RIGIDITY FOR INFINITELY RENORMALIZABLE AREA-PRESERVING MAPS

DENIS GAIDASHEV, TOMAS JOHNSON, AND MARCO MARTENS

Abstract. Area-preserving maps have been observed to undergo a universal period-doubling cascade, analogous to the famous Feigenbaum-Coullet-Tresser period doubling cascade in one-dimensional dynamics. A renormalization approach has been used by Eckmann, Koch and Wittwer in a computer-assisted proof of existence of a conservative renormalization fixed point. Furthermore, it has been shown that *infinitely renormalizable maps* in a neighborhood of this fixed point admit invariant Cantor sets on which the dynamics is "stable" - the Lyapunov exponents vanish on these sets.

Infinite renormalizability exists in several settings in dynamics, for example, in unimodal maps, dissipative Hénon-like maps, and conservative Hénon-like maps. All of these types of maps have associated invariant Cantor sets. The unimodal Cantor sets are rigid: the restrictions of the dynamics to the Cantor sets for any two maps are $C^{1+\alpha}$-conjugate. Although, strongly dissipative Hénon maps can be seen as perturbations of unimodal maps, surprisingly the rigidity breaks down. The Cantor attractors of Hénon maps with different average Jacobians are not smoothly conjugated. It is conjectured that the average Jacobian determines the rigidity class. This conjecture holds when the Jacobian is identically zero, and in this paper we prove that the conjecture also holds for conservative maps close to the conservative renormalization fixed point.

Rigidity is a consequence of an interplay between the decay of geometry and the convergence rate of renormalization towards the fixed point. Therefore, to demonstrate rigidity, we prove that the upper bound on the spectral radius of the action of the renormalization derivative on infinitely renormalizable maps is sufficiently small.

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Date: 2011-04-12.

Stony Brook IMS Preprint #2012/4
April 2012

Introduction

Following the pioneering discovery of the Feigenbaum-Coullet-Tresser period doubling universality in unimodal maps (Feigenbaum 1978), (Feigenbaum 1979), (Tresser and Coullet 1978), universality — independence of the quantifiers of the geometry of orbits and bifurcation cascades in families of maps of the choice of a particular family — has been demonstrated to be a rather generic phenomenon in dynamics.

Universality problems are typically approached via *renormalization*. In a renormalization setting one introduces a *renormalization* operator on a functional space, and demonstrates that this operator has a *hyperbolic fixed point*. This approach has been very successful in one-dimensional dynamics, and has led to explanation of universality in unimodal maps (Epstein 1986, Epstein 1989, Lyubich 1999, Martens 1999), critical circle maps (de Faria 1992, de Faria 1999, Yampolsky 2002, Yampolsky 2003) and holomorphic maps with a Siegel disk (McMullen 1998, Yampolsky 2007, Gaidashev and Yampolsky 2007). There is, however, at present no complete understanding of universality in conservative systems, other than in the case of the universality for systems "near integrability" (Abad *et al* 2000, Abad *et al* 1998, Koch 2002, Koch 2004, Koch 2008, Gaidashev 2005, Kocić 2005, Khanin *et al* 2007).

Period-doubling renormalization for two-dimensional maps has been extensively studied in (Collet *et al* 1980, de Carvalho *et al* 2005, Lyubich and Martens 2011). Specifically, the authors of (de Carvalho *et al* 2005) have considered strongly dissipative Hénon-like maps of the form

$$F(x,y) = (f(x) - \epsilon(x,y), x), \tag{1}$$

where $f(x)$ is a unimodal map (subject to some regularity conditions), and ϵ is small. Whenever the one-dimensional map f is *renormalizable*, one can define a renormalization of F, following (de Carvalho *et al* 2005), as

$$R_{dCLM}[F] = H^{-1} \circ F \circ F|_U \circ H,$$

where U is an appropriate neighborhood of the critical value $v = (f(0), 0)$, and H is an explicit non-linear change of coordinates. (de Carvalho *et al* 2005) demonstrates that the degenerate map $F_*(x,y) = (f_*(x), x)$, where f_* is the Feigenbaum-Collet-Tresser fixed point of one-dimensional renormalization, is a hyperbolic fixed point of R_{dCLM}. Furthermore, according to (de Carvalho *et al* 2005), for any infinitely-renormalizable map of the form (1), there exists a hierarchical family of "pieces" $\{B^n_\sigma\}$, organized by inclusion in a dyadic tree, such that the set

