Dynamics of Certain Smooth One-dimensional Mappings
IV. Asymptotic geometry of Cantor sets

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Dedicated to Professor John Milnor on the occasion of his sixtieth birthday

Abstract

We study hyperbolic mappings depending on a parameter ε. Each of them has an invariant Cantor set. As ε tends to zero, the mapping approaches the boundary of hyperbolicity. We analyze the asymptotics of the gap geometry and the scaling function geometry of the invariant Cantor set as ε goes to zero. For example, in the quadratic case, we show that all the gaps close uniformly with speed $\sqrt{\varepsilon}$. There is a limiting scaling function of the limiting mapping and this scaling function has dense jump discontinuities because the limiting mapping is not expanding. Removing these discontinuities by continuous extension, we show that we obtain the scaling function of the limiting mapping with respect to the Ulam-von Neumann type metric.

Contents

§1 Introduction

Ulam and von Neumann studied the nonlinear self mapping $q(x) = 1 - 2x^2$ of the interval $[-1, 1]$. They observed that $\rho_q = 1/(\pi\sqrt{1 - x^2})$ is the density function of a unique absolutely continuous q-invariant measure (we only consider probability measures). In modern language, this observation follows from making the singular change of metric $|dy| = (2|dx|)/(\pi\sqrt{1 - x^2})$. If we let $y = h(x)$ be the corresponding change of coordinate and $\tilde{q} = h \circ q \circ h^{-1}$, then q becomes $\tilde{q}(y) = 1 - 2|y|$, a piecewise linear mapping with expansion rate 2 on $[-1, 1]$. The dynamics of $\tilde{q}$ is more easily understood.

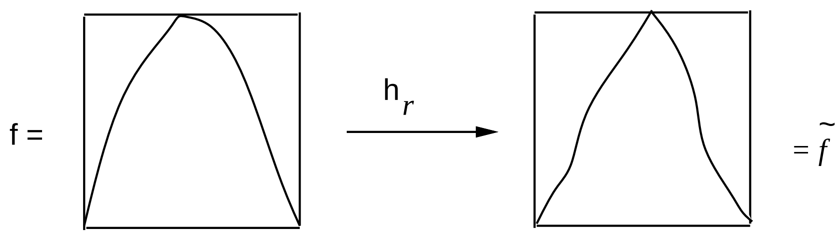


Figure 1a **Figure 1b**

Under quite general conditions, a mapping whose graph looks like the graph shown in Figure 2 has hyperbolic properties. We may say that a mapping whose graph looks like the graph shown in Figure 1a is on the <u>boundary of hyperbolicity</u> (a more precise definition of what we call the boundary of hyperbolicity is given below).

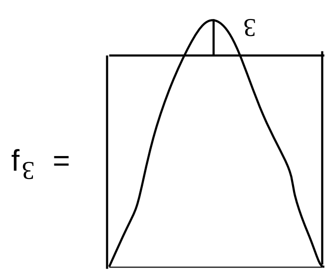


Figure 2

In order to study more general smooth self mappings of the interval with a unique power law critical point, we employ a change of metric similar to the one used by Ulam and von Neumann. The change of metric has singularities of same type at the two boundary points of the interval. It is universal in the sense that it does not depend on particular mapping f, but only on the power law $|x|^\gamma$ at the critical point. Suppose $y = h_\gamma(x)$ is the corresponding change of coordinate on the interval. After this change of coordinate, f becomes $\tilde{f} = h_\gamma \circ f \circ h_\gamma^{-1}$ (see Figure $1b$), which is smooth except at the critical point. The mapping $\tilde{f}$ has nonzero derivative at every point except the critical point. At the critical point, the left and the right derivatives of $\tilde{f}$ exist and are positive and negative, respectively.

A nice feature of the mapping $q(x) = 1 - 2x^2$ is that $\tilde{q}$ is expanding with Hölder continuous derivative, which implies that a certain binary tree of intervals associated with the dynamics of $\tilde{q}$ has bounded geometry (see [J2]). The expanding property does not carry over to our more general setting but the bounded geometry does (see [J2]).

Suppose E_0 is the set consisting of the critical point of f and of the two boundary points of the interval. For every positive integer n, let E_n be the preimage of E_{n-1} under f. The n^{th}-partition η_n of the interval determined by f is the collection of all the subintervals bounded by consecutive points of E_n. Let λ_n be the maximum length of the intervals in the n^{th}-partition.

We say the sequence of nested partitions $\{\eta_n\}_{n=0}^{\infty}$ determined by f decreases exponentially if λ_n decreases exponentially.

We need the following conditions in this paper.

(*) f to be a C^1 self mapping of an interval with a unique power law, $|x|^\gamma$ for some $\gamma > 1$, critical point and to map the critical point to the right endpoint of the interval and both endpoints of the interval to the left endpoint (see Figure 1a),

(**) the derivative of $\tilde{f}$ to be piecewise α-Hölder continuous for some $0 < \alpha \leq 1$ and

(***) the sequence of nested partitions $\{\eta_n\}_{n=0}^{\infty}$ determined by f to decrease exponentially.

The exponential decay of (***) were proved in [J2] under either of the following two hypotheses.

(1) The mapping f is C^3 with nonpositive Schwarzian derivative and expanding at both boundary points of the interval, that is, the absolute values of the derivatives of f at both boundary points are greater than one (see also [Mi]). The Schwarzian derivative of f is $S(f) = f'''/f' - (3/2)(f''/f')^2$.

(2) The mapping $\tilde{f}'$ is piecewise Lipschitz and all the periodic points of f are expanding, that is, the absolute values of the eigenvalues of f at all periodic points are greater than one (see also [Ma]). The eigenvalue of f at a periodic point p of period n of f is $e_p = (f^{\circ n})'(p)$.

We call the set of mappings satisfying (*), (**) and (***) the boundary of hyperbolicity, $\mathcal{BH}$.

The mappings f on $\mathcal{BH}$ are limits of mappings f_ε, which do not keep the interval invariant, but keep invariant a Cantor set Λ_ε having bounded geometry (see Figure 2). We say the mappings like f_ε are hyperbolic (a more precise definition of what we call a hyperbolic mapping is given below). The space of hyperbolic mappings, as well as the asymptotic behavior of these hyperbolic mappings as they approach the boundary of hyperbolicity, is the topic of this paper.

We use $f_{\varepsilon,0}$ and $f_{\varepsilon,1}$ to denote the left and right branches of f_ε, respectively. Let $g_{\varepsilon,0}$ and $g_{\varepsilon,1}$ be the inverses of $f_{\varepsilon,0}$ and $f_{\varepsilon,1}$. For a finite string $w = i_0 \cdots i_n$ of zeroes and ones, we use $g_{\varepsilon,w}$ to denote the composition $g_{\varepsilon,w} = g_{\varepsilon,i_0} \circ \cdots \circ g_{\varepsilon,i_n}$. Define $I_{\varepsilon,w}$ to be the image under $g_{\varepsilon,w}$ of the interval where f_ε is defined (see Figure 3).

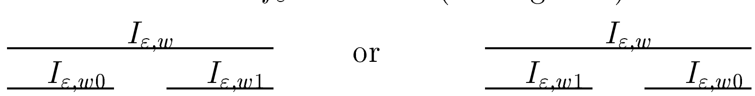


Figure 3

Suppose $\eta_{n,\varepsilon}$ is the collection of $I_{\varepsilon,w}$ for all finite strings w of zeroes and ones of length $n+1$. We use $\lambda_{n,\varepsilon}$ to denote the maximum length of the intervals in $\eta_{n,\varepsilon}$. Notice that the union of all the intervals in $\eta_{n,\varepsilon}$ covers the maximal invariant set of f_ε.

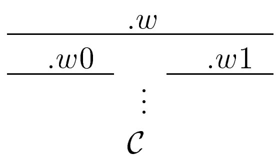


Figure 4a

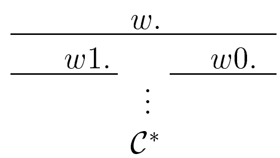


Figure 4b

There are two topologies on the set of all the labellings w. One topology is induced by reading the labellings w from left to right; the other by reading the labellings w from right to left. The limit set of the set of these labellings w in the topology induced by reading the labellings w from right to left is the phase space of the dynamical system f_ε. We call it the topological Cantor set $\mathcal{C}$ (see Figure 4a). Points in $\mathcal{C}$ are one-sided infinite strings of zeroes and ones extending infinitely to the right. If we take the limit set of the labellings w in the topology induced by reading from right to left, we obtain the dual Cantor set $\mathcal{C}^*$. A point in $\mathcal{C}^*$ is called a "dual point" which is one-sided infinite string of zeroes and ones extending infinitely to the left.

The scaling function of f_ε, when it is defined, is a function defined on $\mathcal{C}^*$. Assume $a^* \in \mathcal{C}^*$, so that a^* is a one-sided infinite string of zeroes and ones extending infinitely to the left. Suppose $a^* = (\cdots wi.)$, where w is a finite string of zeroes and ones and i is either zero or one. Note that I_{wi} is a subinterval of I_w. Let $s(wi)$ equal the ratio of the lengths, $|I_{wi}|/|I_w|$. We let $s(a^*)$ be the limit set of $s(wi)$ as the length of w tends to infinity. If this limit set consists of just one number for every $a^* \in \mathcal{C}$, then we say that $s(a^*)$ is the scale of f_ε at a^* and that s is the scaling function of f_ε defined on $\mathcal{C}^*$. Note that the scaling function $s(a^*)$ of f_ε depends on ε. Sometimes we denote it by $s_\varepsilon(a^*)$. The same definition gives the scaling function $s_0(a^*)$ of a mapping f_0 on $\mathcal{BH}$. Since for the mappings f_ε, $\varepsilon \geq 0$, the length of the interval I_w converges to zero uniformly as the length of w approaches infinity, it is obvious that the scaling function is a C^1-invariant. Recall that a smooth invariant is an object associated to f_ε which is the same for f_ε as for $h \circ f_\varepsilon \circ h^{-1}$ whenever h is an orientation preserving C^1-diffeomorphism.

To be sure that the limits defining the scaling function of f_ε actually exist, we need

(i) f_ε to be a mapping from an interval to the real line which maps its unique critical point out of this interval and both endpoints of the interval to the left endpoint (see Figure 2),

(ii) f_ε to be $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ and

(iii) the sequence of maximum lengths $\lambda_{n,\varepsilon}$ determined by f_ε to decrease exponentially.

Parallel to what we do for the boundary of hyperbolicity, we prove the exponential decay of (iii) under either of the following two hypotheses.

(1) The mapping f_ε is C^3 with nonpositive Schwarzian derivative and expanding at both boundary points of the interval.

(2) The mapping f_ε is $C^{1,1}$ and all the periodic points of f_ε are expanding.

We call the set of mappings f_ε satisfying (i), (ii) and (iii) the <u>space of hyperbolic mappings, $\mathcal{H}$</u>. Without loss of generality, we may assume that the interval where f_ε is defined is the interval $[-1, 1]$ and that the critical point of f_ε is zero.

Sullivan [S2] showed that the scaling function is a complete invariant for C^1-conjugacy of mappings in $\mathcal{H}$. It plays the same role that eigenvalues play in the C^2-case (recall that Sullivan [S1] showed that the eigenvalues are complete invariants for C^1-conjugacy of C^2 mappings of $\mathcal{H}$). We examine the asymptotics of the scaling function of f_ε as ε decreases to zero.

$\{f_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0} =$

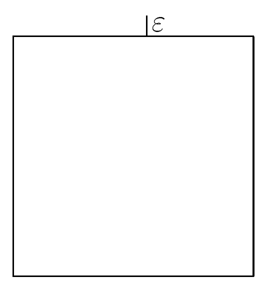


Figure 5

Hereafter, we say that a family $\{f_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ (see Figure 5) of mappings in $\mathcal{BH} \cup \mathcal{H}$ is a <u>good family</u> if it satisfies the following conditions:

1. the family $f_\varepsilon(x)$ is C^1 in both variables ε and x,

2. each f_ε has the same power law $|x|^\gamma$ with $\gamma > 1$ at the critical points and $R^-(x,\varepsilon) = f'_\varepsilon(x)/|x|^{\gamma-1}$ defined on $[-1,0]\times[0,\varepsilon_0]$ and $R^+(x,\varepsilon) = f'_\varepsilon(x)/|x|^{\gamma-1}$ defined on $[0,1]\times[0,\varepsilon_0]$ are continuous,

3. there are positive constants K' and $\alpha' \leq 1$ such that f_ε is $C^{1+\alpha'}$ and the α'-Hölder constant of f'_ε is less than K' for any $0 \leq \varepsilon \leq \varepsilon_0$.

