

QUASISYMMETRIES OF THE BASILICA AND THE THOMPSON GROUP

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ABSTRACT. We give a description of the group of all quasimetric self-maps of the Julia set of $f(z) = z^2 - 1$ that have orientation preserving homeomorphic extensions to the whole plane. More precisely, we prove that this group is the uniform closure of the group generated by the Thompson group of the unit circle and an inversion. Moreover, this result is quantitative in the sense that distortions of the approximating maps are uniformly controlled by the distortion of the given map.

1. INTRODUCTION

Quasimetric geometry of fractal sets has attracted substantial interest in recent years. A natural invariant in this category is the group of quasimetrics of the set. One can roughly classify such a set as “little quasimetric” or “highly quasimetric”, depending on whether this group is finite or infinite. We are interested in this dichotomy for Julia sets of rational maps. In our previous paper [BLM], joint with Mario Bonk, we described a class of little quasimetric Julia sets that are Sierpiński carpets. The goal of this paper is to give an example of a highly quasimetric Julia set, the basilica (see Figure 1), and to describe its group of quasimetrics.

To formulate the main result, let us give quick definitions of the main objects, referring to the main body of the paper for their precise versions.

Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial of degree ≥ 2 . Its *filled Julia set* $\mathcal{K}(f)$ is defined as the set of non-escaping points, and the *Julia set* $\mathcal{J}(f)$ is defined as the boundary of $\mathcal{K}(f)$.

The *basilica* is the filled Julia set of the quadratic polynomial $f: z \mapsto z^2 - 1$.¹ This polynomial has a superattracting cycle $\gamma = \{0, -1\}$ of period two, and $\text{int } \mathcal{K}(f)$ coincides with the basin of this cycle. By

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¹In this paper, we refer informally to the corresponding Julia set \mathcal{J} also as “basilica”.

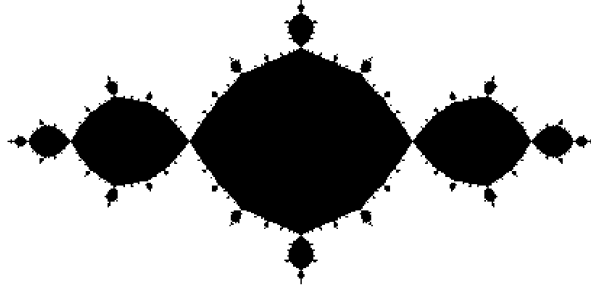


FIGURE 1. The filled Julia set of $f(z) = z^2 - 1$.

definition, the *immediate basin* of γ is the union of two components U_0 and U_{-1} of the basin containing 0 and -1 , respectively. These components are Jordan discs, and the Riemann mapping $\phi_0: U_0 \rightarrow \mathbb{D}$, such that $\phi_0(0) = 0$, $\phi_0'(0) < 0$, brings the return map $f^2: U_0 \rightarrow U_0$ to the monomial form $g: z \mapsto z^2$.

Given a homeomorphism $\eta: [0, \infty) \rightarrow [0, \infty)$, a homeomorphism $h: \mathcal{J}(f) \rightarrow \mathcal{J}(f)$ is called a *quasisymmetry* with *distortion function* η (or η -*quasisymmetry*) if

$$\frac{|h(u) - h(v)|}{|h(u) - h(w)|} \leq \eta\left(\frac{|u - v|}{|u - w|}\right),$$

for all distinct triples u, v , and w in $\mathcal{J}(f)$. In fact, this definition is applicable to a homeomorphism $h: X \rightarrow X$ of any metric space X , with the Euclidean distance replaced by the corresponding metric. Moreover, in the case of the complex plane, $X = \mathbb{C}$, η -quasisymmetry is equivalent to K -quasiconformality, quantitatively (provided the map is normalized at two points). We say that a homeomorphism of $\mathcal{J}(f)$ is *topologically extendable* if it has an extension to an orientation preserving homeomorphism of \mathbb{C} .

The *Thompson group* T is defined as the group of piecewise linear (in the angular coordinate) homeomorphisms of the unit circle \mathbb{T} with breaks at some dyadic points and slopes equal to integer powers of 2. By means of the Riemann mapping ϕ_0 , we can make this group act on ∂U_0 . In this paper, we show that this action admits an extension to an action by quasisymmetries of the basilica Julia set $\mathcal{J} \equiv \mathcal{J}(f)$. We also construct one more quasisymmetry of \mathcal{J} , an involution ι , which permutes the components U_0 and U_{-1} of the immediate basin. Let \hat{T} be the *extended Thompson group* of quasisymmetries of \mathcal{J} generated by T and ι . Our main result asserts that this group generates quantitatively the whole group of topologically extendable quasisymmetries of \mathcal{J} :

Theorem 1.1. *For any distortion function η there exists a distortion function η' , such that for any topologically extendable η -quasisymmetry ξ of the basilica Julia set \mathcal{J} there exists a sequence of η' -quasisymmetries τ_n of \mathcal{J} that belong to the extended Thompson group \hat{T} and uniformly converge to ξ .*

Thus, the basilica is highly quasimetric. As we have already mentioned, this contrasts with the result of [BLM] concerning Sierpiński Julia sets. Note that the standard Sierpiński carpet is little quasimetric as well [BM]. On the other hand, it was shown in [Me] that the “slit carpet” is highly quasimetric, and the corresponding group of quasimetrics bears some similarity with the Thompson group.

Note that the limit sets of non-elementary Kleinian groups are highly symmetric, as they admit infinite groups of Möbius symmetries. Let us say that a compact subset $K \subseteq \widehat{\mathbb{C}}$ of the sphere is *rigid* if all its quasimetrics are Möbius. It was shown in [BKM] that “Schottky sets” of zero area are rigid, contrasting again with our result.

Let us also mention a recent work of J. Belk and B. Forrest [BF] who studied a Thompson-like group of circle homeomorphisms that preserve the invariant lamination of the basilica, and hence descend to homeomorphisms of the basilica itself. Our result implies that these homeomorphisms are quasimetric (and, in fact, form a dense subgroup of basilica quasimetrics). Unlike [BF], we approach the problem from “inside” of the basilica, beginning with the Thompson group acting on the immediate basin.

Let us finally note that our method extends in a straightforward way to hyperbolic Julia sets in the “main molecule” of the Mandelbrot set (obtained from z^2 through a finite cascade of satellite bifurcations), e.g., to the Douady rabbit.

1.1. Notation and terminology. Throughout the paper we denote $\mathbb{N} \cup \{0\}$ by \mathbb{N}_0 . We let \mathbb{D} be the open unit disc in the complex plane \mathbb{C} , and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle in \mathbb{C} .

In what follows, we label a point $z = e(\theta) = e^{2\pi i\theta} \in \mathbb{T}$ by $\theta \in \mathbb{R}/\mathbb{Z}$. In other words, the angular measure on the circle \mathbb{T} is scaled so that its total length is equal to 1, which is standard in dynamics. Points $e((2k + 1)/2^n) \in \mathbb{T}$, $k = 0, 1, \dots, 2^{n-1} - 1$, $n \in \mathbb{N}_0$, are called *dyadic points of level n* . Taking the union of dyadic points of levels $m \leq n$, we obtain 2^n points $e(l/2^n)$, $l = 0, 1, \dots, 2^n - 1$, that tessellate the circle into 2^n (closed) intervals I_l^n . We will refer to these intervals as *dyadic intervals of level n* .

Let g denote the map $z \mapsto z^2$, which is *doubling* in the angular coordinate of \mathbb{T} , i.e., $\theta \mapsto 2\theta \pmod{\mathbb{Z}}$. The dyadic points $z \in \mathbb{T}$ are

dynamically identified as the iterated preimages of the fixed point 1, with the level equal to the smallest $n \in \mathbb{N}_0$ such that $g^n(z) = 1$.

2. DYADIC SUBDIVISION OF BOUNDED FATOU COMPONENTS

The reader can consult [CG, Mi] for a general introduction to the iteration theory of rational functions, and [DH, L] for particular features of the dynamics of quadratic polynomials.

The *Fatou set* is the complement of the Julia set, $\mathcal{F}(f) = \mathbb{C} \setminus \mathcal{J}(f)$. Its connected components are called *Fatou components*. Bounded Fatou components can also be identified as the connected components of the interior of the filled Julia set $\mathcal{K}(f)$.

As we have already mentioned, the *basilica map* $f(z) = z^2 - 1$ is specified by the property that it has a superattracting periodic cycle $\{0, -1\}$ of period two. In particular, it is a postcritically finite hyperbolic map. We let U_0 and U_{-1} be the Fatou components of f that contain 0 and -1 , respectively. Both of these components are Jordan discs. The map f takes U_0 onto U_{-1} as a double branched covering, and it takes U_{-1} back onto U_0 conformally.

Let $\phi_0: (\overline{U_0}, 0, \alpha) \rightarrow (\overline{\mathbb{D}}, 0, 1)$ be the *Böttcher coordinate* of U_0 (which coincides with the appropriately normalized Riemann uniformization of U_0), homeomorphically extended to the boundary. It conjugates the return map $f^2: \overline{U_0} \rightarrow \overline{U_0}$ to the monomial map $g: \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$, $z \mapsto z^2$.

The left-most point of the closure $\overline{U_0}$ on the real line is a fixed point α of f , and it equals to $(1 - \sqrt{5})/2$. It is also the unique point of intersection of $\overline{U_0}$ and $\overline{U_{-1}}$, called the *root* of each of these components. Moreover, α is a *global cut-point* for the Julia set \mathcal{J} : puncturing α out results in breaking \mathcal{J} into two connected components. All the preimages of α under the iterates of f are therefore also global cut-points. The other fixed point of f is $\beta = (1 + \sqrt{5})/2$. It is the right-most point of the intersection of \mathcal{J} with the real line.

Every bounded Fatou component U of f eventually, i.e., under a certain iterate of f , lands in the cycle $\{U_0, U_{-1}\}$. In fact, for each U there exists a unique $n \in \mathbb{N}_0$ such that $f^n: U \rightarrow U_0$ is a conformal map. We call such n the *dynamical distance* from U to U_0 . It follows that all bounded Fatou components of f are Jordan discs as well. Then the map $\phi_U = \phi_0 \circ f^n$ is a conformal map of U onto \mathbb{D} that extends to a homeomorphism $\overline{U} \rightarrow \overline{\mathbb{D}}$. It is called the *Böttcher coordinate* of U . The *root* $\alpha_U \in \partial U$ of U is defined as the preimage of the root $\alpha \in U_0$ under f^n , or equivalently, as the point in ∂U whose Böttcher coordinate $\phi_U(\alpha_U)$ is equal to 1 $\in \mathbb{T}$.

Similarly, we define *dyadic points* on ∂U as points whose Böttcher coordinates are dyadic. Dyadic points of level $\leq n$ induce a *level n dyadic subdivision* of ∂U . It can also be described as follows: for a bounded Fatou component U of dynamical distance $n \geq 0$ from U_0 and for a natural number $m \geq 1$, the full preimage $f^{-m}(\alpha)$ gives the dyadic decomposition of ∂U that consists of 2^{m-n} points. (Here, if $n > m$, then the corresponding set is empty.)