$$\mathcal{C}_F = \bigcap_n \bigcup_\sigma B^n_\sigma$$

is an attracting Cantor set on which F acts as an *adding machine*. Compared to the Feigenbaum-Collet-Tresser one-dimensional renormalization, the new striking feature of the two dimensional renormalization for highly dissipative maps (1), is that *the restriction of the dynamics to this Cantor set is not rigid*. Indeed, if the average Jacobians of F and G are different, for example, $b_F < b_G$, then the conjugacy $F|_{\mathcal{C}_F} \underset{h}{\approx} G|_{\mathcal{C}_G}$ is not smooth, rather it is at best a Hölder continuous function with a definite upper bound on the Hölder exponent: $\alpha \le \frac{1}{2}\left(1+\frac{\log b_G}{\log b_F}\right) < 1$.

The theory has been also generelized to other combinatorial types in (Hazard 2011), and also to three dimensional dissipative Hénon-like maps in (Nam 2011).

Finally, the authors of (de Carvalho *et al* 2005) show that the geometry of these Cantor sets is rather particular: the Cantor sets have universal bounded geometry in "most" places, however there are places in the Cantor set were the geometry is unbounded. Rigidity and universality as we know from one-dimensional dynamics has a probabilistic nature for strongly dissipative Hénon like maps. See (Lyubich and Martens 2011) for a discussion of probabilistic universality and probabilistic rigidity.

It turns out that the period-doubling renormalization for area-preserving maps is very different from the dissipative case.

A universal period-doubling cascade in families of area-preserving maps was observed by several authors in the early 80's (Derrida and Pomeau 1980, Helleman 1980, Benettin *et al* 1980, Bountis 1981, Collet *et al* 1981, Eckmann *et al* 1982). The existence of a hyperbolic fixed point for the period-doubling renormalization operator

$$R_{EKW}[F] = \Lambda_F^{-1} \circ F \circ F \circ \Lambda_F,$$

where $\Lambda_F(x,u) = (\lambda_F x, \mu_F u)$ is an F-dependent *linear* change of coordinates, has been proved with computer-assistance in (Eckmann *et al* 1984).

We have proved in (Gaidashev and Johnson 2009b) that *infinitely renormalizable* maps in a neighborhood of the fixed point of (Eckmann *et al* 1984) admit a "stable" Cantor set, that is the set on which the maximal Lyapunov exponent is zero. We have also shown in the same publication that the conjugacy of stable dynamics is at least bi-Lipschitz on a submanifold of locally infinitely renormalizable maps of a finite codimension.

In this paper we improve the conclusions of (Gaidashev and Johnson 2009b), and prove the following result.

Theorem A. *The stable dynamics of infinitely renormalizable area-preserving maps on the Cantor set $\mathcal{C}_F$ is rigid.*

Specifically, for any two maps F and G in the local stable manifold of the renormalization operator the conjugacy

$$F|_{\mathcal{C}_F} \underset{h}{\approx} G|_{\mathcal{C}_G}$$

is $C^{1+\alpha}$, i.e. h extends to a neighborhood of $\mathcal{C}_F$ as a differentiable map whose derivative is Hölder continuous of exponent α, with

$$\alpha \geq 0.0129241943359375.$$

At the same time, the numerically measured value of the Hölder constant is larger.

$$\alpha > 0.02770.$$

It has been conjectured that the average Jacobian determines the rigidity class of Hénon-like maps. This conjecture holds for C^3 unimodal maps with a non-degenerate critical point, for which the Jacobian is identically zero. In this paper we prove that the conjecture also holds for conservative maps close to the conservative renormalization fixed point. At the same time, the result of (de Carvalho *et al* 2005) states that rigidity does not hold for strongly dissipative Hénon-like maps with different average Jacobians.

An important ingredient of the proof is a new bound on the spectral radius of the renormalization operator. We demonstrate that the spectral radius of the action of DR_{EKW}, evaluated at the Eckmann-Koch-Wittwer fixed point F_{EKW}, restricted to the stable manifold $\mathcal{W}$ of the infinitely renormalizable maps, is equal exactly to the absolute value of the " horizontal" scaling parameter

$$R_{\rm spec}\left(DR_{EKW}[F_{EKW}]|_{\mathcal{W}}\right) = |\lambda_{F_{EKW}}| = 0.2488\ldots.$$

Furthermore, we demonstrate that the single eigenvalue $\lambda_{F_{EKW}}$ in the spectrum of $DR_{EKW}[F_{EKW}]$ corresponds to an eigenvector, generated by a very specific coordinate change.

