A Comparison of Harmonic and Balanced Measures on Cantor Repellors

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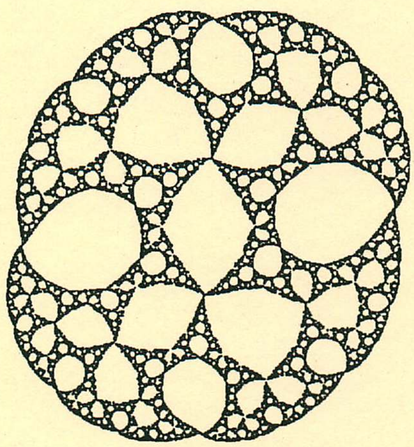

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A comparison of harmonic and balanced measures on Cantor repellors

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Abstract. Let J be a Cantor repellor of a conformal map f. Provided f is polynomial-like or $\mathbb{R}$-symmetric, we prove that harmonic measure on J is equivalent to the measure of maximal entropy if and only if f is conformally equivalent to a polynomial. We also show that this is not true for general Cantor repellors: there is a non-polynomial algebraic function generating a Cantor repellor on which above two measures coincide.

1. Introduction. Harmonic measure in dynamical context appeared for the first time in the Brolin's paper [Br] where it was established that harmonic measure w associated with the unbounded component of the complement of the polynomial Julia set $J(f)$ is *balanced* which means that backward orbits of f are equidistributed with respect to w. Later this balanced measure was interpreted as the unique measure of maximal entropy of f [L], [Ma].

When we have more general conformal dynamical systems, a natural problem of the comparison of these two measure arises. For rational f it was considered by Lopes [Lo] who proved that if $\infty \in \bar{\mathbb{C}} \setminus J(f)$ is a fixed point of f, then it follows from the coincidence of harmonic measure w with the maximal measure that f is a polynomial.

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As a particular case of this theorem, one can consider a Blaschke product fixing ∞:

$$f: z \to \lambda z \prod_{i=1}^{d-1} \frac{z - a_i}{1 - \bar{a}_i z}, \quad |\lambda| = 1, \ |a_i| < 1.$$

Then harmonic measure w is just Lebesgue measure σ on the unit circle $\mathbf{T}$. So, the entropy of σ is equal to $\log d$ if and only if $f: z \to z^d$.

We are going to consider a local setting of the problem when f is defined only in a neighborhood of an invariant compact set $J = J(f)$. The question is to characterize the situation when harmonic and balanced measures are equivalent. It certainly happens if f is conformally conjugate to a polynomial (see Appendix). In this paper we will discuss the reverse problem in the case when J is an expanding Cantor repellor.

Let us pass to precise definitions. Let $U, U_1, U_2, ..., U_d$ be $d+1$ topological discs with piecewise smooth boundaries such that the closures $\bar{U}_i \subset U$, $i = 1, ..., d$. Consider a map $f: \bigcup_{i=1}^{d} U_i \to U$ which is a conformal isomorphism $f_i: U_i \to U$ on each U_i (see Figure 1). By an *(expanding) Cantor repellor* we mean the set

$$J = J(f) = \{x : f^n x \in \bigcup_{i=1}^{d} U_i, \ n = 0, 1, ...\}.$$

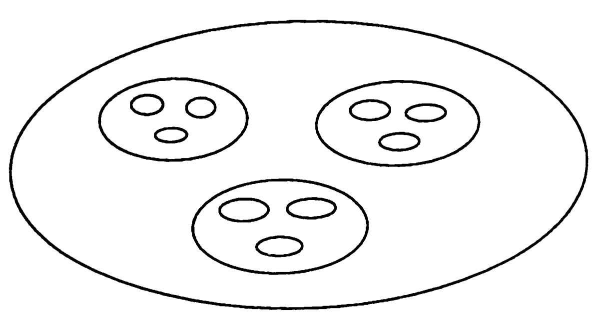

Figure 1.

Let us say that the map f generating the repellor is *symmetric* if each one of the domains U_i is *symmetric* about the real line $\mathbf{R}$, and f preserves $\mathbf{R}$.

Let us say that f is *polynomial-like* if it allows a polynomial-like (in the sense of Douady and Hubbard [DH]) continuation to a bigger domains $V \to W$. This means that V and W are topological discs with piecewise smooth boundaries, with V relatively compact in W, and $f: V \to W$ is a proper of degree d , that is, a branched covering of degree d (see §5 for more details).

Saying that two maps f and g are *(conformally) conjugate* we mean that there is a (conformal) conjugacy in some neighborhoods of the Julia sets (so, actually we are speaking about *germs*).

It is very easy to understand what is the balanced measure (measure of maximal entropy) m in our setting. Namely, m is uniquely determined by the property that for any finite sequence $x_1, ..., x_n$ of symbols $1, ..., d$

$$m\{z : f^i z \in U_{x_i},\ i = 1, ...n\} = 1/d^n.$$

In this paper we will prove the following theorem.

Theorem. Let f be either symmetric or polynomial-like generating an expanding Cantor repellor $J(f)$, and let w be harmonic measure on $J(f)$. Then w is absolutely continuous with respect to the balanced measure m if and only if f is conformally conjugate to a polynomial. □

We were surprised that this statement is not true for general Cantor repellors:

Example. There is an expanding Cantor repellor on which harmonic measure is balanced but which is not conformally equivalent to a polynomial Julia set.

The function f generating this repellor is algebraic. Our considerations actually show that any expanding Cantor repellor on which harmonic measure is balanced is conformally equivalent to an algebraic one. We are going to discuss this phenomenon in a later paper.

The investigation of harmonic measure from the dynamical point of view was started by Carleson [Ca]. He constructed an invariant harmonic measure (that is, a f-invariant measure equivalent to w) as the probability distribution for a stationary sequence of nearly independent random variables. Later the powerful methods of Bowen-Ruelle-Sinai thermodynamical formalism were introduced into the subject (see [MV], [PUZ]). This approach plays a crucial role here as well.

In conclusion let us mention a well-known statement concerning the circle maps which is important for understanding our result.

Lemma 1.1. (compare [SS]). Let $f : \mathbf{T} \to \mathbf{T}$ be an analytic expanding map of the circle. Assume that its maximal measure m is non-singular with respect to the Lebesgue measure σ on $\mathbf{T}$. Then f is analytically conjugate to $z \to z^d$.

Proof. An expanding map of the circle has unique invariant measure μ non-singular with respect to the Lebesgue measure. Moreover, this measure is absolutely continuous with respect to σ with analytic density ρ [K]. In our case $m = \mu$.

