Quasisymmetric Conjugacies Between Unimodal Maps

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Abstract

It is shown that some topological equivalency classes of S-unimodal maps are equal to quasisymmetric conjugacy classes. This includes some infinitely renormalizable polynomials of unbounded type.

1 Introduction

1.1 Quasisymmetric classification of unimodal maps

Unimodal maps. We discuss unimodal maps of the interval. A standard example is the quadratic family

$$x \to ax(1-x)$$

where a is a parameter from the interval $(0, 4]$. Other important classes are maps extendable in an analytic quadratic-like fashion in the sense of [7] and the S-unimodal class where no analytic extension is postulated, instead the map is assumed to have negative Schwarzian derivative. In the following discussion, unless otherwise indicated, we mean maps from the union of these two classes.

Unimodal maps exhibit impressively rich dynamics. The framework for studying them was laid by [18]. There, topological dynamics of unimodal was described in terms of the *kneading sequence*. However, the basic idea the kneading invariant can be traced back to an earlier paper [17].

In some cases, the dynamics of unimodal maps has been well understood. This includes maps with periodic or preperiodic kneading sequences for which the analytic cases were studied in [16] and [6]. In this paper, we confine ourselves to other, or aperiodic, invariants.

Quasisymmetric conjugacies. By the work of [9] we know that two maps with the same aperiodic kneading sequence are topologically conjugate. The conjugating homeomorphism is quasisymmetric if and only if it can be extended to a quasiconformal hoemomorphism of the plane. By the celebrated theorem of [3], this is equivalent to the ratio

$$\frac{g(x+h) - g(x)}{g(x) - g(x-h)}$$

being uniformly bounded for all real x and h so that the relevant points are in the domain of g. Moreover, there exists a quasiconformal extension whose norm is bounded in a uniform way in terms of the supremum of this ratio (which we will call the quasisymmetric norm.)

It is known that quasisymmetric homeomorphisms are Hölder continuous, but usually not absolutely continuous.

In dynamics, the idea of studying quasiconformal (-symmetric) conjugacy classes has been introduced and proven stunningly successful by a series of works by D. Sullivan. A recent work [21] deals directly with unimodal maps and was an inspiration, as well as the starting point of this work.

Various results on quasisymmetric classification. The standing conjecture is that the quasisymmetric conjugacy classes are equal to topological conjugacy classes for aperiodic kneading invariants. This conjecture has so far been proven in three cases.

First, there is a class of infinitely renormalizable maps which was treated in [21].

Secondly, in the Misiurewicz case which means that the critical point is not recurrent the conjecture was proved by [14].

Finally, a recent result of Yoccoz should be mentioned which implies the conjecture for all non-renormalizable polynomials in the analytic polynomial-like class. This work has not yet been circulated; the reader may, however, consult [12].

What this paper contributes. We prove the conjecture for some, not all, non-renormalizable maps in the S-unimodal class. We also show a new approach to renormalizable cases. We prove the conjecture in some infinitely renormalizable cases where it is new even in the polynomial class.

Consequences of our results. A famous consequence is that is that if the conjecture is proven for any kneading sequence in the polynomial class, the corresponding component of the Mandelbrot set reduces to a point. This is proven by the pull-back construction of [21] and a deformation argument of the kind used in [16] and [20].

Another consequence concerns the existence of absolutely continuous invariant measures. The Collet-Eckmann condition (see [5]) is shown to be a topological invariant of non-renormalizable maps in our class. The proof of this remark is in Section 5.3.

The question of topological invariance of the Collet-Eckmann condition in the class of S-unimodal non-renormalizable maps was stated by J. Guckenheimer [11]. There is a related question of whether the existence of an absolutely continuous invariant measure is a topological property for S-unimodal

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maps. This, however, seems harder and at this point we can only state it as a problem.

1.2 Induced maps

Assumptions. The class of functions $\mathcal{C}$ is defined by the conditions:

Definition 1.1 *1. Each $f \in \mathcal{C}$ maps the interval $[-1, 1]$ into itself.*

2. *Functions from $\mathcal{C}$ are three times differentiable and, wherever the first derivative is nonzero, their Schwarzian derivative is non-positive.*

3. *Each function $f \in \mathcal{C}$ can be represented as $h(x^2)$ with h being a diffeomorphism.*

4. *The critical value $h(0)$ is greater than 0.*

These assumptions in particular imply that $f(-a) = f(a)$.

Definition 1.2 *Let α be an aperiodic kneading sequence. Then $\mathcal{C}_\alpha$ is defined to be the set of all maps from $\mathcal{C}$ with this kneading sequence.*

In this paper, we only deal with maps whose kneading sequence is aperiodic.

Assumption 4 implies that there exists a fixed point q for every $f \in \mathcal{C}$ with $q > 0$. We consider the induced map ϕ defined to be the first return map on the interval between q and $-q$. This interval will be called the *fundamental inducing domain* of f.

It is easy to see that the induced map consists of a number of continuous branches all of which except one are monotonic. Also, the construction is topological, by which we mean that if maps f_1 and f_2 are topologically conjugate, the same is true of their induced maps.

Definition 1.3 *Given an interval $I \subset [-1, 1]$ we define a **stopping rule** on I to be a continuous positive integer valued function defined on an open subset of I.*

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Definition 1.4 *An **induced map** of $f \in \mathcal{C}$ on an interval $I \in [-1, 1]$ is a map of the form*
$$x \to f^{s(x)}(x)$$
where $s(x)$ is a stopping rule on I and we mean that the induced map is not defined where the stopping rule is not.

So, induced maps and stopping rules are really the same thing and we will keep in mind that one always determines the other.

Definition 1.5 *A restriction of an induced map φ to a connected component of its domain will be called a **branch** of φ.*

Definition 1.6 *An **induced monotone branch** is an induced map with a constant stopping rule whose domain is an interval and which is monotone.*

Definition 1.7 *An induced monotone branch defined on an interval (a, b) is said to be ϵ-**extendable** if there is an induced monotone branch g with the same stopping rule defined on a larger interval $(c, d) \supset (a, b)$ such that the cross-ratio*
$$\frac{|g(a) - g(c)||g(b) - g(d)|}{|g(a) - g(d)||g(b) - g(c)|}$$
is more than ϵ.

In the future, we will fix a uniform value of ϵ and simply talk of extendable maps. Monotone extendable branches have bounded distortion (see [10]).

Definition 1.8 *A **critical branch** is a branch of the form $g(x^2)$ where g is a monotone branch, defined on a symmetric neighborhood of 0.*

Here, it is understood that the domain of g may very well be larger than the image of the domain of the map by the quadratic map. Hence, our notion of the *image* of a critical branch is non-standard, as we define it to be the image of g.

Definition 1.9 *A critical branch $g(x^2)$ is **extendable** if g is.*

Definition 1.10 *A branch with domain P and stopping rule s is said to be **folding** (extendable) if there is an $\overline{s} < s$ such that*

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- $f^{\overline{s}}$ on P is an induced monotone branch (extendable).

- $f^{s-\overline{s}}$ on $f^{\overline{s}}(P)$ is a critical branch (extendable).

The image of a folding branch is, by definition, equal to the image of the corresponding critical branches.

Definition 1.11 *If ξ is a branch with stopping time s, a settled branch can be defined for any settling time $\overline{s} \leq s$. The settled branch is always $f^{\overline{s}}$, and its domain is equal to the domain of ξ. If $f^{\overline{s}}$ folds on the domain of ξ, we also need to specify the image. By definition, it is equal to the preimage of the image of ξ by $f^{s-\overline{s}}$ (which is well-defined since $f^{s-\overline{s}}$ is invertible on the relevant interval).*

Prefered induced maps. We describe a class of induced maps which have particularly useful properties.

Definition 1.12 *An induced map is called a* **preferred map** *if it has the following properties:*

- *All branches are either monotone or folding and extendable.*

- *All folding branches have the same critical value whose image is not entirely contained in one the external branches.*

- *The branches do not accumulate at the endpoints of the interval, and the external branches are monotone.*

Notational conventions. We will use parallelism in our notations between objects defined for f and similarly defined objects for $\hat{f}$ which automatically receive the same labeling only marked with a hat ˆsign.

Another problem comes from a considerable number of uniform constants which will abound in future arguments. To say that a constant is "uniform" means that it depends only on global distortion properties of maps f and $\hat{f}$. More precisely, it only depends on the C^3 norm of the corresponding map h and the infimum of derivative of h. Uniform constants will be denoted by the letter K with a subscript.

A statement which contains uniform constants should mean that "for each occurrence of a uniform constant, there exists a uniform numerical value

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which makes the statement true." We do not claim that uniform constants denoted with the same letter correspond to a fixed value throughout the paper. Thus, $K_1 > K_1$ would be considered a true statement, though we will use subscripts to avoid such extreme examples.

1.3   Non-renormalizable maps of basic type

Two maps.   From now on we consider a pair of maps, f and $\hat{f}$, both from $\mathcal{C}$ with the same kneading sequence. It is known that under our assumptions they are topologically conjugated so that

$$\hat{f} = h^{-1} \circ f \circ h \ .$$

In this context, we can talk about **equivalent stopping rules** s and $\hat{s}$ if the relation if the domain of s is mapped onto the domain of $\hat{s}$ by the topological conjugacy H, and

$$s = \hat{s} \circ H$$

holds where defined.

Basic construction.   The way we refer to the topological dynamics of our maps is through the basic construction as it stands in [15].

We assume that

On each stage of the construction, the critical value falls into a monotone branch.

Since the basic construction is topological, this is a topological condition. We will refer to it as "basic dynamics."

We do not know how to express this assumption in the language of kneading sequences. However, we note that the basic class is wider that the set discussed in [13]. On the other hand, in the complex analytic case it is narrower than the intermittently recurrent class considered by Yoccoz in his recent work.

The main result.   Theorem 1

Any two maps from $\mathcal{C}$ with basic dynamics are quasisymmetrically conjugate; moreover, the quasisymmetric norm of conjugacy is bounded by a uniform constant.

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Two important structures of a folding map. Any map from $\mathcal{C}$ has two distinguished points: the critical value and the fixed point q inside $(-1, 1)$. Importance of the forward critical orbit is well-known. In particular, its combinatorics defines the topological class of the map.

However, there is another structure worth looking at, and that is the backward orbit of the fixed point. In non-renormalizable cases this orbit is dense, thus any homeomorphism which maps backward preimages of the fixed point of one map onto corresponding points of another map with the same dynamics must be the conjugacy. This is how we build the conjugacy in this work as a limit point of "branchwise equivalences".

The forward critical orbit continues to play an important role in our construction, and the reader may note how both concepts interact in our "critical pull-back" and "marking" operations.

Introducing branchwise equivalences.

Definition 1.13 *A* **branchwise equivalence** *is a triple which comprises two equivalent stopping rules together with a homeomorphism which maps the domains of branches of one map onto the corresponding domains of branches of the other map.*

The homeomorphism from the domain of f to the domain of $\hat{f}$ which is the third component of the branchwise equivalence will also be called a branchwise equivalence, and the induced maps will then be referred to as the "underlying induced maps".

The subset of I on which a branchwise equivalence coincides with the conjugacy will be called its *marked set*. By definition, the marked set contains at least the boundary of the domain of s.

The domains of branches of f^s as well as components of the interior of the complement of the domain of s will be called the *domains* of this branchwise equivalence. We will thus speak of monotone and folding domains, while the last kind will be referred to as *indifferent domains*.

A branchwise equivalence which has no indifferent domains (i.e., the domain of the underlying stopping rule s is dense) will be called *regular*.

The construction that we use up to Section 6 only gives regular branchwise equivalences.

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The strategy of the proof. With every inducing construction there is an associated procedure of refining the domains of branches. This gives a natural way of building up a conjugacy between two maps. With any pair of equivalent induced maps we can associate a branchwise equivalence. If our inducing construction is sufficiently general, as it is the case with the basic construction followed by inducing on all monotone branches, we may hope that the actual topological conjugacy can be found somewhere in the closure of these branchwise equivalences. If so, the only thing remaining is to show that all branchwise equivalences in the class we consider are uniformly quasisymmetric. This is, of course, the hardest part.

Our basic technique will be patching different branchwise equivalences together to get new branchwise equivalences. This kind of procedure cannot be effectively carried out using real maps only. This is one reason why we will complexify our problem and indeed work on the level of quasiconformal extensions of branchwise equivalences.

So first, we are going to define the procedure of inducing and at the same time of constructing branchwise equivalences. We will then check that that construction is sufficiently general, so that in fact the basic construction can be approached using our methods.

Then, real work begins. We will redefine the construction in terms of quasiconformal extensions of branchwise equivalences. The complex procedure will be designed so as to guarantee that complex quasiconformal norms of the maps we construct will be uniformly bounded.

1.4   Renormalizable maps

Statement. Let f now be a renormalizable map. Let I^i, $i > 0$, be the maximal decreasing sequence of restrictive intervals around 0. Then I^i_j denote the orbit of I^i by the map.

We also get a sequence of maps from $\mathcal{C}$. Here, $f_0 := f$ and f_i is the first return map on I^i conjugated by an affine map so that I^i becomes $[-1, 1]$.

We will say that a preferred regular induced map of a renormalizable map is suitable if one of I^1_j is in the domain of a folding branch and mapped into itself by the branch.

