CLASSIFICATION OF INVARIANT FATOU COMPONENTS FOR DISSIPATIVE HÉNON MAPS

MISHA LYUBICH AND HAN PETERS

Abstract. Fatou components for rational functions in the Riemann sphere are very well understood and play an important role in our understanding of one-dimensional dynamics. In higher dimensions the situation is less well understood. In this work we give a classification of invariant Fatou components for moderately dissipative Hénon maps. Most of our methods apply in a much more general setting. In particular we obtain a partial classification of invariant Fatou components for holomorphic endomorphisms of projective space, and we generalize Fatou's Snail Lemma to higher dimensions.

1. Main Result

Fatou components play an important role in our understanding of rational dynamics in the Riemann sphere, and in that setting they have been accurately described. In higher dimensions the description of Fatou components is largely open. In this work we give a classification of invariant Fatou components for moderately dissipative polynomial automorphisms of $\mathbb{C}^2$, and in particular, for moderately dissipative complex Hénon maps

$$(z, w) \mapsto (p(z) - \delta w, z),$$

where $p(z)$ is a polynomial of degree $d \geq 2$.

Theorem 1. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a non-elementary polynomial automorphism of degree $d \geq 2$, and $\delta = \det Df$ be its Jacobian. Assume that f is moderately dissipative, i.e.*

$$|\delta| < \frac{1}{d^2}. \tag{1}$$

Let Ω be an invariant Fatou component of f with bounded forward orbits. Then one of the following three cases is satisfied.

1. *All orbits in Ω converge to an attracting fixed point $p \in \Omega$. The component Ω is biholomorphically equivalent to $\mathbb{C}^2$.*
2. *All orbits in Ω converge to a properly embedded submanifold $\Sigma \subset \Omega$, and Σ is biholomoprphically equivalent to either the unit disk or an annulus. The manifold Σ is invariant under f and f acts on Σ as an irrational rotation.*
3. *All orbits in Ω converge to a fixed point $p \in \partial\Omega$. The eigenvalues λ_1 and λ_2 of $Df(p)$ satisfy $|\lambda_1| < 1$ and $\lambda_2 = 1$, and Ω is biholomorphically equivalent to $\mathbb{C}^2$.*

The only uncertainty in this classification is whether the submanifold in Case (2) can actually be biholomorphically equivalent to an annulus.

Stony Brook IMS Preprint #2012/7
December 2012

A rougher description of invariant Fatou components had been given in the work of Bedford-Smillie, Fornæss-Sibony [10] and Ueda [32], which we will review in the next section. Our contribution here is to prove that if an orbit in Ω converges to the boundary, then all orbits must converge to a single semi-parabolic fixed point.

Remark 2. An important supplement to the above picture is that in cases (1) and (3) the domain Ω contains a "critical point" of f (as long as the multipliers at p are different), i.e., a point of tangency between the strong stable foliation in Ω and the untable manifold of any saddle. This result had been obtained in [7].

Acknowledgement. We thank A. Eremenko, A. Gabrielov, and J. Milnor for insightful comments. The first author was partially supported by the NSF and Balzan-Palis Fellowship. The second author was supported by a SP3-People Marie Curie Actionsgrant in the project Complex Dynamics (FP7-PEOPLE-2009-RG, 248443).

2. Background

In the 1960's the astronomer Michel Hénon suggested that complicated behavior observed in the Poincaré section of the Lorenz model would already occur for the much simpler maps given by

$$(x, y) \to (x^2 + c - \delta y, x), \tag{2}$$

at least for specific parameters c, δ. These maps are now called Hénon maps and have since become one of the most extensively studied dynamical systems, both in the real and in the complex setting.

We will work with a more general definition of Hénon mappings. It was proved by Friedland and Milnor [13] that every polynomial automorphisms of $\mathbb{C}^2$ is affinely conjugate to either an affine map, an elementary map, or a finite compositions of *generalized* Hénon mappings. Here f is a generalized Hénon mapping if f is of the form

$$(z, w) \mapsto (p(z) - \delta w, z), \tag{3}$$

where p is a polynomial and $\delta \in \mathbb{C} \setminus \{0\}$. The dynamical behavior of affine and elementary maps is easy to describe. Therefore we will only look at finite compositions of Hénon maps, and for simplicity we will refer to these maps just as Hénon maps.

Complex Hénon mappings have been studied extensively by Hubbard & Oberste-Vorth [18], [19], [20], Bedford-Smillie [2], [3], Fornæss-Sibony [9] and many other authors. A basic property of Hénon maps that will be useful to us is the existence of the following *filtration*. For $R > 0$ sufficiently large we define

$$W = \{(z, w) \mid \max(|z|, |w|) \leq R\} \tag{4}$$

$$V^+ = \{(z, w) \mid |z| \geq \max(|w|, R)\}, \quad \text{and,} \tag{5}$$

$$V^- = \{(z, w) \mid |w| \geq \max(|z|, R)\}. \tag{6}$$

One easily checks that for $R > 0$ large enough one has $f(V^+) \subset V^+$ and $f^{-1}(V^-) \subset V^-$. Moreover, the orbit of any point in V^+ will converge to the attracting fixed point $[1 : 0 : 0]$ on the line at infinity.

It follows that the escaping set

$$I_\infty = \bigcup_{n \in \mathbb{N}} f^{-n}(V^+) = \{z \in \mathbb{C}^2 \mid \|f^n(z)\| \to \infty\} \tag{7}$$

is one Fatou component, and that for any other Fatou component the forward orbits are bounded. In this article we will, in the Hénon setting, only consider Fatou components with bounded forward orbits.

Let us introduce Fatou components in a more general setting. Let X be a complex manifold, and let $f : X \to X$ be a holomorphic map. We say that $z \in X$ lies in the Fatou set $\mathcal{F}$ of f if the family of iterates $\{f^n\}$ is normal in a neighborhood of z. A connected component of the Fatou set is called a *Fatou component*.

When X is the Riemann sphere and f is a rational function the possible Fatou components have been precisely described. Sullivan [29] proved in 1982 that every Fatou component is (pre-)periodic, and periodic Fatou components had already been classified in the works of Fatou & Julia Siegel and Arnol'd & Herman: an invariant Fatou component is either the basin of an attracting fixed point, a parabolic basin, a Siegel disk or a Herman ring (see Milnor [25]).

Fatou components in two complex variables have been studied by a number of authors. In general there is no reason to believe that all Fatou components are (pre-)periodic, but there has been some progress in describing *periodic* Fatou components. Bedford-Smillie [3] have introduced the notion of a recurrent Fatou component, which we will adopt here.

Definition 3. An invariant Fatou component Ω is called *recurrent* if there exists a point $z \in \Omega$ whose orbit accumulates at a point in Ω.

Normality implies that a Fatou component is recurrent precisely when it contains a recurrent orbit. If Ω is not recurrent then all orbits in Ω converge to $\partial\Omega$. For rational self-maps of the Riemann sphere the recurrent Fatou components are basins of attracting fixed points, Siegel disks and Herman rings, while the only non-recurrent components are basins of parabolic fixed points. Recurrent Fatou components in two complex dimensions have been studied by Bedford-Smillie [3], Fornæss-Sibony [11] and Ueda [32]. The following theorem combines results from [11] and [32].

Theorem 4. *Let f be a Hénon map and suppose that Ω is a recurrent invariant Fatou component. Then either:*

(1) Ω is an attracting basin of some fixed point in Ω, and Ω is biholomorphic to $\mathbb{C}^2$.
(2) there exists a one-dimensional closed complex submanifold Σ of Ω and $f^n(K) \mapsto \Sigma$ for any compact set K in Ω. The Riemann surface Σ is biholomorphic to a disk or an annulus and $f|_\Sigma$ is conjugate to an irrational rotation, or
(3) the domain Ω is a Siegel domain.

Recall that a Fatou component Ω is called a Siegel domain if there exists a sequence of iterates f^{n_j} that converges on Ω to the identity map.

While there are still a number of open questions regarding Theorem 4, for example whether the Riemann surface Σ in Case (2) can really be biholomorphic to an annulus, the recurrent Fatou components are relatively well understood.

The situation was quite different for non-recurrent Fatou components. These components have been studied by Weickert [34] and Jupiter-Lilov [22], but were far less well understood.

3. Outline of the Proof

Let us first describe the main difficulty for dealing with non-recurrent Fatou components. If Ω is a non-recurrent Fatou component then all orbits converge to the boundary $\partial\Omega$. By normality there exists a sequence $\{f^{n_j}\}$ that converges, uniformly on compact subsets of Ω, to a limit map $h : \Omega \to \partial\Omega$. In general the map h is not unique and depends on the sequence (n_j), we will see some examples of this in Section 4. The main difficulty lies in the fact that a priori it is not even clear whether the *limit set* $h(\Omega)$ is always unique.

Theorem 1 follows from several intermediate results, most of which hold in a more general setting. If we do not assume that the Hénon map is moderately dissipative then we still have the following.

Theorem 5. *Let f be a Hénon map and suppose that Ω is a non-recurrent invariant Fatou component. Then there exists a sequence $\{f^{n_j}\}$ that converges uniformly on compact subsets of Ω to a fixed point $p \in \partial\Omega$. If the entire sequence $\{f^n\}$ converges to p then the eigenvalues λ_1 and λ_2 of $Df(p)$ satisfy $|\lambda_1| < 1$ and $\lambda_2 = 1$, and Ω is biholomorphically equivalent to $\mathbb{C}^2$.*

We also obtain results that lie outside the class of Hénon maps, for example the following.