4. there are positive constants K'' and $\alpha'' \leq 1$ such that $f'_\varepsilon(x)/|x|^{\gamma-1}$ defined on $[-1,0]$ and $f'_\varepsilon(x)/|x|^{\gamma-1}$ defined on $[0,1]$ are α''-Hölder continuous and their α''-Hölder constants are less than K'' for any $0 \leq \varepsilon \leq \varepsilon_0$,

5. there are two positive constants C_0 and $\lambda < 1$ such that the maximum lengths $\lambda_{n,\varepsilon}$ satisfy $\lambda_{n,\varepsilon} \leq C_0\lambda^n$ for all positive integers n and all $0 \leq \varepsilon \leq \varepsilon_0$.

A function s defined on $\mathcal{C}^*$ is called Hölder continuous if there are two positive constants C and $\lambda < 1$ such that $|s(a^*) - s(b^*)| \leq C\lambda^n$ for any a^* and b^* in $\mathcal{C}^*$ with the same first n coordinates.

Let $\mathcal{A}$ stand for the countable set of points in $\mathcal{C}^*$ whose coordinates are eventually all zeroes and let $\mathcal{B}$ stand for the complement of $\mathcal{A}$ in $\mathcal{C}^*$.

We can now state the main results of this paper more precisely.

THEOREM A. *Suppose $\{f_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ is a good family. There is a family of Hölder continuous functions $\{s_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ on the dual Cantor set $\mathcal{C}^*$ such that s_ε is the scaling function of f_ε and*

(1) for every $0 < \varepsilon_1 \leq \varepsilon_0$, s_ε converges to s_{ε_1} uniformly on $\mathcal{C}^*$ as ε tends to ε_1,

(2) for every $a^* \in \mathcal{C}^*$, the limit $s_0(a^*)$ of $\{s_\varepsilon(a^*)\}_{0<\varepsilon\leq\varepsilon_0}$ exists as ε tends to 0, the limiting function $s_0(a^*)$ is the scaling function of f_0 and satisfies:

(2.1) s_0 has jump discontinuities at all points in $\mathcal{A}$,

(2.2) s_0 is continuous at all points in $\mathcal{B}$ and the restriction of s_0 to $\mathcal{B}$ is a Hölder continuous function.

Our proof of this theorem depends on a distortion lemma (Lemma 12). We call it the uniform $C^{1+\alpha}$-Denjoy-Koebe distortion lemma.

More generally, every f on the boundary of hyperbolicity $\mathcal{BH}$ has a scaling function (see Figure 6). In fact, we show the following theorem.

THEOREM C. *Suppose f is on $\mathcal{BH}$ and $\tilde{f}$ is again f viewed in the singular metric associated to f. There exist the scaling function s_f of f and the scaling function $s_{\tilde{f}}$ of $\tilde{f}$ and these scaling functions satisfy:*

(a) $s_{\tilde{f}}$ is Hölder continuous on $\mathcal{C}^*$,

(b) s_f has jump discontinuities at all points in $\mathcal{A}$ and s_f is continuous at all points in $\mathcal{B}$,

(c) the restriction of s_f to $\mathcal{B}$ equals the restriction of $s_{\tilde{f}}$ to $\mathcal{B}$.

One example of a result about a scaling function for a mapping on the boundary of hyperbolicity is given in the following proposition (see Figure 6, c=0).

PROPOSITION 2. *Let $q(x) = 1 - 2x^2$ be the mapping of the interval $[-1, 1]$. Then $s_q(a^*) = 1/2$ for all a^* in $\mathcal{B}$ and $s_q(a^*) \neq 1/2$ for all a^* in $\mathcal{A}$.*

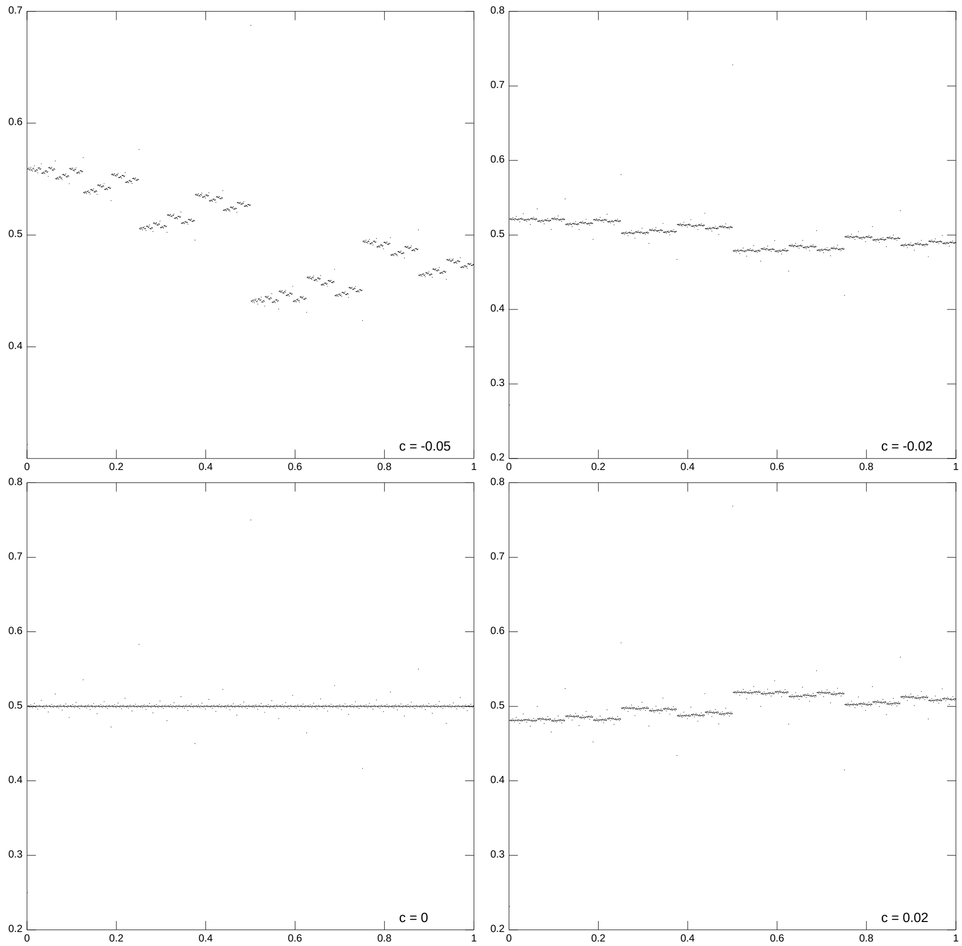
Suppose f_ε is a mapping whose graph looks like the graph shown in Figure 2 and $\{\eta_{n,\varepsilon}\}_{n=0}^{\infty}$ is the sequence determined by f_ε. We suppress the subscript ε when there can be no confusion. For every positive integer n and I_w in η_n, let I_{w0} and I_{w1} be the two intervals in η_{n+1} which are contained in I_w. We call the complement of I_{w0} and I_{w1} in I_w the gap on I_w and denote it by G_w. Let G be the complement of I_0 and I_1. We call G the leading gap and the set of ratios $\{|G_w|/|I_w|\}$ for all finite strings w of zeroes and ones the gap geometry of the maximal invariant set of f_ε. We study the asymptotic dependence on ε of the gap geometry of the family of the maximal invariant sets of f_ε for $0 \leq \varepsilon \leq \varepsilon_0$.

Suppose β is a function defined on $[0, 1]$. We say β determines asymptotically the gap geometry of the maximal invariant sets of f_ε for $0 \leq \varepsilon \leq \varepsilon_0$, if there is a positive constant C such that for all $0 \leq \varepsilon \leq \varepsilon_0$ and all finite strings w of zeroes and ones,

(1) $C^{-1}\beta(\varepsilon) \leq |G_{\varepsilon,w}|/|I_{\varepsilon,w}| \leq C\beta(\varepsilon)$ and

(2) $|I_{\varepsilon,wi}|/|I_{\varepsilon,w}| \geq C^{-1}$, where i is either one or zero.

The constant C is called a determining constant.



The graphs of the scaling functions for
$f_c(x) = -x^2 + 2 + cx^2(4 - x^2)$, *with* $c = -0.05$, -0.02, 0, *and* 0.02.

Figure 6

Suppose $\{f_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ is a good family. Then the sizes of the leading gaps G_ε is of order of $\varepsilon^{\frac{1}{\gamma}}$. Moreover, we prove the following theorem.

THEOREM B. *The family of the maximal invariant sets of f_ε for $0 < \varepsilon \leq \varepsilon_0$ is a family of Cantor sets. Furthermore, the function $\varepsilon^{\frac{1}{\gamma}}$ determines asymptotically the gap geometry of the family of maximal invariant sets of f_ε for $0 < \varepsilon \leq \varepsilon_0$.*

Suppose $HD(\varepsilon)$ is the Hausdorff dimension of the maximal invariant set of f_ε and s_ε is the scaling function of f_ε for $0 \leq \varepsilon \leq \varepsilon_0$. Theorem B has the following two corollaries.

COROLLARY 2. *There is a positive constant C which does not depend on ε such that*

$$0 < HD(\varepsilon) \leq 1 - C\varepsilon^{\frac{1}{\gamma}}$$

for all $0 \leq \varepsilon \leq \varepsilon_0$.

COROLLARY 3. *There is a positive constant C which does not depend on ε such that*

$$1 - C^{-1}\varepsilon^{\frac{1}{\gamma}} < s_\varepsilon((a^*0.)) + s_\varepsilon((a^*1.)) < 1 - C\varepsilon^{\frac{1}{\gamma}}$$

for all $a^* \in \mathcal{C}^*$ and all $0 \leq \varepsilon \leq \varepsilon_0$.

For the family $\{1+\varepsilon-(2+\varepsilon)x^2\}_{0\leq\varepsilon\leq 1}$ in $\mathcal{H}\cup\mathcal{BH}$, we can say even more:

PROPOSITION 3. *There is a constant $C > 0$ which does not depend on ε such that*

$$1 - C^{-1}\sqrt{\varepsilon} \leq HD(\varepsilon) \leq 1 - C\sqrt{\varepsilon}$$

for all $0 \leq \varepsilon \leq 1$.

§2 The Boundary Of Hyperbolicity, $\mathcal{BH}$

The example studied by Ulam and von Neumann [UN] in 1947 is the nonlinear transformation $q(x) = 1 - 2x^2$. They discovered the density function $\rho_q(x) = 1/(\pi\sqrt{1-x^2})$ of the unique absolutely continuous invariant measure for this mapping. From the metric point of view, $dy = 2\rho_q(x)dx$ is a singular metric on the interval $[-1, 1]$. Under the corresponding change of coordinate $y = h(x)$, q becomes a piecewise linear transformation $\tilde{q}(y) = h \circ q \circ h^{-1}(y) = 1 - 2|y|$. The point is that the dynamics of $\tilde{q}$ is more easily understood.

§2.1 The singular change of metric on the interval.

Suppose f is a C^1 self mapping of an interval with a unique critical point c. We always make the following assumptions: *(1)* f is a C^1 self mapping of $[-1, 1]$, *(2)* f is increasing on $[-1, c]$ and decreasing on $[c, 1]$, *(3)* f maps c to 1 and *(4)* f maps -1 and 1 to -1 (see Figure 1a). Without loss of generality, we always assume c equals 0.

Let $r_f(x) = f'(x)/|x|^{\gamma-1}$ for $x \neq 0$. We say f has power law at the critical point if there is some number $\gamma > 1$ such that the limits of $r_f(x)$ as x increases to zero and as x decreases to zero exist and equal nonzero numbers A and $-B$, respectively. For example, f has power law at the critical point if $f(x) = 1 - (A/\gamma)|x|^\gamma$ for negative x close to zero and $f(x) = 1 - (B/\gamma)|x|^\gamma$ for positive x close to zero. We call the ratio of A to $-B$, which is the limit of $f'(-x)/f'(x)$ as x decreases to zero, the asymmetry of f at the critical point (see [J3]). We always assume that f has power law $|x|^\gamma$ for some $\gamma > 1$ at the critical point.

We define the singular metric associated to f to be

$$dy = \frac{dx}{(1-x^2)^{\frac{\gamma-1}{\gamma}}}$$

on $[-1, 1]$ The corresponding change of coordinate on $[-1, 1]$ is $y = h_\gamma(x)$, where

$$h_\gamma(x) = -1 + b\int_{-1}^{x} \frac{dx}{(1-x^2)^{\frac{\gamma-1}{\gamma}}}$$

with $b = 2/\int_{-1}^{1} dx/((1+\varepsilon)^2 - x^2)^{\frac{\gamma-1}{\gamma}}$. The representation of f under the singular metric associated to f is

$$\tilde{f} = h_\gamma \circ f \circ h_\gamma^{-1}.$$

Before we see some properties of f and $\tilde{f}$, we state some definitions.