Our main result here is that:
Theorem 2

Consider two infinitely renormalizable topologically conjugated maps from
$\mathcal{C}$. *Suppose that for each i there exist a quasisymmetric branchwise equivalence between suitable preferred induced maps of f_i and $\hat{f}_i$. If their quasisymmetric norms are uniformly bounded, then f and $\hat{f}$ are quasisymmetrically conjugate and the qs norm of the conjugacy is bounded by a uniform function of the common bound.*

Comment. Theorem 2 may be applied in various situations. If the infinitely renormalizable map has "bounded type" as introduced in [21], the assumption is relatively easy to verify, because all suitable maps are finitely complicated and subject to "bounded geometry".

However, we hope that usefulness of Theorem 2 extends far beyond that. The reader recalls that our approach to the conjugacy in non-renormalizable cases is by building more and more refined branchwise equivalences. The point is that quite often the suitable equivalence can be constructed in our way, and really the fact that the map is renormalizable makes no difference in the construction. Hence, a uniform bound on the quasisymmetric norms follows exactly as in the renormalizable case.

That means, for example, that if all suitable induced maps can be obtained in the basic construction, the conjugacy is uniformly quasisymmetric. This defines a class of infinitely renormalizable maps for which, as far as we are aware, the quasisymmetric conjugacy result is new even in the polynomial case.

Moreover, from the point of view of our approach of refined branchwise equivalence, all that is needed to close the infinitely renormalizable case is a uniform estimate for the construction in remaining non-renormalizable cases, called "box cases" in [15].

Section 6 also contains a theorem for finitely renormalizable maps, which we will not discuss here.

2 Branchwise equivalences.

2.1 Introduction of branchwise equivalences.

We assume that we are given two maps f and $\hat{f}$ which satisfy our assumptions.

We will give the description of our construction as a recursive procedure. That is, we are going to show simple primary objects and define operations allowing us to construct more complicated things from those simplest ones. From now on, we assume that the basic dynamical intervals $(-q, q)$ and $(-\hat{q}, \hat{q})$ have been uniformized by the affine maps from the unit interval. So we will simply assume they are both $[0, 1]$.

Before we continue, we would like to make a comment on the condition that images of folding branches must not be contained in an external branch. A simple observation is that if all other conditions for the map to be preferred are satisfied except for this one, there is a simple way "adjust" the map to become preferred. Namely, we can compose folding branches with that external branch. Since a repelling (pre-)periodic point is an endpoint of the external branch, the image of the fold will become longer. If we continue to compose until the critical value leaves the domain of the external branch, we will get a preferred map.

Boundary-refinement. There is a typical construction which we now describe. We can consider the leftmost branch of the map and compose it with the map itself. Then, we can take the new leftmost branch and again compose it with the original map. As this procedure is repeated, the leftmost branch gets exponentially shorter. If we continue the process to infinity, we get something that will be called *a map infinitely boundary-refined on the left*. Of course, we can also construct maps infinitely boundary-refined refined on the right or on both sides. We could also choose a point x very close to 0 and continue the left boundary refinement until x is no longer in the leftmost branch. This would be the *boundary refinement to the depth of x*. If we start with equivalent induced maps, and the depth of the refinement is determined by topologically equivalent points, then the resulting maps will also be equivalent.

Boundary refinement of a branch. Suppose we are given a preferred induced map. Any *monotone* branch of this map can be boundary-refined as follows: we first boundary-refine the whole map, and then compose the branch with the result of the refinement. Everything we said of refining a map has obvious consequences for this construction. However, there is one particular case we want to discuss. Suppose our monotone branch shares its

right endpoint with a very short folding branch. Very short critical means that the critical value falls into one of the external branches and takes a long time to leave the external branch adjacent to 1.[1]

We will often want to refine the monotone branch so that the domain of the rightmost branch of the result has length uniformly comparable with the length of the adjacent folding branch. If the critical value takes n iterates to leave to rightmost branch, we choose a point in the domain of the monotone branch which hits also stays in the domain of the rightmost branch for n iterates, and its n-th image hits the boundary point of that branch. If we then refine the monotone branch to the depth of the image of this point, then, indeed, the length of the domain of the new branch adjacent to the folding branch is comparable with the length of the domain of the folding branch itself. Also, this construction is topological: if we start with a pair of equivalent maps, we get equivalent maps.

We call it the *refinement to the depth of the adjacent folding branch.*

2.2   How to build branchwise equivalences.

The primary branchwise equivalence.   We need a preferred branchwise equivalence to begin with. We would also like it to satisfy two estimates uniform with respect to the choice of f and $\hat{f}$.

1. For either map, the lengths of the domains of any two adjacent branches are comparable, i.e. their ratio is bounded and bounded away from 0 by a uniform constant.

2. The primary branchwise equivalence is affine inside the domain of any branch, and the identity outside the interval $[-1, 1]$.

3. The quasisymmetric norm of the primary branchwise equivalence is uniformly bounded.

Finding the primary branchwise equivalence.   A reasonable way to start is by looking at the induced maps ϕ and $\hat{\phi}$ as defined in the introduction. They are always preferred. However, the additional estimates may not hold and the only way that can happen is when the central folding branch is

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[1] 1 is q after reparametrization: a repelling fixed point for the induced map.

extremely short and, as a result, the two monotone branches adjacent to it are non-extendable. Each of these non-extendable branches maps onto the whole interval $[0, 1]$, and, thus, they can be composed with the original map, or "refined" as we will often refer to this procedure.

First, we consider the situation when there are at least two monotone branches on each side of the central folding branch. Since the situation is wholly symmetric, it is enough to analyze what happens to the non-extendable branch on the left of the central folding branch. We refine it on the right to the depth of the central critical branch. Now, the branch adjacent to the central branch will already be extendable. But the refinement procedure will also create preimages of the central branch together with its non-extendable neighbors.

So in the next step, we start we our original map again and this time refine the non-extendable branches by composing them with a suitably boundary-refined version of the map obtained on the previous step. As we continue the process to infinity, the non-extendable branches eventually are crammed into a Cantor set (of zero measure.) Note also, that the procedure leaves the external branches of the original map unaffected.

However, if initially there is only one monotone branch on each side, this procedure would lead to a non-preferred induced map in which branches accumulate to 0 and 1.

Fortunately, there is another solution available in this case. If the central branch is very short, the induced map intersects the diagonal at a point q', which is repelling with period 2. We now consider the first return map from the interval $-q', q'$ onto itself. This turns out to a preferred-map with one folding and infinitely many monotone branches. Hence, the previously described construction applies.

Various types of primary branchwise equivalences. In the previous paragraph we indicated how to construct a pair of initial preferred induced maps. Once we have them, we can construct a branchwise equivalence between them. There will be two kinds of branchwise equivalences: marked and unmarked. The unmarked branchwise equivalence by the definition is the identity outside of the interval $[0, 1]$ and is affine in domains of all the branches. The marked equivalence is so designed as to map a point in the domain of one of the monotone branches to its corresponding point under

the topological conjugacy. Typical the point here will belong to the forward critical orbit. To achieve that, we make the branchwise equivalence linear fractional (for example) on the domain of this branch.

Boundary-refined versions. The map defined above can also occur in infinitely many boundary-refined versions. The main technical problem that we encounter with boundary- refined branchwise equivalences is that the banal extension by the identity beyond the unit interval does not work. Simply, as we consider boundary- refinements to growing depth, the quasisymmetric norm deteriorates (unless both f and f' have the same eigenvalue in q), and for the infinite depth boundary-refinement this extension could not be quasisymmetric at all.

So we use another extension. Let us say that we want to obtain the right boundary-refinement of the primary map. We do the boundary-refinement as previously described and construct the branchwise equivalence in the same way we showed in the last paragraph. Then, we extend by the identity to the left of the unit interval and mirror the result about 1 to extend it to the right of 1.

This gives us a quasisymmetric map.

Summary of primary branchwise equivalences. Our future estimates will depend on the following estimates for this primary map:

- The maximum ratio of lengths of any two adjacent domains.

- The ratio of 1 to the lengths of the external branches in case of branchwise equivalences which are not boundary-refined.

- The maximum quasisymmetric norm of any branchwise equivalence.

2.3 Branchwise equivalences build-up

Our next task is to describe how to "refine" primary branchwise equivalences so as to make them approach the actual conjugacy. The process is more or less parallel to the basic construction. However, one important difference is that we want uniformly quasisymmetric maps on all stages of the construction, which something that the basic construction does not provide.

We first describe how to obtain branchwise equivalences which are not boundary- refined.

The first operation: critical pull-back. To perform this operation we need a branchwise equivalence Υ_1. For the underlying pair of induced maps, we pick a pair of corresponding folding branches ψ and $\hat{\psi}$. We assume that these branches are extendable, and that their images are not contained in an external branch. We also need another branchwise equivalence Υ_2 such that the critical value of ψ falls into a monotone branch of Υ_2. [2]

First, we will describe what happens on the level of the associated stopping rules. In terms of induced maps critical pull-back can be expressed as composing ψ with the induced map associated with Υ_2 and the same thing with $\hat{\psi}$ in the other map. However, here is one exception: if the critical value of ψ is in one of the extreme domains of Υ_2, then we continue composing the resulting critical branch with this extreme branch until the critical value leaves its domain. This, however, will not create any new branches.

Then we proceed to define the construction of the branchwise equivalence between the new induced maps.

First, we take a *marked* version of Υ_2. Namely, we require that

$$c(\hat{\psi}) = \Upsilon_2 \circ c(\psi)$$

where the notation $c(\cdot)$ means "the critical value of". Marking means changing the branchwise equivalence Υ_2. We will later describe this process precisely. Right now we only assume that marking does not alter Υ_2 except on the monotone domain which contains the critical value.

The marking ensures that Υ_2 can be lifted by ψ and $\hat{\psi}$ and, by the definition, it is the order-preserving lift that is going to replace Υ_1 inside the domain of ψ.

Outside the domain of ψ, the branchwise equivalence is left unchanged.

Please note that if we have a number of folding branches, the critical refinement on one of them commutes with the critical refinement on any other, and in this sense we can say that the critical refinement can be done concurrently on all folding branches of a given map.

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[2]We will often use the word "branch" to really mean "the domain of a branch". We hope that it will not lead to confusion, while making our text smoother.

The second operation: monotone pull-back. We need a branchwise equivalence Υ_1 and a pair of corresponding extendable monotone branches ξ and $\hat{\xi}$ of the underlying induced maps. We also need another branchwise equivalence Υ_2.

On the level of induced maps, this operation is simply composing ξ and $\hat{\xi}$ with corresponding maps associated with Υ_2.

To get the new branchwise equivalence, we replace Υ_1 on the domain of ξ with $\hat{\xi}^{-1} \circ \Upsilon_2 \circ \xi$ and leave it alone outside of the domain of ξ.

Boundary refined versions. If we use Υ_1 which is not boundary-refined in a pull-back step, then we get a non-boundary-refined map as a result. To obtain a boundary-refined version of the result to a certain depth, we should start with Υ_1 refined to this depth.

A step of the construction. In the preceding paragraph, we described basic elements of the construction. Now, we will show how a step of the basic construction can be mimicked using these techniques.

Our starting point is a preferred branchwise equivalence Υ (we also know how to construct its marked versions.) Our construction will eventually yield another preferred branchwise equivalence, and the change of the folding branches will change in the same way as in the basic construction. However, we will watch that these properties are satisfied, which are considered unimportant in the basic construction:

- The lengths of any two adjacent domains are comparable.

- If two monotone branches share an endpoint, they are always refined simultaneously, and if fact they are subject to an infinite-depth boundary refinement at their common endpoint.

The first critical pull-back and boundary refinement. By assumption, the critical value is in a monotone branch. If it is not too close to an endpoint of the domain, or the adjacent branch is folding, we simply apply critical pull-back on all folding branches. If, however, the critical value is too close to an endpoint, that gives us a map with non-extendable branches. If these non- extendable branches are monotone, we use another procedure. We boundary-refine the branch adjacent to the branch containing the critical

value. The boundary refinement has the depth comparable to the distance of the critical value from the endpoint on one side. What happens on the other side depends on whether the next branch is folding. If it is, there is no additional refinement on that side and this step is completed. If it is monotone, there an infinite- depth boundary refinement on that side. In fact, all consecutive monotone branches are subject to the infinite-depth boundary refinement which will only end when a folding branch is encountered. Then after this process has been completed, the resulting map is pulled-back on all folding branches of Υ, just like in easier case.

The filling-in. The previous step resulted in a non-preferred map, because the folding branches may have different critical values, and also because some folding branches may not be extendable. The following procedure of "filling-in" is completely analogous to what was described under the same heading in the basic construction. If the result of the first critical pull-back is denoted with Υ', the second step differs from the first in the the map we pull-back in Υ', not Υ. If a preliminary sequence of boundary- refinements is needed, we do it on Υ' just like we did on Υ in the first step. Then, the resulting Υ'' is again pulled-back on the folding branches of Υ and so on to infinity. Eventually, all folding branches with the critical value in the old place will disappear, squeezed into a Cantor set, and we get a preferred branchwise equivalence again.

The final refinement of the monotone branches. Our objective is to get a sequence of branchwise equivalences which tend to the topological conjugacy. If only do basic steps as described above, this will not be the case, since some monotone branches, for example the external ones, will never be refined. That is why at some moment we have to stop and refine these "lagging" monotone branches.

