STABILITY AND BIFURCATIONS FOR DISSIPATIVE POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^2$

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Abstract. We study stability and bifurcations in holomorphic families of polynomial automorphisms of $\mathbb{C}^2$. We say that such a family is weakly stable over some parameter domain if periodic orbits do not bifurcate there. We first show that this defines a meaningful notion of stability, which parallels in many ways the classical notion of J-stability in one-dimensional dynamics. In the second part of the paper, we prove that under an assumption of moderate dissipativity, the parameters displaying homoclinic tangencies are dense in the bifurcation locus. This confirms one of Palis' Conjectures in the complex setting. The proof relies on the formalism of semi-parabolic bifurcation and the construction of "critical points" in semi-parabolic basins (which makes use of the classical Denjoy-Carleman-Ahlfors and Wiman Theorems).

Introduction

One of the main goals in the modern theory of dynamical systems is to describe the dynamics of a typical mappings in a representative family. Let us consider for instance the space of C^k diffeomorphisms ($k \geq 1$) of real compact surfaces. It was briefly believed in the 1960's that hyperbolicity was generically satisfied in $\mathrm{Diff}^k(M)$. This hope was discouraged fast, particularly with the discovery by S. Newhouse [Nw1, Nw2] of an open region $\mathcal{N}$ in $\mathrm{Diff}^k(M)$, $k \geq 2$ containing a dense subset of maps that display homoclinic tangencies. Moreover, a generic map in $\mathcal{N}$ has infinitely many sinks. (We will refer to $\mathcal{N}$ as the *Newhouse region*.)

A more refined picture of typical dynamics of diffeomorphisms then gradually emerged. It was articulated by J. Palis as a series of conjectures (see e.g. [Pa], [PT, Chap. 7]). The first conjecture on this list is the following:

Conjecture (Palis). *Every $f \in \mathrm{Diff}^k(M)$, $k \geq 1$, can be C^k-approximated either by a hyperbolic diffeomorphism or by one exhibiting a homoclinic tangency.*

Here "homoclinic tangency" means a tangency between the stable and unstable manifolds of some saddle periodic point. Since hyperbolic diffeomorphisms are structurally stable, this singles out homoclinic tangencies as a basic phenomenon responsible for bifurcations. This conjecture was proven for $k = 1$ by E. Pujals and M. Sambarino [PS], nevertheless it remains wide open for $k > 1$. More generally, there has been an important progress in the understanding of C^1-generic dynamics in the past few years (see [Cr] for a recent overview).

Another situation that has been extensively studied is One-Dimensional Dynamics, both real and complex. In fact, the early Density of Hyperbolicity Conjecture turned out to be true in the real one-dimensional case [Ly3, GS, KSS]. It is conjectured to be true in the complex case as well (this is known as the *Fatou Conjecture*), but this problem is still open.

Consider a holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of rational mappings of degree d on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, parameterized by a complex manifold Λ (which may be the whole space of

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rational mappings of degree d). We say that the family is *J-stable* in a connected open subset $\Omega \subset \Lambda$ if in Ω the dynamics is structurally stable on the Julia set J. Work of R. Mañé, P. Sad and D. Sullivan [MSS] and independently of the second author [Ly1, Ly2] implies that the J-stability locus is dense in Λ. In addition, parameters with preperiodic critical points (which is the one-dimensional counterpart of the homoclinic tangency) are dense in the bifurcation locus. We see that the Fatou Conjecture is reduced to the problem whether J-stability implies hyperbolicity (for a sufficiently generic family, like the whole space of polynomials or rational maps of a given degree).

In this paper we deal with families of polynomial automorphisms of $\mathbb{C}^2$, which shares features with both of the previous settings. S. Friedland and J. Milnor [FM] showed that dynamically interesting automorphisms in $\mathbb{C}^2$ are conjugate to compositions of Hénon mappings $(z, w) \mapsto (aw + p(z), az)$, where a is a non-zero complex number and p is a polynomial of degree at least two. *In what follows, we assume without saying that all automorphisms under consideration are dynamically interesting*, and in particular, they have dynamical degree $d \geq 2$ (see §1 for a review of this notion).

Note that a polynomial automorphism f has constant complex Jacobian $\mathrm{Jac}\, f = \det Df$. So $\mathrm{Jac}\, f$ is a well-defined quantity attached to f. We work in the dissipative setting, and our main results actually require some stronger form of dissipation, namely we need

$$|\mathrm{Jac}\, f| < \frac{1}{d^2}, \quad \text{where } d \text{ is the dynamical degree of } f. \tag{1}$$

We will call such maps *moderately dissipative.*

We denote by J^* the closure of saddle periodic points of f. It is unknown whether J^* is always equal to the "small Julia set" J, which can be defined in classical terms as the locus where both families $\{f^n\}_{n\geq 0}$ and $\{f^n\}_{n\leq 0}$ are not normal.

From one-dimensional holomorphic dynamics we borrow the idea of focusing on J-stability rather than hyperbolicity, and in accordance with the Palis program, we explain bifurcations by the presence of homoclinic tangencies. Our main result is the following, in the spirit of the Palis conjecture (the precise meaning of the terminology "weakly stable" will be explained shortly).

Theorem A. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of moderately dissipative polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. Then weakly stable maps, together with maps exhibiting homoclinic tangencies form a dense subset of Λ.*

It is also true that *weakly stable maps, together with maps that have infinitely many sinks form a dense subset in* Λ. Somewhat surprisingly, this is just an observation obtained by analyzing the one-dimensional argument.

The set of weakly stable parameters λ_0 will be simply referred to as the *stability locus*, and its complement is by definition the *bifurcation locus*. It is worth mentioning here that G. Buzzard [Bu] showed that *the Newhouse region is non-empty* in the space of polynomial automorphisms of sufficiently high degree. It follows that the stability locus is not dense in general.

Let us now discuss the notion of weak stability. To say it briefly, a family of polynomial automorphisms is weakly stable in some open set if periodic points do not bifurcate there. The first part of this paper is devoted to demonstrating that this defines a reasonable notion of stability in this context, parallel to the usual J-stability in dimension 1. In particular we show that in a weakly stable family:

- there are no homoclinic bifurcations, and moreover all homoclinic and heteroclinic intersections can be followed holomorphically;
- the sets J^*, J^-, J^+, K move continuously in the Hausdorff topology;
- connectivity properties of the Julia sets are preserved.

Naturally, these results are based on a generalization to two dimensions of the fundamental idea of *holomorphic motion*. A fundamental problem here is that a holomorphic motion of a set X in higher dimension does not automatically admit an extension to a motion of $\overline{X}$. In practice, we work with a weaker notion of "branched holomorphic motion", in which collisions are allowed. Because of this, we have not been able to prove that weak stability implies structural J^*-stability.

The main point of this paper is to design a mechanism creating homoclinic tangencies from bifurcations of periodic points (for moderately dissipative polynomial automorphisms of $\mathbb{C}^2$). Notice that conversely, the creation of sinks from homoclinic tangencies is classical and goes back to Newhouse [Nw2]. The theory of weak J^*-stability gives a fresh insight into this phenomenon as well.

Let us now formulate a more precise version of Theorem A:

Theorem A′. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of moderately dissipative polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. Then parameters with homoclinic tangencies are dense in the bifurcation locus.*

To understand the strategy of the proof of this theorem, let us first review the one-dimensional result that parameters with preperiodic critical points are dense in the bifurcation locus. The classical proof of this fact, based on Montel's theorem [Le], does not seem to have an analogue in our context.

Let us outline an argument that admits a generalization to dimension two. If λ_0 belongs to the bifurcation locus, then some periodic point changes type near λ_0. In particular there exists λ_1 close to λ_0 such that at λ_1, there is a periodic point p whose multiplier crosses the unit circle at a rational parameter. The theory of parabolic implosion [Lv, Sh] describes how the dynamics in the basin $\mathcal{B}$ of the parabolic point can "implode" for some parameters close to λ_1. In particular, under generic assumptions, and replacing f by some iterate if needed, for well chosen sequences $\lambda_n \to \lambda_1$, $f^n_{\lambda_n}$ converges locally uniformly in $\mathcal{B}$ to some limiting holomorphic function $g : \mathcal{B} \to \mathbb{C}$, which we refer to as a *transit map*. Fix a repelling periodic point q, which necessarily persists as $q(\lambda)$ in the neighborhood of λ_1. By a classical theorem of Fatou, there exists a critical point c in $\mathcal{B}$. The point is that is actually possible to adjust the sequence λ_n so that $g(c) = q$. From this we infer that for large n, there exists λ'_n close to λ_n, such that $f^n_{\lambda'_n}(c(\lambda'_n)) = q(\lambda'_n)$, which is precisely the result that we seek.

To prove Theorem A′, in the second part of the paper we design a two-dimensional generalization of this argument. In the dissipative regime, if some periodic point $p(\lambda)$ bifurcates at λ_0, then one multiplier of $p(\lambda)$ crosses the unit circle while the other stays smaller than 1. If furthermore $p(\lambda_0)$ has a root of unity as multiplier, it is said to be *semi-parabolic*, and we say that $p(\lambda)$ undergoes a *semi-parabolic bifurcation*. Then the proof is divided into two main steps:

- Step 1: prove the existence of "critical points" in the basins of semi-parabolic periodic points.

- Step 2: use “semi-parabolic implosion” to make these critical points leave the basin under small perturbations of λ_0, eventually creating tangencies.

The critical points in Step 1 are defined as follows. Let f be a polynomial automorphism with a semi-parabolic periodic point p, which we may assume is fixed. Then p admits a basin of attraction $\mathcal{B}$, which is endowed with a holomorphic *strong stable foliation*, whose leaves are characterized by the property that points in the same leaf approach one another exponentially fast under iteration. Then by definition a critical point is a point of tangency between the strong stable foliation in $\mathcal{B}$ and the unstable manifold of some saddle periodic point q.

We obtain the following result.

Theorem B. *Let f be a moderately dissipative polynomial automorphism of $\mathbb{C}^2$. Assume that f possesses a semi-parabolic periodic point with basin of attraction $\mathcal{B}$. Then for any saddle periodic point q, every component of $W^u(q) \cap \mathcal{B}$ contains a critical point.*

Notice that this is precisely the place where the assumption on the Jacobian is required. Curiously, the proof relies on the classical theory of entire functions of finite order in one complex variable. The same idea was then used by H. Peters and the second author [LyP] to obtain a nearly complete classification of periodic Fatou components for moderately dissipative polynomial automorphisms of $\mathbb{C}^2$.

Remark. The classical theory of entire functions was first applied to (one-dimensional) polynomial dynamics by Eremenko and Levin [EL].

The second step relies on the construction of transit mappings in the context of semi-parabolic bifurcations. Semi-parabolic points are roughly classified according to the multiplicity of $f - \mathrm{id}$ at the periodic point under consideration. The theory of semi-parabolic implosion was recently developed by Bedford, Smillie and Ueda [BSU] who obtained a satisfactory picture in the multiplicity two case. In particular, it generalizes a theorem of Lavaurs [Lv], thus obtaining a precise description of the transit behaviour in this setting. However, these results depend on certain explicit changes of variables that do not readily extend to the general case.

In our situation we have to deal with semi-parabolic points of arbitrary multiplicity, so we need to develop a more general method. It was inspired by a chapter of the celebrated Orsay Notes by Douady and Hubbard, cheerfully entitled “*un tour de valse*” [DS] (written by Douady and Sentenac).

To be specific, if λ_0 is a parameter at which a semi-parabolic bifurcation occurs, replacing f by some iterate if needed, there exists a sequence of parameters $\lambda_n \to \lambda_0$ such that $f^n_{\lambda_n}$ converges in $\mathcal{B}$ to some holomorphic map $g : \mathcal{B} \to \mathbb{C}^2$. Notice that due to dissipation, g has 1-dimensional image. Though these transit mappings are not as explicit as in [BSU], they can still be well controlled (see Theorem 6.7). If now $q = q(\lambda_0)$ is any saddle point, and c is a critical point in $W^u(q)$, we can adjust the sequence λ_n so that $g(c) \in W^s(q)$. It is then easy to find parameters λ'_n close to λ_n for which $W^u(q(\lambda'_n))$ and $W^s(q(\lambda'_n))$ are tangent, thereby concluding the proof.

The plan of the paper is the following. The first section is devoted to some preliminaries on polynomial automorphisms of $\mathbb{C}^2$. In §2, we define the notion of weak J^*-stability, which is the direct analogue to the one-dimensional notion of J-stability and study the properties of weakly J^*-stable families. In §3 we show that a weakly J^*-stable family is also weakly stable on J^+, J^-, and K. In particular this justifies the use of the more general “weakly stable" terminology. We also prove that a dissipative family of polynomial automorphisms has persistently connected Julia set, then it is weakly stable (Theorem 3.5). This generalizes a well known result in dimension 1. The proof of Theorem A′ occupies §4 to 7. In §4, we recall some basics on semi-parabolic dynamics. The existence of critical points in semi-parabolic basins (Theorem B) is discussed in §5, which also includes some preparatory material on entire functions of finite order. A slight adaptation gives the existence of critical points in attracting basins. Details are given in Appendix A. Semi-parabolic implosion and transit mappings are studied in §6, and finally in §7 we assemble these results to prove Theorem A′.

Throughout the paper we use the following notation: if u and v are two real valued functions, we write $u \asymp v$ (resp $u \lesssim v$) if there exists a constant $C > 0$ such that $\frac{1}{C}u \leq v \leq Cu$ (resp. $u \leq Cv$). The disk in $\mathbb{C}$ of radius r centered at 0 is denoted by $\mathbb{D}_r$. Moreover, $\mathbb{D}$ stands for $\mathbb{D}_1$.

Remark. The results of this paper were first announced at the Balzan-Palis Symposium on Dynamical Systems (IMPA, June 2012) and at the Workshop on Holomorphic Dynamical Systems (Banff, July 2012).

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1. Preliminaries: Dynamical degree

In this section we recall some basics on the dynamics of polynomial automorphisms of $\mathbb{C}^2$, and establish some preparatory results.

Let f be a polynomial automorphism of $\mathbb{C}^2$ with non-trivial dynamics. Non-trivial dynamics means for instance here that f has positive topological entropy, which then equals $\log d$, where $d = \lim_{n\to\infty}(\deg(f^n))^{1/n}$ is the *dynamical degree* of f. According to Friedland and Milnor [FM] this happens if and only if f is conjugate to a product of Hénon mappings $(z, w) \mapsto (aw + p(z), az)$.

We use the following standard notation: $K^\pm$ is the set of points with bounded orbits under $f^{\pm n}$; $J^\pm = \partial K^\pm$ are the forward/backward Julia sets that can be also defined in the usual "normal families" sense; $J = J^+ \cap J^-$ is the Julia set and J^* is the closure of saddle periodic points. We also let $U^\pm = \mathbb{C}^2 \setminus K^\pm$. Recall that it is an open question whether J is always equal to J^*. We denote by $G^{+/-}$ the forward/backward escape rate functions, and by $T^{+/-} = dd^cG^{+/-}$ the stable/unstable currents, and $\mu = T^+ \wedge T^-$ the measure of maximal entropy (see e.g. [BS1, BLS1, FS] for more details on these notions). Recall that μ is equidistributed by saddle periodic orbits and that its support is precisely J^* [BLS1, BLS2].

We will be interested in holomorphic families $(f_\lambda)_{\lambda\in\Lambda}$ of polynomial automorphisms, parameterized by some complex manifold Λ. We put a subscript λ to denote the parameter dependence of the corresponding objects, e.g., J_λ, μ_λ, etc.

The following proposition, which might be known to some experts (see e.g. [Fu], and also [X, Thm 1.6] for the birational case), asserts that as far as we are interested in properties of f_λ which are typical with respect to λ, it is not a restriction to assume that the f_λ are products of Hénon mappings.

Proposition 1.1. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of polynomial automorphisms in $\mathbb{C}^2$, parameterized by a connected complex manifold. There exists a Zariski open set $\Lambda' \subset \Lambda$ and an integer $d \geq 1$ such that for $\lambda \in \Lambda'$, f_λ has dynamical degree d.*

Furthermore, if $d \geq 2$, locally in Λ' we can write

$$f_\lambda = \varphi_\lambda^{-1} \circ h_\lambda^1 \circ \cdots \circ h_\lambda^m \circ \varphi_\lambda$$

where (φ_λ) is a polynomial automorphism and $(h_\lambda^i)_{i=1,\ldots,m}$ are Hénon mappings of degree d_i, with $\sum d_i = d$, all depending holomorphically on λ.

To prove the proposition we need to recall some ideas from [FM]. Fix coordinates (z, w) on $\mathbb{C}^2$. We denote by E the group of automorphisms preserving the family of lines $\{w = C\}$. Such automorphisms are of the form $(z, w) \mapsto (\alpha z + p(w), \beta w + \gamma)$ and will be referred to as *elementary*. (More generally, an automorphisms is elementary if it can be put in this form in some system of coordinates (z, w).) The group of affine automorphisms will be denoted by A. It turns out that the group $\mathrm{Aut}(\mathbb{C}^2)$ of polynomial automorphisms of $\mathbb{C}^2$ is the free product of A and E, amalgamated along their intersection $S := A \cap E$, that is, every $f \in \mathrm{Aut}(\mathbb{C}^2) \setminus S$, can be written as a composition $f = g_k \circ \cdots \circ g_1$, where g_i belongs to $A \setminus S$ or $E \setminus S$. This decomposition is unique, up to simultaneously replacing g_i by $g_i \circ s$ and g_{i-1} by $s^{-1} \circ g_{i-1}$, for some $s \in S$. The degree of such a composition is to equal $\prod \deg(g_i)$ (of course only elementary automorphisms contribute to the degree). One has that $\deg(f^n) = (\deg f)^n$ if and only if f is cyclically reduced, that is the extreme factors g_1 and g_k belong to different subgroups A and E. In general, write

$$f = a_m \circ e_m \circ a_{m-1} \circ e_{m-1} \circ \cdots \circ e_1 \circ a_1, \text{ with } a_i \in A \setminus S \text{ and } e_i \in E \setminus S,$$

with possibly a_m or a_1 equal to the identity. We define the multidegree of f as $(d_m, \ldots, d_1)$ where $d_i = \deg(e_i)$.

Proof. It is clear that there exists a Zariski open set $\Lambda_0 \subset \Lambda$ where the degree is constant, say, equal to d'. If $d' = 1$ there is nothing to prove so assume $d' \geq 2$. A theorem due to Furter asserts that in a connected holomorphic family of polynomial automorphisms, the degree is constant if and only if the multidegree is constant [Fu, Cor. 3]. Hence there exists an integer m such that for every $\lambda \in \Lambda_0$ we can write

$$f_\lambda = a_{m,\lambda} \circ e_{m,\lambda} \circ a_{m-1,\lambda} \circ e_{m-1,\lambda} \circ \cdots \circ e_{1,\lambda} \circ a_{1,\lambda}.$$

We claim that the factors $a_{i,\lambda}$ and $e_{i,\lambda}$ may be chosen to depend holomorphically on λ. This is not obvious since they are not unique. We can deal with the extreme factors a_m and a_1 as in [FM, Lemma 2.4], by observing that the coset space A/S is isomorphic to $\mathbb{P}^1$ and that there is a well defined mapping $f_\lambda \mapsto (a_{1,\lambda}^{-1} S, a_{m,\lambda} S) \in \mathbb{P}^1 \times \mathbb{P}^1$. In a more explicit fashion, this mapping may be expressed as $f_\lambda \mapsto (I(f_\lambda), I(f_\lambda^{-1}))$, where $I(f)$ is the indeterminacy set of f viewed as a rational mapping on $\mathbb{P}^2$, and $\mathbb{P}^1$ is identified to the line at infinity. Since f_λ depends holomorphically on λ, so do $a_{1,\lambda}^{-1} S$ and $a_{m,\lambda} S$, hence absorbing some of the S factors in $e_{m,\lambda}$ and $e_{1,\lambda}$ if necessary, we infer that $a_{1,\lambda}$ and $a_{m,\lambda}$ depend holomorphically in λ. Thus we are left to proving that if f_λ is of the form $f_\lambda = e_{m,\lambda} \circ a_{m-1,\lambda} \circ \cdots \circ e_{1,\lambda}$, then the factors may be chosen to depend holomorphically on λ. By [FM, Lemma 2.10], f_λ admits a *unique* decomposition of the form

$$f_\lambda = (\hat{s}_{m,\lambda} \circ \hat{e}_{m,\lambda}) \circ t \circ \hat{e}_{m-1,\lambda} \circ \cdots \circ t \circ \hat{e}_{1,\lambda},$$

where $\hat{s}_{m,\lambda}$ is affine with diagonal linear part, $\hat{e}_i$ is of the form $(z,w) \mapsto (z + p_i(w), w)$, with $p_i(0) = 0$ and $t(z,w) = (w,z)$. By uniqueness, the factors of this decomposition depend holomorphically on λ (see [Fu, p.909] for details) and our claim is proven.

From this point it is clear that the set of parameters such that f_λ is not cyclically reduced is Zariski closed in Λ_0. Indeed, conjugating f_λ by $a_{1,\lambda}$ we obtain an expression of the form

$$a_{1,\lambda} \circ a_{m,\lambda} \circ e_{m,\lambda} \circ \cdots \circ e_{1,\lambda},$$

which is not cyclically reduced if and only if $a_{1,\lambda} \circ a_{m,\lambda} \in S$, which is an analytic condition. If so, we absorb $a_{1,\lambda} \circ a_{m,\lambda}$ into $e_{m,\lambda}$ and infer that the resulting word is not cyclically reduced iff $e_{1,\lambda} \circ e_{m,\lambda} \in S$, and so on. Iterating this process we obtain a Zariski open set Λ' such that if $\lambda \in \Lambda'$, f_λ is cyclically reduced, and the first part of the proposition is proved.

To establish the second assertion, in Λ' we conjugate f_λ as above to make it cyclically reduced and of the form

$$(t \circ e_k) \circ \cdots \circ (t \circ e_1).$$

Then we argue as in [FM, Theorem 2.6] that a mapping of the form $t \circ e_i$ is affinely conjugate to a Hénon mapping $(z,w) \mapsto (\delta_i z + p_i(w), z)$, which is unique up to finitely many choices if p_i is chosen to be monic and centered. □

Part 1. Holomorphic motions and stability

2. Weak J^*-stability

In this section we will often require an additional –presumably superfluous– assumption. We say that a holomorphic family of polynomial automorphisms is *substantial* if

- either all members of the family are dissipative
- or for any (say saddle) periodic point with eigenvalues α_1, α_2, no relation of the form $\alpha_1^a \alpha_2^b = c$, holds persistently in parameter space, where a, b, c are complex numbers and $|c| = 1$.

As an example, the family of all polynomial automorphisms of dynamical degree d is substantial [BHI, Theorem 1.4]. On the other hand, a family of conservative polynomial automorphisms is not.

For the sake of brevity let us define an *s/u intersection* as a shorthand to "homoclinic or heteroclinic intersection of stable and unstable manifolds of saddle periodic orbits".

Recall that a *holomorphic motion* of a set A, parameterized by Λ is a family of mappings $\varphi_\lambda : A \to \mathbb{C}^2$ such that

- for fixed $a \in A$, $\lambda \mapsto \varphi_\lambda(a)$ is holomorphic;
- for fixed $\lambda \in \Lambda$, $a \mapsto \varphi_\lambda(a)$ is injective;

Holomorphic motions are often *pointed* by fixing a parameter $\lambda_0 \in \Lambda$ and assuming that φ_{λ_0} is the identity mapping.

The total space of a pointed holomorphic motion of A over Λ is a family of disjoint holomorphic graphs over the first coordinate in $\Lambda \times \mathbb{C}^2$. Let us relax this notion as follows:

Definition 2.1. *A branched holomorphic motion over Λ is a family of holomorphic graphs over the first coordinate in $\Lambda \times \mathbb{C}^2$.*

Remark 2.2. This definition bears some similarity with the notion of "analytic multifunction", which was studied by Słodkowski, and others. In particular it appears in [Sl] under the name of "locally trivial analytic multifunction".