Theorem 6. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ and let Ω be a non-recurrent, invariant Fatou component. Suppose that the limit set $h(\Omega)$ is unique. Then $h(\Omega)$ either consists of one point, or $h(\Omega)$ is an injectively immersed Riemann surface, conformally equivalent to either the unit disk, the punctured unit disk or an annulus, and f acts on $h(\Omega)$ as an irrational rotation.*

Notice that both Theorem 5 and Theorem 6 give a precise classification of non-recurrent Fatou components under the assumption that $h(\Omega)$ is unique. It is exactly the uniqueness of $h(\Omega)$ that condition (1) is used for in Theorem 1

We note that in Theorem 6 all Fatou components are known to occur, except for the punctured unit disk, whose possible existence is still an open question. We will see that when the limit map is a single point, the condition on the eigenvalues that holds for Hénon maps does not hold for holomorphic endomorphisms of $\mathbb{P}^2$. Fatou components for holomorphic self-maps of projective space will be discussed in Section 9

Let us now discuss the intermediate results from which Theorems 1, 5 and 6 will follow. The first lemma that we will prove is the following:

Lemma 12. *Let f be a holomorphic endomorphism of a complex manifold X of dimension 2, and let Ω be a non-recurrent Fatou component. Let $h = \lim f^{n_j}$ be a limit map on Ω of generic rank 1. Then $h(\Omega) \subset \partial\Omega$ is an injectively immersed Riemann surface.*

Next we consider the set of all limit maps $h : \Omega \to \partial\Omega$, which we will denote by Γ_f. We introduce a (not necessarily anti-symmetric) partial ordering on Γ (rougly, by divisibility), and prove the following.

Lemma 24. *The invariant minimal subsets of Γ_f are exactly equal to the maximal equivalence classes. It follows that maximal elements exist: there exists an $h \in \Gamma_f$ so that if $k \geq h$ then $h \geq k$.*

The next result is the first that uses properties of Hénon maps.

Lemma 26. *Let f be a Hénon mapping, let $\Omega \subset \mathbb{C}^2$ be a non-recurrent Fatou component and let $h \in \Gamma_f$ be maximal. Then $h(\Omega)$ is a fixed point.*

Lemmas 24 and 26 together give that there must always be a limit map of rank 0, and if the limit set is unique then all orbits converge to this fixed point p. The fact that the eigenvalues λ_1 and λ_2 of $Df(p)$ must satisfy $|\lambda_1| < 1$ and $\lambda_2 = 1$ is a local property that will be proved in this generalization of Fatou's Snail Lemma:

Theorem 27. *Let $f : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ be a germ of a holomorphic mapping that fixes the origin. Suppose that there exists an open set W with $f(W) \cap W \neq \emptyset$ and such that on W the iterates (f^n) converge uniformly to the origin. Further suppose that $\mathrm{D}f(0)$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with $|\lambda_1| = 1$ and $|\lambda_i| < 1$ for $i \geq 2$. Then $\lambda_1 = 1$.*

In the proof of Lemma 26 we will work in a normal hyperbolic setting, comparable to Theorem (4.1) in Hirsch-Pugh-Shub [17]. In this setting the normal submanifold is not necessarily compact, but we will prove uniform estimates on the geometry of the normal submanifold, which will gives us a thickening of the strong stable manifolds to obtain a stable lamination that fills up a neighborhood of the normal submanifold. The existence of strong stable manifolds of definite size near points whose orbits remain in a small neighborhood of a weakly hyperbolic fixed point will also be used in the proofs of Theorem 27 and Lemma 31.

The final step in the proof of Theorem 5 follows from the following result.

Theorem 7. *Let F be a holomorphic automorphism of $\mathbb{C}^2$ with a fixed point p, such that $F'(p)$ has eigenvalues $\{1, \lambda\}$, with $|\lambda| < 1$, and that $F - \mathrm{Id}$ has multiplicity $k+1$ in p. The attracting basin of p has then k components and each component is biholomorphic to $\mathbb{C}^2$.*

This result was first proved by Ueda in the case where $k = 1$ in [30], and generalized to higher multiplicities by Hakim [15]. Note that if the multiplicity $k+1$ is not finite then there must be a curve of fixed points, which cannot occur for Hénon maps, so Theorem 7 does imply the last statement of Theorem 5.

Once Theorem 5 is proved, the proof of Theorem 1 is completed by using the bound on the Jacobian derivative to obtain a growth estimate for the Green's function on stable manifolds. A classical result of Wiman is then used to prove that the limit set must be unique. We note that this is the only point in the proof of Theorem 1 where we use that f is an at least moderately dissipative polynomial automorphism. Theorem 5 holds under the following weaker conditions.

Theorem 8. *Let f be a holomorphic automorphism of a 2-dimensional complex manifold X. Let $\Omega \subset X$ be a non-recurrent Fatou component. If there exists a compact $K \subset X$ so that all orbits in Ω converge to K, and so that f is volume contracting at all points in K, then the conclusions in Theorem 5 hold.*

We also note that here the invertibility of f is only necessary in order to conclude that Ω is biholomorphic to $\mathbb{C}^2$.

The organization of the rest of this paper is the following. In the next section we will describe all known examples of non-recurrent Fatou components in two complex dimensions. In Section 5 we prove that the image of rank 1 limit maps is smooth. In Section 6 we introduce the ordering on the set of limit maps and prove that the maximal limit maps for Hénon maps are fixed points. In Section 7 we prove a higher dimensional version of Fatou's Snail Lemma, which completes the proof of Theorem 5. In Section 8 we discuss the possibility of non-unique limit sets, and prove that for moderately dissipative Hénon maps the limit set is unique, completing the proof of Theorem 1. Finally in Section 9 we consider non-recurrent Fatou components in projective space and prove Theorem 6.

4. Examples of non-recurrent Fatou components

While the only known examples of non-recurrent Fatou components of Hénon maps are parabolic basins, for holomorphic endomorphisms several other examples are known to occur.

For a description of the dynamics of Holomorphic endomorphisms of $\mathbb{P}^2$, see for example the work of Hubbard-Papadopol [21] and Fornæss-Sibony [10], [12]. Fatou components for holomorphic endomorphisms of $\mathbb{P}^2$ are known to be pseudoconvex and Kobayashi hyperbolic [31], [11].

Recurrent Fatou components have been classified for holomorphic endomorphisms and the classification is almost identical to the classification for Hénon mappings.

Theorem 9 (Fornaess-Sibony, Ueda)**.** *Suppose that f is a holomorphic self-map of $\mathbb{P}^2$ of degree $d \geq 2$. Suppose that Ω is an invariant recurrent Fatou component. Then either:*

(1) Ω is an attracting basin of some fixed point in Ω,
(2) there exists a one-dimensional closed complex submanifold Σ of Ω and $f^n(K) \mapsto \Sigma$ for any compact set K in Ω. The Riemann surface Σ is biholomorphic to a disk or an annulus and $f|_\Sigma$ is conjugate to an irrational rotation, or
(3) the domain Ω is a Siegel domain.

Examples of non-recurrent Fatou components in $\mathbb{P}^2$ can easily be constructed by taking a cross product of two polynomials. For example, if p is a polynomial with a parabolic petal, and q is a polynomial *of the same degree* with a Siegel disk, then the map $p \times q : \mathbb{C}^2 \to \mathbb{C}^2$ extends to a holomorphic endomorphism f of $\mathbb{P}^2$. The map f has a Fatou component Ω where all orbits converge to a holomorphic disk, properly embedded in $\partial\Omega$, on which f acts as a rotation.

This idea cannot be applied to products of rational functions, so the product of a parabolic petal and a Herman ring cannot be obtained as easily. This can however be done in the following way.

Example 10. Let g be a rational function that has two distinct Fatou components Ω_1 and Ω_2, where the former is a Herman ring and the latter a parabolic petal. Then $g \times g$ is a holomorphic endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ with a Fatou component $\Omega_1 \times \Omega_2$. Now we use a construction due to Ueda which was also used by Fornæss and Sibony to construct examples of recurrent Fatou components in [11].

Let $\rho : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ be defined by $\rho([z : t], [w : s]) = [zw : ts : zs + wt]$. Then ρ exactly identifies pairs (a, b) and (b, a), and we see that ρ pushes the action of $g \times g$ down to $\mathbb{P}^2$. The new map $f : \mathbb{P}^2 \to \mathbb{P}^2$ has a Fatou component Ω biholomorphic to $\Omega_1 \times \Omega_2$. We see that all orbits converge to an annulus $A \subset \partial\Omega$, and f acts on A as an irrational rotation.

Suppose that the sequence f^{n_j} converges on Ω to a rank 1 limit map $h : \Omega \to \partial\Omega$ as before, and assume that $h(\Omega)$ does not depend on the sequence (n_j). In [34] Weickert showed that the action of f on the limit set $h(\Omega)$ can be lifted to a holomorphic self-map F of the unit disk Δ, and that F is either conjugate to an irrational rotation or $F^n \to \partial\Delta$, locally uniformly on Δ. Weickert gave examples of the former, but stated that he has no examples of the latter nor a proof that it cannot occur. Our Example 10 shows that it can indeed occur. We can think of the universal cover of the annulus A to be an infinite horizontal strip, and it is clear that the lift of a rotation to this strip can be given as a horizontal translation. But that means that for the lift $F : \Delta \to \Delta$ all orbits converge to a point on the boundary $\partial\Delta$. However, note from Theorem 6 that if $h(\Omega)$ is biholomorphic to a disk itself then the action of f is indeed conjugate to a rotation.