The basilica filled Julia set $\mathcal{K} \equiv \mathcal{K}(f)$ has the following combinatorial structure of a (non-locally finite) tree \mathfrak{T} . The vertices of \mathfrak{T} are the bounded Fatou components of f . (In what follows we make no distinction between bounded Fatou components and the vertices of \mathfrak{T} .) Two vertices U and V are connected by an edge E if and only if they *touch*, i.e., their closures intersect. Let $\delta(U)$ stand for the combinatorial distance in this tree from a vertex U to the main vertex U_0 . If two vertices U and V are adjacent, then $|\delta(U) - \delta(V)| = 1$. Under these circumstances, if $\delta(U) = \delta(V) + 1$ then V is called the *principal* vertex of E , while U is called a *satellite* of V .

The *main edge* of \mathfrak{T} , denoted E_0 , is the edge that connects U_0 to U_{-1} via their common root α . The higher level edges are described as follows:

Lemma 2.1. *Let U and V be distinct bounded Fatou components that touch at a point z . Let $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$ be the dynamical distances from U and V to U_0 , respectively, with $n \geq m$. Then U is a satellite of V and the root of U is z .*

Proof. The distances m and n cannot be equal because otherwise we get a contradiction with the fact that f^m is conformal in a neighborhood of z . Application of f^k , $k \leq m$, preserves the differences in dynamical and in combinatorial distances from U and V to U_0 , respectively. This implies that the combinatorial distance from V to U_0 in \mathfrak{T} is one less than that from U to U_0 . Now, $f^m(U)$ is a bounded Fatou component that touches $f^m(V) = U_0$. The point of the intersection is $f^m(z)$, and $n - m \in \mathbb{N}$ is the dynamical distance from $f^m(U)$ to U_0 . Thus $f^{n-m-1}(f^m(U))$ is either U_{-1} or U_1 , the bounded Fatou components that contain -1 and 1 , respectively. In either case we must have $f^{n-m-1}(U_0) = U_0$ because f is at most two to one in each Fatou component. Hence $f^{n-1}(z) = f^{n-m-1}(f^m(z))$ is either α or $-\alpha$, and thus $f^n(z) = \alpha$, i.e., z is the root of U . \square

Thus, every edge E can be labeled by the principal component V and a dyadic number $d \in \mathbb{T}$ representing the point on ∂V where the satellite component U touches V .

3. DYNAMICAL PARTITIONS OF THE JULIA SET \mathcal{J}

Let U_∞ denote the *basin at infinity*, i.e., the unbounded Fatou component of f , and let ϕ_∞ be the *Böttcher coordinate* of U_∞ . Namely, ϕ_∞ is the conformal map of U_∞ onto the complement of the closed unit disc $\mathbb{C} \setminus \overline{\mathbb{D}}$ conjugating f to $g: z \mapsto z^2$.

The *external ray* ρ^θ with angle $\theta \in \mathbb{R}/\mathbb{Z}$ is defined as the pullback of the straight ray $\rho^\theta = \{re(\theta) : 1 < r < \infty\}$ under the Böttcher map ϕ_∞ . The external rays form an invariant foliation with f acting by the angle-doubling: $f(\rho^\theta) = \rho^{2\theta}$.

It is known that the Julia set of a hyperbolic map f is locally connected, so the inverse Böttcher map $\psi_\infty = \phi_\infty^{-1}: \mathbb{D} \rightarrow U_\infty$ extends continuously to the boundary, and so induces a continuous boundary map of \mathbb{T} onto \mathcal{J} . It follows that any ray ρ^θ *lands* at some point $z_\theta \in \mathcal{J}$. For a circle arc $I \subset \mathbb{T}$, we call

$$\psi(I) = \{z_\theta : e(\theta) \in I\}$$

a *Julia arc*.

In the case of the basilica, there exist exactly two rays, $\rho^{\pm 1/3}$, that land at the fixed point α . They bound two (open) sectors, $S_0 \supset U_0$ and $S_\alpha \supset U_{-1}$. The latter sector is also called the *wake rooted at α* . The intersection $\mathcal{K}_\alpha := (\mathcal{K} \cap S_\alpha) \cup \{\alpha\}$ (with the added root) is called the *limb of \mathcal{K} rooted at α* . We denote $\mathcal{J}_\alpha := \partial\mathcal{K}_\alpha$.

This picture can be spread around to the iterated preimages of α . Let $z \in \mathcal{J}$ and $f^n z = \alpha$ for some $n \in \mathbb{N}_0$. We choose the smallest moment n like this. Then z is the landing point of exactly two rays, f^n -preimages of $\rho^{\pm 1/3}$. They bound the unique (open) sector S_z that does not contain U_0 ; it is called the *wake rooted at z* . Moreover, if $n \geq 1$, then f^n conformally maps S_z onto S_0 . Let $\mathcal{K}_z := (S_z \cap \mathcal{K}) \cup \{z\}$ be the *limb of \mathcal{K} rooted at z* , and let $\mathcal{J}_z := \partial\mathcal{K}_z$ be the corresponding Julia arc. Below we may also refer to \mathcal{J}_z as a *limb rooted at z* .

Let \mathcal{R}_n , $n \in \mathbb{N}_0$, be the family of all external rays in U_∞ that land at points of the full preimage $f^{-n}(\alpha)$. Note that each external ray of \mathcal{R}_n lands on the boundaries of exactly two bounded Fatou components, adjacent components of the tree \mathfrak{T} .

We say that distinct external rays ρ_1 and ρ_2 of \mathcal{R}_n are *adjacent* if they are not separated in U_∞ by other external rays of \mathcal{R}_n (in other words, the angles θ_1, θ_2 are adjacent points of the set $g^{-n}\{\pm 1/3\} \subset \mathbb{T}$).

Lemma 3.1. *If $\rho_1, \rho_2 \in \mathcal{R}_n$ are adjacent external rays, then there exists a bounded Fatou component U of f such that ρ_1 and ρ_2 land at boundary points of U .*

Proof. We apply induction. The statement is trivially true for $n = 0$. Assuming that the statement is true for $n - 1$, where $n \geq 1$, let ρ_1 and ρ_2 be two adjacent external rays of \mathcal{R}_n . If we apply f to ρ_1 and ρ_2 , we obtain two external rays of \mathcal{R}_{n-1} that are necessarily adjacent. Otherwise, we would apply f^{-1} to the region in U_∞ between $f(\rho_1)$ and $f(\rho_2)$ that contains a separating external ray to get a contradiction. Note that in the Böttcher coordinate of U_∞ the arc on \mathbb{T} that corresponds to the landing points of ρ_1 and ρ_2 and does not contain the landing points of other rays in \mathcal{R}_n has angular length at most $1/3 < 1/2$, and hence f is conformal in the region of U_∞ between ρ_1 and ρ_2 that does not contain other external rays of \mathcal{R}_n . Therefore, $f(\rho_1)$ and $f(\rho_2)$ land on the boundary of the same bounded Fatou component U . If $U = U_{-1}$, then ρ_1 and ρ_2 must land on the boundary of U_0 , because U_0 is the only preimage of U_{-1} under f . If $U \neq U_{-1}$, any branch of the map f^{-1} restricted to U is conformal onto another bounded Fatou component V , and there are exactly two such branches because f has degree 2. Suppose ρ_1 lands on the boundary of V . The branch f^{-1} that takes U onto V extends conformally across the arc ω on the boundary of U between the landing points of $f(\rho_1)$ and $f(\rho_2)$ into the region R in $\mathbb{C} \setminus \bar{U}$ that satisfies the following properties. The arc ω does not contain other landing points of external rays in \mathcal{R}_{n-1} , and the region R is bounded by ω , $f(\rho_1)$, and $f(\rho_2)$. The image of $f(\rho_2)$ under this branch of f^{-1} is then necessarily the external ray ρ_2 , and we are done. \square

For $n \in \mathbb{N}_0$, let:

- D_n be the n -fold preimage $g^{-n}\{\pm 1/3\} \subset \mathbb{T}$ (consisting of 2^{n+1} points);
- P_n be the tiling of the circle \mathbb{T} by the points of D_n (comprising 2^{n+1} closed arcs $I_k \subset \mathbb{T}$);
- $\Pi_n = \psi_\infty(P_n)$ be the corresponding tiling of \mathcal{J} (comprising 2^{n+1} closed Julia arcs $J_k = \psi_\infty(I_k) \subset \mathcal{J}$ with *endpoints* at preimages of the fixed point α of *level* $\leq n$).

Each set J_k corresponds to a pair of adjacent external rays of \mathcal{R}_n , as follows. Let ρ_1 and ρ_2 be two adjacent external rays of \mathcal{R}_n whose landing points are z_1 and z_2 , respectively. There are two cases that need to be considered: either $z_1 = z_2$ or $z_1 \neq z_2$.

Case 1. If $z = z_1 = z_2$, this point is the root of a unique bounded Fatou component U unless $z = \alpha$, the fixed point of f . This follows from Lemma 2.1. If $z = \alpha$, we choose $U = U_{-1}$. The Julia arc J_k of Π_n that corresponds to ρ_1, ρ_2 is \mathcal{J}_z .

Case 2. Now assume that $z_1 \neq z_2$ and let U be a bounded Fatou component of f whose boundary contains z_1 and z_2 . Such U exists by

Lemma 3.1. Let ω be the arc on ∂U between z_1 and z_2 that contains no other landing points of external rays in \mathcal{R}_n . Let R be the region in $\mathbb{C} \setminus \overline{U}$ that is bounded by ω , ρ_1 , and ρ_2 . The Julia arc J_k that corresponds to the pair ρ_1, ρ_2 is the closure of the intersection of \mathcal{J} with the region R .

It follows immediately from the definition that the map f^n takes each Julia arc J_k of Π_n onto the closure of one of the two connected components of $\mathcal{J} \setminus \{\alpha\}$: either \mathcal{J}_0 that contains the boundary of U_0 or \mathcal{J}_α that contains the boundary of U_{-1} . The first case occurs when the landing points z_1 and z_2 are the same. In this case the map f^n is one to one. In the other case, i.e., $z_1 \neq z_2$, the preimage of each point z of \mathcal{J}_α under f^n is a singleton except if $z = \alpha$, when the preimage consists of two points.

Lemma 3.2. *The set D_n , $n \geq 0$, is the set of 2^{n+1} points on the unit circle such that the angular lengths of complementary intervals alternate between $1/(3 \cdot 2^n)$ and $2/(3 \cdot 2^n)$. In particular, there exists a constant $L \geq 1$ independent of n , and an orientation preserving piecewise linear L -bi-Lipschitz map ψ of \mathbb{T} whose break points are points of D_n , and such that $\psi(D_n)$ is a set of 2^{n+1} points on the unit circle such that all complementary intervals have equal lengths.*

Proof. Let I and J be two adjacent complementary intervals at level n . Application of the map $g(z) = z^2$ doubles the lengths of each one of them and keeps them adjacent. Because for $n = 0$ the points $1/3$ and $2/3$ satisfy the desired property and form a 2-cycle under the dynamics of g , the first part of the claim follows from induction. The second part follows from the observation that in order to make all complementary intervals to have the same length, one needs to scale the complementary intervals of length $1/(3 \cdot 2^n)$ by $3/2$ and the complementary intervals of length $2/(3 \cdot 2^n)$ by $3/4$. In particular, $L = 3/2$. \square

Let us finish this section with a description of a *pinched disk* topological model for the basilica. Let d be the diameter of \mathbb{D} with one endpoint at $e(1/3)$, and let \mathbb{D}_\pm be the corresponding semi-disks. For the sake of definiteness we assume that $e(2/3)$ is contained in \mathbb{D}_- . Let us connect the points $e(1/3)$ and $e(2/3)$ of D_0 with the hyperbolic geodesic $\gamma^0 \subset \overline{\mathbb{D}}$. The points of D_1 split into two symmetric pairs, one contained in $\overline{\mathbb{D}_+}$, the other contained in $\overline{\mathbb{D}_-}$. Each of these pairs can be connected with a hyperbolic geodesic. Since $\{e(1/3), e(2/3)\}$ is a periodic cycle for the map g , one of these geodesics, namely the one contained in $\overline{\mathbb{D}_-}$, is the geodesic γ^0 . We denote the other geodesic by γ^1 . The full preimage of $\partial\gamma^1$ under the doubling map g consists of two pairs of points in D_2 , one contained in \mathbb{D}_+ , the other contained in \mathbb{D}_- .