With $\mathcal{G}$ being such a family, we denote by $\mathcal{G}_\lambda = \{\gamma(\lambda),\ \gamma \in \mathcal{G}\}$, and we say that the sets $\mathcal{G}_\lambda$ are "moving under the branched motion $\mathcal{G}$".

If now $(f_\lambda)_{\lambda\in\Lambda}$ is a family of polynomial automorphisms, we may define a notion of equivariant branched holomorphic motion. Such a branched motion $\mathcal{G}$ over Λ is equivariant if $\gamma \in \mathcal{G}$ implies that $\lambda \mapsto f_\lambda(\gamma(\lambda)) \in \mathcal{G}$ and $\lambda \mapsto f_\lambda^{-1}(\gamma(\lambda)) \in \mathcal{G}$.

Definition 2.3. *A holomorphic family (f_λ) of polynomial automorphisms of $\mathbb{C}^2$ is J^*-stable (resp. weakly J^*-stable) if the sets J^*_λ move under an equivariant (resp. branched) holomorphic motion*

Later on we will see that equivariance is automatically satisfied. The same definition may be applied to J, $J^\pm$, etc. It is easy that if there is a subset $S_\lambda \subset J_\lambda$ which is dense for all parameters and moves under a branched holomorphic motion, then so does J^*_λ (see Lemma 2.8 below). There are (at least) two natural candidates for such a S: a dense set of periodic points or a dense set of s/u intersections. We will see that in addition it is enough that S_{λ_0} is dense in J_{λ_0} for some parameter λ_0.

The following theorem is very much in the spirit of one dimensional dynamics [MSS, Ly1]. It shows that weak J^*-stability is a reasonable notion of stability for polynomial automorphisms.

Theorem 2.4. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. The following are equivalent:*

- *(i.) The sets J^*_λ move under an equivariant branched holomorphic motion over Λ, i.e. (f_λ) is weakly J^*-stable.*
- *(ii.) Every periodic point stays of constant type (saddle, attracting, repelling, indifferent) throughout the family.*
- *(iii.) J^*_λ moves continuously in the Hausdorff topology.*

If furthermore (f_λ) is dissipative, the following two conditions are equivalent, and imply the previous ones:

- *(iv.) The number of attracting cycles is (finite and) locally constant.*
- *(v.) The period of attracting cycles is locally uniformly bounded.*

The following result follows exactly as in dimension 1:

Corollary 2.5. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of dissipative polynomial automorphisms of $\mathbb{C}^2$. If the number of attracting cycles is locally uniformly bounded on Λ, then the locus of weak J^*-stability is open and dense.*

In particular we have the following nice corollary in the spirit of the Palis conjectures. We say that f_λ is a *Newhouse automorphism* if it possesses infinitely many sinks.

Corollary 2.6. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of dissipative polynomial automorphisms of $\mathbb{C}^2$. Then*

$$\{\textit{locally weakly } J^*\textit{-stable parameters}\} \cup \{\textit{Newhouse parameters}\}$$

is dense in Λ.

Proof. Let $B \subset \Lambda$ be the open set where the number of attracting cycles is locally uniformly bounded. By Corollary 2.5, weak J^*-stability is dense in B. Now in B^c, the set

$$U_m = \{\lambda, f_\lambda \text{ possesses at least } m \text{ attracting cycles}\}$$

is relatively open and dense. We conclude by Baire's Theorem. □

According to the work of Buzzard [Bu], it is known that the Newhouse region, i.e. the closure of the set of Newhouse parameters, has non-empty interior in the space of polynomial automorphisms of sufficiently high degree. On the other hand it is an open question whether weak J^* stability implies that there are only finitely many sinks (i.e. whether the conditions *(i.)-(v.)* in Theorem 2.4 are equivalent).

It is also worthwhile to state the following result which will follow from the proof of Theorem 2.4 (see Proposition 2.13 below).

Corollary 2.7. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a weakly stable substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. Then:*

- *the branched motion of the set J^* is unbranched over the set of periodic, homoclinic and heteroclinic points;*
- *homoclinic and heteroclinic tangencies are persistent.*

2.1. **An Extension Lemma.** As in the 1-dimensional setting, we will use extension properties of (branched) holomorphic motions. The classical λ-lemma asserts that a holomorphic motion of $A \subset \mathbb{C}$ extends to $\overline{A}$ and is automatically continuous. Of course neither of these statements is true in higher dimension, whence our use of branched motions. In that respect, the following lemma is obvious.

Lemma 2.8. *If $\mathcal{G}$ is a locally uniformly equicontinuous branched holomorphic motion over Λ, then so is $\overline{\mathcal{G}}$. This holds in particular when $\mathcal{G}_\lambda$ is bounded in $\mathbb{C}^2$, locally uniformly with respect to λ.*

Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of polynomial automorphisms of dynamical degree $d \geq 2$, and denote by $\widehat{f}$ the fibered map $\widehat{f} : (\lambda, z) \mapsto (\lambda, f_\lambda(z))$. If $\mathcal{G}$ is a branched holomorphic motion over Λ, then $\widehat{f}$ acts on $\mathcal{G}$, and equivariance is equivalent to $\widehat{f}$-invariance.

According to Proposition 1.1, (f_λ) is conjugate to a holomorphic family of composition of Hénon mappings. From this it easily follows that the sets K_λ are locally uniformly bounded in $\mathbb{C}^2$. In particular, if $\mathcal{G}$ is a branched motion such that $\mathcal{G}_\lambda \subset K_\lambda$ for all λ, then it is locally uniformly equicontinuous. In the next lemma we show that a similar property holds when $\mathcal{G}_\lambda \subset J^\pm_\lambda \setminus K_\lambda$.

Lemma 2.9. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of polynomial automorphisms of dynamical degree $d \geq 2$. Then the family of holomorphic mappings $\gamma : \Lambda \to \mathbb{C}^2$ such that for every $\lambda \in \Lambda$, $\gamma(\lambda) \in J^+_\lambda \setminus K_\lambda$ (resp. $J^-_\lambda \setminus K_\lambda$) is normal.*

Proof. It is no loss of generality to assume that Λ is the unit disk in $\mathbb{C}$. We treat the case of $J^+_\lambda \setminus K_\lambda$, of course the other one is identical Assume by contradiction that the family of mappings given in the statement is not normal. Then by the Zalcman-Brody lemma there exists a sequence of parameters $\lambda_n \to \lambda_1$, a sequence of reparameterizations $\phi_n : \mathbb{D}_{r_n} \to \Lambda$ and a sequence γ_n with values in $J^+ \setminus K$, such that $\gamma_n \circ \phi_n$ converges to a non-constant entire curve $\zeta : \mathbb{C} \to \overline{J^+_{\lambda_1} \setminus K_{\lambda_1}}$. Now $t \mapsto G^-_{\phi_n(t)}(\gamma_n \circ \phi_n(t))$ is a sequence of positive harmonic functions, thus $G^-_{\lambda_1} \circ \zeta$ is non-negative and harmonic on $\mathbb{C}$, so it is constant. But $J^+ \cap \{G^- = c\}$ is compact, so we get the desired contradiction. $\square$

Though normality is what we need, let us also make a slightly stronger statement:

Lemma 2.10. *Let f be a product of Hénon mappings. For any $R > 0$, the domain*

$$\Omega = U^+ \cap (\{\max(|x|, |y|) < R\} \cup \{|y| < |x|\})$$

is Kobayashi hyperbolic.

Proof. Let us consider a domain

$$Q = \{z = (x, y) \in U^+ : \ |y| < R\,|\varphi^+(z)|\},$$

where φ^+ is the forward Böttcher function. It is well defined since $|\varphi^+|$ is such, and it contains Ω, by increasing R slightly if needed. Let $\widetilde{U}^+$ be the covering of U^+ that makes the Böttcher function φ^+ well defined. Then

$$\widetilde{Q} = \{|Y| < R\,|\Phi|^+\},$$

where capitals mean the lifts to $\tilde{U}$. Then $Z \mapsto (\Phi^+(Z), Y/\Phi^+(Z))$ maps $\widetilde{Q}$ onto the bidisk $(\mathbb{C} \setminus \mathbb{D}) \times \mathbb{D}_R$, with discrete fibers. Since the latter is hyperbolic, so are $\widetilde{Q}$, Q, and Ω (see [Ko, Prop. 1.3.12 and 3.2.9]). □

The main observation of this section is that some *unique* extension results can be obtained for dynamically defined branched holomorphic motions.

Lemma 2.11. *Let $\mathcal{G}$ be a branched holomorphic motion over Λ, such that for every $\lambda \in \Lambda$, $\mathcal{G}_\lambda \subset K_\lambda$ (resp. $\mathcal{G}_\lambda \subset J^+_\lambda \setminus K_\lambda$ or $\mathcal{G}_\lambda \subset J^-_\lambda \setminus K_\lambda$). Assume that (γ_k) is a sequence of graphs in $\mathcal{G}$ such that for some $\lambda_0 \in \Lambda$, $\gamma_k(\lambda_0) \to p(\lambda_0)$, as $k \to \infty$, where $p(\lambda_0)$ belongs to some uniformly hyperbolic invariant compact set E_{λ_0}.*

Then there exists a unique holomorphic continuation $(p(\lambda))_{\lambda\in\Lambda}$ of $p(\lambda_0)$ such that $\gamma_k(\lambda) \to p(\lambda)$ for all λ, as $k \to \infty$. Furthermore $p(\lambda)$ coincides with the natural continuation of p near λ_0 as a point of the hyperbolic set E_λ that dynamically corresponds to E_{λ_0}. In particular if (γ'_k) is any other sequence with $\gamma'_k(\lambda_0) \to p(\lambda_0)$, then $\gamma'_k(\lambda) \to p(\lambda)$.

This holds in particular when $p(\lambda_0)$ is a saddle periodic point (resp. a transverse s/u intersection), in which case $p(\lambda)$ remains periodic (resp. a s/u intersection) throughout $\lambda \in \Lambda$.

The following consequence is obvious.

Corollary 2.12. *Any holomorphic motion of a set $\mathcal{P}$ of saddle points is continuous, and extends continuously to other saddle points and transverse s/u intersections that belong to $\overline{\mathcal{P}_{\lambda_0}}$ for some parameter λ_0.*

Proof of Lemma 2.11. Let us first deal with the case where $\mathcal{G}_\lambda \subset K_\lambda$. Let N be a neighborhood of $\lambda_0 \in \Lambda$ where E_λ persists as a hyperbolic set. Then the point p admits a natural local continuation $(p(\lambda))_{\lambda\in N}$. We claim that in N, $\gamma_k(\lambda) \to p(\lambda)$ when $k \to \infty$. Then the other conclusions of the lemma follow.

Indeed consider any cluster value of the sequence of holomorphic maps $(\gamma_k(\lambda))_{\lambda\in\Lambda}$ (this is possible by the equicontinuity assumption). By our claim it has to coincide with $p(\lambda)$ in N. This in turn allows to define a holomorphic continuation $p(\lambda)$ of p throughout Λ.

If $p(\lambda_0)$ is a periodic point, then by analytic continuation of the identity $f^N_\lambda(p(\lambda)) = p(\lambda)$, $p(\lambda)$ remains periodic in the family.

Likewise, if $p(\lambda_0)$ is a transverse s/u intersection, then for λ close to λ_0, there exists saddle periodic points $p_1(\lambda)$ and $p_2(\lambda)$ such that $p(\lambda) \in W^s(p_1(\lambda)) \cap W^u(p_2(\lambda))$ (with possibly $p_1 = p_2$). In particular $f^n(p(\lambda)) \to p_1(\lambda)$ and $f^{-n}(p(\lambda)) \to p_2(\lambda)$ near the origin. Since $(f^n(q(\lambda)))_{\lambda\in\Lambda}$ and $(f^{-n}(q(\lambda)))_{\lambda\in\Lambda}$ are normal families of holomorphic mappings, we infer that p_1 and p_2 admit holomorphic continuations to Λ, necessarily as periodic points as before, and $f^n(p(\lambda)) \to p_1(\lambda)$ and $f^{-n}(p(\lambda)) \to p_2(\lambda)$ throughout Λ.

It remains to prove our claim that $\gamma_k(\lambda) \to p(\lambda)$ in some neighborhood of λ_0. The observation is that for $\lambda \in N$, the dynamics is locally expansive near $p(\lambda)$, that is: there exists $\delta > 0$, which can be chosen to be uniform in N (reducing N if needed), such that if $q(\lambda)$ is such that $d(f_\lambda^n(q(\lambda)), f_\lambda^n(p(\lambda))) \leq \delta$ for all $n \in \mathbb{Z}$, then $p(\lambda) = q(\lambda)$. Now let q be any cluster value of the sequence of graphs $\gamma_k(\lambda)$, and consider the family $(\widehat{f}^n(q))_{n\geq 0}$. This is a bounded, hence normal, family of graphs (since they are contained in $\bigcup J_\lambda$), and by assumption, $\widehat{f}^n(q)(\lambda_0) = f_{\lambda_0}^n(q(\lambda_0)) = f_{\lambda_0}^n(p(\lambda_0))$. Therefore by equicontinuity, for λ close to λ_0, $f_\lambda^n(q(\lambda))$ remains close to $f_\lambda^n(p(\lambda))$ and we are done.

Assume now that for all $\lambda \in \Lambda$, $\mathcal{G}_\lambda \subset J_\lambda^+ \backslash K_\lambda$. By Lemma 2.9, we can extract a subsequence, still denoted by γ_k, such that γ_k converges to some $\gamma : \Lambda \to \mathbb{C}^2$ with $\gamma(\lambda_0) = p(\lambda_0)$. We claim that γ takes its values in K. Indeed $\lambda \mapsto G_\lambda^-(\gamma_k(\lambda))$ is a sequence of positive harmonic functions, so its limit $\lambda \mapsto G_\lambda^-(\gamma(\lambda))$ is harmonic and non-negative, and we conclude by observing that $G_{\lambda_0}^-(\gamma(\lambda_0)) = 0$, whence $\lambda \mapsto G_\lambda^-(\gamma(\lambda))$ vanishes identically. Then by applying the first part of the proof we deduce that $\gamma(\lambda) = p(\lambda)$, which was the desired result. □

2.2. **Proof of Theorem 2.4.** Let us start with the following statement.

Proposition 2.13. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$, and assume that there exists a branched holomorphic motion $\mathcal{G}$ such that:*

- *for every λ, $\mathcal{G}_\lambda \subset K_\lambda$;*
- *for some λ_0, $\mathcal{G}_{\lambda_0}$ is dense in $J_{\lambda_0}^*$.*

Then:

(a) no periodic point can change its nature (saddle, sink, source, indifferent) in the family (f_λ);
(b) the set of saddle points of (f_λ) moves holomorphically and continuously;
(c) for every λ in Λ, $\overline{\mathcal{G}_\lambda} \supset J_\lambda^*$;
(d) the family $(f_\lambda)_{\lambda\in\Lambda}$ is weakly J^*-stable;
(e) the set of s/u intersections moves holomorphically and continuously;
(f) homoclinic and heteroclinic tangencies are persistent.

It follows from this proposition that $(i.) \Leftrightarrow (ii.)$ in Theorem 2.4. Indeed, if J_λ^* is moving under a branched holomorphic motion, then *i.* above says that the periodic points do not change type. Conversely, assume that the periodic points of (f_λ) stay of constant type throughout Λ, and fix a parameter λ_0. Let $\mathcal{S}_{\lambda_0}$ be the set of saddle points of λ_0, which is dense in $J_{\lambda_0}^*$. By assumption, these periodic points remain of saddle type in the family, so give rise to a branched holomorphic motion $\mathcal{S}_\lambda$ satisfying the assumptions of the theorem. By *iii.*, (f_λ) is weakly J^*- stable.

Here are some consequences, the proof of which will be given afterwards.

Corollary 2.14. *Under the assumptions of the theorem, if there exists a persistent set of saddle points, which is dense for some parameter, then the family is weakly J^*-stable.*

Corollary 2.15. *Under the assumptions of the theorem, if furthermore (f_λ) is J^*-stable in a neighborhood of λ_0 (for instance of f_{λ_0} is uniformly hyperbolic on $J_{\lambda_0}^*$), then for $\lambda \in \Lambda$ there is a semiconjugacy $J_{\lambda_0}^* \to J_\lambda^*$.*

Corollary 2.16. *If a substantial family has the property that its members are topologically conjugate on J^*, and the conjugating map depends continuously on λ, then it is J^*- stable.*

Remark 2.17. Notice that in Proposition 2.13 we do not assume any equivariance for $\mathcal{G}$. This shows that the equivariance assumption is superfluous in Definition 2.3.

Remark 2.18. It follows from Proposition 2.13 and from the density of transverse homoclinic intersections in J^* at every parameter that if all homoclinic intersections can be followed in some $\Omega \subset \Lambda$, then (f_λ) is weakly J^*-stable there. This a priori does *not* imply that weak J^*-stability follows from the absence of homoclinic tangencies. Indeed, if there are no tangencies in Ω, every homoclinic intersection can be followed locally. The point is that intersections may disappear by "slipping off to infinity" inside stable and unstable manifolds. The methods that we develop in Part 2 of the paper may be seen as a way of circumventing this problem.

Proof of Proposition 2.13. Let $\mathcal{S}_{\lambda_0}$ be the set of saddle periodic points of f_{λ_0}. By Lemma 2.11 we can follow holomorphically the points of $\mathcal{S}_{\lambda_0}$ along Λ, giving rise to a branched holomorphic motion $\mathcal{S}$. In the statement of the theorem, we can now replace $\mathcal{G}$ by $\mathcal{S}$. Notice that $\mathcal{S}$ is $\widehat{f}$-invariant.

Let us show [1] that for every λ, $\overline{\mathcal{S}_\lambda} \supset J^*_\lambda$. By [BLS2], if we denote by $\mathcal{S}_{n,\lambda_0}$ the set of saddle orbits with period dividing n, $\#\mathcal{S}_n/d^n \to 1$ and $d^{-n}\sum_{p\in\mathcal{S}_{n,\lambda_0}}\delta_p \to \mu_{\lambda_0}$. Hence $\mathcal{S}_{n,\lambda}$ is a set of periodic points with $\#\mathcal{S}_{n,\lambda} = \#\mathcal{S}_n \sim d^n$. Thus, by the equidistribution theorem for f_λ we get that $d^{-n}\sum_{p\in\mathcal{S}_{n,\lambda}}\delta_p \to \mu_\lambda$, and our claim follows.

Since clearly $\mathcal{S}_\lambda \subset \overline{\mathcal{G}_\lambda}$, we see that $\overline{\mathcal{G}_\lambda} \supset J^*_\lambda$ for $\lambda \in \Lambda$, which proves *(c)*.

To get *(a)* and *(b)* we need the following:

Lemma 2.19. *For every $\lambda \in \Lambda$, every point in $\mathcal{S}_\lambda$ is a saddle point.*

Proof. It is no loss of generality to assume that Λ has dimension 1. Let $q \in \mathcal{S}$. We know that $q(\lambda)$ is periodic for every λ, and of saddle type near λ_0.

Suppose first for simplicity that for every λ, f_λ is dissipative. Then if $q(\lambda)$ bifurcates, it must become a sink in an open set of parameters. But by definition, $q(\lambda) = \lim p_n(\lambda)$, where for λ close to zero we may assume that the $p_n(\lambda)$ are distinct. Thus outside countably many exceptional parameters in Λ, the $p_n(\lambda)$ remain distinct. We conclude that when $q(\lambda)$ is a sink and the parameter is non- exceptional, $q(\lambda)$ is a non-trivial limit of periodic points, which is contradictory.

Let us now address the general case[2]. We start with a saddle periodic point q, that we can follow holomorphically as $(q(\lambda))_{\lambda\in\Lambda}$. We want to show that it cannot change type, i.e. that neither of its eigenvalues crosses the unit circle. Since the eigenvalues of $q(\lambda)$ are not locally constant (this is forbidden by the "substantiality" assumption), at a bifurcating parameter they run through an open arc of the unit circle. Thus we may always assume that we the eigenvalues are far from 1, so that we can follow them holomorphically.

If a bifurcation occurs in the locus where $|\mathrm{Jac}\ f_\lambda| \neq 1$ then a sink or source will be created and we conclude as before. In the remaining case, elliptic points are created. Recall that the possibility of linearizing a periodic point depends on a Diophantine condition on the multipliers. To be specific, a sufficient condition for linearizability is that there exists $\nu > 0$ such that for $j_1, j_2 \geq 1$ and $k = 1, 2$, $\left|\alpha_1^{j_1}\alpha_2^{j_2} - \alpha_k\right| \geq \frac{C}{(j_1+j_2)^\nu}$.

[1]This is not obvious for f_0 could possibly have infinitely many sinks becoming saddles during the deformation

[2]The argument is similar to that of [BLS2, Theorem 3] but the possibility of persistent non-linearizability, e.g. persistent resonance between the eigenvalues, was apparently overlooked there. This is the reason for the additional assumption that (f_λ) is substantial.

Consider a piece C of the curve $\{|\mathrm{Jac}\, f_\lambda| = 1\}$ in parameter space, and a point $\lambda_1 \in C$ where $|\alpha_1| = |\alpha_2| = 1$. We have two possibilities. Either $|\alpha_1| = |\alpha_2| = 1$ along C or not. In the latter case there is a branch of the curve $|\alpha_1| = 1$ having an isolated intersection with C, so we have bifurcations in the dissipative regime and we are done. In the first case, we claim that $q(\lambda)$ cannot be persistently non-linearizable along C. Then, at a parameter where $q(\lambda)$ is linearizable, it is the center of a Siegel ball, and we get a contradiction in the same way as in the dissipative case.

To prove our claim, we consider the functions $\alpha_k(\lambda) = e^{i\theta_k(\lambda)}$, $k = 1, 2$ where λ varies along C. These are real analytic and since the family is substantial, $(\theta_1, \theta_2, 1)$ are linearly independent. It is then a theorem of Schmidt [Sc] (solving a conjecture of Sprindzhuk's, see also [KM]) that for a.e. λ, $(\alpha_1(\lambda), \alpha_2(\lambda))$ is Diophantine. This concludes the proof. □

Let us resume the proof of the theorem. Since $\mathcal{S}_\lambda$ is dense for all λ, we can start with any parameter λ_0, and *(a)* follows from the previous lemma. Item *(b)* then follows from Corollary 2.12, and *(d)* is obvious.

For *(e)*, let q be a transverse point of intersection of $W^s(p_1)$ and $W^u(p_2)$ (for some parameter). The saddle points p_1 and p_2 persist in the family. Since $f_\lambda^n(q(\lambda))$ is a normal family and $f_\lambda^n(q(\lambda)) \to p_1(\lambda)$ for λ close to 0, this holds throughout Λ. In particular $q(\lambda) \in W^s(p_1(\lambda)) \cap W^u(p_2(\lambda))$ for all parameters so it remains an s/u intersection.

From the previous paragraph, we know that all s/u intersections that are transverse at some parameter λ_0 can be followed holomorphically throughout Λ. To establish *(f)*, we show that they stay transverse for all λ, or equivalently, that there are no collisions (notice that tangencies are not a priori incompatible with the fact that intersections are moving holomorphically, due to the possibility of degenerate tangencies).

Consider a pair $q(\lambda_0)$, $q'(\lambda_0)$ of distinct transverse intersections of $W^s(p_1(\lambda_0))$ and $W^u(p_2(\lambda_0))$. We know that q, q' (as well as p_1, p_2) can be followed holomorphically. We have to show that q and q' stay distinct. For this we parameterize $W^u(p_2(\lambda))$ by some $\phi_\lambda : \mathbb{C} \to W^u(p_2(\lambda))$, depending holomorphically on λ (see the comments preceding Lemma 3.2 below), so we may identify $W^u(p_2(\lambda))$ with $\mathbb{C}$. Fix another saddle point $p_3(\lambda)$. Since $W^s(p_3(\lambda_0))$ intersects transversally $W^u(p_1(\lambda_0))$, by the hyperbolic Lambda (or inclination) lemma we get that $q(\lambda_0)$ is the limit, inside $\mathbb{C} \simeq W^u(p_2(\lambda_0))$ of a sequence of transverse intersection points $q_n(\lambda_0)$ of $W^s(p_3(\lambda_0)) \cap W^u(p_2(\lambda_0))$. These intersection points can be followed globally in Λ.