To summarize we now have seen three distinct examples of non-recurrent Fatou components:

(1) all orbits converge to a single fixed point,
(2) all orbits converge to a holomorphic disk, on which f acts as an irrational rotation, and
(3) all orbits converge to an annulus, on which f acts as an irrational rotation.

We note that these are the three components that arise immediately by combining 1-dimensional Fatou components. Is this everything, or can there be 2-dimensional Fatou components that are fundamentally different from combinations of 1-dimensional Fatou components?

5. Smoothness of Rank 1 limit Sets

The main step towards the proof of Lemma 12 is made in Proposition 11 below. We give an elementary analytic proof here, and we will give two more sophisticated but shorter proofs in the appendix.

Proposition 11. *Let $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be the germ of a holomorphic map whose image is singular at the origin in $\mathbb{C}^2$. Let U be a neighborhood of 0 where f is defined. Then there exists an $\epsilon > 0$ so that for every $g : U \to \mathbb{C}^2$ with $\|f - g\|_U \leq \epsilon$ we have that $g(U) \cap f(U) \neq \emptyset$.*

Proof. Without loss of generalization we may assume that U contains a neighborhood of the unit disk Δ. After changing coordinates, both on the the domain and the target, we can further assume that f is of the form

$$f(x) = (f_1(x), f_2(x)) = (x^p, x^q + h.o.t.), \tag{8}$$

where $q > p \geq 2$ (the Puiseux expansion). After a further change of coordinates on the target of the form $(z, w) \mapsto (z, w - \alpha(z))$ we may assume that none of the exponents occuring in $f_2(x)$ are divisible by p.

Let k be the greatest common divisor of all the exponents occuring in f, and write

$$f(x) = \tilde{f}(y) = (y^m, y^n + h.o.t.), \tag{9}$$

where $y = x^k$, $m = p/k$ and $n = q/k$. Then near $(0,0) \in \mathbb{C}^2$ the image of f is given by the equation

$$\prod_{j=1}^{m} (w - \tilde{f}_2((z^{\frac{1}{m}})_j) = w^m + \alpha_{m-2}(z) w^{m-2} + \cdots + \alpha_0(z) = 0, \tag{10}$$

as in the Weierstrass Preperation Theorem. The fact that

$$\alpha_{m-1}(z) = \sum \tilde{f}_2((z^{\frac{1}{m}})_j) \tag{11}$$

vanishes follows from our earlier assumption that none of the exponents in f_2 were divisible by p. Note that for every $j = 0 \cdots m-2$ the coefficient $\alpha_j(z)$ is divisible by z^l, where $l \geq \frac{2q}{p}$. Hence we can rewrite Equation 10 to obtain that a point (z,w) near $(0,0)$ lies in the image of f if and only if

$$\frac{w^m}{z^l r(z,w)} = 1, \tag{12}$$

where we write $r(z,w)$ for the remainder. Now suppose that g is a holomorphic function defined on Δ, close to f in a manner that we will specify shortly. In order to show that $g(\Delta) \cap f(\Delta) \neq \emptyset$ it is sufficient to show that there exists an $x \in \Delta$ such that

$$\phi(x) = \frac{g_2(x)^m}{g_1(x)^l \tilde{r}(x)} = 1. \tag{13}$$

If g is sufficiently close to f then by Rouché's Theorem both the numerator and denominator of ϕ have exactly $\frac{pq}{k}$ zeroes, counting multiplicity. Moreover, the value of $\phi(x)$ will be close to 1 on $\partial\Delta$. Let us for the purpose of contradiction assume that $f(U)$ and $g(U)$ do not intersect. Then it follows that $f(\Delta)$ and $g(\Delta)$ do not intersect for sufficiently small perturbations of g. Hence we may assume that the meromorphic function ϕ has zeroes and poles distinct from each other, exactly q zeroes, each of multiplicity m, and at least p poles of multiplicity at least l.

Let us change coordinates on the target by post-composing ϕ with the map

$$\theta = x \to \frac{x}{x-1}, \tag{14}$$

effectively switching the roles of 1 and ∞ while keeping 0 fixed. Then by our assumption $\tilde{\phi} = \theta \circ \phi$ is bounded and close to infinity on the boundary Δ, still has q zeroes, each of multiplicity m, and takes on the value 1 in at least p points of multiplicity at least l. Let

$$R = \min\{|\tilde{\phi}(x)| \mid x \in \partial\Delta\}. \tag{15}$$

Then by the maximum principle $V = \tilde{\phi}^{-1}(\Delta_R)$ is a disjoint union of finitely many simply connected domains. Since $\tilde{\phi}$ is a branched covering of degree $\frac{pq}{k}$ from V to Δ_R, the Riemann Hurwitz Theorem gives that the Euler characteristic of V satisfies

$$\chi(V) \leq \frac{pq}{k}\chi(\Delta_R) - q(m-1) - p(\frac{2q}{p} - 1) = \frac{pq}{k} - \frac{pq}{k} + q - 2q + p = q - p < 0. \tag{16}$$

But this is a contradiction since the Euler Characteristic of a finite union of disks is positive. $\square$

We stress that in Proposition 11 we consider perturbations of the map f, not of the defining equation for $f(\Delta)$. For example, if $f = (x^2, x^3)$ then the image is given by $\{w^2 - z^3 = 0\}$, which can easily be perturbed to $\{w^2 - z^3 = \epsilon\}$, which does not intersect the original cusp. Note that in this example the topology has changed: the intersection of $\{w^2 - z^3 = \epsilon\}$ with a ball centered at the origin is no longer simply connected.

The following is a direct consequence of Proposition 11.

Lemma 12. *Let f be a holomorphic endomorphism of a complex manifold X of dimension 2, and let Ω be a non-recurrent Fatou component. Let $h = \lim f^{n_j}$ be a limit map on Ω of generic rank 1. Then $h(\Omega) \subset \partial\Omega$ is an injectively immersed Riemann surface.*

Proof. Let $z \in \Sigma = h(\Omega)$ and let $x \in \Omega$ be such that $h(x) = z$. Since h is rank 1 there exist a small disk D through x such that h is non-constant on D and such that for some small neighborhood U of x, the image $h(U)$ equals $h(D)$. We first show that $h(D)$ is smooth for D small enough.

By reparametrizing we may assume that $x = 0$, D is the unit disk, and h maps $0 \in D$ to $(0,0) \in \mathbb{C}^2$. Suppose for the purpose of a contradiction that $h(D)$ is singular at $(0,0)$. After a change of coordinates we can write $h(\zeta) = (\zeta^p, \zeta^q + h.o.t.)$ with $q > p \geq 2$ and q not divisible by p, as in the proof of Proposition 11. But then it follows from Proposition 11 that for j large enough, the set $f^{n_j}(D)$ intersects $h(D)$. This is a contradiction since $f^{n_j}(D) \subset \Omega$ and $h(D) \subset \partial\Omega$. Therefore the image $h(D)$ is a smooth Riemann surface.

Now let $y \in \Omega$ be such that $h(y) = h(x)$. Let V be a small neighborhood of y. Suppose for the purpose of contradiction that the images $h(V)$ and $h(U)$ do not agree as germs. Then we have that $h(U)$ and $h(V)$ are both holomorphic graphs over a straight disk through z, and intersect only in z. It follows that for j large enough $f^{n_j}(U)$ must intersect $h(V)$, and again we have a contradiction. It follows that $h(U)$ and $h(V)$ agree as germs, and $h(\Omega)$ is therefore smooth. □

6. Ordering of limit maps

Let us begin with a useful observation due to Weickert:

Lemma 13 ([34])**.** *If Ω is an invariant Fatou component with a limit map $h = \lim f^{n_j}$, then $h(\Omega)$ is invariant under f. In particular, if $h(\Omega)$ is a point then it is fixed under f.*

Proof. Let $z \in h(\Omega)$ and write $z = h(x)$ for some $x \in \Omega$. Then $f(x) \in \Omega$, and since f and h commute we have $f(z) = f(h(x)) = h(f(x)) \in h(\Omega)$, and the conclusion follows. □

Remark 14. Since we are dealing with invertible maps, the same argument shows that the limit set $h(\Omega)$ is completely invariant under f.

Let us now recall the definition of a Fatou map.

Definition 15. Let f be a holomorphic endomorphism of a complex manifold X. A holomorphic map ϕ from a complex analytic space R into X is called a *Fatou map* for f if $\{f^j \circ \phi\}$ is a normal family.

Lemma 16. *Let X be a 2-dimensional complex manifold and f be a holomorphic endomorphism of X. Let Ω be a non-recurrent Fatou component, and suppose that $\{f^{n_j}\}$ converges uniformly on compact subsets of Ω to a rank 1 limit map $h : \Omega \to \partial\Omega$. Then the inclusion map from $\Sigma_h = h(\Omega)$ into X is a Fatou map.*

Proof. Let us denote the inclusion map by ϕ. Let $m_1, m_2 \dots$ be an increasing sequence of integers and let $K_1 \subset K_2 \subset \dots$ be a compact exhaustion of Ω. For each l we can choose $j(l)$ large enough so that

$$\|f^{n_{j(l)}+m_l}(z) - f^{m_l} \circ h(z)\| < \frac{1}{2^l}, \tag{17}$$

for $z \in K_l$. By the definition of the Fatou component Ω we obtain that a subsequence of $f^{n_{j(l)}+m_l}$ converges uniformly on compact subsets of Ω. But then it follows that there is a subsequence of $f^{m_l} \circ h(z)$ that converges uniformly on compact subsets of Ω, which implies that the corresponding subsequence of $f^{m_l} \circ \phi$ converges uniformly on compact subsets of Σ. □

The normality of the family (f^n) restricted to Σ_h allows us to introduce a natural ordering on the family of limit maps.