DEFINITION 1. *Suppose I is an interval and g is a function on I. We say that*

(a) g is an <u>embedding</u> if g is a homeomorphism from I to $g(I)$,

(b) g is a <u>C^1-embedding</u> if g has a continuous derivative g' on I and the derivative of g at every point in I is not zero, as usual, the derivative of g at each boundary point of I is a one-sided limit,

(c) g is a <u>$C^{1+\alpha}$-embedding</u> for some $0 < \alpha \leq 1$ if g is a C^1-embedding and the derivative g' on I is α-Hölder continuous on I.

If $\alpha = 1$, we usually say g is a $C^{1,1}$-embedding in (c).

Suppose I is an interval and g is an α-Hölder continuous function on I for some $0 < \alpha \leq 1$. There is a positive constant K such that $|g(x) - g(y)| \leq K|x-y|^\alpha$ for all x and y in I. The smallest such K is called the α-Hölder constant of g. If $\alpha = 1$, the smallest such K is usually called the Lipschitz constant of g.

LEMMA 1. *The mapping $\tilde{f}$ is continuous on $[-1,1]$ and the restrictions of $\tilde{f}$ to $[-1,0]$ and to $[0,1]$ are C^1-embeddings.*

Proof. If y is not one of 0, 1 and -1, then $\tilde{f}$ is differentiable at y. Suppose x is the preimage of y under h_γ. By the chain rule,

$$\tilde{f}'(y) = f'(x)(1-x^2)^{\frac{\gamma-1}{\gamma}}/(1-(f(x))^2)^{\frac{\gamma-1}{\gamma}}. \qquad (EQ\ 1.1).$$

Using this equation, we can get that $\tilde{f}'(0-)$ and $\tilde{f}'(0+)$ exist and equal nonzero numbers and that $\tilde{f}'(-1) = (f'(-1))^{\frac{1}{\gamma}}$ and $\tilde{f}'(1) = -(|f'(1)|)^{\frac{1}{\gamma}}$. *QED.*

REMARK 1. The inverse of h_γ is C^1. If the restrictions of r_f to $[-1,0)$ and to $(0,1]$ are α-Hölder continuous for some $0 < \alpha \leq 1$, then the restrictions of $\tilde{f}$ to $[-1,0]$ and to $[0,1]$ are at least $C^{1+\alpha}$ because of (EQ 1.1) (see [J4]).

LEMMA 2. *Suppose $\tilde{f}$ is a continuous self mapping of $[-1,1]$. Assume 0 is the unique turning point, f maps 0 to 1 and maps -1 and 1 to -1 and the restrictions of $\tilde{f}$ to $[-1,0]$ and to $[0,1]$ are C^1-embeddings. Then $f = h_\gamma^{-1} \circ \tilde{f} \circ h_\gamma$ is a C^1 mapping and has the power law $|x|^\gamma$ at the critical point 0 for any $\gamma > 1$.*

Proof. If x is not one of 0, 1 and -1, then f is differentiable at x. Suppose $y = h_\gamma(x)$. By the chain rule,

$$f'(x) = \tilde{f}'(y)(1-(h_\gamma^{-1}\circ\tilde{f}(y))^2)^{\frac{\gamma-1}{\gamma}}/(1-(h_\gamma^{-1}(y))^2)^{\frac{\gamma-1}{\gamma}}. \qquad (EQ\ 1.2).$$

Using this equation, $f'(-1) = (\tilde{f}'(-1))^\gamma$ and $f'(1) = -|\tilde{f}'(1)|^\gamma$, and the limits of $r_f(x)$ as x increases to zero and as x decreases to zero exit and equal nonzero numbers. *QED.*

REMARK 2. The mapping h_γ is $(1/\gamma)$-Hölder continuous. If the restrictions of $\tilde{f}$ to $[-1,0]$ and to $[0,1]$ are $C^{1+\alpha}$ embeddings for some $0 < \alpha \leq 1$, then f is $C^{1+\frac{\alpha}{\gamma}}$ and the restrictions of r_f to $[-1,0)$ and to $(0,1]$ are α/γ-Hölder continuous because of (EQ 1.2) (see [J4]).

§2.2 The definition of the boundary of hyperbolicity

Let f_0 and f_1 be the restrictions of f to $[-1,0]$ and to $[0,1]$, respectively. Then f_0 and f_1 are both embeddings. Let g_0 and g_1 be the inverse of f_0 and f_1. For a finite string $w = i_n \cdots i_0$ of zeroes and ones, let g_w be the composition, $g_w = g_{i_0} \circ \cdots \circ g_{i_n}$. Define I_w to be the image of $[-1,1]$ under g_w. The n^{th}-partition of $[-1,1]$ determined by f is the collection of I_w for all finite strings w of zeroes and ones of length $n+1$. We denote it by $\eta_{n,f}$ or just by η_n when there is no possibility for confusion. We use λ_n to denote the maximum length of the intervals in η_n. The n^{th}-partition $\eta_{n,\tilde{f}}$ of $[-1,1]$ determined by $\tilde{f}$ and the maximum length $\lambda_{n,\tilde{f}}$ of the intervals in $\eta_{n,\tilde{f}}$ are defined similarly.

DEFINITION 2. *We say that the sequence of nested partitions $\{\eta_n\}_{n=0}^\infty$ determined by f decreases exponentially if there are two positive constants C and $\lambda < 1$ such that $\lambda_n \leq C\lambda^n$ for all positive integers n.*

LEMMA 3. *The sequence of nested partitions determined by f decreases exponentially if and only if the sequence of nested partitions determined by $\tilde{f}$ decreases exponentially.*

Proof: Because h_γ is $(1/\gamma)$-Hölder continuous and the inverse of h_γ is C^1, we can easily see this lemma. *QED.*

DEFINITION 3. *The nonlinear mapping f is on the boundary of hyperbolicity, $\mathcal{BH}$, if*

(a) the restrictions of $\tilde{f}$ to $[-1,0]$ and to $[0,1]$ are $C^{1+\alpha}$ embeddings for some $0<\alpha\leq 1$,

(b) the sequence of nested partitions $\{\eta_n\}$ determined by f decreases exponentially.

Next lemma follows from Remark 1 and Remark 2.

LEMMA 4. *The nonlinear mapping f is on $\mathcal{BH}$ if and only if*

(i) f is $C^{1+\alpha}$ for some $0<\alpha\leq 1$ and the restrictions of r_f to $[-1,0)$ and to $(0,1]$ are β-Hölder continuous for some $0<\beta\leq 1$ and

(ii) the sequence of nested partitions determined by f decreases exponentially.

The condition (i) in Lemma 4 and $r_f(0-)=r_f(0+)$ are equivalent to the statement that $f(x)=F(-|x|^\gamma)$ where F is a $C^{1+\alpha}$ diffeomorphism from $[-1,0]$ to $[-1,1]$ (see [J4] for more details).

We give two examples of mappings on $\mathcal{BH}$. The reader may refer to the paper [J2] for the proofs that these two examples are on $\mathcal{BH}$

EXAMPLE 1. *Mappings f such that (1) f is C^3 with nonpositive Schwarzian derivative, (2) f is expanding at both boundary points of $[-1,1]$, that is, $f'(-1)$ and $|f'(1)|$ are greater than one, and (3) the restrictions of r_f to $[-1,0)$ and to $(0,1]$ are α-Hölder continuous for some $0<\alpha\leq 1$.*

The Schwarzian derivative $S(f)$ of f is $S(f)=f'''/f'-(3/2)(f''/f')^2$.

EXAMPLE 2. *Mappings f such that (1) the restrictions of $\tilde{f}$ to $[-1,0]$ and to $[0,1]$ are $C^{1,1}$ embeddings and (2) all the periodic points of f are expanding, that is, the absolute values of the eigenvalues of f at all periodic points are greater than one.*

The eigenvalue of f at a periodic point p of period n of f is $e_f(p)=(f^{\circ n})'(p)$.

§3 The Space Of Hyperbolic Mappings, $\mathcal{H}$

A type of perturbation of $q : x \mapsto 1 - 2x^2$ is a mapping $q_\varepsilon : x \mapsto 1 + \varepsilon - (2 + \varepsilon)x^2$ for a positive number ε. The mapping q_ε maps the critical point out of $[-1, 1]$ and it does not keep $[-1, 1]$ invariant but invariant a Cantor set which has bounded geometry. In this paper, we study the asymptotic behavior of certain mappings like q_ε as they approach the boundary of hyperbolicity (see Figure 5).

§3.1 The definition of the scaling function.

Suppose ε is a positive number and f_ε is a C^1 mapping from $[-1, 1]$ to the real line with a unique critical point c. We always make the following assumptions: (1) f_ε is increasing on $[-1, c]$ and decreasing on $[c, 1]$, (2) $f_\varepsilon(c) = 1 + \varepsilon$ and (3) f_ε maps 1 and -1 to -1 (see Figure 2). Without loss of generality, we always assume c equals 0.

Let $f_{\varepsilon,0}$ and $f_{\varepsilon,1}$ be the restrictions of f_ε to $[-1, 0]$ and to $[0, 1]$. They are two embeddings. Let $g_{\varepsilon,0}$ and $g_{\varepsilon,1}$ be the inverses of $f_{\varepsilon,0}$ and $f_{\varepsilon,1}$. For a finite string $w = i_0 \cdots i_n$ of zeroes and ones, let $g_{\varepsilon,w}$ be the composition $g_{\varepsilon,w} = g_{\varepsilon,i_0} \circ \cdots \circ g_{\varepsilon,i_n}$ and $I_{\varepsilon,w}$ be the image of $[-1, 1]$ under $g_{\varepsilon,w}$. Suppose $\eta_{n,\varepsilon}$ is the collection of $I_{\varepsilon,w}$ for all finite strings w of zeroes and ones of length $n + 1$ and $\lambda_{n,\varepsilon}$ is the maximum length of the intervals in $\eta_{n,\varepsilon}$. The union of the intervals in $\eta_{n,\varepsilon}$ covers the maximal invariant set of f_ε. We always assume that $\lambda_{n,\varepsilon}$ goes to zero as n increases to infinity.

For every interval in $\eta_{n,\varepsilon}$, there is the labelling w where w is the finite string of zeroes and ones such that this interval is the image I_w of $[-1, 1]$ under $g_{\varepsilon,w}$. There are two topologies on the set of all the labellings w . One topology is induced by reading the labellings w from left to right; the other topology is induced by reading the labellings w from right to left.

Suppose we read all the labellings w from left to right and $\mathcal{C}_n = \{w_n | w_n = (.i_0 i_1 \cdots i_n)$, where i_k is either 0 or 1 for $k \geq 0$ and $n \geq 0\}$. Let $\mathcal{C}_n$ have the product topology. The continuous mapping $\sigma_n : \mathcal{C}_{n+1} \mapsto \mathcal{C}_n$ is defined by $\sigma_n((.i_0 i_1 \cdots i_n)) = (.i_1 \cdots i_n)$ for $n \geq 0$. The pairs $\{(\mathcal{C}_n, \sigma_n)\}_{n=0}^{\infty}$ form an inverse limit set. Let $\mathcal{C}$ be the inverse limit of this inverse limit set and σ be the induced mapping on $\mathcal{C}$. We call $\mathcal{C}$ the <u>topological Cantor set</u>. For any a in $\mathcal{C}$, it is an infinite string of zeroes and ones extending to the right, that is, $a = (.i_0 i_1 \cdots)$ where i_k is either zero or one for $k \geq 0$. The mapping σ is the shift mapping on $\mathcal{C}$, that is, σ maps $(.i_0 i_1 \cdots)$ to $(.i_1 \cdots)$. We call $(\mathcal{C}, \sigma)$ the symbolic dynamical system of f_ε because of the following lemma.