We compute the spectral radius of the restriction of the spectrum of $DR_{EKW}[F^*]$ to the stable subspace minus the eigenvalues $\lambda_{F_{EKW}}$, and obtain the following spectral bound, which is of crucial importance to our proof of rigidity.

Theorem B.

$$R_{\rm spec}\left(DR_{EKW}[F^*]|_{\mathcal{W}}\right) \setminus \{\lambda_{F_{EKW}}\} \leq 0.1258544921875.$$

The Cantor set of a renormalization fixed point can be seen as the limit set of the iterated function system generated by two rescalings (see also "presentation function" (Ledrappier and Misiurewicz 1985)). In this context, the pieces of the Cantor set are images of branches of this iterated function system. These branches are compositions of rescalings. The Cantor set of an infinitely renormalizable map is obtained in a similar way. The pieces are also obtained as images of compositions of rescalings. The convergence of renormalization imply that the rescalings in these branches converge exponentially fast to the corresponding rescalings of the renormalization fixed point. In the one-dimensional context the exponential convergence together with the comutativity of derivatives of rescaling is enough to show that the small scale geometry of the Cantor sets is asymptotically the same, that is, to show rigidity.

In the area-preserving case, we still have the exponential convergence. However, derivatives do not commute anymore. This noncommutativity introduces discrepancies between the small scale geometry of the Cantor sets. These discrepancies will disappear on asymptotic scale if there is fast enough convergence of the rescalings, that is, fast enough convergence of renormalizations.

In the dissipative case, the pieces of the Cantor set are also obtained as images of long compositions of rescalings. These rescalings converge exponentially fast to the correponding rescalings of the one-dimensional renormalization fixed point. Although, the the two-dimensional nature of these rescalings decays super exponentially fast, it is still strong enough to let the non commutativity destroy rigidity.

For convenience and readability, the paper is divided in two large logical parts, "Rigidity for Infinitely Renormalizable Maps", and "Spectral Properties of Renormalization". We prove the main result of the paper in the first part; the second part contains a collection of results concerning the renormalization spectrum, out of which the most important is the spectral bounds on the stable and strong stable renormalization manifolds.

Acknowledgment

This work was started during a visit by the authors to the Institut Mittag-Leffler (Djursholm, Sweden) as part of the research program on "Dynamics and PDEs". The hospitality of the institute is gratefully acknowledged. The second author was funded by a postdoctoral fellowship from the Institut Mittag-Leffler, he is currently funded by a postdoctoral fellowship from *Vetenskapsrådet* (the Swedish Research Council). This research is also supported by NSF.

Part 1. Rigidity for Infinitely Renormalizable Maps

1. Renormalization for area-preserving reversible twist maps

An "area-preserving map" will mean an exact symplectic diffeomorphism of a subset of $\mathbb{R}^2$ onto its image.

Recall, that an area-preserving map that satisfies the twist condition

$$\partial_u \left(\pi_x F(x,u)\right) \neq 0$$

everywhere in its domain of definition can be uniquely specified by a generating function S:

$$\left(\begin{array}{c} x \\ -S_1(x,y) \end{array} \right) \overset{F}{\mapsto} \left(\begin{array}{c} y \\ S_2(x,y) \end{array} \right), \quad S_i \equiv \partial_i S. \tag{2}$$

Furthermore, we will assume that F is reversible, that is

$$T \circ F \circ T = F^{-1}, \quad \text{where} \quad T(x,u) = (x,-u). \tag{3}$$

For such maps it follows from (2) that

$$S_1(y,x) = S_2(x,y) \equiv s(x,y),$$

and

$$\left(\begin{array}{c} x \\ -s(y,x) \end{array} \right) \overset{F}{\mapsto} \left(\begin{array}{c} y \\ s(x,y) \end{array} \right). \tag{4}$$

It is this "little" s that will be referred to below as "the generating function". If the equation $-s(y,x)=u$ has a unique differentiable solution $y=y(x,u)$, then the derivative of such a map F is given by the following formula:

$$DF(x,u)=\begin{bmatrix} -\frac{s_2(y(x,u),x)}{s_1(y(x,u),x)} & -\frac{1}{s_1(y(x,u),x)} \\ s_1(x,y(x,u))-s_2(x,y(x,u))\frac{s_2(y(x,u),x)}{s_1(y(x,u),x)} & -\frac{s_2(x,y(x,u))}{s_1(y(x,u),x)} \end{bmatrix}. \tag{5}$$

