

Ergodic theory for smooth one dimensional dynamical systems

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Abstract

In this paper we study measurable dynamics for the widest reasonable class of smooth one dimensional maps. Three principle decompositions are described in this class : decomposition of the global measure-theoretical attractor into primitive ones, ergodic decomposition and Hopf decomposition. For maps with negative Schwarzian derivative this was done in the series of papers [BL1-BL5], but the approach to the general smooth case must be different.

Notations

$X^\circ \equiv \text{int}X$ is the interior of a set X ;

$\overline{X} \equiv \text{cl}X$ is the closure of X ;

$[x, y]$ is a (closed) interval ending at x and y (without assuming $x \leq y$);

$[U, V]$ is the closed convex hull of sets U and V ;

f^n is the n -fold iterate of a map f ;

$\text{orb}(x) = \{f^n x\}_{n=0}^\infty$ is the *orbit (trajectory)* of x ;

$\text{orb}_p(x) = \{f^n x\}_{n=0}^p$;

$\omega(x)$ is the limit set of $\text{orb}(x)$;

$\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers.

1 Statement of the results

In 1985 John Milnor [M] suggested a new approach to measurable dynamics based upon a concept of (measure-theoretical) attractor. He showed that any

smooth dynamical system has a unique global attractor $A(f)$ and stated the problem of decomposing it into minimal ones. Then the minimal attractors would give a view of the structure of typical ω -limit sets.

In the series of papers [BL1-5] the following realization of this program for one dimensional maps with negative Schwarzian derivative was given (alternative approaches in the S -unimodal case were found in [GJ] and [K]). The decomposition of $A(f)$ into “primitive” ones was described (slight modification of the “minimality” property was necessary), and it turned out that it is intimately related to two main measure-theoretical decompositions: ergodic decomposition and Hopf decomposition.

The aim of this paper is to develop this theory under proper smoothness assumptions (without the negative Schwarzian derivative condition).

Let M be a closed interval, and \mathcal{A} denote a class of C^2 -smooth maps $f : M \rightarrow M$ with finitely many non-flat critical points (see §2 for the accurate definition). Denote by λ the Lebesgue measure on M . For an invariant set $A \subset M$ let

$$\begin{aligned} \text{rl}(A) &= \{x \in M : \omega(x) \subset A\} \\ \text{RL}(A) &= \{x \in M : \omega(x) = A\}. \end{aligned}$$

These are two ways of understanding of *the realm of attraction of A*; we need both of them.

By a *limit cycle* we mean a periodic orbit $Z = \{f^k a\}_{k=0}^p$ whose realm of attraction has non-vacuous interior. An interval I is called *periodic of period p* if $f^p I \subset I$. Then

$$\text{orb}_p(I) = \bigcup_{m=0}^p f^m(I)$$

is called a *cycle of I*. If besides $f^p|I$ is monotone, I is called a *periodic homterval*. Any orbit originating in such an interval converges to a cycle, but it can happen that the set of non-limit cycles in $\text{orb}I$ has positive measure. This circumstance forces us to take care of periodic homtervals. Set

$$\Lambda(f) = M \setminus (\cup \text{rl}^\circ(Z_i) \cup (\cup_{n=1}^\infty f^{-n} O_i^\circ)),$$

the union is taken over all limit cycles Z_i and all cycles O_i of periodic homtervals. What we have removed from M is the trivial part of the dynamics.

Theorem 1 *The restriction $f|_{\Lambda(f)}$ has only finitely many ergodic components E_i .*

This means that there is the decomposition $\Lambda_f = \cup_{i=1}^t E_i \pmod{0}$ of Λ_f into the finite union of invariant sets of positive measure such that $f|E_i$ are ergodic (where “mod 0” means that we ignore sets of measure zero).

Now let us introduce an important notion of *attractor* in the sense of Milnor [M]. This means a closed invariant set $A \subset M$ such that

- (i) $\lambda(\text{rl}(A)) > 0$
- (ii) $\lambda(\text{rl}(A) \setminus \text{rl}(A')) > 0$ for any proper closed invariant subset $A' \subset A$.

It is shown in [M] that there is a unique *global attractor* $A(f)$, i.e. an attractor for which $\text{rl}(A(f)) = M \pmod{0}$. The same is true for the restriction of f onto any closed subset $K \subset M$. The corresponding global attractor will be denoted by $A(f|K)$ (if $\lambda(K) = 0$ then $A = \emptyset$).

Clearly, if $\lambda(\text{RL}(A)) > 0$ then A is an attractor. Such attractors we call *primitive*. The ergodic decomposition will allow us to obtain a decomposition of the global attractor into primitive ones. Let us say that an $\text{orb}(x)$ is *absorbed* by an invariant set O if $f^n x \in O$ for some $n \in \mathbb{N}$.

Theorem 2 *There is a decomposition*

$$A(f|\Lambda(f)) = \cup A_k \tag{1.1}$$

of the global attractor into the union of finitely many infinite primitive attractors $A_k \subset \Lambda(f)$. Moreover,

- (i) *For almost all $x \in M$ either $\text{orb}(x)$ is absorbed by a cycle O_i of periodic homtervals, or it tends to a limit cycle Z_j , or $\omega(x) = A_k$ for some k ;*
- (ii) *Each A_k contains a critical point;*
- (iii) *The intersection of any two primitive attractors is at most finite;*
- (iv) *$\text{RL}(A_k) = \text{rl}(A_k) \cap \Lambda(f) = E_k \pmod{0}$ for some ergodic component E_k . This gives one-to-one correspondence between primitive attractors A_k and ergodic components $E_k \subset \Lambda(f)$*

Remark 1.1. It is proved in [MMS] that limit cycles of $f \in \mathcal{A}$ have uniformly bounded periods. Consequently, for *analytic* f the whole number of limit cycles Z_k and cycles O_j of periodic homtervals is finite (note that each O_j either contains a limit cycle or $f^p|O_j \equiv \text{id}$). Moreover, if M is an interval and $f^2 \neq \text{id}$ (or M is a circle and $f^p \neq \text{id}$ for any p), then $A(f)$ can be decomposed into the union of finitely many primitive attractors:

$$A(f) = \cup Z_j \cup A_k.$$

A transformation f of a measure space (X, μ) is called *conservative* if it satisfies the conclusion of the Poincaré Return Theorem: for any measurable subset $Y \subset X$ almost all points $y \in Y$ return to Y infinitely many times. The *conservative kernel* $K(f)$ of f is a maximal measurable set such that $f \upharpoonright K(f)$ is conservative. By *Hopf decomposition* one means the decomposition of M into conservative part $K(f)$ and dissipative part $M \setminus K(f)$. The following result shows that in one dimensional situation it can be described through the notion of attractor.