LEMMA 5. *Suppose Λ_ε is the maximum invariant set of f_ε. There is a homeomorphism h_ε from $\mathcal{C}$ to Λ_ε such that $h_\varepsilon \circ \sigma = f_\varepsilon \circ h_\varepsilon$. In the other words, $(\Lambda_\varepsilon, f_\varepsilon)$ and $(\mathcal{C}, \sigma)$ are conjugate.*

Proof. Suppose $a = (.i_0 i_1 \cdots)$ is any point in $\mathcal{C}$. Let $w_n = (.i_0 \cdots i_n)$ be the first $n+1$ coordinates of a. The intersection of nested intervals $\{I_{w_n}\}_{n=0}^{\infty}$ is nonempty and contains only one point $x(a)$ because the length of I_{w_n} goes to zero as n increases to infinity. Define $h_\varepsilon(a) = x(a)$. Then h_ε is a homeomorphism from $\mathcal{C}$ to Λ_ε and $h_\varepsilon \circ \sigma = f_\varepsilon \circ h_\varepsilon$. *QED.*

Suppose we read all the labellings w from right to left and $\mathcal{C}_n^* = \{w_n^* | w_n^* = (i_n \cdots i_1 i_0.)$, where i_k is either zero or one and $n \geq 0\}$. Let $\mathcal{C}_n^*$ have the product topology. The continuous mapping $\sigma_n^* : \mathcal{C}_{n+1}^* \mapsto \mathcal{C}_n^*$ is defined by $\sigma_n^*((i_n \cdots i_1 i_0.)) = (i_n \cdots i_1.)$ for $n \geq 0$. The pairs $\{(\mathcal{C}_n^*, \sigma_n^*)\}_{n=0}^{\infty}$ also form an inverse limit set. Let $\mathcal{C}^*$ be the inverse limit of this inverse limit set and σ^* be the induced mapping on $\mathcal{C}^*$. We call $\mathcal{C}^*$ the <u>dual Cantor set</u>. Any a^* in $\mathcal{C}^*$ is an infinite string of zeroes and ones extending to the left, that is, $a^* = (\cdots i_1 i_0.)$ where i_k is either zero or one for $k \geq 0$. We call a^* a "dual point" of f_ε. The mapping σ^* is the shift mapping on $\mathcal{C}^*$, that is, σ^* maps $(\cdots i_1 i_0.)$ to $(\cdots i_1.)$. We call $(\mathcal{C}^*, \sigma^*)$ the dual symbolic dynamical system of f_ε.

A sequence $\{x_n\}_{n=0}^{\infty}$ in the maximum invariant set of f_ε is a sequence of backward images of x_0 under f_ε if $f_\varepsilon(x_n) = x_{n-1}$ for all positive integers n. The dual Cantor set will not represent the maximal invariant set of f_ε, but there is a one-to-one corresponding from the dual Cantor set to the set of sequences of backward images of x_0 under f_ε for all points x_0 in the maximal invariant set of f_ε. The <u>scaling function</u> of f_ε is defined on the dual Cantor set $\mathcal{C}^*$ if it exists (see [S2] and [J3]). For the sake of completeness, we give the definition of scaling function as the following.

Suppose a^* is in $\mathcal{C}^*$, so that a^* is an infinite string of zeroes and ones extending to the left. Assume $a^* = (\cdots wi.)$ where w is a finite string of zeroes and ones and i is either zero or one. Note that I_{wi} is a subinterval of I_w. Let $s(wi)$ equal the ratio of the lengths, $|I_{wi}|/|I_w|$.

We let $s(a^*)$ be the limit set of $s(wi)$ as the length of w tends to infinity.

DEFINITION 4. *Suppose f_ε is a mappings in $\mathcal{H}$. If the limit set $s(a^*)$ consists of only one number for a^* in $\mathcal{C}^*$, then we say there is the scale $s(a^*)$ of f_ε at a^*. If there is the scale $s(a^*)$ of f_ε for every a^* in C^*, then we call $s : C^* \mapsto \mathbf{R}^1$ the scaling function of f_ε. Note that the scaling function $s(a^*)$ of f_ε depending on ε. Sometimes we denote it by $s_\varepsilon(a^*)$.*

REMARK 3. For f on $\mathcal{BH}$, we can use the the same arguments as Definition 4 to define the scaling function s_f of f on $\mathcal{C}^*$ by the sequence of nested partitions $\{\eta_n\}_{n=0}^{\infty}$ determined by f.

§3.2 The definition of the space of hyperbolic mappings

Suppose $\{\eta_{\varepsilon,n}\}_{n=0}^{\infty}$ is the sequence determined by f_ε. Just as in Definition 2, we say the sequence $\{\eta_{n,\varepsilon}\}_{n=0}^{\infty}$ determined by f_ε decreases exponentially if $\lambda_{n,\varepsilon}$ decreases exponentially.

DEFINITION 5. The nonlinear mapping f_ε is in the space of hyperbolic mappings, $\mathcal{H}$, if

(a) f_ε is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ and

(b) the sequence $\{\eta_{n,\varepsilon}\}_{n=0}^{\infty}$ determined by f_ε decreases exponentially.

We give two examples of mappings in $\mathcal{H}$. They are similar to Example 1 and Example 2 in §2.

EXAMPLE 3. *Mapping f_ε such that (1) f_ε is a C^3 mapping on $[-1,1]$ with nonpositive Schwarzian derivative and (2) f is expanding at both boundary points of $[-1,1]$, that is, $f'_\varepsilon(-1)$ and $|f'_\varepsilon(1)|$ are greater than one.*

EXAMPLE 4. *Mapping f_ε such that (1) f_ε is $C^{1,1}$ and (2) all the periodic points of f_ε are expanding, that is, the absolute values of all eigenvalues of f_ε at periodic points are greater than one.*

The proofs, that Example 3 and Example 4 are in $\mathcal{H}$, are similar to the proofs of Example 1 and 2 in [J2].

DEFINITION 6. *A function s defined on dual Cantor set C^* is <u>Hölder continuous</u> if there are two positive constants C and $\lambda < 1$ such that $|s(a^*) - s(b^*)| \leq C\lambda^n$ for any a^* and b^* in C^* with the same first n coordinates. We call C a Hölder constant of s.*

LEMMA 6. *Suppose f_ε is in $\mathcal{H}$. There exists a Hölder continuous scaling function s_ε of f_ε.*

Proof. Let d_ε be the minimum value of the restriction of f_ε to the union of $I_{\varepsilon,0}$ and $I_{\varepsilon,1}$. We suppress ε if there can be no confusion. Note that d_ε goes to zero as ε decreases to zero. For any a^* in $\mathcal{C}^*$, we use $w_n i$ to denotes the first $(n+1)$ coordinates of a^* and $s(w_n i)$ to denote the ratio, $|I_{w_n i}|/|I_{w_n}|$. By (b) of Definition 5, we have two positive constants C_0 and $\lambda < 1$ such that $\lambda_{n,\varepsilon} \leq C_0\lambda^n$. Let K be the Hölder constant of f'_ε on $[-1,1]$. Because $s(w_m i) = (|(f^{\circ(m-n)})'(x)|/|(f^{\circ(m-n)})'(y)|)s(w_n i)$ for some x and y in I_{w_m}, by the naive distortion lemma, there is a constant C_ε which equals $C_0K/(d_\varepsilon(1-\lambda^\alpha))$ such that for any $m > n > 0$,

$$|s(w_m i) - s(w_n i)| \leq C_\varepsilon |I_{w_n}|^\alpha.$$

The last inequality implies that the limit of sequence $\{s(w_n i)\}_{n=0}^{+\infty}$ exists as the length of $w_n i$ increases to infinity. We denote this limit by $s(a^*)$ or $s_\varepsilon(a^*)$ if we need to indicate dependence on ε. Let m tend to infinity, then $|s(a^*) - s(w_n i)| \leq C_\varepsilon |I_{\varepsilon,w_n}|^\alpha$ for all positive integers n.

Suppose a^* and b^* are in $\mathcal{C}^*$ with the same first $(n+1)$ coordinates, that is, $a^* = (\cdots w_n i.)$ and $b^* = (\cdots w_n i.)$. Because $|s(a^*) - s(w_n i)| \leq C_\varepsilon |I_{w_n}|^\alpha$ and $|s(b^*) - s(w_n i)| \leq C_\varepsilon |I_{w_n}|^\alpha$, we have that $|s(a^*) - s(b^*)| \leq 2C_\varepsilon |I_{w_n}|^\alpha \leq 2C_0C_\varepsilon\lambda^n$. In other words, s_ε is Hölder continuous on $\mathcal{C}^*$ with a Hölder constant $2C_0C_\varepsilon$. *QED.*

In the next chapter, we will study the asymptotic behavior of scaling function s_ε, as well as the geometry of Cantor set Λ_ε, which is the maximal invariant set of f_ε, for f_ε in $\mathcal{H}$.

§4 Asymptotic Geometry Of Cantor Sets

Suppose f_ε is in $\mathcal{H}$. Let $r_\varepsilon(x) = f'_\varepsilon(x)/|x|^{\gamma-1}$ for nonzero x in $[-1,1]$. We say that f_ε has power law at the critical point if there is some $\gamma > 1$ such that the limits of $r_\varepsilon(x)$ as x increases to zero and as x decreases to zero exist and equal nonzero numbers A_ε and $-B_\varepsilon$, respectively.

We define the <u>smooth metric associated to f_ε</u> to be

$$dy = \frac{dx}{((1+\varepsilon)^2 - x^2)^{\frac{\gamma-1}{\gamma}}}$$

on $[-1,1]$. The corresponding change of coordinate is $y = h_{\gamma,\varepsilon}$ where

$$h_{\gamma,\varepsilon}(x) = -1 + b_\varepsilon \int_{-1}^{x} \frac{dx}{((1+\varepsilon)^2 - x^2)^{\frac{\gamma-1}{\gamma}}}$$

with $b_\varepsilon = 2/\int_{-1}^{1} dx/((1+\varepsilon)^2 - x^2)^{\frac{\gamma-1}{\gamma}}$. The representation of f_ε under the smooth metric associated to f_ε is

$$\tilde{f}_\varepsilon = h_{\gamma,\varepsilon} \circ f_\varepsilon \circ h_{\gamma,\varepsilon}^{-1}.$$

LEMMA 7. *If f_ε has power law $|x|^\gamma$ with $\gamma > 1$, then the mapping $\tilde{f}_\varepsilon$ is continuous on $[-1,1]$ and the restrictions of $\tilde{f}_\varepsilon$ to $[-1,0]$ and to $[0,1]$ are C^1 embeddings.*

REMARK 4. The mapping $h_{\gamma,\varepsilon}$ is a C^∞ diffeomorphism from $[-1,1]$ to itself. If the derivative f'_ε and the restrictions of r_ε to $[-1,0)$ and to $(0,1]$ are α-Hölder continuous for some $0 < \alpha \leq 1$, then the derivative $\tilde{f}_\varepsilon{}'$ is α-Hölder continuous. The α-Hölder constant of $\tilde{f}_\varepsilon$ depends on ε and may go to infinity as ε goes to zero.

LEMMA 8. *Suppose $\tilde{f}_\varepsilon$ is a continuous mapping from $[-1,1]$ to the real line with a unique turning point 0. Suppose $\tilde{f}_\varepsilon$ maps 1 and -1 to -1 and $\tilde{f}_\varepsilon(0) = 1+\varepsilon$. If the restrictions of $\tilde{f}_\varepsilon$ to $[-1,0]$ and to $[0,1]$ are C^1 embeddings, then $f_\varepsilon = h_{\gamma,\varepsilon}^{-1} \circ \tilde{f}_\varepsilon \circ h_{\gamma,\varepsilon}$ for any $\gamma > 1$ is a C^1 mapping from $[-1,1]$ to the real line and has the power law $|x|^\gamma$ at the critical point.*

The proofs of Lemma 7 and Lemma 8 are the same as those of Lemma 1 and 2.

LEMMA 9. *Suppose f_ε has the power law $|x|^\gamma$ with $\gamma > 1$. The restrictions of $\tilde{f}_\varepsilon$ to $[-1,0]$ and $[0,1]$ are $C^{1+\alpha}$ embeddings for some $0 < \alpha \leq 1$ if and only if the restrictions of r_ε to $[-1,0)$ and to $(0,1]$ are α'-Hölder continuous for some $0 < \alpha' \leq 1$.*

The proof of Lemma 9 is similar to Remark 1 and Remark 2.

§4.1 Good families of mappings in $\mathcal{BH} \cup \mathcal{H}$

Suppose $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$ is a family in $\mathcal{H} \cup \mathcal{BH}$ where $f_\varepsilon(0) = 1+\varepsilon$.

