λ-Lemma for families of Riemann surfaces and the critical loci of complex Hénon maps

Tanya Firsova and Mikhail Lyubich

Abstract

We prove a version of the classical λ-lemma for holomorphic families of Riemann surfaces. We then use it to show that critical loci for complex Hénon maps that are small perturbations of quadratic polynomials with Cantor Julia sets are all quasiconformally equivalent.

1 Introduction

A *holomorphic motion* in dimension one is a family of injections $h_\lambda : A \to \hat{\mathbb{C}}$ of some set $A \subset \hat{\mathbb{C}}$ holomorphically depending on a parameter λ (ranging over some complex manifold Λ). It turned out to be one of the most useful tools in one-dimensional complex dynamics. First it was used to prove that a generic rational endomorphism $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is structurally stable (see [9, 12]), and then has found numerous further applications.

Usefulness of holomorphic motions largely comes from their nice extension and regularity properties usually referred to as the *λ-lemma*. The simplest version of the Extension λ-lemma asserts that the holomorphic motion of any subset $X \subset \hat{\mathbb{C}}$ extends to a holomorphic motion of the closure $\bar{X}$ [9, 12]. A more advanced version says that it extends to the whole Riemann sphere over a smaller parameter domain [5, 14]. The strongest version asserts that if Λ is the disk $\mathbb{D} \subset \mathbb{C}$ then the extension is globally defined, over the whole $\mathbb{D}$. Moreover, the maps h_λ are automatically continuous [9, 12] and in fact, quasiconformal [12].

In dimension two, holomorphic motions $h_\lambda : A \to \mathbb{C}^2$, $A \subset \mathbb{C}^2$, do not have such nice properties: in general, they do not admit extension even to the closure $\bar{A}$, and the maps h_λ are not automatically continuous (let alone, quasiconformal). Still, under some circumstances, holomorphic motions turn out to be useful in higher dimensions as well, see [2, 7].

In this paper, we prove a version of the λ-lemma for a class of holomorphic motions in $\mathbb{C}^2$ that naturally arise in the study of complex Hénon maps. Namely, we consider a holomorphic family of Riemann surfaces $S_\lambda \subset \mathbb{C}^2$ that fit into a complex two-dimensional manifold such that the boundaries of S_λ move holomorphically in $\mathbb{C}^2$. We show that under suitable conditions, the holomorphic motion of the boundary can be extended to a holomorphic motion of the surfaces. The proof is based upon Teichmüller Theory.

Stony Brook IMS Preprint #2014/03
May 2014

This work is motivated by study of the geometry of the critical locus $\mathcal{C}$ for the Hénon automorphisms

$$f : (x, y) \mapsto (x^2 + c - ay, x)$$

of $\mathbb{C}^2$. This locus was introduced by Hubbard (see [4]) as the set of tangencies between two dynamically defined foliations outside the "big" Julia set. It was studied in [6, 10] in the case of small perturbations (i.e., with a small Jacobian a) of one-dimensional hyperbolic polynomials $P_c : x \mapsto x^2 + c$. In case when c is outside the Mandelbrot set (and a is small enough), the critical locus has a rich topology described in [6]. Our version of the λ-lemma implies that all these critical loci are quasiconformally equivalent.

2 Background

2.1 Notations

We will use the following notations throughout the paper: Δ for the unit disk, $\mathbb{H}$ for the hyperbolic plane, $\hat{\mathbb{C}}$ for the Riemann sphere.

2.2 λ-lemma

Let M be a complex manifold, and let $\Delta \subset \mathbb{C}$ be a unit disk.

Definition 2.1. *Let $A \subset M$. A holomorphic motion of A over Δ is a map $f : \Delta \times A \to M$ such that:*

1. *For any $a \in A$, the map $\lambda \mapsto f(\lambda, a)$ is holomorphic in Δ;*
2. *For any $\lambda \in \Delta$, the map $a \mapsto f(\lambda, a) =: f_\lambda(a)$ is an injection;*
3. *The map f_0 is the identity on A.*

Holomorphic motions in one-dimensional dynamical context first appeared in [9, 12]. The following simple but important virtues of one-dimensional holomorphic motions are usually referred to as λ-lemma:

Extension λ-lemma ([9, 12])**.** *Let $M = \hat{\mathbb{C}}$, $A \subset \hat{\mathbb{C}}$. Any holomorphic motion $f : \Delta \times A \to \hat{\mathbb{C}}$ extends to a holomorphic motion $\Delta \times \bar{A} \to \hat{\mathbb{C}}$.*

Definition 2.2. *Let (X, d_X), (Y, d_Y) be two metric spaces. A homeomorphism $f : X \to Y$ is said to be η-quasisymmetric, if there exists an increasing continuous function $\eta : [0, \infty) \to [0, \infty)$, such that for any triple of distinct points x, y and z :*

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right)$$

A quasisymmetric map between two open domains is quasiconformal.

Qc λ-lemma ([12]). *Under the circumstances of the Extension λ-lemma, for any $\lambda \in \Delta$, the map $f_\lambda : \bar{A} \to \bar{A}$ is quasisymmetric.*

Later, Bers & Royden [5] and Sullivan & Thurston [14] proved that there exists a universal $\delta > 0$ such that under the circumstances of the Extension λ-lemma, the restriction of f to the parameter disk $\mathbb{D}_\delta$ of radius δ can be extended to a holomorphis motion $\Delta_\delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ ("BRST λ-lemma"). Though this version of the λ-lemma will be sufficient for our dynamical applications, let us also state the strongest version asserting that δ is actually equal to 1:

Slodkowski's λ-lemma. *Let $A \subset \hat{\mathbb{C}}$. Any holomorphic motion $f : \Delta \times A \to \hat{\mathbb{C}}$ extends to a holomorphic motion $\Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.*

In what follows, we will use the same notation f for the extended holomorphic motion.