Now let us consider a homeomorphism $h : \mathbf{T} \to \mathbf{T}$ conjugating f to $z \to z^d$. It carries the balanced measure m of f into the balanced measure σ of $z \to z^d$. Hence, on the universal covering of $\mathbf{T}$ we have

$$h(x) = h(0) + \int_0^x \rho(t)dt,$$

and we are done. □

Our proof of the Theorem reduces the case $w \sim m$ on a Cantor repellor $J(f)$ to the circle case by using a "circle model" for a Cantor repellor. In the general setting of Cantor repellors our circle construction is based upon the assumption $w \sim m$. In the case of a polynomial-like Cantor repellor one can use the Douady-Hubbard circle model which is independent of the assumption $w \sim m$.

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2. Gibbs property of harmonic measure

2.1. We refer to [Bo] or [EL] for the exposition of the theory of Gibbs measures, and here we state only the facts we need for our goals.

On a Cantor repellor $J = J(f)$ the dynamical system $f : J \to J$ is naturally topologically conjugate to the shift $T : \Sigma_d^+ \to \Sigma_d^+$ on the space of one-sided sequences in d symbols. Providing $\sum_d^+$ with the natural metric $(p(\bar{x}, \bar{y}) = \frac{1}{2^n}$, where n is the first moment for which $x_n \neq y_n)$, we see that the conjugacy $h : J \to \sum_d^+$ is Hölder continuous in both directions. So, the class of Hölder function is well-defined if we identify J and $\sum_d^+$ via h.

For any f-quasi-invariant measure ν on J one can consider its Jacobian

$$\mathcal{G}_\nu(z) = \frac{df^*\nu}{d\nu}(z),$$

that is the Radon-Nicodim derivative of f with respect to ν. The function $\psi_\nu \stackrel{def}{=} -\log \mathcal{G}_\nu(z)$ will be called the potential of ν. Hölder regularity of the potential yields a proper Gibbs theory several facts of which we are going to state now:

1) By a Gibbs measure on $J(f)$ we mean an f-invariant measure ν with Hölder potential $\psi_\nu = -\log \mathcal{G}_\nu$.

2) If η is an f-quasi-invariant measure with Hölder potential ψ_η, then there exists the unique f-invariant measure ν absolutely continuous with respect to η . This measure is a Gibbs measure and log $\frac{d\nu}{d\eta}$ is a Hölder function. Moreover, μ is ergodic.

3) In the above situation ψ_η and ψ_ν satisfy the cohomology equation

$$\psi_\eta = \psi_\nu + \gamma \circ f - \gamma$$

with a Hölder function γ. Actually, $\gamma = \log \frac{d\eta}{d\nu}$.

2.2. Estimates of the Jacobian of harmonic measure of a Cantor repellor.

For a cylinder $X = x_1, ..., x_n \subset \sum_d^+$ let

$$Q_X = f_{x_1}^{-1} \circ \cdots \circ f_{x_n}^{-1} U = U_{x_1} \cap f^{-1} U_{x_2} \cap \cdots \cap f^{-(n-1)} U_{x_n},$$

and $\nu(X) = \nu(Q_X)$ for any measure ν on J. We will also adopt the following natural notation: for two cylinders $X = x_1, ..., x_n$, $Y = y_1, ..., y_m$ by XY we mean $x_1, ..., x_n y_1, ..., y_m$. For $X = x_1, ..., x_n$, $|X| = n$.

The crucial estimate for harmonic measure on Cantor repellors was established in [Ca] and [MV].

Proposition 2.1. For a Cantor repellor J there exists C and $q \in (0,1)$ such that for any X, Y, Z

$$\left|\log\left(\frac{w(XYZ)}{w(XY)} : \frac{w(YZ)}{w(Y)}\right)\right| \le Cq^{|Y|}. \ \square \tag{2.1}$$

This proposition was based on the following result, which we will also need later.

Lemma 2.2. Let Ω be a bounded domain, A_i, B_i $(i = 1, ..., n)$ be Jordan domains such that

$$A_1 \supset B_1 \supset A_2 \supset B_2 \supset \cdots \supset A_n \supset B_n,$$

and $A_i \setminus B_i \subset \Omega$. Suppose that $A_i \setminus B_i$ are topological annuli and their modules are bounded from below by $\rho > 0$. If u and v are two positive harmonic functions on Ω vanishing on $A_1 \cap \partial\Omega$, then for any $\zeta,\ z \in \Omega \cap B_n$

$$\left|\frac{u(z)}{v(z)} : \frac{u(\zeta)}{v(\zeta)} - 1\right| \le Cq^n, \tag{2.2}$$

where $c = c(\rho) < \infty$, $q = q(\rho) \in (0,1)$. $\square$.

Using (2.1) with $X = x_1$, $Y = x_2, ..., x_n$, $Z = x_{n+1}$ we get

$$\left|\log \frac{w(x_1, ..., x_n)}{w(x_2, ..., x_n)} - \log \frac{w(x_1, ..., x_n x_{n+1})}{w(x_2, ..., x_n x_{n+1})}\right| \le Cq^n.$$

Thus, the potential

$$\psi_w(x) = \lim \log \frac{\omega(x_1 ... x_n)}{\omega(x_2 ... x_n)}$$

exists for every $x = (x_1, x_2 ...) \in J$ and is Hölder continuous. So, we are in position to apply the theory of Gibbs measures .

Proposition 2.3. There exists a unique f-invariant measure μ absolutely continuous with w. This measure is ergodic and its potential ψ_μ is Hölder continuous (so μ is a

Gibbs measure) Moreover, the logarithm of the Radon-Nikodim derivative $\log \frac{d\mu}{dw}$ is Hölder continuous. □

Remark. This is the same "invariant harmonic measure" which was constructed by Carleson [Ca] for the "standard" Cantor set.

In what follows we will use $G(fz)/G(z),\ z \in \bigcup_{i=1}^{d} U_i$, as a natural extension of the Jacobian of harmonic measure on a Cantor repellor $J(f)$. The next two Propositions show that it is really an extension of J_w as well as some of its properties.