We claim that $q_n(\lambda)$ converges locally uniformly to $q(\lambda)$ in Λ (again here we work in $\mathbb{C}$). Indeed notice first that by Montel's theorem q_n is a normal family, since locally we can follow any finite set of transverse intersections of $W^s(p_1(\lambda)) \cap W^u(p_2(\lambda))$, and $q_n(\lambda)$ stays disjoint from them. Then we argue that $q_n(\lambda_0)$ converges to $q(\lambda_0)$ while $q_n(\lambda)$ is disjoint from $q(\lambda)$, so by Hurwitz' theorem $q_n(\lambda)$ converges to $q(\lambda)$.

To conclude the argument, observe that if $q(\lambda)$ had a collision with $q'(\lambda)$ at some parameter λ_1, then by the argument principle, $q'(\lambda)$ would collide with $q_n(\lambda)$ close to λ_1, which is impossible because they belong to different stable manifolds.

Hence, item *(f)* is established, which finishes the proof of the proposition. □

Conclusion of the proof of Theorem 2.4. We already know that *(ii.)* and *(i.)* are equivalent. Let us deal with condition *(iii.)*. It follows from the equicontinuity of the family of graphs, that if (f_λ) is weakly J^* stable, $\lambda \mapsto J^*_\lambda$ is continuous for the Hausdorff topology. So *(i.)* implies *(iii.)*.

Conversely, *(iii.)* implies *(ii.)*. Indeed if a periodic point changes type, then arguing as in Lemma 2.19, we see that for some λ, a multiplier of the cycle must cross the unit circle at a linearizable parameter. So at this parameter a Siegel ball or Siegel/attracting basin is created, and the corresponding periodic orbit jumps outside J^*, thus preventing continuity of J^*.

To conclude the proof we show the (rather obvious) chain of implications:

$$(iv.) \Rightarrow (v.) \Rightarrow (i.) + \text{ the number of non-saddle cycles is finite } \Rightarrow (iv.).$$

Indeed $(iv.) \Rightarrow (v.)$ is clear. Next, if *(v.)* holds, then all periodic points of sufficiently high prime period are (necessarily persistent) saddles, so by Corollary 2.14, the family is weakly J^*-stable. Therefore, all periodic points are of constant type, hence *(iv.)* holds. □

Proof of Corollary 2.15. By Lemma 2.11 we can follow holomorphically all points of J^*_0. The motion is continuous near the origin, so by analytic continuation we easily deduce that it is continuous throughout Λ. Likewise, the compatibility between the motion and the dynamics holds near the origin so it holds everywhere, hence it defines a global semiconjugacy. □

Proof of Corollary 2.16. Fix a parameter $0 \in \Lambda$. For every λ there is a conjugacy $h_\lambda : J^*_0 \to J^*_\lambda$. If p_0 is a saddle point for f_0, then $h_\lambda(p_0) = p_\lambda$ is a continuously moving periodic point which is a limit of periodic points for all λ. By Lemma 2.19, p_λ is a saddle for all parameters. In particular the assumptions of the stability theorem hold, and the family f_λ is weakly J^*- stable, thus all saddle points move holomorphically. To conclude, let $p \in J^*$ be any point, and let a sequence of saddle points $p_n \to p$. Then for all λ, $h_\lambda(p_n) \to h_\lambda(p)$, so $\lambda \mapsto h_\lambda(p)$ is holomorphic. Finally, it is obvious that the motion is injective since h_λ is a homeomorphism. □

3. Extension of a branched motion

In this section we capitalize on the idea that in a weakly J^*-stable family of polynomial automorphisms, we can apply the one-dimensional theory of holomorphic motions inside stable and unstable manifolds. In §3.2 we show that connectivity properties of Julia sets are preserved in a weakly J^*-stable family. Conversely, if in some dissipative family (f_λ), the Julia set is persistently connected, then the family is weakly J^*-stable (Theorem 3.5). In §3.3 we prove that in a weakly J^*-stable family, the equivariant branched holomorphic motion of J^* actually extends to $J^+ \cup J^-$.

3.1. **Motion of $\partial(W^u(p) \cap K^+)$.** Let f be a polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 2$ and p be a saddle periodic point. Then $W^u(p)$ is biholomorphic to $\mathbb{C}$ and dense in J^-. In particular there are two distinct topologies on $W^u(p)$: the one induced by the isomorphism with $\mathbb{C}$, which we refer to as the *intrinsic topology*, and the topology induced from $\mathbb{C}^2$. Viewed as subsets of $\mathbb{C}$, the components of $W^u(p) \cap K^+$ are simply connected closed subsets, which may be bounded or unbounded. Notice that, with our current state of knowledge, nothing prevents a component of $W^u(p) \cap K^+$ with non-empty interior from being fully contained in J or even in J^*.

The structure of the boundary of components is described in the next lemma. We denote by $H(p)$ the set of homoclinic intersections of $W^u(p)$ and $W^s(p)$. Respectively, $H^{\mathrm{tr}}(p)$ will stand for the subset of transverse homoclinic intersection.

Lemma 3.1. *Relative to the intrinsic topology in $W^u(p) \simeq \mathbb{C}$ we have that*

$$\partial(W^u(p) \cap K^+) = \overline{H(p)} = \overline{H^{\rm tr}(p)}.$$

Proof. This is very similar to [BLS1, §9] (see also the proof of [Du, Cor. 1.9]). We freely use the formalism of laminar currents and Pesin boxes, the reader is referred to [BLS1] for details.

Let $x \in \partial(W^u(p) \cap K^+)$ and let us show that $x \in \overline{H^{\rm tr}(p)}$. Since the dynamical Green function G^+ admits a non-trivial minimum at x, if $\Delta \subset W^u(p)$ is a disk containing x, then G^+ is not harmonic in Δ, i.e. $T^+ \wedge [\Delta] > 0$. Let ψ be a cut-off function in Δ, with $\psi = 1$ near x. By [BS3, Thm 1.6], $d^{-n}(f^n)_*(\psi[\Delta]) \to cT^-$ as $n \to \infty$, with $c = \int \psi[\Delta] \wedge T^+ > 0$. We now argue exactly as in [BLS1, Lemma 9.1]. Let P be a Pesin box of positive μ-measure, and S^+ be the uniformly laminar current made of the local stable manifolds $W^u_{\rm loc}(x)$, $x \in P$, with transverse measure given by the unstable conditionals of μ. Then $0 < S^+ \leq T^+$ so S^+ has continuous potential [BLS1, Lemma 8.2]. It follows that $d^{-n}(f^n)_*(\psi[\Delta]) \wedge S^+ \to cT^- \wedge S^+ > 0$ and we conclude that for large n, $f^n(\Delta)$ admits transverse intersection points with $W^s_{\rm loc}(x)$, for some $x \in P$ (the transversality comes from [BLS1, Lemma 6.4]).

We claim that iterating a bit further, the iterates of Δ intersect $W^s_{\rm loc}(x)$, for *every* $x \in P$. Indeed, for y sufficiently close to x, $f^n(\Delta)$ intersects $W^s_{\rm loc}(y)$. By Poincaré recurrence, we can assume that there exists an infinite sequence $(n_j)_{j\geq 1}$, such that $f^{n_j}(y) \in P$. By the Stable Manifold Theorem, for large j, $f^{n+n_j}(\Delta)$ contains a disk close to $W^u_{\rm loc}(f^{n_j}(y))$, and we are done.

We now apply exactly the same argument to a neighborhood of p in $W^s(p)$. In this way we get disks in $W^s(p)$, arbitrary close to $W^s_{\rm loc}(z)$ for some $z \in P$, from which we get that $f^{n+n_j}(\Delta)$, hence Δ, intersects $W^s(p)$ transversely, and we are done.

The fact that $\overline{H(p)} = \overline{H^{\rm tr}(p)}$ was established in $\mathbb{C}^2$ in [BLS1, Prop. 9.8], but the proof works in the intrinsic topology of $W^u(p)$ as well. For convenience let us recall the argument: since $W^u(p)$ admits non-trivial transverse intersections with $W^s(p)$, it follows from the hyperbolic Lambda lemma that every disk $\Delta \subset W^u(p)$ is the limit of an infinite sequence $\Delta_n \subset W^u(p)$. Hence the result follows from the instability of non-transverse intersections [BLS1, Lemma 6.4].

To conclude the proof, let us show that $H^{\rm tr}(p) \subset \partial(W^u(p) \cap K^+)$. Observe first that $p \in \partial(W^u(p) \cap K^+)$. Indeed otherwise p would lie in the interior of $W^u(p) \cap K^+$, hence by invariance we would conclude that $W^u(p) \subset K^+$, a contradiction. Let now $x \in H^{\rm tr}(p) \subset W^u(p) \cap K^+$ and Δ be a small disk around x. By the hyperbolic Lambda lemma, $f^n(\Delta)$ contains graphs arbitrary close to a neighborhood of p in $W^u(p)$, so $f^n(\Delta)$ must intersect U^+ and we are done. □

Let now $(f_\lambda)_{\lambda\in\Lambda}$ be a weakly J^*-stable family of polynomial automorphisms. Fix a holomorphically moving saddle point (p_λ). Then there exists a holomorphic family of parameterizations $\psi^u_\lambda : \mathbb{C} \to W^u(p_\lambda)$ of $W^u(p_\lambda)$, with $\psi^u_\lambda(0) = p_\lambda$. Indeed, if for $\lambda = \lambda_0$, we fix any transverse intersection q of $W^u(p_{\lambda_0})$ and $W^s(p_{\lambda_0})$, then this intersection can be followed throughout λ, and we normalize ψ^u_λ by declaring that $\psi^u_\lambda(1) = q(\lambda)$.

The following easy lemma is the starting point for most of the results in this section.

Lemma 3.2. *Let (f_λ) be a weakly J^*-stable family of polynomial automorphisms, (p_λ) be a holomorphically moving saddle point and (ψ^u_λ) be a holomorphic family of parameterizations* *of $W^u(p_\lambda)$, as above. Then $(\psi^u_\lambda)^{-1}(\partial(W^u(p_\lambda)\cap K^+_\lambda))$ moves holomorphically in $\mathbb{C}$, where the boundary ∂ is taken relative to the intrinsic topology.*

Proof. By Proposition 2.13 *(e)* and *(f)*, homoclinic intersections move holomorphically and without collisions. Therefore the result follows directly from Lemma 3.1 and the ordinary λ-lemma. □

3.2. **Persistently connected Julia sets.** Let us start with a few general comments on connectivity of Julia sets of polynomial automorphisms.

Of course, if J is totally disconnected, then so is J^*. Moreover, by [BS3], every component of K intersects J^*, so in particular if K is totally disconnected, then $J = J^* = K$. It follows from general topology that if J is totally disconnected then so is K: indeed the boundary of a non-trivial continuum cannot be totally disconnected. On the other hand it is unclear whether total disconnectedness of J^* implies that J is totally disconnected.

By [BS6], J is disconnected iff f is stably and unstably disconnected (which means, by definition that $U^\pm \cap W^u(p)$ is simply connected for some (and then any) saddle point p). It is not difficult to see that K is disconnected in this case (an inclination lemma argument). Likewise, it follows from [BS6] that J is connected iff J^* is connected. We also remark that a dissipative map is always stably disconnected [BS6, Cor. 7.4].

We start by observing that the connectedness of J is preserved in weakly J^*-stable families.

Proposition 3.3. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a weakly J^*-stable substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. Then if for some parameter λ_0, J_{λ_0} is connected, then J_λ is connected for all λ.*

Proof. Let us show that disconnectedness of J is preserved in a weakly J^*-stable family. So assume that for some λ, f_λ is stably and unstably disconnected, so for every saddle point p, $W^u(p)\cap K^+$ and $W^s(p)\cap K^-$ admit compact components. By Lemma 3.2, if C_{λ_0} is any compact component of $W^u(p_{\lambda_0})\cap K^+_{\lambda_0}$, ∂C_{λ_0} moves holomorphically as the parameter evolves, without colliding with the other components, so its continuation $(\partial C)_\lambda$ bounds a compact component of $W^u(p_\lambda)\cap K^+_\lambda$. Hence f_λ is unstably disconnected at all parameters, and the same argument shows that it remains stably disconnected as well. We conclude that J_λ is disconnected for every $\lambda \in \Lambda$. □

In the same way we obtain that if $(f_\lambda)_{\lambda\in\Lambda}$ is weakly J^*-stable and if for some parameter λ_0, J_{λ_0} is totally disconnected, then for all λ, f_λ is stably and unstably totally disconnected. However it seems to be unknown whether stable and unstable total disconnectedness implies that J or J^* is totally disconnected.

We will now show that conversely, preservation of connectivity properties implies stability. Our arguments require dissipativity. The first result is a fairly straightforward consequence of Lemma 3.2.

Proposition 3.4. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of dissipative polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. If J^*_λ is totally disconnected for all λ, then (f_λ) is weakly J^*-stable.*

Proof. Indeed, if J^* is totally disconnected, then if p is any saddle periodic point, $W^u(p)\cap J^*$ is totally disconnected. On the other hand it is well known that any attracting periodic orbit produces a component of $W^u(p)\cap K^+$ with non-empty interior. Let us recall the argument: the basin of a sink is biholomorphic to $\mathbb{C}^2$, so it contains entire curves. To any such entire curve, we may attach a positive closed current by the Ahlfors trick (see e.g. [Nv, §7.4 pp.

349-350]). Now since K^+ supports a unique positive closed current of mass 1 [Si], this current must be T^+. Then we argue as in Lemma 3.1: we fix a Pesin box and construct a local current $S^- \leq T^-$ made of local Pesin stable manifolds. Since S^- has continuous potential, the entire curve must intersect it, more precisely we get a transverse intersection with some local stable leaf $W^u_{\rm loc}(x)$. Finally, $W^u(p)$ contains disks arbitrary close to $W^u_{\rm loc}(x)$ and we are done.

We conclude that if J^*_λ is totally disconnected for all λ (and more generally if all components of $J^*_\lambda \cap W^u(p)$ have empty interior) there are no sinks, thus the family is weakly J^*-stable. □

Our next result asserts that persistent connectedness of J also implies stability. As for the analogous statement for polynomials in $\mathbb{C}$, the argument is ultimately based on the absence of "escaping critical points".

Theorem 3.5. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of dissipative polynomial automorphisms of dynamical degree $d \geq 2$, and assume that for every $\lambda \in \Lambda$, the Julia set J_λ is connected. Then the family (f_λ) is weakly J^*-stable.*

Notice that by Proposition 1.1, we may normalize the family so that f_λ is a product a Hénon mappings, depending holomorphically on λ. The following structure theorem for $J^- \setminus K^+$ was obtained in [BS6]: if f is a composition of Hénon mappings such that J^* (hence J) is connected, then $J^- \setminus K^+$ is a lamination, whose leaves are universal coverings over $\mathbb{C} \setminus \overline{\mathbb{D}}$ under the Böttcher function φ^+, which in this case is well-defined on $J^- \setminus K^+$. We will refer to $J^- \setminus K^+$ as the *solenoid* S.

Theorem 3.5 will follow from the following result of independent interest.

Proposition 3.6. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of dissipative polynomial automorphisms of dynamical degree $d \geq 2$, and assume that for every $\lambda \in \Lambda$, the Julia set J_λ is connected. Then the solenoids S_λ move under an equivariant holomorphic motion that preserves the level sets of the Böttcher function φ^+_λ.*

Proof of Proposition 3.6. Fix $\lambda_0 \in \Lambda$. We will show that S_λ moves holomorphically in some neighborhood of λ_0. Pick a saddle point p_0 for f_{λ_0}. This saddle point persists in some neighborhood of λ. Consider the unstable manifold $W^u(p)$, which is parameterized by some $\psi : \mathbb{C} \to W^u(p)$. There exists some neighborhood of λ where p_0 as well as ψ^u can be followed holomorphically.

By [BS6, Theorem 4.11], applied to the normalized counting (atomic) measure ν on the orbit of p, we get that $W^u(p_0) \setminus K^+$ has only finitely many connected components , and by [BS6, Theorem 2.1], each of them is dense in the solenoid. At some parameter λ_0, let $\mathcal{O}_{\lambda_0}$ be such a component and $z_0 = \psi^u_{\lambda_0}(t_0) \in \mathcal{O}_{\lambda_0}$. For λ close to λ_0, $z_\lambda = \psi^u_\lambda(t_0)$ escapes under f_λ so $\mathcal{O}_{\lambda_0}$ (viewed as a subset of $\mathbb{C}$) persists in some neighborhood Ω of λ_0, which we may assume is biholomorphic to a ball. We denote by $\mathcal{O}_\lambda$ the corresponding component. Now look at the countable set $\mathcal{O}_\lambda \cap \{\varphi^+_\lambda = \varphi^+_{\lambda_0}(z_0)\}$. We claim that these points move holomorphically in λ as λ ranges through Ω. Since $\mathcal{O}_\lambda \cap \{\varphi^+_\lambda = \varphi^+_{\lambda_0}(z_0)\}$ is a dense subset of the transversal $\{\varphi^+_\lambda = \varphi^+_{\lambda_0}(z_0)\} \cap J^-_\lambda$, we deduce that $\{\varphi^+_\lambda = \varphi^+_{\lambda_0}(z_0)\} \cap J^-_\lambda$ moves holomorphically with λ (here we are applying the ordinary λ-lemma in the one dimensional submanifold $\{\varphi^+_\lambda = \varphi^+_{\lambda_0}(z_0)\}$). Since this reasoning is valid in any fiber $S_\lambda \cap \left\{\varphi^+_\lambda = c\right\}$ of the solenoid, we conclude that S_λ moves holomorphically, as asserted (the equivariance is obvious).

It remains to prove our claim. At the parameter λ_0 write $\mathcal{O} \cap \{\varphi^+ = \varphi^+_{\lambda_0}(z_0)\} = \{z_n\}$. Transversality implies that for every n, z_n can be followed holomorphically as a solution of

$\varphi^+_\lambda = \varphi^+_{\lambda_0}(z_0)$ in some neighborhood of λ_0. The point is to show that this neighborhood can be chosen to be uniform with n (see also Remark 2.18). Following [BS6], introduce the symbolic solenoid

$$\Sigma = \left\{ u = (u_j) \in (\mathbb{C} \setminus \mathbb{D})^{\mathbb{Z}},\ u_j^d = u_{j+1} \right\},$$

which is naturally a foliated space. The Böttcher function φ^+_λ can actually be extended to $\mathcal{O}_\lambda$ so we can define a mapping $\Phi_\lambda : J^-_\lambda \setminus K^+_\lambda \to \Sigma$ by the formula

$$\Phi_\lambda(x) = (\varphi^+_\lambda(f^n(x)))_{n\in\mathbb{Z}},$$

which clearly depends holomorphically on λ. Bedford and Smillie prove that Φ_λ is a holomorphic bijection onto a leaf L of Σ.

Let now $z_n \in \mathcal{O} \cap \left\{ \varphi^+ = \varphi^+_{\lambda_0}(z_0) \right\}$, and let U be a neighborhood of λ_0 where z_n can be followed holomorphically as $z_n(\lambda)$. Notice that for $\lambda \in U$, $\Phi_\lambda(z_n(\lambda))$ is constant. Indeed, $\varphi_\lambda(z_n(\lambda)) = \varphi^+_{\lambda_0}(z_0)$ is constant by definition, therefore the set of possible values of $\Phi_\lambda(z_n(\lambda))$ is discrete in the leaf L, hence the result. We thus see that we can extend holomorphically the map $\lambda \mapsto z_n(\lambda)$ throughout Ω by simply putting $z_n(\lambda) = \Phi_\lambda^{-1}\Phi_{\lambda_0}(z_n)$. This completes the proof. $\square$

Proof of Theorem 3.5. By Proposition 3.6, each fiber $J^-_\lambda \cap \left\{\varphi^+_\lambda = \varphi^+_\lambda(z_0)\right\}$ moves holomorphically with λ. Now we observe that for every parameter λ, $f_\lambda^{-n}(\{\varphi^+_\lambda = c\} \cap J^-_\lambda)$ clusters on the whole of J^*_λ as $n \to \infty$: indeed, for μ-a.e. $x \in J^*$, $W^u(x)$ intersects $\{\varphi^+ = c\}$. In addition, $(f_\lambda^{-n}(\{\varphi^+_\lambda = \varphi^+_\lambda(z_0)\} \cap J^-_\lambda))_{n\geq 0}$ is locally uniformly bounded in $\mathbb{C}^2$. We conclude that the set of cluster values of the preimages of the holomorphically moving points $\{\varphi^+_\lambda = \varphi^+_{\lambda_0}(z_0)\} \cap J^-_\lambda$ form a branched holomorphic motion relating the J^*_λ and we are done (recall that by Remark 2.17 we needn't check that this motion is equivariant). $\square$

3.3. **Extension to the big Julia set $J^+ \cup J^-$.** Recall that a family (f_λ) is said to be X-stable if the sets X_λ move under an equivariant branched holomorphic motion. We now prove the equivalence of several notions of weak stability.

Theorem 3.7. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of dissipative polynomial automorphisms of dynamical degree $d \geq 2$. The following properties are equivalent:*

(i) $(f_\lambda)_{\lambda\in\Lambda}$ *is weakly J^*-stable.*
(ii) $(f_\lambda)_{\lambda\in\Lambda}$ *is weakly J^--stable.*
(iii) $(f_\lambda)_{\lambda\in\Lambda}$ *is weakly J^+-stable.*
(iv) $(f_\lambda)_{\lambda\in\Lambda}$ *is weakly K-stable.*
(v) $(f_\lambda)_{\lambda\in\Lambda}$ *is weakly $J^- \cap U^+$-stable (resp. $J^+ \cap U^-$-stable).*

Moreover, if $(f_\lambda)_{\lambda\in\Lambda}$ is weakly J-stable, then the properties (i)-(v) *hold.*

Remark 3.8. We do not know if weak stability implies weak J-stability since we cannot rule out a scenario where under a branched motion of K, a point in J moves out to $\operatorname{Int} K^+$.

Proof. We start by proving that (i) implies (ii) (of course (i) implies (iii) is similar). So let us assume that $(f_\lambda)_{\lambda\in\Lambda}$ is a weakly J^*-stable family of dissipative polynomial automorphisms. Fix a holomorphically moving saddle point p_λ, and consider a holomorphic family of parameterizations $\psi^u_\lambda : \mathbb{C} \to W^u(p_\lambda)$. By Lemma 3.2, $(\psi^u_\lambda)^{-1}(\partial(W^u(p_\lambda) \cap K^+))$ moves holomorphically. Working locally, we apply the Bers-Royden canonical extension of holomorphic motions [BR] to obtain a holomorphic motion of the whole $\mathbb{C}$. The extension being canonical, it is automatically equivariant (the dynamics here is just multiplication by the unstable multiplier), so in this way we get an equivariant holomorphic motion of $W^u(p_\lambda)$ in $\mathbb{C}^2$. For $z \in W^u(p_{\lambda_0})$, we denote its continuation by $z(\lambda)$.

The main step of the proof is to show that the induced family of mappings $\Lambda \to \mathbb{C}^2$ is normal. For this, we need the following lemma.