2.3 Elements of Teichmüller Theory

We assume that the reader is familiar with the basics of Teichmüller Theory. To set up terminology and notation, we recall some basic definitions and statements and refer to [11] for details.

Given a base Riemann surface S, let $\mathcal{QC}(S)$ stand for the set of all Riemann surfaces quasiconformally equivalent to S.

Definition 2.3. *Let $X_1, X_2 \in \mathcal{QC}(S)$, and let $\phi_i : S \to X_i$ be quasiconformal mappings. The pairs (X_1, ϕ_1) and (X_2, ϕ_2) are called Teichmüller equivalent if there exists a conformal isomorphism $\alpha : X_1 \to X_2$ such that ϕ_2 is homotopic to $\alpha \circ \phi_1$ relative to the ideal boundary $I(S)$. The class of equivalent pairs is called a marked by S Riemann surface.*[1]

Definition 2.4. *The Teichmüller space $\mathcal{T}(S)$ modeled on S is the space of marked by S Riemann surfaces.*

The space $\mathcal{T}(S)$ can be endowed with a natural *Teichmüller metric*.

Any marked Riemann surface $(\tilde{S}, \psi) \in \mathcal{T}(S)$ defines an isometry

$$\psi_* : \mathcal{T}(S) \to \mathcal{T}(\tilde{S}), \quad \psi_* : (X, \phi) \to (X, \phi \circ \psi^{-1}), \tag{1}$$

called a *change of the base point* of the Teichmüller space.

Definition 2.5. *A Beltrami form μ on S is a measurable $(-1, 1)$-differential form with $|\mu(z)| < 1$ a.e. It is called bounded if $\|\mu\|_\infty < 1$.*

Locally, μ can be represented as $\mu(z)\dfrac{d\bar{z}}{dz}$, where $\mu(z)$ is a measurable function with $|\mu(z)| < 1$ a.e. (Notice that the latter condition is independent of the choice of the local coordinate.)

[1] Somewhat informally, we will use notation (X, ϕ), or just X, for the equivalence class.

Any Beltrami form μ determines a *conformal structure* on S, i.e., the class of metrics conformally equivalent to $dz + \mu(z)\, d\bar{z}$. (In what follows, Beltrami forms and the corresponding conformal structures will be freely identified.) The standard structure σ corresponds to $\mu \equiv 0$.

Let $\mathcal{M}(S)$ be the space of bounded Beltrami forms on S. It is identified with the unit ball in the complex Banach space $L^\infty(S)$, from which it inherits a natural complex structure.

Any quasiconformal map $f : S \to X$ induces the pullback

$$f^* : \mathcal{M}(X) \to \mathcal{M}(S). \tag{2}$$

Measurable Riemann Mapping Theorem. *Let μ be a bounded Beltrami form on S with $\|\mu\|_\infty = k < 1$. Then there exists a Riemann surface $S_\mu \in \mathcal{QC}(S)$ and a K-quasiconformal map $f_\mu : S \to S_\mu$ with $K = (1+k)/(1-k)$ such that $f_\mu^* \sigma = \mu$. Moreover, it is unique up to postcomposition with some conformal map $h : S_\mu \to S'_\mu$.*

Analytically, $f = f_\mu$ gives a solution to the *Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}. \tag{3}$$

By the Measurable Riemann Mapping Theorem, there is a natural projection

$$\Phi_S : \mathcal{M}(S) \to \mathcal{T}(S).$$

The pullback operator from equation (2) descends to $f^* : \mathcal{T}(X) \to \mathcal{T}(S)$. It is the inverse of the change of the base point f_*.

Theorem 2.1. *There exists a unique complex structure on $\mathcal{T}(S)$ such that the projection Φ_S is holomorphic.*

Notice that the change of the base point (1) is a biholomorphism $\mathcal{T}(S) \to \mathcal{T}(\tilde{S})$, so the complex structure on the Teichmüller space is independent of the choice of S.

Proposition 2.1 (Slodkowski's λ-lemma restated [11])**.** *Every holomorphic map $\gamma : \Delta \to \mathcal{T}(S)$ lifts to a holomorphic map $\tilde{\gamma} : \Delta \to \mathcal{M}(S)$.*

Let S be a hyperbolic Riemann surface, and let $p : \mathbb{H} \to S$ be its universal covering with the group of deck transformations Γ.

Lemma 2.1 ([11])**.** *Let ν be an infinitesimal Beltrami form on S, then $\nu \in$ Ker $d\Phi_S$ if and only if $p^*\nu = \bar{\partial}\eta$, where η is a continuous Γ-invariant vector field on $\bar{\mathbb{H}}$ such that the distributional derivative $\bar{\partial}\eta$ has bounded L^∞-norm and $\eta = 0$ on $\bar{\mathbb{R}}$.*

Corollary 2.1. *Assume that S is a bounded type Riemann surface with the boundary $\partial S = \gamma^1 \cup \ldots \gamma^n$, where γ^i are smooth Jordan curves. Let ν be an infinitesimal Beltrami form. Then $\nu \in$ Ker $d\Phi_S$ if and only if $\nu = \bar{\partial}\xi$, where ξ a continuous vector field on S such that the distributional derivative $\bar{\partial}\xi$ has bounded L^∞ norm and $\xi = 0$ on ∂S.*

Proof. Let $p^{-1}(\xi)$ be a lift of the vector field ξ to $\mathbb{H}$. Let D be a fundamental domain of the group Γ. The vector field ξ vanishes on the boundary. Therefore, $p^{-1}(\xi)|_D$ is bounded in the hyperbolic metric. Since Möbius transformations preserve the the hyperbolic metric, $p^{-1}(\xi)$ is bounded in hyperbolic metric on $\mathbb{H}$. Thus, it vanishes on the boundary in the Euclidean metric. $\square$