Proposition 2.4. For any cylinder X and any $z \in Q_X$

$$\left|\log\left(\frac{G(fz)}{G(z)} : \frac{w(fX)}{w(X)}\right) - 1\right| \le Cq^{|X|}. \tag{2.3}$$

Proposition 2.5. $-\log \frac{G(fz)}{G(z)}$ is a Hölder continuation of $\psi_w = -\log \mathcal{G}_w$ onto $\bigcup_{i=1}^{d} U_i$. □

Proof of Proposition 2.4. Let us fix $i = 1, ..., d$, put $\Omega = U_i,\quad u = G,\quad v = G \circ f_i$ on Ω, and use Lemma 2.2. in this setting. Then we get

$$\left|\frac{G(fz)}{G(z)} : \frac{G(f\zeta)}{G(\zeta)} - 1\right| \le Cq^{|X|},\quad \forall z,\ \zeta \in Q_X. \tag{2.4}$$

Now recall (see [HK]) that for any disc D_{z_0,r_0} with $z \in J$ we have

$$\frac{1}{2\pi r_0}\int_{\partial D_{z_0,r_0}} G(\zeta)|d\zeta| = \int_0^{r_0} \frac{w(J \cap \{|z - z_0| \le s\})}{s}\, ds. \tag{2.5}$$

Using (2.1), (2.4) and (2.5) we obtain the assertion of Proposition 2.4. □

Proof of Proposition 2.5. It follows immediately from Proposition 2.4 and the Hölder continuity of the potential ψ_w. □

In what follows we will use the notation

$$\mathcal{G}(z) = \frac{G(fz)}{G(z)}, \quad z \in \bigcup_{i=1}^{d} U_i. \tag{2.6}$$

Lemma 2.6. Let us consider two topological discs U_1, U, such that the closure $\bar{U}_1 \subset U$, and a conformal isomorphism $g: \ U_1 \to U$. Let ψ be a Hölder function in U_1. Then there exists a unique (up to an additive constant) Hölder solution of cohomology equation:

$$\gamma(gz) - \gamma(z) = \psi(z), \ z \in U_1. \tag{2.7}$$

Proof. Let z_0 be a fixed point of g. Then the function

$$\gamma(z) \stackrel{def}{=} \sum_{n=1}^{\infty} [\psi(g^{-n}z) - \psi(z_0)]$$

will give us a solution of (2.7) provided that the series converges. The series is convergent because the Hölder property of ψ ensures that $\psi(g^{-n}z) - \psi(z_0)$ exponentially decrease as $n \to \infty$. It is clear now that γ is Hölder continuous too.

In order to prove uniqueness up to a constant let us consider the cohomology equation $\gamma(gz) - \gamma(z) = 0$. It follows that γ is constant along the orbits of g^{-1}. These orbits accumulate on z_0 and hence $\gamma(z) \equiv \gamma(z_0)$. □

3. Factorization of the Green function

3.1. In what follows we will make use of two simple technical lemmas. Let $\tilde{U}$ be a closed topological disc, such that $\bigcup_{i=1}^{d} U_i \subset \tilde{U} \subset U$. Recall that $\mathcal{G}(z)$ (see (2.6)) is Hölder continuous on $\bigcup_{i=1}^{d} U_i$.

Lemma 3.1. There exist constants $C \in (0, \infty)$, $\epsilon > 0$ such that for any integer n and any $i = 1, ..., d$

$$|\mathcal{G}(f_i^{-n} z') - \mathcal{G}(f_i^{-n} z'')| \leq C|z' - z''|^{\epsilon} \tag{3.1}$$

for any $z', z'' \in \tilde{U}$. □

Proof. Since f_i^{-n} is a normal family of functions, one can find $C = C(\tilde{U})$ independent of n, and such that

$$|f_i^{-n} z' - f_i^{-n} z''| \leq C|z' - z''|, \quad \forall n, \ i = 1, ..., d.$$

Now the lemma follows from Proposition 2.5 and this fact. □

Lemma 3.2. Let $\sum \epsilon_n(x)$ be a series of Hölder continuous function (on a metric space) with uniformly bounded Hölder norms. Also let us suppose that $||\epsilon_n||_\infty \leq Cq^n$, $q \in (0, 1)$. Then the sum of this series is Hölder continuous with, probably, worse exponent). □

Proof. Let $S(x) = \sum \epsilon_n(x)$. Then

$$|S(x') - S(x'')| \leq Cn|x' - x''|^{\epsilon} + \sum_{m=n+1}^{\infty} (|\epsilon_m(x')| + |\epsilon_m(x'')|)$$
$$\leq Cn|x' - x''|^{\epsilon} + 2Cq^{n+1}(1-q)^{-1}.$$

The choice of $n = \left[\log \frac{1}{|x' - x''|}\right]$ finishes the proof. Note that the Hölder exponent for S is worse than those for ϵ_n. □

3.2. Now let us assume that harmonic measure w and balanced measure m are non-singular. Then the ergodicity of the Gibbs measures implies that μ coincides with m. So,

$$\phi_\mu(x) \equiv \phi_m(x) = -\log d, \ x \in J,$$

and hence

$$-\phi_w(x) = \log d + \gamma(fx) - \gamma(x), \ x \in J, \tag{3.2}$$

where $\gamma = \log \frac{d\mu}{dw}$ is a Hölder function.

Now let us fix $i = 1, ..., d$ and let p_i be the fixed point of $f_i : U_i \to U$. Fixing an arbitrary point $z \in \tilde{U}$ we consider its backward orbit $\{z_{-n}\}_{n=1}^{\infty}$ converging to p_i; $z_{-n} = f_i^{-n}(z)$. It follows from Proposition 2.1, 2.5 and (3.2) that

$$\begin{aligned}\log \tfrac{G(z_{-(n-1)})}{G(z_{-n})} &= \log \mathcal{G}_w(p_i) + O(q^n) \\ &= -\phi_w(p_i) + 0(q^h) = \log d + 0(q^n).\end{aligned} \tag{3.3}$$

Hence, the series

$$\sum_{n=1}^{\infty} \log \frac{G(z_{-(n-1)})}{dG(z_{-n})} = \lim_{n\to\infty} \log \frac{G(z)}{d^n G(z_{-n})} \tag{3.4}$$

is convergent. Lemma 3.1, 3.2 and the estimate (3.3) show that its limit is a Hölder function on $\tilde{U}$.

So, we can consider the following harmonic function defined on $\tilde{U}$

$$\tau_i(z) = \lim_{n\to\infty} d^n G(f_i^{-n} z), \ z \in \tilde{U}. \tag{3.5}$$

Clearly, τ_i satisfies the functional equation

$$\tau_i(fz) = d\tau_i(z), \ z \in \tilde{U}_i = f_i^{-1}\tilde{U}. \tag{3.6}$$

Let us divide $G(z)$ by $\tau_i(z)$ in $\tilde{U}$:

$$G(z) = \tau_i(z)e^{\gamma_i(z)}. \tag{3.7}$$

Then γ_i is given by the series (3.4), in particular, γ_i is Hölder continuous. On $\tilde{U}_i$ this function γ_i satisfies the cohomology equation:

$$\log \mathcal{G}(z) = \log d + \gamma_i(fz) - \gamma_i(z).$$

Restricting this onto $J \cap \tilde{U}_i$ and comparing with (3.2) we conclude by the uniqueness part of Lemma 2.6 that $\gamma_i(x) = \gamma(x) + c_i,\ x \in J$.