Connecting each of these pairs with a hyperbolic geodesic, we obtain two new geodesics in $D_2 \setminus D_1$, denoted by γ_+^2 and γ_-^2 . By the same procedure, for each geodesic γ_ϵ^2 , $\epsilon = \pm$, we obtain two new geodesics (“pullbacks” of γ_ϵ^2), one in \mathbb{D}_+ and the other contained in \mathbb{D}_- , which we denote by $\gamma_{(\epsilon_1 \epsilon_2)}^3$, $\epsilon_i \in \pm$, $i = 1, 2$. Proceeding this way, for each level $n \in \mathbb{N}$, we can construct 2^{n-1} hyperbolic geodesics $\gamma_{\bar{\epsilon}}^n$, $\bar{\epsilon} = (\epsilon_1 \dots \epsilon_{n-1})$, $\epsilon_i \in \pm$, paring the points of $D_n \setminus D_{n-1}$. It is easy to show that all geodesics $\gamma_{\bar{\epsilon}}^n$ are disjoint. One can also show that together they form a closed subset Q of $\overline{\mathbb{D}}$. This subset (endowed with a partition into the geodesics $\gamma_{\bar{\epsilon}}^n$) is called the *basilica lamination*; see Figure 2.

Let us consider an equivalence relation $\underset{f}{\sim}$ on \mathbb{C} whose classes are either the geodesics of the basilica lamination or single points.

Theorem 3.3. [L, Theorem 24.33] *The quotient of $(\mathbb{C}, \overline{\mathbb{D}})$ by the equivalence relation $\underset{f}{\sim}$ is homeomorphic to $(\mathbb{C}, \mathcal{K})$. Moreover, this homeomorphism coincides with the inverse Böttcher coordinate ψ_∞ on $\mathbb{C} \setminus \overline{\mathbb{D}}$ and sends the geodesics $\gamma_{\bar{\epsilon}}^n$ to the corresponding level n preimages of the α -fixed point.*

The equivalence relation $\underset{f}{\sim}$ induces an equivalence relation on the circle \mathbb{T} that pairs points of $D_\infty := \bigcup D_n$, where $\theta_1 \underset{f}{\sim} \theta_2$ if and only if the rays ρ^{θ_1} and ρ^{θ_2} land at the same point of \mathcal{J} . We will refer to it as the *basilica lamination of \mathbb{T}* . (This “lamination” has zero-dimensional leaves.) Obviously, the basilica lamination of \mathbb{T} contains the same amount of information as the basilica geodesic lamination of $\overline{\mathbb{D}}$.

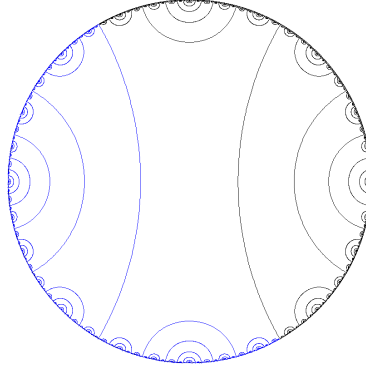
For the sake of reference, let us state a simple lemma:

Lemma 3.4. *Any homeomorphism $\xi: U_\infty \cup \mathcal{J} \rightarrow U_\infty \cup \mathcal{J}$ lifts (by means of the extended inverse Böttcher coordinate $\psi_\infty: \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow U_\infty \cup \mathcal{J}$) to a homeomorphism $h: \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ that preserves the basilica lamination of \mathbb{T} , and whose restriction ξ_∞ to \mathbb{T} satisfies*

$$(3.1) \quad \psi_\infty \circ \xi_\infty = \xi \circ \psi_\infty.$$

Conversely, any homeomorphism $h: \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ that preserves the basilica lamination of \mathbb{T} descends to a homeomorphism $\xi: U_\infty \cup \mathcal{J} \rightarrow U_\infty \cup \mathcal{J}$.

Proof. On $\mathbb{C} \setminus \overline{\mathbb{D}}$, the map h is defined as $\phi_\infty \circ \xi \circ \psi_\infty$. By the Carathéodory Theory, the closure $\mathbb{C} \setminus \overline{\mathbb{D}}$ is naturally homeomorphic to the prime end compactification $\text{cl}^C U_\infty$ of U_∞ . The definition of prime ends easily implies that any homeomorphism $\xi: U_\infty \cup \mathcal{J} \rightarrow U_\infty \cup \mathcal{J}$ induces a

FIGURE 2. The basilica lamination².

homeomorphism of $\text{cl}^C U_\infty$. It follows that h extends continuously to the unit circle \mathbb{T} . This extension respects the basilica lamination of \mathbb{T} since ξ respects the landing pairing between the rays. Equation 3.1 follows by equating the boundary values of $\psi_\infty \circ h$ and $\xi \circ \psi_\infty$.

Conversely, any homeomorphism $h: \mathbb{C} \setminus \mathbb{D} \rightarrow \mathbb{C} \setminus \mathbb{D}$ that preserves the basilica lamination on \mathbb{T} descends to a homeomorphism of the quotient $(\mathbb{C} \setminus \mathbb{D}) / \sim_f$. By Theorem 3.3, the latter is naturally homeomorphic to $U_\infty \cup \mathcal{J}$, providing a desired homeomorphism ξ . \square

See [Th, D, L] for a detailed discussion of geodesic laminations and pinched models.

4. LOCAL PROPERTIES OF QUASISYMMETRIES OF \mathcal{J}

Before we proceed, we briefly recall basic definitions and facts on quasiconformal and quasymmetric maps. For more background one can consult [Vä, AIM, He].

A homeomorphism $f: U \rightarrow \tilde{U}$ between open regions in the plane \mathbb{C} or the Riemann sphere $\hat{\mathbb{C}}$ is called *quasiconformal* if f is in the Sobolev space $W_{\text{loc}}^{1,2}$ and if there exists a constant $K \geq 1$ such that the (formal) Jacobi matrix Df satisfies

$$\|Df(z)\|^2 \leq K \det(Df(z))$$

for almost every $z \in U$. In this case we say f is *K -quasiconformal*; the constant K is called *dilatation* of f . The condition $f \in W_{\text{loc}}^{1,2}$ means

²Source: https://commons.wikimedia.org/wiki/File:Basilica_lamination.png

that the first distributional partial derivatives of f are locally in L^2 . A quasiconformal map is necessarily orientation preserving.

If (X, d_X) and (Y, d_Y) are metric spaces, a homeomorphism $f: X \rightarrow Y$ is called *quasisymmetric* or a *quasisymmetry* if there exists a homeomorphism $\eta: [0, \infty) \rightarrow [0, \infty)$ such that

$$\frac{d_Y(f(u), f(v))}{d_Y(f(u), f(w))} \leq \eta\left(\frac{d_X(u, v)}{d_X(u, w)}\right),$$

for every triple of distinct points $u, v, w \in X$. If we want to emphasize the dependence on the *distortion function* η , we say that f is η -*quasisymmetric*, or η -*quasisymmetry*.

Suppose U and V are subregions of $\widehat{\mathbb{C}}$. Then every orientation-preserving η -quasisymmetric homeomorphism $f: U \rightarrow V$ is K -quasiconformal with K that depends only on η . Conversely, every properly normalized quasiconformal homeomorphism $f: U \rightarrow V$ is *locally quasisymmetric*, i.e., for every compact set $M \subset U$, the restriction $f|_M: M \rightarrow f(M)$ is a quasisymmetry, quantitatively, i.e., η depends only on K and the relative distance between M and ∂U ; see [AIM, p. 58, Theorem 3.4.1 and p. 71, Theorem 3.6.2] and [He, Theorem 11.14]. This is referred to as the *egg yolk principle*.

If we have a family of maps with the same dilatation or distortion function, then we say that the family is *uniform*. E.g., a family of homeomorphisms is *uniformly quasisymmetric* if there exists a homeomorphism $\eta: [0, \infty) \rightarrow [0, \infty)$ such that each map from the family is η -quasisymmetric. Inverses and compositions of quasiconformal or quasisymmetric maps are quantitatively quasiconformal or quasisymmetric, respectively.

According to the Ahlfors–Beurling theorem [BA], each orientation preserving η -quasisymmetric map $h: \mathbb{T} \rightarrow \mathbb{T}$ has a K -quasiconformal extension H to the whole complex plane, where K depends only on η . Conversely, the homeomorphic extension h of each K -quasiconformal homeomorphism $H: \mathbb{D} \rightarrow \mathbb{D}$ or $H: \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ to \mathbb{T} is η -quasisymmetric for some η that depends only on K .

Let ξ be an orientation preserving homeomorphism of \mathbb{C} that leaves \mathcal{J} invariant. Let U, V be bounded Fatou components of f such that $\xi: U \rightarrow V$. Then the homeomorphism $\xi_{U,V} = \phi_V \circ \xi \circ \psi_U: \mathbb{T} \rightarrow \mathbb{T}$, where $\psi_U = \phi_U^{-1}$, preserves the set of dyadic points. Indeed, this follows from the fact that ξ preserves the set of global cut-points of \mathcal{J} . By Lemma 3.4, the global homeomorphism ξ also induces a homeomorphism ξ_∞ of \mathbb{T} that satisfies $\psi_\infty \circ \xi_\infty = \xi \circ \psi_\infty$. For the same reason as above, the induced map ξ_∞ preserves the set $D_\infty = \cup_{n=0}^\infty D_n$, where D_n is the set of all preimages of $\{1/3, 2/3\}$ under g^n , with $g(z) = z^2$.

Let ξ be a topologically extendable η -quasisymmetric map of \mathcal{J} . Let U and V be bounded Fatou components of f such that $\xi(U) = V$. In this case the maps $\xi_{U,V}$, where U runs over all bounded Fatou components, are uniform quasisymmetries, i.e, they are η' -quasisymmetries with η' that depends only on η . This follows from an elementary fact that all the bounded Fatou components are uniform quasidisks, which, in turn, is a consequence of the hyperbolicity of f . In particular, each $\xi_{U,V}$ has a K -quasiconformal extension to \mathbb{C} , where K depends only on η . The following lemma shows that the same holds for the map ξ_∞ .