The group Γ is Fuchsian, so it acts on the whole Riemann sphere $\hat{\mathbb{C}}$. Let $\mathcal{M}^\Gamma(\hat{\mathbb{C}}) \subset \mathcal{M}(\hat{\mathbb{C}})$ be the space of Γ-invariant Beltrami forms on $\hat{\mathbb{C}}$. We can map $\mathcal{M}(S)$ to $\mathcal{M}^\Gamma(\mathbb{C})$ by lifting $\mu \in \mathcal{M}(S)$ to the Beltrami form $\hat{\mu} = p^*\mu$ on $\mathbb{H}$ and then extending it by 0 to the rest of $\hat{\mathbb{C}}$. By the Measurable Riemann Mapping Theorem, there exists a unique solution $f_{\hat{\mu}} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of Beltrami equation (3) for $\hat{\mu}$, fixing $0, 1$ and ∞. It conjugates the Fuchsian group Γ to a quasi-Fuchsian group Γ_μ preserving the quasidisk $f_\mu(\mathbb{H})$. Hence it induces a quasiconformal map $S \to S_\mu$ (for which we will keep the same notation f_μ).

Consider the map

$$\Psi : \mathcal{M}(S) \times \hat{\mathbb{C}} \to \mathcal{M}(S) \times \hat{\mathbb{C}}, \quad (\mu, z) \mapsto (\mu, f_{\hat{\mu}}(z)).$$

The image $\Psi(\mathcal{M}(S) \times \mathbb{H})$ is an open subset of $\mathcal{M}(S) \times \hat{\mathbb{C}}$ called the *Bers fiber space*. Fiberwise actions of quasi-Fuchsian groups Γ_μ induce an action of Γ on the Bers fiber space.

Definition 2.6. *The quotient $\Psi(\mathcal{M}(S) \times \mathbb{H})/\Gamma$ is called the Universal Curve over $\mathcal{M}(S)$.*

3 λ-Lemma for families of Riemann surfaces

Let us consider a complex 3-fold $\Delta \times \mathbb{C}^2$, and let $\pi_1 : \Delta \times \mathbb{C}^2 \to \Delta$ be the natural projection to Δ. Let $\bar{\mathcal{S}} \subset \Delta \times \mathbb{C}^2$ be a complex 2-fold with boundary such that $\pi_1 : \bar{\mathcal{S}} \to \Delta$ is a smooth locally trivial fibration with fibers $\bar{S}_\lambda$. We assume that the fibers $\bar{S}_\lambda$ are compact Riemann surfaces with boundary $\partial S_\lambda = \gamma_\lambda^1 \cup \cdots \cup \gamma_\lambda^n$, where the γ_λ^i are smooth Jordan curves that move holomorphically over Δ. Intrinsic interior of $\bar{\mathcal{S}}$ is a complex 2-fold $\mathcal{S} = \bar{\mathcal{S}} \setminus \partial\mathcal{S}$ that fibers over Δ. The fibers are open Riemann surfaces

$$S_\lambda = \operatorname{int} \bar{S}_\lambda = \bar{S}_\lambda \setminus \partial \bar{S}_\lambda.$$

Note that since Δ is contractible, the fibration $\pi_1 : \mathcal{S} \to \Delta$ is globally trivial in the smooth category.

Theorem 3.1. *Let $f : \Delta \times \partial S_0 \to \mathbb{C}^2$ be a holomorphic motion of ∂S_0 over Δ, and let $f_\lambda(z) = f(\lambda, z)$, $\operatorname{Im} f_\lambda = \partial S_\lambda$. Moreover, assume that the maps $f_\lambda : \partial S_0 \to \partial S_\lambda$ are diffeomorphisms. Then there exists a holomorphic motion $\tilde{f}$ of $\bar{S}_0$ over Δ, such that*

1. *$f = \tilde{f}|_{\partial S_0}$;*
2. *for any $\lambda \in \Delta$, $\operatorname{Im} \tilde{f}_\lambda = S_\lambda$.*

We will show that a family S_λ can be realized as a holomorphic curve in the Universal Curve over the Teichmüller space $\mathcal{T}(S_0)$.

Let us first extend the holomorphic motion f to a smooth motion of $S_0 \to S_\lambda$ over Δ, for which we will use the same notation f_λ as for the original motion. It defines a smooth curve $\tau_\lambda := (S_\lambda, f_\lambda)$ in the Teichmüller space $\mathcal{T}(S_0)$.

Lemma 3.1. *The elements $\tau_\lambda \in \mathcal{T}(S_0)$ do not depend on the choice of extension.*

Proof. Let f_λ and g_λ be two extensions as above. Then

$$g_\lambda^{-1} \circ f_\lambda : \bar{S}_0 \to \bar{S}_0, \quad g_\lambda^{-1} \circ f_\lambda|_{\partial S_0} = \mathrm{Id}, \quad \lambda \in \Delta.$$

Hence the maps $g_\lambda^{-1} \circ f_\lambda$ are homotopic to identity rel ∂S_0, and thus define the same element of the Teichmuüller space $\mathcal{T}(S_0)$. $\square$

Lemma 3.2. *There exists a holomorphic 1-form ω on S_0 that extends smoothly to the boundary and $\omega(z) \neq 0$ for all $z \in \bar{S}_0$.*

Proof. Let R be a Shottky double cover of S_0 [1]. There is a holomorphic embedding $\phi : S_0 \to R$ such that ϕ extends smoothly to the boundary ∂S_0. By Riemann-Roch theorem, we can take a meromorphic form u on R such that zeroes and poles of u belong to $R \backslash \bar{S}_0$. The form $\omega = u|_{S_0}$ is a desired holomorphic 1-form. $\square$

Theorem 3.2. *The curve τ_λ is an analytic curve in $\mathcal{T}(S_0)$.*

Proof. Let us show that $\dfrac{\partial \tau_\lambda}{\partial \bar{\lambda}} = 0$.

