ON THE GEOMETRY OF BIFURCATION CURRENTS FOR QUADRATIC RATIONAL MAPS.

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Abstract. We describe the behaviour at infinity of the bifurcation current in the moduli space of quadratic rational maps. To this purpose, we extend it to some closed, positive $(1,1)$-current on a two-dimensional complex projective space and then compute the Lelong numbers and the self-intersection of the extended current.

1. Introduction.

For any holomorphic family $(f_\lambda)_{\lambda\in M}$ of degree d rational maps on $\mathbb{P}^1$, the bifurcation locus is the subset of the parameter space M where the Julia sets J_λ of f_λ does not move continuously with λ. In their seminal paper [MSS], Mañé, Sad and Sullivan have shown that the bifurcation locus is nowhere dense in M and coincides with the closure of the set of parameters for which f_λ admits a non-persistent neutral cycle.
For the quadratic polynomial family $\left(z^2+\lambda\right)_{\lambda\in\mathbb{C}}$, the bifurcation locus is the boundary of the Mandelbrot set and, in particular, is bounded. The situation is much more complicated for quadratic rational maps. Their moduli space $\mathcal{M}_2$ can be identified with the complement in a complex projective space $\mathbb{P}^2$ of a line at infinity $\mathbb{L}_\infty$ around which the behaviour of the bifurcation locus is far to be completely understood.
The first results in this direction are due to J. Milnor [M] who studied the curves

$$\mathrm{Per}_n(w) := \{[f]\in\mathcal{M}_2 \text{ s.t. } f \text{ has a } n\text{-cycle of multiplier } w\}$$

in the projective space $\mathbb{P}^2$ (see Proposition 2.3). Sharper results have then been obtained by A. Epstein [E] and yields a precise description of the intersections of $\mathrm{Per}_n(w)$ with $\mathbb{L}_\infty$ (see Proposition 4.1). It should be stressed that the deformations of the Mandelbrot set, provided by intersecting the bifurcation locus with the *lines* $\mathrm{Per}_1(t)$ for $|t|<1$ (see Proposition 2.4), play a central role in these investigations. In particular, the work of C. Petersen [P] on the collapsing of limbs (see the beginning of the fourth section) is crucial in Epstein's proof.

Currents not only provide an appropriate framework for studying bifurcations from a measure-theoretic viewpoint, but are also very well suited to investigate the asymptotic distribution of the hypersurfaces $\mathrm{Per}_n(w)$. Let us recall that L. DeMarco has proved that the bifurcation locus of any holomorphic family supports a closed, positive $(1,1)$-current [DeM1]. This current is denoted T_{bif} and called *bifurcation current*. We refer the reader to the survey [Du] or the lecture notes [Be] for a report on recent results involving bifurcation currents and further references.
The link between the bifurcation current T_{bif} and the hypersurfaces $\mathrm{Per}_n(w)$ relies on the fact that the Lyapunov exponent $L(\lambda)$ of f_λ (with respect to its maximal entropy measure) is a global potential for T_{bif} (see [DeM2] or [BB1]). G. Bassanelli and the first author have

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indeed deduced from this property the convergence of the weighted integration currents on $\mathrm{Per}_n(w)$

$$\lim_n \frac{1}{d^n}[\mathrm{Per}_n(w)] = T_{\mathrm{bif}}$$

for $|w| < 1$ or for any $w \in \mathbb{C}$ when the hyperbolic parameters are dense in M (see [BB2]).

The present paper is devoted to the study of the behaviour at infinity of the bifurcation current in the moduli space $\mathcal{M}_2$ of quadratic rational maps. We first show that this current extends naturally to a closed, positive $(1,1)$-current $\hat{T}_{\mathrm{bif}}$ on $\mathbb{P}^2$. We relate this current with the hypersurfaces $\mathrm{Per}_n(w)$ and precise its support (see Theorem 3.1). More precisely, we show that its support intersects the line $\mathbb{L}_\infty$ along a segment $\mathbb{L}_{\infty,\mathrm{bif}}$, which rationals $\{\infty_{p/q}\}_{p/q\in\mathbb{Q}^*}$ are the intersection points of the curves $\mathrm{Per}_n(w)$ and the line $\mathbb{L}_\infty$.
We then use Epstein's result to compute the Lelong numbers of $\hat{T}_{\mathrm{bif}}$ at any point of the line at infinity $\mathbb{L}_\infty$ and get a precise description of the measure $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty]$ (see Theorem 4.2). Namely, we prove that this measure is totally discrete and supported by the points $\infty_{p/q}$'s. The ideal point $\infty_{p/q}$ also appears to be the point at which the limbs $\mathcal{L}_{p/q}$ and $\mathcal{L}_{-p/q}$ of the Mandelbrot set collapse under the motion given by the intersection of the bifurcation locus and the lines $\mathrm{Per}_1(t)$, when $t \to e^{\pm 2i\pi p/q}$. We prove that the dirac mass of the measure $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty]$ at the point $\infty_{p/q}$ coincides with the Lelong number of $\hat{T}_{\mathrm{bif}}$ at $\infty_{p/q}$ and is precisely given by the mass of $\mathcal{L}_{p/q} \cup \mathcal{L}_{-p/q}$ for the bifurcation measure of the family $(z^2+c)_{c\in\mathbb{C}}$.
We finally describe the support of the so-called bifurcation measure $T_{\mathrm{bif}} \wedge T_{\mathrm{bif}}$ and compare this measure with $\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}}$ (see Propositions 5.1 and 5.2). We show that, in contrast to the case of cubic polynomials, the support of the bifurcation measure is unbounded. Let us stress out that the intersection $\hat{T}_{\mathrm{bif}} \wedge \hat{T}_{\mathrm{bif}}$ at infinity is more complicated than expected regarding the complete description of $\hat{T}_{\mathrm{bif}}$ at infinity provided by Theorem 4.2.