Lemma 4.1. *Let ξ be a topologically extendable η -quasisymmetric map of \mathcal{J} . Let ξ_∞ be the induced map on the unit circle \mathbb{T} that satisfies the semi-conjugation $\psi_\infty \circ \xi_\infty = \xi \circ \psi_\infty$. Then ξ_∞ is an η' -quasisymmetric homeomorphism of \mathbb{T} with η' that depends only on η . In particular, ξ_∞ has a K -quasiconformal extension to \mathbb{C} , where K depends only on η .*

Proof. It is enough to show that there exists $C > 0$ that depends only on η , such that if I and J are two adjacent non-overlapping arcs on \mathbb{T} that have the same lengths, then

$$\text{diam}(\xi_\infty(I))/C \leq \text{diam}(\xi_\infty(J)) \leq C \text{diam}(\xi_\infty(I)).$$

To prove this, we first show that there exist constants $C_1, C'_1 > 0$, such that if I and J are two adjacent non-overlapping arcs on \mathbb{T} , then

$$\text{diam}(I)/C_1 \leq \text{diam}(J) \leq C_1 \text{diam}(I)$$

if and only if

$$\text{diam}(I')/C'_1 \leq \text{diam}(J') \leq C'_1 \text{diam}(I'),$$

where for an interval I in \mathbb{T} we denote by I' the corresponding Julia arc, i.e., $I' = \psi_\infty(I)$; we use the same convention in the rest of this proof. In this statement, for the necessary part C_1 is given and C'_1 depends only on C_1 , and for the sufficiency part it is the other way around.

Indeed, since ψ_∞ does not collapse arcs of \mathbb{T} to points, it follows that if I is such an arc with $\text{diam}(I) \geq \epsilon > 0$, then there exists $\delta > 0$ that depends only on ϵ , such that $\text{diam}(I') \geq \delta$. Conversely, the uniform continuity of ψ_∞ implies that if $\delta > 0$ is given such that $\text{diam}(I') \geq \delta$, then there exists $\epsilon > 0$ that depends only on δ , with $\text{diam}(I) \geq \epsilon$.

Now, if $\text{diam}(I)/C_1 \leq \text{diam}(J) \leq C_1 \text{diam}(I)$, then there exists a constant $\epsilon_0 > 0$ that depends only on C_1 , such that for some $n \in \mathbb{N}_0$ we have

$$\text{diam}(I_n), \text{diam}(J_n) \geq \epsilon_0,$$

where $I_n = g^n(I)$ and $J_n = g^n(J)$. From the above it follows that this happens if and only if

$$\text{diam}(I'_n), \text{diam}(J'_n) \geq \delta_0.$$

Here, for the “if” part, ϵ_0 depends only on δ_0 , and for the “only if” part δ_0 depends only on ϵ_0 . Thus, there exist constants $C_2, C'_2 > 0$ that depend only on ϵ_0 (or only on δ_0), such that

$$\text{diam}(I_n)/C_2 \leq \text{diam}(J_n) \leq C_2 \text{diam}(I_n)$$

and

$$\text{diam}(I'_n)/C'_2 \leq \text{diam}(J'_n) \leq C'_2 \text{diam}(I'_n).$$

Note that $I'_n = f^n(I')$ and $J'_n = f^n(J')$. We may and will assume that n is chosen not too large, so that I'_n and J'_n are properly contained in an open set where the appropriate branch of f^{-n} is well-defined and conformal. The egg yolk principle now implies that there exists a constant $C'_1 > 0$ that depends only on C_2, C'_2 , and hence only on C_1 , such that

$$\text{diam}(I')/C'_1 \leq \text{diam}(J') \leq C'_1 \text{diam}(I').$$

The converse implication, namely that the last inequalities imply

$$\text{diam}(I)/C_1 \leq \text{diam}(J) \leq C_1 \text{diam}(I),$$

for some $C_1 > 0$ that depends only on C'_1 , follows the same lines with the egg yolk principle applied to f^n rather than to f^{-n} .

We are now ready to finish the proof. If I and J are two adjacent non-overlapping arcs of \mathbb{T} that have the same lengths, then for some absolute constant $C'_1 > 0$ we have

$$\text{diam}(I')/C'_1 \leq \text{diam}(J') \leq C'_1 \text{diam}(I').$$

Since ξ is η -quasisymmetric, there exists a constant $C'_2 > 0$ that depends only on η and C'_1 , such that

$$\text{diam}(\xi(I'))/C'_2 \leq \text{diam}(\xi(J')) \leq C'_2 \text{diam}(\xi(I')).$$

But, for each interval I in \mathbb{T} , it follows from $\psi_\infty \circ \xi_\infty = \xi \circ \psi_\infty$ that $\xi(I') = (\xi_\infty(I))'$, and so

$$\text{diam}((\xi_\infty(I))')/C'_2 \leq \text{diam}((\xi_\infty(J))') \leq C'_2 \text{diam}((\xi_\infty(I))').$$

Now we apply the above claim to conclude that there exists a constant $C > 0$ that depends only on C'_2 with

$$\text{diam}(\xi_\infty(I))/C \leq \text{diam}(\xi_\infty(J)) \leq C \text{diam}(\xi_\infty(I)),$$

and the lemma follows. \square

5. THOMPSON GROUP ACTION ON \mathcal{J}

5.0.1. *Thompson groups.* The *Thompson group* F is a group of orientation preserving piecewise linear homeomorphisms of the closed interval $[0, 1]$ whose break points, i.e., points of non-differentiability, are dyadic points and such that on intervals of linearity the slopes are integer powers of 2. See [CFP] for background on the group F as well as the Thompson group T of the unit circle, defined below. It follows immediately that the elements of F preserve the set of dyadic points. The group F is generated by

$$A(t) = \begin{cases} t/2, & 0 \leq t \leq 1/2, \\ t - 1/4, & 1/2 \leq t \leq 3/4, \\ 2t - 1, & 3/4 \leq t \leq 1, \end{cases}$$

and

$$B(t) = \begin{cases} t, & 0 \leq t \leq 1/2, \\ t/2 + 1/4, & 1/2 \leq t \leq 3/4, \\ t - 1/8, & 3/4 \leq t \leq 7/8, \\ 2t - 1, & 7/8 \leq t \leq 1. \end{cases}$$

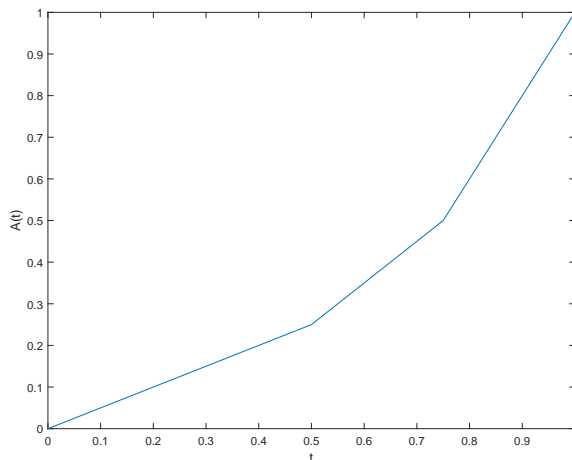


FIGURE 3. The graph of $A(t)$.

Similarly, the *Thompson group* T is the group of orientation preserving piecewise linear (in the angular metric) homeomorphisms of the unit circle \mathbb{T} that preserve the set of dyadic points, whose break points are dyadic points, and the slopes on intervals of linearity are integer powers of 2. Elements of the Thompson group F induce in the

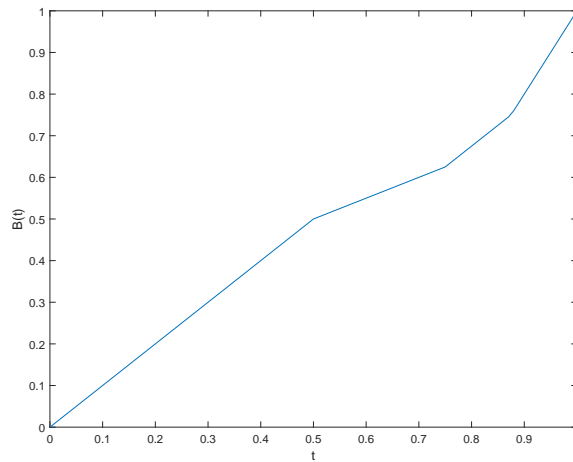


FIGURE 4. The graph of $B(t)$.

obvious way elements of T that fix $1 \in \mathbb{T}$. In addition to the elements induced by A and B defined above, the group T is generated by

$$C(t) = \begin{cases} t/2 + 3/4, & 0 \leq t \leq 1/2, \\ 2t - 1, & 1/2 \leq t \leq 3/4, \\ t - 1/4, & 3/4 \leq t \leq 1. \end{cases}$$

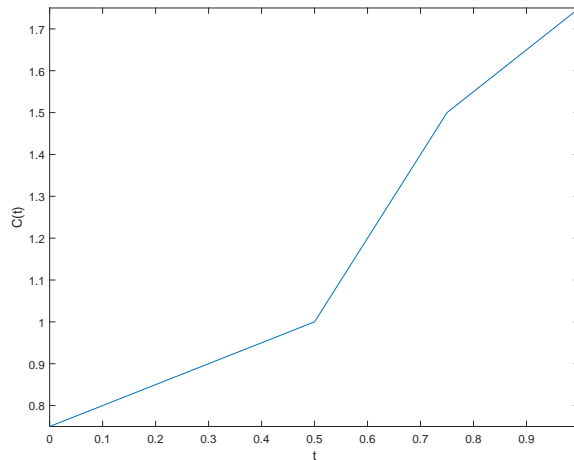


FIGURE 5. The graph of $C(t)$.

5.0.2. *Pseudo-group* Υ_g . For further reference, let us formulate two elementary lemmas:

Lemma 5.1. *Let I_0 and I_1 be two non-empty open arcs on \mathbb{T} so that the endpoints are dyadic. Then there exists an orientation preserving piecewise linear map τ from I_0 onto I_1 whose break points are dyadic points and such that on each interval of linearity, the slope of τ in the angular metric is 2^n for some $n \in \mathbb{Z}$.*

Proof. First, we may assume by applying rotations by dyadic numbers that both I_0 and I_1 have a common endpoint at 1, and they can be identified with intervals in $[0, 1]$ with one endpoint at 0. Then the other endpoints of I_0 and I_1 are dyadic points $d_0, d_1 \in (0, 1]$, respectively. We may assume that $d_0 = k/2^n, d_1 = m/2^n$ and $k < m$. We now apply the identity to the first $k - 1$ intervals of length $1/2^n$ comprising $(0, d_0]$, and scale the last such interval by 2. All the slopes of such a map on $(0, d_0]$ are integer powers of 2 and this map takes $(0, d_0]$ to the interval $(0, d_0 + 1/2^n]$. Inductive process now finishes the proof. \square

Recall that g stands for the doubling map of the circle. Given a path $\gamma: [0, 1] \rightarrow \mathbb{T}$, a moment $m \in \mathbb{N}_0$, and a preimage $z_0 \in g^{-m}(\gamma(0))$, we can uniquely lift the path γ by g^m to a path $\delta: [0, 1] \rightarrow \mathbb{T}$ with $\delta(0) = z_0$. Informally, we will refer to the corresponding analytic continuation $\delta(t) = g^{-m}(\gamma(t))$ of g^{-m} along γ as the *branch* of g^{-m} on γ that *starts* at z_0 .