Fix some $\lambda_0 \in \Delta$. Consider the map $f_\lambda \circ f_{\lambda_0}^{-1} : S_{\lambda_0} \to S_\lambda$. This map defines a family μ_λ of Beltrami forms on S_{λ_0}:

$$\mu_\lambda = \frac{\overline{\partial}\left(f_\lambda \circ f_{\lambda_0}^{-1}\right)}{\partial\left(f_\lambda \circ f_{\lambda_0}^{-1}\right)} \in \mathcal{M}(S_{\lambda_0})$$

Consider the projection map

$$\Phi_{\lambda_0} : \mathcal{M}(S_{\lambda_0}) \to \mathcal{T}(S_{\lambda_0})$$

The map $(f_{\lambda_0})^*$ provides an isomorphism between $\mathcal{T}(S_{\lambda_0})$ and $\mathcal{T}(S_0)$. Moreover,

$$(f_{\lambda_0})^* \circ \Phi_{\lambda_0} : \mathcal{M}(S_{\lambda_0}) \to \mathcal{T}(S_0),$$
$$(f_{\lambda_0})^* \circ \Phi_{\lambda_0}(\mu_\lambda) = \tau_\lambda.$$

Then we have:

$$\left.\frac{\partial \tau_\lambda}{\partial \bar{\lambda}}\right|_{\lambda=\lambda_0} = df_{\lambda_0}^* \circ d\Phi_{\lambda_0} \left.\frac{\partial \mu_\lambda}{\partial \bar{\lambda}}\right|_{\lambda=\lambda_0}$$

Let us show that $\dfrac{\partial \mu_\lambda}{\partial \overline{\lambda}}(\lambda_0) \in \text{Ker } d\Phi_{\lambda_0}$. To simplify the notations, we assume below $\lambda_0 = 0$. We construct a vector field ξ on S_0, such that $\dfrac{\partial \mu_\lambda}{\partial \overline{\lambda}}(0) = \overline{\partial}\xi$, and $\xi = 0$ on ∂S_0 and apply Corollary 2.1. Let $\nu := \dfrac{\partial \mu_\lambda}{\partial \lambda}(0)$, $\kappa := \dfrac{\partial \mu_\lambda}{\partial \overline{\lambda}}(0)$. Since $\mu_0 = 0$,

$$\mu_\lambda = \lambda\nu + \bar{\lambda}\kappa + o(\lambda, \bar{\lambda}).$$

Let $(g_1, g_2) : S_0 \to \mathbb{C}^2$ be the defining functions of the Riemann surface S_0. The functions g_1, g_2 extend smoothly to the boundary, and

$$f_\lambda = \begin{pmatrix} g_1 + \lambda u_1 + \bar{\lambda} v_1 + o(\lambda, \bar{\lambda}) \\ g_2 + \lambda u_2 + \bar{\lambda} v_2 + o(\lambda, \bar{\lambda}) \end{pmatrix}$$

Since f_λ is a holomorphic motion on the boundary, functions v_1 and v_2 are equal to zero on the boundary. Let w be a local coordinate on S_0, $\partial f = \frac{\partial f}{\partial w} dw$, $\bar{\partial} f = \frac{\partial f}{\partial \bar{w}} d\bar{w}$. By Lemma 3.2 there is a holomorphic non-zero 1-form ω on S_0 that extends smoothly to the boundary ∂S_0.

The functions g_1 and g_2 are holomorphic. Thus, $\partial g_1 = h_1\omega$, $\partial g_2 = h_2\omega$, where h_1, h_2 are holomorphic functions on S_0 that extend smoothly to ∂S_0.

$$\partial f_\lambda = \begin{pmatrix} h_1\omega + \lambda\partial u_1 + \bar{\lambda}\partial v_1 + \dots \\ h_2\omega + \lambda\partial u_2 + \bar{\lambda}\partial v_2 + \dots \end{pmatrix} \quad \bar{\partial} f_\lambda = \begin{pmatrix} \lambda\bar{\partial} u_1 + \bar{\lambda}\bar{\partial} v_1 + \dots \\ \lambda\bar{\partial} u_2 + \bar{\lambda}\bar{\partial} v_2 + \dots \end{pmatrix}$$

$$\mu_\lambda \partial f_\lambda = \bar{\partial} f_\lambda$$

$$(\lambda\nu + \bar{\lambda}\kappa + \dots) \begin{pmatrix} h_1\omega + \dots \\ h_2\omega + \dots \end{pmatrix} = \begin{pmatrix} \lambda\bar{\partial} u_1 + \bar{\lambda}\bar{\partial} v_1 + \dots \\ \lambda\bar{\partial} u_2 + \bar{\lambda}\bar{\partial} v_2 + \dots \end{pmatrix}$$

Therefore, $\kappa \begin{pmatrix} h_1\omega \\ h_2\omega \end{pmatrix} = \begin{pmatrix} \bar{\partial} v_1 \\ \bar{\partial} v_2 \end{pmatrix}$. It follows from [15] that the space of maximal ideals in the algebra A of holomorphic functions on S_0 that extend continuously to the boundary is isomorphic to $\bar{S}_0$. The functions h_1, h_2 do not have common zeroes on $\bar{S}_0$. So the ideal generated by h_1 and h_2 coincide with A, in particular function 1 belong to the ideal. Hence there exists a pair of holomorphic functions s_1 and s_2 on S_0 that extend continuously to ∂S_0 so that $s_1 h_1 + s_2 h_2 = 1$. Let η be a holomorphic vector field on S_0, such that $\omega(\eta) = 1$. Since ω extends smoothly to ∂S_0, η extends smoothly to ∂S_0. Set

$$\xi = (s_1 v_1 + s_2 v_2)\eta,$$

then $\kappa = \bar{\partial}\xi$. Functions v_1 and v_2 are smooth in $\bar{S}_0$, so $\bar{\partial} v_1$ and $\bar{\partial} v_2$ are bounded in L^∞-norm. They are also equal to 0 on the boundary of S_0, so by Corollary 2.1 $\kappa \in \text{Ker}\, d\Phi_{\lambda_0}$. □

Proof of Theorem 3.1: By Slodkowski's λ-lemma, there exists a holomorphic family ν_λ on S_0, so that $\Phi_{\lambda_0}\nu_\lambda = \tau_\lambda$. Notice that $\mathcal{S}$ is the preimage of the family $\{\nu_\lambda |\ \lambda \in \Delta\}$ in the Universal Curve over $\mathcal{M}(S_0)$.