Notations. The complex plane will be denoted $\mathbb{C}$ and the Euclidean unit disc in $\mathbb{C}$ will be denoted $\mathbb{D}$. The k-dimensional complex projective space is denoted $\mathbb{P}^k$. A ball of radius r centered at x in some metric space is denoted $B(x,r)$.

2. **Preliminaries.**

2.1. **Lelong numbers and currents in $\mathbb{P}^2$.**

Let T be a closed, positive, $(1,1)$-current in $\mathbb{P}^2$. The *mass* of T on an open set $U \subset \mathbb{P}^2$ is given by

$$\|T\|_U := \int_U T \wedge \omega$$

where ω is the Fubini-Study form on $\mathbb{P}^2$. When V is an algebraic curve in $\mathbb{P}^2$ and $[V]$ is the integration current on V, then $\|[V]\|_{\mathbb{P}^2} = \deg(V)$.

For any $x \in \mathbb{P}^2$, the *Lelong number* of T at x is given by

$$\nu(T,x) := \lim_{r\to 0} \frac{1}{r^2}\|T\|_{B(x,r)}.$$

These numbers somehow measure the singularities of T. If V is an algebraic curve in $\mathbb{P}^2$, then $\nu([V],x)$ is the multiplicity of V at x. We will mainly use the two following properties of Lelong numbers (see [Dem] Proposition 5.12 page 160 and Corollary 7.9 page 169).

Theorem 2.1 (Demailly). (1) *If $T_n \longrightarrow T$, then for any $x \in \mathbb{P}^2$,*

$$\nu(T,x) \geq \limsup_{n\to+\infty} \nu(T_n,x).$$

(2) *If T_1 and T_2 are such that $T_1 \wedge T_2$ is well-defined, then for any $x \in \mathbb{P}^2$,*

$$\nu(T_1 \wedge T_2, x) \geq \nu(T_1,x)\cdot\nu(T_2,x).$$

2.2. The moduli space $\mathcal{M}_2$ of quadratic rational maps.

The space Rat_2 of quadratic rational maps on $\mathbb{P}^1$ may be viewed as a Zarisky-open subset of $\mathbb{P}^5$ on which the group of Möbius transformations acts by conjugation. The *moduli space* $\mathcal{M}_2$ is, by definition, the quotient resulting from this action. Denote by α,β,γ the multipliers of the three fixed points of $f \in \mathrm{Rat}_2$ and $\sigma_1 = \alpha+\beta+\gamma$, $\sigma_2 = \alpha\beta+\alpha\gamma+\beta\gamma$ and $\sigma_3 = \alpha\beta\gamma$ the symmetric functions of these multipliers. Milnor has proved that (σ_1,σ_2) is a good parametrization of $\mathcal{M}_2$ (see [M]).

Theorem 2.2 (Milnor). *The map $[f] \in \mathcal{M}_2 \longmapsto (\sigma_1,\sigma_2) \in \mathbb{C}^2$ is a canonical biholomorphism.*

To study the curves

$$\mathrm{Per}_n(w) := \{[f] \in \mathcal{M}_2 \text{ s.t. } f \text{ has a } n\text{-cycle of multiplier } w\},$$

it is useful to compactify $\mathcal{M}_2$. In this context, it turns out that the projective compactification

$$[f] \in \mathcal{M}_2 \hookrightarrow [\sigma_1 : \sigma_2 : 1] \in \mathbb{P}^2$$

is suitable. Denote by $\mathbb{L}_\infty$ the line at infinity of $\mathcal{M}_2$, i.e.

$$\mathbb{L}_\infty = \{[\sigma_1 : \sigma_2 : 0] \in \mathbb{P}^2 \ / \ (\sigma_1,\sigma_2) \in \mathbb{C}^2 \setminus \{0\}\}.$$

Milnor has described the behavior of $\mathrm{Per}_n(w)$ at infinity as follows (see [M] Lemmas 3.4 and section 4).