DEFINITION 7. *The family $\{f_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$ is a good family if it satisfies the following conditions:*

1. *the mapping* $F(x,\varepsilon) = f_\varepsilon(x)$ *is* C^1 *in both variables* x *in* $[-1,1]$ *and* ε *in* $[0,\varepsilon_0]$,

2. *each* f_ε *has the same power law* $|x|^\gamma$ *with* $\gamma > 1$ *at the critical points and the functions* $R^-(x,\varepsilon) = r_\varepsilon(x)$ *defined on* $[-1,0] \times [0,\varepsilon_0]$ *and* $R^+(x,\varepsilon) = r_\varepsilon(x)$ *defined on* $[0,1] \times [0,\varepsilon_0]$ *are continuous,*

3. *there are positive constants* K' *and* $\alpha' \leq 1$ *such that* f_ε *is* $C^{1+\alpha'}$ *and the* α'*-Hölder constant of* f'_ε *is less than* K' *for any* $0 \leq \varepsilon \leq \varepsilon_0$,

4. *there are positive constants* K'' *and* $\alpha'' \leq 1$ *such that the restrictions of* r_ε *to* $[-1,0)$ *and to* $(0,1]$ *are* α''*-Hölder continuous and the* α''*-Hölder constants of these restrictions are less than* K'' *for any* $0 \leq \varepsilon \leq 1$,

5. *there are two positive constants* C_0 *and* $\lambda < 1$ *such that* $\lambda_{n,\varepsilon} \leq C_0\lambda^n$ *for all positive integers* n *and* $0 \leq \varepsilon \leq \varepsilon_0$.

Let α *be the minimum of* α' *and* α''.

An example of a good family in $\mathcal{BH} \cup \mathcal{H}$ follows the following proposition.

PROPOSITION 1. *Assume that (a)* $\{F_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ *is a family of* C^3 *embeddings on* $[-1,0]$ *with nonpositive Schwarzian derivatives, (b)* F_ε *fixes* -1 *and maps* 0 *to* $1+\varepsilon$ *for any* $0 \leq \varepsilon \leq \varepsilon_0$*, (c) the derivative* $F'_\varepsilon(-1)$ *of* F_ε *at* -1 *is greater than* $1/\gamma$ *and (d)* $G(x,\varepsilon) = F_\varepsilon(x)$ *is* C^2 *on* $[-1,1]\times[0,\varepsilon_0]$*. If* $f_\varepsilon(x) = F_\varepsilon(-|x|^\gamma)$*, then the family* $\{F_\varepsilon(-|x|^\gamma)\}_{0\leq\varepsilon\leq\varepsilon_0}$ *is a good family.*

To prove this proposition, we only need to check the condition (5) in Definition 7. The condition (5) is a direct consequence of the following lemmas.

The first lemma is the C^3-Koebe distortion lemma. Suppose I and J are two intervals and g is a C^3 diffeomorphism from I to J. A measure of the nonlinearity of g is the function $n(g) = g''/g'$. If the absolute value of $n(g)$ on I is bounded above by a positive constant C, then the distortion $|g'(x)|/|g'(y)|$ of g at any pair x and y in I is bounded above by $exp(C|x-y|)$. Suppose $d_I(x)$ is the distance from x to the boundary of I.

LEMMA 10 (the C^3 Koebe distortion lemma). *Suppose g has nonnegative Scharzian derivative. Then $n(g)(x)$ is bounded above by $2/d_I(x)$ for any x in I.*

Proof. See, for example, [J1] or [J5].

LEMMA 11. *Suppose $\{f_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ is the family in Proposition 1 and $\{\eta_{n,\varepsilon}\}_{n=0}^{\infty}$ is the sequence determined by f_ε for every $0\leq\varepsilon\leq\varepsilon_0$. There is a positive constant C which does not depend on parameter ε such that for any $0\leq\varepsilon\leq\varepsilon_0$ and any pair (J,I) with $J\subset I$, $J\in\eta_{n+1,\varepsilon}$ and $I\in\eta_{n,\varepsilon}$, $|J|/|I|\geq C$.*

Proof. We suppress ε if there can be no confusion. For any $0\leq\varepsilon\leq\varepsilon_0$, the first partition η_1 contains four intervals I_{00}, I_{01}, I_{11} and I_{10}. There is a positive constant C_1 which does not depend on ε such that the lengths of the left interval I_{00} and the right interval I_{10} are greater than C_1. The C^3-Koebe distortion lemma says $n(g_w)(x)\leq 2/d_{[-1,1]}(x)$ for any finite string w of zeroes and ones. Moreover, $n(g_w)(x)\leq 2/C_1$ if x is in the union of two middle intervals I_{01} and I_{11}. We also can find a constant $\tau>1$ which does not depend on ε such that $|f'_\varepsilon(x)|\geq\tau$ for all x in the union of the left interval I_{00} and the right interval I_{10}. Now the proof just follows the proof of Example 1 in [J2]. *QED.*

§3.2 Asymptotic scaling function geometry of Cantor sets

Let $\mathcal{A}$ stand the countable set of points in $\mathcal{C}^*$ whose coordinates are eventually all zeroes and let $\mathcal{B}$ stand the complement of $\mathcal{A}$ in $\mathcal{C}^*$.

THEOREM A. *Suppose $\{f_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ is a good family. There is a family of Hölder continuous functions $\{s_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ on the dual Cantor set $\mathcal{C}^*$ such that s_ε is the scaling function of f_ε for any $0<\varepsilon_0\leq\varepsilon_0$, and*

1. *for every $0<\varepsilon_1\leq\varepsilon_0$, s_ε converges to s_{ε_1} uniformly on $\mathcal{C}^*$ as ε tends to ε_1,*

2. *for every $a^*\in\mathcal{C}^*$, the limit $s_0(a^*)$ of $\{s_\varepsilon(a^*)\}_{0<\varepsilon\leq\varepsilon_0}$ as ε decreases to zero exists, the limiting function $s_0(a^*)$ is the scaling function of f_0 and satisfies:*

2.1. s_0 *has jump discontinuities at all points in* $\mathcal{A}$*,*

2.2. s_0 *is continuous at all points in* $\mathcal{B}$ *and the restriction of* s_0 *to* $\mathcal{B}$ *is a Hölder continuous function.*

We will prove this theorem through several lemmas. The first lemma is similar to the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma in [J1]. We call it the uniform $C^{1+\alpha}$-Denjoy-Koebe distortion lemma.

Suppose $\{f_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ is a family of mappings in $\mathcal{H}\cup\mathcal{BH}$ and satisfies the conditions (1)–(4) in Definition 7. We suppress ε when there can be no confusion. For each $0\leq\varepsilon\leq\varepsilon_0$, η_1 contains four intervals I_{00}, I_{01}, I_{11} and I_{10}. Suppose x and y are in one of these four intervals and J_0 is the interval bounded by x and y. Let $\theta(x,y)=\{J_0,J_1,\cdots\}$ be a sequence of backward images of J_0 under f_ε, that means, the restriction of f_ε to J_n embeds J_n onto J_{n-1} for any positive integer n. Let g_n be the inverse of the restriction of the n^{th} iterate of f_ε to J_n. Let d_{xy} be the distance from $\{x,y\}$ to $\{-1,1\}$. Define the distortion of the n^{th} iterate of f_ε at x and y along $\theta(x,y)$ to be the ratio $|g_n'(x)|/|g_n'(y)|$.

Lemma 12. (the uniform $C^{1+\alpha}$-Denjoy-Koebe distortion lemma.) *There are positive constants A, B and C such that for any $0\leq\varepsilon\leq\varepsilon_0$, any x and y in one of the intervals in $\eta_{\varepsilon,1}$ and any sequence of backward images $\theta_\varepsilon(x,y)=\{J_{\varepsilon,0},J_{\varepsilon,1},\cdots\}$ of $J_{\varepsilon,0}$ under f_ε, the distortion of the n^{th} iterate of f_ε at x and y along $\theta_\varepsilon(x,y)$ satisfies*

$$\frac{|g_{\varepsilon,n}'(x)|}{|g_{\varepsilon,n}'(y)|}\leq\exp((A+B\sum_{i=1}^{n}|J_{\varepsilon,i}|+\frac{C|J_{\varepsilon,0}|}{d_{xy}})\sum_{i=1}^{n}|J_{\varepsilon,i}|^{\alpha})$$

for every positive integer n.

Proof. For every $0\leq\varepsilon\leq\varepsilon_0$, η_2 contains eight intervals. The two of them which close to 0 are I_{010} and I_{110}. Suppose $I_{010}=[a,b]$ and $I_{110}=[c,d]$. We call $[a,0]$ and $[0,d]$ the middle intervals, $[-1,a]$ the left interval and $[d,1]$ the right interval (see Figure 7).

$\underline{I_{\varepsilon,000}}$ $\underline{I_{\varepsilon,001}}$ $\underline{I_{\varepsilon,011}}$ $\underline{I_{\varepsilon,010}}$ $\underset{0}{.}$ $\underline{I_{\varepsilon,110}}$ $\underline{I_{\varepsilon,111}}$ $\underline{I_{\varepsilon,101}}$ $\underline{I_{\varepsilon,100}}$

left middle right

Figure 7

By the condition (1) in Definition 7, there is a positive constant C_1 which does not depend on ε such that the lengths of the left interval and the right interval are greater than C_1.

By the conditions (1) and (3) in Definition 7, there are positives constants c_1 and K_1 which do not depend on ε such that the minimum value of $|f_\varepsilon'|$ on the union of the left and right intervals is greater than c_1 and the α'-Holder constants of the restrictions of f_ε' to the left interval and to the right interval are less than K_1.

Suppose y is not one of 0, 1 and -1 and x is the preimage of y under $h_{\gamma,\varepsilon}$. By the chain rule,

$$\tilde{f}_\varepsilon'(y) = \frac{f_\varepsilon'(x)((1+\varepsilon)^2 - x^2)^{\frac{\gamma-1}{\gamma}}}{((1+\varepsilon)^2 - (f_\varepsilon(x))^2)^{\frac{\gamma-1}{\gamma}}}.$$

This equation and the conditions (1) and (4) in Definition 7 imply the restrictions of $\tilde{f}_\varepsilon$ to $[-1,0]$ and to $[0,1]$ are $C^{1+\alpha''}$ embeddings. Moreover, there are constants c_2 and K_2 which do not depend on ε such that the minimum value of $|\tilde{f}_\varepsilon'|$ on the image of every one of the middle intervals under $h_{\gamma,\varepsilon}$ is greater then c_2 and the α''-Hölder constant of the restriction of $\tilde{f}_\varepsilon'$ to the image of every one of the middle intervals is less than K_2.

The restriction of $h_{\gamma,\varepsilon}'$ to the union of the middle intervals is Lipschitz continuous. There are positive constants c_3 and K_3 which do not depend on ε such that the minimum value of restriction of $h_{\gamma,\varepsilon}'$ to the union of middle intervals is greater then c_3 and the Lipschitz constant of such restriction is less than K_3.

Let x_i and y_i be the images of x and y under $g_{\varepsilon,i}$. Notice that this implies that x_i and y_i lie in the same interval of η_{i+1} for $i \geq 0$. For every integer $n > 0$, $g_{\varepsilon,n}'(x)/g_{\varepsilon,n}'(y)$ equals $(f_\varepsilon^{\circ n})'(y_n)/(f_\varepsilon^{\circ n})'(x_n)$. By the chain rule, the ratio $(f_\varepsilon^{\circ n})'(y_n)/(f_\varepsilon^{\circ n})'(x_n)$ equals the product of ratios $f_\varepsilon'(y_{n-i})/f_\varepsilon'(x_{n-i})$ where i runs from 0 to $n-1$. This product can be factored into two products,

$$\prod_{x_i, y_i \in LR} f_\varepsilon'(y_i)/f_\varepsilon'(x_i) \ \ and \ \ \prod_{x_i, y_i \in M} f_\varepsilon'(y_i)/f_\varepsilon'(x_i).$$

Here LR stands for the union of the left and right intervals and M stands for the union of the two middle intervals. We factor the product $\prod_{x_i,y_i\in M} f_\varepsilon'(y_i)/f_\varepsilon'(x_i)$ into three factors,

$$\prod_{x_i,y_i\in M}\frac{f_\varepsilon'(y_i)}{f_\varepsilon'(x_i)}=\prod_{x_i,y_i\in M}\frac{h_{\gamma,\varepsilon}'(y_i)}{h_{\gamma,\varepsilon}'(x_i)}\cdot\prod_{x_i,y_i\in M}\frac{\tilde{f}_\varepsilon'(h_{\gamma,\varepsilon}(y_i))}{\tilde{f}_\varepsilon'(h_{\gamma,\varepsilon}(x_i))}\cdot\prod_{x_i,y_i\in M}\frac{h_{\gamma,\varepsilon}'(f(x_i))}{h_{\gamma,\varepsilon}'(f(y_i))}.$$

The third factor of them can be factored again into two products,

$$\prod_{x_i,y_i\in M}\frac{(1+f_\varepsilon(y_i))^{\frac{\gamma-1}{\gamma}}}{(1+f_\varepsilon(x_i))^{\frac{\gamma-1}{\gamma}}}\quad \textit{and}\quad \prod_{x_i,y_i\in M}\frac{(1-f_\varepsilon(y_i))^{\frac{\gamma-1}{\gamma}}}{(1-f_\varepsilon(x_i))^{\frac{\gamma-1}{\gamma}}}.$$

Now just following the arguments in the proof of the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma in [J1], we can estimate every factors. We then put all estimations together to get $A=K_1/c_1+(K_3^\alpha K_2)/c_2+K_3/c_3+(\gamma-1)/\gamma$, $B=(\gamma-1)/(\gamma C_1)$ and $C=(\gamma-1)/\gamma$. The constants A, B and C do not depend on the parameter ε. *QED.*

COROLLARY 1. *If the family $\{f_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ is a good family, then there are positive constants D and E such that for any $0\leq\varepsilon\leq\varepsilon_0$, any x and y in one of the intervals in $\eta_{1,\varepsilon}$ and any sequence $\theta_\varepsilon(x,y)=\{J_{\varepsilon,0},J_{\varepsilon,1},\cdots\}$ of backward images of $J_{\varepsilon,0}$ under f_ε, the distortion of the n^{th} iterate of f_ε along $\theta_\varepsilon(x,y)$ satisfies*

$$\frac{|g_{\varepsilon,n}'(x)|}{|g_{\varepsilon,n}'(y)|}\leq \exp(D+\frac{E}{d_{xy}})|J_{\varepsilon,0}|^\alpha$$

for every positive integer n.