Lemma 3.9. *Let $(f_\lambda)_{\lambda\in\Lambda}$ is a weakly J^*-stable family of dissipative polynomial automorphisms as above. Then the Bers-Royden holomorphic motion of $W^u(p_\lambda)$ preserves the decomposition $\mathbb{C}^2 = K^+ \cup U^+$.*

Proof. For $\lambda_0 \in \Lambda$, let C_{λ_0} be a connected component of $(\psi^u_{\lambda_0})^{-1}(W^u(p_{\lambda_0}) \cap K^+_{\lambda_0})$ and C_λ its image under the Bers-Royden motion. We have to show that for every $\lambda \in \Lambda$, C_λ is a connected component of $(\psi^u_\lambda)^{-1}(W^u(p_\lambda) \cap K^+_\lambda)$. The topological structure, as well as the bounded/unbounded character of C_λ, is preserved under the motion. From Lemma 3.1 we know that for every λ, $\partial C_\lambda \subset J^*_\lambda$. So, we need to show that for every $\lambda \in \Lambda$, $\mathrm{Int}(C_\lambda) \subset K^+_\lambda$. If Ω_{λ_0} is a bounded connected component of C_{λ_0}, it is an immediate consequence of the Maximum principle applied to G^+ that $\Omega_\lambda \subset K^+_\lambda$, so the interesting case is when Ω_{λ_0} is unbounded.

Notice that by the Maximum Principle again, any connected component of Ω_{λ_0} is simply connected, so the same holds for Ω_λ.

Assume by contradiction that for some parameter λ_1, $\Omega_{\lambda_1} \cap U^+_{\lambda_1} \neq \emptyset$. We first claim that $\Omega_{\lambda_1} \subset U^+_{\lambda_1}$. Indeed otherwise, at $\lambda = \lambda_1$, Ω_{λ_1} intersects $\partial(W^u(p) \cap K^+) = \overline{H(p)}$. Since the holomorphic motion preserves $H(p)$, the same holds at λ_0, but this contradicts the fact that Ω_{λ_0} is contained in K^+. Thus Ω_{λ_1} is a simply connected component of $W^u(p_{\lambda_1}) \cap U^+$. The existence of such a component implies that f_{λ_1} is unstably connected [BS6, Theorem 0.1], so J_{λ_1} is connected. Since the connectedness of Julia sets is preserved in weakly J^* stable families, f_{λ_0} is unstably connected, too. On the other hand, for families of polynomial automorphisms with persistently connected Julia sets, we have shown in Proposition 3.6 that every component of $W^u(p) \setminus K^+$ can be followed holomorphically. This contradiction completes the proof. $\square$

It follows from this result that if $z \in W^u(p) \cap K^+$, then $z(\lambda) \in K_\lambda$ for all λ, so it is locally uniformly bounded in $\mathbb{C}^2$. On the other hand, if $z \in W^u(p) \setminus K^+$, then the same property holds for all λ, and the normality follows from Lemma 2.9

The implication (i)$\Rightarrow$(ii) now easily follows: simply take the closure of the normal family of graphs $z(\lambda)$, where z ranges in $W^u(p)$. Since for every λ, $W^u(p_\lambda)$ is dense in J^-_λ, we get a branched holomorphic motion of J^-, as required.

Let us prove that any item (ii), (iii) or (iv), as well as J stability, implies (i). Under (iv), it follows directly from Lemma 2.11 that if $z(\lambda)$ is a graph of the branched holomorphic motion of K such that $z(\lambda_0)$ is a saddle periodic point, then $z(\lambda)$ is periodic for all $\lambda \in \Lambda$. In particular $z(\lambda)$ locally coincides with the continuation of $z(\lambda_0)$ as a saddle point, therefore it defines a global continuation of $z(\lambda_0)$ as a periodic point. Weak J^*-stability then follows by Proposition 2.13.

Under (iii) (or similarly (ii)), consider such a graph $z(\lambda)$. By Proposition 1.1, we may normalize the family so that f_λ is a product of Hénon mappings. It follows that on a given compact subset of Λ, for sufficiently large R, $J^+ \cap \mathbb{D}^2_R$ is forward invariant. In particular, the family of graphs $(f_\lambda^n(z(\lambda)))_{n\geq 0}$ is normal[3]. Let γ be any of its cluster values. Then $\gamma(\lambda)\in K_\lambda$ and $\gamma(\lambda_0)=z(\lambda_0)$ is a saddle point. As above, from Lemma 2.11 we conclude that $\gamma(\lambda)$ coincides with the local continuation of this saddle point, and weak J^* stability follows.

Let us now show that (i) implies (iv). By the first part of the proof we know that if f is weakly J^* stable, the branched motion of J^* admits an equivariant extension to J^-, which in addition is a normal family. It is enough to show that this extension preserves $K=K^+\cap J^-$. But if $z(\lambda)$ is a graph of the branched motion of J^- such that for some λ_0, $z(\lambda_0)$ belongs to $K^+_{\lambda_0}$, then by equicontinuity, the iterates $(f_\lambda^n(z(\lambda)))_{n\geq 0}$ are locally uniformly bounded, that is, $z(\lambda)\in K^+_\lambda$, as desired.

It remains to deal with condition (v). We've just seen that under (i), the branched motion of J^- obtained by taking the closure of the Bers Royden extension preserves the decomposition $K^+\cup U^+$, hence (v) holds. Conversely, if $(f_\lambda)_{\lambda\in\Lambda}$ is weakly $J^-\cap U^+$-stable, then we obtain a branched holomorphic motion of J^* by observing that due to Lemma 3.1, $J^*\subset\overline{J^-\cap U^+}$, and applying Lemma 2.11. Therefore (v) implies (i), and the proof is complete. □

From now on a family satisfying (i)-(v) of Theorem 3.7 will be simply referred to as *weakly stable.*

Corollary 3.10. *If $(f_\lambda)_{\lambda\in\Lambda}$ is weakly stable, there exists an equivariant branched holomorphic motion of $J^+\cup J^-$ which is a normal family and preserves stable and unstable manifolds of saddle periodic points. In particular $J^+\cup J^-$ moves continuously in the Hausdorff topology.*

Proof. The existence of a branched motion of $J^+\cup J^-$ which is a normal family follows from the proof that (i) implies (ii) Theorem 3.7. That such a motion preserves stable and unstable manifolds was established in the proof of the converse implication. □

Remark 3.11. It is not difficult to show that for $X=J^+$ or K, (f_λ) is weakly stable is equivalent to the continuity of $\lambda\mapsto X_\lambda$ in the Hausdorff topology (for J^* this was done in Theorem 2.4). On the other hand this is false for J^-. Indeed in the dissipative setting $K^-=J^-$, and it is classical that K^- moves upper semi-continuously while J^- moves lower semi-continuously.

According to [LyP], non-wandering components of $\mathrm{Int}\,(K^+)$ of a moderately dissipative polynomial automorphism $f:\mathbb{C}^2\to\mathbb{C}^2$ can be classified as attracting, parabolic, or rotation basins. (For components of $\mathrm{Int}\,(W^u(p)\cap K^+)$, there is one more theoretical option: they can be contained in the small Julia set J.) We cannot rule out that some of these components change type under a branched holomorphic motion of the Julia set (compare Remark 3.8). However, Lemma 2.11 implies:

Proposition 3.12. *Under a branched holomorphic motion over a weakly stable domain, if for some parameter λ_0, z_0 is a point in $J_0^-\cap K_0^+$ belonging to the basin of attraction of a sink q_0 then for every λ, $z(\lambda)$ stays in the basin of $q(\lambda)$.*

Proof. We know that q_0 persists as a sink $q(\lambda)$ throughout the stability domain. It is clear that $f_\lambda^n(z(\lambda))\to q(\lambda)$ in the neighborhood of λ_0. Since in addition, $(f_\lambda^n(z(\lambda)))_{n\geq 0}$ is a normal family, this convergence persists by analytic continuation. □

[3]Notice that we did not assume any equicontinuity in the definition of weak J^+-stability

Part 2. Semi-parabolic implosion and homoclinic tangencies

4. Semi-parabolic dynamics

In this paragraph we collect some basic facts about semi-parabolic dynamics: basins, petals, etc., following the work of Ueda [U1, U2], Hakim [H] (see also [BSU]). Let f be a polynomial automorphism of $\mathbb{C}^2$. A periodic point p is *semi-parabolic* if its multipliers are 1 (or more generally a root of unity) and b with $|b| < 1$. Notice that this forces f to be dissipative. Replacing f by f^q for some $q \geq 1$ we can assume that p is fixed. Then, if we denote by $k+1$ the multiplicity at 0 of $f - \mathrm{id}$ (which is finite because f has no curve of fixed points), there exist local coordinates (x, y) in the neighborhood of p such that $p = (0, 0)$ and f is of the form

$$(x, y) \mapsto (x + x^{k+1} + Cx^{2k+1} + x^{2k+2}g(x, y), by + xh(x, y)), \tag{2}$$

where g and h are holomorphic near the origin and C is a complex number (see [H, Prop. 2.3]). Notice that in these coordinates, $\{x = 0\} = W^{ss}_{\mathrm{loc}}(0)$ is the local (strong) stable manifold of 0, and $f|_{\{x=0\}}$ is linear. For $r > 0$, the flower-shaped open set $\left\{x, \ \left|x^k + r^k\right| < r^k\right\}$ admits k connected components, which will be denoted by $P^\iota_{r,j}$, $0 \leq j \leq k-1$. Then for small $\eta > 0$, the domains $\mathcal{B}_{r,j,\eta} := P^\iota_{r,j} \times \mathbb{D}_\eta$ are attracted to the the origin under iteration. Finally, let $\mathcal{B}_j = \bigcup_{n\geq 0} f^{-n}(\mathcal{B}_{r,j,\eta})$. The open sets $\mathcal{B}_j$ are biholomorphic to $\mathbb{C}^2$ and are the components of the basin of attraction of p in $\mathbb{C}^2$.

To be more specific, in $\mathcal{B}_{r,j,\eta}$ we change coordinates by letting $(z, w) = ((kx^k)^{-1}, y)$, so that in the new coordinates, f assumes a form

$$(z, w) \longmapsto \left(z - 1 + \frac{c}{z} + O\left(\frac{1}{|z|^{1+1/k}}\right), bw + O\left(\frac{1}{|z|^{1/k}}\right)\right),$$

where c is a complex number depending on C. Notice that in the new coordinates, $\mathcal{B}_{r,j,\eta}$ corresponds to a region of the form $\{\mathrm{Re}(z) < -M\} \times \mathbb{D}_\eta$, with $M = (2kr^k)^{-1}$. Therefore if we set

$$w^\iota(x, y) = w^\iota(x) = \frac{1}{kx^k} + c\log\frac{1}{kx^k}$$

we infer that the limit

$$\varphi^\iota(x, y) = \lim_{n\to\infty} (w^\iota(f^n(x, y)) + n)$$

exists and satisfies the functional equation $\varphi^\iota \circ f = \varphi^\iota - 1$ (beware that this normalization differs from the references mentioned above). In addition, $\varphi^\iota - w^\iota$ is a bounded holomorphic function in $\mathcal{B}_{r,j,\eta}$. In the paper, the letter ι will stand for "incoming" and o for "outgoing", following a convenient notation from [BSU].

It easily follows that in the original coordinates, if $(x, y) \in \mathcal{B}$ then $f^n(x, y) = (x_n, y_n)$ with $x_n \sim (kn)^{-1/k}$ and $y_n = O(n^{-1/k})$, see [H, Prop. 3.1] (beware that y_n needlt be exponentially small).

Fix a component $\mathcal{B} = \mathcal{B}_j$ of the basin of attraction. By the iteration we can extend φ^ι to $\mathcal{B}$. It turns out that $\varphi^\iota : \mathcal{B} \to \mathbb{C}$ is a fibration [H, Thm 1.3], and that there exists a function $\phi_2 : \mathcal{B} \to \mathbb{C}$ such that $\Phi = (\varphi^\iota, \phi_2) : \mathcal{B} \to \mathbb{C}^2$ is a biholomorphism.

The following result is similar to [BSU, Thm 1.2], and its proof will be left to the reader.

Proposition 4.1. *If p_1 and p_2 are points in $\mathcal{B}$ such that $\varphi^\iota(p_1) = \varphi^\iota(p_2)$ then*

$$\lim_{n\to+\infty} \frac{1}{n} \log \mathrm{dist}(f^n(p_1), f^n(p_2)) = \log|b| < 0.$$

On the other hand, if $\varphi^\iota(p_1) \neq \varphi^\iota(p_2)$ *then* $\operatorname{dist}(f^n(p_1), f^n(p_2))$ *decreases like* $n^{-(1+1/k)}$.

From now on we will refer to the foliation $\left\{\varphi^\iota = C^{st}\right\}$ as the *strong stable foliation* in $\mathcal{B}$, and it will be denoted by $\mathcal{F}^{ss}$. Its structure near the origin is easy to describe. Indeed since $\varphi^\iota - w^\iota$ is bounded near the origin, it follows from Rouché's theorem that the leaf of $\mathcal{F}^{ss}$ through $(x_0, 0)$ in $\mathcal{B}_{r,j,\eta}$ is a vertical graph, whose distance to the line $x = x_0$ tends to 0 as $x_0 \to 0$. In particular $\mathcal{F}^{ss}$ extends continuously to $\mathcal{B}_{r,j,\eta} \cup \{x = 0\} = \mathcal{B}_{r,j,\eta} \cup W^{ss}_{\mathrm{loc}}(0)$ by adding $W^{ss}_{\mathrm{loc}}(0)$ as a leaf.

On the "outgoing" side, there is also a notion of "repelling petal", which is defined as follows. With coordinates as in (2), denote by $P^o_{r,j}$, $0 \leq j \leq k-1$, the connected components of $\left\{x, \ \left|x^k - r^k\right| < r^k\right\}$. Then for small $r, \eta > 0$ the set $\Sigma_{j,\mathrm{loc}}$ defined by

$$\Sigma_{j,\mathrm{loc}} = \left\{q \in P^o_{r,j} \times \mathbb{D}_\eta : \ \forall n \geq 0, f^{-n}(q) \in P^o_{r,j} \times \mathbb{D}_\eta \text{ and } f^{-n}(q) \underset{n\to\infty}{\longrightarrow} 0\right\}.$$

Then $\Sigma_{j,\mathrm{loc}}$ is a graph $\{y = \psi(x)\}$ over the first coordinate, which extends continuously to the origin by putting $\psi(0) = 0$ (this extension cannot be made holomorphic, see [BSU, Prop. 1.3]). This is stated in [U2, Thm 11.1] only for $k = 1$, however the proof relies on a more general result [U2, Lemma 11.2] which allows to treat the general case as well. As usual we extend the petal globally to $\mathbb{C}^2$ by letting

$$\Sigma_j = \bigcup_{n\geq 0} f^n\left(\Sigma_{j,\mathrm{loc}}\right).$$

Then Σ_j is biholomorphic to $\mathbb{C}$ (this is stated only for $k = 1$ in [U2, Thm 11.6] but the adaptation to the general case is straightforward). The union $\bigcup_j \Sigma_j$ is the set of points converging to the semi-parabolic point p under backward iteration, referred to as the *asymptotic curve* in [U2] and [BSU]. We will also call it the *unstable manifold* of p.

5. Critical points in basins

In this section we prove the existence of critical points in semi-parabolic basins for sufficiently dissipative maps. Let f be a polynomial automorphism with a semi-parabolic basin $\mathcal{B}$. Recall that by a critical point, we mean a point of tangency between the strong stable foliation in $\mathcal{B}$ and the unstable manifold of some saddle periodic point. The argument is based on a refined version of some classical properties of entire functions of finite order: see §5.1. The proof of Theorem B comes in §5.2. These results will be generalized to attracting basins in Appendix A.

5.1. **Entire functions of finite order.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. The *order* of f is defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,f)}{\log r}, \text{ where } M(r,f) = \max\left\{|f(z)|, \ |z| = r\right\}.$$

The class of entire functions of finite order is well-known to display a number of remarkable properties, some of which we recall now. We say that $a \in \mathbb{C}$ is an *asymptotic value* of f if there exists a continuous path $\gamma : [0, \infty) \to \mathbb{C}$ tending to infinity such that $f(\gamma(t)) \to a$ as $t \to \infty$. The famous Denjoy-Carleman-Ahlfors Theorem asserts that if f is an entire function of order $\rho < \infty$, then it admits at most 2ρ distinct asymptotic values (see e.g. [GO, Chap. 5] or [Ln]). Another essentially equivalent formulation is that for every $R > 0$, the open set $\{z, |f(z)| > R\}$ admits at most $\max(2\rho, 1)$ connected components.

When $\rho < \frac{1}{2}$, we see that f has no asymptotic values. One can actually be more precise in this case. Indeed, Wiman's theorem [GO, Chap. 5, Thm 1.3] asserts that there exists a sequence of circles $\{|z| = r_n\}$ with radii $r_n \to \infty$ such that $\min\{|f(z)|, |z| = r_n\} \to \infty$.

To prove the existence of critical point in semi-parabolic basins we will require a slight generalization of the Denjoy-Carleman-Ahlfors theorem on asymptotic values. We say that a is an *ε-approximate asymptotic value* of f if there exists a continuous path $\gamma : [0, \infty) \to \mathbb{C}$ tending to infinity such that $\limsup_{t\to\infty} |f(\gamma(t)) - a| < \varepsilon$. The statement is as follows:

Theorem 5.1. *Let f be an entire function of finite order. Assume that f admits n distinct ε-approximate asymptotic values $(a_i)_{i=1,\dots,n}$, with $\varepsilon < \min_{i\neq j} \frac{|a_i - a_j|}{5}$.*

Then the order of f is at least $n/2$.

To prove the theorem let us first recall the classical Phragmen-Lindelöf Principle.

Proposition 5.2. *Let D be an unbounded domain in $\mathbb{C}$. Let f be a bounded holomorphic function on D, such that $\limsup_{\partial D \ni z\to\infty} |f| \le \delta$. Then $\limsup_{D\ni z\to\infty} |f| \le \delta$.*

The following result is a version of a classical Lindelöf Theorem (the argument below is adadpted from [Ln, Thm 12.2.2]).

Theorem 5.3. *Let D be a simply connected unbounded domain in $\mathbb{C}$, whose boundary consists of two simple curves γ_1, γ_2 both tending to infinity, and disjoint apart from their common starting point. Let f be holomorphic on D and continuous on ∂D, and assume that when z goes to infinity along γ_i, f has the property that $\limsup_{t\to\infty} |f(\gamma_i(t)) - a| < \varepsilon$, with $\varepsilon < \frac{|a_1 - a_2|}{5}$. Then f is unbounded on D.*

Proof. Assume by contradiction that f is bounded, and let $g(z) = (f(z) - a_1)(f(z) - a_2)$. Then g is bounded on D, and $\limsup |g(z)| \le \delta$, as $z \to \infty$ along ∂D, for some $\delta < \frac{6}{5} |a_1 - a_2| \varepsilon$. It follows that $\limsup_{D\ni z\to\infty} |g| \le \delta$.

Now for every $R > 0$ there exists a curve Γ in D joining γ_1 and γ_2 and staying at distance at least R from the origin. If R is large enough, we then have that $|g| < \frac{6}{5} |a_1 - a_2| \varepsilon$ along Γ. Furthermore at $\Gamma \cap \gamma_1$ (resp. $\Gamma \cap \gamma_2$), f is ε-close to a_1 (resp. a_2). So there exists $z_0 \in \Gamma$ such that $|f(z_0) - a_1| = |f(z_0) - a_2| \ge \frac{|a_1 - a_2|}{2}$. We infer that

$$|g(z_0)| \ge \frac{|a_1 - a_2|^2}{4} \ge \frac{5}{4} |a_1 - a_2| \varepsilon,$$

a contradiction. □

Proof of Theorem 5.1. (compare [Ln, Cor. 14.2.3]) By assumption there are n curves γ_i going to infinity along which f ε-approximately converges to a_i. We may assume that all these curves are simple, start from 0, and intersect only at 0. We reassign the indices so that the curves are arranged in clockwise order. By the previous theorem f must be unbounded in the domain comprised between γ_i and γ_{i+1} (here we put $\gamma_{n+1} = \gamma_1$). Therefore the order of f is at least $n/2$ by the ordinary Denjoy-Carleman-Ahlfors Theorem. □

Remark 5.4. Alex Eremenko has pointed out to us the following version of the Denjoy-Carleman-Ahlfors Theorem. Let f be an entire function bounded in the left-half plane and outside horizontal strips. Let

$$M(x) = \max_{\operatorname{Re} z = x} |f(z)|$$

and

$$\rho = \rho(f) = \limsup_{x\to+\infty} \frac{\log M(x)}{x}.$$

Then f admits at most 2ρ asymptotic values. [This follows from the subharmonic version of the DCA Theorem applied to the function $u(z) = \log^+(f(\log z)/M)$, where M is the supremum of $|f|$ outside the half-strip $\Pi = \{\operatorname{Re} f > 1,\ |\operatorname{Im}| f < \pi\}$, and $\log z$ is the principal value of the logarithm in $\mathbb{C}\setminus\mathbb{R}_-$.]

This Theorem admits an ε-approximate version similar to Theorem 5.1.

5.2. **Semi-parabolic basins.** In this paragraph we prove Theorem B. We first recall that stable and unstable manifolds of saddle points, as well as strong stable manifolds of semi-parabolic points are entire curves, whose parameterizations are defined dynamically. More precisely, if p is a fixed point with an expanding eigenvalue κ^u, then the associated stable manifold is parameterized by an entire function $\psi^s : \mathbb{C} \to \mathbb{C}^2$ satisfying $\psi^u(0) = p$ and $f \circ \psi^u(t) = \psi^u(\kappa^u t)$ for every $t \in \mathbb{C}$, and similarly for a contracting eigenvalue. Let us now make an easy but important observation (see [EL] in the one-dimensional setting and [BS6, J] in our setting).

Lemma 5.5. *Let q be a fixed point of a polynomial automorphism of dynamical degree $d \geq 2$ with an expanding eigenvalue κ^u, and $\psi^u : \mathbb{C} \to \mathbb{C}^2$ is a parameterization of the associated unstable manifold as above. Then the coordinates of ψ^u are entire functions of finite order*

$$\rho = \frac{\log d}{\log|\kappa^u|}.$$

Notice that any other parameterization has the same order, since two parameterizations differ from an affine map of $\mathbb{C}$. So we may speak of the order of the unstable manifold $W^u(q)$.

In the semi-parabolic case the result specializes as follows:

Corollary 5.6. *If p is a semi-parabolic periodic point for a polynomial automorphism of dynamical degree $d \geq 2$, then the order of $W^{ss}(p)$ is*

$$\frac{\log d}{\log|\mathrm{Jac}\ f|^{-1}}.$$

Proof. Apply Lemma 5.5 to f^{-k}, where k is the period of p. □

Our use of this corollary will be the following.

Corollary 5.7. *Let f be a polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 2$, with a semi-parabolic periodic point p. Assume that the Jacobian of f satisfies $|\mathrm{Jac}\ f| < \frac{1}{d^2}$. Then the connected component of p in $W^{ss}(p) \cap J^-$ is $\{p\}$.*

Proof. The assumption on the Jacobian together with the previous corollary imply that the order of $W^{ss}(p)$ is smaller than $\frac{1}{2}$. Then by Wiman's theorem there exists a sequence of circles $\{|t| = r_n\}$ such that the second coordinate (say) of ψ^s satisfies $\min_{\{|t|=r_n\}} |\psi^s(t)| \to \infty$. Since K is bounded, these circles must be eventually disjoint from K. So we infer that the connected component of p in $(\psi^s)^{-1}(J^-)$ is bounded in $\mathbb{C}$. Since in addition the component is invariant under multiplication by κ^s, we are done. □

Theorem B now clearly follows from the previous corollary, together with the following result.

Proposition 5.8. *Let f be a polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 2$, possessing a semi-parabolic periodic point with basin of attraction $\mathcal{B}$. Assume that the connected component of p in $W^{ss}(p) \cap J^-$ is reduced to $\{p\}$. Then for every saddle periodic point q, every component of $W^u(q) \cap \mathcal{B}$ contains a critical point.*

Notice that in the situation of the proposition, for the same reasons as in Proposition 3.4, $W^u(q) \cap \mathcal{B}$ is never empty.