Proposition 2.3 (Milnor). (1) *For any $w \in \mathbb{C}$ the curve $\mathrm{Per}_1(w)$ is a line in $\mathbb{P}^2$ whose equation in $\mathbb{C}^2$ is $(w^2+1)\lambda_1 - w\lambda_2 - (w^3+2) = 0$ and which intersects the line at infinity $\mathbb{L}_\infty$ at the point $[w : w^2+1 : 0]$.*

(2) *For any $n \geq 2$ and any $w \in \mathbb{C}$, the curve $\mathrm{Per}_n(w)$ is an algebraic curve in $\mathbb{P}^2$ whose degree equals the number $d(n) \sim 2^{n-1}$ of n-hyperbolic components of the Mandelbrot set. In addition, the intersection $\mathrm{Per}_n(w) \cap \mathbb{L}_\infty$ is contained in the set $\{[1 : u+1/u : 0] \in \mathbb{P}^2 \ / \ u^q = 1 \text{ with } q \leq n\}$.*

For reasons which will appear to be clear later, we shall denote by $\mathbb{L}_{\infty,\mathrm{bif}}$ the subset of $\mathbb{L}_\infty$ defined by

$$\mathbb{L}_{\infty,\mathrm{bif}} := \{[1 : e^{i\theta} + 1/e^{i\theta} : 0] \in \mathbb{P}^2/\theta \in [0,2\pi]\}.$$

Golberg and Keen showed how the Mandelbrot set $\mathbf{M}$ determines the connectedness locus for quadratic rational maps having an attracting fixed point (see [GK] Section 1 and [BB2] Theorem 5.4 for a potential theoretic approach).

Theorem 2.4 (Goldberg-Keen, Bassanelli-Berteloot). *There exists a holomorphic motion $\sigma : \mathbb{D} \times \mathrm{Per}_1(0) \longrightarrow \mathbb{C}^2$ such that $\mathbf{M}_t := \sigma_t(\mathbf{M}) \Subset \mathrm{Per}_1(t) \cap \mathbb{C}^2$ is the connectedness locus of the line $\mathrm{Per}_1(t)$. Moreover, if $|w| \leq 1$ and $n \geq 1$, then $\mathrm{Per}_1(t) \cap \mathrm{Per}_n(w) \subset \mathbf{M}_t$.*

2.3. Bifurcation currents in $\mathcal{M}_2$.

Every rational map f of degree $d \geq 2$ on the Riemann sphere admits a maximal entropy measure μ_f. The Lyapunov exponent of f with respect to this measure is defined by

$$L(f) = \int_{\mathbb{P}^1} \log |f'| \mu_f.$$

It turns out that the function $L : \mathrm{Rat}_2 \longrightarrow L(f)$ is *p.s.h* and continuous on Rat_2. We still denote by L the function induced on $\mathcal{M}_2$. The *bifurcation current* T_{bif} is a closed positive (1, 1)-current on $\mathcal{M}_2$ which may be defined by

$$T_{\mathrm{bif}} = dd^c L.$$

As it has been shown by DeMarco [DeM2], the support of T_{bif} concides with the bifurcation locus of the family in the classical sense of Mañe-Sad-Sullivan (see also [BB1], Theorem 5.2). Using properties of the Lyapounov exponent, Bassanelli and the first author proved that the curves $\mathrm{Per}_n(0)$ equidistribute the bifurcation current (see [BB2]).

Theorem 2.5 (Bassanelli-Berteloot)**.** *The sequence* $2^{-n}[\mathrm{Per}_n(0)]$ *converges to* T_{bif} *in the sense of currents in* $\mathcal{M}_2$*. Moreover,* $2^{-n}[\mathrm{Per}_n(0)]|_{\mathrm{Per}_1(0)}$ *converges weakly to* $\Delta(L|_{\mathrm{Per}_1(0)}) = \frac{1}{2}\mu_{\mathbf{M}}$*, where* $\mu_{\mathbf{M}}$ *is the harmonic measure of* $\mathbf{M}$.

As the function L is continuous, one can define the *bifurcation measure* μ_{bif} of the moduli space $\mathcal{M}_2$ as the Monge-Ampère mass of the function L, i.e.

$$\mu_{\mathrm{bif}} := (dd^c L)^2 := dd^c\,(L dd^c L).$$

This measure has been introduced by Bassanelli and the first author [BB1]. Buff and Epstein also studied it in [BE]. Recall that a rational map f is said to be strictly poscritically finite if each critical point of f is preperiodic but not periodic. We denote by $\mathcal{SPCF}$ the set of classes of quadratic strictly postcritically finite rational maps. We also denote by $\mathcal{S}$ the set of *Shishikura* maps, i.e. $\mathcal{S} := \{[f_0] \in \mathcal{M}_2 \ / \ f_0$ has 2 distinct neutral cycles$\}$. Combining the work of Bassanelli and the first author with that of Buff and Epstein, we have the following.

Theorem 2.6 (Bassanelli-Berteloot, Buff-Epstein)**.**

$$\mathrm{supp}(\mu_{\mathrm{bif}}) = \overline{\mathcal{SPCF}} = \overline{\mathcal{S}}.$$

These results are still valid in moduli spaces of degree d rational maps for any $d \geq 2$. Notice that Buff and the second author [BG] have proved that flexible Lattès maps belong to $\mathrm{supp}(\mu_{\mathrm{bif}})$.

3. Extension of the bifurcation current to Milnor's compactification.

As the current T_{bif} is a weak limit of weighted integration currents on curves which are actually defined on $\mathbb{P}^2$, one may expect to naturally extend it to $\mathbb{P}^2$. We will show how this is possible and prove some basic properties of the extended current. More precisely, we establish the following result which may be considered as a measure-theoretic counterpart of Milnor's description of the bifurcation locus in $\mathcal{M}_2$.