Theorem 3 *The global attractor $A(f)$ coincides mod 0 with the conservative kernel $K(f)$.*

An invariant set $K \subset M$ is said to be *topologically minimal* if $\omega(x) = K$ for all $x \in K$. By a *Cantor attractor* we mean an attractor which is a Cantor set.

Theorem 4 *Let A be a Cantor primitive attractor. Then*

- (i) *the restriction $f \upharpoonright A$ is topologically minimal;*
- (ii) *topological entropy $h(f \upharpoonright A)$ is equal to zero;*
- (iii) *there is a critical point $c \in A$ such that $A = \omega(c)$.*
- (iv) *A does not intersect any other sets of the decomposition (1.1).*

Remark 1.2. Till now it is unknown if there are Cantor attractors different from Feigenbaum-like ones (see the next section for the definition).

Theorem 5 *Let A be a primitive attractor. Then one of the following possibilities holds:*

- A1. *A is a limit cycle;*
- A2. *A is a cycle of transitive intervals;*
- A3. *A is a Cantor attractor.*

Corollary 1.1. *The number of infinite primitive attractors A_k (which is equal to the number of ergodic components $E_k \subset \Lambda(f)$) does not exceed the number of critical points in $\Lambda(f)$.*

Let us say that A is a *minimal* attractor if there are no smaller attractors $A' \subset A$.

Corollary 1.2. *Any primitive attractor A is minimal except only one case: A is a cycle of transitive intervals whose boundary ∂A contains a parabolic limit cycle.*

Remark 1.3. The whole above theory holds for maps of the circle as well but the statements need minor modifications concerning immersions of the circle. Let us mention also that ergodicity of circle diffeomorphisms was proved quite long ago by M. Herman [H] and A. Katok (see [KSF]).

Let us describe the structure of the paper. Section 2 contains preliminaries on topological one dimensional dynamics (including the principle concept of a chain of intervals) and distortion lemmas. Sections 3,4 are technical ones. The former contains an estimate of the intersection multiplicity of a monotone chain of intervals. The latter explains how to control density moving along a chain of intervals (it needs a concept of “ $D(X, \epsilon)$ broken lines”). Getting together all these tools, in the last Section 5 we prove the main results.

This paper is a natural continuation of the series [BL1-BL5]. However, the approach here differs from that in previous ones. In the case of negative Schwarzian derivative we started from the description of primitive attractors, and then pass to ergodic decomposition etc. In the general smooth case considered here the way is opposite (and it is the only way known to the author): the starting point is ergodic decomposition while the primitive attractors can be described only in the very end.

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2 Preliminaries

Let M be a finite union of disjoint closed intervals (it is convenient for technical reasons to consider non-connected M as well). Remember that \mathcal{A} denotes a class of C^2 -smooth self-maps of the manifold M with non-flat critical points. The latter means that there are C^2 – *charts* around each critical point c and the corresponding critical value fc in which f is reduced to the form $x \mapsto \sigma|x|^r$ with real $r \geq 2$ and a sign $\sigma \in \{+1, -1\}$ which may depend on the sign($x - c$).

Remark 2.1. The smoothness conditions on the map f are determined by the range of validity of so called Koebe Principle (see below). The precise regularity for it ($C^{1+Zygmund}$) was established by Sullivan [S]. So, this seems to be the widest reasonable regularity for our theory.

Denote by $C(f)$ the set of critical points of f . There are critical points of two types: *extrema* or (*turning points*) and *inflection points*. Let d denote the number of extrema. Points of the set $C(f) \cup \partial M$ will be called *singular*.

There is a natural involution τ in a neighbourhood of any extremum c , namely $\tau(x) = x'$ if $f(x) = f(x')$. It follows from non-flatness that τ is smooth with $\tau'(c) = -1$.

Actually, by C^2 -smooth conjugacy the map $f \in \mathcal{A}$ can be reduced to $\sigma|x - c|^r + b$ in a neighbourhood of any critical point c . *In what follows we will suppose that it is the case.* Then τ is reduced to the standard isometric reflection with respect to c .

Let us make also the following convention:

$$f(\partial M) \subset \partial M. \quad (2.1)$$

It is possible because of the following *surgery*. Let us include M into a compact one dimensional manifold \tilde{M} such that $\partial\tilde{M} \cap M = \emptyset$. Then f can be continued to a map $\tilde{f} \in \mathcal{A}$ of \tilde{M} in such a way that $C(\tilde{f}) = C(f)$ and (2.1) holds for $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$.

Now we need more definitions. An invariant closed set $R \subset M$ will be called *transitive* if it contains a dense orbit.

A set R will be called a *basic set* (see [B1, B2]) if it is a set of all points belonging to a cycle of intervals $O = orb(I)$ of period p and satisfying the following property: for any open interval $J \subset O$ intersecting R and for any compact subset $K \subset \text{int}O$ there is a N such that

$$\bigcup_{k=n}^{n+p} f^k J \supset K \quad (2.2)$$

for all $n \geq N$. Clearly, a basic set is closed and invariant. There are basic sets of three types : periodic orbits, cycles of intervals and Cantor basic sets.

By a *Feigenbaum-like attractor* we mean an invariant Cantor set $F \subset M$ of the following structure:

$$F = \bigcap_{n=1}^{\infty} orb_{p_n}(I_n)$$

where $I_1 \supset I_2 \supset \dots$ is a nested sequence of periodic intervals of periods p_n such that $p_n \rightarrow \infty$.

The following Topological Structural Theorem follows from the pure topological considerations (see [Sh, JR, Ho, B1, B2]) and the absence of wandering intervals (see [G1], [Y], [L], [BL7], [MMS] and references there).

Theorem A. *For any $x \in M$ one of the following possibilities holds:*

- (0) $\text{orb}(x)$ is absorbed by a cycle of a periodic homterval;
- (i) $\text{orb}(x)$ tends to a limit cycle;
- (ii) $\text{orb}(x)$ is absorbed by a basic set;
- (iii) $\omega(x)$ is a Feigenbaum-like attractor.

Now let R be either a basic set or a Feigenbaum-like attractor. Restricting f onto an appropriate cycle of intervals and using the above surgery we can localize f with respect to R in the following sense:

L0. $R \subset \text{int}M$.

L1. All critical points belong to R .

L2. There are no limit cycles in $\text{int}M$.

Maps satisfying L1-L2 we will call R -local (or just *local*). There are local maps of two types: *finitely renormalizable* when R is a basic set, and *infinitely renormalizable* when R is a Feigenbaum-like attractor.

It is easy to see that the mixing property (2.2) together with L2 yield *the sensitive dependence to initial conditions* on a local basic set R in the following sense. There exists $\gamma(R) > 0$ with the following property: $\forall \tau \exists N$ such that for any closed interval J intersecting R , $\lambda(J) \geq \tau$, we have $\lambda(f^n J) > \gamma(R)$, $n \geq N$.

The following easy but useful Proposition was stated in [L].