So we can normalize γ_i and τ_i saving (3.6), (3.7) and the harmonicity of τ_i in such a way that

$$\gamma_i(x) \equiv \gamma(x),\ x \in J,\ i = 1, ..., d. \tag{3.8}$$

4. Removing the singularities of $\tau_i - \tau_j$

Now let us show that actually all functions γ_i coincide not only on $J(f)$ but on the whole domain $\tilde{U}$. First we note that

$$|\tau_i(z) - \tau_j(z)| = G(z)|e^{\gamma_i(z)} - e^{\gamma_j(z)}| \leq CG(z)d(z, J)^\eta \tag{4.1}$$

Functions τ_i, τ_j are harmonic and positive in $U \setminus J$ and subharmonic and non-negative on U. Our first goal is to prove that their Riesz measures on J are the same. For this purpose we need the following two results.

Lemma 4.1. Given a Cantor repellor $J(f)$ there exist constants c_1, c_2, c_3, c_4 such that

1) $c_i \mathrm{dist}(\partial Q_X, J) \leq \mathrm{length}\,(\partial Q_X) \leq c_2 \mathrm{dist}(\partial Q_X, J(f))$,

2) $\forall z \in \partial Q_X,\ c_3 w(X) \leq G(z) \leq c_4 w(X)$..

Proof. First assertion is an immediate consequence of the Koebe distortion theorem. The second follows from Proposition 2.4 and estimate (2.5). □

Lemma 4.2. Let $J(f)$ be a Cantor repellor, V be its neighborhood and u_1, u_2 be two harmonic in $V \setminus J$ function, which are subharmonic in V. Suppose also that

$$u_1(z) - u_2(z) = o(G(z)), \quad z \to J. \tag{4.2}$$

Then $u_1 - u_2$ is harmonic in V.

Proof. Using the Riesz representation for functions u_1 and u_2 subharmonic in V we can write

$$u_1(z) = h_1(z) - \int_J \log \tfrac{1}{|z-\xi|} d\mu_1(\xi) = h_1 - U^{\mu_1},$$

$$u_2(z) = h_2(z) - \int_J \log \tfrac{1}{|z-\xi|} d\mu_2(\xi) = h_2 - U^{\mu_2},$$

where h_1, h_2 are harmonic in V. Now let us consider $u(z) = u_1(z) - u_2(z)$ and $\Phi(z) = \partial u(z)$ $\left(\partial \stackrel{def}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right)$. Then

$$\Phi(z) = A(z) + c\int_J \frac{d\mu_1(\xi)}{z-\xi} - c\int \frac{d\mu_2(\xi)}{z-\xi}, \tag{4.3}$$

where $A \stackrel{def}{=} \partial(h_1 - h_2)$ is analytic in V and c is a constant. Now let us use the Cauchy formula in $V \setminus \bigcup_{|X|=n} Q_X$:

$$\Phi(z) = \frac{1}{2\pi i}\int_{\partial V} \frac{\Phi(\xi)}{\xi - z} d\xi + \sum_{|X|=n} \frac{1}{2\pi i}\int_{\partial Q_X} \frac{\Phi(\xi) d\xi}{\xi - z} = I + \sum.$$

As u is harmonic in $V \setminus J$ a trivial estimate

$$|\Phi(z)| \leq |\nabla u(z)| \leq o(d(z)^{-1}G(z))$$

follows from (4.2). Combining this with Lemma 4.1 we get the estimate (with $\epsilon_n \to 0$).

$$\begin{aligned} |\Sigma| &\leq C\epsilon_n \sum_{|X|=n} \int_{\partial Q_X} G(z)d(z)^{-1}|dz| \\ &\leq C\epsilon_n \sum_{|X|=n} w(X) = C\epsilon_n \to 0. \end{aligned}$$

So $\Sigma = 0$ and Φ can be extended as a holomorphic function in the whole V. In particular for almost every square S lying in V

$$\int_{\partial S} \Phi(z)dz = 0,$$

and, thus, $\mu_1 = \mu_2$ (see (4.3)). We conclude that

$$u = u_1 - u_2 = h_1 - h_2$$

and the lemma is proved. □

Estimate (4.1) and Lemma 4.2 show that all functions $\tau_i - \tau_j$, $i, j = 1, ..., d$, are harmonic in U. Suppose that $\tau_i - \tau_j \not\equiv 0$ for a pair i, j. Let us denote $Z_{ij} = \{z \in U : \tau_i(z) - \tau_j(z) = 0\}$. We have just proved that $\tau_i - \tau_j$ is harmonic and so real analytic in U. Thus, Z_{ij} consists locally of finite union of real analytic curves. But

$$\tau_i|J = \tau_j|J = G|J \equiv 0,$$

and so $J \subset Z_{ij}$.

If J is not contained in a finite union of real analytic curves (4.4) then we have already come to a contradiction and $\tau_j \equiv \tau_i$. It is clear that (4.4) is true as a rule but now we need to cope with the opposite case: J is covered by a finite number of real analytic curves $\Gamma_1, ..., \Gamma_m$. Without loss of generality we may assume that each of these curves contains infinitely many points of J. Now it is clear that

$$f_i^{-1}(\Gamma_s) \subset \bigcup_{t=1}^{m} \Gamma_t, \quad i = 1, ..., d; \; s = 1, ..., m. \tag{4.5}$$

These curves may intersect only in finite number of points inside $\tilde{U}$ (namely, only in points where $\nabla(\tau_i - \tau_j) = 0$). Now (4.5) shows that they do not intersect at all. Using (4.5) again we see that $m = 1$ - we have only one real analytic curve, containing J. We call this curve Γ and note that $f_i(\Gamma)$ intersects Γ in infinitely many points, so

$$f_i \Gamma = \Gamma, \quad i = 1, ..., d. \tag{4.6}$$

Let U_Γ be a thin neighborhood of Γ in which the reflection with respect to Γ, $z \longrightarrow z^*$, is defined. Then (4.6) shows that

$$f_i z^* = (f_i z)^*, \quad z \in f_i^{-1} U_\Gamma, \quad i = 1, ..., d. \tag{4.7}$$

Let us consider $\hat{\tau}_i(z) \overset{def}{=} \tau_i(z) + \tau_i(z^*)$, $z \in U_\Gamma$. The analog of (3.6) holds:

$$\hat{\tau}_i(fz) = d\hat{\tau}_i(z), \quad z \in f_n^{-1} U_\Gamma, \tag{4.8}$$

which is manifest from (4.7) and (3.6). Each $\hat{\tau}_i$ is harmonic and $\star$-symmetric in U_Γ, so

$$\frac{\partial \hat{\tau}_i}{\partial n}(z) \equiv 0, \quad z \in \Gamma, \quad i = 1, ..., d, \tag{4.9}$$

where n is the unit normal to Γ. On the other hand

$$\hat{\tau}_i(z) - \hat{\tau}_j(z) = 2(\tau_i(z) - \tau_j(z)) = 0, \quad z \in \Gamma, \tag{4.10}$$

as Γ was defined as a subset of Z_{ij}. Now (4.9) and (4.10) imply that an analytic function $\partial(\hat{\tau}_i - \hat{\tau}_j)$ vanishes on Γ, and thus, vanishes on U_Γ. The same is evidently true for $\bar{\partial}(\hat{\tau}_i - \hat{\tau}_j)$, which means that $\hat{\tau}_i - \hat{\tau}_j \equiv \text{const}$ in U_Γ. Taking (4.10) into account we have $\hat{\tau}_i \equiv \hat{\tau}_j$ in U_Γ.