Theorem 3.1. *There exists a closed positive* $(1,1)$*-current* $\hat{T}_{\mathrm{bif}}$ *on* $\mathbb{P}^2$ *whose mass equals* $1/2$ *and such that:*

1. $\hat{T}_{\mathrm{bif}}|_{\mathbb{C}^2} = T_{\mathrm{bif}}$,
2. $2^{-n}[\mathrm{Per}_n(0)]$ *converges to* $\hat{T}_{\mathrm{bif}}$ *in the sense of currents on* $\mathbb{P}^2$*,*
3. $\mathrm{supp}(\hat{T}_{\mathrm{bif}}) = \mathrm{supp}(T_{\mathrm{bif}}) \cup \mathbb{L}_{\infty,\mathrm{bif}}$.

We shall use the two following lemmas which are of independant interest. The first one is essentially due to Milnor (see Theorem 4.2 in [M] or Theorem 2.3 in [BB2]). The second one will be proved at the end of the section.

Lemma 3.2. $\|[\mathrm{Per}_n(0)]\|_{\mathbb{P}^2} \sim 2^{n-1}$.

Lemma 3.3. $\overline{\bigcup_{n\geq 1} \mathrm{Per}_n(0)} \cap \mathbb{L}_\infty = \mathbb{L}_{\infty,\mathrm{bif}}$.

Proof. We first justify the existence of $\hat{T}_{\mathrm{bif}}$. According to the Skoda-El Mir Theorem (see [Dem] Theorem 2.3 page 139), the trivial extension of T_{bif} through the line at infinity $\mathbb{L}_\infty$ is a closed positive $(1,1)$-current on $\mathbb{P}^2$ if T_{bif} has locally bounded mass near $\mathbb{L}_\infty$. The Lemma 3.2, combined with the fact that $2^{-n}[\mathrm{Per}_n(0)]$ converges to T_{bif} on $\mathbb{C}^2$ (see Theorem 2.5), immediatly yields $\|T_{\mathrm{bif}}\|_{\mathbb{C}^2} \leq 1/2$. The extension $\hat{T}_{\mathrm{bif}}$ of T_{bif} therefore exists and $\|\hat{T}_{\mathrm{bif}}\|_{\mathbb{P}^2} \leq 1/2$.

Let us now prove that $2^{-n}[\mathrm{Per}_n(0)]$ converges to $\hat{T}_{\mathrm{bif}}$ on $\mathbb{P}^2$ and $\|\hat{T}_{\mathrm{bif}}\|_{\mathbb{P}^2} = 1/2$. By Lemma 3.2, $(2^{-n}[\mathrm{Per}_n(0)])_{n\geq 1}$ is a sequence of closed positive $(1,1)$-currents with uniformly bounded mass on $\mathbb{P}^2$. According to the compactness porperties of such families of currents, it suffices to show that $\hat{T}_{\mathrm{bif}}$ is the only limit value of $2^{-n}[\mathrm{Per}_n(0)]$.

Assume that $2^{-n_k}[\mathrm{Per}_{n_k}(0)]$ converges to T on $\mathbb{P}^2$. By Siu's Theorem (see [Dem] Theorem 8.16 page 181), one has $T = S + \alpha[\mathbb{L}_\infty]$, where S has no mass on $\mathbb{L}_\infty$. But, by Lemma 3.3, the line $\mathbb{L}_\infty$ is not contained in $\mathrm{supp}(T)$. Thus $\alpha = 0$ and T has no mass on $\mathbb{L}_\infty$. Since $T|_{\mathbb{C}^2} = \lim_k 2^{-n_k}[\mathrm{Per}_{n_k}(0)] = T_{\mathrm{bif}}$, this shows that T is actually the trivial extension of T_{bif} through $\mathbb{L}_\infty$ and therefore, according to the first part of the proof, equals $\hat{T}_{\mathrm{bif}}$.

Now, Lemma 3.2 immediatly yields $\|\hat{T}_{\mathrm{bif}}\|_{\mathbb{P}^2} = 1/2$. □

Proof of Lemma 3.3. Let $\sigma : \mathrm{Per}_1(0) \times \mathbb{D} \longrightarrow \bigcup_{|u|<1} \mathrm{Per}_1(u)$ be the holomorphic motion given by Theorem 2.4. Let us set $\mathbf{M}_u := \sigma_u(\mathbf{M})$. As $\mathbf{M}$ is compact and σ is continuous one has

$$\bigcup_{|u|\leq r<1} \mathbf{M}_u \Subset \mathbb{C}^2 \text{ for all } 0 < r < 1.$$

Let us also recall that $\mathrm{Per}_n(0) \cap \mathrm{Per}_1(u) \subset \mathbf{M}_u$ for $n \geq 2$ and $u \in \mathbb{D}$.