Proposition 2.1. *Let J be an interval whose orbit does not tend to a limit cycle. Then*

$$\inf_{0 < m < \infty} \lambda(f^m(J)) > 0.$$

Let us fix two constants $\xi > 0$ and $\eta > 0$ till the end of the paper. Let η be so small that η -neighbourhoods of critical points don't intersect, and the involution τ is well-defined in the η -neighborhoods of extrema. Then choose an ξ by Proposition 2.1 in such a way that for any interval J containing a

critical point

$$\lambda(J) > \eta \Rightarrow \lambda(f^m(J)) > \xi \quad (2.3)$$

Now let us introduce a notion of a *maximal chain of intervals* which is a key to polymodal maps (see [L]). By a *chain of intervals* \mathbb{I} we mean just a sequence of intervals $\{I_m\}_{m=1}^n$ such that $fI_m \subset I_{m+1}, m = 0, 1, \dots, n$. The chain is called *maximal* if I_m are the maximal intervals satisfying this property.

Let $I_n \subset M^\circ$ and $\lambda(I_n) < \xi$. Then we have for the maximal chain \mathbb{I} and $1 \leq m < n$ that (take into account (2.1)) either $f : I_m \rightarrow I_{m+1}$ is a homeomorphism, or I_m is symmetric with respect to some extremum, and $f(\partial I_m) \subset \partial I_{m+1}$

In what follows I denotes a closed interval such that $I \subset M^\circ$ and $\lambda(I) < \xi$.

Let $x \in M$ and $f^n(x) \in I$. The main way of constructing maximal chains of intervals is a *pull-back* of I along the $\text{orb}_n(x)$. Namely, set $I_n = I$, and I_m be the maximal interval containing x_m for which $f^{n-m}I_m \subset I, m = 0, 1, \dots, n-1$.

Define the *order* $\text{ord } \mathbb{I} \equiv \text{ord}(n, x, I)$ of the chain (pull-back) \mathbb{I} as the number of intervals I_m containing extrema. If f^n monotonously map I_0 onto I_n we say that \mathbb{I} is a *monotone* chain (pull-back).

Proposition 2.2 [L]. Consider a local map f . Let n be the first moment when $\text{orb}(x)$ passes through I . Consider a pull-back $\mathbb{I} = \{I_m\}_{m=0}^n$ of I along $\text{orb}_n(x)$, and let $\{I_{m_i}\}_{i=1}^{\nu}$ be the intervals of the chain containing extrema. Then we have for $i > d$ (where d is the number of extrema)

- (i) $I_{m_{i-d}} \supset I_{m_i}$ and I_{m_i} is periodic;
- (ii) m_i is the first moment when $\text{orb}(x)$ passes through $\text{int}I_{m_{i-d}}$.
- (iii) f^{m_i} monotonously maps a neighbourhood of x onto an appropriate half of the interval $I_{m_{i-d}}$.

Corollary 2.1. In the above situation provided I is non-periodic, we have $\text{ord } \mathbb{I} \leq d$. In particular, it is the case if f is finitely renormalizable and I is small enough.

Now let us describe the analytical tools of the paper in the form of two Distortion Lemmas. Both of them follow from *the Koebe Principle* in one dimensional dynamics intensively exploited in recent works, see [Y], [G2], [MS], [L], [Sw], [S] ...

In what follows we assume that $f \in \mathcal{A}$, f^n is *monotone* (perhaps, with critical points) on an interval J and denote by \mathbb{J} the monotone chain of intervals $\{f^l J\}_{l=0}^n$.

Denote by $\mu = \text{mult}\mathbb{J}$ the *intersection multiplicity* of \mathbb{J} (i.e. the maximal number of intervals from \mathbb{J} with non-empty intersection). For a measurable set X let

$$\text{dens}(X|I) = \lambda(X \cap I)/\lambda(I),$$

$$\text{Dens}_a(X|I) \equiv \text{Dens}(x|[a, b]) = \sup_{y \in (a, b)} \text{dens}(X|[a, y]).$$

In what follows assume that $X \subset I$.

Three Interval Distortion Lemma (see [MS], [BL6]). *Consider an interval $I \subset J^n$, and let J^+ and J^- be the components of $J \setminus I$. Suppose*

$$\lambda(f^n J^+ / \lambda(f^n I)) \geq \delta.$$

Then there are positive constants $\sigma = \sigma_\mu(\delta)$ and $q(\epsilon) = q_\mu(\epsilon, \delta)$, $q(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$, such that

- (i) $\lambda(J^+) / \lambda(I) \geq \sigma_\mu(\delta)$.
- (ii) $\text{dens}(f^n X | f^n I) \geq 1 - \epsilon \Rightarrow \text{dens}(X|I) \geq 1 - q(\epsilon)$.

Two Interval Distortion Lemma. (see [BL6]). *Divide J by a point a into two subintervals L and R , $b = f^n a$. Assume $\lambda(f^n L) / \lambda(f^n R) \leq K$. Then*

$$\text{Dens}_a(X|L) \geq 1 - \delta \Rightarrow \text{Dens}_b(f^n X | f^n L) \geq 1 - \alpha_\mu(\delta, K)$$

where $\alpha_\mu(\delta, K) \rightarrow 0$ as $\delta \rightarrow 0$, K fixed.

3 An estimate of the intersection multiplicity

A technique of estimating the intersection multiplicity of a monotone pull-back of an interval I was developed by A. Blokh (see [BL7], §2.3) in order to generalize the results of [L] onto the smooth case. The interval I was supposed to be symmetric around an extremum. Here we will develop the technique for an arbitrary I (concentrating only over new points).

Let us pass to the main definition. Consider an interval $I = [a, b]$ and a point $x \in M$ which does not lie in a basin of a limit cycle. Let $x(n) = f^n x$ be the first point of $\text{orb}(x)$ lying in (a, b) . Assume that there are $p, r \in \mathbb{N}$ and a point $v \in (a, b)$ for $m = n - (r - 1)p$ and $J = [x(m), v]$ the following properties hold

- D1. J° contains $x(n)$ and exactly one of the points a, b , say a ;
- D2. f^p is an orientation preserving homeomorphism of J onto $[x(m + p), b]$;
- D3. J° contains points $x(n + ip)$, $|i| \leq r - 2$, and no other points of the $\text{orb}_{n+(r-2)p} x$.

Then we say that $x(n)$ belongs to a *multiple collection* $\{x(n + ip)\}_{i=-r+1}^{r-2}$. A number r is called the *depth* of $x(n)$ in the collection. Let us denote by *depth* $\text{dp}_a(n, x, I) \equiv \text{dp}_a(n)$ the maximal depth of $x(n)$ in all multiple collections containing it (if there are no such collections set $\text{dp}_a(n) = 0$). Set

$$\text{dp}(n) \equiv \text{dp}(x, n, I) = \max\{\text{dp}_a(n), \text{dp}_b(n)\}. \quad (3.1)$$

Remark 3.1. Observe that if $\text{dp}_a(n) > 1$ then f^p move all points of J toward b . Indeed, it is true for the endpoint $x(m)$. If it fails for some point of J then J contains a fixed point α . Hence, f^p maps monotonously the interval $[x(m), \alpha]$ into itself. Then $\text{orb}(x)$ should converge to a cycle contradicting the assumption.