This is a little bit similar to the boundary-refinement sequence described in the previous paragraph. We refine all monotone branches which are not contained in the external primary branches. We all also refine some branches contained in the external primary branches if it is necessary to preserve the rules of the chain boundary refinement. At the points of tangency with folding branches, we refine to the depth so chosen that the new adjacent branch is always shorter than the folding branch, but still in a uniformly bounded

away from 0 ratio. Between adjacent monotone branches the refinement is infinitely deep. In the next step, we refine all new monotone branches except for those on which the previous refinement stopped (i.e. the ones adjacent to the folding branches of the original map.) As we repeat this process, we eventually wind up with monotone branches all smaller than the longest folding branch of the original map. But, that tends to zero in the basic construction, so indeed we get a sequence of maps that tend to the conjugacy on the set between the primary external branches.

Marking conditions.

Definition 2.1 *A* marking condition *is a choice of an infinite ray which starts in a monotone domain of f.*

Reduced versions of branchwise equivalences. Given a ray and a branchwise equivalence v constructed in the way just described, we want to consider a reduced version of this equivalence with respect to the ray. To define the "reduced version" we need to exactly mean a branchwise equivalence which coincides with v on all branches of some primary branchwise equivalence which are intersected by the ray, why all other branches of that branchwise equivalence have been refined at most twice.

It is clear that reduced version can always be constructed. For primary branchwise equivalences, they can be the same. Now, every other branchwise equivalence, as we have seen, is created by subsequent refinements of the primary branchwise equivalence. Thus, we can simply skip the refinements of the branches which are not intersected by the ray. There is one exception to this rule, if the primary branch which contains the ray's end is boundary-refined so that new branches accumulate at the endpoint not covered by the ray. Then, the next adjacent branch has to be boundary-refined, too. That is why we allowed two refinements in the definition.

The marking. The *marking* determined by the ray is an operator which changes the branchwise equivalence on the branch which contains the end only, so that the end gets mapped onto its conjugate point.

This is not really a definition, since there are certainly many way in which one could mark in this sense. We postpone the precise definition until next

section. Right now we only need to believe that the marking operation has been defined.

Marked versions of branchwise equivalences. The *marked version* of a branchwise equivalence with respect to a marking condition is, first, reduced accordingly to this condition. Secondly, is marked in the sense of the previous paragraph.

A brief explanation. Marked versions of branchwise equivalences will be used to be pulled-back by critical branches. A careful reader has likely guessed the meaning of the ray's end, which is at the critical value. Thus, marking ensures that the pull-back is well-defined.

The interpretation of the direction of the ray is that it covers the image of the folding branch. Thus, it makes no difference how the branchwise equivalence is defined beyond the ray. However, when we complexify the procedure, that region will have a preimage beyond the real line. So, the idea is to make the branchwise equivalence as simple as possible in that region, and avoid future trouble.

Summary.

Proposition 1 *If the construction starts with a primary branchwise equivalence as described, continues through an arbitrary number of steps, and ends with a final refinement, the result is a preferred branchwise equivalence with the following uniform geometric estimates:*

- *The ratio of any two adjacent domains is bounded.*

- *The lengths of the external branches are bounded away from* 0.

Proof:
We only give an outline. The argument for the first part was discussed. To see that the second part is true, we notice that if an external branch of the primary map is refined, the resulting external branch will never be refined. Indeed, the only possibility that could happen is a chain boundary- refinement. However, by our construction, the image of a folding branch is never contained in an external branch of the primary map. Also, the refinement of the primary external branch must have created folding branches. But,

if there are folding branches between the tip of the folding branch and the external branch, the chain boundary refinement will stumble on them and will never reach the new external branch.

$$\square$$

If we can prove that this map is uniformly qs, our main theorem will follow. To prove that the conjugacy is qs on the whole interval we can use arguments similar to those in [15].

Thus, our main theorem reduces to the following:

Prove that all branchwise equivalence obtained in the procedure described above are uniformly quasisymmetric.

This is what the balance of the paper is about.

3 Complexification of induced maps

3.1 Introductory remarks

Why complexification? There are two basic reasons why we want to work with a complexified version of our problem. First is that the critical pull-back is hard to handle using the real variable methods only. This is an operation which involves two maps with unbounded real distortion. True, their effects should cancel, but there is no good way to account for that if we confine ourselves to the real line.. On the other hand, since the quadratic polynomial is analytic it has a null impact on quasiconformal distortions.

Another advantage of using quasiconformal maps was mentioned at the end of the previous paragraph. The point is that is easier to paste quasi-conformal maps. We simply need to check that the result is continuous and the quasiconformal distortion is bounded. Also, quasiconformal distortion is something which can be localized. To make this point more clear let us consider two real quasisymmetric maps: one is the identity to the left of zero, the other to the right. It is intuitively obvious that since their "quasisymmetric distortions" are supported in different regions, the quasisymmetric norm of the composition should be more like a maximum than the sum of the norms. But there is no correct and convenient way to express this kind of intuition other than in terms of quasiconformal maps. The support of quasiconformal distortion is a well-defined notion and the fact the distortions are supported in different regions can be used correctly.

The strategy. In this section, we will extend branches of induced maps to quasiconformal mappings of the whole plane. We will require these extensions to have special properties and, in fact, the main technical burden of our work is going to be in that part.

In the next section, we will be able extend the branchwise equivalences first for all branches more or less independently, and only on a small set around each branch. We will show that the "piecewise extensions" obtained in this way are uniformly quasiconformal where defined.

Finally, we will show that the piecewise extensions can glued and that only involves bounded quasiconformal distortion.

3.2   Extensions of individual branches

The objective of this passage is to show how a branch can be extended to the whole plane. It is important that at this moment we regard the branch as a separate entity. That means, we will not care about whether our extension is consistent with other branches.

Simple extensions.

Tangent extension. Suppose we have a monotone extendable branch ξ defined on an interval J. Let us extend ξ on the whole line using affine maps so that the resulting map is differentiable. By the definition, the **tangent extension** of ξ is the tangent map of the function defined in this way, where tangent spaces are identified with vertical lines.

It should be mentioned here that the idea of tangent extensions and their properties were discussed in D. Sullivan's lectures in New York in the fall of 1989.

The qc distortion of tangent extensions can be computed in an elementary way, and the result can be expressed as follows:

Fact 3.1 *Consider a monotone extendable branch ξ and its tangent extension $\mathcal{EX}(\xi)$. Rescale by an affine mapping so that the length of the domain of ξ becomes* 1. *Then, the conformal distortion at (x, y) equals*

$$\frac{\partial_{\bar{z}} \mathcal{EX}(\xi)}{\partial_z \mathcal{EX}(\xi)} = \frac{iy \cdot f''/f'(x)}{2 - iy \cdot f''/f'(x)} \; .$$

Local extensions of branches.

Definition 3.1 *Consider a monotone branch ξ. It is defined as an iterate of f on its domain, and the iterate of f can be represented as an alternating composition of maps h and the quadratics. The* local extension *of ξ is defined as the corresponding composition of maps of the plane, where transformations h restricted the images of the domain have been replaced by their tangent extensions, and the quadratics have been extended analytically.*

Definition 3.2 *Definition 3.1 is extended on folding branches as follows. By definition 1.10, a folding branch is a composition of one monotone branch, a quadratic polynomial, and another monotone branch. To obtain its local extension, we extend the monotone branches locally, and the quadratic analytically.*

   Problems with local extensions.   Eventually, we will cut off small pieces of local extensions around the domains on the real line and will glue them into a global quasiconformal map. But in order to do that, we at least need to know that "small pieces around the real domains" map to "small pieces around the real images" and not in a totally weird way. Our next, highly technical section, is devoted proving the suitable estimates.

3.3   Local extensions in the proximity of the real line

Normalized monotone branches.

Definition 3.3 *Suppose that we have diffeomorphism written as a composition*

$$\xi = h_n \circ Q_n \circ \cdots \circ h_1 \circ Q_n$$

where h_i are negative Schwarzian maps, while Q_i are quadratic polynomials with critical points c_i respectively. Furthermore, we assume that all maps are automorphisms of the unit interval, and that the composition is defined, and still a diffeomorphism, on a larger interval J so that $\xi(J) = (-\epsilon, 1+\epsilon)$ where ϵ is a constant between 0 and 1 to be specified soon.

   We will call this composition a normalized monotone branch.

   Correspondingly, we can consider its local extension, in which maps h_i are extended tangentially, whereas the polynomials are extended analytically.

   We will use the notation $J_0 := J$ and $J_i = h_i(Q_i(J_{i-1}))$.

Correspondence between the local extensions of branches and local extensions of normalized branches. Let us consider a monotone extendable branch with a stopping rule s. Look at all s images of the domain. They can all be affinely rescaled to become the unit interval. Thus, we get a corresponding normalized branch. Its local extension again is related to the local extension of the branch by affine maps. Also, the extendability condition is satisfied with a uniform ϵ.

There is an important estimate for normalized monotone branches which come from monotone extendable branches in the way just described.

Fact 3.2 *If an abstract monotone branch θ comes from an extendable monotone branch, then $S\theta > -K_1$.*

Proof:
It is a standard fact that this follows from extendability. See [8], Proposition 2, for the proof of a very similar statement.

$\square$

The relation of standard and analytic extensions of the quadratics. If the polynomials Q_i were extended tangentially, then local extensions would be easy to understand. So, it is natural to study a relation between the analytic extension of a quadratic map and the tangent extension of the same map. We have a Lemma:

Lemma 3.1 *Let us consider a quadratic map F from the unit interval into itself, with the critical point in $c \notin I$. For a point $z = (x, y)$, $0 < x < 1$, the y coordinate of the image is the same for both the tangent and the analytic extension. The x coordinates differ not more than*

$$K_1 \frac{y^2}{(dist(c, I))^2} \ .$$

Proof:
This elementary geometry.

$\square$

Perturbations of normalized monotone branches. If we pick a point $z = (x, y)$ and look at its image by the local extension of $h_i \circ Q_i$, we discover that the x-coordinate of the image is not equal to $h_i \circ Q_i(x)$. However, the discrepancy is rather small as indicated by Lemma 3.1. Thus, to follow the x-coordinates of the images of a point by consecutive maps from the composition, we need to perturb the normalized monotone branch. This gives rise to the following object.

Definition 3.4 *If $\xi = h_n \circ Q_n \circ \cdots \circ h_1 \circ Q_1$ is a normalized local branch, its perturbation is any composition*

$$\overline{\xi} = h_n \circ g_n \circ Q_n \circ \cdots \circ h_1 \circ g_1 \circ Q_1$$

where each g_i is an orientation-preserving homography that fixes $Q_i(J_{i-1})$.

If g_i fixes (a, b), it is uniquely characterized by the number

$$\Delta_i = \log \frac{(x - a)(g_i(x) - b)}{(g_i(x) - a)(b - x)}$$

which is independent of the choice x.

Before we consider more closely the correspondence between local extensions and perturbations, here is a the property of perturbations which will be of interest to us.

Lemma 3.2 *Consider a perturbation $\overline{\xi}$ and a point x so that $\overline{\xi}(x) \in [0, 1]$. Then, $\overline{\xi}'(x)$ is bounded away from 0 by a constant. Moreover, provided $\sum_{i=1}^{n} |\Delta_i|$ is sufficiently small, it is also bounded in uniform way.*

Proof:
Denote $J_0 = (a, b)$.

The first claim follows immediately because the infinitesimal cross-ratio

$$\frac{(b - a)dx}{(x - a)(b - x - dx)}$$

is increased by $\overline{\xi}$ (to see why, consult [19]).

The second claim will follow if we can show that $\overline{\xi}^{-1}(0, 1)$ has uniformly large length.

Consider the sequence

$$u_i = Q^{-1}_{n-i+1} \circ g^{-1}_{n-i+1} \circ h^{-1}_{n-i+1} \circ \cdots \circ Q^{-1}_n \circ g^{-1}_n \circ h^{-1}_n(0) \ .$$

If $J_{n-i} = (a', b')$. We claim that

$$|\log \frac{(0 - a')(b' - u_i)}{(u_i - a')(b - 0)}| \leq \sum_{n-i+1}^{n} |\Delta_i| \ .$$

Indeed, this is immediately seen true by induction if one keeps in mind that $Q^{-1}_i \circ h^{-1}_{i-1}$ contracts the "Poincaré metric" on the images of J. The Poincaré metric on an interval is the conformally invariant metric on the disc whose diameter is the interval. It is classically known that it can be represented as the logarithm of some cross-ratio, thus it is expanded by negative Schwarzian maps and contracted by positive Schwarzian maps.

The same argument can be applied to the preimages of 1. Since the Poincaré length of the interval $(0, 1)$ inside J_n is more than $-2 \log \epsilon$, it is enough for $\sum_{i=1}^{n} |\Delta_i|$ to be less than $- \log \epsilon/2$ in order to ensure that the Poincaré length of $\overline{\xi}^{-1}(0, 1)$ is definite.

$\square$

Orbits by the local extension and perturbations. We consider a sequence $z_0 = (x_0, y_0)$ and

$$z_i = (x_i, y_i) := h_i \circ Q_i(z_{i-1}) \ .$$

If also $0 < x_i < 1$ for every i between 0 and $n - 1$ inclusively, we call this sequence an *orbit*.