We now proceed by contradiction and assume that there exists

$$\zeta \in \left(\overline{\textstyle\bigcup_n \mathrm{Per}_n(0)} \cap \mathbb{L}_\infty\right) \setminus \mathbb{L}_{\infty,\mathrm{bif}}.$$

By Proposition 2.3, $\zeta \in \mathrm{Per}_1(u_0)$ for some $u_0 \in \mathbb{D}$. Let us pick $\lambda_k \in \mathrm{Per}_{n_k}(0)$ such that $\lambda_k \to \zeta$. Then there exists $u_k \in \mathbb{D}$ such that $u_k \to u_0$ and $\lambda_k \in \mathrm{Per}_1(u_k)$, which is impossible since

$$\lambda_k \in \mathrm{Per}_{n_k}(0) \cap \mathrm{Per}_1(u_k) \subset \mathbf{M}_{u_k} \subset \bigcup_{|u|<r} \mathbf{M}_u$$

for some $|u_0| < r < 1$. □

4. **Lelong numbers of the bifurcation current at infinity.**

The aim of the present section is to compute the Lelong numbers of $\hat{T}_{\mathrm{bif}}$ at any point. This is related to previous works of Petersen ([P]) and Epstein ([E]) which we briefly describe.

Let ♡ be the main cardioid of the Mandelbrot set and $\mathcal{L}_{p/q}$ the *p/q-limb* of $\mathbf{M}$ (see [Br] page 84). Denote by $d_{p/q}(n)$ the number of n-hyperbolic component of $\mathcal{L}_{p/q}$ and set

$$D_{p/q}(n) = \begin{cases} d_{p/q}(n) & \text{if } p/q = 1/2, \\ 2d_{p/q}(n) & \text{otherwise.} \end{cases}$$

Let σ be the holomorphic motion of $\mathrm{Per}_1(0)$ given by Theorem 2.4 and

$$\infty_{p/q} := [1 : 2\cos(2\pi p/q) : 0]$$

if $1 \le p \le q/2$ and $p \wedge q = 1$. Petersen has proved that the limb $\sigma_u(\mathcal{L}_{p/q})$ of $\mathbf{M}_u$ disappears when u tends non-tangentially to $e^{-2i\pi p/q}$. Using this result, Epstein has precisely described the intersection $\mathrm{Per}_n(w) \cap \mathbb{L}_\infty$. He namely proved the following:

Proposition 4.1 (Epstein). *For any $w \in \mathbb{C}$ and any $n \ge 2$,*

$$[\mathrm{Per}_n(w)] \wedge [\mathbb{L}_\infty] = \sum_{\substack{1 \le p \le q/2 \le n/2 \\ p \wedge q = 1}} \nu([\mathrm{Per}_n(w)], \infty_{p/q}) \delta_{\infty_{p/q}} = \sum_{\substack{1 \le p \le q/2 \le n/2 \\ p \wedge q = 1}} D_{p/q}(n) \delta_{\infty_{p/q}}.$$

From the above Proposition and by using the previous section, we deduce the following.

Theorem 4.2. (1) *The Lelong numbers of $\hat{T}_{\mathrm{bif}}$ are given by*

$$\nu(\hat{T}_{\mathrm{bif}}, a) = \begin{cases} 1/6 & \text{if } a = \infty_{1/2}, \\ 1/(2^q - 1) & \text{if } a = \infty_{p/q} \text{ and } , q \ge 3 \\ 0 & \text{if } a \in \mathbb{P}^2 \setminus \{\infty_{p/q} \ / \ p \wedge q = 1, \ 1 \le p \le q/2\}. \end{cases}$$

(2) *The measure $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty]$ is discrete and given by*

$$\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty] = \sum_{\substack{1 \le p \le q/2 \\ p \wedge q = 1}} \nu(\hat{T}_{\mathrm{bif}}, \infty_{p/q}) \delta_{\infty_{p/q}}.$$

The proof requires the two following lemmas. The first one is a consequence of Theorem 3.1 and the second one relies on a simple computation. They will be proved at the end of the section.

Lemma 4.3. *The measure $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty]$ is a well-defined positive measure on $\mathbb{P}^2$ of mass 1/2.*

Lemma 4.4. *Let* $\mu := \frac{1}{6}\delta_{\infty_{1/2}} + \sum_{\substack{1 \le p \le q/2 \\ p \wedge q = 1, \ q \ge 3}} \frac{1}{2^q - 1} \delta_{\infty_{p/q}}$, *then μ has mass* $1/2$.

Proof. First observe that $\nu(\hat{T}_{\mathrm{bif}}, a) = 0$ when $a \notin \mathbb{L}_{\infty,\mathrm{bif}}$, since then $\hat{T}_{\mathrm{bif}}$ has a continuous potential in a neighborhood of a. Let us now pick $1 \le p \le q/2$ such that $p \wedge q = 1$. By item (3) of Theorem 3.1 and Theorem 2.1, we have

$$\nu(\hat{T}_{\mathrm{bif}}, \infty_{p/q}) \ge \limsup_{n \to \infty} \nu(2^{-n}[\mathrm{Per}_n(0)], \infty_{p/q}).$$

Proposition 4.1 states that $\nu([\mathrm{Per}_n(0)], \infty_{p/q}) = D_{p/q}(n)$ is the number of n-hyperbolic components of the union $\mathcal{L}_{p/q} \cup \mathcal{L}_{-p/q}$ of limbs of the Mandelbrot set. Thus, by Theorem 2.5,

$$2^{-n}\nu([\mathrm{Per}_n(0)], \infty_{p/q}) \longrightarrow \tfrac{1}{2}\mu_{\mathbf{M}}(\mathcal{L}_{p/q} \cup \mathcal{L}_{-p/q}).$$