□

4 Application to dynamics

4.1 Background on Hénon maps

Complex Hénon maps are biholomorphisms $f_\lambda : \mathbb{C}^2 \to \mathbb{C}^2$ of the form

$$f_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix},$$

where $\lambda = (a, c)$, $a \in \mathbb{C}^*$, $c \in \mathbb{C}$.

In the one-dimensional holomorphic dynamics, the global phase portrait is to a large extent determined by the behavior of the critical points. Being diffeomorphisms, Hénon maps do not have critical points in the usual sense. However, they possess an interesting analogous object, the *critical locus*.

Let us recall the following dynamically significant sets:

$$U^+_\lambda = \{(x, y) : \ f^n_\lambda(x, y) \to \infty \text{ as } n \to +\infty\}, \quad K^+_\lambda = \mathbb{C}^2 \backslash U^+_\lambda, \quad J^+_\lambda = \partial K^+_\lambda,$$

$$U^-_\lambda = \{(x, y) : \ f^{-n}_\lambda(x, y) \to \infty \text{ as } n \to +\infty\}, \quad K^-_\lambda = \mathbb{C}^2 \backslash U^-_\lambda, \quad J^-_\lambda = \partial K^-_\lambda,$$

$$J_\lambda = J^+_\lambda \cap J^-_\lambda.$$

Domains U^+_λ and U^-_λ are called *(forward and backward) escape loci*; J_λ is called the *Julia set* of the Hénon map.

In the one-dimensional polynomial dynamics, critical points of the polynomial are critical points of the Green's function on the complement of the filled Julia set. For a complex Hénon map, one can define the *forward and backward Green's functions* that measure the escape rate of the orbits under forward and backward iterations of the map [8]:

$$G^+_\lambda(x, y) = \lim_{n\to\infty} \frac{\log^+ |f^n_\lambda(x, y)|}{2^n},$$

$$G^-_\lambda(x, y) = \lim_{n\to\infty} \frac{\log^+ |f^{-n}_\lambda(x, y)|}{2^n} + \log |a|.$$

Let $p_c(x) = x^2 + c$. When $a \to 0$, Hénon maps degenerate to a 1-dimensional map $x \mapsto p_c(x)$, acting on parabola $x = p_c(y)$. When $a \to 0$, the Green's functions G^+_λ converge to $G^+_{(0,c)}(x, y) = G_{p_c}(x)$, where $G_{p_c}(x)$ is the Green's function of the map $x \mapsto p_c(x)$. The functions G^+_λ, G^-_λ are pluriharmonic on the escape loci U^+_λ, U^-_λ respectively. Therefore, their level sets are foliated by Riemann surfaces. We denote by $\mathcal{F}^+_\lambda$, $\mathcal{F}^-_\lambda$ the corresponding foliations. These Riemann surfaces are in fact copies of $\mathbb{C}$ [8].

There are also analogues $\phi_{\lambda,+}$, $\phi_{\lambda,-}$ of the *Böttcher coordinates*. The function $\phi_{\lambda,+}$ is well defined and holomorphic in a neighborhood V^+_λ of $(x = \infty, y = 0)$ in the $\hat{\mathbb{C}}^2$-compactification of $\mathbb{C}^2$, and $\phi_{\lambda,+} \sim x$ as $x \to \infty$. Moreover, it semiconjugates f to $z \mapsto z^2$, $\phi_{\lambda,+}(f_\lambda) = \phi^2_{\lambda,+}$.

In V_λ^+, the foliation $\mathcal{F}_\lambda^+$ consists of the level sets of $\phi_{\lambda,+}$. It can be propagated to the rest of U_λ^+ by the dynamics. One can also extend $\phi_{\lambda,+}$ to U_λ^+ as a multi-valued function, and then use any branch of it to define $\mathcal{F}_\lambda^+$. Moreover, any branch is related to the Green's function by $G_\lambda^+ = \log|\phi_{\lambda,+}|$.

The function $\phi_{\lambda,-}$ is defined in an analogous way.

4.2 Critical Locus

Definition 4.1. *The critical locus $\mathcal{C}_\lambda$ is the set of tangencies between foliations $\mathcal{F}_\lambda^+$ and $\mathcal{F}_\lambda^-$.*

The critical locus is given by the zeroes of the 2-form

$$w = d\log\phi_{\lambda,+} \wedge d\log\phi_{\lambda,-}.$$

It is a non-empty proper analytic subset of $U_\lambda^+ \cap U_\lambda^-$ which is invariant under the maps f_λ, f_λ^{-1}.

Lyubich and Robertson ([10]) gave a description of the critical locus for Hénon mappings

$$(x, y) \mapsto (p(x) - ay, x),$$

where $p(x)$ is a hyperbolic polynomial with the connected Julia set, a is sufficiently small. They showed that for each critical point c of p there is a component the critical locus that is asymptotic to the line $y = c$. The rest of the components are iterates of these ones, and each is a punctured disk. In this case, all critical loci are obviously conformally equivalent.

A topological description of the critical locus for complex Hénon maps that are perturbations of quadratic polynomials with disconnected Julia sets is given in [6]. The critical locus is a connected Riemann surface with rich topology. It is composed of countably many Riemann spheres S_n with holes, that are connected to each other by handles. There are 2^{k-1} handles between S_n and S_{n+k}. On each sphere S_n the handles accumulate to two Cantor sets.