With every orbit we can associate the unique perturbation which satisfies

$$x_i = h_i \circ g_i \circ Q_i(x_{i-1})$$

for all i from the relevant range. One can check that Δ_i is bounded proportionally to the discrepancy between the tangent and analytic extension of Q_i at z_{i-1}, thus by $K_1 \frac{y^2_{i-1}}{\text{dist}_{([0,1], c_i)}}$ according to Lemma 3.1. Therefore, the crucial sum $\sum_{i=1}^{n} |\Delta_i|$ can be estimated by

$$\sum_{i=1}^{n} |\Delta_i| \leq K_1 \left(\max\{y_{i:0 \leq i < n}\} \right)^2 \sum_{i=0}^{n-1} \frac{1}{(\text{dist}([0, 1], c_i))^2} \ .$$

On the other hand, we can calculate that for any $x \in (0, 1)$

$$S\xi(x) \leq - \sum_{i=0}^{n-1} \frac{K_2}{(\text{dist}([0,1], c_i))^2} \ .$$

Finally, as a consequence of Fact 3.2 we obtain

$$\sum_{i=1}^{n} |\Delta_i| \leq K_3 (\max\{y_i : 0 \leq i < n\})^2 \ .$$

That the perturbation contains some information about the orbit is evident from our next lemma.

Lemma 3.3 *Consider an orbit z_i and the corresponding perturbation $\overline{\xi}$. We have an estimate*

$$|\log \frac{y_n}{y_0} - \log \overline{\xi}'(x_0)| \leq - \sum_{i=0}^{n-1} \frac{K_2}{(dist([0,1], c_i))^2} \ .$$

Proof:
Observe that
$$y_i / y_{i-1} = \frac{(h_i \circ g_i \circ Q_i)'(x_{i-1})}{g_i'(Q_i(x_{i-1}))} \ .$$
So, the difference is bounded by the sum of logarithms of g_i' at the appropriate points. However, $|\log g_i'(Q_i(x_{i-1}))|$ can be estimated according to Lemma 3.1, and then the sum can be bounded using Fact 3.2.

$\square$

For an orbit, let Y denote the maximum of y_i with $0 \leq i < n$. The essence of our results obtained so far is in the following lemma:

Lemma 3.4 *Provided $Y < K_1$,*

$$|\log y_n / y_0| \leq K_2 \ .$$

Proof:
This is simply a summary of Lemmas 3.2 and 3.3 together with our estimate of $\sum_{i=0}^{n-1} \Delta_i$.

$\square$

The basic result. We now have all necessary tools to quickly prove the main result of this section.

For a point $z = (x, y)$ with $0 < x < 1$, let the **height** of z mean $y / \min(x, 1 - x)$.

Proposition 2 *If the height of some z is less than K_1, a uniform constant, and $z_n = (x_n, y_n)$ is the image of z under the local extension of a normalized monotone branch, then $0 < x' < 1$ and the ratio of heights of z_n and z, as well as y_n / y is uniformly bounded from both sides.*

Proof:
The condition for the ratio y_n / y is similar to the claim of Lemma 3.4, but there are two things that we need to check. First of all, we need to show that z defines an orbit, that is x_i is always between 0 and 1. Secondly, we need to bound Y in terms of y_0.

Working to eliminate Y. In the following two lemmas we simply assume that $0 < x_i < 1$ for $0 \le i < n$.

Lemma 3.5 *For a suitable constant K_1, if $y_0 < K_1$, then $y_n < K_2 y_0$.*

Proof:
This is a simple corollary to Lemma 3.4. The trick is to look at the first i for which $y_i > \exp(K_{2,L.\ 3.4}) y_0$. Provided y_0 is small, Lemma 3.4 applied to the composition cut off at $n := i$ gives a contradiction.

$\square$

Lemma 3.6 *Provided $y_0 < K_1$,*

$$|\log y_n / y_0| \le K_2 \ .$$

Proof:
Follows immediately from Lemmas 3.4 and 3.5.

$\square$

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Working to prove that z_i is an orbit. Next, we have to investigate how x_i depends on x_0. Fortunately, this is reduced to a one-dimensional question about perturbations. We introduce a family of perturbations $\overline{\xi}_t$. If Δ_i the difference between the x-coordinate of $Q_i(z_{i-1})$ and $Q_i(x_i)$, then $\overline{\xi}_t$ corresponds to the sequence $t\Delta_i$. Hence, $\overline{\xi}_0 = \xi$ while $\overline{\xi}_1 = \overline{\xi}$. We also get sequences $x_i(t)$ whose definition is natural. We further denote

$$\overline{\xi}_t^i := h_n \circ g_{n,t} \circ Q_n \circ \cdots \circ h_i \circ g_{i,t} \circ Q_i$$

Fact 3.3

$$\frac{dx_n(t)}{dt} = \sum_{i=1}^{n} \frac{d\overline{\xi}_t^i}{dx}(x_i(t) \cdot h'_{i-1}(h_{i-1}^{-1}(x_i(t))\Delta_i \; .$$

Proof:
Elementary.

$\square$

Our next lemma is analogous to Lemma 3.4 except that the x-coordinate is now involved.

Lemma 3.7 *For any $K_1 > 0$, a $K_2 > 0$ can be chosen so that if the height of z less than K_2 and $0 < x_i(t) < 1$ for all $0 \le i < n$ and $0 \le t \le \tau \le 1$, then $|x_n(\tau) - x_n(0)| < K_1 \min(x_0, 1 - x_0)$.*

Proof:
If the height of z is small, Lemma 3.6 applies. This allows us to estimate $\sum_{i=0}^{n} \Delta_i$ by $K_2 K_3 \min(x_0, 1 - x_0)^2$. Subsequently,, Lemma 3.2 can be applied to estimate the derivatives in the formula of Fact 3.3 by constants. Then, that formula immediately yields the claim.

$\square$

Finally, we notice that Lemma 3.7 remains true even when the assumption of $x_i(t)$ being between 0 and 1 has been removed. To this end, we note that $\min(x_n(0), 1 - x_n(0))/\min(x_0(0), 1 - x_0(0))$ is bounded away from 0. That follows directly from extendability of the monotone branch. So, $K_{1,L.\ 3.7}$ can be chosen so as to ensure that $|x_n(\tau) - x_n(0)| < \min(x_n(0), 1 - x_n(0))$. Then, we consider an arbitrary z which satisfies other assumptions of Lemma 3.7 and look for the lowest $\tau \le 1$ and lowest i so that $x_i(\tau) \notin (0, 1)$. Then Lemma 3.7 applied to the composition cut off at $n := i$ gives a contradiction.
Proposition 2 follows directly.

$\square$

3.4 Global extensions

Diamond neighborhoods.

Definition 3.5 *For an interval, its* diamond neighborhood *of size a is defined to be an open quadrilateral bounded by the set of points with height a.*

We have a lemma which is a simple corollary to Proposition 1:

Lemma 3.8 *Fix an $\alpha \leq K_{1,P.2}$. Then, there is a positive function Γ so that, for any extendable branch $\xi = h_2 \circ Q \circ h_1$, the local extension of ξ maps the size $\Gamma(\alpha)$ diamond neighborhood of the domain of ξ into the size α diamond neighborhood of the image of h_2.*

Proof:
First, assume that $\xi = h_1$, i.e. ξ is monotone. Then, the claim follows immediately from Proposition 2. Next, consider the case of $\xi = Q \circ h_1$. We see in an elementary way that Q will not extend a diamond neighborhood of bounded size to much. To conclude the argument, we apply Proposition 2 to h_2.

$\square$

Definition 3.6 *The diamond neighborhood of size $\min(1/2, \Gamma(K_{1,P.2}))$, in the notations of Lemma 3.8 of the domain of any branch will be called the* large diamond neighborhood *of that branch.*

Definition 3.7 *A diamond-like neighborhood of an interval on the real line is any open neighborhood of the interior of the interval. The size of a diamond-like neighborhood is defined to the size of the largest diamond neighborhood of that interval contained in the diamond-like neighborhood.*

Introduction of global extensions.

Postulates.   We will construct a new type of complex extension of the branch, called a *global extension*. The idea is to make the same as the local extension close to the real domain of the branch, but change it far from that domain in order to make glueing possible with extensions of other branches.

Thus, formally, the global extension of a branch ξ will satisfy these postulates:

1. The global extension is the same as the local extension on the large diamond neighborhood of ξ.

2. If the domain of ξ is (a, b), the global extension is affine outside of the rectangle with vertices at $a + (b - a)i$, $b + (b - a)i$, $b - (b - a)i$, and $a - (b - a)i$.

3. The global extension is uniformly qc function.

The construction.   We will sketch the construction of global extensions from local extensions. First, we consider the case of a monotone branch and its corresponding normalized monotone branch. We choose approximate maps so that the extension remains local on the diamonds of size $K_{1,P,2}$ around each intermediate image of the domain, while it is tangent outside of the diamond neighborhood of unit height. We leave without a proof that such adjusting map can be constructed with complex distortion of the order of $\text{dist}((0, 1), c_i)^{-2}$. This, in view of Fact 3.2, ensures that the complex distortion of the composition will be uniformly bounded.

Having thus constructed a map which is a tangent extension outside a diamond of unit height, it is easy to build a global extension, and we leave it to the reader.

In the case of $\xi = h_2 \circ Q \circ h_1$, we can build global extensions of both monotone branches h_1 and h_2, and the suitable extension of Q can be constructed in an elementary way.

A convention.   In the future will work mostly with global extensions. So, if we say "an extension of the branch" we mean the global extension. In formulas, the global extension of ξ will appear as $\mathcal{EX}(\xi)$.

Better qc estimates for global extensions. We will show a lemma about the qc distortion inside large diamond neighborhoods.

Lemma 3.9 *Consider a monotone extendable branch. Let D mean the total length of all intermediate images of the domain of this branch. The complex distortion of the extension of this branch at a point z inside the large neighborhood is bounded by a constant multiplied by D and by the ratio of the distance from z to the line to the length of the domain of the branch.*

Proof:
Consider the corresponding normalized branch. The key estimate is Fact 3.1. The complex distortion of the composition is bounded by the sum of complex distortions of maps h_i along the orbit of z.

According to Fact 3.1, each contribution is bounded proportionally to $y_i h_i''/h_i'(x_i)$. The quantity y_i is roughly constant according to Proposition 2, and so it is comparable to the ratio of distance from z to the line by the length of the domain.

To estimate estimate the "nonlinearity ratio" h_i''/h_i' we need to remember that h_i is just an affine rescaling of h restricted to the i-th image of the domain. Clearly, h''/h' is uniformly bounded, and affine rescaling multiplies the nonlinearity ratio by the derivative of the rescaling map, which is equal to the length of the domain of h_i in our case.

$\square$

Lemma 3.9 is useful provided that we can give a good estimate of D. Fortunately, it is so. Since there is a lot of expansion in our inducing construction, we will be able to show that in interesting cases the D is just proportional to the length of the image of the branch.

Lemma 3.9 also has obvious consequences for folding branches, since they can be written as

$$h_1 \circ Q \circ h_2$$

with h_1, h_2 monotone and Q analytic. Lemma 3.9 may be applied to both monotone branches separately.

4 Complexified branchwise equivalences

4.1 Basic properties and constructions

Primary branchwise equivalences. In the section on real branchwise equivalences we described how to construct primary branchwise equivalences on the real line. The main objective of this section is to describe the process of their complexification. However, even before we do that, we wish to discuss an important technical principle of the construction.

Arbitrary fineness principle. Roughly speaking, we may assume that the longest domain of the primary induced map is as short as it suits us. More precisely, we claim

Fact 4.1 *For any $\alpha > 0$ primary induced maps can be constructed so that the lengths of the branches do not exceed α, and the qs norm of the corresponding branchwise equivalence is bounded by $\Gamma(\alpha)$ where K is the function of α only.*

Proof:
We have shown how to construct uniformly qs primary branchwise equivalences without the fineness requirement. If we want a finer branchwise equivalence, we need to refine by pull-backs. If we simultaneously refine all the branches by pull-backs, the lengths of the domains will decrease by a fixed factor. So, only a finite number of such steps will be needed to attain the specified fineness. Each step, however, increases the qs norm by a bounded amount, which can be seen by arguments analogous to those given in the Addendum. Also, a less tedious complex way will shown later to see that.

$\square$

In the future, we will often assume that "the primary branchwise equivalence is sufficiently fine." That, in effect, means that the α which occurs in Fact 4.1 will be chosen many times. However, since this paper will hopefully end up having a finite length, a positive minimum will still exist.

Primary extensions of branchwise equivalences. Here, we list list the desired properties of complex primary branchwise equivalences. We assume that the domains of both induced maps have been normalized by affine maps to become $[-1, 1]$.

1. On the real line, the map is a branchwise equivalence as described in the Branchwise Equivalences section.

2. The map is is the identity outside of the ball of radius 3 centered at 0 on the plane.

3. on the diamond neighborhood of size 1 around each branch, the map is affine.

4. The map is quasiconformal, and its qc norm can be bounded by a uniform function of the qs norm of the underlying real branchwise equivalence.