On the other hand, Bullett and Sentenac have proved that $\mu_{\mathbf{M}}(\mathcal{L}_{p/q}) = \mu_{\mathbf{M}}(\mathcal{L}_{-p/q}) = \frac{1}{2^q - 1}$ (see [BS]). This gives $\nu(\hat{T}_{\mathrm{bif}}, \infty_{p/q}) \ge \frac{1}{2^q-1}$ if $p/q \neq 1/2$ and $\nu(\hat{T}_{\mathrm{bif}}, \infty_{1/2}) \ge \frac{1}{2(2^2-1)} = \frac{1}{6}$.

Let $a \in \mathbb{L}_\infty$. By Theorem 2.1 , we have $\nu(\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty], a) \ge \nu(\hat{T}_{\mathrm{bif}}, a)\nu([\mathbb{L}_\infty], a)$. As $\mathbb{L}_\infty$ is a line, we get $\nu(\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty], a) \ge \nu(\hat{T}_{\mathrm{bif}}, a)$. For $a = \infty_{p/q}$, we thus have $\nu(\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty], \infty_{p/q}) \ge \frac{1}{2^q-1}$ if $q \ge 3$ and $\nu(\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty], \infty_{1/2}) \ge \frac{1}{6}$, which we restate as

$$\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty] \geq \mu.$$

Since, according to Lemmas 4.3 and 4.4, the positive measures $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty]$ and μ have the same mass, this yields $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty] = \mu$. We thus get point (2) and $\nu(\hat{T}_{\mathrm{bif}}, \infty_{p/q}) = \frac{1}{2^q-1}$ for $q \geq 3$ and $\nu(\hat{T}_{\mathrm{bif}}, \infty_{1/2}) = \frac{1}{6}$.

From $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty] \geq \nu(\hat{T}_{\mathrm{bif}}, a)\delta_a$ for any $a \in \mathbb{L}_\infty$ and $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty] = \mu$ we get $\nu(\hat{T}_{\mathrm{bif}}, a) = 0$ for $a \neq \infty_{p/q}$. □

Proof of Lemma 4.3. Let us first remark that, by Theorem 3.1, $\mathrm{supp}(\hat{T}_{\mathrm{bif}}) \cap \mathbb{L}_\infty = \mathbb{L}_{\infty,\mathrm{bif}}$ and let us decompose $\mathbb{P}^2$ as the disjoint union of the line $\mathrm{Per}_1(0)$ and (a copy of) $\mathbb{C}^2$. Let us stress that with these notations one has $\mathbb{L}_\infty \setminus \mathrm{Per}_1(0) \cap \mathbb{L}_\infty \subset \mathbb{C}^2$. By Proposition 2.3, the set $\mathbb{L}_{\infty,\mathrm{bif}}$ is compact in $\mathbb{C}^2$. By definition, the current $\hat{T}_{\mathrm{bif}}$ has a potential u in $\mathbb{C}^2$ which is continuous in $\mathbb{C}^2 \setminus \mathbb{L}_\infty$. By item (3) of Theorem 3.1, this potential is actually continuous in $\mathbb{C}^2 \setminus \mathbb{L}_{\infty,\mathrm{bif}}$. Let B_1 be a ball of $\mathbb{C}^2$ containing $\mathbb{L}_{\infty,\mathrm{bif}}$ and $(B_i)_{i\geq 2}$ be a covering of $\mathbb{C}^2 \setminus B_1$ by balls such that $\overline{B_i} \cap \mathbb{L}_{\infty,\mathrm{bif}} = \emptyset$ for all $i \geq 2$.

As $\{u \text{ is unbounded}\} \subset \mathbb{L}_{\infty,\mathrm{bif}}$ and $\mathbb{L}_{\infty,\mathrm{bif}} \cap \partial B_i = \emptyset$ and B_i is pseudoconvex for any $i \geq 1$, a result of Demailly asserts that $\hat{T}_{\mathrm{bif}}|_{\mathbb{C}^2} \wedge [\mathbb{L}_\infty] = dd^c u \wedge [\mathbb{L}_\infty]$ is well-defined (see [Dem] Proposition 4.1 page 150). Since, by Proposition 2.3 and Theorem 4.2, $\mathbb{L}_\infty$ and $\mathrm{supp}(\hat{T}_{\mathrm{bif}})$ don't intersect in a neighborhood of $\mathrm{Per}_1(0)$, the measure $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty]$ is a well-defined positive measure on $\mathbb{P}^2$.