The moral of our consideration is whether (4.4) holds or does not hold we can find a neighborhood V of the set J and a positive harmonic function τ on $V \setminus J$ such that

$$\tau(fz) = d\tau(z), \quad z \in f^{-1}V. \tag{4.11}$$

It is worthwhile to emphasize the difference between (3.6) and (4.11). In (4.11) the same function τ serves for all branches $f_n^{-1}, ..., f_d^{-1}$ of f^{-1}. Note, that either $\tau = \tau_1 = \cdots = \tau_d$ or $\tau = \hat{\tau}_1 = \cdots = \hat{\tau}_d$, the first option taking place e.g. if $\dim J(f) > 1$.

5. Polynomial-like mappings: use of the circle model

5.1. In this section we will prove the Theorem for polynomial-like maps. *A polynomial-like map of degree* d is a triple (f, U, W), where U and W are topological discs with piecewise smooth boundaries, with U relatively compact in W, and $f : U \to W$ is a complex analytic map, proper of degree d (that is, a branched covering of degree d). If $f : U \to W$ is polynomial-like one can consider a *filled-in Julia set*

$$K(f) = \bigcap_{n \geq 0} f^{-n}(U),$$

the set of $z \in U$ such that f^n is defined and belongs to U for all $n \in N$. *The Julia set* of f is $J(f) = \partial K(f)$.

The following well-known statement gives a necessary and sufficient condition for $K(f)$ to be an expanding Cantor repellor (see [F], [DH]).

Proposition 5.1. The set $K(f)$ is connected if and only if all the critical points of f belong to $K(f)$. If none of the critical points belongs to $K(f)$ then $K(f) = J(f)$ is a Cantor set. □

5.2. The circular model of Douady and Hubbard associates to a polynomial-like map f an expanding real analytic endomorphism F of the unit circle $\mathbf{T}$ of degree d (defined up to analytic conjugacy). If f is a polynomial, then $F : z \mapsto z^d$. Polynomial-like maps f_1 and f_2 are called *externally equivalent* if corresponding circle endomorphisms F_1 and F_2 are analytically conjugate.

The Douady - Hubbard Theorem [DH]. Let f be a polynomial-like mapping of degree d. Then f is conformally equivalent to a polynomial if and only if it is externally equivalent to $z \mapsto z^d$.

Now let us construct the external map F corresponding to a polynomial-like map $f : U \to W$. Let us consider a narrow annulus I_0 with analytic boundary whose outer boundary coincides with ∂U. Consider the annulus $X_0 = (W \setminus U) \cup I_0$. The annulus I_0 is adjacent to its inner boundary, while the annulus $E_0 \equiv f I_0$ is adjacent to its outer boundary.

Consider now a sequence of annuli X_n of moduli $d^{-n} \text{mod} X_0$, and a chain of d-sheeted coverings $\rho_n : X_n \to X_{n-1}$. Let $\pi_n : X_n \to X_0$, I_n and E_n be the π_n-preimages of I_0 and E_0 correspondingly. Then there is the following commutative diagram

$$\begin{array}{ccc} E_n & \xrightarrow{i_n} & I_{n-1} \\ \pi_n \downarrow & & \downarrow \pi_{n-1} \\ E_0 & \xleftarrow[f]{} & I_0 \end{array}$$

with an analytic diffeomorphism i_n between "outer" annulus E_n and "inner" annulus I_{n-1}. By means of these diffeomorphisms we can glue all X_n together, and construct a doubly-connected Riemann surface A_+ (the notation will become clear in a little while).

Consider also a smaller Riemann surface $A'_+ \subset A_+$ obtaining by gluing together the annuli X_n for $1 \leq n < \infty$. The coverings $\rho_n : X_n \to X_{n-1}$ correctly induce a d-sheeted covering $F : A'_+ \to A_+$.

Let us show now that $\mathrm{mod}(A_+) < \infty$. Indeed, otherwise A_+ and A'_+ are conformal punctured disks, and F allows analytical continuation to the puncture (say, 0) as $z \mapsto az^d + ..., d > 1$. It contradicts the fact that all orbits of F escape A'_+.

Hence, A_+ has a conformal representation as an annulus $\{z : 1 < |z| < R\}$, and in this representation A'_+ becomes a subannulus whose inner boundary coincides with $\mathbf{T}$ while the outer is an analytic Jordan curve in A_+.

The covering $F : A'_+ \to A_+$ must be continuous up to the circle $\mathbf{T}$, and by the Schwarz reflection principle can be analytically continued to the covering of symmetric annuli $A' \to A$.

So, we have a real analytic map F_f of the circle. Standard considerations with Poincaré metric shows that it is expanding on $\mathbf{T}$. The construction is completed.

5.3. Let us assume for simplicity that $\deg f = d = 2$, and that the critical point c of

f lies in $U \setminus \overline{f^{-1}U}$. Then $f^{-1}U$ consists of two components U_1 and U_2, and $f : U_i \to U$ is a conformal isomorphism.

Provided harmonic and maximal measures are not mutually singular, we have constructed in §4 a harmonic in $U \setminus J$ function τ with the property

$$\tau(fz) = 2\tau(z). \tag{5.1}$$

for $z \in U_1 \cup U_2$. We wish to extend τ as positive harmonic on $W \setminus J$ satisfying (5.1) for all $z \in U$. Let γ be a real analytic cut of $W \setminus U$ from $f(c)$ to ∂W. Let $O \stackrel{def}{=} W \setminus \gamma$. Then in O there are two univalent branches of f^{-1} : f_1^{-1}, f_2^{-1}.

Put $u_i = 2\tau \circ f_i^{-1}$, $i = 1, 2$. These are two positive harmonic functions on $O \setminus J$. By (5.1) $u_1 = \tau = u_2$ on U. So $u_1 \equiv u_2$ on O, which means that going around the point $f(c)$ does not change the value of $u = 2\tau \circ f_1^{-1}$. Thus, u is positive and harmonic in $W \setminus J$, and satisfies (5.1) in the whole domain U.