Proof. The condition (5) in Definition 7 implies $\sum_{i=0}^n |J_{\varepsilon,i}|$ is less that $C_2=2C_0/(1-\lambda)$ and $\sum_{i=0}^n |J_{\varepsilon,i}|^\alpha$ is less that $P=2^\alpha|J_{\varepsilon,0}|^\alpha C_0/(1-\lambda^\alpha)=|J_{\varepsilon,0}|^\alpha C_3$. We use D to denote $(A+BC_2)C_3$ and E to denote CC_3, where A, B and C are the constants in the previous lemma. Now it is easy to show this corollary from the previous lemma. *QED.*

Before we prove more lemmas, we study asymptotic behavior of the maximal invariant set Λ_ε of f_ε in $\mathcal{H}$ when f_ε approaches $\mathcal{BH}$.

§4.3 Determination of the geometry of Cantor set by the leading gap.

Suppose f_ε is a mapping in $\mathcal{H}$. We suppress ε when there can be no confusion. Let Λ be the maximal invariant set of f_ε and $\{\eta_n\}_{n=0}^{\infty}$ be the sequence determined by f_ε. For any positive integer n and any I_w in η_n, let I_{w0} and I_{w1} be the two intervals in η_{n+1} which are contained in I_w. We call the complement of I_{w0} and I_{w1} in I_w the <u>gap</u> on I_w and denote it by G_w. Let G be the complement of I_0 and I_1 in $[-1,1]$. We call G the <u>leading gap</u>.

DEFINITION 8. *We call the set of ratios, $\{|G_{\varepsilon,w}|/|I_{\varepsilon,w}|\}$, for all finite strings w of zeroes and ones the <u>gap geometry</u> of Λ_ε or f_ε.*

Suppose $\{f_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ is a family in $\mathcal{H}\cup\mathcal{BH}$ and $\{\Lambda_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ is the family of the corresponding maximum invariant sets.

DEFINITION 9. *Suppose β is a function defined on $[0,\varepsilon_0]$. We say β determines asymptotically the gap geometry of $\{\Lambda_\varepsilon\}_{0<\varepsilon\leq 1}$ if there is a positive constant C such that for all $0\leq\varepsilon\leq\varepsilon_0$, all finite strings w of zeroes and ones and $i=0$ or 1,*

1. $C^{-1}\beta(\varepsilon)\leq|G_{\varepsilon,w}|/|I_{\varepsilon,w}|\leq C\beta(\varepsilon)$,
2. $|I_{\varepsilon,wi}|/|I_{\varepsilon,w}|\geq C^{-1}$.

The constant C is called a determining constant.

THEOREM B. *Suppose $\{f_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ is a good family. Then the family $\{\Lambda_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ is a family of Cantor sets. Moreover, the function $\varepsilon^{\frac{1}{\gamma}}$ on $[0,\varepsilon_0]$ determines asymptotically the gap geometry of $\{\Lambda_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$.*

Proof. For each $0\leq\varepsilon\leq\varepsilon_0$, η_1 contains four interval I_{00}, I_{01}, I_{11} and I_{10}. We call I_{00} the left interval, I_{10} the right interval and I_{01} and I_{11} the middle intervals. We also call η_1 the first level and η_2 the second level.

By the conditions (2) in Definition 7, there is a positive constant C_1 which does not depend on ε such that $C_1^{-1}\varepsilon^{\frac{1}{\gamma}}\leq|G_0|\leq C_1\varepsilon^{\frac{1}{\gamma}}$.

By condition (1) in Definition 7, there is a positive constant C_2 which does not depend on ε such that for any triple (G,J,I) where I is an interval in the first level or η_0, G is the gap on I and $J\subset I$ is an interval in the second interval or the first level, $|J|/|I|\geq C_2$ and $C_2^{-1}\varepsilon^{\frac{1}{\gamma}}\leq|G|/|I|\leq C_2\varepsilon^{\frac{1}{\gamma}}$.

For any integer $n > 1$ and any triple (G, J, I) where I is in η_n, G is the gap on I and $J \subset I$ is in η_{n+1}, let G_i, J_i and I_i be the images of G, J and I under the i^{th} iterate of f_ε for $0 \leq i \leq n-1$. Then I_{n-1} is in the first level, G_{n-1} is the gap on I_{n-1} and $J_{n-1} \subset I_{n-1}$ is in the second level.

We divide the possible itineraries of the sequence of triples $\{(G_i, J_i, I_i)\}_{i=0}^{n-1}$ into two cases. The first case is that no one of I_i is in the union of the middle intervals. The second case is that some of I_i is in one of the middle intervals.

In the first case, G_i, J_i and I_i are in the union of the left interval and the right interval for all $0 \leq i < n$. By the conditions (1) and (3) in Definition 7, there is a positive constant c which does not depend on ε such that the minimum value of the restriction of f'_ε to the union of the left and right intervals is greater than c. Suppose λ, C_0, α' and K' are the constants in the conditions (3) and (5) in Definition 7. By the naive distortion lemma (see [J1]), there is a constant C_3 which equals $C_2 \exp(K'C_0/(c(1-\lambda^{\alpha'})))$ such that $|J|/|I| \geq C_3^{-1}$ and $C_3^{-1}\varepsilon^{\frac{1}{\gamma}} \leq |G|/|I| \leq C_3\varepsilon^{\frac{1}{\gamma}}$ because G, J and I are the images of G_{n-1}, J_{n-1} and I_{n-1} under the $(n-1)^{th}$ iterate of f_ε.

In the second case, let m be the largest positive integer such that I_m is in one of the middle intervals. We can divide this case into two subcases according to m. One is that m is $n-1$. The other is that m is less than $n-1$.

If $m = n-1$, then I_{n-1} is one of the middle intervals. By the condition (1) in Definition 7, there is a positive constant C_4 which does not depend on ε such that the lengths of the left interval and the right interval are greater then C_4. By Corollary 1, there is a constant C_5 which equals $C_2 exp(D + E/C_4)$ and does not depend on ε such that $|J|/|I| \geq C_5^{-1}$ and $C_5^{-1}\varepsilon^{\frac{1}{\gamma}} \leq |G|/|I| \leq C_5\varepsilon^{\frac{1}{\gamma}}$ because the restriction of the $(n-1)^{th}$ iterate of f to I embeds I to I_{n-1} and the distance from I_{n-1} to $\{-1, 1\}$ is greater than C_4.

If $m < n-1$, then I_i is in the union of the left and right intervals for $m < i \leq n-1$. Because I_{n-1} is one of the left and the right intervals, I_{m+1} has 1 as a boundary and I_m is the one closing 0 in $\eta_{n-m,\varepsilon}$. For the sequence $I_{m+1}, \cdots, I_{n-1}$, no one of them is in the union of the middle intervals. By the same arguments as those in the first case imply that $|J_{m+1}|/|I_{m+1}| \geq C_3^{-1}$ and $C_3^{-1}\varepsilon^{\frac{1}{\gamma}} \leq |G_{m+1}|/|I_{m+1}| \leq C_3\varepsilon^{\frac{1}{\gamma}}$. For the sequence $I_0, \cdots, I_m$, the last one I_m is in one of the middle intervals. Similar arguments to those in the subcase $m = n - 1$ imply that $|J|/|I| \geq C_6^{-1}|J_m|/|I_m|$ and $C_6^{-1}|G_m|/|I_m| \leq |G|/|I| \leq C_6|G_m|/|I_m|$, where $C_6 = exp(D + E/C_4)$. Because the restriction of f_ε to I_m is comparable with the power law mapping $|x|^\gamma$ uniformly on ε by the condition (2) in Definition 7, we may assume $f_\varepsilon|I_m = 1 + \varepsilon - |x|^\gamma$. There is a positive constant C_7 which does not depend on ε such that $|J_m|/|I_m| \geq C_7^{-1}|J_{m+1}|/|I_{m+1}|$ and $C_7^{-1}|G_{m+1}|/|I_{m+1}| \leq |G_m|/|I_m| \leq C_7|G_{m+1}|/|I_{m+1}|$. We get that $|J|/|I| \geq (C_6C_7C_3)^{-1}$ and $(C_6C_7C_3)^{-1}\varepsilon^{\frac{1}{\gamma}} \leq |G|/|I| \leq (C_6C_7C_3)\varepsilon^{\frac{1}{\gamma}}$.

Let C be $C_7C_6C_3$. It is a determining constant of the gap geometry of $\{\Lambda_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$. *QED.*

Let $HD(\varepsilon)$ be the Hausdorff dimension of Λ_ε. An immediate consequence of Theorem B is the following corollary.

Corollary 2. *There is a positive constant C which does not depend on ε such that*

$$0 < HD(\varepsilon) \leq 1 - C\varepsilon^{\frac{1}{\gamma}}$$

for all $0 \leq \varepsilon \leq \varepsilon_0$.

Corollary 3. *There is a positive constant C which does not depend on ε such that*

$$C^{-1}\varepsilon^{\frac{1}{\gamma}} \leq s_\varepsilon((a^*0.)) + s_\varepsilon((a^*1.)) \leq C\varepsilon^{\frac{1}{\gamma}}$$

for all a^* in $\mathcal{C}^*$ and $0 \leq \varepsilon \leq \varepsilon_0$.

Proof. For any $0 < \varepsilon \leq \varepsilon_0$ and any $a^* \in \mathcal{C}^*$, $s(w1) + s(w0) = |G_w|/|I_w|$ where $a^* = (\cdots w.)$ and G_w is the gap on I_w. Now this corollary is a consequence of Theorem A and Lemma 5. *QED.*

§4.4 The proof of Theorem A

Let us go on to prove Theorem A. Suppose $\{f_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ is a good family. For any a^* in $\mathcal{C}^*$, we always use w_ni to denote the first $(n+1)$ coordinates of a^*, that is, $a^* = (\cdots w_ni.)$, and use $s(\varepsilon, w_ni)$ to denote the ratio of lengths, $|I_{\varepsilon,w_ni}|/|I_{\varepsilon,w_n}|$. Suppose s_ε is the scaling function of f_ε for $0 < \varepsilon \leq \varepsilon_0$. We suppress ε when there can be no confusion.

LEMMA A1. *For each* $0 < \varepsilon_1 \leq \varepsilon_0$*,* s_ε *converges to* s_{ε_1} *uniformly on* $\mathcal{C}^*$ *as* ε *tends to* ε_1*.*

Proof. Suppose C_ε is the constant obtained in Lemma 5. By the condition (1) in Definition 7, C_ε is continuous on $0 < \varepsilon \leq \varepsilon_0$. We can find a positive number δ such that $C_\varepsilon \leq 2C_{\varepsilon_1}$ for all ε in $(\varepsilon_1 - \delta, \varepsilon_1 + \delta)$.