Finally, Bézout Theorem asserts that $\|\hat{T}_{\mathrm{bif}} \wedge [\mathbb{L}_\infty]\|_{\mathbb{P}^2} = \|\hat{T}_{\mathrm{bif}}\|_{\mathbb{P}^2} \cdot \|[\mathbb{L}_\infty]\|_{\mathbb{P}^2} = 1/2$. □

Proof of Lemma 4.4. Denote by $\phi(n)$ the Euler function. As the sets $\{p \text{ s.t. } 1 \leq p \leq q/2,\ p \wedge q = 1\}$ and $\{q - p \text{ s.t. } 1 \leq p \leq q/2,\ p \wedge q = 1\}$ have same cardinality, we get

$$\begin{aligned}
\mu(\mathbb{P}^2) &= \frac{1}{6} + \sum_{\substack{1\leq p\leq q/2,\\ p\wedge q=1,\ q\geq 3}} \frac{1}{2^q-1} = \frac{1}{2(2^2-1)} + \frac{1}{2} \sum_{\substack{1\leq p<q,\\ p\wedge q=1,\ q\geq 3}} \frac{1}{2^q-1} \\
&= \frac{1}{2}\sum_{q\geq 2}\left(\sum_{\substack{1\leq p<q,\\ p\wedge q=1}} \frac{1}{2^q-1}\right) = \frac{1}{2}\sum_{q\geq 2}\frac{\phi(q)}{2^q-1}.
\end{aligned}$$

A classical result (see [HW] Theorem 309 page 258) asserts that the series $\sum_{n\geq 1} \phi(n)\frac{x^n}{1-x^n}$ locally uniformly converges on $\mathbb{D}$ and that its sum is $\frac{x}{(1-x)^2}$. Therefore,

$$\mu(\mathbb{P}^2) = \frac{1}{2}\left(\sum_{q\geq 1}\frac{\phi(q)}{2^q-1} - \phi(1)\right) = \frac{1}{2}\left(\sum_{q\geq 1}\frac{\phi(q)(\frac{1}{2})^q}{1-(\frac{1}{2})^q} - \phi(1)\right) = \frac{1}{2}.$$

□

5. Behavior of the bifurcation measure near infinity.

One often compares the moduli space $\mathcal{M}_2$ of quadratic rational maps with the moduli space $\mathcal{P}_3$ of cubic polynomials. In this section, we enlight some important differences between these two spaces. We first show that the bifurcation measure is not compactly supported in $\mathcal{M}_2$.

Proposition 5.1. *The cluster set of the support of μ_{bif} in $\mathbb{P}^2$ is precisely $\mathbb{L}_{\infty,\mathrm{bif}}$.*

Proof. By Theorem 3.1, it suffices to show that $\mathbb{L}_{\infty,\mathrm{bif}}$ is accumulated by points of $\mathrm{supp}(\mu_{\mathrm{bif}})$. Recall that $\mathrm{Per}_1(e^{2i\pi\nu})\cap\mathbb{L}_\infty = \{[1:2\cos(2\pi\nu):0]\}$ for any $0\leq\nu\leq 1$ (see Proposition 2.3). Let us fix $0<\theta_0<1$. For $\theta>0$ small enough, the lines $\mathrm{Per}_1(e^{2i\pi\theta_0})$ and $\mathrm{Per}_1(e^{2i\pi(\theta-\theta_0)})$ do not intersect on $\mathbb{L}_\infty$ and therefore,

$$\{\lambda(\theta)\} := \mathrm{Per}_1(e^{2i\pi\theta_0})\cap\mathrm{Per}_1(e^{2i\pi(\theta-\theta_0)})\subset\mathbb{C}^2.$$

Since $\lambda(\theta)$ has two distinct neutral fixed points, it belongs to $\mathrm{supp}(\mu_{\mathrm{bif}})$ (see Theorem 2.6). It remains to check that $\lim_{\theta\to 0}\lambda(\theta) = [1:2\cos(2\pi\theta_0):0]$. By the holomorphic index formula (see [M]), the multiplier $\gamma(\theta)$ of the third fixed point of $\lambda(\theta)$ is given by

$$\gamma(\theta) = \frac{2-(e^{2i\pi\theta_0}+e^{2i\pi(\theta-\theta_0)})}{1-e^{2i\pi\theta}}.$$

As $\lim_{\theta\to 0}|\gamma(\theta)| = +\infty$, one has $\lim_{\theta\to 0}\|\lambda(\theta)\| = +\infty$. The conclusion follows, since $\lambda(\theta)$ stays on $\mathrm{Per}_1(e^{2i\pi\theta_0})$ and $\mathrm{Per}_1(e^{2i\pi\theta_0})\cap\mathbb{L}_\infty = \{[1:2\cos(2\pi\theta_0):0]\}$. □

We would like to extend μ_{bif} as a reasonable bifurcation measure on $\mathbb{P}^2$. To this aim, we compare the trivial extension of μ_{bif} to $\mathbb{P}^2$ with $\hat{\mu}_{\mathrm{bif}} := (\hat{T}_{\mathrm{bif}})^2$. We prove the following.

Proposition 5.2. *There exists a positive measure μ_∞ supported by $\mathbb{L}_{\infty,\mathrm{bif}}$ such that*

$$\hat{\mu}_{\mathrm{bif}} = \mu_{\mathrm{bif}} + \frac{1}{36}\delta_{\infty_{1/2}} + \sum_{\substack{1\leq p\leq q/2\\ p\wedge q=1,\, q\geq 3}} \frac{1}{(2^q-1)^2}\delta_{\infty_{p/q}} + \mu_\infty.$$

Les us stress that our previous results ensure the existence of μ_∞ as a non-negative measure. A recent result due to Kiwi and Rees concerning the number of (n,m)-hyperbolic components of $\mathcal{M}_2$ allows to see that μ_∞ is actually a positive measure.