We are ready to formulate the main result of this paper:

Theorem 4.1. *The critical loci of the Hénon maps that are small perturbations of quadratic polynomials with disconnected Julia sets are quasiconformally equivalent.*

4.3 Topological description of the critical locus

In this section we will give, following [6], a precise description of the critical locus.

Let $\mathcal{A}$ be the space of one-sided sequences of 0's and 1's ("infinite strings"), and let $\mathcal{A}^n$ be the space of n-strings of 0's and 1's.

Let us describe *truncated spheres* that will serve as the building blocks for the critical locus. Consider a 2-sphere $S \equiv S^2$ and a pair of disjoint Cantor sets

$\Sigma, \Theta \subset S$. Let us fix a nest of *figure-eight curves* Γ^n_α and L^n_α, $n = 0, 1, 2, \ldots$, $\alpha \in \mathcal{A}^n$, respectively generating these Cantor sets in the following natural way[2].

Let us start with a single figure-eight curve Γ^0 bounding two domains D^1_0 and D^1_1 (with an arbitrary assignment of labeling). The curve $\Gamma^1_0 \subset D^1_0$ bounds two domains D^2_{00} and D^2_{01} compactly contained in D^1_0 (with an arbitrary assignment of the second label) , and similarly, $\Gamma^1_1 \subset D^1_1$ bounds two domains D^2_{10} and D^2_{11} inside D^1_1, etc. See Figure 1.

We assume that $\bigcup_\alpha D^n_\alpha \supset \Sigma$ and $\operatorname{diam} D^n_\alpha \to 0$ as $n \to \infty$ (uniformly in $\alpha \in \mathcal{A}^n$), so for each sequence $\alpha \in \mathcal{A}$, there is a unique point

$$\sigma_\alpha = \bigcap_{n=1}^{\infty} \overline{D^n_{\alpha_n}} \in \Sigma,$$

where $\alpha_n \subset \mathcal{A}^n$ is the initial n-string of α. That gives us a one-to-one coding of points $\sigma \in \Sigma$ by sequences $\alpha \in \mathcal{A}$.

Similarly, Θ is generated by a hierarchical nest of figure-eights L^n_α. We assume that these two nests are *disjoint* in the sense that figure-eight L^0 lies in the unbounded component of $\mathbb{C} \setminus \Gamma^0$, and the other way around.

The singular points σ^n_α and θ^n_α of the figure-eights Γ^n_α and L^n_α respectively are called their *centers*. For each figure-eight Γ^n_α, select a disk $V^n_\alpha \ni \sigma^n_\alpha$ whose closure is disjoint from all other figure-eights Γ^m_β and from L^0. Then select a disk $U^n_\alpha \ni \theta^n_\alpha$ with similar properties for each figure-eight L^n_α. Moreover, make these choices so that the closures of all these disks are pairwise disjoint.

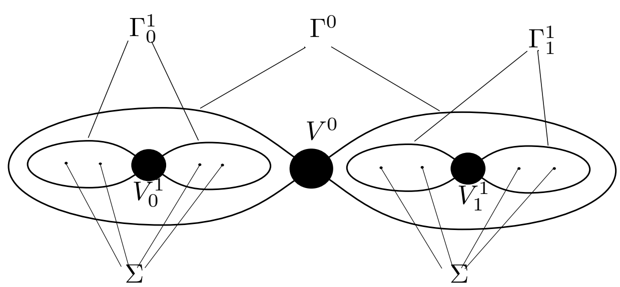

Figure 1: The geometry of a truncated sphere

For each $n \in \mathbb{N}$, $\alpha \in \mathcal{A}^n$, we choose a homeomorphism h^n_α between the boundaries of V^n_α and U^n_α. Finally, we mark a point $p \in S$ in the exterior of both figure-eights and the disks $\bar{U}^0$, $\bar{V}^0$. With all these choices in hand, we call

$$S \setminus X, \quad \text{where } X := \Sigma \cup \Theta \cup \{p\} \bigcup_n \left(\bigcup_{\alpha \in \mathcal{A}^n} U^n_\alpha \cup V^n_\alpha \right),$$

a *truncated sphere*. Note that for any two truncated spheres $S \setminus X$ and $S' \setminus X'$ there is a homeomorphism $(S, X) \to (S', X')$ that restricts to the natural homeomorphisms between the corresponding marked sets.

[2]For $n = 0$, we let $\mathcal{A}^0 = \emptyset$.

Theorem 4.2. *Assume that the quadratic polynomial $x \mapsto x^2 + c$ has disconnected Julia set. Then there exists $\delta > 0$ such that for any $|a| < \delta$ the critical locus of the Hénon map*

$$f_\lambda : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}$$

is a non-singular Riemann surface that admits the following topological model. Take countably many copies $S_m \setminus X_m$, $m \in \mathbb{Z}$, of the truncated sphere $S \setminus X$, and glue the boundary of V_α^n of S_k to the boundary of U_α^n of S_{n+k+1} by means of the homeomorphism h_α^n. The model map acts by translating $S_n \setminus X_n$ to $S_{n+1} \setminus X_{n+1}$.