Remark 3.2. Observe also that $\min\{\text{dp}_a(n), \text{dp}_b(n)\} = 1$. Indeed, denote by r_a, p_a, J_a the data corresponding to a , and use the similar notations for b . Assume that $r_b \geq r_a > 1$. Then $x(n + p_a) \in [a, x(n)] \subset J_b$ contradicting D3.

Lemma 3.1. *Under circumstances described above let H be an interval ending at x such that f^n monotonously maps H onto $[x(n), b]$. Assume $\text{dp}(n) \geq 2$. Then*

$$\text{mult}\{f^k H\}_{k=0}^n \leq 2\text{dp}_a(n).$$

Proof. Let a point y belong to κ of the intervals $H_\kappa \equiv f^\kappa H$, r of them lying on the one side of y and $\kappa - r$ on the other. We are going to prove that $r \leq \text{dp}(n)$ and $\kappa - r \leq \text{dp}(n)$, which implies the required. Clearly, we can restrict ourselves to the estimate of r .

Let $[x(i_1), y] \supset \dots \supset [x(i_r), y]$, $[x(i_k), y] \subset H_{i_k}$. Then we have (see [BL7])

- (i) $i_1 < \dots < i_r$;
- (ii) $x(i_k)$ are the only points of the $\text{orb}_n(x)$ lying in $K \equiv [x(i_1), x(i_r)]$.

Applying f^{n-i_r} we can assume that $i_r = n$. Denote $i_1 = m$. Assume also for definiteness that $a < b$.

By the assumption, all points $x(i_k)$ lie outside (a, b) . In fact, they should lie to the left of a . Indeed, otherwise $H_m \supset [x(m), x(n)] \supset [x(n), b]$. But f^{n-m} maps monotonously H_m onto $[x(n), b]$. So, $\text{orb}(x)$ would have converged to a limit cycle.

Now set $p = n - i_{r-1}$, and apply f^p to $K = [x(m), x(n)]$. Since $K \subset H_m$ and $p < n - m$, f^p is monotone on K . Since f^p maps $[x(i_{r-1}), x(n)] \subset H_{i_{r-1}}$ into $H_n = [x(n), b]$ preserving orientation, we conclude that $f^p|_K$ preserves orientation. Now it follows from the above property (ii) that (see [BL7])

$$i_{k+1} = i_k + p, \quad k = 1, \dots, r - 1.$$

Further, let v be the right endpoint of the interval H_{n-p} . Denote $J = [x(m), v]$. Clearly, J satisfies properties D1-D2 of the definition of a multiple collection. Let us check that it satisfies D3 as well.

First, $x(n - ip) \in K \subset J$ for $i = 1, \dots, r - 1$. Further, $x(n + ip) \in f^{ip}H_{n-ip} = [x(n), b]$ for $i = 1, \dots, r - 1$. Moreover, for $i < (r - 1)$ we have $x(n + ip) \in f^{ip}H_{n-(i+1)p} = [x(n - p), v]$, so $x(n + ip) \in J^\circ$.

Let $x(l) \in J^\circ$, $l \leq n + (r - 2)p$, $l \neq n + ip$ for $|i| \leq r - 2$. Then $l > n$. Indeed, $J \subset (a, b) \cup K$. If $l \leq n$ then $x(l)$ does not belong to (a, b) by the assumption and does not belong to K by the above property (ii).

So, $l = n + k$ where $0 < k < p(r - 2)$. Hence, $s \equiv m + k < n - p$, and x_s lies outside $[x(m), b]$ according to what has been proved right now. Hence, the interval $f^k K = [x(l), x(s)]$ contains one of the points $x(m), b$. But it cannot contain $x(m)$ because otherwise

$$(x(n), b) \supset \text{int}(f^{n-m}K) = \text{int}(f^{n-m-k}(f^k K)) \ni x(n - k).$$

Consequently, $T \equiv [x(l), x(s)] \ni b$. Moreover, f^p maps T monotonously, orientation preserving and without fixed points onto $[x(l + p), x(s + p)]$. Since $x(s + p) \in T$, f^p moves all points of T to the left. In particular, b is moved to the left.

Further, f^p is also a monotonous map on the interval $[a, v)$ which has with T a common point $x(l)$. Hence, f^p is monotonous on $[a, b]$. Since a is

moved to the right and b to the left, all orbits in $[a, b]$ converge to limit cycles contradicting the assumptions.

So, the interval J satisfies all properties $D1 - D3$ and hence $\text{dp}_a(n) \geq r$.

Corollary 3.1. *Under the above circumstances let $\mathbb{I} = \{f^k I\}_{k=0}^n$ be the monotone pull-back of $I = I_n$ along $\text{orb}_n(x)$. Then*

$$\text{mult } \mathbb{I} \leq 2(\text{dp}(n) + 1)$$

Proof. Divide I_0 into two intervals H^+ and H^- ending at x , and apply the lemma to these intervals taking into account Remark 3.1.

4 Transportation of broken lines

Here we are going to prove the series of density lemmas. In what follows f is assumed to be R -local, X denotes an invariant set of positive measure, $I = [a, b] \subset \text{int}M$.

By a *broken line* beginning at x and ending at y we mean a sequence of points $\mathcal{L} = \{x_k\}_{k=0}^n$ such that $x_0 = x, x_n = y$; the intervals $[x_k, x_{k+1}]$ are the *links* of the broken line (a link can be degenerate, i.e. $x_k = x_{k+1}$). We say that \mathcal{L} is a *proper* broken line if all links are non-degenerate and

$$[x_{k-1}, x_k] \subset [x_k, x_{k+1}], \quad k = 1, \dots, n - 1$$

We say that \mathcal{L} satisfies $D(X, \epsilon)$ -*property* if for any non-degenerate link $[x_k, x_{k+1}]$ we have $\text{Dens}(X|[x_k, x_{k+1}]) \geq 1 - \epsilon$ for $k = 1, 2, \dots, n - 1$. Any $D(X, \epsilon)$ -broken line can be easily turned into a proper $D(X, \epsilon)$ -broken line with the same beginning and end .