Now let us construct a harmonic solution of the equation

$$u(Fz) = 2u(z). \tag{5.2}$$

in the annulus A_+. By the construction, the annulus X_0 is naturally embedded into A_+. Set $u|X_0 = \tau$. Then (5.2) holds in the annulus I_0. Using it we can pull u back and spread it over the whole annulus A_+.

By (5.2) the Riesz measure ν of u, $\nu = \Delta u$, is equidistributed with respect to F and so it is the maximal measure of F. Let us show that it is absolutely continuous with respect to Lebesgue measure σ. To this end let us consider also the odd extension of u onto the the annulus A:

$$v(z) \stackrel{def}{=} \begin{cases} u(z), & z \in A_+; \\ -u\left(\frac{1}{\bar{z}}\right), & z \in A \setminus A_+. \end{cases}$$

This function is harmonic in A. Hence, the normal derivative $\partial u/\partial n = \partial v/\partial n$ is real analytic on $\mathbf{T}$. Now the Green formula yields that

$$\nu \equiv \Delta u = \frac{\partial u}{\partial n} d\sigma.$$

So, ν is absolutely continuous with respect to the Lebesgue measure, and application of Lemma 1.1 completes the second proof of the Theorem in the case of polynomial-like maps.

6. Cantor repellors generating by symmetric maps.

In this section we will prove the Theorem for symmetric Cantor repellors. To make the exposition easier we restrict ourselves to the case of degree 2.

6.1. Renormalization. Consider a symmetric degree two map $f : \bigcup_{i=1}^{2} U_i \to U$. Let τ be the subharmonic function in a connected symmetric neighborhood $V \subset U$ of $J(f)$ satisfying (4.11). Take a central gap $I \equiv I^0$ in our Cantor set, and let I_k^n, $k = 1, ...2^n$ be its n-fold preimages which we refer to as *gaps of rank* n. Each gap I_k^n contains a unique τ-critical point c_k^n "of rank n", $c \equiv c^0$, and τ has no other critical points (see [W]). So, the map f is well-defined at all τ-critical points of rank $n > 0$. Together with equation (4.11) this yields that

$$\tau(c_k^n) = b/2^n, \quad b \equiv \tau(c). \tag{6.1}$$

Select $t \in (b/2^n, b/2^{n-1})$ to be so small that the set $Y^t = \{z \in V : \tau(z) < t\}$ is compactly contained in V. Then one can see from the combinatorics of the critical points that Y^t consists of 2^n disjoint topological discs Y_i^t with analytic boundaries such that $Y_i^t \cap J(f)$ is a dynamical cylinder of $J(f)$ of rank n. Let us consider a component $W \equiv Y_i^t$ of Y^t, and two components W_1 and W_2 of $Y^{t/2}$ contained in W.

Let f_0^{-n} be the branch of the inverse function which maps $J(f)$ into W. Define topological disks N_i, $i = 1, 2$ as the components of $f^{-n}U_i \cap W_i$ containing $W_i \cap J(f)$. Now let us define on $N_1 \cup N_2$ a renormalization of our map f in the following way:

$$g|N_i = f_0^{-n} \circ f^{n+1}|N_i.$$

Clearly, g is a symmetric map generating an expanding Cantor repellor $J(g) = J(f) \cap W$. This map is actually conformally conjugate to f by the conjugacy $f^{n-1} : N_1 \cup N_2 \to U_1 \cup U_2$. So, we can replace f by its renormalization g. The advantage which we have gained is that now the harmonic function τ is well-defined on the topological disk $W \supset J(g)$ such that $\tau|\partial W \equiv const$.

6.2. Now we are going to carry out a local conformal change of variable which turns τ into the Green function of the complement of $J(g)$. Let us normalize τ in the following way:

$$\int_{\partial W} \frac{\partial \tau}{\partial n} ds = 2\pi. \qquad (6.2)$$

Consider a harmonic conjugate τ^* for τ. It follows from (6.2) that the analytic function

$$\phi(z) = \exp(\tau(z) + i\tau^*(z))$$

is single-valued in a narrow annulus A adjacent to ∂W from inside. Moreover, if we select a τ-level line as the inner boundary of A then ϕ conformally maps A onto a round annulus $\{z : r < |z| < R\}$. It follows that we can continue ϕ to a diffeomorphism $\mathbb{C}\setminus W \to \{z : |z| \geq r\}$ which is identical near ∞ (we preserve the same notation for the continuation). Consider also a smooth continuation $\log|\phi(z)|$ of τ equal to $\log|z|$ near ∞ (which will be also denoted by τ).

Let σ be the standard conformal structure on the plane $\mathbb{C}$. Let us consider now a conformal structure $\mu = \phi^*(\sigma)$ (μ is standard in W). By the Smooth Riemann Mapping Theorem, there is a diffeomorphism $\psi : \mathbb{C} \to \mathbb{C}$ normalized by the condition

$$\psi(z) \sim z \text{ near } \infty, \tag{6.3}$$

and such that $\psi_*(\mu) = \sigma$. Since ψ is analytic near $J(g)$, the map $p = \psi \circ g \circ \psi^{-1}$ is conformally conjugate to g.

Furthermore, the function $\phi \circ \psi^{-1}$ is a (multi-valued) analytic function on $\mathbb{C}\setminus J(p)$ such that $\log|\phi \circ \psi^{-1}|$ is single-valued. Hence the function $\kappa = \tau \circ \psi^{-1}$ is a single-valued harmonic function in $\mathbb{C}\setminus J(p)$. Because of the normalization (6.3), $\kappa \sim \log|z|$ near ∞, and hence it is the Green function of $\mathbb{C}\setminus J(p)$.

6.3. Analytic continuation to a polynomial. To make notations easier let us pretend now that $p = f$ is our original endomorphism, and $\tau = \kappa$ is the Green function of $\mathbb{C}\setminus J(f)$. So, τ is globally defined, and we can use the functional equation $\tau(pz) = 2\tau(z)$ in order to continue f to a polynomial.

To this end consider the multi-valued analytic function ϕ associated with τ, and satisfying the functional equation $\phi(pz) = \phi(z)^2$ near $J(f)$. This equation gives us a way to continue f as $\phi^{-1} \circ \phi(z)^2$.

Let $Q \subset W$ and $Q_i \subset W_i$ be the smallest intervals containing $J(f)$ and $J(f) \cap W_i$ correspondingly. The ϕ gives a univalent map of a slitted topological disk $W \setminus Q$ onto a "hedgehog domain" H lying in the annulus $\{z : 1 < |z| < e^R\}$ (see Figure 2). By (6.1) this hedgehog has 2^{n+1} needles of length $b/2^n$ and arguments $(\tau^*(c) + \pi i)/2^n$, $i = 0, 1, ..., 2^{n+1} - 1$ (compare with Levin and Sodin [LS] who introduced the hedgehog construction to dynamics).