In particular, we can consider an arc $I \subset \mathbb{T}$, a curve $g^n: I \rightarrow \mathbb{T}$, and a branch of some g^{-m} on this curve. We obtain a composition $\xi = g^{-m} \circ g^n$ on I . If the image $\tilde{I} := \xi(I)$ is also an arc on \mathbb{T} (i.e., $\xi: I \rightarrow \mathbb{T}$ does not “wrap around the circle”), then $\xi: I \rightarrow \tilde{I}$ is a linear (in the angular coordinate) diffeomorphism with slope 2^{n-m} . Let us denote the pseudo-group of such diffeomorphisms by Υ_g .

Lemma 5.2. *If I and \tilde{I} are arcs of \mathbb{T} whose endpoints are dyadic, and if ξ is a linear map of I onto \tilde{I} such that the slope of ξ in the angular coordinate is an integer power of 2, then ξ belongs to the pseudo-group Υ_g .*

Proof. Let $I = [a, b]$ and $\tilde{I} = [\tilde{a}, \tilde{b}]$. Since a is assumed to be dyadic, we can find $n \in \mathbb{N}_0$ such that $g^n(a) = 1$. Let us consider the branch of g^{-n} on the path $g^n: I \rightarrow \mathbb{T}$ that starts at 1. The composition $r = g^{-n} \circ g^n: I \rightarrow \mathbb{T}$ is a linear (in the angular coordinate) map with slope 1. Hence it rotates I to a dyadic interval $J = [1, e(\theta)]$, $\theta \in [0, 1]$.

Likewise, we can find a moment $\tilde{n} \in \mathbb{N}_0$ and a branch of $g^{-\tilde{n}}$ such that the composition $\tilde{r} = g^{-\tilde{n}} \circ g^{\tilde{n}}: \tilde{I} \rightarrow \mathbb{T}$ rotates \tilde{I} to a dyadic interval $\tilde{J} = [1, e(\tilde{\theta})]$, $\tilde{\theta} \in [0, 1]$. Moreover, $\tilde{\theta} = 2^k \theta$, where 2^k is the slope of ξ on I , $k \in \mathbb{Z}$. Hence $\tilde{J} = g^k(J)$, where, in the case when k is negative, the branch of g^k is chosen so that the point 1 is fixed.

The desired map can now be written as $\tilde{r}^{-1} \circ g^k \circ r$. □

Let us say that a piecewise linear homeomorphism $\mathbb{T} \rightarrow \mathbb{T}$ is *piecewise dynamical* if its restriction to every interval of linearity belongs to the pseudo-group Υ_g .

Corollary 5.3. *Any element of the Thompson group T is piecewise dynamical.*

For instance, the A -generator of the Thompson group can be obtained as follows:

$$A = \begin{cases} g^{-1}, & 0 \leq t \leq 1/2, \quad \text{where the branch } g^{-1} \text{ fixes } 1, \\ g^{-2} \circ g^2, & 1/2 \leq t \leq 3/4, \quad \text{where } g^{-2}: (0, 1) \rightarrow (1/4, 1/2), \\ g, & 3/4 \leq t \leq 1. \end{cases}$$

5.0.3. *Piecewise dynamical action of the Thompson group on \mathcal{J} .* Similarly to Υ_g , let us consider the *dynamical pseudo-group* Υ_f comprising local isomorphisms $\psi = f^{-m} \circ f^n: V \rightarrow W$, where $m, n \in \mathbb{N}_0$, V is simply connected, $f^n|_V$ is univalent, and $f^{-m} \circ f^n|_V$ is a well defined analytic branch of the algebraic function $f^{-m} \circ f^n$. Note that in the log-Böttcher coordinate $u = \log \phi_\infty(z)$, the map ψ is affine with slope 2^{n-m} , $u \mapsto 2^{n-m}u + c$, $c \in \mathbb{R}$.

Let us say that a homeomorphism $\xi: J \rightarrow \tilde{J}$ between Julia arcs is *piecewise dynamical* if J can be decomposed into finitely many Julia arcs J_i that share only global cut-points, so that each restriction $\xi|_{J_i}$, extended to some neighborhood W_i , belongs to the pseudo-group Υ_f . If $J = \tilde{J} = \mathcal{J}$, piecewise dynamical homeomorphisms $\xi: \mathcal{J} \rightarrow \mathcal{J}$ form a group, and we denote it by $\text{Dyn}(\mathcal{J})$.

Let us also say that a map $h: U \rightarrow V$ between two bounded Fatou components *respects the Böttcher coordinate* if it is the identity in the Böttcher coordinates of these components, i.e.,

$$\phi_V \circ h \circ \psi_U = \text{id}: \mathbb{D} \rightarrow \mathbb{D}.$$

Lemma 5.4. *The Thompson group T induces the group of piecewise dynamical quasimorphisms of \mathcal{J} that keep the central Fatou component ∂U_0 invariant. Moreover, they admit global quasiconformal extensions to \mathbb{C} that respect the Böttcher coordinates of all non-central bounded Fatou components.*

Proof. Let ξ_0 be an element of T acting on ∂U_0 piecewise linearly in the (inner) angular coordinate. We want to extend it to an orientation preserving quasimetric homeomorphism of \mathcal{J} (actually, to a global quasiconformal map of $(\mathbb{C}, \mathcal{J})$).

Let $I \subseteq \partial U_0$ be an arc of linearity for ξ_0 . By Corollary 5.3, $\xi_0|_I = f^{-m} \circ f^n$ for some $n, m \in \mathbb{N}_0$. Hence ξ_0 admits an analytic extension $\hat{\xi}_0$ to a neighborhood V of I as an element of the dynamical pseudo-group Υ_f .

Let us puncture out from I all dyadic points of level $\leq n$. Take any complementary (open) interval $J \subseteq I$. Let Ω_J be the region bounded by J and two external rays landing at ∂J (where the rays are selected so that Ω_J does not intersect the wakes attached to ∂J). Since $f^n(\Omega_J)$ is disjoint from $U_0 \cup U_{-1}$, all branches of f^{-m} are well defined on it, so $\hat{\xi}_0$ extends analytically to Ω_J (mapping it to some $\Omega_{\tilde{J}}$). We restrict this map to the intersection of Ω_J with the filled Julia set \mathcal{K} and extend it continuously to the closure \mathcal{K}_J of this set.

In this way, we extend ξ_0 to all $\mathcal{K} \setminus U_0$, except finitely many limbs attached to ∂U_0 . Let S_z be one of the wakes containing such a limb. Its root $z \in \partial U_0$ is a dyadic point of some level k . Let $\tilde{z} = \xi_0(z)$. Since ξ_0 is Thompson, \tilde{z} is also dyadic (of some level \tilde{k}), so there is a wake $S_{\tilde{z}}$ rooted at \tilde{z} .

Recall that S_0 is the sector bounded by $\rho^{\pm \frac{1}{3}}$ that contains U_0 . Then f^k and $f^{\tilde{k}}$ conformally map S_z and $S_{\tilde{z}}$, respectively, onto S_0 . (If k or \tilde{k} is 0, then f takes S_z , respectively $S_{\tilde{z}}$, to S_0 .)

Thus, we obtain a map $\xi_z = f^{-\tilde{k}} \circ f^k: S \rightarrow \tilde{S}$ of pseudo-group Υ_f . Restricting it to the filled Julia set, we obtain a homeomorphism $\xi_z: \mathcal{K}_z \rightarrow \mathcal{K}_{\tilde{z}}$ between the corresponding limbs. Putting these maps together, we obtain an extension of ξ_0 to a homeomorphism $\xi: \mathcal{K} \setminus U_0 \rightarrow \mathcal{K} \setminus U_0$ that restricts to an element in $\text{Dyn}(\mathcal{J})$.

Let us show that the map ξ admits a global quasiconformal extension to the complex plane. By definition, it has a conformal extension to each non-central bounded Fatou component as a conformal map $f^{-m} \circ f^n$. Since ∂U_0 is a quasicircle and $\xi_0: \partial U_0 \rightarrow \partial U_0$ is piecewise linear in the inner angular coordinate, it is quasisymmetric. Hence it admits a quasiconformal extension to U_0 (Ahlfors-Beurling extension [BA]).

We now extend ξ to U_∞ . Let J_i be the arcs of ∂U_0 considered above, and let z_i be their boundary dyadic points. Let $\mathcal{Y}_i = \partial \mathcal{K}_{J_i}$ and $\mathcal{Z}_i = \partial \mathcal{K}_{z_i}$ be the corresponding Julia arcs. In the Böttcher coordinate, these Julia arcs correspond to some arcs Y_i and Z_i tessellating the circle \mathbb{T} .

Since ξ belongs to the dynamical pseudo-group Υ_f on each \mathcal{Y}_i and \mathcal{Z}_i , it induces linear (in the outer angular coordinate of the circle \mathbb{T}) maps $Y_i \rightarrow \tilde{Y}_i$ and $Z_i \rightarrow \tilde{Z}_i$. Moreover, since ξ_0 preserves the cyclic order in which the arcs J_i and points z_i appear on ∂U_0 , these maps form a single piecewise linear homeomorphism $h: \mathbb{T} \rightarrow \mathbb{T}$. Applying

the Ahlfors-Beurling extension once again, we obtain a quasiconformal extension \hat{h} of h to U_∞ .

Note that h preserves the basilica lamination on \mathbb{T} . It is so on

$$\mathbb{T} \setminus \bigcup \partial Z_i \quad (= \mathbb{T} \setminus \bigcup \partial Y_i)$$

since ξ admits local homeomorphic extensions to \mathbb{C} near any point of $\mathcal{J} \setminus \{z_i\}$. It is also true at the points of $\bigcup \partial Z_i$. Indeed, the lamination pairs the boundary points of each Z_i (corresponding to the rays enclosing the limb \mathcal{Z}_i). Since ξ maps the limb \mathcal{Z}_i to another limb $\tilde{\mathcal{Z}}_i$, the boundary points of $\tilde{Z}_i = h(Z_i)$ correspond to the rays enclosing $\tilde{\mathcal{Z}}_i$. So, they are paired by the lamination. Hence \hat{h} descends to a homeomorphism of $(\mathbb{C}, \mathcal{K})$ providing us with the desired quasiconformal extension of ξ to \mathbb{C} . Here we also need the fact that the Julia set \mathcal{J} is removable for quasiconformal maps [Jo, Corollary 2 on p. 5 and Remark on p. 3]. We continue to denote this extension by ξ .

It remains to show that ξ respects the Böttcher coordinates of all non-central bounded Fatou components. In each such component U it has the form $f^{-m} \circ f^n$ and maps it to another such component \tilde{U} . The claim now follows from the fact that the Böttcher coordinate of a bounded Fatou component with dynamical distance k to U_0 is $\phi_0 \circ f^k$. Indeed, since in the construction above n is chosen so that $f^n(U)$ does not contain the postcritical points, we have that n is at most the dynamical distance from U to U_0 . If $l \in \mathbb{N}_0$ is such that $n+l$ is the dynamical distance from U to U_0 , then for the branch of f^{-l} that takes U_0 to $f^n(U)$, the map ξ in U is given by $f^{-m} \circ f^{-l} \circ f^{n+l}$. We conclude that $m+l$ is the dynamical distance from \tilde{U} to U_0 and for an appropriate branch of $f^{-(m+l)}$ the map ξ in U is given by

$$f^{-(m+l)} \circ f^{n+l}.$$

This is equivalent to saying that ξ respects the Böttcher coordinates. □

5.0.4. *Thompson-like action.* The Thompson group T is based on dyadic points, iterated preimages of the fixed point 1 under the doubling map g . More generally, one can consider iterated preimages of any periodic cycle α and define the associated *Thompson-like* group T_α . I.e., the elements of T_α preserve the set D_α of all preimages of the elements of α under the iterates of the map g , the break points are at D_α , and the slopes are integer powers of 2. The above discussion readily extends to this setting, and in particular, we have the following counterpart of Corollary 5.3.