Definition 17. Given a non-recurrent Fatou component Ω for f, we define

$$\Gamma_f := \{h : \Omega \to \partial\Omega \mid \exists n_1, n_2, \ldots : f^{n_j}|_{\Omega} \to h\}. \tag{18}$$

Remark 18. The set Γ_f, equipped with the topology of uniform convergence on compact subsets of Ω, is compact. The map f acts on Γ_f both by pre- and post-composition, which by commutativity of the iterates of f induce the same action on Γ_f. It is well known that for any continuous group action on a compact set there exists a minimal invariant subset. These subsets are exactly the maximal equivalence classes with respect to the ordering that we introduce in this section.

Remark 19. The set Γ_f is remeniscent of the *Sushkevich kernel* for almost periodic semigroup actions, compare [23].

Now let $h \in \Gamma_f$ be a rank 1 limit map with image $\Sigma_h \subset \partial\Omega$. We have seen in Lemma 16 that the inclusion map $\Sigma \to X$ is a Fatou map. Let f^{m_l} be a convergent subsequence on Σ, converging to a map $\phi : \Sigma \to \phi(\Sigma)$.

Lemma 20. *The map $\phi \circ h$ is an element of Γ_f.*

Proof. Write $h = \lim f^{n_j}$. Then there exist $j(l)$ large enough such that the iterates $f^{m_l + n_{j(l)}}$ converge to $\phi \circ h$, uniformly on compact subsets of Ω. □

We would like to say that in the setting of Lemma 20 that $k = \phi \circ h$ is larger than h. Instead we will use the following equivalent definition.

Definition 21. Let $h = \lim f^{n_j} \in \Gamma_f$ and let $k \in \Gamma_f$. If there exists a sequence (m_l) in $\mathbb{Z}^+$ such that for any sufficiently large $\{j(l)\}$ we have that

$$k = \lim f^{m_l + n_{j(l)}}, \tag{19}$$

uniformly on compact subsets of Ω, then we say that $k \geq h$. This ordering is reflexive and transitive, but not necessarily anti-symmetric. For example, if f acts as a rotation on $h(\Omega)$, then Γ_f is equivalent to S^1, and $k \leq h$ for any $h, k \in \Gamma_f$. By considering h and k equivalent if $h \geq k$ and $k \geq h$ we do obtain a partial ordering on the equivalence classes.

Lemma 22. *Let $h, k \in \Gamma_f$, and write $\Sigma_h = h(\Omega)$. Then $k \geq h$ if and only if there exists an sequence $\{m_l\}$ in $\mathbb{Z}^+$ such that f^{m_l} converges on Σ_h to $\phi : \Sigma_h \to \partial\Omega$ and $k = \phi \circ h$.*

Proof. Let us write $h = \lim f^{n_j}$. If $k = \phi \circ h$, with $\phi = \lim f^{m_l}$, then it follows immediately that for $j(l)$ large enough we have that $k = \lim f^{m_l + n_{j(l)}}$, so $k \geq h$.

To prove the other direction, suppose that $k \geq h$, which by our definition means that there exist a sequence $\{m_l\}$ in $\mathbb{Z}^+$ so that for $j(l)$ large enough $k = f^{m_l + n_{j(l)}}$. Since the inclusion map from Σ_h into X is a Fatou map, we can restrict to a subsequence of $\{m_l\}$ if necessary so that f^{m_l} converges on Σ to a map $\phi : \Sigma_h \to \partial\Omega$. It follows that $\phi \circ h = k$. □

Lemma 23. *If $h, k \in \Gamma_f$ and $k \geq h$, then $k(\Omega) \subset \overline{h(\Omega)}$. Moreover, if $x, y \in \Omega$ are such that $h(x) = h(y)$ then $k(x) = k(y)$.*

We note that Lemma 23 follows immediately from Lemma 22.

Lemma 24. *The invariant minimal subsets of Γ_f are exactly equal to the maximal equivalence classes. In particular, there exists an $h \in \Gamma_f$ so that if $k \geq h$ then $h \geq k$.*

Proof. Suppose that $\omega \subset \Gamma_f$ is a maximal equivalence class. It follows immediately from the definition of the ordering that the closure of the orbit under f of any $h \in \omega$ is equal to ω, so ω is a minimal invariant subset.

On the other hand, suppose that $V \subset \Gamma_f$ is a minimal invariant subset, let $h \in V$ and suppose that there exists an $k \in \Gamma_f$ with $k \geq h$. By the minimality of V the map h must lie in the closure of the orbit of k under f, so by Lemma 22 we have that $h \geq k$. Hence h lies in a maximal equivalence class ω, which as we have seen is also a minimal invariant subset. But then ω must equal V, which completes the proof. □

We prove the following.

Lemma 25. *Suppose that h_{max} lies in a maximal equivalence class in Γ_f. Suppose that h has rank 1 and write $\Sigma_h = h_{max}(\Omega)$. Then the tangential derivatives along Σ_h of the maps (f^n) are uniformly bounded away from 0 and ∞ on compact subsets of Σ_h.*

Proof. By Lemmas 12 and 13, Σ_h is an invariant injectively immersed Riemann surface, so we can indeed talk about tangential derivatives. Let K be a compact subset of Σ_h. Since the inclusion map is a Fatou map, we immediately obtain that on K the modulus of the tangential derivatives is uniformly bounded from above. Suppose for the purpose of contradiction that moduli of the tangential derivatives is not bounded away from 0. Then we can a sequence of iterates (f^{n_j}) and a sequence of points a_j so that $a_j \to a \in K$ and such that the tangential derivative of f^{n_j} at a_j goes to 0. By restricting to a convergent subsequence we obtain a limit map

$$\phi = \lim f^{n_j}|_{\Sigma_h}, \tag{20}$$

with $\phi'(a) = 0$. By Lemma 22 we have that $\phi \circ h_{max} = k \in \Gamma_f$, and $k \geq h_{max}$. Hence by Lemma 12 $\phi(\Sigma_h) = k(\Omega)$ is smooth, which implies that $\phi : \Sigma_h \to \sigma_k$ is $d : 1$ near the point a, for some $d > 1$. But then it follows from Lemma 23 that k is strictly larger than h, which contradicts the maximality of h. □

We will now restate and prove Lemma 26, formulated in Section 3, which will conclude the proof of the first part of Theorem 5.

Lemma 26. *Let f be a Hénon mapping, let $\Omega \subset \mathbb{C}^2$ be a non-recurrent Fatou component and let $h \in \Gamma_f$ be a maximal limit map. Then $h(\Omega)$ consists of a single fixed point.*

Proof. If h has rank 0 then the desired follows from Lemma 13. So assume for the purpose of a contradiction that h has rank 1. Recall from Lemma 12 that $\Sigma_h = h(\Omega)$ is an injectively immersed Riemann surface, and from Lemma 25 that on any compact subset of Σ_h the tangential derivatives of the family $\{f^n\}$ are uniformly bounded away from 0 and ∞.

Recall from [2] that Σ_h is a bounded set, contained in the polydisk

$$W = \{(z, w) \in \mathbb{C}^2 \mid |z|, |w| \leq R\}, \tag{21}$$

for some sufficiently large R depending on f.

Since Ω is non-recurrent, the Jacobian derivative δ of f necessarily satisfies $|\delta| < 1$. Let $z \in \Sigma_h$ and let D be a relatively compact holomorphic disk in Σ_h, centered at z. For $x \in \Sigma_h$ and $m > 0$ we define the tangent cone

$$C_x^m = \{v = v_1 + v_2 \in T_x(\Sigma_h) \oplus N_x(\Sigma_h) \mid |v_2| \le m|v_1|\}. \tag{22}$$

Since Ω lies in the compact set W, the map f is volume contracting and the tangential derivatives of the iterates f^j along Σ are uniformly bounded from above and away from 0, we can find $N \in \mathbb{N}$ and $0 < m' < m$ so that

$$df^N(C_x^m) \subset C_{f^N(x)}^{m'}, \tag{23}$$

for every integer n and every $x \in f^n(D)$.

Note that since the sequence of iterates (f^n) restricted to Σ_h is a normal family, the second derivatives of f^n along Σ_h are bounded from above on D, uniformly over n. Hence we can make D smaller if necessary so that for every n the holomorphic disk $f^n(D)$ is a graph over the the complex line through $f^n(z)$ tangent to Σ_h. It follows that, by again decreasing D if necessary, we can extend the invariant cone fields C^m and $C^{m'}$ on each $f^n(D)$ to a bidisk centered at $f^n(z)$ whose radii are independent of n. Here the axes of these bidisks are the tangent and normal directions of Σ_h at $f^n(z)$.

By standard construction (see for example [17]) there exists through every $x \in D$ a strong stable manifold $W^s(x)$. Moreover, the uniform size of the bidisks guarantees that the stable manifolds through $x \in D$ extend almost vertically through the bidisk containing D. Hence we obtain a stable foliation, filling a neighborhood of D. But since the sequence of iterates is normal when restricted to D, the iterates also form a normal family in the union of these stable manifolds, which implies that Σ_h does not lie on the boundary of the Fatou set but in the interior. This contradicts our hypothesis and completes the proof. $\square$

Of course Lemma 26 does not rule out the existence of rank 1 limit maps. It does follow that given any rank 1 limit map $h \in \Gamma_f$, there exists a sequence f^{n_j} that converges uniformly on compact subsets of $\Sigma_h = h(\Omega)$ to a fixed point in $\overline{\Sigma}_h$.

