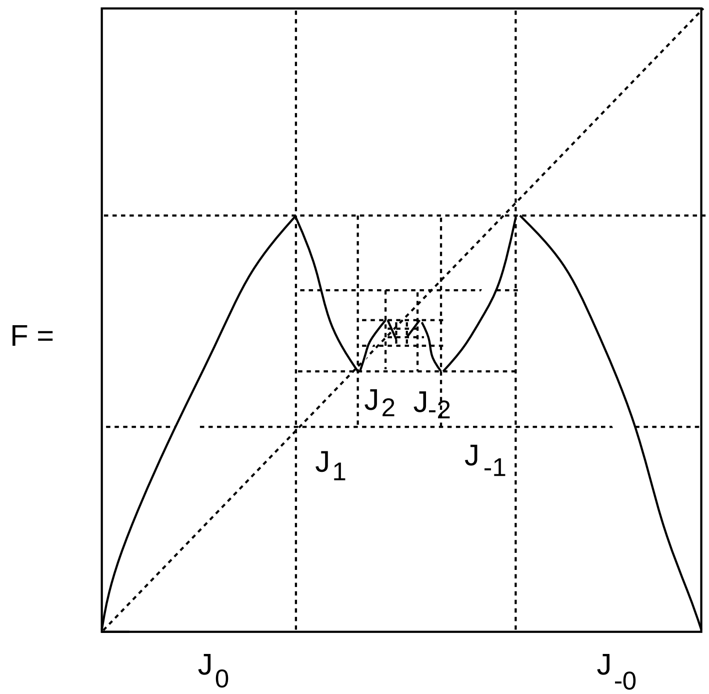

Figure 3

Let $g_n = (F|J_n)^{-1}$ and $g_{-n} = (F|J_{-n})^{-1}$, $n = 0, 1, \cdots$, be the inverse branches of F. Suppose $w = i_0 i_1 \cdots i_{k-1}$ is a finite sequence of $\mathbf{Z} \cup \{-0\}$. We say it is admissible if the range J_{i_l} of g_{i_l} is contained in the domain $F_{i_{l-1}}(J_{i_{l-1}})$ of $g_{i_{l-1}}$ for $l = 1, \cdots, k-1$. For an admissible sequence $w = i_0 i_1 \cdots i_{k-1}$, we can define a map $g_w = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{k-1}}$. We use $D(g_w)$ to denote the domain of g_w and use $|D(g_w)|$ to denote the length of the interval $D(g_w)$.

Definition 1. *We say the induced Markov map F has bounded distortion property if there is a constant $C_5 = C_5(f) > 0$ such that*

(a) $|J_n| / \cup_{l=n+1}^{\infty} |J_l| \geq C_5$ and $|J_{-n}| / \cup_{l=n+1}^{\infty} |J_{-l}| \geq C_5$ for all the integers $n \geq 0$, and

(b) $|\Big(N(g_w)\Big)(x)| \leq C_5 / |D(g_w)|$ for all x in $D(g_w)$ and all admissible w.

The reason we give this definition is the following lemma. First, we note that if g_w has the property (b) in Definition 1, then distortion of g_w

$$\frac{|g_w(x)|}{|g_w(y)|} \leq \exp(C_5)$$

for all x and y in $D(g_w)$.

Lemma 5. *Suppose f and g in $\mathcal{U}$ are two Feigenbaum maps and H is the conjugacy between f and g. If both of the induced Markov maps F and G have bounded distortion property, then*

H is quasisymmetric.

Proof. It can be proved by almost the same arguments as that we used in the paper [11]. For more details of the proof, the reader may refer to [11].

Suppose g is a C^3-diffeomorphism with nonnegative Schwarzian derivative defined on J. We say g has a *Koebe space* $C > 0$ around J if g can be extended to an interval $I \supset J$ as a C^3-diffeomorphism with nonnegative Schwarzian derivative, and moreover, the minimum of the ratios $|L|/|J|$ and $|R|/|J|$ is greater than C where L and R are the connected components of $I \setminus J$.

LEMMA 6. *Suppose g is a C^3-diffeomorphism with nonnegative Schwarzian derivative defined on J. Moreover, suppose g has a Koebe space $C > 0$ around J. Then the nonlinearity $N(g)$ satisfies*

$$|\Big(N(g)\Big)(x)| \leq \frac{C'}{|J|}$$

where $C' = C'(C) > 0$ is a constant.

Proof. It is a well-known lemma. We call it the C^3-Koebe distortion lemma. See, for example [2, 8, 14, 10, 17, etc.].

Now the proof of Theorem 1 concentrates on the next lemma.

LEMMA 7. *Suppose $f = h \circ Q_t$, for some $t > 1$, in $\mathcal{U}$ is a Feigenbaum map and F is the Markov map induced by f. Then F has the bounded distortion property.*

Proof. The condition (a) in Definition 1 is assured by Lemma 2. We only need to check the condition (b) in Definition 1.

Suppose n is an integer $n \neq 0$, then g_n can be extended to the interval $\Omega_{|n|} = L_{|n|} \cup M_{|n|-1}$ as a C^3-diffeomorphism with nonnegative Schwarzian derivative (because h is C^3-diffeomorphism with nonpositive Schwarzian derivative). For g_0 and g_{-0}, without loss of generality, we may assume that they can be extended to the interval $(-\infty, -1] \cup L_0$. Lemma 1 and Lemma 2 now assured that the map g_n has a definite Koebe space $C_7 > 0$ around its domain for all n in $\mathbf{Z} \cup \{-0\}$. We note that the intervals $\Omega_{|n|}$ are nested for $|n| = 0, 1, \cdots$.

Suppose $w = i_0 i_1 \cdots i_{k-1}$ is an admissible sequence of $\mathbf{Z} \cup \{-0\}$ and $g_w = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{k-1}}$. By the definition of an admissible sequence, one can check that (see Figure 1)

$$|i_0| \leq |i_1| \leq \cdots \leq |i_{k-1}|.$$

Hence g_w can be extended to the domain $\Omega_{|i_{k-1}|}$ as a C^3-diffeomorphism with nonnegative

Schwarzian derivative. Again the map g_w has the Koebe space C_7 around its domain. This implies that

$$|\Big(N(g_w)\Big)(x)| \leq \frac{C_8}{|D(g_w)|}$$

for all x in $D(g_w)$ where $C_8 = C_8(C_7) > 0$ is a constant. This completes the proof of Lemma 7.

The arguments in Lemma 1 to Lemma 7 give the proof of Theorem 1.

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