Remark 5.9. The theorem remains true if case when q is a semi-parabolic point rather than a saddle (in particular, when $q = p$). The proof is the same except it makes use of the version of the Denjoy-Carleman-Ahlfors Theorem stated in Remark 5.4

Proof. Let q be as in the statement of the proposition, and ψ^u as above be a parameterization of $W^u(q)$. Translating the coordinates and iterating if needed we may assume that the semi-parabolic point is fixed and equal to $0 \in \mathbb{C}^2$. Also we may assume that q is fixed. Let $\Omega \subset \mathbb{C}$ be a component of $(\psi^u)^{-1}(\mathcal{B})$ –which must be non-empty, as already observed. By the maximum principle, Ω is biholomorphic to a disk. Observe first that $\varphi^\iota \circ \psi^u : \Omega \to \mathbb{C}$ cannot be constant for otherwise $W^u(q)$ would coincide with a strong stable leaf. We argue by contradiction, so assume that Ω contains no critical point. Then $\varphi^\iota \circ \psi^u : \Omega \to \mathbb{C}$ is a locally univalent map. Since $\varphi^\iota \circ \psi^u$ cannot be a covering, it must possess asymptotic values, that is, there exists a path $\gamma : [0, \infty) \to \Omega$, tending to infinity in Ω such that $\varphi^\iota \circ \psi^u(\gamma(t))$ has a well defined limit in $\mathbb{C}$ as $t \to \infty$ (see [Nv, p. 284] or [GK, Lemma 1.2] for a modern presentation).

Recall that there exists a function $\phi_2 : \mathcal{B} \to \mathbb{C}$ such that $\Phi := (\varphi^\iota, \phi_2) : \mathcal{B} \to \mathbb{C}^2$ is a biholomorphism. At this point the proof splits into two cases.

Case 1: $\phi_2 \circ \psi^u(\gamma)$ is unbounded.

Observe that this case must occur when $\Phi \circ \psi^u : \Omega \to \mathbb{C}^2$ is proper, which happens for instance when Ω is relatively compact in $\mathbb{C}$. Consider a domain $\mathcal{B}_{r,j,\eta}$ as in §4, corresponding to the basin $\mathcal{B}$ in which the strong stable foliation is made of vertical graphs, clustering at $W^{ss}_{\rm loc}(0) = \{x = 0\}$. For sufficiently large n, the connected component of $f^n(\psi^u(\gamma)) \cap \mathcal{B}_{r,j,\eta}$ containing $f^n(\psi^u(\gamma(0)))$ is a path contained in a vertical leaf, which by our unboundedness assumption goes up to the boundary of $\mathcal{B}_{r,j,\eta}$. Taking a cluster value of this sequence of subsets for the Hausdorff topology, we obtain a closed connected subset of $W^{ss}_{\rm loc}(0) \cap J^-$, containing 0, and touching the boundary, hence not reduced to a point. This contradicts Corollary 5.7, and finishes the proof in this case.

Case 2: $\phi_2 \circ \psi^u(\gamma)$ is bounded.

A first observation is that under this assumption, the path γ must go to infinity in $\mathbb{C}$. Indeed otherwise let (t_n) be a sequence such that $\gamma(t_n)$ converges to $\zeta \in \partial\Omega \subset \mathbb{C}$. Then $\psi^u(\gamma(t_n))$ converges to $\psi^u(\zeta) \notin \mathcal{B}$, contradicting the fact that $\Phi(\psi^u(\gamma(t_n)))$ stays bounded in $\mathbb{C}^2$. In particular γ is an asymptotic path for the entire mapping ψ^u

For the sake of explanation, let assume first for simplicity that $\phi_2 \circ \psi^u(\gamma(t))$ admits a limit as $t \to \infty$. Thus $\Phi \circ \psi^u(\gamma(t))$ converges in $\mathbb{C}^2$, and $\psi^u(\gamma(t))$ converges to some limiting point $\omega \in \mathcal{B}$, which must be an asymptotic value of ψ^u (that is, both coordinates are asymptotic values of the coordinate functions of ψ^u). By the invariance of $W^u(p)$, all iterates $f^n(\omega)$, $n \in \mathbb{Z}$, are asymptotic values of ψ^u. Since ψ^u has finite order, this contradicts the Denjoy-Carleman-Ahlfors theorem.

In the general case we use Theorem 5.1 instead. Let K be the cluster set of $\psi^u(\gamma)$, which is a compact subset of $\mathcal{B}$, contained in a leaf $\left\{\varphi^\iota = C^{st}\right\}$ of the strong stable foliation. Let us study the shape of $f^n(K)$. When n is large enough, $f^n(K)$ is contained in a small neighborhood of the origin, inside a local strong stable leaf. Perform a *linear* change of coordinates so that in the new coordinates, f expresses as $f(x,y) = (x, by) + h.o.t.$. These coordinates are tangent (at 0), but not equal, to the adapted local coordinates of §4. In a bidisk near 0, the strong stable foliation is made of vertical graphs and $W^{ss}_{\mathrm{loc}}(0)$ is a vertical graph tangent to the y-axis. Let $(x_0, y_0) \in K$ and $(x_n, y_n) = f^n(x_0, y_0)$. Then we infer that $x_n \sim (kn)^{-1/k}$ and $f^n(K)$ is a subset of the leaf $\mathcal{F}^{ss}(x_n, y_n)$ of size exponentially small with n. Denoting by $\mathrm{pr}_1 : (x, y) \mapsto y$ the first projection, we deduce that $\mathrm{pr}_1(f^n(K))$ is a set of exponentially small diameter about x_n. It follows that for every integer k there exists $\varepsilon > 0$ and integers $n_1, \ldots, n_k$ such that the sets $\mathrm{pr}_1(f^{n_j}(K))$, $1 \leq j \leq k$ are of diameter smaller than ε, and 5ε-apart from each other. Notice that the sets $\mathrm{pr}_1(f^{n_j}(K))$ are ε-approximate asymptotic values of $\mathrm{pr}_1 \circ \psi^u$ in the sense of Theorem 5.1. Now since pr_1 is linear, $\mathrm{pr}_1 \circ \psi^u$ is an entire function of finite order. Thus we obtain a contradiction with Theorem 5.1, and the proof is complete. □

6. Semi-parabolic bifurcations and transit mappings

In this section we develop an analogue of the "tour de valse" of Douady and Sentenac [DS] in the context of semi-parabolic implosion. When the family can be put in the form

$$f_\varepsilon(x,y) = (x + (x^2+\varepsilon^2)\alpha_\varepsilon(x,y), b_\varepsilon(x)y + (x^2+\varepsilon^2)\beta_\varepsilon(x,y)),$$

(this corresponds to the case $k = 1$ in (3) below), we may directly appeal to the results of Bedford, Smillie and Ueda [BSU]. In our setting, however, we have to deal with periodic points and bifurcations of a more general nature, and it is unclear how to extend the results of [BSU].

Consider a family $(f_\lambda)_{\lambda\in\Lambda}$ of dissipative polynomial automorphisms of $\mathbb{C}^2$, with a periodic point changing type, that is, one multiplier crosses the unit circle. It is no loss of generality to assume that Λ is the unit disk.

Let us start with a few standard reductions. Recall that for polynomial automorphisms all periodic points are isolated. Replacing f_λ by some iterate, we may assume the bifurcating periodic point is fixed. Passing to a branched cover of Λ if necessary, the fixed point moves holomorphically, so we assume it is equal to $0 \in \mathbb{C}^2$. Since f_λ is dissipative the multipliers depend holomorphically on λ, and we denote them by ρ_λ and b_λ, with $\rho_0 = e^{2\pi i\frac{p}{q}}$ and $|b_\lambda| < 1$ for all $\lambda \in \Lambda$. We further assume that ρ_λ crosses $\partial\mathbb{D}$ with non-zero speed, i.e. $\frac{\partial\rho}{\partial\lambda}\big|_{\lambda=0} \neq 0$.

6.1. Good local coordinates.

The first step is to find adapted local coordinates. which is a parameterized version of the discussion in §4. This is analogous to Proposition 1 in [DS].

Proposition 6.1. *If (f_λ) is as above, then for λ sufficiently close to 0, there exists a local change of coordinates $(x,y) = \varphi_\lambda(z,w)$ in which f^q_λ takes the form*

$$f^q_\lambda(x,y) = (\rho^q_\lambda x + x^{k+1} + x^{k+2}g_\lambda(x,y), b^q_\lambda y + x h_\lambda(x,y)) \tag{3}$$

with g_λ and h_λ holomorphic and $h_\lambda(0,0) = 0$. Moreover $k = \nu q$ for some integer $\nu \geq 1$.

Proof. Since we are working locally near $(0,0) \in \Lambda \times \mathbb{C}^2$, we freely reduce the domains of definition in (λ, x, y) when necessary. We will also feel free to use to the same symbols for coordinates after successive changes of variable.

Recall from §4 that there exists an integer k and local coordinates in which f_0^q is of the form

$$(x,y)\mapsto (x+x^{k+1}+x^{k+2}g(x,y), b_0^q y + xh(x,y)). \tag{4}$$

Then f^q admits k (open) attracting petals and k repelling petals, which are permuted by f. These petals approach 0 at certain directions permuted by the differential $D_0 f$, so necessarily $k=\nu q$ for some nonzero integer ν.

From now on for notational ease we replace f^q by f, that is we assume $\rho_0=1$. Since the differential $D_0 f_\lambda$ is diagonalizable, there exists a (λ-dependent) linear change of coordinates so that $f_\lambda(x,y)$ takes the form $(\rho_\lambda x, b_\lambda y)+h.o.t.$ There exists a local strong stable manifold tangent to the y-axis; we change coordinates so that it becomes $\{x=0\}$. By the Schröder theorem we can linearize $f_\lambda|_{\{x=0\}}$, holomorphically in λ. Hence in the new coordinates, $f_\lambda(0,y)=(0,b_\lambda y)$, so that

$$f_\lambda(x,y)=(\rho_\lambda x(1+O(x)), b_\lambda y + xh_\lambda(x,y)).$$

Of course $h_\lambda(0,0)=0$ since the linear part is $(\rho_\lambda x, b_\lambda y)$. From now on all changes of variables will be "horizontal", i.e. of the type $(x,y)\mapsto (x(1+O(x,y)),y)$, so the form of the second coordinate persists and we focus on the first one.

Express $f_\lambda(x,y)$ as

$$f_\lambda(x,y)=(\rho_\lambda a_1(\lambda,y)x+a_2(\lambda,y)x^2+\cdots+a_j(\lambda,y)x^j+\cdots, b_\lambda y+O_{\lambda,y}(x)), \tag{5}$$

where the a_j are holomorphic and $a_1(\lambda,0)=1$. To start with, for $\lambda=0$, we put f_0 in form (4), so that for $j\leq k$, $a_j(0,y)=0$ and $a_{k+1}(0,y)=1$.

The first task is to arrange so that $a_1\equiv 1$. This is similar to [U1]. For this we look for a change of coordinates of the form $(X,Y)=(x\varphi_\lambda(y),y)$, with $\varphi_\lambda(0)=1$. Using the notation $f_\lambda(x,y)=(x_1,y_1)$ (and similarly in the (X,Y) variables), we infer that

$$X_1=\rho_\lambda a_1(\lambda,Y)\frac{\varphi_\lambda(Y_1)}{\varphi_\lambda(Y)}X+O(X^2)=\rho_\lambda a_1(\lambda,Y)\frac{\varphi_\lambda(b_\lambda Y)}{\varphi_\lambda(Y)}X+O(X^2).$$

Therefore we see that to obtain the desired form, it is enough to choose

$$\varphi_\lambda(y)=\prod_{n=0}^{\infty} a_1(\lambda, b_\lambda^n y),$$

which is locally a convergent product since $a_1(\lambda,y)=1+O_\lambda(y)$ and $|b_\lambda|<1$. Notice also that for $\lambda=0$, $\varphi_0(y)=1$ so the change of variables is the identity. In particular f_0 remains of the form (4).

We then argue by induction. So assume that we have found coordinates such that for some $j\leq k$, $a_2(\lambda,y)=\cdots=a_{j-1}(\lambda,y)=0$, and f_0 remains under the form (4) . Put

$$(X,Y)=\left(x+\frac{a_j(\lambda,y)}{\rho_\lambda-\rho_\lambda^j}x^j, y\right).$$

Notice that since $a_j(0)=0$ and $\rho_\lambda-1$ has a simple root at the origin, the change of coordinates is also well defined at $\lambda=0$. Now, for $\lambda\neq 0$, since the term $a_j(\lambda,y)$ is non-resonant, a classical explicit computation shows that in disappears in the new coordinates. Hence by continuity the same holds for $\lambda=0$.

Moreover, for $\lambda = 0$ the change of coordinates is of the form $(X, Y) = (x + A_j(y)x^j, y)$, so $(x, y) = (X - A_j(Y)X^j + h.o.t., Y)$. In the new coordinates we obtain

$$\begin{aligned} X_1 = x_1 + A_j(y_1)x_1^j &= x + x^{k+1} + O(x^{k+2}) + A_j(y)(x + x^{k+1} + O(x^{k+2}))^j \\ &= x + A_j(y)x^j + x^{k+1} + O(x^{k+2}) \\ &= X + X^{k+1} + O(X^{k+2}), \end{aligned}$$

so f_0 remains of form (4) (observe that $j \le k$ is used here).

Hence by induction we arrive at a situation where the first coordinate of $f_\lambda(x, y)$ is of the form $\rho_\lambda x + a_k(\lambda, y)x^k + O(x^{k+1})$, with $a_k(0, y) = 1$, and the desired form follows by putting $(X, Y) = (a_k(\lambda, y)^{\frac{1}{k-1}} x, y)$. □

Remark 6.2. Observe that the normal form (2) is more precise than the one that we obtain here for f_0. Indeed, as opposed to the case $\lambda = 0$, we cannot in general kill the terms $x^{k+2}, \cdots, x^{2k}$ in the first coordinate of (3).

In fact, the vanishing of these terms for $\lambda = 0$ is incompatible with keeping (f_λ) in form (3). Indeed, the change of variables required to kill these terms at $\lambda = 0$ is of the form $(x, y) \mapsto (x + \alpha_\lambda(y)x^j, y + h.o.t.)$, where $\alpha_0(0) \neq 0$ and $j \le k$ (compare [Be, Thm 6.5.7]). If $\rho_\lambda \neq 1$ for some λ, it brings back a non-zero term x^j in the first coordinate of f_λ.

On the other hand, if by chance the terms x^j, $j = k+2 \le k \le 2k$, vanish, we can reduce ourselves to [BSU] by letting $(x', y') = (k\rho_\lambda^{k-1}x^k, y)$ and then $(x'', y'') = (x' + (1-\rho^k)/2, y')$. It is the presence of these extra non-vanishing terms that prevents us from using [BSU] directly.

6.2. **Transit mappings in the one-dimensional case: un tour de valse.** To fix the ideas, let us establish the statement we need in the one-dimensional case first. This is a refined version of [DS]. Consider a holomorphic family $(f_\lambda)_{\lambda \in \Lambda}$ of mappings defined in some neighborhood of the origin in $\mathbb{C}$, of the form

$$f_\lambda(x) = \rho_\lambda x + x^{k+1} + x^{k+2} g_\lambda(x), \tag{6}$$

with g holomorphic. As before Λ is the unit disk. We assume that $\rho_0 = 1$ and $\frac{\partial \rho}{\partial \lambda}(0) \neq 0$ (this amounts to replacing f_λ by its q^{th} iterate in (3)).

Recall that for $\lambda = 0$ the repelling and attracting directions are respectively defined by the property that $(1 + x^k) \in \mathbb{R}^{+/-}$. We fix two consecutive such directions with respective angles 0 and $\frac{\pi}{k}$, and non-overlapping sectors about them by putting

$$S^\iota = \left\{ \arg x \in \left(-\frac{5\pi}{4k}, -\frac{3\pi}{4k} \right) \right\} \text{ and } S^o = \left\{ \arg x \in \left(-\frac{\pi}{4k}, \frac{\pi}{4k} \right) \right\}.$$

The result is as follows.

Theorem 6.3. *Let f_λ be as in (6) and $S^{\iota/o}$ be as above. There exists a neighborhood V of the origin in $\mathbb{C}$ with the following property: if Q^ι and Q^o are open topological disks with $Q^\iota \Subset S^\iota \cap V$ and $Q^o \Subset S^o \cap V$, then for every neighborhood W of 0 in Λ, there exists an integer N and a radius r such that if $n \ge N$ there exists a holomorphic map $\lambda_n : Q^\iota \times Q^o \to W$ such that for every $(z^\iota, z^o) \in Q^\iota \times Q^o$, $f^n_{\lambda_n(z^\iota, z^o)}$ is a well defined univalent function on $B(z^\iota, r)$, with $f^n_{\lambda_n}(z^\iota) = z^o$ and $\left| (f^n_{\lambda_n})' - 1 \right| \le \frac{1}{5}$.*

To prove the theorem we work in the new coordinate $z = \frac{\rho_\lambda^{k+1}}{kx^k}$. Notice that for $\lambda = 0$, the change of variables maps the sector $\left\{ \arg x \in \left(-\frac{3\pi}{2k}, \frac{\pi}{2k} \right) \right\}$ onto $\mathbb{C} \setminus i\mathbb{R}^-$, hence for small enough

λ, S^ι and S^o are contained in $\left(\frac{\rho_\lambda^{k+1}}{kx^k}\right)^{-1}(\mathbb{C}\setminus i\mathbb{R}^-)$. In the new coordinates, S^ι and S^o are respectively perturbations of the sectors $\left\{\arg z \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)\right\}$ and $\left\{\arg z \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)\right\}$.

Using the the classical notation $x_1 = f_\lambda(x)$ (and similarly for z), we infer that

$$\begin{aligned} z_1 &= \frac{\rho_\lambda^{k+1}}{kx_1^k} = \frac{\rho_\lambda^{k+1}}{k(f_\lambda(x))^k} = \frac{\rho_\lambda^{k+1}}{k\rho_\lambda^k x^k\left(1+\frac{x^k}{\rho_\lambda}+O(x^{k+1})\right)^k} \\ &= \frac{\rho_\lambda^{k+1}}{k\rho_\lambda^k x^k}\left(1-\frac{kx^k}{\rho_\lambda}+O(x^{k+1})\right) = \frac{\rho_\lambda}{kx^k}-1+O(x) \\ &= \rho_\lambda^{-k}z-1+\eta_\lambda(z), \text{ with } \eta_\lambda(z) = O\left(\frac{1}{|z|^{1/k}}\right) \text{ as } z\to\infty, \text{ uniformly in } \lambda\in\Lambda. \end{aligned}$$

The exponent $1/k$ will play a special role in the estimates to come, so for notational ease, from now on we put $\gamma = 1/k$. We also change coordinates in the parameter space by putting $u = \rho_\lambda^{-k} - 1$, so that u now ranges in some neighborhood W of the origin, and our mapping writes as

$$f_u(z) = (1+u)z - 1 + \eta_u(z), \text{ with } \eta_u(z) = O\left(\frac{1}{|z|^\gamma}\right).$$

In these coordinates, f_u is defined in an open set Ω_R of the form

$$\Omega_R = \left\{z,\ |z| > R, \arg(z) \in \left(\frac{-3\pi}{8}, \frac{11\pi}{8}\right)\right\}, \tag{7}$$

for some $R = R_0$. Its complement is shaded on Figure 1 and will be referred to as the "forbidden region". We also pick two bounded open topological disks Q^ι and Q^o such that $Q^\iota \Subset S^\iota \cap \Omega_R$ and $Q^o \Subset S^o \cap \Omega_R$, where $R \geq R_0$ is to be fixed later (this corresponds to the choice of the neighborhood V in the statement of the theorem).

We fix a constant M such that for every parameter $u \in W$ and every $z \in \Omega_{R_0}$,

$$|\eta_u(z)| \leq \frac{M}{|z|^{1/k}} = \frac{M}{|z|^\gamma} \text{ and } |\eta_u'(z)| \leq \frac{M}{|z|^{1+\gamma}}. \tag{8}$$

We will let u vary in a small subset of W, of the form $W_n = B\left(-\frac{2\pi i}{n}, \frac{1}{n^{1+\gamma/2}}\right)$. Notice that for $u \in W_n$ we have that

$$|1+u| = 1 + \frac{2\pi^2}{n^2} + o\left(\frac{1}{n^2}\right) \text{ and } \arg(1+u) = -\frac{2\pi}{n} + O\left(\frac{1}{n^{1+\gamma/2}}\right).$$

In the following we always consider n so large that $n^{\gamma/2} > 100$, $1 - \frac{30}{n^2} \leq |1+u| \leq 1 + \frac{30}{n^2}$ and $\left|\arg(1+u) + \frac{2\pi}{n}\right| \leq \frac{1}{100n}$.

To understand the argument better, it is instructive to think of f_u as a perturbation of the affine map $\ell_u : z \mapsto (1+u)z - 1$. When $u \in W_n$ and n is large, ℓ_u is approximately a rotation by angle $-\frac{2\pi}{n}$ centered at $\frac{1}{u}$. Notice also that $\frac{1}{u}$ is close to $\frac{in}{2\pi}$ (see Figure 1).

To fix the idea, let us first analyze the linear case, dealing with ℓ_u instead of f_u.

Proposition 6.4. *With notation as above, there exists an integer N, and a radius r such that if $n \geq N$ then for every $(z^\iota, z^o) \in Q^\iota \times Q^o$, there exists a parameter $u = u(z^\iota, z^o) \in W_n$, depending holomorphically on (z^ι, z^o) and such that*

- *$\ell_u^n(z^i) = z^o$;*
- *for every $z \in B(z^\iota, r)$ the iterates $\ell_u^j(z)$, $j = 1, \ldots, n$ do not enter the forbidden region;*
- *$|(\ell_u^n)' - 1| \leq \frac{1}{5}$ on $B(z^\iota, r)$.*

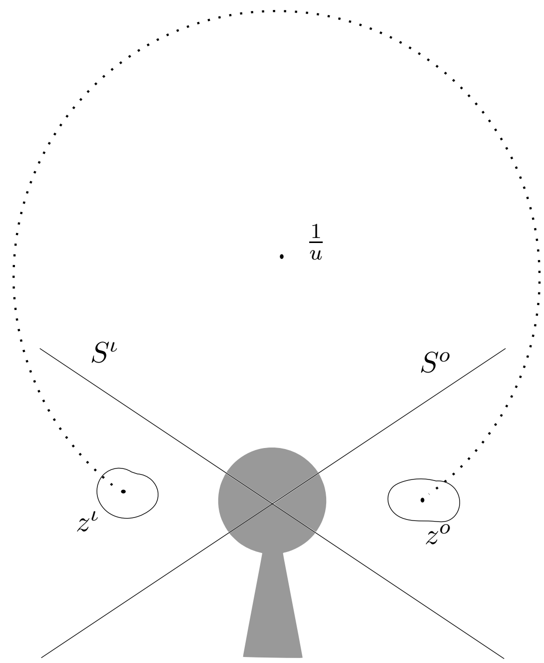

FIGURE 1. Schematic view of the orbit connecting z^ι to z^o. It shadows the arc of a circle passing through the "gate" between the fixed points $1/u$ and ∞. The forbidden region is shaded.

Proof. Let $l = \lfloor n/2 \rfloor$ and $m = n - l$. As said above, $\ell_u(z) = (1+u)\left(z - \frac{1}{u}\right) + \frac{1}{u}$ has its fixpoint at $\frac{1}{u}$. Write $u = \frac{-2\pi i}{n} + \frac{v}{n^{1+\gamma/2}}$, with $v \in \mathbb{D}$. For $j \leq n$ we have that

$$\begin{aligned}(1+u)^j = \exp(j \log(1+u)) &= \exp\left(j \log\left(1 - \frac{2\pi i}{n} + \frac{v}{n^{1+\gamma/2}}\right)\right) \\ &= \exp\left(j\left(-\frac{2\pi i}{n} + \frac{v}{n^{1+\gamma/2}} + O\left(\frac{1}{n^2}\right)\right)\right) \\ &= \exp\left(-\frac{2j\pi i}{n} + \frac{jv}{n^{1+\gamma/2}} + O\left(\frac{j}{n^2}\right)\right),\end{aligned}$$

in particular for $j = n$

$$(1+u)^n = 1 + \frac{v}{n^{\gamma/2}} + O\left(\frac{1}{n}\right), \tag{9}$$

where the $O(\cdot)$ is uniform with respect to $v \in \mathbb{D}$.