7. Parabolic Basins

Let f be a Hénon map with a non-recurrent Fatou component Ω, and suppose that $\{f^n\}$ converges uniformly on compact subsets of Ω to a point $p \in \Gamma_f$. Since the Jacobian derivative of f must be strictly less than 1 in absolute value, at least one of the eigenvalues must have modulus strictly less than 1. As p lies in the Julia set, it cannot be an attractive fixed point, and since there is uniform convergence to p on an open subset, p cannot be a hyperbolic fixed point. It follows that the other eigenvalue must have modulus exactly equal to 1. In fact, we now prove that the other eigenvalue must be equal to 1.

Theorem 27. *Let $f : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ be a germ of a holomorphic mapping that fixes the origin. Suppose that there exists an open set W with $f(W) \cap W \neq \emptyset$ and such that on W the iterates (f^n) converges uniformly to the origin. Further suppose that $\mathrm{D}f(0)$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with $|\lambda_1| = 1$ and $|\lambda_i| < 1$ for $i \ge 2$. Then $\lambda_1 = 1$.*

Proof. After changing coordinates we may assume that the eigen vector corresponding to λ_1 is $(1, 0, \ldots, 0)$ and all other eigen vectors of $\mathrm{D}f(0)$ are orthogonal to $(1, 0, \ldots, 0)$.

Recall that we have a strong stable manifold through 0 of complex dimension $k-1$. Since this strong stable manifold is locally a graph over the hyperplane $\{z_1 = 0\}$ we can locally change coordinates such that the local stable manifold of 0 is equal to the hyperplane $\{z_1 = 0\}$. We denote by C a (not necessarily unique) $\mathcal{C}^\infty$ center manifold through 0. Since there may not be a holomorphic center manifold we cannot assume that there is a holomorphic change of coordinates that maps the center manifold to the z_1-plane, but we do have that C is tangent to the z_1-plane at 0.

Let V be a small ball centered at 0 so that $f|_V$ is invertible and acts weakly hyperbolically. That is, we choose V small enough such that $\{(v, v') \in T(V) \mid |v| \geq \|v'\|\}$ is an invariant cone field. We denote by K the subset of points in V whose orbits stay in V, so in particular for every $z \in \Omega$ we have that $f^n(z) \in K$ for large enough $n \in \mathbb{N}$. By choosing V small enough we have that for every point $z \in K$ we locally have a strong stable manifold through z that extends to the boundary of V. It follows from the invariant cone field that the strong stable manifolds are *close to vertical*, that is, for every pair $(x, x'), (y, y')$ in a local stable manifold we have that $\|x' - y'\| < |x - y|$.

Let $w \in W$ be such that $f(w)$ also lies in W. Then for some $N \in \mathbb{N}$ we have that $f^n(W) \subset V$ for all $n \geq N$. We write $W' = f^N(W)$.

Then through every point in W' we have a local stable manifold as noted before. Since the strong stable manifolds are close to vertical they must intersect the z_1-plane in a unique point. We denote by $h : K \to \mathbb{C}_{z_1}$ the holonomy map induced by the strong stable manifolds: h maps a point $z \in K$ to the intersection of the strong stable manifold through z with the z_1-plane. Then $U = h(W')$ is a connected subset of $\Omega \cap \mathbb{C}_{z_1}$ that contains the points $u = h(f^N(w))$ and $v = h(f^{N+1}(w))$. Note that v is not necessarily equal to $f(u)$ since the z_1-plane is not invariant, but v lies in the strong stable manifold through $f(u)$. Also note that U cannot contain 0: if U did contain 0 then Ω would contain a neighborhood of 0 which would contradict our assumptions.

Let ψ be a conformal map from the unit disk Δ to U that maps 0 to u, and define

$$\phi_n(\zeta) = \frac{\pi_1 \circ f^n \circ \psi(\zeta)}{\pi_1 \circ f^n(u)}, \tag{24}$$

where π_1 is the (straight) projection onto the z_1-plane.

Note that since $f^n(U)$ avoids the strong stable manifold $W^s(0)$, the maps ϕ_n are well-defined and $\phi(\zeta) \neq 0$ for any $\zeta \in \Delta$. It is clear that $\phi_n(0) = 1$ for every n, and it follows from the invariant cone field for f in V that $f^n(\Omega)$ is "almost horizontal" and the map ϕ_n is univalent. It follows that the family $\{\phi_n\}$ is normal. Moreover, we note that $\phi_n(\psi^{-1}(v)) \to \lambda_1$, and by Hurwitz Theorem it follows that any limit map of the family must be univalent. Then we must a constant $\epsilon > 0$ such that $|\phi_n'(0)| > \epsilon$ for any $n \in \mathbb{N}$. Hence by the Koebe $\frac{1}{4}$-Theorem we have that

$$\mathrm{d}(\phi_n(0), \partial(\phi_n(\Delta))) > \frac{1}{4}\epsilon, \tag{25}$$

and hence

$$\mathrm{d}(\pi_1(f^n(u)), \partial(\pi_1 f^n(\Omega))) > \tilde{\epsilon}|f^n(u)|. \tag{26}$$

Now let $j \in \mathbb{N}$ be such that $\mathrm{d}(\lambda_1^j, 1) < \frac{\tilde{\epsilon}}{2}$.

Let us write $f^n(u) = u_n = (x_n, y_n)$, with $x_n \in \mathbb{C}_{z_1}$ and $y_n \in \mathbb{C}^{k-1}$. For large n we claim that $\|y_n\| < |x_n|^2$. Indeed, w lies in the stable manifold of a point in the invariant center manifold C, and the distance $\mathrm{d}(u_n, C)$ decreases exponentially. The claim follows from the fact that C is tangent to the z_1-plane at 0. Hence it follows from the fact that the local stable manifolds in V are vertical and the uniform convergence of f^n on U that for large n we have

$$\mathrm{d}(\pi_1(x_n), h(x_n)) < \frac{1}{8}\tilde{\epsilon}|u_n|. \tag{27}$$

It follows from the fact that the local stable manifolds through u_n are almost vertical and $u_n \to 0$ that for large n we have that

$$\mathrm{d}(h(u_n), h(u_{n+j})) < \frac{2}{3}\tilde{\epsilon}|u_n|, \tag{28}$$

while

$$d(h(u_n), \partial(hf^n(U))) > \frac{3}{4}\tilde{\epsilon}|u_n|. \tag{29}$$

It follows from estimates (28) and (29) that $h(u_{n+j})$ lies in $h(f^n(U))$ and hence that $V \cap \Omega \cap \mathbb{C}_{z_1}$ contains a Jordan curve that winds around the origin. The vertical foliation through this Jordan curve plus the maximum principle now give us that Ω contains a neighborhood of 0, which is in contradiction with the assumption that $0 \in \partial(\Omega)$. □

Lemma 26 and Theorem 27 combined with Theorem 7 together imply Theorem 5. Let us note that the dynamical behavior of a germ $f : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ with eigenvalues $\lambda_1, \ldots, \lambda_k$ satisfying $\lambda_1^p = 1$ and $|\lambda_i| < 1$ for $i \geq 2$ is very well understood due to the following result of Di Giuseppe [6].

Theorem 28 (Di Giuseppe)**.** *Let $f = (f_1, \ldots, f_k)$ be a semi-hyperbolic germ as above. Then up to holomorphic conjugacy one of the following is satisfied:*

$$(i)\ f_1(z) = \lambda_1 z_1;$$
$$(ii)\ f_1(z) = \lambda_1 z_1 + a_k z_1^{kq+1} + o(\|z\|^{kq+1}), \text{with } a_k \neq 0 \text{ and } k \geq 1.$$

Moreover, in case (i), f is locally topologically conjugate at the origin with $g := df_0$; and in case (ii), f is locally topologically conjugate at the origin with $g(z) := (\lambda_1 + z_1^{kq+1}, \lambda_2 z_2, \ldots, \lambda_k z_k)$.

Di Giuseppe's result actually applies in greater generality: $|\lambda_i| > 1$ is allowed for $i \geq 2$ as long as there is *quasi-absence of resonance.* In our setting quasi-absence of resonance is automatically satisfied.

8. Uniqueness of the limit set

Let f be a holomorphic endomorphism of a complex manifold X, and let Ω be a non-recurrent Fatou component for f.

Question 29. Given a holomorphic endomorphism of a complex manifold X with a non-recurrent Fatou component. Is it possible for the limit set $h(\Omega)$ to depend on the map $h \in \Gamma_f$?

Although non-uniqueness may be hard to imagine for Hénon mappings or holomorphic endomorphisms of $\mathbb{P}^2$, the following construction gives an affirmative answer to the above question. The complex manifold will be be a domain in $\mathbb{C}^2$.

Theorem 30. *There exists an open connected set $D \subset \mathbb{C}^2$ and a map $f : D \to D$ such that $\{f^n\}$ is normal on D, all orbits $f^n(z)$ converge to ∂D and $h(D)$ depends on $h \in \Gamma_f$.*

Proof. We let

$$f(z,w) = (p(z), q_z(w)), \tag{30}$$

where

$$p(z) = z - z^3 + h.o.t. \tag{31}$$

is such that

$$p(\frac{1}{\sqrt{u}}) = \frac{1}{\sqrt{u+1}}, \tag{32}$$

for u in the strip $S = \{0 < \mathrm{Im}(u) < 2\pi\}$. We define the map $q_z(w)$ by

$$q_z(w) = \frac{\phi(p(z))}{\phi(z)} w, \tag{33}$$

where

$$\phi(z) = 1 - e^{i\frac{1}{z}}. \tag{34}$$

The domain D is given by $\Omega \times \mathbb{C}$, where Ω contains the z-values that correspond to u-values in the half-strip S. Note that $\phi(z) \neq 0$ for any $z \in \Omega$, so the map f is well defined.