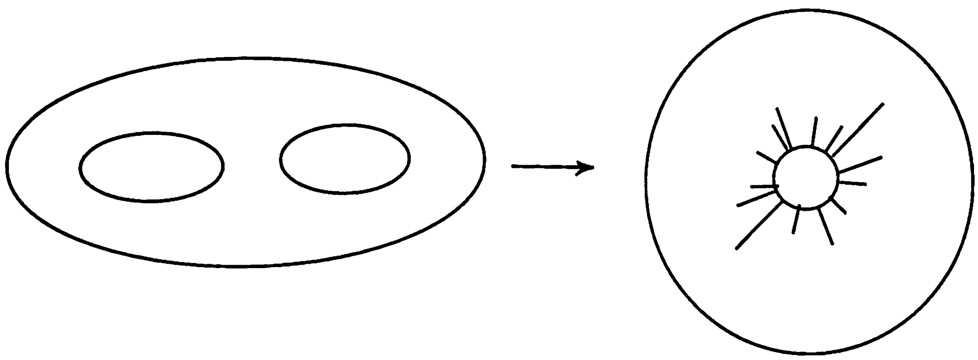

Figure 2.

But looking at (6.1) again we see that the map $\phi(z)^2$ univalently maps the slitted domain $W_i \setminus Q_i$ onto the *same* hedgehog domain. Hence $\phi^{-1} \circ \phi(z)^2$ univalently maps $W_i \setminus Q_i$ onto $W \setminus Q$. But near Q_i this map coincides with f. So, we have analytic continuation of f to univalent maps $W_i \to W$, $i = 1, 2$.

Further, the ϕ^2 gives a double covering of $\bar{\mathbb{C}} \setminus (W_1 \cup W_2)$ onto the round disk $\{z : |z| \geq e^R\}$, while ϕ maps univalently $\bar{\mathbb{C}} \setminus W$ onto the same disk. Hence, $\phi^{-1} \circ \phi(z)^2$ is a double covering of $\bar{\mathbb{C}} \setminus (W_1 \cup W_2)$ onto $\bar{\mathbb{C}} \setminus W$.

Finally, we see that the $\phi^{-1} \circ \phi(z)^2$ gives an analytic continuation of f to a double covering of the whole complex plane, and that is what we need.

7. An example of a non-polynomial repellor on which harmonic measure is balanced.

Let us start with the Zhukovskii map $\Phi : z \mapsto \frac{1}{2}(z + 1/z)$ which univalently maps the complement of the unit disk onto the complement of the interval [-1,1]. Consider a map $h^0(z) = \Phi(i\Phi^{-1}(z/i))$ which univalently maps $\mathbb{C}\setminus[-i, i]$ onto $\mathbb{C}\setminus[-1, 1]$. Globally h^0 is well-defined on the two-sheeted Riemann surface over $\mathbb{C}$ branched at $\{+i, -i\}$. The h^0 maps these branched points at the same point 0. Moreover h^0 has two critical points lying over 0 and mapped into the critical values 1 and -1.

Consider now two open topological disks with smooth boundaries W_1 and W_2 in $\mathbb{C}$ containing $[-i, i]$ and [-1,1], and such that

$$h^0 : W_1\setminus[-i, i] \to W_2\setminus[-1, 1].$$

Let us rescale these domains by real affine maps $z \mapsto z/\lambda + a_i$ with a big $\lambda >> 0$ in such a way that the closures of new domains U_i lie in $W_2\setminus[-1, 1]$ on opposite sides of the slit [-1,1] (see Figure 3). Denote by $I_i \subset U_i$ the rescaled intervals $[-i, i]$ and [-1,1]. A rescaling of h_0 gives us an algebraic function h mapping $U_1\setminus I_1$ onto $U_2\setminus I_2$.

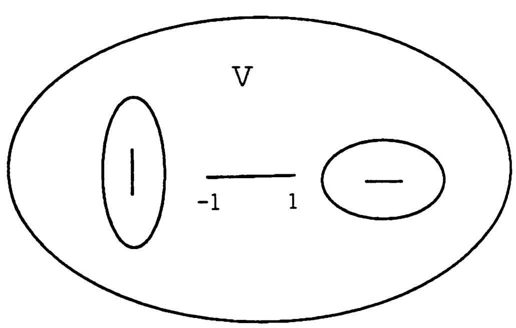

Figure 3.

Set $V \equiv W_2$, and define a map $f : (U_1 \setminus I_1) \cup U_2 \to V$ in the following way. The restriction $f|U_2$ is just an affine map inverse to the rescaling map, while $f|(U_1 \setminus I_1) = (f|U_2) \circ h$. Note that f is well-defined on the banks of the slit I_1, maps it onto [-1,1], and carries the branched points ∂I_1 at the same point 0.

In order to see that f generates a Cantor repellor in the sense of §1, let us consider an open topological disk $\tilde{V} \subset V \setminus [-1,1]$ containing $\mathrm{cl}\, U_1 \cup \mathrm{cl}\, U_2$. Let $\tilde{U}_i \subset U_i \setminus I_i$ be the preimages of $\tilde{V}$. Then $f : \tilde{U}_1 \cup \tilde{U}_2 \to \tilde{V}$ is a weakly polynomial-like map generating a Cantor repellor J.

Let us show now that f is quasi-conformally conjugate to a map g analytic in the whole complex plane with one slit, and looking like $z \mapsto z^2$ in a neighborhood of ∞. To this end consider a topological disk $D \supset \mathrm{cl}\, V$ with a smooth boundary, and continue f smoothly to V in such a way that f is a double covering of $V \setminus (I_1 \cup I_2)$ over $D \setminus [-1,1]$, with the critical point at 0, and such that $f(x) = f(-x)$ for $x \in [-1,1]$. Then stick to ∂V the map $z \mapsto z^2$ as Douady and Hubbard did in [DH]. More specifically, take an $R > 1$ and a diffeomorphism $\psi : \mathbb{C} \setminus V \to \{z : |z| > R\}$ such that ψ is identical near ∞ and $\psi(z)^2 = \psi(fz)$ for $z \in \partial V$. Continue now f to $\mathbb{C} \setminus V$ as $\psi^{-1} \circ \psi(z)^2$. For the reason which will become clear later let us select the ψ in such a way that

$$\log|\psi(f0)| > \log|\psi(f1)| > \frac{1}{2}\log|\psi(f0)|. \tag{7.1}$$

We have constructed a smooth map f on the slitted sphere $S^2 \setminus I_1$ which is analytic in a neighborhoods of $J(f)$ and ∞, and coincides with $z \mapsto z^2$ near ∞. Moreover, f carries corresponding points on opposite banks of the slit I_1 into 0-symmetric points of [-1,1], and hence $f(fz)$ is a single-valued function in a neighborhood of I_1.