Corollary 5.5. *Any element of the Thompson-like group T_α is piecewise dynamical.*

In what follows, we let $\alpha = \{1/3, 2/3\}$. Let T_α^b be the subgroup of the Thompson-like group T_α consisting of homeomorphisms $\mathbb{T} \rightarrow \mathbb{T}$ that preserve the basilica lamination.

Lemma 5.6. *Any piecewise dynamical homeomorphism $\xi: \mathcal{J} \rightarrow \mathcal{J}$ induces a homeomorphism $\xi_\infty: \mathbb{T} \rightarrow \mathbb{T}$ that belongs to T_α^b . Moreover, the map $\text{Dyn}(\mathcal{J}) \rightarrow T_\alpha^b$, $\xi \mapsto \xi_\infty$, is a group isomorphism.*

Proof. It follows from the definition that any piecewise dynamical homeomorphism ξ of \mathcal{J} has a homeomorphic extension to $\overline{U_\infty}$, and therefore, by Lemma 3.4, the induced homeomorphism ξ_∞ of \mathbb{T} preserves the basilica lamination. The rest of the properties of ξ_∞ follow from the assumption that ξ is piecewise dynamical.

Conversely, any such map ξ_∞ extends to a homeomorphism of $\mathbb{C} \setminus \mathbb{D}$ and hence descends to a homeomorphism ξ of \mathcal{J} by Lemma 3.4. The homeomorphism ξ has to be piecewise dynamical because such is ξ_∞ according to Corollary 5.5.

The statement about the map $\xi \mapsto \xi_\infty$ being an isomorphism follows immediately from the relation

$$\psi_\infty \circ \xi_\infty = \xi \circ \psi_\infty.$$

□

One element of T that will be useful in what follows is the rotation $\rho(z) = -z$. It is given by $f^{-2} \circ f^2$, where the branch of f^{-2} is chosen so that $f^{-2}(\alpha) = -\alpha$.

6. INVERSION OF \mathcal{J}

In this section we define an inversion ι of \mathcal{J} that along with the action of the Thompson group T on \mathcal{J} generates a group whose elements approximate every quasisymmetric self-map of \mathcal{J} quantitatively. We recall that $\mathcal{J}_\alpha = \partial\mathcal{K}_\alpha$, where \mathcal{K}_α is the limb rooted at α that contains the boundary ∂U_{-1} of the bounded Fatou component U_{-1} containing -1 . Also, $\mathcal{J}_0 = (\mathcal{J} \setminus \mathcal{J}_\alpha) \cup \{\alpha\}$.

Lemma 6.1. *There exists a quasisymmetric homeomorphism ι of \mathcal{J} that fixes α , interchanges \mathcal{J}_α and \mathcal{J}_0 , and satisfies*

$$\iota^2 = \text{id}.$$

Moreover, ι has a global quasiconformal extension to \mathbb{C} and respects the Böttcher coordinates of all bounded Fatou components.

Proof. We define ι by $\iota = f$ on \mathcal{J}_α , and $\iota = f^{-1}$ on \mathcal{J}_0 . Here we choose the branch of f^{-1} that fixes α . We prove that so defined ι is quasiasymmetric by showing that it extends to a quasiconformal map of \mathbb{C} .

In each bounded component of the Fatou set whose closure intersects \mathcal{J}_α we extend ι by $\iota = f$, and in each such component whose closure intersects \mathcal{J}_0 we extend it by $\iota = f^{-1}$. In particular, it is immediate that if U is any bounded Fatou component and $\partial\tilde{U} = \iota(\partial U)$, then this extension of ι respects the Böttcher coordinates of U and \tilde{U} .

The map ι has obvious conformal extensions to both wakes in \mathbb{C} bounded by the two external rays in U_∞ landing at α . These extensions are conjugate by ϕ_∞ to the maps $g(z) = z^2$ for $1/3 \leq e(\theta) \leq 2/3$ and $z \mapsto -\sqrt{z}$ for $-1/3 \leq e(\theta) \leq 1/3$, respectively, where the principal branch of the square root is selected. We can use the Ahlfors–Beurling extension [BA] to extend this piecewise linear map to a quasiconformal map of $\mathbb{C} \setminus \overline{\mathbb{D}}$ onto itself. This quasiconformal extension is conjugated back by ϕ_∞ to a quasiconformal map of U_∞ onto itself that agrees with ι on the boundary of U_∞ .

Combining the above quasiconformal extension of ι with the conformal extensions into bounded Fatou components, we obtain a homeomorphism of \mathbb{C} that is quasiconformal on the Fatou set. But the Julia set of a postcritically finite polynomial is removable for quasiconformal maps [Jo], and so we get a quasiconformal map of \mathbb{C} . Such a map is quasiasymmetric, and hence ι is quasiasymmetric on \mathcal{J} . \square

We call the homeomorphism ι the *inversion* with respect to α . The composition $\sigma = \rho \circ \iota$ with the rotation ρ , given by $\rho(z) = -z$, acts as a *shift* to the right by one on the infinite chain \mathcal{C} of bounded Fatou components that intersect the real line. This follows from the fact that σ preserves \mathcal{C} and takes ∂U_{-1} to ∂U_0 . Moreover, σ respects the Böttcher coordinates of all bounded Fatou components, except that $\sigma: U_{-1} \rightarrow U_0$ is 180°-rotation in the Böttcher coordinates. (In general, we will say that a map $h: U \rightarrow V$ between bounded Fatou components *rotates* the Böttcher coordinate if $\phi_U \circ h \circ \psi_V$ is a rotation of the disk \mathbb{D} .)

These remarks allow us to derive the following property:

Lemma 6.2. *The group \hat{T} generated by the Thompson group T and ι acts transitively on the vertices of \mathfrak{T} . Moreover, let U be an arbitrary bounded Fatou component, and let p be the shortest combinatorial chain in \mathfrak{T} that joins U_0 to U . Then there exists $t \in \hat{T}$ with the following properties:*

- (i) $t(U) = U_0$ and t respects the Böttcher coordinate;
- (ii) for any bounded Fatou component $V \notin p$, the map $t: V \rightarrow t(V)$ respects the Böttcher coordinate;
- (iii) for any bounded Fatou component $V \in p$, the map $t: V \rightarrow t(V)$ rotates the Böttcher coordinate by some dyadic angle.

Proof. Suppose that $p = \{U_0, U_1, \dots, U_k = U\}$. We will argue by induction on k . The base $k = 0$ is obvious. In general, the component U_1 touches U_0 , and so the base point of U_1 corresponds to a dyadic point on ∂U_0 . The group T contains rotations by any dyadic angle, and hence acts transitively on the dyadic points of ∂U_0 . Thus there exists an element $t_1 \in T$ such that $t_1(U_1) = U_{-1}$. The elements of T preserve the combinatorial distance on \mathfrak{T} , and therefore $t_1(U)$ has the same combinatorial distance k to U_0 as U . Now we apply the shift σ to take U_{-1} to U_0 . The combinatorial distance from $\sigma \circ t_1(U)$ to U_0 is $k - 1$, and we can apply the inductive hypothesis. Let $t_2 \in \hat{T}$ be an element given by the inductive hypothesis that takes $\sigma \circ t_1(U)$ to U_0 .

The map $t' = t_2 \circ \sigma \circ t_1 \in \hat{T}$ takes U to U_0 . Moreover, $\sigma \circ t_1$ respects the Böttcher coordinates of all non-central bounded Fatou components except U_1 , on which it rotates the coordinate by 180° . On U_0 itself, it rotates the Böttcher coordinate by some dyadic angle. By the inductive assumption, $t' \in \hat{T}$ respects the Böttcher coordinate for all $V \notin p$ and rotates it for all $V \in p$ by some dyadic angles. Finally, by postcomposing t' with a rotation γ in T , we can ensure that $t = \gamma \circ t': U \rightarrow U_0$ respects the Böttcher coordinate as well. \square

7. PROOF OF THEOREM 1.1

Let $n \in \mathbb{N}$ be arbitrary and let $\Pi_n = \{J_k, k = 1, 2, \dots, 2^{n+1}\}$ be the partition of \mathcal{J} by Julia arcs described in Section 3. We replace the map ξ restricted to each J_k by a piecewise dynamical map as follows.

Lemma 7.1. *Let ξ be a topologically extendable η -quasisymmetric map of \mathcal{J} . Then there exist $N \in \mathbb{N}_0$ and a finite family \mathcal{F} of Julia arcs that depend only on η and have the following property. For all $n \geq N$, for any $J \in \Pi_n$, if $\tilde{J} = \xi(J)$, then there exists $M \in \mathbb{N}_0$ with $\tilde{\Lambda} = f^M(\tilde{J}) \in \mathcal{F}$. Moreover, $\tilde{\Lambda}$ is the closure of a connected component of $\mathcal{J} \setminus \{\tilde{z}_1, \tilde{z}_2\}$ for some global cut-points $\tilde{z}_1, \tilde{z}_2 \in \partial U_0$, and f^{-M} is conformal in a neighborhood of $\tilde{\Lambda}$.*

Proof. We first note that ξ has a K -quasiconformal extension to \mathbb{C} , where K depends only on η . Indeed, it follows from Lemma 4.1 that

the homeomorphism ξ_∞ of \mathbb{T} defined by

$$\psi_\infty \circ \xi_\infty = \xi \circ \psi_\infty$$

has a K -quasiconformal extension to $\mathbb{C} \setminus \overline{\mathbb{D}}$, where K depends only on η . Since ϕ_∞ is conformal in U_∞ , it implies that ξ has an extension to U_∞ as a K -quasiconformal homeomorphism. Likewise, the discussion preceding Lemma 4.1 implies that ξ has K -quasiconformal extension to each bounded Fatou component, perhaps with a different K but that depends only on η . By [Jo], the Julia set \mathcal{J} is removable for quasiconformal maps, and therefore the claim follows.

Assume that the Julia arc J in the statement of the lemma is determined by adjacent external rays $\rho_1, \rho_2 \in R_n$ with landing points z_1, z_2 , respectively. There are two cases to consider: $z_1 = z_2$ and $z_1 \neq z_2$.