Lemma 4.1. *Let a point x don't converge to a limit cycle, n be the first moment when $\text{orb}(x)$ passes through $I^\circ = (a, b)$. Assume that there is an interval $H \ni x$ monotonously mapped onto I by f^n . Let H^+ and H^- be the closures of the components of $H \setminus \{x\}$. Then $\forall \epsilon \exists \delta$ such that if*

$$\text{Dens}_x(X|H^+) > 1 - \delta \text{ or } \text{Dens}_x(X|H^-) > 1 - \delta$$

then there exists $D(X, \epsilon)$ broken line beginning at $x_n \equiv f^n x$ and ending at an endpoint of I .

Proof. Let us make the following conventions: $\text{dp}_b(n, x, I) = 1$ (see Remark 3.1) and $f^n H^- = [a, x_n]$. Denote $r = \text{dp}_a(n, x, I)$. Then by Corollary 3.1 we have $\text{mult}\{f^l H\}_{l=0}^n \leq 2r + 2$.

Let $\epsilon > 0$. Let us choose a big number K (so that $(1 + K^{-1})(1 - \epsilon) < 1$); then choose $\sigma = \sigma_6(1/2K)$ by the Three Interval Distortion Lemma, and then find a small number δ satisfying the following inequalities:

$$\frac{1 - \alpha_6(\delta, 2K)}{1 + K^{-1}} > 1 - \epsilon, \quad \alpha_6(\delta, \sigma^{-1}) < \epsilon$$

where α is taken from Two Interval Distortion Lemma.

Assume first that $r \leq 2$. If $|a - x_n| \leq K|b - x_n|$ then by the Two Interval Distortion Lemma we get $\text{Dens}(X|[x_n, a]) \geq 1 - \alpha_6(\delta, K) > 1 - \epsilon$.

Otherwise consider an interval $[y, x_n] \subset [a, x_n]$ such that

$$|y - x_n| = K|b - x_n|.$$

Then $\text{Dens}(X|[x_n, y]) > 1 - \epsilon$, and for any $w \in [x_n, b]$

$$\text{dens}(X|[y, w]) \geq \text{dens}(X|[y, x_n]) \frac{|y - x_n|}{|y - b|} \geq \frac{1 - \alpha_6(\delta, K)}{1 + K^{-1}} > 1 - \epsilon. \quad (4.1)$$

It follows that the two-linked broken line $\{x_n, y, b\}$ can be turned into a $D(X, \epsilon)$ broken line $\{x_n, \bar{y}, b\}$ with $\bar{y} \in [y, x_n]$ (perhaps, $\bar{y} = x_n$). To this end it is enough to set \bar{y} to be the nearest to x_n point of $[y, x_n]$ satisfying (4.1).

Assume from now on that $r \geq 3$, and set

$$w_i = x(n + ip) \in I, \quad i = 0, 1, \dots, r - 2, \quad w_{-1} = a, \quad w_r = b.$$

Denote by M_i the intervals on which these points divide I , $i = 0, 1, \dots, r - 1$. In particular, $M_0 = [a, x(n)]$, $M_{r-1} = [w_{r-1}, b]$. Now consider two cases:

I. $\text{Dens}_x(X|H^-) > 1 - \delta$. Then let us consider two subcases:

(i) $\lambda(M_{k+1}) \geq K^{-1}\lambda(M_k)$ for some $k \in [0, r - 2]$. Let us take the first such k . Assume $k = 0$. Then consider the interval $G \subset H$ mapped onto $M_0 \cup M_1$ by f^n . It is easy to see that $\text{dp}(n, x, M_0 \cup M_1) = 2$. Applying the Two Interval Distortion Lemma to $f^n|G$ we get

$$\text{Dens}(X|[x(n), a]) > 1 - \alpha_6(\delta, K) > 1 - \epsilon.$$

So, $\{x(n), a\}$ is a $D(X, \epsilon)$ one-linked broken line.

Now let $k > 1$. Then we have

$$\lambda(M_{k+1}) \geq \frac{1}{K}\lambda(M_k) \text{ and } \lambda(M_{k-1}) \geq \lambda(M_k).$$

Let $k < r - 2$. Applying the Three Interval Distortion Lemma to f^{kp} with the central interval M_0 , we get $\lambda(M_1) \geq \sigma\lambda(M_0)$. Now we can apply the Two Interval Distortion Lemma to f^n as above. It gives

$$\text{Dens}(X|[x(n), a]) > 1 - \alpha_6(\delta, \sigma^{-1}) > 1 - \epsilon,$$

and we are done.

Finally, let $k = r - 2$. Then consider an interval $N_1 = [w_0, z]$, $w_1 \leq z < w_2$, monotonously mapped onto M_{r-2} by $f^{(r-2)p}$. Replacing M_1 by N_1 in the above argument we will get the same conclusion.

(ii) Now assume $\lambda(M_{k+1}) \leq K^{-1}\lambda(M_k)$ for all $k \in [0, r - 2]$. Then for any such k we can construct a two-linked $D(X, \epsilon)$ broken line beginning at w_k and ending at w_{k+1} (see the above argument for $r \leq 2$). Getting together these lines we obtain a $D(X, \epsilon)$ broken line beginning at $x(n)$ and ending at b .

II. $\text{Dens}_x(X|H^+) > 1 - \delta$. Again let us consider two subcases:

(i) $\lambda(M_{k+1}) \geq 2K\lambda(M_k)$ for some $k \in [0, r - 2]$.

If $k = 0$ then we have a two-linked $D(X, \epsilon)$ broken line in $M_0 \cup M_1$ beginning at $x(n)$ and ending at a (see the argument for $r \leq 2$).

Otherwise consider the first moment k for which $\lambda(M_{k+1}) \geq 2K\lambda(M_k)$. By the same reason as above we have a two-linked $D(f^{n+kp}(X \cap H), \epsilon)$ broken line in $M_k \cup M_{k+1}$ beginning at w_k and ending at w_{k-1} , and such that the length of the first link of the line does not exceed $K\lambda(M_{k+1})$. So, this line lies deeply inside the interval $[M_{i-1}, M_{i+1}]$. Hence we can pull it back to w_0 by f^{kp} with bounded distortion (it needs more careful selection of the constants which we leave to the reader), and then act as in case I.

(ii) Assume $\lambda(M_{k+1}) \leq 2K\lambda(M_k)$ for all $k \in [0, r - 2]$.

Then $\text{Dens}(X|[w_k, w_{k+1}]) > 1 - \epsilon$ for all $k = 0, \dots, r - 2$, and hence $\text{Dens}(X|[x(n), b]) > 1 - \epsilon$. The Lemma is proved. \square

Lemma 4.2. *Let a point $x \in M$ don't converge to a cycle, n be the first moment for which $f^n x \in I^\circ$. Consider the pull-back $\mathbb{I} = \{I_m\}_{m=0}^n$ of $I = I_n$ along $\text{orb}(x)$, $\nu = \text{ord } \mathbb{I}$. Then $\forall \epsilon > 0 \exists \delta = \delta(\nu, \epsilon) > 0$ such that if there is a $D(X, \delta)$ broken line beginning at x and ending at ∂I_0 then there is a $D(X, \epsilon)$ broken line beginning at $f^n x$ and ending at ∂I .*

Remark 4.1. If I is non-periodic then by Proposition 2.2 we can use this lemma with $\nu \leq d$. In particular it is the case when x belongs to a basic set R and I is small enough.