Simple geometric considerations (see [DS]) then show that for $j \leq \lceil n/2 \rceil$ $\ell_u^j(z^\iota)$ (resp. $\ell_u^{-j}(z^o)$) do not enter the forbidden area.

Let us prove that there exists $u(z^\iota, z^o)$, depending holomorphically on $(z^\iota, z^o) \in Q^\iota \times Q^o$ and such that $\ell_u^l(z^\iota) = \ell_u^{-m}(z^o)$. Then for such a parameter, by connecting the two pieces of orbits $1, \ldots, l$ and $l+1, \ldots, n$, we infer that the iterates $\ell_u^j(z^\iota)$ do not enter the forbidden area for $1 \leq j \leq n$ and since ℓ_u is affine, the control of the derivative follows from (9).

To prove this, consider the expression

$$\frac{\ell_u^l(z^\iota) - \frac{1}{u}}{\ell_u^{-m}(z^o) - \frac{1}{u}} = (1+u)^n \frac{z^\iota - \frac{1}{u}}{z^o - \frac{1}{u}}.$$

A simple computation shows that

$$\frac{z^\iota - \frac{1}{u}}{z^o - \frac{1}{u}} = 1 + (z^\iota - z^o)\frac{2\pi i}{n} + O\left(\frac{1}{n^{1+\gamma/2}}\right).$$

Therefore by (9), we infer that

$$\frac{\ell_u^l(z^\iota) - \frac{1}{u}}{\ell_u^{-m}(z^o) - \frac{1}{u}} = 1 + \frac{v}{n^{\gamma/2}} + O\left(\frac{1}{n}\right), \tag{10}$$

where the $O(\cdot)$ is uniform with respect to $v \in \mathbb{D}$, $z^\iota \in Q^\iota$ and $z^o \in Q^o$. Thus when n is large enough the quantity in (10) winds once around 1 as v turns once around $\partial\mathbb{D}$, and the result follows from the Argument Principle. □

We now turn to f_u. Let us start with a technical lemma.

Lemma 6.5. *Fix $R \geq (10^5 kM)^k$. With notation as above, there exists an integer $N = N(R)$ depending only on R such that if $n \geq N$, $u \in W_n$ and if $z^\iota \in Q^\iota$ then for every $1 \leq j \leq \lceil \frac{n}{2} \rceil$:*

(i) *$f_u^j(z^\iota)$ stays outside the forbidden area;*
(ii) *$\left| f_u^j(z^\iota) - \frac{1}{u} \right| \geq \frac{n}{10}$;*
(iii) *writing $f_u^j(z^\iota) = z_j = x_j + iy_j$ we have that either $x_j \leq x_0 - \frac{j}{10}$ or $y_j \geq \frac{n}{10}$. In particular $|z_j| \geq \min\left(\frac{|z_0|}{2} + \frac{j}{10}, \frac{n}{10}\right)$.*

The same results holds for $f_u^{-j}(z^o)$, when $z^o \in Q^o$ (in that case the last condition needs to be replaced by "either $x_j \geq x_0 + \frac{j}{10}$ or $y_j \geq \frac{n}{10}$")

Proof. We first deal with the assertions (i) and (ii). We argue by induction so assume the result holds for $j \leq k-1$, for some $k \leq \lceil \frac{n}{2} \rceil$. Let us write

$$\frac{f_u^j(z^\iota) - \frac{1}{u}}{f_u^{j-1}(z^\iota) - \frac{1}{u}} = (1+u) + \frac{\eta_u(f_u^{j-1}(z^\iota))}{f_u^{j-1}(z^\iota) - \frac{1}{u}},$$

so that

$$\frac{f_u^k(z^\iota) - \frac{1}{u}}{z^\iota - \frac{1}{u}} = (1+u)^k \prod_{j=0}^{k} \left(1 + \frac{\eta_u(f_u^j(z^\iota))}{(1+u)(f_u^j(z^\iota) - \frac{1}{u})}\right). \tag{11}$$

Considering the modulus of this expression, we see that

$$\begin{aligned}\left| f_u^k(z^\iota) - \frac{1}{u} \right| &\geq \left| z^\iota - \frac{1}{u} \right| \left(1 - \frac{30}{n^2}\right)^{\lceil \frac{n}{2} \rceil} \prod_{j=0}^{k-1} \left(1 - \frac{M}{(0.9)R^\gamma \left| f_u^j(z^\iota) - \frac{1}{u} \right|}\right) \\ &\geq \left| z^\iota - \frac{1}{u} \right| \left(1 - \frac{30}{n^2}\right)^{\lceil \frac{n}{2} \rceil} \left(1 - \frac{10M}{(0.9)R^\gamma n}\right)^{\lceil \frac{n}{2} \rceil},\end{aligned}$$

where the first estimate follows from bound (8) on η_u and the second estimate follows from the induction hypothesis. Since $\left(1-\frac{30}{n^2}\right)^{n/2} \to 1$ as $n \to \infty$, by our choice of R we see that when $n \geq N(R)$,

$$\left|f_u^k(z^\iota) - \frac{1}{u}\right| \geq \frac{9}{10}\left|z^\iota - \frac{1}{u}\right| \geq \frac{9}{10} d\left(\frac{1}{u}, S^\iota\right) \geq \frac{9}{10}\frac{n}{2\sqrt{2}\pi} \geq \frac{n}{10},$$

which proves (ii).

To prove that $f_u^k(z^\iota)$ does not enter the forbidden region, we look at the argument of $f_u^k(z^\iota) - \frac{1}{u}$. Recall that $\left|\arg(1+u) + \frac{2\pi}{n}\right| \leq \frac{1}{100n}$ so by (11)

$$\left|\arg\left(\frac{f_u^k(z^\iota) - \frac{1}{u}}{z^\iota - \frac{1}{u}}\right) - \left(\frac{-2k\pi}{n}\right)\right| \leq \frac{k}{100n} + \sum_{j=0}^{k-1}\left|\arg\left(1 + \frac{\eta_u(f_u^j(z^\iota))}{(1+u)(f_u^j(z^\iota) - \frac{1}{u})}\right)\right|.$$

With our choice of R, $\left|\frac{\eta_u(f_u^j(z^\iota))}{(1+u)(f_u^j(z^\iota) - \frac{1}{u})}\right| \leq \frac{1}{200n}$, so since $\log(1+z) = z + h.o.t.$, when n is large enough, we infer that

$$\left|\arg\left(1 + \frac{\eta_u(f_u^j(z^\iota))}{(1+u)(f_u^j(z^\iota) - \frac{1}{u})}\right)\right| \leq \frac{1}{100n}.$$

Thus we obtain that

$$\left|\arg\left(\frac{f_u^k(z^\iota) - \frac{1}{u}}{z^\iota - \frac{1}{u}}\right) - \left(\frac{-2k\pi}{n}\right)\right| \leq \frac{k}{100n} + \frac{k}{100n} \leq \frac{k}{50n},$$

therefore arguing geometrically we see that $f_u^k(z^\iota)$ stays outside the forbidden region. The induction step is complete proving (i).

To establish (iii), let us first observe that due to the the above estimate on the argument, when $j \leq \lceil \frac{n}{2} \rceil$, $\arg\left(\frac{f_u^j(z^\iota) - \frac{1}{u}}{z^\iota - \frac{1}{u}}\right)$ is equal to $\frac{-2j\pi}{n}$, up to an error of at most $\frac{1}{50}$. Expressing in coordinates, we see that

$$x_{j+1} = x_j - 1 + \frac{2\pi}{n} y_j + \varepsilon_j \text{ and } y_{j+1} = y_j - \frac{2\pi}{n} x_j + \varepsilon'_j,$$

with

$$|\varepsilon_j|, |\varepsilon'_j| \leq \max\left(\frac{1}{n^{1+\gamma/2}}, \frac{M}{|z_j|^\gamma}\right) \leq \frac{1}{1000}$$

because $n^{\gamma/2} \geq 100$ and by the previous step, $z_j \in \Omega_R$. We see that, as soon as $y_j \leq \frac{n}{10}$, we have that $x_{j+1} \leq x_j - \frac{1}{10}$. Now when y_j reaches $\frac{n}{10}$, and until j is as large as $\frac{n}{4}$ (a time at which y_j is approximately equal to $\frac{n}{2\pi}$), since $\left|z_j - \frac{1}{u}\right| \geq \frac{9}{10}\left|z_0 - \frac{1}{u}\right|$, by expressing the distance in coordinates and using the estimate on the argument, we infer that $x_j \leq -\frac{n}{100}$ therefore $y_{j+1} \geq y_j$. The result follows.

The argument for $f_u^{-j}(z^o)$, $1 \leq j \leq \lceil \frac{n}{2} \rceil$ is similar, and is left to the reader. □

Proof of Theorem 6.3. We argue as in Proposition 6.4. As before let $l = \lfloor n/2 \rfloor$ and $m = n-l$. Using (11) with $k = l$, we obtain

$$\frac{f_u^l(z^\iota) - \frac{1}{u}}{z^\iota - \frac{1}{u}} = (1+u)^l \prod_{j=1}^{l} \left(1 + \frac{\eta_u(f_u^{j-1}(z^\iota))}{(1+u)(f_u^{j-1}(z^\iota) - \frac{1}{u})}\right).$$

Hence, using Lemma 6.5 together with the inequality $\left|\prod(1+x_j) - 1\right| \leq \exp\sum|x_j| - 1$, we infer that

$$\left|\frac{f_u^l(z^\iota) - \frac{1}{u}}{(1+u)^l\left(z^\iota - \frac{1}{u}\right)} - 1\right| = \left|\frac{f_u^l(z^\iota) - \frac{1}{u}}{\ell_u^l(z^\iota) - \frac{1}{u}} - 1\right| \leq \exp\left(\sum_{j=0}^{l} \frac{10M}{(0.9)n\min(\frac{\lfloor z^\iota \rfloor}{2} + \frac{j}{10}, \frac{n}{10})^\gamma}\right) - 1$$

$$\leq \exp\left(\sum_{j=0}^{\lceil \frac{n}{2} \rceil} \frac{100M}{9n\min(\frac{j}{10}+1, \frac{n}{10})^\gamma}\right) - 1 \leq \exp\left(\frac{1000M}{n^\gamma}\right) - 1 = O\left(\frac{1}{n^\gamma}\right), \tag{12}$$

where in the last inequality we use an elementary estimate

$$\sum_{j=0}^{\lceil \frac{n}{2} \rceil} \frac{1}{\min(\frac{j}{10}+1, \frac{n}{10})^\gamma} \leq 50n^{1-\gamma}.$$

Doing the same with $f_u^{-m}(z^o)$ we get that

$$\frac{f_u^l(z^\iota) - \frac{1}{u}}{f_u^{-m}(z^o) - \frac{1}{u}} \left(\frac{\ell_u^l(z^\iota) - \frac{1}{u}}{\ell_u^{-m}(z^o) - \frac{1}{u}}\right)^{-1} = 1 + O\left(\frac{1}{n^\gamma}\right).$$

Thus, from (10) we deduce that

$$\frac{f_u^l(z^\iota) - \frac{1}{u}}{f_u^{-m}(z^o) - \frac{1}{u}} = 1 + \frac{v}{n^{\gamma/2}} + O\left(\frac{1}{n^\gamma}\right),$$

where the $O(\cdot)$ is uniform with respect to $(v, z^\iota, z^o) \in \mathbb{D} \times Q^\iota \times Q^o$. Therefore we conclude that if n is large enough, when u winds once around ∂W_n (i.e. v winds once around $\partial\mathbb{D}$), the curve $u \mapsto \dfrac{f_u^l(z^\iota) - 1/u}{f_u^{-m}(z^o) - 1/u}$ winds once around 1, so by the Argument Principle, there exists a unique $u = u(z^\iota, z^o) \in W_n$ (thus, depending holomorphically on (z^ι, z^o)), such that $f_u^l(z^\iota) = f_u^{-m}(z^o)$. Given such a u, we see that the iterates $f_u^j(z^\iota)$ $1 \leq j \leq n$ stay outside the forbidden region, and $f_u^n(z^\iota) = z^o$.

Let us now estimate the derivative $(f_u^l)'(z)$, for $z \in Q^\iota$ (this is place where we need to be precise on the value of R). Recall that for $1 \leq j \leq l$, $f_u^j(z)$ is well-defined, and write

$$(f_u^l)'(z) = \prod_{j=1}^{l} (f_u)'(f_u^{j-1}(z)), \text{ where } (f_u)'(z) = 1 + u + \eta_u'(z),\ \left|\eta_u'(z)\right| \leq \frac{M}{|z|^{1+\frac{1}{k}}}.$$

So we get that

$$(f_u^l)'(z) = (1+u)^l \prod_{j=1}^{l} \left(1 + \frac{\eta_u'(f_u^{j-1}(z))}{1+u}\right).$$

Our choice of u and l implies that $0.99 \leq \left|(1+u)^l\right| \leq 1.01$ for large n, while

$$\left|\prod_{j=1}^{l}\left(1+\frac{\eta_u'(f_u^{j-1}(z))}{1+u}\right)-1\right| \leq \exp\left(\sum_{j=1}^{\lceil\frac{n}{2}\rceil}\frac{10M}{9(\min(\frac{R}{2}+\frac{j}{10},\frac{n}{10}))^{1+\gamma}}\right)-1. \tag{13}$$

For large n (depending only on R) we have that

$$\sum_{j=1}^{\lceil\frac{n}{2}\rceil}\frac{1}{(\min(\frac{R}{2}+\frac{j}{10},\frac{n}{10}))^{1+\gamma}} \leq \sum_{j=\lfloor\frac{R}{2}\rfloor}^{\infty}\frac{10^{1+\gamma}}{j^{1+\gamma}}+\sum_{j=\lfloor\frac{n}{10}\rfloor}^{\lceil\frac{n}{2}\rceil}\left(\frac{n}{10}\right)^{-(1+\gamma)} \leq 200kR^{-\gamma}. \tag{14}$$

Since $R \geq (10^5kM)^k$, by (14) we infer that the right hand side of (13) is smaller than $\frac{1}{400}$, and finally we conclude that when $z \in Q^\iota$ and n is large enough, $\left|(f_u^l)'(z)-1\right| \leq \frac{1}{50}$.

The following lemma is a classical consequence of the usual Contraction Mapping Principle, and will be left to the reader:

Lemma 6.6. *Let f be a holomorphic function on $\mathbb{D}_r$ such that $|f'-1| \leq a < 1$ on $D(0,r)$. Then f is injective on $\mathbb{D}_r$ and*

$$D(f(0),(1-a)r) \subset f(\mathbb{D}_r) \subset D(f(0),(1+a)r).$$

From this we deduce that there exists $r>0$ independent on n such that f_u^l is univalent on $B(z^\iota,r)$, and its image contains $B(f^l(z^\iota),r)$. Likewise, there exists $r>0$ such that f_u^{-m} is univalent on $B(z^o,r)$, with derivative $\frac{1}{50}$-close to 1. Thus we conclude that f_u^l maps univalently $B(z^\iota,\frac{r}{2})$ into $B(z^o,r)$, and its derivative satisfies $|(f_u^n)'(z)-1| \leq \frac{1}{5}$ This completes the proof of the theorem. □

6.3. **Transit mappings in dimension 2.** We return to the two-dimensional setting. The treatment will be based on the observation that in a two-dimensional thickening of the domain Ω_R, the maps f_λ admit a *dominated splitting*, i.e., they have a horizontal cone field invariant under the forward dynamics, and moreover, they are contracting in the vertical direction.

Let us first fix some notation. As before the parameter space Λ is the unit disk. Changing coordinates and passing to an iterate if needed, by Proposition 6.1 we may assume that $(f_\lambda)_{\lambda\in\Lambda}$ is a holomorphic family of germs of diffeomorphisms in $(\mathbb{C}^2,0)$ of the form

$$f_\lambda(x,y) = (\rho_\lambda x + x^{k+1} + x^{k+2}g_\lambda(x,y), b_\lambda y + xh_\lambda(x,y), \tag{15}$$

where $\rho_0 = 1$, $\frac{d\rho}{d\lambda}(0) \neq 0$, and $|b_\lambda| \leq b < 1$ for all λ.

As in the one-dimensional case, we consider two consecutive sectors $S^\iota = \left\{\arg x \in \left(-\frac{5\pi}{4k},-\frac{3\pi}{4k}\right)\right\}$ and $S^o = \left\{\arg x \in \left(-\frac{\pi}{4k},\frac{\pi}{4k}\right)\right\}$. For $\lambda=0$, consider a bidisk $V = V_1\times V_2$ around the origin such that:

- $(S^\iota \cap V_1)\times V_2$ is attracted to the origin under forward iteration;
- there exists a local repelling petal $\Sigma \subset V_1\times V_2$, which is a graph over $S^o\cap V_1$, defined by the property that every orbit converging to 0 under backward iteration in $(S^o\cap V_1)\times V_2$ belongs to Σ.

Theorem 6.7. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be as above. There exists a bidisk $V = V_1\times V_2$ around $0\in\mathbb{C}^2$ such that if $Q^\iota \Subset (S^\iota\cap V_1)\times V_2$, $Q^o \Subset \Sigma$, and $\mathcal{F}$ is a germ of holomorphic foliation transverse to Σ along Q^o, then for every neighborhood W of 0 in Λ, there exists an integer N and a radius r such that if $n\geq N$, there exists a holomorphic map $\lambda_n : Q^\iota\times Q^o \to W$ such that for* *every $(p^\iota, p^o) \in Q^\iota \times Q^o$, for $\lambda_n = \lambda_n(p^\iota, p^o)$, there exists a bidisk $D^2(p^\iota, r)$ around p^ι, and a neighborhood $D_\Sigma(p^o, r) = B(p^o, r) \cap \Sigma$ of p^o in Σ, such that the following properties hold:*

- *$f^n_{\lambda_n}(p^\iota)$ belongs to $\mathcal{F}(p^o)$, the leaf of $\mathcal{F}$ through p^o;*
- *the preimage of $\mathcal{F}$ under $f^n_{\lambda_n}$ defines a holomorphic foliation $\mathcal{F}^{-n}$ of $D^2(p^\iota, r)$ by vertical graphs along which $f^n_{\lambda_n}$ contracts by a factor b^n;*
- *the derivative of $f^n_{\lambda_n}$ along any horizontal line in $D^2(p^\iota, r)$ satisfies $\left|\frac{\partial f^n_{\lambda_n}}{\partial z} - 1\right| \leq \frac{1}{5}$.*

To prove the theorem, we consider the dynamics of f_λ in a domain of the form

$$\left\{\arg(x) \in \left(\frac{-3\pi}{2k}, \frac{\pi}{2k}\right)\right\} \times \mathbb{D}_{s_0}$$

and as in the previous section we change coordinates by putting $(z, w) = (\frac{\rho_\lambda^{k+1}}{kx^k}, y)$, and $u = \rho_\lambda^{-k} - 1$. In the new coordinates, u ranges in some small neighborhood W of the origin and f_u is defined in a domain of the form $\Omega_{R_0} \times \mathbb{D}_{s_0}$, where Ω_R is as in (7), $R_0 \geq 1$, $s_0 \leq 1$, and its expression becomes

$$f_u(z, w) = ((1 + u)z - 1 + \eta_u(z, w), b_u w + \theta_u(z, w)), \tag{16}$$

where $\eta_u(z, w)$ and $\theta_u(z, w)$ are of the form $\frac{1}{z^{1/k}}\varphi_u(\frac{1}{z^{1/k}}, w)$, with φ_u holomorphic in the neighborhood of the origin. In the new coordinates,

$$S^\iota = \left\{z,\ \arg(z) \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)\right\} \text{ and } S^o = \left\{z,\ \arg(z) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)\right\}.$$

As above we let $W = W_n = D\left(-\frac{2\pi i}{n}, \frac{1}{n^{1+\gamma/2}}\right)$ (recall that $\gamma = 1/k$).

We will gradually adjust the parameters R and s. We fix M such that for $(z, w) \in \Omega_{R_0} \times \mathbb{D}_s$ and $u \in W$,

$$|\eta_u(z, w)|,\ \left|\frac{\partial \eta_u}{\partial w}(z, w)\right|,\ |\theta_u(z, w)|,\ \left|\frac{\partial \theta_u}{\partial w}(z, w)\right| \leq \frac{M}{|z|^\gamma}, \text{ and } \left|\frac{\partial \eta_u}{\partial z}(z, w)\right| \leq \frac{M}{|z|^{1+\gamma}}.$$

Due to dissipation, there is now an asymmetry between positive and negative iterates. The idea of the construction of the transition mapping is now to pull back $n/2$ times a leaf of the foliation $\mathcal{F}$ from the "outgoing" region Q^o and to push forward $n/2$ times a point from the "incoming" region Q^i, and use the Argument Principle to make the image of the point belong to the preimage of the leaf.

We will first prove Theorem 6.7 under a seemingly stronger assumption that the foliation $\mathcal{F}$ is composed of graphs over the second coordinate in $\Omega_R \times \mathbb{D}_s$, with slope bounded by 1/100. We start by showing that the backward graph transform is well defined for such vertical graphs on an appropriate subregion of $\Omega_{R_0} \times D_{s_0}$ (as long as R_0 is large and s_0 is small). This is a standard technique for maps with dominated splitting, which is e.g. used to construct the strong stable foliation on forward invariant regions (this is not the case we are dealing with here).

Lemma 6.8. *Let $\widetilde{\Omega}_R = \left\{\zeta \in \Omega_R : D(\frac{1+\zeta}{1+u}, 1) \subset \Omega_R\right\}$. There exists R_0 and s_0 such that if $R \geq R_0$ and $s \leq s_0$, then if Γ is a vertical graph of slope $\leq 1/100$ in $\widetilde{\Omega}_R \times \mathbb{D}_s$ then $f_u^{-1}(\Gamma) \cap (\Omega_R \times \mathbb{D}_s)$ is a vertical graph in $\Omega_R \times \mathbb{D}_s$ of slope $\leq 1/100$.*

Proof. Take $\Gamma = \{z = \psi(w)\}$ with $\psi(0) \in \widetilde{\Omega}_R$ and $|\psi'| \leq 1/100$. Then $f^{-1}(\Gamma)$ admits an equation of the form $\Psi(z,w) = 0$, where

$$\Psi(z,w) = (1+u)z - 1 + \eta_u(z,w) - \psi(b_u w + \theta_u(z,w)).$$

For $w = 0$, Rouché's theorem implies that for $R \geq R_0$, there exists z such that

$$\Psi(z,0) = 0 \text{ and } \left| z - \frac{1+\psi(0)}{1+u} \right| \leq \frac{2M}{|z|^\gamma}. \tag{17}$$

In addition we have

$$\frac{\partial \Psi}{\partial z} = 1 + u + O(R^{-\gamma}) \text{ and } \left| \frac{\partial \Psi}{\partial w} \right| \leq \frac{b}{100} + O(R^{-\gamma}).$$

Thus, the result follows from the Implicit Function Theorem. □

From now on the parameter $s = s_0$ will be fixed, and for notational simplicity we denote the second factor $\mathbb{D}_{s_0}$ by D. Let $Q^\iota \Subset S^\iota \times D$. We will now state two different counterparts of Lemma 6.5: one for push-forwards, and the other one for pullbacks. For $p^\iota = (z^\iota, w^\iota) \in Q^\iota$, we denote by $p^\iota_j = (z^\iota_j, w^\iota_j)$ its j^{th} iterate under f_u.

Lemma 6.9. *With notation as above, fix $R \geq \max(M^k(1-b)^{-k}s_0^{-k}, (10^5 kM)^k)$. Then there exists an integer $N = N(R)$ such that if $n \geq N$, $p^\iota \in Q^\iota \times D$ and $u \in W_n$ then for every $1 \leq j \leq \lceil \frac{n}{2} \rceil$ we have that*

(i) *$f_u^j(p^\iota)$ belongs to $\Omega_R \times D$*
(ii) $\left| z^\iota_j - \dfrac{1}{u} \right| \geq \dfrac{n}{10}$;
(iii) $\left| z^\iota_j \right| \geq \min\left(\dfrac{|z_0|}{2} + \dfrac{j}{10}, \dfrac{n}{10} \right)$.