If $z = z_1 = z_2$, then z is the root of a bounded Fatou component U such that $\partial U \in J$, and $J = J_z = \partial \mathcal{K}_z$, where \mathcal{K}_z is the limb rooted at z . Since any homeomorphism of \mathcal{J} preserves the set of global cut-points, $\xi(z)$ is such a point. Moreover, from [He, Proposition 10.8] applied to the quasisymmetry ξ it follows that there exists $N \in \mathbb{N}_0$ that depends only on K such that if $n \geq N$, then $J \in \Pi_n$ implies that the Julia arc \tilde{J} does not contain ∂U_0 . Roughly speaking, [He, Proposition 10.8] states that if A and B are overlapping sets in a metric space with the diameter of A being smaller than the diameter of B , then an application of a quasisymmetric map with controlled distortion cannot spoil this relationship between the diameters of the corresponding images of A and B too much, quantitatively. In particular, it follows from the proof of Lemma 2.1 that $\xi(z)$ is the root of the bounded Fatou component \tilde{U} such that $\partial \tilde{U} = \xi(\partial U)$. Let $M \in \mathbb{N}_0$ be the dynamical distance from \tilde{U} to U_0 . Then $\tilde{\Lambda} = f^M(\tilde{J}) = \mathcal{J}_0$, and the claim follows in this case, with \mathcal{F} consisting of the single element \mathcal{J}_0 . We have $\tilde{z}_1 = \tilde{z}_2 = \alpha$. The fact that f^{-M} is conformal in a neighborhood of \mathcal{J}_0 follows from the assumption that M is the dynamical distance from \tilde{U} to U_0 .

Now assume that $z_1 \neq z_2$. We know from Lemma 3.1 that there exists a bounded Fatou component U such that $z_1, z_2 \in \partial U$. As above, there exists $N \in \mathbb{N}_0$ that depends only on K such that if $n \geq N$, then for $J \in \Pi_n$ we have that \tilde{J} does not contain ∂U_0 . Let $\mathcal{J}_{z_1} = \partial \mathcal{K}_{z_1}$ and $\mathcal{J}_{z_2} = \partial \mathcal{K}_{z_2}$, where $\mathcal{K}_{z_1}, \mathcal{K}_{z_2}$ are the two limbs that are attached to J at z_1 and z_2 , respectively.

The egg yolk principle applied to an appropriate branch of f^{-n} implies that $\text{diam} \mathcal{J}_{z_i} \geq c \text{diam} J$, $i = 1, 2$, for some $c > 0$ that does not depend on n or J . Let $\tilde{\mathcal{J}}_i = \xi(\mathcal{J}_{z_i})$, $i = 1, 2$. Since ξ is quasisymmetric, an application of [He, Proposition 10.8] gives that $\text{diam} \tilde{\mathcal{J}}_i \geq$

$\tilde{c} \operatorname{diam} \tilde{\mathcal{J}}$, $i = 1, 2$, where the constant \tilde{c} depends only on K . Since f is hyperbolic, there exist constants $C = C(f) > 0$ and $M \in \mathbb{N}_0$ with the following properties. First, for $\tilde{\Lambda} = f^M(\tilde{\mathcal{J}})$ we have $\operatorname{diam}(\tilde{\Lambda}) \geq C$. Then, there exist a point $p \in \tilde{\Lambda}$ and $r > 0$, such that $\tilde{\Lambda}$ is contained in the disc $B(p, r/4)$. Finally, the disc $B(p, r)$ does not contain 0 and 1, i.e., the postcritical points of f . In particular, $r \geq 2C$. Since there are only finitely many large limbs of \mathcal{J} , the claim about the existence of a finite family \mathcal{F} would follow if we show that there is a lower bound depending only on K for $\operatorname{diam} f^M(\tilde{\mathcal{J}}_i)$, $i = 1, 2$. If both $f^M(\tilde{\mathcal{J}}_1)$ and $f^M(\tilde{\mathcal{J}}_2)$ are not contained in $B(p, r/2)$, then

$$\operatorname{diam} f^M(\tilde{\mathcal{J}}_i) \geq r/4 \geq C/2, \quad i = 1, 2.$$

If $f^M(\tilde{\mathcal{J}}_i)$ is contained in $B(p, r/2)$ for some $i = 1, 2$, then the claim follows from the egg yolk principle applied to f^{-M} in the disc $B(p, r)$, and the fact that $\operatorname{diam} \tilde{\mathcal{J}}_i \geq \tilde{c} \operatorname{diam} \tilde{\mathcal{J}}$, $i = 1, 2$.

The claim that, in the case $z_1 \neq z_2$, the map f^{-M} is conformal in a neighborhood of $\tilde{\Lambda}$ follows from the assumption that $\tilde{\Lambda}$ is contained in $B(p, r/4)$ and $B(p, r)$ does not contain either 0 or 1.

Finally, $\tilde{\Lambda}$ is the closure of a connected component of $\mathcal{J} \setminus \{\tilde{z}_1, \tilde{z}_2\}$, where $\tilde{z}_1 = f^M(\xi(z_1))$, $\tilde{z}_2 = f^M(\xi(z_2))$, and these points belong to the boundary of a bounded Fatou component V . The component V does not have to be U_0 . However, since $\operatorname{diam}(\tilde{\Lambda}) \geq C = C(f)$, the dynamical distance $m \in \mathbb{N}_0$ from V to U_0 depends only on C . The set $f^m(\tilde{\Lambda})$ is a connected component of $\mathcal{J} \setminus \{f^m(\tilde{z}_1), f^m(\tilde{z}_2)\}$, and $f^m(\tilde{z}_1), f^m(\tilde{z}_2)$ belong to ∂U_0 . Moreover, if we replace each element $\tilde{\Lambda} \in \mathcal{F}$ by $f^m(\tilde{\Lambda})$, all the other properties will be unchanged since m depends only on C , and hence only on f . This completes the proof. \square

Lemma 7.2. *Let $\Lambda = \mathcal{J}_0$, and let $\tilde{\Lambda}$ be a Julia arc from the finite family \mathcal{F} guaranteed by Lemma 7.1. Then there exists a piecewise dynamical homeomorphism \mathcal{T} of $\Lambda \setminus \{\alpha\}$ onto $\tilde{\Lambda} \setminus \{z_1, z_2\}$.*

Proof. If, in the notation of Lemma 7.1, $\tilde{z}_1 = \tilde{z}_2$, the present lemma is trivial. Otherwise, the intersection of $\tilde{\Lambda}$ with ∂U_0 is an arc with dyadic endpoints. In the Böttcher coordinate of U_0 the intersection $\Lambda \cap (\partial U_0 \setminus \{\alpha\})$ corresponds to the open arc $I_0 = \mathbb{T} \setminus \{1\}$, and the intersection $\tilde{\Lambda} \cap (\partial U_0 \setminus \{\tilde{z}_1, \tilde{z}_2\})$ corresponds to an open arc I_1 on \mathbb{T} . Thus we can apply Lemma 5.1 to get a piecewise linear map τ from I_0 onto I_1 with only dyadic break points and with slopes that are integer powers of 2. The map τ corresponds to a piecewise dynamical map \mathcal{T} from Λ onto $\tilde{\Lambda}$ as described in Section 5, specifically in Lemma 5.4. \square

Let $\Lambda = \mathcal{J}_0$. For n large as in Lemma 7.1, we consider a tiling of \mathcal{J} into the Julia arcs J_i , $i = 1, 2, \dots, 2^{n+1}$, of Π_n . For each $i = 1, 2, \dots, 2^{n+1}$, there exists $n_i \in \{n, n+1\}$ such that f^{n_i} maps J_i onto Λ . Let $\tilde{J}_i = \xi(J_i)$, and $\tilde{\Lambda}_i = f^{M_i}(\tilde{J}_i)$, where M_i is the constant from Lemma 7.1 corresponding to \tilde{J}_i . Now, let us replace the map ξ on $J_i \setminus \{z_1^i, z_2^i\}$, where z_1^i, z_2^i are the endpoints of J_i , by the map $\tau_i = f^{-M_i} \circ \mathcal{T}_i \circ f^{n_i}$, where \mathcal{T}_i is the piecewise dynamical map from $\Lambda \setminus \{\alpha\}$ onto $\text{int } \tilde{\Lambda}_i$ that comes from Lemma 7.2. Pasting these 2^{n+1} maps together, we obtain a global piecewise dynamical map $\tau \in \text{Dyn}(\mathcal{J})$.

We next prove that τ is quasismetric on \mathcal{J} with a controlled distortion function η by showing that it has a quasiconformal extension to the whole plane with a controlled dilatation K . We do this by showing that τ can be quasiconformally extended into each bounded Fatou component and into the basin at infinity, so that the dilatation is controlled. Let U be an arbitrary bounded Fatou component of f . If the dynamical distance m from U to U_0 is greater than n , the map τ has a conformal extension into U . Indeed, it is defined in ∂U as a single element of the pseudo-group Υ_f . Assume now that m is at most n . Then the boundary of U is partitioned by its intersection with the elements from Π_n into the dyadic intervals of level $n - m$. Let t be the piecewise dynamical map of ∂U that is the restriction of τ to ∂U . It agrees with ξ at the endpoints of the dyadic intervals above. Now we need two more lemmas concerning circle maps.

For a homeomorphism ξ of \mathbb{T} and a finite set $E \subset \mathbb{T}$ we denote by ξ_E the linear interpolation of the restriction $\xi|_E$ of ξ to E .

Lemma 7.3. *For any distortion homeomorphism η , there exists a distortion homeomorphism η' with the following property. If ξ is an η -quasisymmetric map of the unit circle \mathbb{T} and E is a finite subset of \mathbb{T} such that all the complementary intervals of E in \mathbb{T} have the same angular length, then ξ_E is η' -quasisymmetric.*

Proof. Let d denote the angular distance on \mathbb{T} . Since ξ is η -quasisymmetric, [He, Proposition 10.8] gives that there exists a constant $L \geq 1$ that depends only on η such that for any three distinct points $o, p, q \in E$ with $d(o, p) = d(o, q)$, we have

$$(7.1) \quad d(\xi(o), \xi(p)) \leq Ld(\xi(o), \xi(q)).$$

Let a denote the angular length of each complementary interval of E . To verify that ξ_E is quasismetric with a distortion function that depends only on η , it is enough to check that there exists a constant $C \geq 1$ that depends only on η and has the following property. If o, p , and q are arbitrary points on \mathbb{T} with $d(o, p) = d(o, q) = \delta$, for

some $\delta > 0$, then $d(\xi_E(o), \xi_E(p)) \leq Cd(\xi_E(o), \xi_E(q))$. We consider the following cases.

Case 1: $\delta \leq a$ and o, p, q are contained in the same complementary interval of E . This case is trivial and $C = 1$ because ξ_E is linear on each such interval.

Case 2: $\delta \leq a$ and o, p, q are not contained in the same complementary interval. We assume that o and p are contained in the closure of the same complementary interval I of E and q is in the adjacent interval I' . Let us assume that ξ_E scales I by s and it scales I' by s' . We know from (7.1) that $s/L \leq s' \leq Ls$. Let $c \in E$ be the common point of the intervals I and I' . Then

$$d(\xi_E(o), \xi_E(q)) = d(\xi_E(o), \xi_E(c)) + d(\xi_E(c), \xi_E(q)) = sd(o, c) + s'd(c, q).$$

For the last expression we have

$$sd(o, q)/L \leq sd(o, c) + s'd(c, q) \leq Lsd(o, q).$$

Combining this with the assumptions that $d(o, p) = d(o, q)$ and that ξ_E scales I by s , we conclude that

$$d(\xi_E(o), \xi_E(p))/L \leq d(\xi_E(o), \xi_E(q)) \leq Ld(\xi_E(o), \xi_E(p)),$$

and so $C = L$ in this case.

Case 3: $a < \delta \leq 3a$. This case reduces to a repeated application of the proof of Case 2 at most 3 times. We conclude that $C = L + L^2 + L^3$ works in this case.