The reader may note that these postulates leave us with the freedom to define the map inside the diamonds of size 1, as long as the map is qc and and affine on the boundary. This freedom will be used for marking.

The complexification lemma.

Lemma 4.1 *Suppose that a qs branchwise equivalence υ is given between two induced maps or their boundary-refined derivatives which satisfy the property that the ratio of any two adjacent branches is uniformly bounded. If υ is the identity outside of $(-3, 3)$, then its primary extension can be constructed.*

Proof:
Draw a circle of radius 3 from 0. Consider a Jordan curve Γ which consists of the upper half of the circle, the upper halves of the boundaries of size 1 diamond neighborhoods of branches, and the connecting pieces of the real line from -3 to -1 and from 1 to 3.

An analogous curve $\hat{\Gamma}$ exists in the phase space of $\hat{f}$. Define a homeomorphism G of Γ onto $\hat{\Gamma}$ which is the identity on the circle and the pieces of the real line, and affine on the boundary of each diamond. The lemma will clearly follow if we prove that this homeomorphism can be extended to the region encompassed by Γ.

We notice that Γ is a uniform quasicircle. An easy way to see that is by noticing that the three point property of Ahlfors (see [1]). The homeomorphism G is also quasisymmetric in the sense that if distances $|x - y|$ and $|x - z|$ are comparable, so are the corresponding distances in the images.

Moreover, the quasisymmetric norm of G is bounded in terms of the qs norm of the map on the real line.

Next, Γ and $\hat{\Gamma}$ can be uniformized to the round circle by a qc map, and the counterpart of G on the boundary quasisymmetric. Since it can be extended in the classical way (see [3]), and pulled back to the inside of Γ, the lemma follows.

$\square$

Thus, primary extensions of primary branchwise equivalences exist, and are uniformly quasiconformal. In the future, when we talk of complex primary branchwise equivalences, we mean exactly primary extensions of primary branchwise equivalences.

Admissible extensions of branchwise equivalences. Now we will define the class of complex extensions with desirable properties, which will be called *admissible extensions*.

Postulates. A complex extension Υ of a branchwise equivalence is admissible if it satisfies the conditions listed below.

1. Admissible extensions are the identity outside of the ball $B(0,3)$.

2. They are quasiconformal.

3. For every branch ξ with stopping time s, we can choose an integer $\overline{s} \leq s$ so that the corresponding settled branch $\overline{\xi}$ allows us to represent Υ by the formula:
$$\Upsilon = \hat{f}^{-\overline{s}} \circ A \circ f^{\overline{s}}.$$

   The formula is supposed to valid on some maximal diamond-like neighborhood of the domain of ξ, which we will call a *close neighborhood* of that domain and denote with $D(\xi)$. The map A is supposed to be quasiconformal and affine outside of the image of $D(\xi)$ by $\overline{\xi}$. Moreover, if ξ is monotone, A is simply affine.

The primary branchwise equivalences are admissible with $\overline{s} = 0$ on all branches.

The norm of admissible extensions. By the norm of an admissible branchwise equivalence we will mean the maximum of its quasiconformal norm and the reciprocals of the sizes of its close neighborhoods.

4.2 Complex pull-backs.

Complex marking. We will show how to mark an admissible complex branchwise equivalence. The idea is to use the last property of admissible equivalences and change A inside the image of $D(\xi)$ only. Remember, that we only mark monotone branches, thus A is affine. By Proposition 2, the image of $D(\xi)$ by $\overline{\xi}$ contains a diamond D' on size comparable to the size of $D(\xi)$. So, we change A only inside D' to make it linear-fractional inside the diamond of half the size of D'.

It is easy to observe that the quasiconformal norm of such a map depends on two estimates: the nonlinearity of the linear-fractional map on the real line, and the smallness of the size of D'.

The first bound always holds:

Fact 4.2 *Consider any point contained in the domain of a monotone branch, and its image by conjugacy. Uniformize this domain, and the corresponding domain of $\hat{f}$ by the unit interval using affine maps. Then, the Poincaré distance between the point and its image is uniformly bounded.*

Proof:
It is enough to consider the conjugacy near the boundary of a monotone branch.

As monotone branches have uniformly bounded distortion, the question reduces the primary boundary-refined branchwise equivalence. By construction, the dynamically defined objects scale exponentially near q and $-q$ and the exponential rate depends on the eigenvalue at q. This implies that the conjugacy moves points around only by bounded Poincaré distances. See [14] for a detailed analysis in a closely related case.

$\square$

The second estimate depends directly on the norm of the branchwise equivalence being refined. We will address that issue later.

Simple pull-backs. We assume that an admissible branchwise equivalence Υ_1 is given. Also, a branch ξ is chosen in the corresponding induced map. Also, another admissible branchwise equivalence Υ_2 is given so that if ξ is folding, its critical value is a monotone domain of Υ_2 and Υ_2 is marked by the ray which starts at the critical value and covers the image of the interval by ξ.

We want to refine ξ by pulling-back Υ_2. This will be a multi-step process.

We now show the first step in which we construct the correct map on and close to the domain of ξ, but do not concern ourselves with how this map matches the global Υ_1.

Extension of Υ_1 on the close neighborhood. Since Υ_1 is admissible, on $D(\xi)$ it can be represented as

$$\hat{f}^{-\overline{s}} \circ A \circ f^{\overline{s}} \, .$$

On the real line, the pull-back of Υ_2, assuming appropriate marking, can be written as

$$\hat{\xi}^{-1} \circ \Upsilon_2 \circ f^{s-\overline{s}} \circ A^{-1} \circ \tilde{f}^{\overline{s}} \circ \Upsilon_1 = G \circ \Upsilon_1$$

where G is just a new notation for the complicated composition in front of Υ_1. Moreover, this formula makes sense on the whole plane, and on $D(\xi)$ it gives the same as

$$\hat{\xi}^{-1} \circ \Upsilon_2 \circ \xi \, .$$

The map G will be called the *simple pull-back* of Υ_2 by ξ.

The map G is the identity beyond the ball centered at the midpoint of the domain of ξ, of radius three times the length of the domain. To see that, we notice that the composition

$$f^{s-\overline{s}} \circ A^{-1} \circ \hat{f}^{\overline{s}}$$

is affine beyond the preimage of that ball by definition of global extensions and the requirement imposed on A by admissibility of Υ_1. Similarly,

$$\hat{\xi}^{-1}$$

is affine beyond that ball. Finally, Υ_2 is the identity except on the ball, again by admissibility.

Where we stand with the construction. To define the new branch-wise equivalence by $G \circ \Upsilon_1$ is not quite a good idea since G is not required to be the identity on the line everywhere beyond the domain of ξ, and, in fact, could not be for a boundary-refined Υ_2. So, we will have to change the simple pull-back a little bit to take care of this problem.

In our construction the final result of the refinement will still be $G \circ \Upsilon_1$ on the large diamond neighborhood of ξ. So, the resulting map is going to be admissible.

Finally, it should be pointed out that G "lives" in the extension of the phase space of $\hat{f}$ unlike branchwise equivalences which go from the phase space of f to the phase space of $\hat{f}$.

Hexagonal extensions. We will now change the simple pull-back to get another map called the "hexagonal extension". The hexagonal extension is not a homeomorphism of the entire plane, but only of some hexagon around the domain being refined. On the other hand, it is easy to extend by the identity if Υ_2 is not boundary-refined, or glued with an analogous map around the adjacent domain of the chain refinement.

Angular squeezing. We start with a simple extension G.

We will describe the procedure in polar coordinates around 0. We define a map
$$\mathcal{S}_0 : (-\pi, \pi) \to (-\pi/3, \pi/3)$$
which keeps everything inside the arc $-\pi/4, \pi/4$ fixed, and squeezes the sectors $(-\pi, -\pi/4)$ and $(\pi/4, \pi)$ diffeomorphically into $-\pi/2 + 0.001, -\pi/4$ and $\pi/4, \pi/2 - 0.001$ respectively. If we assume the distance from 0 is unchanged, this defines through polar coordinates the map also denoted by $\mathcal{S}_0$ which is quasiconformal and can be extended to a multivalued function through the negative numbers.

An analogous procedure can be carried out around 1 and the resulting map is to be denoted $\mathcal{S}_1$. Then, we may consider the map
$$\rho_1 = \mathcal{S}_0 \circ \mathcal{S}_1 \circ G \circ \mathcal{S}_1^{-1} \circ \mathcal{S}_0^{-1} \ .$$

Vertical squeezing. We would like to modify the map in such a way that all the above listed properties remain true and the last one holds with $|Imz| < 1$.

This can be easily done by introducing a diffeomorphism $\mathcal{V}$ which is the identity inside the strip $-0.5 < \Im z < 0.5$. The strip $-1 < \Im z < 1$ which gets mapped onto $-2 < \Im < 2$ and outside that strip the map is a shift by 1 vertically. Obviously, a quasiconformal map with these properties exists.

So then we may consider

$$\rho_2 := \mathcal{V}^{-1} \circ \rho_1 \circ \mathcal{V}$$
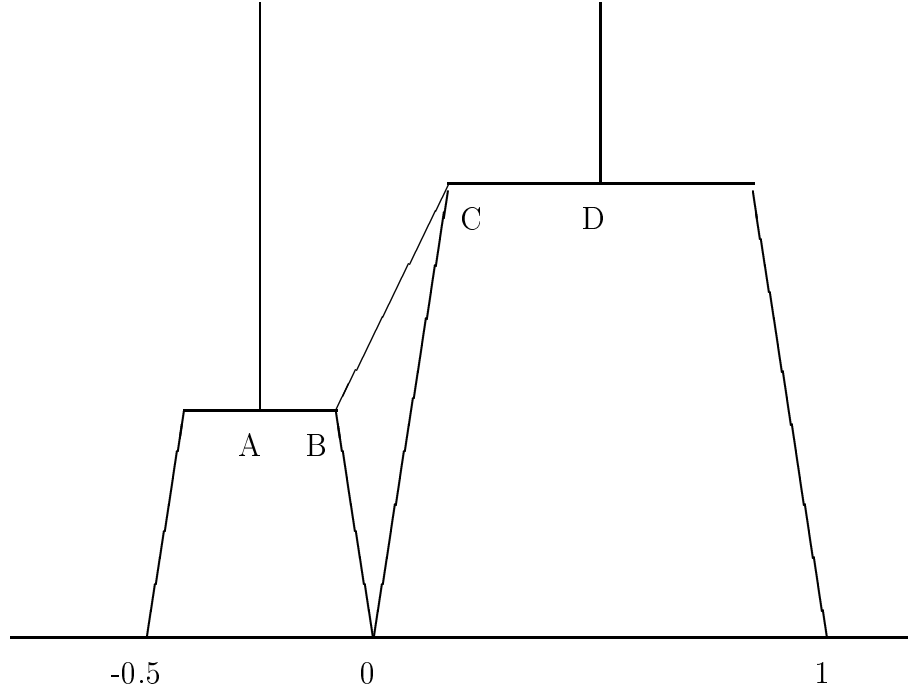
which does what we wanted.

A definition and comments on hexagonal extensions. A hexagonal extension, denoted by G_h is defined to be ρ_2 restricted to the hexagon with vertices $0, 0.01 - i, 0.99 - i, 1, 0.99 + i, 0.01 + i$. It is easily verified that indeed this hexagon is inside the domain of ρ_2.

The hexagonal extension G_h has a number of properties which will be important for us and can be verified straightforwardly:

- It is the same as G inside the diamond with vertices $0, 0.5 - 0.5i, 1, 0.5 + 0.5i$, in particular on the large diamond neighborhood of the domain of $\hat{\xi}$.

- The map can be continuously extended and then the boundary of its domain is mapped onto itself.

- The map is quasiconformal.

- It is the identity outside the region $|Imz| < 2$.

- G_h is the identity on the top and bottom edges, and also on its entire boundary if Υ_2 was not boundary-refined.

Pasting the neighbors in the chain boundary-refinement. We are ready to describe how to complexify a chain boundary refinement.

First, can construct hexagonal extensions for all members of the chain. The next thing to do is to glue the neighbors together. Namely, we assume that one interval of the chain is $(0, 1)$ and its neighbor is $(-a, 0)$. All this looks as follows:

The drawing shows only half of the actual picture, which is symmetric with respect to the real axis. The maps originally are only defined within the two hexagons. We will define their **glueing map** which will extend their set-theoretical sum. In addition to the union of the hexagons, the glueing map is defined in the whole infinite strip between the lines $\Re = -0.25$ and $\Im = 0.5$. Moreover, it is assumed that the glueing map is the identity everywhere above the line joining points A, B, C, D. Of course, it is also symmetric with respect to the real axis. Finally, it is uniformly quasiconformal.[3]

Proposition 3 *A glueing map with these desired properties can always be constructed.*

Proof:
Since the lengths of any two adjacent branches in our construction are com-

---

[3]The word "uniformly" meaning that the bound for the QC norm should not depend on a particular choice of branches, only on two maps f and $\hat{f}$.

parable within uniform constants (see Proposition 1), the map on $B0C$ is intrinsically quasisymmetric. We leave it to the reader to complete the proof.

$$\square$$

The quasiconformal implementation of the chain boundary refinement. To implement the chain boundary refinement we glue together all neighbors using glueing maps and then take a set-theoretical sum of all glueing maps. This map is then extended by the identity on the whole complex plane. The complexified version of the chain boundary refinement is then composition with that map, called the *refining map*.