Proof. Arguing exactly as in the proof of Lemma 4.3, one sees that $\hat{\mu}_{\mathrm{bif}} = \hat{T}_{\mathrm{bif}}\wedge\hat{T}_{\mathrm{bif}}$ is a well-defined positive measure on $\mathbb{P}^2$. By definition, $\hat{\mu}_{\mathrm{bif}}$ and μ_{bif} coincide on $\mathbb{C}^2$. To prove the existence of μ_∞, it thus only remains to justify that

$$\hat{\mu}_{\mathrm{bif}} \geq \mu_{\mathrm{bif}} + \frac{1}{36}\delta_{\infty_{1/2}} + \sum_{\substack{1\leq p\leq q/2\\ p\wedge q=1,\, q\geq 3}} \frac{1}{(2^q-1)^2}\delta_{\infty_{p/q}}.$$

By Theorem 2.1 and Theorem 4.2, we have

$$\nu(\hat{\mu}_{\mathrm{bif}},\infty_{p/q}) \geq \nu(\hat{T}_{\mathrm{bif}},\infty_{p/q})^2 = \frac{1}{(2^q-1)^2},\ \text{when } q\geq 3$$

and $\nu(\hat{\mu}_{\mathrm{bif}},\infty_{1/2})\geq 1/36$. The existence of μ_∞ then follows, since

$$\hat{\mu}_{\mathrm{bif}}\big|_{\mathbb{L}_\infty} \geq \sum_{\substack{1\leq p\leq q/2\\ p\wedge q=1}} \nu(\hat{\mu}_{\mathrm{bif}},\infty_{p/q})\delta_{\infty_{p/q}}.$$

Let us now show that $\mu_\infty>0$. One easily deduce from the convergence of $2^{-n}[\mathrm{Per}_n(0)]$ towards T_{bif} in any family (see [BB2]) that

$$\mu_{\mathrm{bif}} = \lim_m\lim_n 2^{-(n+m)}[\mathrm{Per}_n(0)\cap\mathrm{Per}_m(0)]$$

on $\mathbb{C}^2$. Thus $\mu_{\mathrm{bif}}(\mathbb{C}^2)\leq\limsup_m\limsup_n 2^{-(n+m)}[\mathrm{Per}_n(0)\cap\mathrm{Per}_m(0)](\mathbb{C}^2)$. Kiwi and Rees proved recently that the number of (n,m)-hyperbolic components in $\mathbb{C}^2$ is at most

$$\left(\frac{5}{48}2^n - \frac{1}{8}\sum_{q=2}^{n}\frac{\phi(q)\nu_q(n)}{2^q-1}\right)2^m + \mathcal{O}(2^m),$$

where $\nu_q(n) \sim 2^{n-1}/(2^q-1)$ and $\nu_q(n) \geq 2^{n-1}/(2^q-1) - 1/2$ (see [KR] Theorem 1.1). A standard transversality statement asserts that this number actually coincides with $[\mathrm{Per}_n(0)\cap \mathrm{Per}_m(0)](\mathbb{C}^2)$ (see [BB2] Theorem 5.2). Thus

$$\mu_{\mathrm{bif}}(\mathbb{C}^2) \leq \frac{5}{48} - \frac{1}{16}\sum_{q\geq 2}\frac{\phi(q)}{(2^q-1)^2}.$$

Let us now proceed by contradiction, assuming that $\mu_\infty = 0$. Since $\hat{T}_{\mathrm{bif}}$ has mass $1/2$, Bézout Theorem gives $\|\hat{\mu}_{\mathrm{bif}}\|_{\mathbb{P}^2} = 1/4$. Therefore,

$$\frac{1}{4} = \|\mu_{\mathrm{bif}}\| + \frac{1}{36} + \sum_{\substack{1\leq p\leq q/2\\ p\wedge q=1, q\geq 3}}\frac{1}{(2^q-1)^2} = \|\mu_{\mathrm{bif}}\| + \frac{1}{2}\sum_{q\geq 2}\frac{\phi(q)}{(2^q-1)^2} - \frac{1}{36},$$

which yields $\frac{25}{56} \leq \sum_{q\geq 2}\frac{\phi(q)}{(2^q-1)^2}$. We then have

$$\frac{25}{56} \leq \sum_{q\geq 2}\frac{\phi(q)}{(2^q-1)^2} \leq \frac{\phi(2)}{6} + \frac{\phi(3)}{28} + \frac{1}{8}\sum_{q\geq 4}\frac{\phi(q)}{2^q-1}$$

which is impossible, since $\sum_{q\geq 1}\frac{\phi(q)}{2^q-1} = 2$ (see proof of Lemma 3.2), $\phi(2) = 1$ and $\phi(3) = 2$. $\square$

Remark 5.1. *To underline the contrast with the moduli space of cubic polynomials $\mathcal{P}_3$, we recall that the bifurcation measure is compactly supported and coincides with $\hat{\mu}_{\mathrm{bif}}$ in $\mathcal{P}_3$.*

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