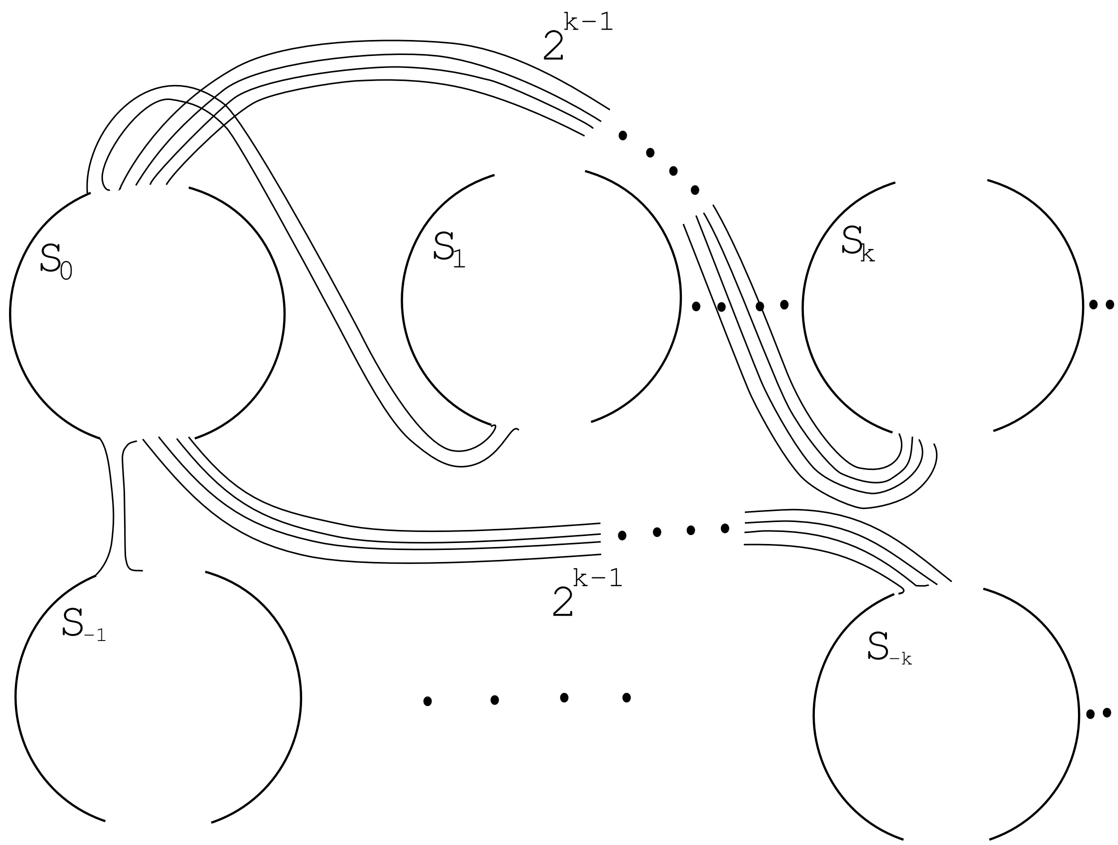

Figure 2: Critical Locus.

4.4 Proof of Theorem 4.1

In [6] we gave a detailed description of the position of the critical locus $\mathcal{C}_\lambda$ in $\mathbb{C}^2$ for $\lambda \in \Lambda$, where Λ is a set of parameters of a small perturbation of quadratic polynomials with disconnected Julia set. Below we fix a parameter $\lambda_0 = (a_0, c_0) \in \Lambda$ and use the description from [6] to construct a holomorphic motion of the critical loci $\mathcal{C}_\lambda$, for λ that belong to 1-parameter family in a neighborhood of λ_0. Let us first describe a fundamental domain of the critical locus in $\mathbb{C}^2$. Let

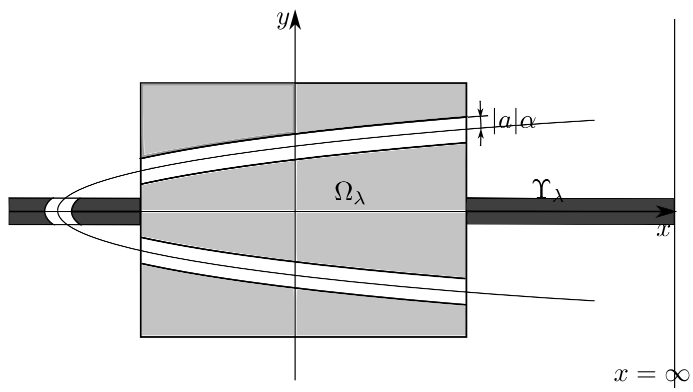

Figure 3: Domains Ω_λ and Υ_λ

$$\Omega_\lambda = \{(x, y) \in \mathbb{C}^2 : \quad G_a^+ \le r, \quad |y| \le \alpha, \quad |p_c(y) - x| > |a|\alpha\}$$

$$\Upsilon_\lambda = \{(x, y) \in \mathbb{C}^2 : \ G_a^+(x, y) \ge r, \quad |y| \le \epsilon\}.$$

When $a \to 0$, domains Ω_λ converge in Hausdorff topology to $\Omega_{(0,c)}$. In [6] we choose r, α and ϵ, depending on c, so that for c' close to c and a small enough, $\mathcal{C}_\lambda \cap (\Omega_\lambda \cup \Upsilon_\lambda)$ form a fundamental domain for the map f_λ on the critical locus. We further cut $\Omega_\lambda \cap U_\lambda^+$ into subdomains Ω_λ^α, where α goes over all finite diadic strings.

We recursively encode the n-th preimages ξ_α of 0 under the map $z \mapsto z^2 + c$ by diadic n-strings α. We assume that 0 itself is parametrized by $\emptyset$. Let α^0, $\alpha^1 \in \mathcal{A}^{n+1}$ be the strings obtained by adding 0, 1 correspondingly to α on the right. We encode preimages of ξ_α by α^0 and α^1. Since each connected component of

$$\{\frac{r}{2^{n+1}} \le G_{p_c} \le \frac{r}{2^n}\}$$

contains a unique n-preimage of the critical point, they are encoded by diadic n-strings as well.

$$\Omega_{(0,c)}^\alpha = \{\text{a connected component of } \{\frac{r}{2^{n+1}} \le G_{(0,c)}^+ \le \frac{r}{2^n}\} \cap \Omega_{(0,c)}\}$$

that contains a line $x = \xi_\alpha$, $\alpha \in \mathcal{A}^n$.