Case 4: $\delta > 3a$. Let I_o, I_p , and I_q be the closures of the complementary intervals of E that contain o, p , and q , respectively. Since ξ_E agrees with ξ at the endpoints of each such interval, there exist o', p' , and q' in I_o, I_p , and I_q , respectively, such that $\xi_E(o) = \xi(o'), \xi_E(p) = \xi(p')$, and $\xi_E(q) = \xi(q')$. Now,

$$\begin{aligned} \frac{d(\xi_E(o), \xi_E(p))}{d(\xi_E(o), \xi_E(q))} &= \frac{d(\xi(o'), \xi(p'))}{d(\xi(o'), \xi(q'))} \leq \eta \left(\frac{d(o', p')}{d(o', q')} \right) \\ &\leq \eta \left(\frac{\delta + 2a}{\delta - 2a} \right) \leq \eta(5), \end{aligned}$$

and the claim follows in this case with $C = \eta(5)$. \square

Lemma 7.4. *Let \mathcal{L} be a finite family of orientation preserving piecewise linear homeomorphisms between intervals of \mathbb{T} . Let ξ be an orientation preserving homeomorphism of \mathbb{T} and let E be a finite subset of \mathbb{T} such that all complementary intervals of E in \mathbb{T} have the same length. Suppose that t is an orientation preserving piecewise linear homeomorphism of \mathbb{T} that agrees with ξ on the set E . Moreover, assume that for each complementary interval I of E there are $M, n \in \mathbb{N}_0$*

with $g^M \circ t \circ g^{-n}$ being an element of \mathcal{L} defined on $g^n(I)$. Then there exists a constant $L \geq 1$ that depends only on \mathcal{L} such that the map $\xi_E^{-1} \circ t$ is L -bi-Lipschitz.

Proof. The assumption implies, in particular, that there is a finite family \mathcal{L}_0 of orientation preserving linear maps between intervals of \mathbb{T} , that depends only on \mathcal{L} , such that for each complementary interval I of E in \mathbb{T} the map $g^M \circ \xi_E \circ g^{-n}$ is an element of \mathcal{L}_0 . Therefore the map $g^n \circ \xi_E^{-1} \circ t \circ g^{-n}$ is a homeomorphism of $g^n(I)$ that belongs to a finite family, depending only on \mathcal{L} , of orientation preserving piecewise linear homeomorphisms. Hence there exists $L \geq 1$ that depends only on \mathcal{L} such that $g^n \circ \xi_E^{-1} \circ t \circ g^{-n}$ is an L -bi-Lipschitz homeomorphism of $g^n(I)$. Since g is the scaling map (by the factor 2) in the angular metric, the map $\xi_E^{-1} \circ t$ is L -bi-Lipschitz on each I , and therefore on all of \mathbb{T} . \square

Lemmas 7.3 and 7.4 imply that the restriction of the map τ to the boundary of each bounded Fatou component U has a K' -quasiconformal extension to U , where K' depends only on η . The set E in Lemma 7.3 is the set of dyadic points at level $n - m$, where m is the dynamical distance from U to U_0 . The finite family \mathcal{L} in Lemma 7.4 comes from Lemmas 7.1 and 7.2. Indeed, Lemma 7.1 guarantees the existence of a finite family \mathcal{F} of subsets of \mathcal{J} such that for any Julia arc $J \in \Pi_n$ we have $\tilde{\Lambda} = f^M(\xi(J)) \in \mathcal{F}$ for some $M \in \mathbb{N}_0$. Lemma 7.2 then gives a piecewise dynamical map \mathcal{T} from $\Lambda \setminus \{\alpha\}$ onto $\tilde{\Lambda} \setminus \{z_1, z_2\}$, in the notations of that lemma. Because the family \mathcal{F} is finite, the family of such maps \mathcal{T} is finite. Now, if the defining external rays of J land on the boundary of U and if we assume for simplicity that $\Lambda = f^n(J)$ rather than $\Lambda = f^{n+1}(J)$, then we have $\mathcal{T} = f^M \circ \tau \circ f^{-n}$, where the inverse branches of f are chosen appropriately. Passing to the Böttcher coordinates of U and $\tilde{U} = \xi(U)$ we conclude that the map $g^{M-\tilde{m}} \circ t \circ g^{-(n-m)}$, belongs to a finite family \mathcal{L} , where t is the conjugate map of the map τ by ϕ_U , and \tilde{m} is the dynamical distance from \tilde{U} to U_0 . Since ξ_E is η' -quasisymmetric and $\xi_E^{-1} \circ t$ is L -bi-Lipschitz, we conclude that $t = \xi_E(\xi_E^{-1} \circ t)$ is $\eta' \circ L^2$ -quasisymmetric, where L^2 is the scaling map by L^2 . Thus, the Ahlfors–Beurling extension [BA] gives that there exists $K' \geq 1$ that depends only on η' and L , and hence only on η , such that the map t has a K' -quasiconformal extension into \mathbb{D} . Conjugating back via Böttcher coordinates we conclude that τ has a K' -quasiconformal extension into every bounded Fatou component U .

To deal with the unbounded component U_∞ , we first apply Lemma 3.2 and then proceed in the same way as for bounded Fatou components, i.e., using Lemmas 7.3 and 7.4. The finite set E in this case is the

set $\psi(D_n)$, where ψ is the bi-Lipschitz map guaranteed by Lemma 3.2. The crucial difference is that, unlike the case of bounded Fatou components, the boundary of U_∞ touches itself and one has to be careful to preserve the basilica lamination. By Lemma 3.4, any topologically extendable homeomorphism ξ of \mathcal{J} induces a homeomorphism ξ_∞ of \mathbb{T} that preserves the basilica lamination. By Lemma 4.1, ξ_∞ is η' -quasisymmetric (with η' depending only on η). As ξ is replaced by a piecewise dynamical map τ , the map ξ_∞ is replaced by a Thompson-like element τ_∞ satisfying

$$\psi_\infty \circ \tau_\infty = \tau \circ \psi_\infty.$$

Such a map τ_∞ necessarily respects the basilica lamination. By Lemmas 7.3 and 7.4, the map τ_∞ is η'' -quasisymmetric with η'' that depends only on η' and hence only on η . Therefore it extends to a K'' -quasiconformal homeomorphism of $\mathbb{C} \setminus \mathbb{D}$ (with K'' depending only on η'' , and ultimately, only on η).

Thus, the map τ has a K'' -quasiconformal extension to U_∞ as well, where K'' depends only on η . Putting this together, we obtain a K -quasiconformal extension of τ from \mathcal{J} into each Fatou component of f , where $K = \max\{K', K''\}$. We denote this extension by τ as well. As above, the polynomial f is postcritically finite, and, according to [Jo], the Julia set \mathcal{J} is removable for quasiconformal maps. The map τ is hence K -quasiconformal in the whole complex plane, and therefore η' -quasisymmetric for some η' that depends only on η . Its restriction τ to \mathcal{J} is thus also η' -quasisymmetric.

Lemma 7.5. *Any piecewise dynamical map τ of \mathcal{J} belongs to the group \hat{T} generated by T and ι .*

Proof. It is proved in Lemma 6.2 that the group \hat{T} acts transitively on the set of bounded Fatou components. Suppose that $\tau(\partial U_0) = \partial \tilde{U}_0$ and let $g \in \hat{T}$ be such that $g(\partial U_0) = \partial \tilde{U}_0$. Then $g^{-1} \circ \tau$ keeps ∂U_0 invariant, and so, without loss of generality, we assume that τ keeps ∂U_0 invariant. Note that Lemma 2.1 implies that such τ sends the root of each $U \neq U_0$ to the root of $\tilde{U} = \tau(U)$.

Let U be any bounded Fatou component of f and $\tilde{U} = \tau(U)$. Let ϕ_U and $\phi_{\tilde{U}}$ be the Böttcher coordinates of U and \tilde{U} , respectively. Since τ is assumed to be piecewise dynamical, the restriction of the map $\phi_{\tilde{U}} \circ \tau \circ \psi_U$ to the unit circle is an element of the Thompson group T of the unit circle. Here $\psi_U = \phi_U^{-1}$. Moreover, this map respects the Böttcher coordinate for all but finitely many bounded Fatou components U . This follows immediately from the fact that there are only finitely many global cut-points used in the definition of a piecewise dynamical map.

Let t be the smallest subtree of \mathfrak{T} that contains U_0 and such that τ respects the Böttcher coordinate of every bounded Fatou component U that is not a vertex of t . Let k be the number of vertices of t . We prove by induction on k that there is an element g of \hat{T} such that $\tau = g \in \hat{T}$. If $k = 1$, then there exists $\theta \in T$ such that $\theta^{-1} \circ \tau$ is the identity element on ∂U_0 . Moreover, since θ respects the Böttcher coordinate of each bounded Fatou component $U \neq U_0$, then $\theta^{-1} \circ \tau$ is necessarily the identity in the Böttcher coordinate of every U , and so it is the identity.

Now suppose the result is true for $k - 1$. Let U be a leaf of t , i.e., a degree one vertex, and let \tilde{U} be the leaf of \tilde{t} that corresponds to U under τ . Let p be the unique path from U_0 to U in t . By Lemma 6.2, there is an element $g_U \in \hat{T}$ such that $g_U(\partial U) = \partial U_0$ and g_U respects the Böttcher coordinates of all bounded Fatou components V with the possible exception of V being a vertex of p . Likewise, there is an element $g_{\tilde{U}} \in \hat{T}$ that has the same properties with respect to \tilde{U} and \tilde{t} . Also, g_U and $g_{\tilde{U}}$ respect the Böttcher coordinates of U and \tilde{U} , respectively.

Then the restriction of $g_{\tilde{U}} \circ \tau \circ g_U^{-1}$ to ∂U_0 equals to the restriction of an element $\theta_U \in T$, and hence $g_{\tilde{U}} \circ \tau \circ g_U^{-1} \circ \theta_U^{-1}$ is the identity on ∂U_0 . Moreover, since τ takes the root of U to the root of \tilde{U} , the map θ_U fixes the root α of U_0 . The map $g = g_U^{-1} \circ \theta_U \circ g_{\tilde{U}}$ is then an element of \hat{T} that keeps U invariant, fixes the root of U , and respects the Böttcher coordinate of every $V \neq U$. In the Böttcher coordinates of U and \tilde{U} , the map $\tau \circ g^{-1}$ is the identity. Also, it is the identity in Böttcher coordinates of every vertex V that is not in t . Now we can apply the induction. \square

Recall that in [BF] the authors studied the Thompson-like group T_α^b of piecewise linear homeomorphisms of the unit circle that preserve the basilica lamination. Obviously, our group \hat{T} generated by T and ι is isomorphic to a subgroup of T_α^b . The following is an immediate corollary of Lemmas 5.6 and 7.5.

Corollary 7.6. *The groups \hat{T} and T_α^b are isomorphic.*

To finish the proof of Theorem 1.1 we need to show that τ approaches ξ on \mathcal{J} uniformly as n goes to infinity. Indeed, as one moves further away from the vertex of \mathfrak{T} that corresponds to U_0 , the diameters of the corresponding bounded Fatou components go to 0. This follows from the fact that f is hyperbolic. Moreover, the diameter of each $J_k \in \Pi_n$ goes to 0 as $n \rightarrow \infty$. This, along with the fact that τ agrees with ξ at all the preimages of α under f^n , finishes the proof of Theorem 1.1. \square

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