Let K be a compact subset of Ω. Then $p^n(z)$ converges to the origin along the positive real axis, uniformly for $z \in K$. Moreover, we have that

$$\max_{z,z' \in K} \mathrm{dist}(\frac{1}{p^n(z)}, \frac{1}{p^n(z')}) \to 0. \tag{35}$$

Hence given any increasing sequence in $\mathbb{N}$ we can find a subsequence $\{n_j\}$ for which

$$e^{i\frac{1}{p^{n_j}(z)}} \to 0, \tag{36}$$

with $|\theta| = 1$, uniformly for $z \in K$. Given that

$$w_n = \frac{\phi(f^{n+1}(z))}{\phi(f^n(z))} \cdots \frac{\phi(f(z))}{\phi(z)} w = \frac{\phi(f^{n+1}(z))}{\phi(z)} w_0, \tag{37}$$

it follows that $\{f^n\}$ is normal on $\Omega \times \mathbb{C}$. The first coordinate z_n will always converge to the origin, so all the orbits (z_n, w_n) converge to ∂D.

To see that $h(D)$ depends on the sequence $\{n_j\}$, note that by choosing the $n_j \in \mathbb{N}$ appropriately we can make sure that $e^{i\frac{1}{z_{n_j}}}$ converges to 1 for some initial (z_0, w_0). It follows that (z_{n_j}, w_{n_j}) converges to $(0,0)$ for any starting point (z,w) in a compact subset of D.

On the other hand, we can make sure that $e^{i\frac{1}{z_{n_j}}}$ converges to -1 for some point z_0. In this case the map h will satisfy $h(z,w) = (0, \frac{2}{z} w)$ and the image of h is the complex line $\{0\} \times \mathbb{C}$, showing that the image $h(D)$ depends on the subsequence $\{f^{n_j}\}$. $\square$

We note that the above construction is quite different from a Fatou component of a Hénon map or a holomorphic map of projective space. The map f is not defined on the boundary of D, and f cannot be extended holomorphically to any neighborhood of 0.

Let us return to our Hénon map f with a non-recurrent Fatou component Ω. If $h(\Omega)$ depends on $h \in \Gamma_f$, then by Lemma 26 there must exist both an $k \in \Gamma$ of rank 0 (for example any maximal h), and an $h \in \Gamma$ of rank 1. The latter follows from the following classical argument, already used in higher dimensions by Jupiter and Lilov in [22]. The union of the limit sets must be connected, the image of a rank 0 limit map is a fixed point, and a Hénon map has only finitely many fixed points. Therefore if all limit maps have rank 0, the limit map is unique. The same argument holds for holomorphic endomorphisms, the fact that these maps have only finitely many fixed points was proved by Fornæss and Sibony in [10].

So let us suppose that there exist limit maps in Γ_f both with rank 0 and with rank 1. Let us call k a limit map of rank 0. Then $p = k(\Omega)$ is a fixed point, and since f is dissipative, p is either a hyperbolic fixed point or a semi-attracting fixed point. In either case there exists a strong stable manifold $W^s(p)$, which is biholomorphic to the complex plane.

Lemma 31. *Let f be a Hénon map and Ω be an invariant non-recurrent Fatou component with bounded forward orbits. Suppose that a subsequence f^{n_j} converges uniformly on compact subsets of Ω to the fixed point p, which necessarily has a strong stable manifold $W^s(p)$. Further suppose that the limit set $\{p\}$ is not unique. Then there exists a sequence of iterates f^{m_l} that converges uniformly on compact subsets of Ω to a rank 1 limit map $h : \Omega \to W^s(p)$.*

Proof. Since f has only finitely many fixed points, we can find an $\epsilon > 0$ so that $\overline{B_{2\epsilon}(p)}$ contains no fixed points besides p. Further decrease ϵ if necessary so that (1) there exists a dominated splitting on $B_{2\epsilon}(p)$ and (2) there exists a $z \in \Omega$ whose orbit enters and leaves $B_{2\epsilon}(p)$ infinitely often. The dominated splitting guarantees the existence of a strong stable manifold in B_ϵ through every point whose orbits remains in B_ϵ. These strong stable manifolds lie *properly* in B_ϵ.

Let $0 < \epsilon\prime < \epsilon$ be such that $f^{-1}(B_{\epsilon\prime}(p)) \subset B_\epsilon(p)$. Then there exist increasing integers $m_1, m_2, \ldots$ and $k_1, k_2, \ldots$ such that for every integer l

(1) $f^{m_l}(z) \in B_\epsilon(p) \setminus B_{\epsilon\prime}(p)$,
(2) $f^{m_l+j}(z) \in B_\epsilon(p)$ for $j = \{1, \ldots, k_l\}$, and
(3) $f^{m_l+k_l}(z) \in B_{\frac{1}{2^l}}(p)$.

By restricting to a subsequence if necessary we may assume that f^{m_l} converges uniformly on compact subsets of Ω to a limit map $h : \Omega \to \partial\Omega$. Denote $x = h(z)$. Note that x must lie in $\overline{B_\epsilon(p) \setminus B_{\epsilon\prime}(p)}$, and hence cannot be a fixed point. Therefore h must have rank 1.

Note that the orbit of x must remain in $\overline{B}_\epsilon$. If not, there could not be points $f^{m_l}(z)$ arbitrarily close to x whose orbits approach p arbitrarily nearly before leaving $\overline{B}_\epsilon$. Since the orbit of x remains in $\overline{B}_\epsilon$, the dominated splitting guarantees the existence of a unique strong stable manifold $W^s(x)$.

The limit set $h(\Omega)$ either intersects $W^s(x)$ locally only in x (where by locally we refer to the topology on the Riemann surfaces $h(\Omega)$ and $W^s(x)$), or locally coincides with $W^s(x)$. By normality of the family f^n restricted to $h(\Omega)$, the tangential derivatives of the family f^n must be uniformly bounded, locally in $h(\Omega)$. Therefore it follows that there is a neighborhood of x in $h(\Omega)$ for which the orbits stay in $B_{2\epsilon}(p)$, which implies the existence of strong stable manifolds through any point in $h(\Omega)$. Hence if $W^s(x)$ and $h(\Omega)$ intersect locally only in x, the strong stable manifolds fill some neighborhood of x. It follows that the orbits of all points in a small neighborhood of x remain in $B_{2\epsilon}(p)$, which contradicts the fact that x is the limit of the sequence $f^{m_l}(z)$. Hence $h(\Omega)$ and $W^s(x)$ must locally coincide.

By Lemma 13, $h(\Omega)$ is invariant under f, and by Lemma 26 there must be a sequence of iterates on $h(\Omega)$ that converges to a fixed point, which in this case can only be p. This is not possible unless $W^s(x) = W^s(p)$. Hence $h(\Omega)$ must be contained in $W^s(p)$. □

Lemma 32. *Let $h \in \Gamma_f$ be of rank 1 so that $h(\Omega) \subset W^s(p)$. Then $h(\Omega)$ lies in $J = J^+ \cap J^-$.*

Proof. Since $h(\Omega)$ lies in $W^s(p)$, it is clear that $h(\Omega)$ lies in K^+. It is equally clear that no point of $h(\Omega)$ can lie in the interior of K^+, as it would imply normality of the sequence of iterates in a neighborhood of the point. Therefore $h(\Omega)$ lies in J^+.

Let $x \in h(\Omega)$, and let $z \in \Omega$ be such that $h(z) = x$. Suppose for the purpose of contradiction that x does not lie in K^-. Then there is an $N \in \mathbb{N}$ such that $f^{-N}(x) \in V^-$. Write $h = f^{n_j}$. Then we see that for j large enough $f^{-N}(f^{n_j}(z)) = f^{n_j - N}(z)$ also lies in V^-. But this is a contradiction, as $n_j - N \to \infty$ as $j \to \infty$. Therefore $h(\Omega) \subset K^-$. Since f is dissipative, $J^- = K^-$ and we are done. □

The following Lemma completes the proof of Theorem 1.

Lemma 33. *Let f be a Hénon map of degree d, whose Jacobian determinant δ satisfies*

$$|\delta| < \frac{1}{d^2}, \tag{38}$$

and let p be a fixed point that is not attracting. Suppose that Ω is an invariant non-recurrent Fatou component with a limit map $h = \lim f^{n_j}$ that maps Ω into $W^s(p)$. Then $h(\Omega) = \{p\}$.

Finally, we will make use of a subharmonic version of the classical result by Wiman [33] (see also [14], Ch. V, Thm 1.3) which is also a particular case of a (subharmonic version) of the Denjoy-Carleman-Ahlfors Theorem, see [28] and [16], §4.6. It had been recently used in the same context by Dujardin and the first author in [7], see Remark 2 in the Introduction. (In the context of one-dimensional complex dynamics, it had been earlier used by Eremenko and Levin [8].)