Proof. It follows from expression (16) for f_u that if $\frac{M}{R^\gamma} < (1-b)s_0$ and $(z,w) \in \Omega_R \times D$, then the second coordinate of $f_u(z,w)$ belongs to D. So we only need to focus on the first coordinate. By (16) we have that

$$\frac{z^\iota_{j+1} - \frac{1}{u}}{z^\iota_j - \frac{1}{u}} = 1 + u + \frac{\eta_u(f_u^j(p^\iota))}{z^\iota_j - \frac{1}{u}}, \tag{18}$$

with $\left| \eta_u(f_u^j(p^\iota)) \right| \leq \frac{M}{R^\gamma}$ as soon as $f_u^j(p^\iota) \in \Omega_R \times D$. Then the proof is identical to that of Lemma 6.5. □

We now deal with pullbacks. Given $p^o \in Q^o$, we consider a holomorphic foliation $\mathcal{F}$ by vertical graphs of slope bounded by $1/100$ in a neighborhood of p^o, and by $\mathcal{F}(p)$ the leaf through p. Starting from $\mathcal{F}_0 = \mathcal{F}(p)$, by applying successive graph transforms we inductively define $\mathcal{F}_{-j-1} = f_u^{-1}(\mathcal{F}_{-j}) \cap (\Omega_R \times D)$. We also let $\zeta_{-j} = \mathcal{F}_{-j} \cap \{w = 0\}$.

Lemma 6.10. *Let R be as in Lemma 6.9. There exists an integer $N = N(R)$ such that if $n \geq N$, if $p \in Q^o$ and $u \in W_n$ then for every $1 \leq j \leq \lceil \frac{n}{2} \rceil$ we have*

(i) *$\mathcal{F}_{-j}(p)$ is a well-defined vertical graph in $\Omega_R \times D$, with slope bounded by $1/100$;*
(ii) $\left| \zeta_{-j} - \dfrac{1}{u} \right| \geq \dfrac{n}{10}$;
(iii) $|\zeta_{-j}| \geq \min\left(\dfrac{|z_0|}{2} + \dfrac{j}{10}, \dfrac{n}{10} \right)$.

Proof. From (17) we infer that

$$\left|\zeta_{-j-1} - \frac{1+\zeta_{-j}}{1+u}\right| \leq \frac{2M}{|\zeta_{-j}|^\gamma},$$

that is,

$$\frac{\zeta_{-j-1} - \frac{1}{u}}{\zeta_{-j} - \frac{1}{u}} = \frac{1}{1+u} + \frac{\varepsilon_j}{\zeta_{-j} - \frac{1}{u}}, \text{ with } |\varepsilon_j| \leq \frac{2M}{|\zeta_{-j}|^\gamma}, \tag{19}$$

so as before the result follows exactly as in the one-dimensional case (for (i) we also use Lemma 6.8). □

Pick now an R satisfying all the above requirements. Let as above $p^\iota = (z^\iota, w^\iota) \in Q^\iota$, $p^o \in Q^o$, and let $\mathcal{F}^o$ be the leaf of $\mathcal{F}$ through p^o. Let $l = \lfloor n/2 \rfloor$ and $m = n - l$. By Lemma 6.9, for $1 \leq j \leq l$ $f_u^j(p^\iota) \in \Omega_R$. Then, using (18) exactly as in (12) (i.e. by taking the product from 0 to $l-1$) we deduce that

$$\left|\frac{z_l^\iota - \frac{1}{u}}{(1+u)^l(z^\iota - \frac{1}{u})} - 1\right| = O\left(\frac{1}{n^\gamma}\right).$$

On the pullback side, recall that $\mathcal{F}_{-j}(p^o)$ denotes the j^{th} graph transform of $\mathcal{F}_0$, and let $\zeta_{-j} = \mathcal{F}_{-j}(p^o) \cap \{w = 0\}$. Using (19) and taking the product from $j = 0$ to $j = -m+1$ we obtain:

$$\left|(1+u)^m \frac{\zeta_{-m} - \frac{1}{u}}{\zeta_0 - \frac{1}{u}} - 1\right| = O\left(\frac{1}{n^\gamma}\right).$$

Thus, writing $u = \frac{-2\pi i}{n} + \frac{v}{n^{1+\gamma/2}}$, from the two previous displayed equations together with (9), we obtain that

$$\frac{z_l^\iota - \frac{1}{u}}{\zeta_{-m} - \frac{1}{u}} = 1 + \frac{v}{n^{\gamma/2}} + O\left(\frac{1}{n^\gamma}\right). \tag{20}$$

Now express the graph $\mathcal{F}_{-m}$ as $z = \psi(w)$, with $\psi(0) = \zeta_{-m}$. Since $|\psi(w_l^\iota) - \psi(0)| \leq 1/100$, we infer that

$$\frac{\psi(w_l^\iota) - \frac{1}{u}}{\zeta_{-m} - \frac{1}{u}} = 1 + O\left(\frac{1}{n}\right),$$

so from (20) we finally deduce that

$$\frac{z_l^\iota - \frac{1}{u}}{\psi(w_l^\iota) - \frac{1}{u}} = 1 + \frac{v}{n^{\gamma/2}} + O\left(\frac{1}{n^\gamma}\right).$$

By the Argument Principle we conclude that for every $(p^\iota, p^o) \in Q^\iota \times Q^o$, there exists a unique (hence, depending holomorphically on (p^ι, p^o)) $u = u(p^\iota, p^o) \in W_n$ such that $\psi(w_l^\iota) = z_l^\iota$, that is, $f_u^l(p^\iota) \in \mathcal{F}_{-m}(p^o)$.

For this parameter u we can pull back $\mathcal{F}_{-m}(p^o)$ under f_u^m, thus obtaining a vertical graph $\mathcal{F}_{-n}(p^o)$ through p^ι. It is clear that the derivative df^n contracts exponentially along this graph, more precisely $\left\|d(f^n|_{\mathcal{F}_{-n}(p^o)})\right\| \lesssim b^n$. Indeed, the tangent vectors to $\mathcal{F}_{-n}(p^o)$ remain in a cone field close to the vertical under iteration, and the second factor gets contracted at rate b.

From now on the parameter u is fixed. To simplify notation we drop the subscript u and write $f_u^j = (f_1^j, f_2^j)$ We will prove at the same time that f_u^n is defined in a fixed domain around

p^ι and estimate its derivatives. Let $r > 0$ be such that all iterates f^j, $1 \leq j \leq n$ are well defined on $D(z^\iota, r) \times \{w^\iota\}$, and for all $j \leq n$, $\left\|f^j(p^\iota) - f^j(z, w^\iota)\right\|$ is bounded by, say, 1. For the moment r depends on n.

The estimate we need is contained in the following lemma. Denote $f^j(z, w^\iota)$ by (z_j, w_j^ι).

Lemma 6.11. *For r as above, let $K = 2(1-b)^{-1}M$. Then for every $z \in D(z^\iota, r)$, and every $1 \leq j \leq n$*

$$\frac{\partial f_1^j}{\partial z}(z, w^\iota) = (1+u)^j \prod_{i=1}^{j}(1+\delta_i), \text{ with } |\delta_i| \leq \frac{K}{|z_{i-1}|^{1+\gamma}}, \text{ and } \left|\frac{\partial f_2^j}{\partial z}(z, w^\iota)\right| \leq \frac{K}{|z_{j-1}|^{1+\gamma}}$$

As a preliminary observation, notice that if $R \geq (10^6 k(1-b)^{-1}M)^k$, and δ_i is as in the statement of the lemma and n is large enough, then for every $1 \leq j \leq n$,

$$\left|(1+u)^j \prod_{i=1}^{j}(1+\delta_i) - 1\right| \leq \frac{1}{5}. \tag{21}$$

Indeed, this follows from the proof of Theorem 6.3 (see (13) and (14) there; also if $j \geq l$ we need to split the product at l and to estimate separately the two terms).

Proof. We argue by induction on j. The result holds true for $j = 1$. So assume that it holds for some j. We compute

$$\begin{aligned}\frac{\partial f_1^{j+1}}{\partial z}(z, w^\iota) &= \left((1+u) + \frac{\partial \eta_u}{\partial z}(z_j, w_j^\iota)\right)\frac{\partial f_1^j}{\partial z}(z, w^\iota) + \frac{\partial f_2^j}{\partial z}(z, w^\iota)\frac{\partial \eta_u}{\partial w}(z_j, w_j^\iota)\\ &= (1+u)^{j+1}\prod_{i=1}^{j}(1+\delta_i)\left(1 + \frac{1}{1+u}\frac{\partial \eta_u}{\partial z}(z_j, w_j^\iota)\right) + \frac{\partial f_2^j}{\partial z}(z, w^\iota)\frac{\partial \eta_u}{\partial w}(z_j, w_j^\iota).\end{aligned}$$

By the induction hypothesis,

$$\left|\frac{\partial f_2^j}{\partial z}(z, w^\iota)\frac{\partial \eta_u}{\partial w}(z_j, w_j^\iota)\right| \leq \frac{K}{|z_{j-1}|^{1+\gamma}}\frac{M}{|z_j|^\gamma}.$$

Since $\frac{z_{j+1}}{z_j}$ is close to $1+u$ and $(1+u)^{j+1}\prod_{i=1}^{j}(1+\delta_i)$ is close to 1, we can write

$$\frac{\partial f_1^{j+1}}{\partial z}(z, w^\iota) = (1+u)^{j+1}\prod_{i=1}^{j}(1+\delta_i)\left(1 + \frac{1}{1+u}\frac{\partial \eta_u}{\partial z}(z_j, w_j^\iota) + \delta\right), \text{ with } |\delta| \leq \frac{2KM}{|z_j|^{1+2\gamma}}.$$

Thus if we put $\delta_{j+1} = \frac{1}{1+u}\frac{\partial \eta_u}{\partial z}(z_j, w_j^\iota) + \delta$ we get that

$$|\delta_{j+1}| \leq \left(\frac{M}{|1+u|} + \frac{2KM}{R^\gamma}\right)\frac{1}{|z_j|^{1+\gamma}},$$

which, from the choice of R and K is not greater than $\frac{K}{|z_j|^{1+\gamma}}$.

To get the bound on the derivative of f_2^{j+1}, we write

$$\frac{\partial f_2^{j+1}}{\partial z}(z, w^\iota) = b_u\frac{\partial f_2^j}{\partial z}(z, w^\iota) + \frac{\partial f_1^j}{\partial z}(z, w^\iota)\frac{\partial \theta_u}{\partial z}(z_j, w_j^\iota) + \frac{\partial f_2^j}{\partial z}(z, w^\iota)\frac{\partial \theta_u}{\partial w}(z_j, w_j^\iota),$$

and we get that

$$\left|\frac{\partial f_2^{j+1}}{\partial z}(z, w^\iota)\right| \leq b\frac{K}{|z_{j-1}|^{1+\gamma}} + \frac{6}{5}\frac{M}{|z_j|^{1+\gamma}} + \frac{K}{|z_{j-1}|^{1+\gamma}}\frac{M}{|z_j|^\gamma}$$
$$\leq \frac{K}{|z_j|^{1+\gamma}}\left(b\left|\frac{z_j}{z_{j-1}}\right|^{1+\gamma} + \frac{6M}{5K} + \frac{M}{R^\gamma}\left|\frac{z_j}{z_{j-1}}\right|^{1+\gamma}\right).$$

To conclude, we observe that when n is large enough, due to the choice of R and K, the expression within parentheses is smaller than 1. The proof of the lemma is complete. □

We are now in position to conclude the proof of Theorem 6.7. Let r_0 be the supremum of the radii $r > 0$ such that f^j is well-defined, and $f^j(z, w^\iota)$ stays at distance at most 1 from $f^j(z^\iota, w^\iota)$ for $1 \leq j \leq n$. By the above lemma and (21), $r_0 \geq \frac{2}{3}$. Then the image of $D(z^i, r_0) \times \{w^\iota\}$ under f^n is a graph over some neighborhood of z^o, which by Lemma 6.6 must contain $D(z^o, \frac{1}{2})$. Now since the repelling petal Σ is a graph (relative to the first coordinate) over $\{z,\ \mathrm{Re}(z) > R\}$, we infer that for $p \in B(p^o, \frac{1}{4}) \cap \Sigma$, $f^n(D(z^i, r_0) \times \{w^\iota\})$ intersects $\mathcal{F}(p)$ close to p. Therefore we can pull back $\mathcal{F}(p)$ under f^n to get a vertical graph intersecting $D(z^i, r_0) \times \{w^\iota\}$ along which (for the same reasons as before) the derivative of f^n along is smaller than b^n, and the proof is complete.

What remains to be done is to remove the simplifying assumption that $\mathcal{F}$ is a foliation by vertical graphs in $\Omega_R \times D$. For this we simply iterate backwards and use the previous analysis to show that we for k large enough, $f^{-k}(\mathcal{F})|_{\Omega_R \times D}$ is made of vertical graphs of slope $\leq 1/100$. Indeed, let Δ be a germ of a holomorphic disk transverse to Σ at $p^o = (z_0, w_0) \in \Omega_R \times D$. Let $f_0^{-k}(p) = (z_{-k}, w_{-k})$ which (in our coordinates) converges to infinity by staying in $\Omega_R \times D$. Applying the reasoning of Lemma 6.11 for $u = 0$ (together with (21)) shows that for every $w \in D$, $f^k(D(z_{-k}, \frac{2}{3}) \times \{w\})$ is a horizontal graph over $D(z_0, \frac{1}{2})$, which is exponentially close to Σ due to vertical contraction. Thus when k is large enough, it intersects Δ exactly in one point, and the result follows.

7. Proof of the main theorem on homoclinic tangencies

7.1. **Creating tangencies between horizontal and vertical moving curves.** Here we explain how to obtain a tangency between two holomorphically moving complex curves by using only "soft complex analysis", i.e. basically the Argument Principle. We work in the unit bidisk $\mathbb{B} = \mathbb{D}^2$. A subvariety V in $\mathbb{B}$ (or current, etc.) is *horizontal* if there exists some $\varepsilon > 0$ such that $V \subset \mathbb{D} \times \mathbb{D}_{1-\varepsilon}$. Vertical objects are defined similarly.

Following [HO], we define the *horizontal (resp. vertical) Poincaré cone field* as the set of tangent vectors $v = (v_1, v_2) \in T_x\mathbb{B} \simeq \mathbb{C}^2$ such that $|v_1|_{\mathrm{Poin}} > |v_2|_{\mathrm{Poin}}$ (resp. $|v_2|_{\mathrm{Poin}} > |v_2|_{\mathrm{Poin}}$), where $|\cdot|_{\mathrm{Poin}}$ denotes the Poincaré metric in $\mathbb{D}$. The contraction property of the Poincaré metric implies that if Γ is a horizontal graph in $\mathbb{B}$, then for every $x \in \Gamma$, $T_x\Gamma$ is contained in the horizontal Poincaré cone field.

A horizontal manifold (or subvariety) V in $\mathbb{B}$ has a *degree*, which is the degree of the branched cover $\pi_1 : V \to \mathbb{D}$ (here of course π_1 is the first coordinate projection). If V is irreducible and $d > 1$ then $\pi_1|_V$ must have critical points (indeed otherwise it would be a non-trivial covering of the unit disk). In particular, it admits tangent vectors in the vertical Poincaré cone field.

By definition a holomorphic family of submanifolds $(V_\lambda)_{\lambda\in\Lambda}$ of a complex manifold M is the data of a codimension 1 analytic set (which might be singular) $\widehat{V}\subset\Lambda\times M$ such that for every $\lambda\in\Lambda$, $V_\lambda=\widehat{V}\cap(\{\lambda\}\times\mathbb{B})$.

Here is the precise statement.

Proposition 7.1. *Let $(V_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of horizontal submanifolds of degree k in $\mathbb{B}$, parameterized by a connected Stein manifold Λ. We assume that:*

- (i) *There exists a compact subset $\Lambda_0\Subset\Lambda$ such that if $\lambda\notin\Lambda_0$, V_λ is the union of k graphs.*
- (ii) *There exists $\lambda_0\in\Lambda$ such that V_{λ_0} is not the union of graphs.*

Then, if $(W_\lambda)_{\lambda\in\Lambda}$ is any holomorphic family of vertical graphs in $\mathbb{B}$, there exists $\lambda_1\in\Lambda$ such that V_{λ_1} and W_{λ_1} admit a point of tangency.

Using the above remarks, we see that condition (ii) could be replaced by "there exists $x\in\mathbb{B}$ and $\lambda_0\in\Lambda$ such that $T_xV_{\lambda_0}$ is contained in the vertical Poincaré cone field".

Proof. To simplify the exposition, we assume that Λ is the unit disk (we will use the result in that case only). The proof in the general case is similar.

Notice first that if λ is close to $\partial\Lambda$, then there are no tangencies between V_λ and W_λ. Indeed the tangent vectors to V_λ and W_λ belong to disjoint cone fields. In particular, reducing Λ a little bit if needed, we may assume that the V_λ (resp. W_λ) are uniformly horizontal (resp. vertical), that is, that they are contained in $\mathbb{D}\times\mathbb{D}_{1-\varepsilon}$ (resp. $\mathbb{D}_{1-\varepsilon}\times\mathbb{D}$) for some fixed $\varepsilon>0$.

If $V\subset\mathbb{B}$ is a smooth holomorphic curve, we let $\mathbb{P}TV$ be its lift (which is also a holomorphic curve) to the projectivized tangent bundle $\mathbb{P}T\mathbb{B}\simeq\mathbb{B}\times\mathbb{P}^1$. Notice that since V is smooth, $\mathbb{P}TV$ intersects every $\mathbb{P}^1$ fiber at a single point. If $(V_\lambda)_{\lambda\in\Lambda}$ is a holomorphic family of submanifolds, we obtain in this way a holomorphic family of submanifolds $(\mathbb{P}TV_\lambda)_{\lambda\in\Lambda}$ in $\mathbb{B}\times\mathbb{P}^1$. In other words, there exists a subvariety of $\Lambda\times\mathbb{B}\times\mathbb{P}^1$, of dimension 2, which we denote $\widehat{\mathbb{P}TV}$ such that for every $\lambda\in\Lambda$,

$$\widehat{\mathbb{P}TV}\cap(\{\lambda\}\times\mathbb{B}\times\mathbb{P}^1)=\mathbb{P}TV_\lambda.$$

Let now $W=(W_\lambda)_{\lambda\in\Lambda}$ be any holomorphic family of vertical graphs in $\mathbb{B}$. An intersection point between $\widehat{\mathbb{P}TV}$ and $\widehat{\mathbb{P}TW}$ corresponds to a parameter λ_0 at which V_{λ_0} and W_{λ_0} are tangent. We claim that then $\widehat{\mathbb{P}TV}\cap\widehat{\mathbb{P}TW}$ has dimension 0 (if non-empty). In particular, the varieties $\widehat{\mathbb{P}TV}$ and $\widehat{\mathbb{P}TW}$ intersect properly in $\mathbb{B}\times\mathbb{P}^1\times\Lambda$. Observe first that this intersection is compactly supported in $\Lambda\times\mathbb{B}\times\mathbb{P}^1$, indeed:

- as observed above, there are no tangencies between V_λ and W_λ when λ is close to $\partial\Lambda$;
- the intersection points between V_λ and W_λ are contained in $\mathbb{D}^2_{1-\varepsilon}$ for some $\varepsilon>0$.

By the Maximum Principle, the projection of $\widehat{\mathbb{P}TV}\cap\widehat{\mathbb{P}TW}$ to $\Lambda\times\mathbb{B}$ is a finite set. Hence any component of $\widehat{\mathbb{P}TV}\cap\widehat{\mathbb{P}TW}$ of positive dimension is contained in a $\mathbb{P}^1$ fiber, which is impossible by definition of the lifts $\widehat{\mathbb{P}TV}$ and $\widehat{\mathbb{P}TW}$. This proves our claim.

By assumption, there exists λ_0 such that V_{λ_0} admits a vertical tangent vector, hence a tangency with some vertical line L. Let $\widehat{\mathbb{P}TL}\subset\Lambda\times\mathbb{B}\times\mathbb{P}^1$ be the surface corresponding to the trivial family where L is fixed. Then $\widehat{\mathbb{P}TV}\cap\widehat{\mathbb{P}TL}$ is non-empty, therefore it is a finite set.

We can now deform L to W through some holomorphic family $(W_{\lambda,s})$ of vertical graphs with $W_{\lambda,0}=L$ and $W_{\lambda,1}=W_\lambda$, and s ranges in some neighborhood $\mathbb{D}_{1+\varepsilon}$ of the closed unit disk. For this we can simply take a linear interpolation. In this way we obtain a holomorphic deformation $(\widehat{\mathbb{P}TW_s})_{s\in\overline{\mathbb{D}}_\varepsilon}$ from $\widehat{\mathbb{P}TL}$ to $\widehat{\mathbb{P}TW}$, parameterized by a neighborhood of the unit disk.

To conclude that $\widehat{\mathbb{P}TV} \cap \widehat{\mathbb{P}TW} \neq \emptyset$, we argue that the set of parameters $s \in \mathbb{D}_{1+\varepsilon}$ such that $\widehat{\mathbb{P}TV} \cap \widehat{\mathbb{P}TW_s} \neq \emptyset$ is open in $\mathbb{D}_{1+\varepsilon}$ by the continuity of the intersection index of properly intersecting analytic sets of complementary dimensions (see [Ch, Prop. 2 p.141]) and closed because intersection points stay compactly contained in $\mathbb{B}$. This completes the proof. □

7.2. **From critical points to tangencies.** In this section we settle the second main step of Theorem A′. If p_λ is a holomorphically varying periodic point for a holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of dissipative polynomial automorphisms of $\mathbb{C}^2$, we say that p_λ *undergoes a non-degenerate semi-parabolic bifurcation at* λ_0 if one of the multipliers ρ_λ of p_λ satisfies $\rho_{\lambda_0} = 1$ and $\frac{\partial\rho}{\partial\lambda}|_{\lambda=\lambda_0}$ is a submersion $\Lambda \to \mathbb{C}$.

Proposition 7.2. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of dissipative polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree d with positive entropy, parameterized by the unit disk. Assume that:*

- *there exists a holomorphically varying periodic point p_λ which admits a non-degenerate semi-parabolic bifurcation at λ_0;*
- *for $\lambda = 0$, there is a critical point in one of the basins of attraction of p_λ.*

Then λ_0 can be approximated by parameters possessing homoclinic tangencies.

Proof. Without loss of generality we may assume that Λ is the unit disk, $\lambda_0 = 0$, and p_λ is fixed. Normalize the situation so that f_λ is locally of the form (15). Conjugating by a rotation, we may assume that the critical point lies in the basin $\mathcal{B}$ corresponding to the attracting direction $\left\{(x,0),\ \arg(x) = -\frac{\pi}{k}\right\}$. Let q_0 be a saddle point such that $W^u(q_0)$ admits a point of tangency with the strong stable foliation in $\mathcal{B}$. (Then, by the hyperbolic λ-lemma (see e.g. [PT, Thm 2, p. 155]) and the fact that any pair of saddle points have transverse heteroclinic intersections, any saddle point would do.) Let t be such a point of tangency.

The global repelling petal Σ in the direction of $\{(x,0),\ \arg(x) = 0\}$ is an immersed curve biholomorphic to $\mathbb{C}$ ([U1], see §4). Hence, using the theory of laminar currents, exactly as in [BLS1], it admits transversal intersections with $W^s(q_0)$. We fix a transverse intersection point $m \in \Sigma \cap W^s(q_0)$.