Proof. *Step 1.* First assume that $\nu = \text{ord } \mathbb{I} = 0$, so f^n monotonously maps $H \equiv I_0$ onto I . Let $\mathcal{L} = \{x_0 = x, \dots, x_{m-1}, x_m\}$ be a given $D(X, \delta)$ broken line beginning at x and ending at an endpoint $x_m \in \partial H$ of H . Let us check the required by induction in m . The base of induction $m = 1$ is given by Lemma 4.1. Without loss of generality we can consider that \mathcal{L} is proper. Then the $m - 1$ - linked broken line $\mathcal{T} = \{x_0, \dots, x_{m-1}\}$ is contained in the interval $[a, x_{m-1}]$ where $a \in \partial H$. By the induction assumption, there exists a $D(X, \epsilon)$ broken line \mathcal{R} beginning at $f^n x$ and ending at either ∂I or $f^n x_{m-1}$. In the former case we are done. In the latter case construct by Lemma 4.1 a $D(X, \epsilon)$ broken line \mathcal{R}' beginning at $f^n x_{m-1}$ and ending at ∂I . Getting together \mathcal{R} and \mathcal{R}' we obtain the required broken line.

Step 2. Let c be an extremum, J be a short c -symmetric interval containing a $D(X, \epsilon)$ broken line \mathcal{T} . Then \mathcal{T} can be reconstructed into a $D(X, \rho(\epsilon))$ broken line \mathcal{T}' with the same beginning and end and containing in $\text{cl}(J \setminus \{c\})$. Moreover, $\rho(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. The reconstruction is described (implicitly) in [BL4].

Step 3. Consider the intervals $I_{n(1)}, \dots, I_{n(\nu)}$ of the chain \mathbb{I} containing extrema. By Step 1 we have a $D(X, \epsilon_1)$ broken line in $I_{n(1)}$ beginning at $f^{n(1)}$ and ending at $\partial I_{n(1)}$. By Step 2 we can change it to $D(X, \epsilon'_1)$ broken line whose interior does not contain c . Now apply f to this line using obvious local estimates in a neighborhood of c . Then we will get a $D(X, \epsilon''_1)$ broken line in $I_{n(1)+1}$.

Proceeding in the same manner from $n(1)$ to $n(2)$, from $n(2)$ to $n(3)$ etc., we will get the required broken line. \square

Lemma 4.3. *Let f be finitely renormalizable. Let x be a density point of*

an invariant set X absorbed by a basic set R , $\epsilon > 0$. Then there exist $\gamma > 0$ and a natural N with the following property. For any interval I of length $< \gamma$ omitting N subsequent points of $\text{orb}(x)$ there is an interval $J \subset I \setminus \omega(x)$ such that $\text{dens}(X|L_i) \geq 1 - \epsilon$ for each component L_i of $I \setminus J$, $i = 1, 2$.

Proof. Clearly, we can assume that $x \in R$. Now let us select several constants using notations fixed in §2. Let $\lambda(I) < \min\{\eta, \gamma(R)\} \equiv \gamma$.

Choose $\delta_1 = \delta(d, \epsilon)$ and $\delta_2 = \delta(d, \delta_1)$ by Lemma 4.2 (recall that d is the number of extrema). Since x is a density point of X , there is $\rho > 0$ such that

$$\text{Dens}(X|[x, a]) > 1 - \delta_2$$

if $|x - a| \leq \rho$. By Proposition 2.1, there is a $\tau > 0$ such that for any interval $T \ni x$ of length $> \rho$ we have

$$\lambda(f^n(T)) > \tau, \quad n = 0, 1, 2, \dots$$

Finally, by sensitive dependence to initial conditions, there exists an N such that for any closed interval J intersecting R with $\lambda(J) \geq \tau$ we have: $\text{diam } f^n(J) > \gamma$ for all $n \geq N$.

Denote $x_m = f^m x$. Let I omit N subsequent points of $\text{orb}(x)$ beginning with x_n , and $l \geq N$ be the first moment for which $x_{n+l} \in I$. Consider the pull-back $\mathbb{I} = \{I_m\}_{m=n}^{n+l}$ of I along $\text{orb}_{n+l}(x_n)$, $I_n \equiv K$. By the choice of N , we have

$$\lambda(K) < \tau. \tag{4.2}$$

Our nearest goal is to construct a $D(X, \delta_1)$ -broken line beginning at x_n and ending at ∂K . Set

$$\mathcal{P} = (\text{orb}_{n-1}(x) \cap K^\circ) \cup \partial K.$$

Let $T \subset K$ be the smallest interval containing x_n and ending at points of $\mathcal{P}(n)$.

Consider the pull-back $\{T_m\}_{m=0}^n$ of T along $\text{orb}_n(x)$. It follows from (4.2) that $\lambda(T_0) < \rho$. Hence, X is thick in $T_0 \equiv [a, b]$:

$$\text{Dens}(X|[x, a]) > 1 - \delta_2, \quad \text{Dens}(X|[x, b]) > 1 - \delta_2.$$

So, the intervals $[x, a]$ and $[x, b]$ can be considered as $D(X, \delta)$ -broken lines (with one links). By Lemma 4.2, there is a $D(X, \delta_1)$ -broken line \mathcal{L}_0 beginning at x_n and ending at ∂T .

If \mathcal{L}_0 is ended at ∂K , we are done. Otherwise it ends at a point $x_{n(1)} \in K$ with $n(1) < n$. Handling $x_{n(1)}$ in the same manner, we will find a $D(X, \delta_1)$ -broken line beginning at $x_{n(1)}$ and ending at either ∂K or $x_{n(2)}$ with $n(2) < n(1)$.

Proceeding in such a manner, we construct a sequence of $D(X, \delta_1)$ -broken lines $\mathcal{L}_0, \dots, \mathcal{L}_k$ such that \mathcal{L}_{i+1} starts at an endpoint of \mathcal{L}_i , $i = 0, 1, \dots, k-1$, \mathcal{L}_0 starts at $x(n)$ and \mathcal{L}_k ends at ∂K . Putting together these lines, we get the required broken line $\mathcal{L} = \cup \mathcal{L}_i$.

Now let us consider the map $f^l : K \rightarrow I$. By Lemma 4.2, we get a $D(X, \epsilon)$ -broken line \mathcal{Y} beginning at x_{n+l} and ending at ∂I . Reconstruct it into a proper $D(X, \epsilon)$ -broken line and consider its last link S . This interval contains x_{n+l} , ends at ∂I and $\text{dens}(X|S) > 1 - \epsilon$.