As in [DH], consider now an f-invariant conformal structure in $\bar{\mathbb{C}}\setminus D$, and pull it back by iterates of f. We will obtain an invariant measurable conformal structure μ on $\mathbb{C}$ with bounded dilatation (since f is conformal in a neighborhood of J). By the Measurable Riemann Theorem, find a quasi-conformal homeomorphism ψ, conformal near ∞ and tangent to id at ∞ which push μ to a standard conformal structure. It conjugates f to a map g analytic on the slitted sphere (the slit γ comes from I_1). Moreover, $g(gz)$ is single-valued in a neighborhood of the slit, and $g(z) \sim z^2$ at ∞.

Let us show now that harmonic measure on $J(g)$ is balanced. To this end consider the Brolin function

$$G(z) = \lim_{n\to\infty} \frac{1}{2^n} \log |g^n z|. \tag{7.2}$$

It is well-defined and harmonic in $\mathbb{C}\setminus J(g)$ since the functions $g^n z$ are eventually single-valued in a neighborhood of any point $z \in \mathbb{C}\setminus J(g)$ (the limit exists since f looks like $z \mapsto z^2$ near ∞). Moreover, clearly G is non-negative, has a $\log |z|$-singularity at ∞, and satisfies the functional equation

$$G(gz) = 2G(z). \tag{7.3}$$

Hence, $G(z) \to 0$ as $z \to J(p)$ which proves that G is the Green function of $\mathbb{C}\setminus J(g)$ with the pole at ∞. Taking the Laplacian of the above functional equation we conclude that harmonic measure on $J(g)$ is balanced.

What remains to show is that the map g is not locally conformally conjugate to a polynomial. To be more definite, let us carry all the above construction so that f preserves the real line (*warning*: however, f is not symmetric in the sense of §1 since there is no domain of its definition consisting of two $\mathbf{R}$-symmetric topological disks.) Assume that ψ locally conjugate f to a polynomial p. Consider the Green function τ of $\mathbf{C}\setminus J(p)$, and pull it into a neighborhood of $J(g)$, $G_1 = \tau \circ \psi$. Since both G and G_1 satisfy equation (7.3), $\Delta G = \Delta G_1$ =balanced measure on $J(g)$. Hence, $G = G_1$ (see Appendix).

In particular, G and G_1 have the same critical values in Z. Now let us see at the critical values of G. Let π be the conjugacy between f and g, and q_1 and q_2 be the centers of the intervals I_1 and I_2. By (7.3) the critical points of G coincide with the preimages of the critical points of g, that is with $\pi(0)$ and the preimages of $\pi(q_1)$ and $\pi(q_2)$. But

$$G(\pi(q_1)) = \frac{1}{2}G(\pi(1)) = \frac{1}{4}\log|\psi(f1)|,$$
$$G(\pi(q_2)) = \frac{1}{2}G(\pi(0)) = \frac{1}{4}\log|\psi(f0)|.$$

Taking preimages of $\pi(q_i)$ lying in the neighborhood Z, we conclude from here, estimates (7.1) and equation (7.3) that G has in Z critical values whose ratio is not an integer power of 2. On the other hand, ratio of any two critical values of G_1 is an integer power of 2. This contradiction completes our construction.

8. An open question

The positive answer to the following question would immediately imply our Theorem in the polynomial-like case (and a number of other consequences):

Does harmonic measure class on a polynomial-like Cantor repellor go to Lebesgue measure class on the circle when we apply the Douady-Hubbard circular model?

Is there a natural circular model for every expanding Cantor repellor?

9. Appendix: harmonic measure dictionary.

9.1. Any subharmonic function on an open subset $\mathcal{O}$ of $\bar{\mathbb{C}}$ generates positive σ–finite measure $\mu_u = \Delta u$ called its *Riesz measure* (here the Laplasian Δ is understood in the sense of distributions). This measure behaves naturally under conformal changes of variable (so, actually it can be defined on any Riemann surface).

In what follows we assume that $K \subset \mathbb{C}$ is *filled-in*, that is, $\mathbb{C} \setminus K$ is connected. If K is sufficiently thick (of *positive capacity*) then one can construct the *Green function* G, that is, a non-negative subharmonic function on $\mathbb{C}$, harmonic on $\mathbb{C} \setminus K$, vanishing on K, and having the asymptotics $G(z) \sim \log |z|$ near ∞. Its Riesz measure is called *harmonic measure* on K. Knowing harmonic measure ω, one can restore the Green function as its logarithmic potential:

$$G(z) = c + \int \log |z - \zeta| d\omega(\zeta)$$

with $c = 1/cap(K)$.

Now let u be any non-negative subharmonic functions defined on a neighborhood $N = N_u$ of K, harmonic on N and vanishing on K. Let us denote this class of functions by $\mathcal{U}_K$.

Proposition 9.1. The Riesz measure μ_u of a function $u \in \mathcal{U}_K$ is equivalent to harmonic measure w with the density bounded away from zero and ∞.

Proof. There are positive constants A and B such that $A \cdot G \leq u \leq B \cdot G$ on the boundary of the neighborhood N. As the same estimate trivially holds on K, the

Maximum Principle extends it to the whole neighborhood N. It follows that the function $v = u - A \cdot G$ is subharmonic on N. Hence, Δv is positive measure, so that $\mu_u \leq A\omega$. Similarly, $\mu_u \geq \omega$. □

Let us call the *harmonic measure class* on K the class of measures equivalent to harmonic measure on K. A homeomorphism between two compact sets $\phi : K \to K'$ will be called conformal if it allows a conformal continuation into a neighborhood of K. Now the natural property of the Riesz measures yields

Corollary. The harmonic measure class is preserved by conformal homeomorphisms.

9.2. On formula (2.5). This formula is valid in a more general setting. Let u be a subharmonic function with the Riesz mass μ. Let $n(t)$ be the mass in $|z| \leq t$. We define

$$N(r) = \int_0^r (n(t)/t)dt.$$

The following formula generalizes (2.5). And the usual Jensen formula is easily recognizable in it (see e.g. [HK]) :

$$\frac{1}{2\pi} \int_0^{2\pi} u(r \exp i\theta)d\theta = N(r) + u(0)$$

Actually this last formula is immediate consequence of the Green's theorem applied to a pair of functions: u and $v(z) = \log(r/|z|)$.

Now we can apply this formula to get the following result.

Proposition 9.2. Let K be a regular compact on the plane and let u, v be two non-negative functions in a neighborhood O of K, such that u and v are positive and harmonic in $O \setminus K$, and vanish on K (so they are subharmonic in O).

Suppose also that their Riesz measure are equal. Then $u = v$. □

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