Fix a neighborhood W of 0 in Λ. Iterating t forward and m backward a few times, we may assume that both points are close to 0. Theorem 6.7 thus provides us with an integer N and a radius r, so that for $n \geq N$, a transition mapping $f^n_{\lambda_n}$ is defined from $D^2(t,r)$ to $D^2(m,r)$ such that $f^n_{\lambda_n}(t) = m$. Let Γ^u_λ be the component of $W^u_\lambda(q_\lambda) \cap D^2(t,r)$ containing t. Reducing W if necessary, Γ^u_λ can be followed holomorphically as $\lambda \in W$.

To make the situation visually clearer, we consider adapted coordinates close to t and m. These changes of coordinates have bounded derivatives. Abusing slightly, we declare that in the new coordinates, the neighborhoods remain of size r. Near t we choose (z^ι, w^ι) so that $t = (0,0)$, the strong stable foliation $\mathcal{F}^{ss}$ becomes the vertical foliation $\left\{z^\iota = C^{st}\right\}$ and for $\lambda \in W$, Γ^u_λ is a horizontal manifold in $D^2(t,r)$ of some degree $d \geq 2$, which is transverse to $\mathcal{F}^{ss}$ outside t. Near m we choose local coordinates (z^o, w^o) such that $m = (0,0)$ $\Sigma^0 = \{w^o = 0\}$, and the component of $W^s(q_\lambda)$ containing m is $\{z^o = 0\}$. Denote by $\mathcal{F}$ the vertical foliation in the target bidisk $D^2(m,r)$.

For $|z^o| \leq r$, let us consider the parameter $\lambda_n = \lambda_n(t, z^o)$ given by Theorem 6.7 such that the first coordinate of $f^n_{\lambda_n}(t)$ is z^o. For every such parameter, by Theorem 6.7 $f^n_{\lambda_n}$ realizes a crossed mapping of degree 1 [HO] from $D^2(t,r)$ to $D(z^o, \frac{r}{2}) \times \mathbb{D}_r$. So when $|z^o| \leq \frac{r}{4}$ we get by restriction a crossed mapping from $D^2(t,r)$ to $\mathbb{D}_{\frac{r}{4}} \times \mathbb{D}_r$. In particular, we infer that for $|z^o| \leq \frac{r}{4}$, $f^n_{\lambda_n}(\Gamma^u_{\lambda_n})$ is a horizontal submanifold of degree d in $\mathbb{D}_{\frac{r}{4}} \times \mathbb{D}_r$.

To conclude the argument, let us show that when z^o ranges in $\mathbb{D}_{\frac{r}{4}}$ and n is large, the family of curves $f^n_{\lambda_n}(\Gamma^u_{\lambda_n})$ satisfies the assumptions of Proposition 7.1 in $\mathbb{D}_{\frac{r}{8}} \times \mathbb{D}_r$.

The first observation is that the preimage $\mathcal{F}^{-n}_{\lambda_n}$ of $\mathcal{F}$ under $f^n_{\lambda_n}$ converges to the strong stable foliation associated to f_0 in $D^2(t,r)$, uniformly in z^o. Indeed, we know that the leaves of $\mathcal{F}_{-n}$ are graphs with bounded geometry over some fixed direction, and the $f^n_{\lambda_n}$ contract exponentially along these graphs, with uniform bounds. So any cluster limit of $\mathcal{F}^{-n}$ must be $\mathcal{F}^{ss}(f_0)$, which proves our claim.

Now, when $|z^o| = \frac{r}{4}$, for every z such that $|z| < \frac{3r}{16}$, when n is large, for $\lambda_n = \lambda_n(t, z^o)$ $f^{-n}_{\lambda_n}(\{z\} \times \mathbb{D}_r)$ is close to a leaf of $\mathcal{F}^{ss}$ which intersects $\mathbb{D}_r \times \{0\}$ transversely at a distance $\geq \frac{r}{8}$ from t. Therefore, $f^{-n}_{\lambda_n}\big(\mathbb{D}_{\frac{3r}{16}} \times \mathbb{D}_r\big) \cap \Gamma^u_{\lambda_n}$ is the union of d graphs, over a disk of radius greater than $\frac{3r}{16} \cdot \frac{4}{5} = \frac{3r}{20}$ by Lemma 6.6. Pushing by $f^n_{\lambda_n}$ and applying the lemma again, we conclude that $f^n_{\lambda_n}(\Gamma^u_{\lambda_n})$ is a union of d graphs over $\mathbb{D}_{\frac{3r}{25}}$.

On the contrary, when $z^o = 0$, for $\lambda_n = \lambda_n(t, 0)$, $\mathrm{pr}_1(f^n_{\lambda_n}) = 0$. Since $\mathcal{F}^{-n}$ converges to $\mathcal{F}^{\mathrm{ss}}$, when n is large $\Gamma^u_{\lambda_n}$ has a tangency with $\mathcal{F}^{-n}$ close to t, hence $f^n_{\lambda_n}(\Gamma^u_{\lambda_n})$ has a vertical tangency close to m.

We see that the assumptions of Proposition 7.1 are satisfied so there exists a parameter $\lambda_n = \lambda_n(t, z^o)$ such that $f^n_{\lambda_n}(\Gamma^u_{\lambda_n})$ has a tangency with $W^s(q_{\lambda_n})$ close to m, and we are done. □

Proof of Theorem A′. Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$ with a bifurcation at λ_0. By Proposition 1.1 we may assume that the f_λ are products of Hénon mappings. Then close to λ_0 there is a periodic point with a multiplier ρ crossing the unit circle. Without loss of generality we may assume that $\dim(\Lambda) = 1$. Hence there exists λ_1 close to λ_0 such that at λ_1, the multiplier is a root of unity, different from 1, so that this periodic point p_λ can be followed holomorphically close to λ_1. In addition we may assume that $\frac{\partial\rho}{\partial\lambda}|_{\lambda=\lambda_1} \neq 0$. Replacing f_λ by f^k_λ for some k, we may assume that p_λ is fixed and $\rho_{\lambda_1} = 1$ (we keep the same notation for the new multiplier, which is the k^{th} power of the previous one) . Notice that for the new multiplier we still have that $\frac{\partial\rho}{\partial\lambda}|_{\lambda=\lambda_1} \neq 0$.

Since the condition that $|\mathrm{Jac}\, f_\lambda| < \deg(f_\lambda)^{-2}$ is preserved under iteration, Theorem B asserts that there is a critical point in every component of the attracting basin of p_{λ_1}. Thus the result follows from Proposition 7.2. □

Appendix A. Attracting basins

The methods of §5.2 also give the existence of "critical points" in attracting basins, under certain minor hypotheses (that are needed to even define the critical points). Though these results are not used in the paper, they are interesting on their own right.

Let f be a polynomial automorphism of dynamical degree $d \geq 2$ with an attracting point p. As usual, we may assume that p is fixed, and we denote by $\mathcal{B}$ its basin of attraction. It is classical that there is a local holomorphic change of coordinates which puts f in a simple normal form (this result goes apparently back to Lattès [La]). Let κ_1 and κ_2 be the eigenvalues of Df_p, ordered so that $0 < |\kappa_2| \leq |\kappa_1| < 1$. We say that (κ_1, κ_2) is resonant if there exists an integer $i \geq 1$ such that $\kappa_2 = \kappa_1^i$ (notice that $i = 1$ is allowed). Then there exists a local change of coordinates near p such that in the new coordinates (z_1, z_2), f expresses as

$$f(z_1, z_2) = \begin{cases} (\kappa_1 z_1, \kappa_2 z_2) \text{ if } (\kappa_1, \kappa_2) \text{ is not resonant,} \\ (\kappa_1 z_1, \kappa_2 z_2 + \alpha z_1^i) \text{ otherwise, where } i \text{ is as above, and } \alpha \in \{0, 1\} . \end{cases}$$

In any case, we see that the vertical foliation $\{z_1 = C\}$ is invariant under f. If $|\kappa_2| < |\kappa_1|$ this is the "strong stable foliation", characterized by the property that points in the same leaf approach each other at the fastest possible rate κ_2^n. As before, it will be denoted by $\mathcal{F}^{ss}$. Using the dynamics, the coordinates (z_1, z_2) extend to the basin and define a biholomorphism $\mathcal{B} \simeq \mathbb{C}^2$. In the non-resonant (i.e. linearizable) case, the foliation $\{z_2 = C\}$ is invariant as well. We then simply refer to $\{z_1 = C\}$ and $\{z_2 = C\}$ as the invariant coordinate foliations in $\mathcal{B}$.

We give two statements on the existence of critical points. The first one parallels Theorem B.

Theorem A.1. *Let f be a polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 2$, possessing an attracting point p, whose eigenvalues satisfy $0 < |\kappa_2| < |\kappa_1| < 1$, with basin of attraction $\mathcal{B}$. Assume that $|\mathrm{Jac}\, f| < d^{-4}$, or more generally that the connected component of p in $W^{ss}(p) \cap J^-$ is $\{p\}$. Then for every saddle periodic point q, every component of $W^u(q) \cap \mathcal{B}$ contains a critical point, that is, a point of tangency with the strong stable foliation in $\mathcal{B}$.*

The second statement concerns the hyperbolic case.

Theorem A.2. *Let f be a polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 2$, possessing an attracting point p with basin $\mathcal{B}$. Assume that f is uniformly hyperbolic on J, and fix any saddle periodic point q.*

If the eigenvalues of p satisfy $|\kappa_2| < |\kappa_1|$ (resp. are non-resonant), then every component of $W^u(q) \cap \mathcal{B}$ admits a tangency with the strong stable foliation of $\mathcal{B}$ (resp. with both invariant coordinate foliations).

Here is an interesting geometric consequence. Recall that if f is dissipative and hyperbolic, J^+ is (uniquely) laminated by stable manifolds. Let us denote by $W^s(J)$ this lamination. It is natural to wonder whether the strong stable foliation in $\mathcal{B}$ matches continuously with the lamination of J^+ (recall that $\partial\mathcal{B} = J^+$). The existence of critical points implies that this is never the case (compare [BS7, Cor. A.2]).

Corollary A.3. *Let f be as in the previous theorem, in particular f is hyperbolic on J. Then if p is an attracting point with eigenvalues $|\kappa_2| < |\kappa_1|$ and basin $\mathcal{B}$, then for every $x \in J$, $W^s(J) \cup \mathcal{F}^{ss}(\mathcal{B})$ does not define a lamination near x. If p is linearizable, the same holds for both invariant coordinate foliations.*

Proof. Let us deal with the case where $|\kappa_1| < |\kappa_2|$. It is enough to assume that x is a saddle periodic point. Hyperbolicity implies that $W^u(J)$ and $W^s(J)$ are transverse near x, so if $W^s(J) \cup \mathcal{F}^{ss}(\mathcal{B})$ is a lamination near x, $\mathcal{F}^{ss}(\mathcal{B})$ must be transverse to $W^u(J)$ near x. On the other hand, there exist critical points on $W^u(x)$ arbitrary close to x (obtained from the previous ones by iterating backwards). This contradiction finishes the proof. □

Proof of Theorems A.1 and A.2. This is very similar to Proposition 5.8 so the proof is merely sketched. Let us first deal with the case where $|\kappa_2| < |\kappa_1|$, with f hyperbolic or not. Let $\pi_1 : \mathcal{B} \to \mathbb{C}$ be the projection along the strong stable foliation. In the coordinates (z_1, z_2), it simply expresses as $(z_1, z_2) \mapsto z_1$. Assume by contradiction that there is no critical point in Ω. Then $\pi_1 \circ \psi^u : \Omega \setminus (\psi^u)^{-1}(W^{ss}(p)) \to \mathbb{C}^*$ is a locally univalent map. Since it cannot be a covering it must possess an asymptotic value, hence there is a diverging path γ in Ω such that the limit $\lim_{t\to\infty} \pi_1 \circ \psi^u(\gamma(t)) = \omega$ exists in $\mathbb{C}^*$. Let $\pi_2 : \mathcal{B} \to \mathbb{C}$ be the second coordinate projection. As before, we split the argument according to the bounded or unbounded character of $\pi_2 \circ \psi^u(\gamma)$.

If $\pi_2 \circ \psi^u(\gamma)$ is unbounded, we iterate forward and take cluster values to create an unbounded component C of $J^- \cap W^{ss}(p)$ containing p. Now if $|\mathrm{Jac}\, f| < d^{-4}$, then $|\kappa_2| < d^{-2}$, so by Corollary 5.7, the component of p in $W^{ss}(p) \cap J^-$ is a point, and we get a contradiction.

If f is hyperbolic we argue as follows: in $\mathcal{B} \setminus \{p\}$, J^- is laminated by unstable manifolds. In particular by [BLS1, Lemma 6.4] the set of tangencies between $W^{ss}(p)$ and the unstable lamination is discrete. Pick $c \in C \setminus \{p\}$ such that $W^{ss}(p)$ and the unstable lamination are transverse near c. There exist coordinates (x, y) close to c in which $W^{ss}(p)$ is $\{x = 0\}$, $c = (0, 0)$, and the leaves of the unstable lamination close to c are horizontal in the unit bidisk $\mathbb{B}$. By construction, there is a sequence of integers n_j such that $f^{n_j}(\psi^u(\gamma))$ has a component C_j, vertically contained in $\mathbb{B}$, touching the boundary, and passing close to c. On the other hand C_j must be contained in a leaf of the unstable foliation so we get a contradiction.

If $\pi_2 \circ \psi^u(\gamma)$ is bounded, then as before the path γ must be unbounded in $W^u(q)$. Let E be the cluster set of $\psi^u(\gamma)$, which is a compact subset of the strong stable leaf $\{z_1 = \omega\}$. If $|\kappa_2| < |\kappa_1|$, then as in Proposition 5.8 we make a linear change of coordinates close to p such that in the new coordinates, f expresses as $f(x, y) = (\kappa_1 x, \kappa_2 y) + h.o.t.$ We see that in these coordinates, $\mathrm{pr}_1(f^n(E))$ is a set of diameter $\lesssim \kappa_2^n$ about $x_n \sim c\kappa_1^n$, which leads to a contradiction with Theorem 5.1, exactly as in Proposition 5.8.

It remains to treat the case where f is hyperbolic, p is linearizable, and we look for tangencies with any of the invariant coordinate foliations. We argue exactly as before, with (π_1, π_2) being the linearizing coordinate projections, in any order, and keep the same notation. The case where $\pi_2 \circ \psi^u(\gamma)$ is unbounded is dealt with exactly as above. If now $\pi_2 \circ \psi^u(\gamma)$ is bounded and E denotes its cluster set in the leaf $\{z_1 = \omega\}$, we observe that as in the unbounded case, the laminar structure of J^- outside p forces E to be reduced to a point. Therefore ψ^u admits an asymptotic value in $\mathcal{B} \setminus \{p\}$ and the contradiction arises by iterating and applying the ordinary Denjoy-Carleman-Ahlfors theorem. □

References

[Be] Beardon, Alan F. *Iteration of rational functions. Complex analytic dynamical systems.* Graduate Texts in Mathematics, 132. Springer-Verlag, New York, 1991.

[BLS1] Bedford, Eric; Lyubich, Mikhail; Smillie, John. *Polynomial diffeomorphisms of $\mathbb{C}^2$. IV. The measure of maximal entropy and laminar currents.* Invent. Math. 112 (1993), 77–125.

[BLS2] Bedford, Eric; Lyubich, Mikhail; Smillie, John. *Distribution of periodic points of polynomial diffeomorphisms of $\mathbb{C}^2$.* Invent. Math. 114 (1993), 277-288.

[BS1] Bedford, Eric; Smillie, John. *Polynomial diffeomorphisms of $\mathbb{C}^2$: currents, equilibrium measure and hyperbolicity.* Invent. Math. 103 (1991), 69–99.

[BS3] Bedford, Eric; Smillie, John. *Polynomial diffeomorphisms of $\mathbb{C}^2$. III. Ergodicity, exponents and entropy of the equilibrium measure.* Math. Ann. 294 (1992), 395–420.

[BS6] Bedford, Eric; Smillie, John. *Polynomial diffeomorphisms of $\mathbb{C}^2$. VI. Connectivity of J.* Ann. of Math. (2) 148 (1998), 695–735.

[BS7] Bedford, Eric; Smillie, John. *Polynomial diffeomorphisms of $\mathbb{C}^2$. VII. Hyperbolicity and external rays.* Ann. Sci. École Norm. Sup. (4) 32 (1999), 455–497.

[BSU] Bedford, Eric; Smillie, John; Ueda, Tetsuo. *Parabolic bifurcations in complex dimension two.* Preprint (2011).

[BR] Bers, Lipman; Royden, Halsey L. *Holomorphic families of injections.* Acta Math. 157 (1986), 259–286.

[Bu] Buzzard, Gregery T. *Infinitely many periodic attractors for holomorphic maps of 2 variables.* Ann. of Math. (2) 145 (1997), 389–417.

[BHI] Buzzard, Gregery T.; Hruska, Suzanne Lynch; Ilyashenko, Yulij. *Kupka-Smale theorem for polynomial automorphisms of $\mathbb{C}^2$ and persistence of heteroclinic intersections.* Invent. Math. 161 (2005), 45–89.

[Ch] Chirka, Evgueny M. *Complex analytic sets.* Mathematics and its Applications (Soviet Series), 46. Kluwer Academic Publishers Group, Dordrecht, 1989.

[Cr] Crovisier, Sylvain. *Perturbation de la dynamique de difféomorphismes en petite régularité.* Astérisque, to appear.

[DS] Douady, Adrien; Sentenac, Pierrette. *Un tour de valse,* in *Étude dynamique des polynômes complexes, Partie II.* , by Adrien Douady and John H. Hubbard. Publications Mathématiques d'Orsay, 1985.

[Du] Dujardin, Romain. *Some remarks on the connectivity of Julia sets for 2-dimensional diffeomorphisms,* in *Complex Dynamics,* 63-84, Contemp. Math., 396, Amer. Math. Soc., Providence, RI, 2006.

[EL] Eremenko, ALexandre; Levin, Gennadi. *On periodic points of a polynomial.*, English translation: Ukrainian Mth. J., 41 (1989), 1258–1262.

[FS] Fornæss, John Erik; Sibony, Nessim *Complex Hénon mappings in $\mathbb{C}^2$ and Fatou-Bieberbach domains.* Duke Math. J. 65 (1992), 345–380.

[FM] Friedland, Shmuel; Milnor, John. *Dynamical properties of plane polynomial automorphisms.* Ergodic Theory Dynam. Systems 9 (1989), 67–99.

[GK] Goldberg, Lisa R.; Keen, Linda. *A finiteness theorem for a dynamical class of entire functions.* Ergodic Theory Dynam. Systems 6 (1986), no. 2, 183–192.

[GO] Goldberg, Anatoly A. and Ostrovskii, Iossif V. *Value distribution of meromorphic functions.* Translations of Math. Monographs, v. 236, AMS, 2008.

[GS] J. Graczyk and G. Swiatek. *Generic hyperbolicity in the logistic family.* Annals of Math. 146 (1997), 1–52.

[Fu] Furter, Jean-Philippe. *Plane polynomial automorphisms of fixed multidegree.* Math. Ann. 343 (2009), no. 4, 901–920.

[H] Hakim, Monique. *Attracting domains for semi-attractive transformations of* $\mathbf{C}^p$. Publ. Mat. 38 (1994), no. 2, 479–499.

[HO] Hubbard, John H.; Oberste-Vorth, Ralph W. *Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials.* Real and complex dynamical systems, 89–132, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 464, Kluwer Acad. Publ., Dordrecht, 1995.

[J] Jin, Teisuke. *Unstable manifolds and the Yoccoz inequality for complex Hénon mappings.* Indiana Univ. Math. J. 52 (2003) 727–751.

[KM] Kleinbock, Dmitry; Margulis, Gregory A. *Flows on homogeneous spaces and Diophantine approximation on manifolds.* Ann. of Math. (2) 148 (1998), 339–360.

[Ko] Kobayashi, Shoshichi. *Hyperbolic complex spaces.* Grundlehren der Mathematischen Wissenschaften, 318. Springer-Verlag, Berlin, 1998.

[La] Lattès, Samuel. *Sur les formes réduites des transformations ponctuelles dans le domaine d'un point double.* Bull. Soc. Math. Fr. 39 (1911), 309–345.

[Ln] Jim K. Langley, *Postgraduate notes on complex analysis.* Available online at `http://www.maths.nottingham.ac.uk/personal/jkl/pg1.pdf`.

[Lv] Lavaurs, Pierre. *Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques.* Thèse, Université Paris-Sud, 1989.

[KSS] O. Kozlovski, W. Shen and S. van Strien. *Density of hyperbolicity in dimension one.* Annals of Math., 166 (2007), 145–182.

[Le] G. M. Levin. *Irregular values of the parameter of a family of polynomial mappings.* Russian Math. Surveys 36:6 (1981), 189–190.

[Ly1] Lyubich, Mikhail. *Some typical properties of the dynamics of rational mappings.* Russian Math. Surveys 38 (1983), no. 5, 154–155.

[Ly2] Lyubich, Mikhail. *An analysis of stability of the dynamics of rational functions.* Teoriya Funk., Funk. Anal. & Prilozh., , no 42 (1984), 72–91 (Russian). English translation: Selecta Mathematica Sovetica, 9 (1990), 69–90.

[Ly3] Lyubich, Mikhail. *Dynamics of quadratic polynomials*, I-II. Acta Math., 178 (1997), 185–297.

[LyP] Lyubich, Mikhail; Peters, Han. *Classification of invariant Fatou components for dissipative Hénon maps.* Preprint IMS at Stony Brook, no 7 (2012).

[MSS] Mañé, Ricardo; Sad, Paulo; Sullivan, Dennis. *On the dynamics of rational maps.* Ann. Sci. École Norm. Sup. 16 (1983), 193–217.

[Nv] Nevalinna, Rolf. *Analytic functions.* Grundlehren der Mathematischen Wissenschaften, 162, Springer-Verlag, New York-Berlin, 1970.

[Nw1] Newhouse, Sheldon. *Non-density of Axiom $A(a)$ on S^2.* in *Global Analysis,* Proc. Symp. Pure Math. XIV, 191–203, AMS, Providence (1970).

[Nw2] Newhouse, Sheldon. *Diffeomorphisms with infinitely many sinks.* Topology 13 (1974), 9–18.

[Pa] Palis, Jacob. *A global view of dynamics and a Conjecture of the denseness of finitude of attractors.* In: Asterisque volume in honor of Douady's 60th birthday "Géométrie complexe et systémes dynamiques", 261 (2000), 335–348.

[PT] Palis, Jacob; Takens, Floris. *Hyperbolicity and Sensitive-Chaotic Dynamics at Homoclinic Bifurcations.* Cambridge Univ. Press, 1993

[PS] Pujals, Enrique R.; Sambarino, Martín *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms.* Ann. of Math. (2) 151 (2000), no. 3, 961–1023.

[Sc] Schmidt, Wolfgang M. *Metrische Sätze über simultane Approximation abhängiger Grössen.* Monatsh. Math. 68 (1964), 154–166.

[Sh] Shishikura, Mitsuhiro. *Bifurcation of parabolic fixed points.* The Mandelbrot set, theme and variations, 325–363, London Math. Soc. Lecture Note Ser., 274, Cambridge Univ. Press, Cambridge, 2000.

[Si] Sibony, Nessim. *Dynamique des applications rationnelles de $\mathbb{P}^k$.* Dynamique et géométrie complexes (Lyon, 1997), Panoramas et Synthèses, 8, 1999.

[Sl] Słodkowski, Zbigniew. *Approximation of analytic multifunctions.* Proc. Amer. Math. Soc. 105 (1989), no. 2, 387–396.

[U1] Ueda, Tetsuo. *Local structure of analytic transformations of two complex variables. I.* J. Math. Kyoto Univ. 26 (1986), no. 2, 233–261

[U2] Ueda, Tetsuo. *Local structure of analytic transformations of two complex variables. II.* J. Math. Kyoto Univ. 31 (1991), no. 3, 695–711.

[X] Xie, Junyi. *Periodic points of birational maps on projective surfaces.* Preprint (2011), `math.AG:1106.1825`.

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