Replace the interval I by $I_1 = I \setminus S$ and consider the first moment $l_1 > n$ when $x_{n+l(1)} \in I_1$. Clearly, $l_1 > l$. Repeating the previous argument, we get an interval $S_1 \subset I_1$ containing $x_{n+l(1)}$, ending at ∂I_1 and such that $\text{dens}(X|I_1) > 1 - \epsilon$.

Set $I_2 = I_1 \setminus S_1$ and proceed in the same manner. We result with a nested sequence of intervals $I \supset I_1 \supset I_2 \supset \dots$ whose intersection J satisfies the required properties. \square

Corollary 4.1. *Let f be finitely renormalizable. Let I be an interval centered at a point $a \in \omega(x)$, I^+ and I^- be the components of $I \setminus \{a\}$. Then*

$$\max(\text{dens}(X|I^+), \text{dens}(X|I^-)) \rightarrow 1 \text{ as } \lambda(I) \rightarrow 0.$$

Lemma 4.4. *Let I be a non-periodic interval symmetric around an extremum c , the set X be τ -symmetric. Then $\forall \epsilon \exists \delta$ such that*

$$\lambda(I) < \delta \Rightarrow \text{dens}(X|I) > 1 - \epsilon.$$

Proof. Observe that if f is finitely renormalizable, then it follows immediately from Corollary 4.1. So, a new information we will get only in infinitely renormalizable case.

Let $F_0 \equiv I$, l_1 be the first moment when $\text{orb}(x)$ passes through F_0 . Now define inductively $F_k = [x(l_k), \tau(x(l_k))]$, and l_{k+1} as the first moment when $\text{orb}(x)$ passes through the interval F_k .

Denote by M_k^- the component of $\text{cl}(F_{k-1} \setminus F_k)$ containing $x(l_k)$, and by M_k^+ the component of $\text{cl}(F_k \setminus F_{k+1})$ containing $x(l_k)$. Clearly, it is enough to show that for any k there is an interval J_k such that $M_k^+ \subset J_k \subset F_k$ and $\text{dens}(X|J_k) > 1 - \epsilon$.

Denote $o_k = \text{ord}(n, x, F_k)$.

If $o_k < 2d$ then we have such an interval by Lemma 4.2. Otherwise by Proposition 2.2 there exists an interval $H_{k-1} \ni x$ monotonously mapped by f^{l_k} onto $M_k^- \cup M_k^+$. Clearly, $\text{dp}(n, x, M_k^- \cup M_k^+) \leq 2$, and we can apply the Two Interval Distortion Lemma to $f^n|H_n$.

If $\lambda(M_k^+) \leq \lambda(M_k^-)$ then we conclude that X is thick in M_k^+ as required. Otherwise X is thick in M_k^- . Now let us pass from $x(l_k)$ to $x(l_{k+1})$. It follows from Proposition 2.2 that $\text{ord}(l_{k+1} - l_k, x(l_k), F_k) \leq d$. Hence, the existence of the required interval J_k follows again from Lemma 4.2. \square

5 Proof of the main results.

Let us start with the following theorem proved by Guckenheimer [G] in the case of negative Schwarzian derivative, and by Mañé [Ma] in the general smooth case (see also [vS]).

Theorem B. *For almost every $x \in \Lambda(f)$ the limit set $\omega(x)$ contains a critical point.*

Problem. Is it true that $\omega(x)$ contains an *extremum* for almost every $x \in \Lambda(f)$?

Proof of Theorem 1. Denote by X_c the set $\{x : \omega(x) \ni c\}$. By Theorem A,

$$\bigcup_{c \in \mathcal{C}(f)} X_c = M \pmod{0}$$

Let X be any completely invariant subset of X_c of positive measure. In the finitely renormalizable case we have by Corollary 4.1

- (i) If c is an extremum then $\text{dens}X|c = 1$;
- (ii) If c is a reflection point then $\text{dens}X|c \geq 1/2$.

Hence, in the first case $f|X_c$ is ergodic, and in the second case X_c contains at most two ergodic components.

In the infinitely renormalizable case we can select c as an extremum and apply Lemma 4.4. \square

Proof of Theorem 2. Let us associate to any ergodic component $E \subset \Lambda(f)$ the following attractor A (cf [M]). Consider the family of neighborhoods U such that the orbits of almost all points $x \in E$ pass through U only finitely many times. Let $\{U_i\}_{i=1}^\infty$ be a countable basis of this family. Set

$$A = M \setminus (\cup_{i=1}^\infty U_i).$$

Clearly, A is a closed invariant set. Let us check that

$$E \subset \text{RL}(A) \pmod{0}, \tag{5.1}$$

Indeed, let V be a neighbourhood of a point $a \in A$. Let us consider the set

$$E_V = \{x \in E : \text{orb}(x) \text{ passes through } V \text{ infinitely many times}\}.$$

By definition, $\lambda(E_V) > 0$. Since E_V is completely invariant, ergodicity yields $E_V = E \pmod{0}$. Taking a countable basis of neighbourhoods V_i of A , we obtain: $\omega(x) = A$ for almost all $x \in E$ which is equivalent to (5.1).

Inclusion (5.1) implies that A is a primitive attractor. So, we have constructed finitely many primitive attractors A_k corresponding to ergodic components E_k . Since

$$\text{rl}(\cup A_k) \supset \cup E_k \supset \Lambda(f) \pmod{0},$$

$\cup A_k$ is a global attractor for $f|_{\Lambda(f)}$, and we have the decomposition (1.1).

Statements (i) and (ii) are also clear now: the former follows from (5.1), the latter from Theorem B. Let us prove (iii).

Let E_i and E_j be two ergodic components in $\Lambda(f)$, A_i and A_j be the corresponding primitive attractors. Assume $A_i \cap A_j$ is infinite. Then we can pick up two close points $a_1, a_2 \in A_i \cap A_j$. Let us consider an interval $I = [a_0, a_3]$ containing $[a_1, a_2]$ and such that

$$|a_0 - a_1| = |a_1 - a_2| = |a_2 - a_3|.$$

Applying Corollary 4.1, it is easy to see that each set E_i, E_j is thick at least in two of three intervals $[a_0, a_1], [a_1, a_2], [a_2, a_3]$. Hence, both E_i, E_j are thick in one of these intervals, which is impossible.