From the proof of Lemma 5, we have that for any a^* in $\mathcal{C}^*$, $|s_\varepsilon(a^*) - s(\varepsilon, w_n i)| \leq C_\varepsilon |I_{\varepsilon,w_n}|^{\alpha'}$ and $|s_{\varepsilon_1}(a^*) - s(\varepsilon_1, w_n i)| \leq C_{\varepsilon_1} |I_{\varepsilon_1,w_n}|^{\alpha'}$. Because we may write $|s_\varepsilon(a^*) - s_{\varepsilon_1}(a^*)|$ in $|s_\varepsilon(a^*) - s(\varepsilon, w_n i) + s(\varepsilon, w_n i) - s(\varepsilon_1, w_n i) + s(\varepsilon_1, w_n i) - s_{\varepsilon_1}(a^*)|$, this implies that

$$|s_\varepsilon(a^*) - s_{\varepsilon_1}(a^*)| \leq 2C_{\varepsilon_1}|I_{\varepsilon,w_n}|^{\alpha'} + C_{\varepsilon_1}|I_{\varepsilon_1,w_n}|^{\alpha'} + |s(\varepsilon, w_n i) - s(\varepsilon_1, w_n i)|$$

for all $n > 0$ and ε in $(\varepsilon_1 - \delta, \varepsilon_1 - \delta)$. Now the last inequality and the conditions (1) and (5) in Definition 7 imply this lemma. *QED.*

LEMMA A2. *For every* a^* *in* $\mathcal{C}^*$*, the limit of* $\{s_\varepsilon(a^*)\}_{0<\varepsilon\leq\varepsilon_0}$ *exists as* ε *decreases to zero.*

Proof. For every $0 \leq \varepsilon \leq \varepsilon_0$, $\eta_{1,\varepsilon}$ contains four intervals $I_{\varepsilon,00}$, $I_{\varepsilon,01}$, $I_{\varepsilon,11}$ and $I_{\varepsilon,10}$. We call $I_{\varepsilon,01}$ and $I_{\varepsilon,11}$ the middle intervals, $I_{\varepsilon,00}$ the left interval and $I_{\varepsilon,10}$ the right interval.

For a^* in C^*, we may arrange it into two cases according to its coordinates. The first case is that the coordinates of a^* are eventually all zeroes. The second case is that there are infinite many ones in the coordinates of a^*.

In the first case, we can find a positive integer N such that I_{ε,w_n} is in the left interval for every $n \geq N$. By the condition (1) in Definition 7, there is a positive constant c which does not depend on ε such that the minimum value of $|f'_\varepsilon|$ on the left interval is greater than c. By the naive distortion lemma and and the conditions (3) and (5) in Definition 7 and similar arguments to the proof of Lemma A1, there is a constant C_1, which equals $exp(K'C_0/(c(1-\lambda^{\alpha'}))$, such that

$$|s_\varepsilon(a^*) - s_{\varepsilon'}(a^*)| \leq C_1(|I_{\varepsilon,w_n}|^{\alpha'} + |I_{\varepsilon',w_n}|^{\alpha'}) + |s(\varepsilon, w_n i) - s(\varepsilon', w_n i)|$$

for all ε and ε' in $(0, \varepsilon_0]$ and $n \geq N$. Now we can show that the limit of $\{s_\varepsilon(a^*)\}_{0<\varepsilon\leq\varepsilon_0}$ as ε decreases to zero exists.

In the second case, by the condition (1) in Definition 7, there is a positive constant C_2 which does not depend on ε such that the lengths of the left interval and the right interval are greater than C_2 for any $0 \leq \varepsilon \leq \varepsilon_0$. By Corollary 1, there is a constant $C_3 = D + E/C_2$ which does not depend on ε such that if I_{ε,w_n} is in one of the middle intervals, then $|s(\varepsilon, w_m i) - s(\varepsilon, w_n i)| \leq C_3 |I_{\varepsilon,w_n}|^\alpha$ for all ε in $(0, \varepsilon_0]$ and $m > n > 0$ because $s(\varepsilon, w_m i) = (|(f^{\circ(m-n)})'(x)|/|(f^{\circ(m-n)})'(y)|) s(\varepsilon, w_n i)$ for some x and y in I_{ε,w_m}. Moreover, let m increases to infinity, then $|s_\varepsilon(a^*) - s(\varepsilon, w_n i)| \leq C_3 |I_{\varepsilon,w_n}|^\alpha$ if I_{ε,w_n} is in one of the middle intervals.

If the n^{th} coordinate of a^* is one, then for any ε and ε' in $(0, \varepsilon_0]$, I_{ε,w_n} is in one of the middle intervals for ε and I_{ε',w_n} is in one of the middle intervals for ε'. Then

$$|s_\varepsilon(a^*) - s_{\varepsilon'}(a^*)| \leq |s(\varepsilon, w_n i) - s(\varepsilon', w_n i)| + C_3(|I_{\varepsilon,w_n}|^\alpha + |I_{\varepsilon',w_n}|^\alpha).$$

Because there are infinite many ones in the coordinates of a^*, the last inequality implies that the limit of $\{s_\varepsilon(a^*)\}_{0<\varepsilon\leq\varepsilon_0}$ as ε decreases to zero exists. *QED.*

Let $s_0(a^*)$ be the limit of $\{s_\varepsilon(a^*)\}_{0<\varepsilon\leq\varepsilon_0}$ as ε decreases to zero. Then s_0 defines a function on $\mathcal{C}^*$.

LEMMA A3. The limiting function s_0 is the scaling function of f_0.

Proof. The proof is similar to the proof of Lemma A1. Let us outline the proof. There are four intervals in the first partition η_1 determined by f_0. We call the one adjacent to -1 the left interval, the two adjacent to 0 the middle intervals and the one adjacent to 1 the right interval. For any a^* in $\mathcal{C}^*$, its coordinates either are eventually all zeroes or contains infinite many ones.

If the coordinates of a^* are eventually all zeroes, then all I_{w_n} are eventually in the left interval. There is a positive constant C_1 such that for any $m > n > 0$, $|s(w_m i) - s(w_n i)| \leq C_1 |I_{w_n}|^{\alpha'}$ by the naive distortion lemma.

If there are infinite many ones in the coordinates of a^*. Suppose the n^{th} coordinate of a^* is one. Then I_{w_n} is in one of the middle interval. There is positive constant C_2 such that for any $m > n > 0$, $|s(w_m i) - s(w_n i)| \leq C_2 |I_{w_n}|^\alpha$ by the Corollary 1.

In both of cases, the limit of $\{s(w_n i)\}_{n=0}^\infty$ as the length of $w_n i$ increases to infinity exists. The scaling function of f_0 exists. Allowed

$\varepsilon = 0$ in the proof of Lemma A2, we can show that this scaling function is s_0. *QED.*

LEMMA A4. *The scaling function s_0 has jump discontinuities at all points in $\mathcal{A}$.*

Proof. Suppose a^* is in $\mathcal{A}$ and $a^* = (0_\infty wi.)$ where 0_∞ is the one-sided infinite string of zeroes extending to the left, w is a finite string of zeroes and ones and i is either zero or one. Let 0_n be the finite string of zeroes of length n. The interval $I_{0_n w}$ is eventually in the left interval I_{00}. We use b_n to denote the length of $I_{0_n w}$ and a_n or a_n' to denote the length of $I_{0_n wi}$. Let c_n be the distance from $I_{0_n w}$ to -1 (see Figure 8).

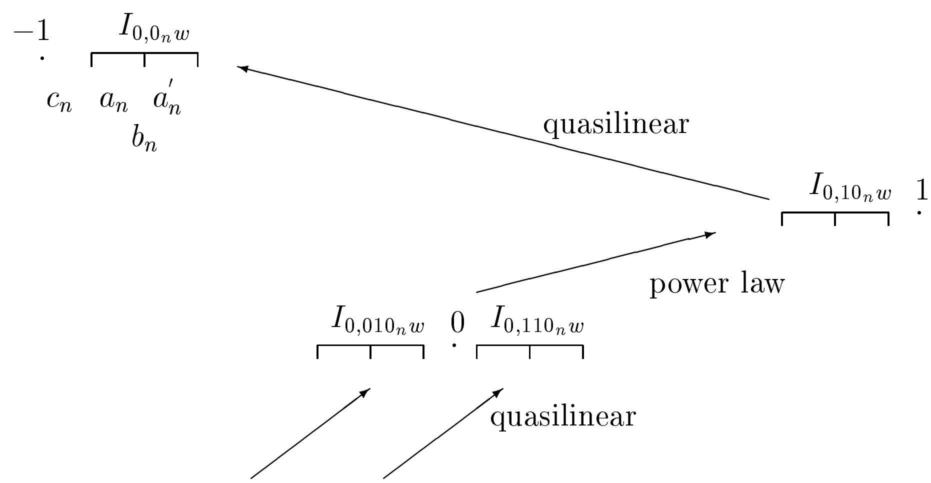


Figure 8

Let j be either zero or one. Because f_0 has the power law $|x|^\gamma$ with $\gamma > 1$ at the critical point and $I_{j10_n w}$ close to the critical point, the limit of $\{s(j10_n wi)\}_{n=0}^\infty$ equals the limit of

$$s_{n,1} = \frac{(a_n + c_n)^{\frac{1}{\gamma}} - c_n^{\frac{1}{\gamma}}}{(b_n + c_n)^{\frac{1}{\gamma}} - c_n^{\frac{1}{\gamma}}}$$

or the limit of

$$s_{n,2} = \frac{(b_n + c_n)^{\frac{1}{\gamma}} - (a_n + c_n)^{\frac{1}{\gamma}}}{(b_n + c_n)^{\frac{1}{\gamma}} - c_n^{\frac{1}{\gamma}}}$$

as n increases to infinity if the limits of $s_{n,1}$ and $s_{n,2}$ as n increases infinity exist.

Because the minimum value of the restriction of f_0 to the left interval I_{00} is positive, by using the naive distortion lemma, we can show the limit of $\{b_n/c_n\}_{n=0}^{\infty}$ and the limit $\{a_n/c_n\}_{n=0}^{\infty}$ as n increases to infinity exist. We use $\tau_1(a^*)$ and $\tau_2(a^*)$ to denote these limit, respectively. Now we conclude that the limit of $\{s(j10_nwi)\}_{n=0}^{\infty}$ as n increases to infinity exists and

$$\lim_{n\mapsto+\infty} s(j10_nwi) = \frac{(1+\tau_2(a^*))^{\frac{1}{\gamma}}-1}{(1+\tau_1(a^*))^{\frac{1}{\gamma}}-1} \quad or$$

$$\lim_{n\mapsto+\infty} s(j10_nwi) = \frac{(1+\tau_1(a^*))^{\frac{1}{\gamma}}-(1+\tau_2(a^*))^{\frac{1}{\gamma}}}{(1+\tau_1(a^*))^{\frac{1}{\gamma}}-1}.$$

Because I_{j10_nwi} is in one of the middle intervals, I_{01} and I_{11}. from the proof of Lemma A3, the error of $s(j10_ni)$ to $s_0(b^*)$ can be estimated by $|I_{j10_nwi}|^{\alpha}$, that is, there is a positive constant C_2 such that

$$|s_0(b^*)-s(j10_nwi)| \leq C_2|I_{j10_nwi}|^{\alpha}$$

for any $b^*=(\cdots j10_nwi.)$ in $\mathcal{C}^*$. Now we can get that the limit of $s_0(b^*)$ as $b^*\neq a^*$ tends to a^* exists and

$$\lim_{b^*\neq a^*, b^*\mapsto a^*} s_0(b^*) = \frac{(1+\tau_2(a^*))^{\frac{1}{\gamma}}-1}{(1+\tau_1(a^*))^{\frac{1}{\gamma}}-1} \quad or$$

$$\lim_{b^*\neq a^*, b^*\mapsto a^*} s_0(b^*) = \frac{(1+\tau_1(a^*))^{\frac{1}{\gamma}}-(1+\tau_2(a^*))^{\frac{1}{\gamma}}}{(1+\tau_1(a^*))^{\frac{1}{\gamma}}-1}.$$

Because $s_0(a^*)=\tau_2(a^*)/\tau_1(a^*)$, the limit of $s_0(b^*)$ as $b^*\neq a^*$ tends to a^* does not equal $s_0(a^*)$. In other words, s_0 has jump discontinuity at a^*. *QED.*

LEMMA A5. *The scaling function s_0 is continuous at all points in $\mathcal{B}$.*

Proof. Suppose a^* is in $\mathcal{B}$. Let b^* be any point in $\mathcal{C}^*$ with the same first $(n+1)^{th}$ coordinates w_ni as that of a^*. If the n^{th} coordinate of a^* is one, then I_{w_n} is in one of the middle intervals I_{01} and I_{11}. The errors from $s_0(a^*)$ and $s_0(b^*)$ to $s(w_n i)$ can be estimated by $|I_{w_n}|^\alpha$, that is, there is a positive constant C such that $|s_0(a^*) - s(w_n i)|$ and $|s_0(b^*) - s(w_n i)|$ are less than $C|I_{0,w_n}|^\alpha$. Then $|s_0(a^*) - s_0(b^*)| \leq 2C|I_{w_n}|^\alpha$. Because there are infinite many ones in the coordinates of a^*, the limit of $\{s_0(b^*)\}$ as b^* tends to a^* exists and $\lim_{b^* \mapsto a^*} s_0(b^*) = s_0(a^*)$. In other words, s_0 is continuous at a^*. *QED.*

Suppose $\tilde{f}_0 = h_\gamma \circ f_0 \circ h_\gamma^{-1}$ is again the representation of f_0 under the singular metric associated to f_0.