Theorem 34 (Wiman)**.** *Let g be a non-constant subharmonic function on $\mathbb{C}$ whose order of growth is less than 1/2, i.e., $g(z) = O(|z|^\rho)$ for some $\rho < \frac{1}{2}$. Then all components of $\{g = 0\}$ are bounded.*

Proof of Lemma 33. Suppose for the purpose of a contradiction that $h(\Omega)$ is not equal to $\{p\}$, in which case h must have rank 1. By the classical Poincaré Theorem, the restriction $f|\,W^s(p)$ is globally linearizable, i.e., there exists a a biholomorphism $\Psi : \mathbb{C} \to W^s(p)$ that satisfies the functional equation

$$\Psi(\lambda^{-1}\zeta) = f^{-1}(\Psi(\zeta)), \tag{39}$$

where λ is the stable multiplier of p. Note that by our assumption,

$$\lambda \leq \delta < \frac{1}{d^2} \tag{40}$$

Let us now consider the *backward Green function* $G_- : \mathbb{C}^2 \to \mathbb{R}$

$$G_-(z) = \lim_{n \to +\infty} \frac{1}{d^n} \log \|f^{-n} z\|.$$

It is a non-negative plurisubharmonic function vanishing on K_- and satisfying the functional equation

$$G_-(f^{-1}z) = d\,G_-(z). \tag{41}$$

Let us restrict this function to $W^s(p)$ and pull it back to $\mathbb{C}$, i.e., let $g = G_- \circ \Psi$. This is a non-negative subharmonic function on $\mathbb{C}$ vanishing on

$$\Psi^{-1}(K_-) \supset \Psi^{-1}(h(\Omega)) =: \Lambda.$$

Note that Λ is connected and by (39) is invariant under the scaling $z \mapsto \lambda^{-1}z$. Hence it is an unbounded continuum in $\mathbb{C}$.

Moreover, by (39 and (41), it satisfies the functional equation

$$g(\lambda^{-1}z) = d\,g(z).$$

It follows that

$$g(z) = O(|z|^\rho), \quad \text{with } \rho = -\frac{\log d}{\log \lambda} < 1/2,$$

where the last estimate follows from (40).

By the Wiman Theorem, all components of $\{g = 0\}$ are bounded, contradicting unboundedness of Λ. □

Remark 35. As we have alluded earlier, the original Wiman Theorem was concerned with entire (rather than subharmonic) functions. In fact, it can be directly used in our context as well, using the coordinate function w instead of the Green function G^-.

9. Holomorphic endomorphisms of projective space

In this section we give a description of Fatou components under the assumption that the image of the limit map $h = \lim f^{n_j}$ is independent of the sequence (n_j).

Before we prove Theorem 6, let us note that for holomorphic endomorphisms of $\mathbb{P}^2$ we cannot expect the same description as for Hénon maps in Theorem 5. First of all, we have already seen in Example 10 a Fatou component Ω for which all orbits converge to an invariant disk or annulus lying in the boundary of Ω. Also, even in the case where all orbits converge to a point $p \in \partial\Omega$, we cannot expect the eigenvalues λ_1 and λ_2 of $Df(p)$ to satisfy $|\lambda_1| < 1$ and $\lambda_2 = 1$. Of course, if one of the eigenvalues λ_1 satisfies $|\lambda_1| < 1$, then $|\lambda_2| = 1$ and Theorem 27 implies that $\lambda_2 = 1$. However, by taking the cross product of two polynomials of the same degree that both have a parabolic fixed point, we can obtain an example of a Fatou components where all orbits converge to a point $p \in \partial\Omega$ with $Df(p) = \mathrm{Id}$.

Naively one might then expect that at least one of the eigenvalues has to equal 1, but the examples below show that this does not hold either. The first map has a *quasi-parabolic* fixed point, as studied by Bracci and Molino in [4].

Example 36. Let us first construct an example where both eigenvalues have modulus 1 but one of the eigenvalues is not equal to 1. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be given by

$$f(z, w) = (z(1 + z), \lambda w(1 + z)), \tag{42}$$

where $|\lambda| = 1$. Let us write $(z_n, w_n) = f^n(z, w)$. It is clear that z_n only depends on z_0, and

$$z_n = z_0 \prod_{i=o}^{n-1} (1 - z_i). \tag{43}$$

Similarly,

$$w_n = w_0 \prod_{i=o}^{n-1} (1 - z_i). \tag{44}$$

So we see that if z_n converges to 0 then so will w_n, and the basin of the origin is given by $\Omega \times \mathbb{C}$, where $\Omega \subset \mathbb{C}$ is the parabolic basin of the map $z \mapsto z(1 + z)$.

This map does not extend holomorphically to $\mathbb{P}^2$, but we can consider the following modification.

$$f(z, w) = (z(1 + z) + z^d, \lambda w(1 + z) + w^d), \tag{45}$$

where $d \geq 3$. It is easy to see that the dynamical behavior near the origin is similar, yet this map extends holomorphically to $\mathbb{P}^2$. Hence we see that for any $|\lambda| = 1$ there exist a holomorphic endomorphism of $\mathbb{P}^2$ with a parabolic basin and corresponding eigenvalues equal to 1 and λ.

With a little more effort we can construct a parabolic basin in $\mathbb{P}^2$ where neither eigen value equals 1.

Example 37. Again we first consider a selfmap of $\mathbb{C}^2$.

$$f(z, w) = (e^{i\theta} z(1 + zw), e^{-i\theta} w(1 + zw)). \tag{46}$$

We claim that for any $\theta \in \mathbb{R}$ this map has a parabolic basin at the origin. Notice first that in a neighborhood of the origin, the axes are completely invariant. We also have that $z_{n+1}w_{n+1} = z_n w_n(1 + z_n w_n)$. Writing y for zw we obtain

$$y_n = y_0 \prod_{i=0}^{n-1} (1 + y_i)^2. \tag{47}$$

We also have that

$$z_n = z_0 \prod_{i=0}^{n-1} (1 + y_i), \tag{48}$$

and

$$w_n = w_0 \prod_{i=0}^{n-1} (1 + y_i). \tag{49}$$

We therefore see that if $y_n \to 0$ then $z_n, w_n \to 0$ as well, and we conclude that the origin has a parabolic basin.

Again the map f does not extend to a holomorphic endomorphism of $\mathbb{P}^2$, but we can consider a similar modification

$$F(z, w) = f(z, w) + (z^d, w^d), \tag{50}$$

where $d \geq 7$ and the map does extend to $\mathbb{P}^2$.

Lemma 38. *The holomorphic map F has an open set of orbits converging uniformly to the origin.*

Proof. Let $(z_0, w_0) \neq 0$ be such that the following induction hypotheses are satisfied for some $\epsilon > 0$:

$$(i)\ \ |z_0| < \epsilon, \quad |w_0| < \epsilon,$$
$$(ii)\ \ |\frac{z_0^2}{w_0}| < \epsilon, \quad |\frac{w_0^2}{z_0}| < \epsilon,$$
$$(iii)\ \ |\mathrm{Re}(z_0 w_0)| < 0, \quad |\mathrm{Im}(z_0 w_0)| < \epsilon |\mathrm{Re}(z_0 w_0)|.$$

A straightforward calculation shows that, as long as ϵ is small enough, the hypotheses are also satisfied for (z_n, w_n), and (again assuming that ϵ is small enough) that z_n and w_n both converge to 0. □

We now restate and prove Theorem 6.

Theorem 6. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ and let Ω be a non-recurrent, invariant Fatou component. Suppose that the limit set $h(\Omega)$ is unique. Then $h(\Omega)$ either consist of one point p, or $h(\Omega)$ is a injectively immersed Riemann surface, conformally equivalent to either the unit disk, the punctured unit disk or an annulus, and f acts on $h(\Omega)$ as an irrational rotation.*

Proof. If $h(\Omega)$ is a point then we are done, so we may assume that h has rank 1. By Lemma 12 the image $\Sigma_h = h(\Omega)$ is a injectively immersed Riemann surface, invariant under f. By the work of Weickert [34], Σ_h is hyperbolic. We claim that every orbit in Σ_h is recurrent in the topology of Σ_h. If not there would be an orbit in Σ accumulating on a point in $\partial\Omega \setminus \Sigma_h$, which by Lemma 16 would imply that $h(\Omega)$ is not unique.

A hyperbolic Riemann surface either has a diskrete automorphism group, or is biholomorphic to the unit disk, the punctured unit disk or an annulus. By Lemma 24 we may assume that h is minimal. Hence $f : \Sigma \to \Sigma_h$ is an automorphism, and as we noted above its action on Σ_h is recurrent. As was shown in [10], a holomorphic endomorphism of $\mathbb{P}^2$ can only have finitely many fixed points. Therefore the automorphism group of Σ_h cannot be diskrete. Hence Ω must be the unit disk, the punctured unit disk or an annulus, and f acts on Σ_h as an irrational rotation. □

We note that it is unknown whether $h(\Omega)$ can be equivalent to a punctured disk.

10. Appendix: Perturbations of Singular Riemann Surfaces

We give two alternative proofs of Proposition 11, which we first restate.

Proposition 11. *Let $f : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be the germ of a holomorphic map whose image is singular at the origin in $\mathbb{C}^2$. Let U be a neighborhood of 0 where f is defined. Then there exists an $\epsilon > 0$ so that for every $g : U \to \mathbb{C}^2$ with $\|f - g\|_U \leq \epsilon$ we have that $g(U) \cap f(U) \neq \emptyset$.*

Geometric proof. The following argument is a more geometric presentation of the argument given in the main body of the paper. To fix the idea, let us assume that the image of f is the standard cusp $\{w^2 = z^3\}$. Let us include into into a holomorphic foliation $\mathcal{F}$ with leaves

$$L_\lambda = \{w^2 = \lambda z^3\}, \quad \lambda \in \hat{\mathbb{C}}.$$

Let us puncture out 0, and consider the space $\mathcal{O}$ of leaves in the punctured neighborhood of the origin. This space has a natural Riemann orbifold structure (supported on the sphere) whose local charts are obtained by taking local transverals to $\mathcal{F}$ and slicing the leaves to it. There are two orbifold points on $\mathcal{O}$: the leaf $w = 0$ is an orbifold point of order 3 and the leaf $z = 0$ is an orbifold point of order 2. So, the Euler characteristic of $\mathcal{O}$ is equal to $1/2 + 1/3 < 1$.