Now let us pass to (iv). By (5.1),

$$E_k \subset \text{RL}(A_k) \subset \text{rl}(A_k) \cap \Lambda(f).$$

So, it is enough to prove that the set $(\text{rl}(A_k) \cap \Lambda(f)) \setminus E_k$ has zero measure. But otherwise there is another ergodic component $E_i \subset \Lambda(f) \cap \text{rl}(A_k)$. The attractor A_i corresponding to E_i is infinite and is contained in A_k , contradicting to what has been proved above. \square

Proof of Theorem 3. It follows from Corollary 4.1 in the same way as in [BL5]. \square

Proof of Theorem 4. (i) Let K be a closed invariant set in A . We are going to prove that $K = A$. Assume it is not the case. Then there is a closed interval I centered at a point $a \in A$ and such that $I \cap K = \emptyset$, $\lambda(I) < \gamma$ where γ is taken from Lemma 4.3. Let a divide I into semi-intervals I^+ and I^- . Since A is a Cantor attractor, there is an invariant set $X \subset \text{RL}(A)$ of positive measure such that

$$\text{dens}(X|I^+) < 1 - \epsilon \quad \text{and} \quad \text{dens}(X|I^-) < 1 - \epsilon. \quad (5.2)$$

Let N be from Lemma 4.3. Find a neighborhood U of K such that $\text{orb}_N(x) \cap I = \emptyset$ for any $x \in U$. Let us pick up a density point $x \in X$ such that $\omega(x) = A$. Then $f^n x \in U$ for some n , hence I omits N subsequent points of $\text{orb}(x)$, and we can apply Lemma 4.3. But its conclusion contradicts (5.2).

(ii) follows from Corollary 4.1. in the same way as in [BL5].

(iii) is immediate from (i) and Theorem 2(ii).

(iv) follows from (i) and Theorem 2(iii). \square

Proof of Theorem 5. If cases A1 and A3 does not hold then A has non-empty interior. Since A is primitive, it should be transitive. It is easy to show that a closed invariant transitive set must be a cycle of intervals, so A2 holds. \square

Proof of Corollary 1.1. Associate to an infinite primitive attractor A a critical point $c(A)$ belonging to A . Moreover, if A is a cycle of intervals, let $c(A) \in A^\circ$. It follows from Theorem 4(iv) that this correspondence is injective, and we are done. \square

Proof of Corollary 1.2. It easily follows from Theorem 4(iv) and Theorem 5. \square

References

- [B1] A. M. Blokh. Decomposition of dynamical systems on an interval. Russian Math. Surveys, v.83 (1983), N5, 133-134.
- [B2] A.M. Blokh. On the dynamical systems on one-dimensional branched manifolds, I,II,III. Theory of Functions, Functional Analysis and Applications (Kharkov), v.46 (1986), 8-18; v.47 (1987), 67-77; v. 48 (1987), 32-46.
- [BL1] A. M. Blokh, M. Yu. Lyubich. Attractors of maps of the interval. Functional Analysis and Applications, v. 21 (1987), 70-71.
- [BL2] A. M. Blokh, M. Yu. Lyubich. Attractors of maps of the interval. Banach Center Publications (of the Dynamical Systems Semester held in Warsaw, 1986), v.23 (1989).
- [BL3] A. M. Blokh, M. Yu. Lyubich. Ergodic properties of maps of the interval. Functional Analysis and Applications, v.23 (1989), N1, 59-60.
- [BL4] A. M. Blokh, M. Yu. Lyubich. On the decomposition of one dimensional dynamical systems into ergodic components. Algebra and Analysis (Translated in English as "Leningrad Math. Journal"), v.1 (1989), N1, 128-145.
- [BL5] A. M. Blokh, M. Yu. Lyubich. Measurable dynamics of S-unimodal maps of the interval. Preprint Institute for Math. Sci., Stony Brook, 1990/2.
- [BL6] A. M. Blokh, M. Yu. Lyubich. Measure and dimension of solenoidal attractors of one dimensional dynamical systems. Comm. Math. Physics, v.1 (1990), 573-583.
- [BL7] A. M. Blokh, M. Yu. Lyubich. Non-existence of wandering intervals and structure of topological attractors of one-dimensional dynamical systems. II. The smooth case. Ergodic Theory and Dynamical Systems, v.9 (1989), N4, 751-758.
- [G1] J. Guckenheimer. Sensitive dependence to initial conditions for one-dimensional maps. Comm. Math. Phys., v.70 (1979), 133-160.

- [G2] J. Guckenheimer. Limit sets of S-unimodal maps with zero entropy. *Comm. Math. Phys.*, v.110 (1987), 655-659
- [GJ] J. Guckenheimer, S. Johnson. Distortion of S-unimodal maps, to appear in *Annals Math.*
- [JR] L. Jonker, D. Rand. Bifurcations in one dimension. I. The non-wandering set. *Inventiones Math.*, v.62 (1981), 347-365.
- [H] M. Herman. Sur la conjugation différentiable de difféomorphismes du cercle à la rotation. *Publ. Math. IHES*, v.49 (1979), 5-233.
- [Ho] F. Hofbauer. The structure of piecewise monotonic transformations. *Ergodic Theory and Dynamical Systems*, v.1 (1981), 159-178.
- [K] G.Keller. Exponents, attractors and Hopf decomposition for interval maps. Preprint, 1988.
- [KSF] I. P. Kornfel'd, Ya. G. Sinai, S. V. Fomin. *Ergodic Theory*. "Nauka", Moscow, 1980.
- [L] M. Yu. Lyubich. Non-existence of wandering intervals and structure of topological attractors of one-dimensional dynamical systems. I. The smooth case. *Ergodic Theory and Dynamical Systems*, v.9 (1989), N4, 737-750.
- [M] J. Milnor. On the concept of attractor. *Comm. Math. Phys.*, v.99 (1985), 177-195.
- [Ma] R. Mañé. Hyperbolicity, sinks and measure in one-dimensional dynamics. *Comm. Math. Phys.*, v.100 (1985), 495-524.
- [MS] W. de Melo, S.J. van Strien. A structure theorem in one dimensional dynamics. Preprint, 1986.
- [MMS] M. Martens, W. de Melo, S. J. van Strien. Julia-Fatou-Sullivan theory for real one-dimensional dynamics. Preprint, 1988.
- [MMSS] M. Martens, W. de Melo, S. J. van Strien, D. Sullivan. Bounded geometry and measure of the attracting Cantor sets of quadratic-like maps. Preprint, 1988.

- [S] D. Sullivan. On the structure of infinitely many dynamical systems nested inside or outside a given one. Preprint IHES/M/90/75.
- [vS] S. J. van Strien. Hyperbolicity and invariant measures for general C^2 interval maps satisfying Misiurewicz condition. Preprint, 1987.
- [Sh] A. N. Sharkovskii. A partially ordered system of attracting sets. Dokl. Akad. Nauk SSSR, v. 170 (1966), N6, 1276-1278.
- [Sw] G. Swiatek. Bounded distortion properties of one-dimensional maps. Preprint IMS, SUNY, Stony Brook, 1990/10.
- [Y] J.-C. Yoccoz. Il n'y a pas de contre-exemple de Denjoy analytiques. C.R.Acad. Sci. Paris, ser. Mat., v.289 (1984), 141-144,