If the chain ends at a folding branch, we can put G_h equal to the identity on this folding branch, glue it with the last hexagon of the chain, and extend by the identity beyond the folding branch.

Summary. We can describe the complex realization of two main pull-back operations: the pull-back of on a single branch, and the chain pull-back. In both cases, the new branchwise equivalence Υ_3 can be written as $\overline{G_h} \circ \Upsilon_1$ where $\overline{G_h}$ is called the refining map. The construction of the refining map is more complicated in the case of a chain refinement and has been described above. If only one branch is refined by pulling back a non-boundary-refined Υ_2, then the refining map is just the extension of the hexagonal extension by the identity outside of the hexagon.

Regions of the refining map. Here, we assume that inside every close neighborhood a diamond neighborhood has been chosen, called *small diamond*. We will later explain how to specify small diamonds.

Given a refining map, we can split the plane into three regions:

- The pull-back region which is the union of images by Υ_1 of the close neighborhoods of the branches being refined.

- The trivial region which consists of all points whose whole neighborhoods are fixed by the map.

- The glueing region which is the rest.

The push-forward map. On the component of the pull-back region around the domain of a branch whose stopping time is s, we have the push-forward map defined simply as $\mathcal{EX}(f^s)$. The image should be thought to belong to the domain of Υ_2. Thus, formally the push-forward "map" is a pair: the map itself, and Υ_2.

4.3 Filling-in.

The structure of the complex filling-in. In the Branchwise Equivalences section we described the filling-in on the level of real branchwise equivalences as a limit of the sequence of critical pull-backs. This allows for an immediate extension of the procedure, since we have already defined complex realizations of critical pull-backs. However, certain questions emerge because of the infinite nature of this process. The most important thing that we need to prove is that the limit exists. Also, there is an issue of whether the limit map is going to be quasiconformal or even a homeomorphism. We prove a lemma which immediately implies the existence of a limit. The estimates will not be tackled until the next section.

Formal complex description. We are given an admissible branchwise equivalence Υ marked by its own critical value so that the ray covers the image of the interval by the folding branch. We assume that, if necessary, some branches of Υ have been boundary-refined and, therefore, all monotone branches obtained as a result of the critical pull-back will be uniformly extendable.

We build the sequence of branchwise equivalences defined inductively as follows:

1.

$$\Upsilon_0 := \Upsilon \ .$$

2. Υ_{i+1} is the result of a simultaneous critical pull-back of Υ_i onto all critical branches whose domains are covered by the marking ray.

A fineness requirement for Υ. Consider two quantities: α, the length of the longest domain of a branch of Υ, and β, the infimum of the sizes of

small diamonds around the branches of Υ. A fineness requirement is that the ratio $\frac{\beta}{\alpha}$ be sufficiently large. How large we need will specified in the proof of the next lemma.

The existence of a limit.

Lemma 4.2 *Let Υ satisfy a suitable fineness requirement, to be specified on the course of the proof. Then the sequence Υ_i converges everywhere. Moreover, for every point z not in the real line, its image $\Upsilon_i(z)$ stabilizes after a finite number of steps.*

Proof:
We observe that all Υ_i are the same as Υ_0 except on some diamond neighborhood of the interval $[-1, 1]$ whose size is proportional to the length of the longest folding branch of Υ. Indeed, each Υ_{i+1} is obtained by the critical pull-back onto the folding branches of Υ. However, the refining map for each pull-back is the identity except on the hexagon of the diameter comparable to the domain length.

From these two observations, we infer that a fineness requirement can be chosen so that the image of the push-forward map for any branch of Υ being refined contains the region in which Υ_i and Υ_0 differ. The possibility of that follows from Proposition 2.

Then, it follows immediately that Υ_{i+1} is the same as Υ_1 except on the preimage of the pull-back region of the refining map. Since the map being refined is always Υ, the push-forward is fixed. Thus, we see that if z is in the domain of the push-forward map, the sequence $\Upsilon_i(z)$ stabilizes if and only if the sequence $\Upsilon_i(\rho(z))$ stabilizes where ρ stands for the push-forward map. Thus, the lemma will be proven if we show that ρ can only have finitely many iterates for any z not on the real line.

This is so because, again if Υ is sufficiently fine, we see that the distance of $\rho(z)$ from the real line is either larger by a fixed constant than the distance from from z to the line, or $\rho(z)$ is no longer in the domain of ρ. To this end, we have to examine the folding branch whose small diamond contains z. It can be written as $h_1 \circ Q \circ h_2$. The monotone branches h_1 and h_2 roughly preserve distances from the real line relative to the domain and the image. Thus, since the domain is small and the image is the whole interval $[-1, 1]$, the distance indeed grows unless Q decreases it a lot. That can only happen

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if $h_1(z)$ is close to the imaginary line. But then $\rho(z)$ will be close to the line, but its projection onto the line will be beyond the marking ray. Thus, $\rho(z)$ will be in the region where Υ_i is no different from Υ_0.

$\square$

Complex distortion of the push-forward map. Consider a filling-in construction, a point z not on the real line, and and integer k. Let Υ_∞ mean the limiting branchwise equivalence. By the push-forward step we mean the following procedure. Find the largest i so that $\Upsilon_j(z)$ is the same same as $\Upsilon_k(z)$ for all $k > j \geq i$. Assume that $i > 1$. Then, as argued in the proof of Lemma 4.2, $\Upsilon(z)$ is in the pull-back region. Thus, the push-forward map can applied to find z_1 We also put $k_1 = i - 1$. Denote this push-forward map with ζ_1. We can the repeat the push-forward step with $z := z_1$ and $k := k_1$, and thus construct the sequences up to z_l and k_l. The construction may end when $i_l = 0$ or 1. In the latter case z_l is in the glueing region.

We can then compose the push-forward maps $\zeta_l \circ \cdots \circ zeta_1$. An important property of our construction is that

$$\Upsilon_k = (\hat{\zeta}_l \circ \cdots \circ \hat{\zeta}_1)^{-1} \circ \Upsilon_{i_l} \circ \zeta_l \circ \cdots \circ\ zeta_1$$

on a neighborhood of z. In the future, we will need this lemma:

Lemma 4.3 *For any z, the complex distortion of the corresponding composition on z' in a neighborhood of z $\zeta_l \circ \cdots \circ\ zeta_1$ is bounded in a uniform way proportionally to the y-coordinate of z_l. Υ.*

Proof:
All maps ζ_i are local extensions of folding branches of Υ. We claim that the complex distortion of the composition at any point z' in a neighborhood of z where the composition is defined is bounded proportionally to the sum of the y-coordinates of points z_i. Indeed, by Lemma 3.9, the complex distortion of ζ_i can be bounded proportionally to the sum of y-coordinates of z_i and z_{i-1}.

But, as in the proof of Lemma 4.2, we argue that the y-coordinates grow exponentially with the push-forward step. The lemma follows.

$\square$

5 Estimates of conformal distortion

5.1 Global description of the construction

Postulates. In the previous section we learned how to realize complex steps of individual pull-back, chain refinement and filling-in. Thus, we already know that the construction described in the Branchwise Equivalences section could be traced by these complex procedures. We are now ready for our most challenging task of estimating the conformal distortion of the maps we get.

We start with an abstract approach by defining an admissible complex construction.

An admissible complex construction. We begin with a primary complex branchwise equivalence which we will need in four versions: non-boundary-refined, fully boundary-refined, and boundary-refined on each side. We still reserve the right to choose these primary branchwise equivalences suitably fine.

Then, we proceed to build more branchwise equivalences by these steps used in an arbitrary order:

- A monotone or critical pull-back on a single branch.

- A simultaneous chain pull-back.

- A filling-in as described in the previous section.

In addition, we assume that the construction is conducted so the length ratio of any pair of adjacent domains is always uniformly bounded. Also, we inductively define "irregularity" of a point on the real line as follows. All points of the primary equivalences receive irregularity 0. If Υ_1 is refined by pulling back Υ_2, the irregularity at a point is equal to the sum of its irregularity with respect to Υ_1 and the irregularity of its push-forward image relative Υ_2 increased by 1 if the point is an endpoint of a branch being refined and Υ_2 is not boundary refined on the side of the push-forward image of the point. point.

We will later add one more assumption, but we need to preparations to state it clearly.

We notice that extensions of real branchwise equivalences built in the Branchwise Equivalences section can be obtained in an admissible construction. The estimate on the length ratio of adjacent domains follows directly from Proposition 1 and the second property is also provided by the real construction.

From our construction, we see that $D(\zeta_i)$ are restricted by two conditions: that they must be inside $D(\xi)$, and also inside the preimages of $D(\zeta_i)$.

The tree. The complex construction is quite complicated, and trees can be used to describe and better understand it. We now understand the construction as a set of branchwise equivalences where each comes with the prescription for how to build it from other branchwise equivalences so that it is possible to ultimately reduce it to the primary branchwise equivalences.

A vertex of the tree is the following triple: a point z of the complex plane, a branchwise equivalence Υ_1 and its refining map G.

Each vertex may have up to two daughters: one "left" and one "up". If $\Upsilon_1(z)$ is not in the pull-back region of G, we look at how Υ_1 was constructed. If Υ_1 is primary, there are no daughters. Otherwise, it is equal to $G_0 \circ \Upsilon_0$. Then (z, Υ_0, G_0) is the left daughter, and still there is no up daughter.

If $\Upsilon_1(z)$ is in the pull-back region, we look at its push-forward image $\rho(z)$ in the domain of Υ_2. We find the left daughter as in the previous case, and there is an up daughter, too. If Υ_2 was obtained as $G_3 \circ \Upsilon_3$, the up daughter is $(\rho(z), \Upsilon_3, G_3)$.

Thus, given one vertex a tree can be built according to these rules.

Degrees of branches. Given a tree, we introduce the *degree* of a folding branch. By definition, it is 0 for all branches of the primary map. Whenever a new central branch is created, its degree grows by 1 compared with the degree of the old central branch. Finally, the degree of a folding branch which is the preimage of some central branch is equal to the degree of that central branch.

Clearly, the length of the domain decreases exponentially fast with the degree and the ratio can be controlled by choosing the primary map.

Restrictions on admissible constructions. We make two more assumptions about our admissible constructions.

- Given any Υ and its refining map G, the tree built from (z, Υ, G) is finite except for z from a set of zero measure.

- In any tree of the construction if we follow a branch up, the degrees of folding branches being refined form a non-increasing sequence. The degrees of consecutive branches can be equal only if they belong to consecutive branchwise equivalences in a filling-in step.

From now on when we speak of admissible constructions, we mean that these two properties hold as well.

Push-forward along vertical branches. In the previous section, we defined the push-forward map on any component of the pull-back region. Now, the push-forward map may be associated with a vertical edge in the tree of some point. Indeed, choose the triple $(Upsilon, G, z)$ with any z in the domain of the push-forward map, and the map itself is defined. Analogously, the definition can be extended so that we can push-forward along any vertical branch. The definition, which now depends on the choice of some tree, goes simply by composing the push-forward maps which correspond to consecutive vertical edges. The map is defined wherever the composition is.

Small diamonds.

Bounded sizes of close neighborhoods.

Lemma 5.1 *The sizes of close neighborhoods in an admissible construction are uniformly bounded away from* 0.

Proof:
This is clearly true of primary branchwise equivalences whose close neighborhoods have unit size. If we examine close neighborhoods of branches obtained in a pull-back operation, we notice that they certainly contain preimages of the close neighborhoods of the branches being pulled back intersected with the large diamond around the branch being refined. From this, it follows by induction that for any branch with stopping time s all points whose forward orbits stay in large diamonds around intermediate images are in the close neighborhood. This set contains a diamond of fixed size by Proposition 2.

$\square$

These diamonds of fixed size will be called *small diamonds.*

We notice that, in addition to being contained in close neighborhoods, small diamonds have this property:

If z is in the small diamond of a branch created by refinement of a branchwise equivalence Υ by a refining map G, any push-forward map constructed for the vertical branch starting at (Υ, G, z) is defined on the whole small diamond.

Actually, the defining formula of close neighborhoods may be regarded as a corollary from this "push-forward" property when the push-forward goes all the way up to the primary equivalence.

Bounded conformal distortion of marking. When we defined complex marking, the question was left unanswered of the complex distortion of the modified map A. Now, in view of Fact 4.2 and Lemma 5.1 we see that the modification can be done in a uniformly quasiconformal fashion.

The choice of primary branchwise equivalences. We can choose the primary branches short enough so that the following is satisfied:

- Any complexified branchwise equivalence coincides with the primary branchwise equivalence except on inside a diamond neighborhood of the unit interval. The size of that neighborhood can be made arbitrarily small by choosing suitably short branches in the primary map.

- Suppose that a branch ζ is being refined. Subsequently, consider a chain refinement which involves any branch created inside ζ, possibly after many steps. Then, the corresponding refining map is the identity outside the small diamond neighborhood of ζ, unless a branch inside ζ adjacent to an endpoint of ζ is involved in the chain refinement.

  In particular, if the original refinement was a pull-back of a boundary-refined map, no future refinements inside of ζ will ever affect the region outside of the diamond.