Let $[\lambda]$ be the point of $\mathcal{O}$ corresponding to the leaf L_λ. The function g naturally induces a holomorphic orbifold map $g : U \to \mathcal{O}$ that does not assume value $[\lambda = 1]$ but whose boundary values are close to this point. It follows that g is proper over $\mathcal{O} \setminus B$, where B is a small neighborhood of $[1]$. Hence g is an orbifold branched covering over $\mathcal{O} \setminus B$ of some degree d. By the orbifold Riemann-Hurwitz Theorem, the Euler characteristic of $g^{-1}(\mathcal{O} \setminus B)$ is at most $d \cdot \chi(\mathcal{O} \setminus B) < 0$. On the other hand, it follows from the Maximum Principle that $g^{-1}(\mathcal{O} \setminus B)$ is the union of disks, which has a positive Euler characteristic.

In general, consider the foliation with leaves $\{w^m = \lambda z^l r(z, w)\}$ (using notation of (12)). The space of its leaves in the punctured neighborhood of 0 is an orbifold with Euler characteristic leass than 1 (as it has at least two orbifold points, and one of them has order at least 3). The rest of the argument is the same. □

Topological proof The following argument was proposed to us by Gabrielov and Milnor.

We recall a few facts that can be found in [24]. Let us consider a small closed ball $B = \mathbb{B}_\epsilon$ in $\mathbb{C}^2$ bounded by a 3-sphere $S = \partial B$. We may assume that $V := f(U) \cap B$ lies properly in B, and we denote its intersection with S by γ. If ϵ is chosen sufficiently small then γ is a non-trivial knot in S. Moreover, there exists a retraction $\pi : B \setminus V \to S \setminus \gamma$.

Let us consider the component U_0 of $g^{-1}(B)$ containing 0. Then $g : U_0 \to B$ is a singular 2-cell bounded by a knot η in S. Since $g(U_0)$ is disjoint from V, we can retract it by π to $S \setminus \gamma$. We obtain a singular 2-cell Δ in $S \setminus \gamma$ bounded by η.

Since g is close to f, the knots γ and η are parallel. [Two disjoint knots are called *parallel* if they bound an embedded annulus $S^1 \times [0, 1] \to S$. It is easy to see that if both knots are smooth and η is a small C^1-perturbation of γ, then the knots are parallel.] Let A be the embedded annulus bounded by $\gamma \cup \eta$. Then the sum $D := \Delta + A$ is a singular 2-cell bounded by γ. We can apply to it the following classical result, proved by Papakyriakopoulos in [26]:

Lemma 39 (Dehn's Lemma). *Let D be a singular 2-cell in S^3 bounded by a knot γ that has an annular neighborhood in D. Then there exists an emebedded 2-cell D' bounded by γ.*[*]

It follows that the knot γ is trivial – contradiction. □

References

[1] Abate, M. *Iteration theory, compactly divergent sequences and commuting holomorphic maps*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **18** (1991), no. 2, 167–191.

[2] Bedford and E. Smillie, J. *Polynomial diffeomorphisms of $\mathbb{C}^2$: currents, equilibrium measure and hyperbolicity*, Invent. Math. **103** (1991), no. 1, 69–99.

[3] Bedford and E. Smillie, J. *Polynomial diffeomorphisms of $\mathbb{C}^2$: II. Stable manifolds and recurrence*, J. Amer. Math. Soc. **4** (1991), no. 4, 657–679.

[4] Bracci, F. and Molino, L. *The dynamics near quasi-parabolic fixed points of holomorphic diffeomorphisms in $\mathbb{C}^2$*, Amer. J. Math. **126** (2004), no. 3, 671–686.

[5] Denjoy, A. *Sur l'itération des fonctions analitiques*, C. R. Acad. Sci. Paris **182** (1926), 255–257.

[*]This is a slight modification of the original statement of Dehn's lemma that refers to a piecewise linear knot and singular disk.

[6] Di Giuseppe, P. *Topological Classification of Holomorphic, Semi-Hyperbolic Germs, in "Quasi-Absence" of Resonances*, available on arXiv:math/0501397.
[7] Dujardin, R. and Lyubich, M. *Stability and bifurcations for polynomial automorphisms of* $\mathbb{C}^2$. Manuscript in preparation.
[8] Eremenko, A. and Levin, G. *On periodic points of a polynomial.*, English translation: Ukrainian Mth. J., **41** (1989), no. 11, 1258–1262.
[9] Fornæss, J. -E. and Sibony, N. *Complex Hénon mappings in* $\mathbb{C}^2$ *and Fatou-Bieberbach domains*, Duke Math. J. **65**, (1992), no. 2, 345–380.
[10] Fornæss, J. E. and Sibony, N. *Complex Dynamics in Higher Dimension. I.*, Asterisque **222** (1991), 201–231.
[11] Fornæss, J. E. and Sibony, N. *Classification of recurrent domains for some holomorphic maps*, Math. Ann. **301** (1995), no. 4, 813–820.
[12] Fornæss, J. -E.. and Sibony, N. *Complex Dynamics in Higher Dimension II*, Modern Methods in complex Analysis, Ann Studies **137** (1995), 135–182.
[13] Friendland, S. and Milnor, J. *Dynamical properties of plane polynomial automorphisms*, Ergodic Theory Dynam. Systems **9** (1989), no. 1, 67–99.
[14] Goldberg, A.A. and Ostrovskii, I.V. Value distribution of meromorphic functions. Translations of Math. Monographs, v. 236, AMS, 2008.
[15] Hakim, M. *Attracting domains for semi-attractive transformations of* $\mathbb{C}^p$, Publ. Math. **38** (1994), 479–499.
[16] Hayman, W.K. and P.B. Kennedy. Subharmonic functions, vol I. Academic Press, 1976.
[17] Hirsch M. W., Pugh C. C. and Shub M. *Invariant Manifolds*, Bull. Amer. Math. Soc. **76** (1970), no. 5, 1015-1019.
[18] Hubbard, J.H. "The Hénon mapping in the complex domain" in *Chaotic Dynamics and Fractals,* ed. by M.F. Barnsely and S.G. Demko. Academic, Orlando, Florida, (1986).
[19] Hubbard, J. H. and Oberste-Vorth, R. W. *Hénon mappings in the complex domain. I. The global topology of dynamical space*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 5–46.
[20] Hubbard, J. H. and Oberste-Vorth, R. W. *Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **464** (1995), 89–132.
[21] Hubbard, J.H., Papadopol, P. *Superattractive fixed points in* $\mathbb{C}^n$, Indiana Univ. Math. J. **43**, (1994), no. 1, 321–365.
[22] Jupiter, D. and Lilov, K. *Invariant Fatou components of automorphisms of* $\mathbb{C}^2$, Far East J. Dyn. Syst. **6** (2003), no. 1, 49–65.
[23] Lyubich, M. and Lyubich, Yu. *Spectral theory for almost periodic representations of semigroups.* Ukrainian Math. J. **36** (1985), 474–478. (Translation from Russian.)
[24] Milnor, J. *Singular points of complex hypersurfaces*, Annals of Mathematics Studies **61**, Princeton University Press, (1968).
[25] Milnor, J. *Dynamics in One Complex Variable*, Annals of Mathematical Studies **160**, Princeton University Press, (1999).
[26] Papakyriakopoulos, C. D. *On Dehn's lemma and the asphericity of knots*, Ann. of Math. **66** (1957), no. 2, 1–26.
[27] Peters, H. and Zeager, C. *Tautness and Fatou components in* $\mathbb{P}^2$, J. Geom. Anal. **22** (2012), no. 4, 934941.
[28] M. Sodin. Lars Ahlfors' thesis. Israel mathematical Conference Proceedings, v. 14 (2000).
[29] Sullivan, D. *Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. **122** (1985) 401–418.
[30] Ueda, T. *Local structure of analytic transformations of two complex variables I*, J. Math Kyoto Univ. **26** (1986), no. 2, 233 - 261.
[31] Ueda, T. *Fatou sets in complex dynamics on projective spaces*, J. Math. Soc. Japan **46** (1994), 545-555.
[32] Ueda, T. *Holomorphic maps on projective spaces and continuations of Fatou maps*, Michigan Math J. **56** (2008), no. 1, 145-153.
[33] Wiman, A. *Sur une extention d'un théoréme de M. Hadamard*, Arkiv för Math., Astr. och Fys. **2** (1905), no. 14, 1–5.
[34] Weickert, B. J. *Nonwandering, nonrecurrent Fatou components in* $\mathbb{P}^2$, Pac. J. Math. **211** (2003), no. 2, 391–397.
[35] Wolff, J. *Sur l'itération des fonctions holomorphes dans une région, et dont les valuers appartiennent à cette région*, C. R. Acad. Sci. Paris **182** (1926), 42–43.

[36] Wolff, J. *Sur l'itération des fonctions bornées*, C. R. Acad. Sci. Paris **182** (1926), 200–201.

Misha Lyubich
Institute for Mathematical Sciences
Stony Brook University
Stony Brook
USA
Han Peters,
Korteweg de Vries Instituut voor Wiskunde
Universiteit van Amsterdam
P.O. Box 94248
1090 GE AMSTERDAM
Netherlands
h.peters@uva.nl