By the choice of r in [6], the connected components of

$$\{\frac{r}{2^{n+1}} \le G_\lambda^+ \le \frac{r}{2^n}\} \cap \Omega_\lambda$$

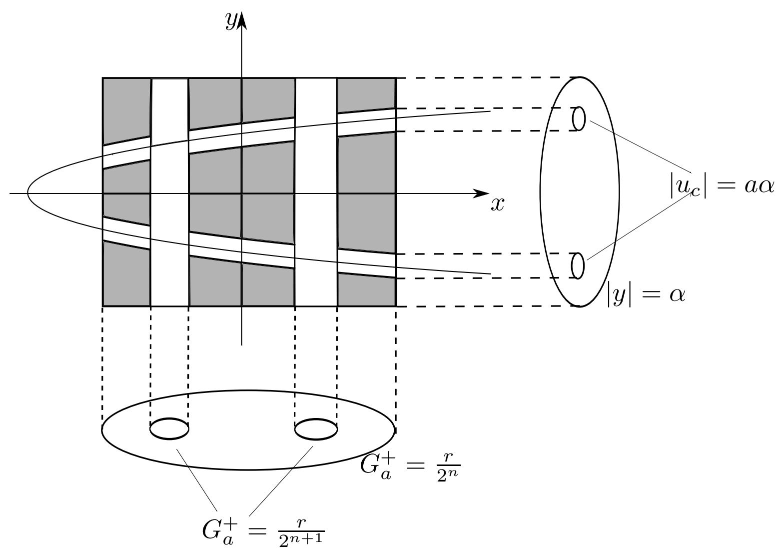

Figure 4: Domain Ω_a^α

depend continuously on a in the Hausdorff topology. We denote by Ω_λ^α continuation of $\Omega_{(0,c)}^\alpha$.

Let $u_c = y^2 + c - x$.

Lemma 4.1 ([6, Lemma 11.4])**.** *In Ω_λ^α, where $\alpha \in \mathcal{A}^n$, $n = 0, 1, \dots,$ the critical locus is a connected sum of two disks D_1 and D_2 with two holes each. The boundary of D_1 belongs to $\{|y| = \alpha\}$, and the holes of D_1 have boundaries on $\{|u_c| = |a|\alpha\}$. The boundary of D_2 belongs to $\{G_\lambda^+ = \frac{r}{2^n}\}$ and the holes to $\{G_\lambda^+ = \frac{r}{2^{n+1}}\}$.*

A holomorphic motion near the boundaries $\{G_\lambda^+ = 2^{-n}r\}$ and $\{G_\lambda^+ = 2^{-(n+1)}\, r\}$ is defined so that it preserves the values of the functions $\phi_{\lambda,+}^{2^n}$ and $\phi_{\lambda,+}^{2^{n+1}}$ respectively. Similarly, the holomorphic motion of the boundaries $\{|y| = \alpha\}$ and $\{|u_c| = |a|\alpha\}$ preserves the values of y and u_c.

We apply Theorem 3.1 to the piece of the critical locus inside Ω_λ^α and extend the holomorphic motion to the interior.

Lemma 4.2 ([6, Lemma 13.1])**.** *There exists δ such that $\forall |a| < \delta$ the critical locus $\mathcal{C}_\lambda$ in Υ_λ is a punctured disk, with a hole removed. The puncture is at the point $(\infty, 0)$, the boundary of the hole belongs to $\{|p_c(y) - x| = |a|\alpha\}$.*

We apply Theorem 3.1 to $\mathcal{C}_\lambda \cap \Upsilon_\lambda$ and extend the holomorphic motion to the interior. We propogate the holomorphic to the rest of $\mathcal{C}_\lambda$ by dynamics. The space Λ is path connected. Therefore, the critical loci $\mathcal{C}_\lambda$ for all maps that are small perturbations of quadratic polynomials with disconnected Julia set are quasiconformally equivalent.

References

[1] L. Ahlfors and L. Sario, *Riemann surfaces*, Princeton University Press, 1960.

[2] G. Bassanelli and F. Berteloot, *Lyapunov Exponents, bifurcation currents and laminations in bifurcation locus*, Math.Ann. **345** (2009), 1-23.

[3] E. Bedford and J. Smillie, *Polynomial diffeomorphism of $\mathbb{C}^2$. V. Critical points and Lyapunov exponents*, J. Geom. Anal. **8(3)** (1998), 349-383.

[4] ———, *Polynomial diffeomorphisms of $\mathbb{C}^2$. VI. Connectivity of J*, Ann. of Math. **148(2)** (1998), 695-735.

[5] L. Bers and H. Royden, *Holomorphic families of injections*, Acta Math. **157** (1986), 259-286.

[6] T. Firsova, *Critical locus for complex Hénon maps*, Indiana Univ. Math. J. **61** (2012), 1603-1641.

[7] R. Dujardin and M. Lyubich, *Stability and bifurcations for dissipative polynomial automorphisms of $\mathbb{C}^2$*, Preprint IMS at Stony Brook **13-01** (2013).

[8] J. H. Hubbard and R. W. Oberste-Vorth, *Hénon mappings in the complex domain. I. The global topology of dynamical space*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 5-46.

[9] M. Lyubich, *Some typical properties of the dynamics of rational maps*, Russian Math. Surveys **38** (1983), 154-155.

[10] M. Lyubich and J. Robertson, *The critical locus and rigidity of foliations of complex Hénon maps*, Unpublished manuscript (2004).

[11] J. H. Hubbard, *Teichmüller Theory and Applications to Geometry, Topology and Dynamics*, Vol. 1, Matrix Editions, 2006.

[12] R. Mañé, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. **16** (1983), 193-217.

[13] Z. Slodkowski, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math Society **111** (1991), 347-355.

[14] D. Sullivan and W. Thurston, *Extending holomorphic motions*, Acta Math **157** (1986), 243-257.

[15] M. Voichick, *Ideals and invariant subspaces of analytic functions*, Transactions of Am. Math. Soc. **111** (1964), 493-512.