Quasiconformal estimates for the refining map. Clearly, the quasiconformal distortion of the map is null on the trivial region, while on the pull-back region it strictly depends on the properties of the maps being pulled back. We want the following estimate:

The quasiconformal distortion is uniformly bounded on the glueing region.

Let us think for a moment that only monotone branches are being refined. Then, it is sufficient to choose a suitably fine primary branchwise equivalence. Indeed, we showed in section 3 that the glueing operations only contribute a uniformly bounded distortion. The only issue is to show the potentially unbounded distortion coming from the pullback itself is supported inside the pull-back region. We noted the the potentially unbounded distortion of the map being pulled back is supported inside a diamond, which can be made tiny by choosing the primary equivalence appropriately. Thus, it will also be a tiny diamond after the monotone pull-back, and we can choose the constants so that in fact it fits inside the pull-back region. The critical pull-back poses a problem, though. There will be a part of the preimage of the diamond which sticks out.

This is why we construct marked maps in a special way, so that we do not refine the primary branches whose preimages are going to be imaginary. Then, the same argument which we have used for monotone pull-backs still applies.

5.2   Estimates

Rough distortion. We will estimate quasiconformal distortion in terms of a "combinatorial" object that we call *rough distortion*. The definition is as follows:

Definition 5.1 *For any branchwise equivalence, we define an integer valued function on the plane. For the primary branchwise equivalence it is identically 0. For a refining map, it is 0 in the trivial region, the same as at the corresponding points of the pulled-back map in the pull-back region, and 1 in the glueing region. Finally, after a pull-back operation, the value of the function is the sum of its value for the map being refined and for the refining map at the image.*

The function so defined is called the rough distortion.

Remark: Thus, precisely speaking, the rough distortion depends not just on the branchwise equivalence, but also on the way it was obtained (although this way is in fact unique in our construction.) Since all our arguments are recursive, that makes no difference.

Complex distortion bounded by rough distortion. Here is an important lemma.

Proposition 4 *There is a function $Q(n)$ so that for any branchwise equivalence if the rough distortion at a point is n, the quasiconformal distortion is bounded by $Q(n)$.*

Proof:
We will prove that by induction with respect to the rough distortion. Fix your attention on the map being refined and some point z. We look for the last refinement step that changed the map in a neighborhood of z. For that step, z cannot be in the trivial region. In z is in the glueing region, we are done. Indeed, the rough distortion must have grown by 1 compared with the map being refined. So, this cannot happen at all at the initial step of the induction (rough distortion equal to 0), otherwise an estimate follows from the fact that the complex distortion of the glueing map in the glueing region is uniformly bounded.

So, the real problem occurs if z is in the pull-back region. Then, we consider its push-forward image z' in some branchwise equivalence. If z' again is in the pull-back region of some refinement, we can iterate the procedure. Thus, we get a sequence of points $z_0 := z, \ldots, z_k$ of images by consecutive push-forward maps, and $\Upsilon_k(z_k)$ is no longer in the pull-back region. Let $\zeta_0, \ldots, \zeta_{k-1}$ denote the consecutive push-forward maps. Since the glueing regions are open, the composition $\zeta_{k-1} \circ \cdots \circ \zeta_0$ is defined on an neighborhood of z. To complete the proof of the proposition, it will be enough if show that the complex norm of this composition is uniformly bounded. Indeed, by the properties of the pull-back region, the branchwise equivalence on a neighborhood of z is given by

$$(\hat{\zeta_{k-1}} \circ \cdots \circ \hat{\zeta_0})^{-1} \circ \Upsilon_k \circ \zeta_{k-1} \circ \cdots \circ \zeta_0 \ .$$

As we already noted, the complex distortion of Υ_k at z_k can be bounded from induction.

Every point z_i for $i < k$ can be associated with a branch, namely the only in whose small diamond it is. We call this branch ξ.

Next, we seek out sequences of critical pull-backs which correspond to a filling-in operation. We regard the push-forward map which corresponds to the filling-in (see previous section) as just one map.

So, we get subsequences indexed by i_j. Maps ζ_{i_j} are now of three types: local extensions of monotone and folding branches, and push-forward maps of the filling-in.

Then, we observe that the complex distortion of the whole composition at z is bounded in a uniform fashion proportionally to the sum of lengths of ξ_{i_j}. Indeed, look at the ζ_{i_j} at z_{i_j}. The height of z_{i_j} relative the domain ξ_{i_j} as well of $z_{i_{j+1}}$ relative the domain $\xi_{i_{j+1}}$ is bounded for any $i_{j+1} < k$ by virtue of both points being in their respective small diamonds.

The map ζ_{i_j} is nothing else but the local extension of a branch being refined. If it is monotone, or of the form

$$h_1 \circ Q \circ h_2$$

where h_1 and h_2 are monotone so that Lemma 3.9 can be used to bound the complex distortion of the extensions of monotone branches. The result is that the distortion of ζ_{i_j} is bounded is bounded by the sum of lengths of the domains of ξ_{i_j} and $\xi_{i_{j+1}}$. If ζ_{i_j} a filling-in push-forward, the estimate follows directly form Lemma 4.3.

This reasoning does not apply to ζ_{k-1}, but its contribution is also uniformly bounded. Thus, we only need to sum up the contributions for all i to the bound proportional to the sum of lengths of the domains.

This reduces the problem to the real line.

Next, we pick a subsequence of j, which we denote with j_l. An index j enters this subsequence unless ζ_{i_j} is monotone. We claim the sum of lengths of all domains is bounded proportionally to the sum of lengths of domains of $\xi_{i_{j_l}}$. Indeed, consider the domains of ξ_{i_m} with $j_l < m \leq j_{l+1}$. Since ζ_{i_m} are monotone except for $m = j_{l+1}$, the lengths of ξ_{i_m} increase exponentially with m. Thus, the total is bounded by the last term, which gives our claim.

Finally, the lengths of the domains of $\xi_{i_{j_l}}$ grow exponentially with l as a direct consequence of admissibility of the construction, namely, that the degrees of folding branches decrease up any vertical branch the tree unless consecutive vertices belong to the same pull-back operation.

The proposition follows.

□

Thus, it remains to show that the rough distortion is uniformly bounded, which is what we do next.

Boundedness of the rough distortion.

Lemma 5.2 *Inside the small diamond neighborhoods the rough distortion is 0.*

Proof:
By the definition, the rough distortion at a point inside the large diamond is the same as the rough distortion at its image by push-forward map. But the small diamond was defined by the property that the push-forward map can be iterated all the way, and for the primary branchwise equivalence the rough distortion is 0.

□

We look for the simplest branchwise equivalence so that the rough distortion at a point z is k. That means that when the branchwise equivalence was created, neither the pulled-back map nor the refined map had points with rough distortion k. Consider the map being refined. Clearly, the image of z is in the glueing region. Then, look at the branch directly below z. Call this branch ζ. If z is above the boundary of two branches, take any of them. Observe that z cannot be above the Cantor set, since such points are fixed by subsequent construction.

Since the image of z was in the glueing region, the height of z with respect to ζ is bounded away from 0 and infinity. Then look for the refinement step when ζ was created. Unless ζ was adjacent to the endpoint of the branch then being refined, z was in the small diamond. If z is in the small diamond of the branch being refined, we can push both z and ζ forward and look at the corresponding objects (we continue to call them z and ζ.) Then, we can repeat the procedure. Thus, we arrive at one of two possible outcomes: either we can push forward to the primary map, or at some point ζ is adjacent to the boundary of the branch being refined, and moreover the ζ is outside of the small diamond neighborhood of that branch. In the first situation the rough distortion at ζ is 0, so k is 1 and we are done. Let us consider

the branch being refined and call it ζ_1. Clearly, Z must also be the end of the whole chain, because otherwise the adjacent refinements would both be boundary-refined and there would be branch adjacent to z. The point z must be very close to an endpoint od ζ_1 which we call Z compared with the length of ζ_1. How close again depends on the choice of the primary map. The rough distortion at z is now at least $k-2$. We use the same argument to z and ζ_1. That means that we either can push them forward to the primary map, or a push-forward image coincides with an endpoint of the branch being refined, in which case the rough distortion distortion may drop by 1. However, our construction ensures that among the push-forward images of any point at most two are endpoints of chains.

Thus, we can get at most one repetition of this situation. So, $k \leq 3$.

By Proposition 4 this concludes the proof of the main theorem.

5.3   Invariance of the Collet-Eckmann condition

The construction of [13] provides maps with absolutely continuous invariant measures which are all basic. Such maps constitute a positive measure set of parameter values for typical one-parameter families of S-unimodal maps. As proved by Benedicks and Carleson (see [2]) the same is true for maps satisfying Collet-Eckmann condition which can be written as

$$Df^n(0) > ab^n a > 0, b > 1$$

for every $n > 0$.

Theorem1 implies that:

Corollary   *For maps from $\mathcal{C}$ with basic dynamics, the Collet-Eckmann condition is a topological invariant.*

Proof of the Corollary   The basic construction of [15] results in a partition of $(-q, q)$ into domains of monotone branches f_i which are uniformly extendable and so are all their compositions.

Next, we notice that if $f_i = f^{n_i}$, then all compositions f^j, $j \leq n_i$ are also extendable from the domain of f_i. Indeed, the "space" around the image of f^j is the preimage of the space around the image of f^{n_i} by a negative Schwarzian map, hence it is large.

Thus, the derivatives in the Collet-Eckmann condition are approximated up to a multiplicative constant by ratios of lengths of dynamically defined intervals.

But the qs conjugacy is also Hölder continuous and so exponential decreasing of such ratios is preserved.

6 Renormalizable polynomials

In this section, we extend our results to a certain class of renormalizable S-unimodal maps, including some infinitely renormalizable ones for which the result is new even in the polynomial case.

6.1 Statement of the problem

Restrictive induced maps.

Definition 6.1 *Suppose that ϕ is a preferred induced map or is unimodal, that is, consists of one folding branch. Suppose that the critical value of ϕ is in the domain of a folding branch ψ, moreover, under the iterations of ϕ the critical orbit forever stays inside the domain of ψ. If that happens, we say that ϕ is a restrictive induced map.*

Lemma 6.1 *If ϕ is a restrictive induced map, then the underlying f has a restrictive interval. If h_1 means the natural diffeomorphism from the domain of ψ onto the central domain of ϕ, then $h_1 \circ \phi$ is the first return map on this interval.*

Proof:
Standard.

$\square$

Remark. Suitable induced maps mentioned in the introduction are restrictive induced maps in the sense of Definition 6.1.

Renormalization. Let ϕ be a restrictive induced map, I be the restrictive interval from Lemma 6.1, and f_1 be the first return map onto I. Which rescale I affinely so that it becomes the unit interval. The first return map gives us some unimodal endomorphism of $[0, 1]$, which we will also call f_1. The basis of argument is this:

Fact 6.1 *Under our assumptions, $f_1 \in \mathcal{C}$. Moreover, if $f_1 = h_1(x^2)$ the distortion of h_1 is bounded in a uniform way.*

This follows directly from Theorem 1 in [21]. An equivalent result of [4] should also be noted.

Matching sequences of restrictive induced maps. Let $\varphi_0, \varphi_1, \varphi_\omega$ be a sequence of induced maps, either finite or infinite. All of them are restrictive except for the last one, φ_ω, if it exists. Also, we assume that φ_0 is an induced map on the standard domain $[-q, q]$ for some interval map f.

We say that a sequence which satisfies all these properties is a *matching sequence of restrictive induced maps* if φ_{i+1} and φ_i are related as follows:

associate a map f_1 with φ_i as in the previous paragraph. Then φ_{i+1} is an induced map on the standard domain for f_1.

The main result.

Proposition 5 *Let $\varphi_0, \ldots, \varphi_\omega$ be a matching sequence of restrictive induced maps for f, and $\hat{\varphi}_0, \ldots, \hat{\varphi}_\omega$ be an analogous sequence for a topologically conjugate map $\hat{f}$. We allow ω to be infinite.*

We assume that all φ_i with the possible exception of φ_ω are regular. Suppose, further, that admissible complexified branchwise equivalences Υ_i are given between φ_i and $\hat{\varphi}_i$ for which small diamonds can be chosen with uniform size, and which are uniformly quasiconformal.

If ω is finite, then Υ_ω is assumed to be conjugacy.

Then, f and $\hat{f}$ are quasisymmetrically conjugate. Moreover, the qs norm of the conjugacy is bounded by a continuous function of the small diamond size and the supremum of qc norms.

The rest of this section will devoted to the proof of Proposition 5. Before we tackle the proof, we would like to give a couple of simple corollaries.

Corollaries. First of all, Proposition 5 can be used in the "basic-renormalizable" case. This case can be characterized by the requirement that all maps of a matching sequence of restrictive induced maps can be obtained by the basic construction, that is, in the process of their construction the critical value of the intermediate preferred induced maps never falls into a folding domain. By the results of previous sections, complexified branchwise equivalences can be constructed which satisfy the assumptions of Proposition 5. So, our main theorem can be extended on the basic-renormalizable case.

This also includes a "basic-finitely renormalizable" case in which the non-renormalizable map obtained in the last box is also assumed to be basic. Then, we use Theorem 1 and Proposition 5 with finite ω to prove quasisymmetric conjugacy.