LEMMA A6. *There is a Hölder continuous scaling function $\tilde{s}_0$ of $\tilde{f}_0$ and the restriction of s_0 to $\mathcal{B}$ equals the restriction of $\tilde{s}_0$ to $\mathcal{B}$. In particular, the restriction of s_0 to $\mathcal{B}$ is Hölder continuous on $\mathcal{B}$.*

Proof. By using similar arguments to the proof of Lemma 6, we can show that there is a Hölder continuous scaling function $\tilde{s}_0$ of $\tilde{f}_0$.

The restriction of h_γ to the union of the middle intervals I_{10} and I_{11} is a C^1 embedding. For any a^* in $\mathcal{B}$, if the n^{th} coordinate of a^* is one, then I_{w_n} is in one of the middle intervals. We use $|h_\gamma(I_{w_n i})|/|h_\gamma(I_{w_n})|$ and $|I_{w_n i}|/|I_{w_n}|$ to approach $\tilde{s}_0(a^*)$ and $s_0(a^*)$, respectively. Now we can show that $\tilde{s}_0(a^*) = s_0(a^*)$ because there are infinite many ones in the coordinates of a^*. *QED.*

Lemma 6 and Lemma A1 to A6 give the proof of Theorem A.

§4.5 Scaling functions of mappings on $\mathcal{BH}$

More generally, we have the following theorem.

THEOREM C. *Suppose f is on $\mathcal{BH}$ and $\tilde{f}$ is the representation of f under the singular metric associated to f. There exist the scaling function $s_{\tilde{f}}$ of $\tilde{f}$ and the scaling function s_f of f and these scaling functions satisfy that*

(a) $s_{\tilde{f}}$ is Hölder continuous on $\mathcal{C}^*$,

(b) s_f has jump discontinuities at all points in $\mathcal{A}$ and s_f is continuous at all points in $\mathcal{B}$,

(c) the restriction of s_f to $\mathcal{B}$ equals the restriction of $s_{\tilde{f}}$ to $\mathcal{B}$.

Proof. The proof is the same as the proofs of Lemma A3 to Lemma A6. *QED.*

Suppose $\mathcal{S} = \{s_f | f \in \mathcal{BH}\}$. We can use s_f to compute the eigenvalues of f at all periodic points (see [J3]) and the power law at the critical point, that is,

$$\gamma = \frac{\log s_f((0_\infty .))}{\log(\lim_{b^* \in \mathcal{B}, b^* \mapsto 0_\infty} s_f(b^*))}.$$

The absolute value of the asymmetry of f at the critical point is $|sv_f| = \lim_{n \mapsto +\infty} |I_{010_n}|/|I_{110_n}|$.

An example of a scaling function in $\mathcal{S}$ is the following proposition.

PROPOSITION 2. *Let $q : x \mapsto 1 - 2x^2$. Then $s_q(a^*) = 1/2$ for $a^* \in \mathcal{B}$ and $s_q(a^*) \neq 1/2$ for $a^* \in \mathcal{A}$.*

Proof. Recall that $\tilde{q}(y) = 1 - 2|y|$. The proof of this proposition just follows the proof of Lemma A3 and Lemma A6. *QED.*

§4.6 The Hausdorff dimension of the maximal invariant set of q_ε.

From Proposition 2, we can observe more on the Hausdorff dimension of the maximal invariant set of $q_\varepsilon(x) = 1 + \varepsilon - (2 + \varepsilon)x^2$. Suppose Λ_ε is the maximal invariant set of q_ε and $HD(\varepsilon)$ is the Hausdorff dimension of Λ_ε.

PROPOSITION 3. *There is a positive constant C which does not dependent on ε such that*

$$1 - C^{-1}\sqrt{\varepsilon} \leq HD(\varepsilon) \leq 1 - C\sqrt{\varepsilon}$$

for all $0 \leq \varepsilon \leq 1$.

Proof. Suppose $dy = dx/\sqrt{(1+\varepsilon)^2 - x^2}$ is the metric associated to q_ε on $[-1, 1]$, $y = h_{2,\varepsilon}(x)$ is the corresponding change of coordinate and $\tilde{q}_\varepsilon = h_{2,\varepsilon} \circ q_\varepsilon \circ h_{2,\varepsilon}^{-1}$. We suppress ε when there can be no confusion.

Let ε be in $[0, 1]$. We call I_{0_4} and I_{10_4} the end intervals. Recall that I_{0_4} is the interval in η_3 adjacent to -1 and I_{10_4} is the interval in η_4 adjacent to 1. We call the complement of the interiors of the end intervals the middle interval.

Suppose y is in $[-1, 1]$ and x is the preimage of y under h_2. The nonlinearity of $\tilde{q}$ at y is

$$n(\tilde{q})(y) = \frac{\varepsilon(1+\varepsilon)}{(2(1+\varepsilon)-(2+\varepsilon)x^2)\sqrt{(1+\varepsilon)^2-x^2}}.$$

We can find a positive constant C_1 which does not depend on ε such that

$$|n(\tilde{q})(y)| \leq C_1\varepsilon$$

for any y in the image of the middle interval under h_2.

Let $\tilde{\Lambda}$ be the maximal invariant set of $\tilde{q}$. It is diffeomorphic to Λ. The sets $\tilde{\Lambda}$ and Λ have the same Hausdorff dimension. We use $\tilde{I}$ to denote the image of I under h_2 and computer the Hausdorff dimension of $\tilde{\Lambda}$.

By direct computations, there is a positive constant C_2 which does not depend on ε such that for any $I_w \in \eta_3$,

$$1 - C_2^{-1}\varepsilon \leq |\tilde{I}_{w0}|/|\tilde{I}_{w1}| \leq 1 + C_2\varepsilon.$$

Suppose w is a finite string of zeroes and ones. We call a piece string of consecutive zeros in w a zero element. We call the maximum length of all the zero elements in w the zero-length of w. If the length of w is greater than 4 and I_w is in the union of the end intervals, then the zero-length of w has to be greater then or equal to 4. Using this fact, we can show that for any finite string w of zeros and ones, if the length of w is greater than 4 and the zero-length of w is less than 4, the image under the i^{th} iterate of q for any $0 < i \leq n-4$ is in the middle interval for any $0 < i \leq n-4$. For any finite string w of zeroes and ones satisfying that the length of w is greater then 4 and the zero-length of w is less than 3, the image of I_w under the $(n-4)^{th}$ iterate of q_ε is in η_3 and $|\tilde{q}_\varepsilon^{\circ(n-4)}(\xi)|/|\tilde{q}_\varepsilon^{\circ(n-4)}(\theta)|$ is bounded above by $\exp(C_1\varepsilon)$ for any ξ and θ in $\tilde{I}_w$. We can find a positive constant C_3 which does note depend on ε such that

$$1 - C_3\varepsilon \leq \frac{|\tilde{I}_{w0}|}{|\tilde{I}_{w1}|} \leq 1 + C_3\varepsilon$$

for any finite string w of zeroes and ones satisfying that the length of w is greater then 4 and the zero-length of w is less than 3.

The restriction of the nonlinearity, $n(h_2)(x) = x/((1+\varepsilon)^2 - x^2)$, to the middle interval is bounded above by a positive constant C_4 which does not depend on ε. For any finite string wi of zeroes and ones, we use $\tilde{s}(wi)$ to denote the ratio $|\tilde{I}_{wi}|/|\tilde{I}_w|$. A direct consequence of Theorem B is that there is a positive constant C_5 which does not depend on ε such that $\tilde{s}(w0) + \tilde{s}(w1) \geq 1 - C_5\sqrt{\varepsilon}$ for all finite string w satisfying that the length of w is greater then 4 and the zero-length is less than 3. Moreover, there is positive constant C_6 which does not depend on ε such that $\tilde{s}(wi) \geq (1/2)(1 - C_6\sqrt{\varepsilon})$ where $i = 0$ or 1.

Let $S_n = \sum |\tilde{I}_w|^\delta$ where sum is over all finite string w satisfying that the zero-length is less than 3. For $w = i_0 i_1 \cdots i_n$, let $w_k = i_0 \cdots i_k$ for $4 \leq k \leq n$. We can write S_n in

$$\sum (\tilde{s}(w_n)\tilde{s}(w_{n-1}) \cdots \tilde{s}(w_4))^\delta |\tilde{I}_{w_4}|^\delta.$$

Suppose C_7 is the minimum length of the intervals in η_3. Then S_n is greater then $C_7((1/2)(1-C_6\sqrt{\varepsilon}))^{\delta(n-4)} N_n$, where N_n is the total number of finite strings of zeroes and ones which satisfies that the zero-length are less than 3.

It is easy to check that $N_2 = 4$ and $N_n = 2N_{n-1} - 1$ for any $n > 2$. We can find a positive constant C_8 such that S_n is greater than $C_8((1 - C_6\sqrt{\varepsilon})/2)^{n\delta} 2^n$ for $n \geq 4$.

Let $\delta_0 = \log 2/(\log 2 - \log(1 - C_6\sqrt{\varepsilon}))$. Then $2((1 - C_6\sqrt{\varepsilon})/2)^{\delta_0} = 1$ and $HD(\varepsilon)$ is greater than or equals to δ_0. Here δ_0 is bounded below by $1 - sqrt\varepsilon$ for a positive constant C which does not depend on ε.

Another side of the inequality in Proposition 3 comes from Corollary 2. *QED.*

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References

[GS] *J. Guckenheimer and Stewart Johnson*, Distortion of S-Unimodal maps, [1988], Cornell University, Preprint.

[J1] *Y. Jiang*, Dynamics of Certain Smooth One-dimensional mappings, I. The $C^{1+\alpha}$-Denjoy-Koebe distortion lemma. Preprint, IMS at Stony Brook.

[J2] *Y. Jiang*, Dynamics of Certain Smooth One-dimensional mappings, II. Geometrically finite one-dimensional mappings. Preprint, IMS at Stony Brook.

[J3] *Y. Jiang*, Dynamics of Certain Smooth One-dimensional mappings, III. Scaling function geometry.

[J4] *Y. Jiang*, Local normalization of one dimensional maps, [1989], *June, 1989, IHES, Preprint.*

[J5] *Y. Jiang*, Generalizaed Ulam-von Neumann Transiformations, Thesis, CUNY, Graduate Center.

[J6] *Y. Jiang*, Scaling Structures for Circle Mappings, [1989], *Preprint.*

[Ma] *R. Mañe*. Hyperbolicity, Sinks and Measure in One Dimensional Dynamics, [1985], *Comm. in Math. Phys.* **100***, 495-524* and *Erratum Comm. in Math. Phys.* **112***, (1987) 721-724*

[Mi] *M. Misiurewicz*, Absolutely Continuous Measures for Certain Maps of An Interval, [1981], *Inst. Hautes. Études. Sci. Publ. Math. 1978, No.* **53***, 17 - 51.*

[PS] *T. Pignataro and D. Sullivan*, Ground State and Lowest Eigenvalue of the Laplacian for Non-compact Hyperbolic Surfaces, [1988], *Preprint.*

[L] *R. de la Llave*, Invariant for Smooth Conjugacy of Hyperbolic Dynamical System II, [1987], *Comm. in Math. Phys.* **109***, 369-378.*

[S1] *D. Sullivan*, Quasiconformal Homeomorphisms in Dynamics, Topology, and Geometry, [1986], *Proceedings of the International Congress of Mathematicians, Berkeley, California, USA.*

[S2] *D. Sullivan*, Differentiable Structure on Fractal Like Sets Determined by Intrinsic Scaling Functions on Dual Cantor Sets, [1988], *The Proceedings of Symposia in Pure Mathematics, Vol.* **48**.

[S3] *D. Sullivan*, Bounded Structure of Infinitely Renormalizable Mappings, [1989], *Universality in Chaos, 2^{nd} Edition, Adam Hilger, Bristol, England* .

[UN] *S. M. Ulam and J. von Neumann*, On the Combinations of Stochastic and Deterministic Processes, [1947], *Bull. Amer. Math. Soc. 53, 1120.*