Also, the case of Feigenbaum, or "bounded type", maps considered in [21] is reduced to Fact 6.1. The "bounded type" assumption means that the number of branches of all maps from the matching sequence of restrictive induced maps is uniformly bounded. But then, their sizes must be uniformly comparable as an easy consequence of Fact 6.1, and maps Υ_i can be constructed "by hand" to satisfy the hypotheses of Proposition 5.

6.2 A single matching step

Not surprisingly, the idea of the proof of Proposition 5 is to somehow imprint the structure given by Υ_{i+1} into the restrictive which lives somewhere in Υ_i and continue with this process. To do this in a way that, given our results about admissible constructions, will automatically ensure a bounded qc norm of the result, we need to "prepare" Υ_i for this operation. Our matching step proposition is about that.

The matching step proposition.

Proposition 6 *Suppose that we have a regular restrictive induced map φ an the corresponding admissible complexified branchwise equivalence Υ with uniformly large small diamonds.*

Then, an admissible complex boundary-refined branchwise equivalence Υ' can be built which satisfies the following conditions:

- *The quasiconformal norm of Υ' is bounded by a uniform function of the qc norm of Υ.*

- *All branches are monotone and map onto the fundamental inducing domain of the first return map on the restrictive interval.*

- *The marked set comprises the complement of the union of the domains of the branches.*

Derivation of Proposition 5. We will show how Proposition 5 follows from Proposition 6.

If the matching sequence is infinite, we choose ω in an arbitrary fashion. We consider the following complex construction.

Maps $\Upsilon'_0, \ldots, \Upsilon'_{\omega-1}$ and a boundary-refined version of Υ_ω, called Υ'_ω, are primary where Υ'_i, $i < \omega$, means the map which corresponds to Υ_i by Proposition 6.

We construct a sequence Υ^i defined inductively. Υ^ω is equal to Υ'_ω. For $0 \leq i < \omega$, Υ^i is form by the pull-back of Υ^{i+1} onto all branches of Υ'_i. It should be noted that this operation can be realized as a simultaneous chain monotone refinement, usually with many chains.

Since this is an admissible construction, the qc norm of Υ^0 is bounded uniformly as a function of the maximum of norms of Υ'_i, hence of Υ_i.

If Υ_ω was a conjugacy, Υ^0 is, too. Otherwise, the matching sequence is infinite. We notice that the Υ^0 in that case coincides with the conjugacy except on the interior of all preimages of the restrictive interval of φ_ω. But ω can be chosen arbitrarily, and, as it grows, the complement of this set grows to a dense set (tends to the whole interval in the Hausdorff distance uniformly with ω). Hence, the corresponding maps Υ^0 tend to the conjugacy on the line in the C^0 topology. Even though it is not obvious that they converge everywhere, they are a normal family, since they are uniformly continuous and all identical except on a compact set.

Proposition 5 follows.

An outline of the construction. Let ψ mean the folding branch which fixes an image of the restrictive interval.

We will first describe how to construct Υ' on the real line. We will do so in familiar terms of pull-backs so that the complexification of this procedure will be easy.

Suppose that φ is not unimodal.

First, we want to pull the structure defined by Υ into the domain of the branch ψ. We notice that each point of the line which is outside of the restrictive interval will be mapped outside of the domain of ψ under some number of iterates of ψ. We can consider sets of points for which the number of iterates required to escape from the domain of ψ is fixed. Each such set clearly consists of two intervals symmetric with respect to the critical point. The endpoints of these sets form two symmetric sequences accumulating at the endpoints of the restrictive interval, which will be called *outer staircases*. Consequently, the connected components of these sets will be called steps.

This allows us to construct an induced map from the complement of the restrictive interval in the domain of ψ to the outside of the domain ψ with branches defined on the steps of the outer staircases. That means, we can pull-back Υ to the inside of the domain of ψ.

Next, we construct the *inner staircases*. We notice that every point inside the restrictive interval but outside of the fundamental inducing domain inside it is mapped into the fundamental inducing domain eventually. Again, we can consider the sets on which the time required to get to the fundamental inducing domain is fixed, and so we get the steps of a pair of symmetric inner staircases.

So far, we have obtained a branchwise equivalence which has one in-different domain equal the restrictive interval and besides has extendable monotone and folding branches. Denote it with Υ^1. The folding branches are all preimages of ψ. We now refine the folding branches.

This can be done in the usual filling-in way.

We conclude with refinement of remaining monotone branches analogous to the final refinement step in the basic case. Since there are no folding branches left, we can destroy all monotone branches. Thus, we will be left with indifferent branches only, all of which are certain preimages of the re-strictive interval. So, at least topologically, we obtain a good candidate for Υ' on the line.

In the case when φ is unimodal, the outer staircase cannot be constructed. Instead we build the inner staircases twice. The first step is as described. For the second step, we notice that the restrictive interval is the same as the fundamental inducing domain of ψ. So, the inner staircases can be built again.

This completes the real description of the matching step. What remains is to define the complex version of this procedure and do estimates.

Outer staircases.   Unless we explicitly indicate otherwise, the assumption is that φ is not unimodal.

Pulling-in outer staircases from far away.   Suppose that the domain of ψ is very short compared with the length of the the domain of φ. This means that the domain of ψ is extremely large compared with the restrictive interval. This unbounded situation leads to certain difficulties and is dealt with in our next lemma.

Lemma 6.2 *One can construct a map* Υ^1 *which is an admissible complex branchwise equivalence and its qc norm, as well as the sizes of its small diamonds are uniformly related to the analogous estimates for* Υ. *In addition, an integer* i *can be chosen so that the following conditions are satisfied:*

- *The functional equation*

$$\Upsilon \circ \psi^j = \hat{\psi}^j \Upsilon^1$$

  holds for any $0 \leq j \leq i$ *whenever the left-hand side is defined.*

- *The length of the interval which consists of points whose* i *consecutive images by* ψ *remain in the domain of* ψ *forms a uniformly bounded ratio with the length of the restrictive interval.*

Proof:
We rescale affinely so that the restrictive intervals become $[-1, 1]$ in both maps. Denote the domains of ψ and $\hat{\psi}$ with P and $\hat{P}$ respectively. Then, ψ can be represented as $h(x^2)$ where h''/h' is very small provided that $|P|$ is large. We can assume that $|P|$ is large, since otherwise we can take $\Upsilon^1 := \Upsilon$ to satisfy the claim of our lemma. Thus, assuming that $|P|$ is large enough, we can find a uniform r so that the preimages of $B(0, r)$ by ψ, $(\hat{\psi})$ and $z \to z^2$ are all inside $B(0, r/2)$. Also, we can have $B(0, r)$ contained in the small diamond around the domain of ψ. Next, we choose the largest i so that $[-r, r] \subset \psi^{-i}(P)$.

Then, we change ψ and $\hat{\psi}$. We will only describe what is done to ψ. Outside of $B(0, r)$, ψ coincides with its standard extension. Inside the preimage of $B(0, r)$ by $z \to z^2$ it is $z \to z^2$. In between, it can be interpolated by a bounded distortion smooth 2-1 local diffeomorphism. We leave to the reader

to convince himself it is possible to construct such a map. Also, look up [21] where a similar situation is considered. The modified extension will be denoted with ψ'.

Next, we pull-back Υ by ψ' and $\hat{\psi}'$ exactly i times. That is, if Υ_0 is taken equal to Υ, then Υ_{j+1} is Υ refined by pulling-back Υ_j onto the domain of ψ. This perhaps requires a little further clarification, since in Section 4 we only defined the pull-back by branches and under the assumption that the critical branch was in a monotone domain. Here, we mean the the simple pull-back is obtained by the same formula that would be used to pull-back by ψ, only ψ is replaced by ψ'. Note that no extra marking is required as the critical value of ψ is at 0 and 0 is preserved automatically. This determines the map inside the small diamond. Outside of it, the refining map is corrected in the usual way. Note that the middle branch of the map so constructed is not "folding" since it is of degree 2^i rather than quadratic. Hence, it does not satisfy our definition of the folding branch and must be considered indifferent. Υ_i constructed in this way can be taken as Υ^1.

Now we need to check whether Υ^1 has all the properties claimed in the Lemma. To see admissibility and the functional equation condition, we note that all branches of any Υ_j and their small diamonds are in the region where ψ coincides with ψ'. Thus, the same arguments as in Section 4 can be used to prove admissibility, and the functional equation is also evidently true.

The last condition easily follows from the fact that r can be chosen in a uniform fashion.

So, what remains is estimates of the qc norm Υ_i. First, we note that the qc distortion of Υ_j for any $j \le i$ at points not inside $B(0, r)$ is bounded as a uniform function of the qc norm of Υ. We notice that Υ_j in the complement of $B(0, r)$ is the pull-back of Υ by unmodified ψ. So, as usual, we can use the fact that the distances of push-forward images from the line grow exponentially, thus the total distortion added is bounded.

Points inside $B(0, r)$ are pull-backs of points outside of $B(0, r)$ by ψ'. But, ψ' is conformal inside $B(0, r) \setminus \psi'^{-1}(B(0, r))$ and quasiconformal inside $B(0, r)$. Also, only one push-forward image is inside the region where the map is not conformal. So, again, only bounded distortion is acquired.

$\square$

Comment. One should be aware that the situation handled in Lemma 6.1 is not a bounded pull-back situation. The proportions of the preimages of the domains on the last step constructed may be arbitrarily different from the proportions on the zeroth step. For example, even if Υ is not boundary-refined, it is not true that the preimage of the outermost domain constitutes any fixed part of the last step.

The staircase construction. We take Υ^1 obtained in Lemma 6.1 and restrict our attention to its restriction to the real line, denoted with v_1. We rely on the fact that v_1 is a quasisymmetric map and its qs norm is uniformly bounded in terms of the quasiconformal norm of Υ^1.

For a while, we will be working with real methods.

Completion of outer staircases. We will construct a real map v_2 from the domain of φ to the domain of $\hat{\varphi}$ with following properties:

- The map v_2 coincides with v_1 outside of the domain of ψ. Also, it satisfies
$$v_2 \circ \psi^j = \hat{\psi}^j \circ v_2$$
on the complement of the restrictive interval provided that ψ^j is defined.

- Inside the restrictive interval, it is the "inner staircase equivalence", that is, all endpoints of the inner staircase steps are mapped onto the corresponding points.

- Its qs norm is uniformly bounded as a function of the qc norm of Υ.

Outer staircases constructed in Lemma6.1 connect the boundary points of the domain of ψ to the i-th steps which are in the close neighborhood of the restrictive interval. Also, the i-th steps are the corresponding fundamental domains for the inverses of ψ in the proximity of the boundary of the restrictive interval.

From Fact 6.1, the derivative of ψ at the boundary of the restrictive interval is uniformely bounded away from one.

Then, it is straightforward to see that the equivariant correspondence between infinite outer staircases which uniquely extends v_1 from the i-th steps is uniformely qs.

Inside the restrictive interval, the map is already determined on the end-points of steps, and can be extended in an equivariant way onto each step of the inner staircase.

Re-complexification. We want to construct an admissible complex extension of v_2 which is regarded as a branchwise equivalence. This can readily be done by Lemma 4.1. The result will be called Υ^2. Note that Υ^2 is "primary" in the sense that all settling times are 1. As to the stopping times, they have been defined by the refinement procedure outside of the restrictive interval. However, we also want to regard steps of the inner staircase as domains of branches. The stopping time on a step is going to correspond to the iterate of ψ which maps this step onto the fundamental inducing domain inside the restrictive interval.

Remarks on the staircase construction. The map Υ^2 represents the first important step of matching in the case when φ is not unimodal. We have built both inner and outer staircases and they fit together. Moreover, we introduced the structure of an induced map in the outer staircase, i.e. it is divided into the domains of monotone branches and folding branches which are copies of ψ. In future, they will be refined and eventually all taken out.

The construction of Υ' when φ is unimodal. In this case, the construction is quite elementary. There is no outer staircase, so only the inner staircase is considered. The real map Υ_2 maps the steps of one inner staircase onto the corresponding steps. Then, it is extended beyond the restrictive interval so that it is affine outside of an interval twice its size, and is uniformly quasisymmetric. Then, regard it is a boundary-refined branchwise equivalence, and construct Υ^2 as its admissible extension by Lemma 4.1.

Next, we construct the map Υ_3 which is completely analogous to Υ_2, except that it now map the inner staircase inside formed by preimages of the fundamental inducing domain of φ and not of the first return map on its restrictive interval. Note that the branches of Υ_3 map in a monotone fashion onto the restrictive interval. Thus, we can pull-back Υ_2 on them, which a usual chain monotone pull-back. The result is Υ'. That it has the desired properties is clear.

Final filling of the outer staircase. It remains to construct Υ' in the case when φ is not unimodal.

We need to fill all monotone and secondary folding branches left outside of the restrictive interval.

Filling-in of secondary folding branches. We perform a filling-in of the folding branches of Υ_3. This only requires one-time marking of the primary branchwise equivalence, hence presents no problem.

Final refinement. Then, we apply the final refinement construction to fill all monotone branches. Since there is no obstacle presented by folding branches, the final refinement can be continued until all monotone branches have disappeared in the limit.

So, we have obtained Υ' in the non-unimodal case. Its topological properties are evident, and the fact that the qc norm is suitably bounded follows from out previous estimates.

Proposition 6 has been proven.

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