The period-doubling phenomenon can be illustrated with the area-preserving Hénon family (cf. (Bountis 1981)) :

$$H_a(x,u)=(-u+1-ax^2,x).$$

Maps H_a have a fixed point $((-1+\sqrt{1+a})/a,(-1+\sqrt{1+a})/a)$ which is stable (elliptic) for $-1<a<3$. When $a_1=3$ this fixed point becomes hyperbolic: the eigenvalues of the linearization of the map at the fixed point bifurcate through -1 and become real. At the same time a stable orbit of period two is "born" with $H_a(x_\pm,x_\mp)=(x_\mp,x_\pm)$, $x_\pm=(1\pm\sqrt{a-3})/a$. This orbit, in turn, becomes hyperbolic at $a_2=4$, giving birth to a period 4 stable orbit. Generally, there exists a sequence of parameter values a_k, at which the orbit of period 2^{k-1} turns unstable, while at the same time a stable orbit of period 2^k is born. The parameter values a_k accumulate on some a_∞. The crucial observation is that the accumulation rate

$$\lim_{k\to\infty}\frac{a_k-a_{k-1}}{a_{k+1}-a_k}=8.721... \tag{6}$$

is universal for a large class of families, not necessarily Hénon.

Furthermore, the 2^k periodic orbits scale asymptotically with two scaling parameters

$$\lambda=-0.249\ldots,\quad \mu=0.061\ldots \tag{7}$$

To explain how orbits scale with λ and μ we will follow (Bountis 1981). Consider an interval (a_k,a_{k+1}) of parameter values in a "typical" family F_a. For any value $\alpha\in(a_k,a_{k+1})$ the map F_α possesses a stable periodic orbit of period 2^k. We fix some α_k within the interval (a_k,a_{k+1}) in some consistent way; for instance, by requiring that $DF_{\alpha_k}^{2^k}$ at a point in the stable 2^k-periodic orbit is conjugate, via a diffeomorphism H_k, to a rotation with some fixed rotation number r. Let p'_k be some unstable periodic point in the 2^{k-1}-periodic orbit, and let p_k be the further of the two stable 2^k-periodic points that bifurcated from p'_k. Denote with $d_k=|p'_k-p_k|$, the distance between p_k and p'_k. The new elliptic point p_k is surrounded by (infinitesimal) invariant ellipses; let c_k be the distance between p_k and p'_k in t he direction of the minor semi-axis of an invariant ellipse surrounding p_k, see Figure 1. Then,

$$\frac{1}{\lambda}=-\lim_{k\to\infty}\frac{d_k}{d_{k+1}},\quad \frac{\lambda}{\mu}=-\lim_{k\to\infty}\frac{\rho_k}{\rho_{k+1}},\quad \frac{1}{\lambda^2}=\lim_{k\to\infty}\frac{c_k}{c_{k+1}},$$

where ρ_k is the ratio of the smaller and larger eigenvalues of $DH_k(p_k)$.

This universality can be explained rigorously if one shows that the *renormalization* operator

$$R_{EKW}[F]=\Lambda_F^{-1}\circ F\circ F\circ\Lambda_F, \tag{8}$$

where Λ_F is some F-dependent coordinate transformation, has a fixed point, and the derivative of this operator is hyperbolic at this fixed point.

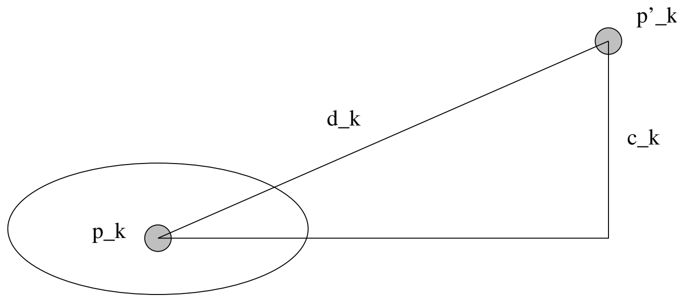

FIGURE 1. The geometry of the period doubling. p_k is the further elliptic point that has bifurcated from the hyperbolic point p'_k.

It has been argued in (Collet *et al* 1981) that Λ_F is a diagonal linear transformation. Furthermore, such Λ_F has been used in (Eckmann *et al* 1982) and (Eckmann *et al* 1984) in a computer assisted proof of existence of a reversible renormalization fixed point F_{EKW} and hyperbolicity of the operator R_{EKW}.

We will now derive an equation for the generating function of the renormalized map $\Lambda_F^{-1} \circ F \circ F \circ \Lambda_F$.

Applying a reversible F twice we get

$$\begin{pmatrix} x' \\ -s(Z,x') \end{pmatrix} \stackrel{F}{\mapsto} \begin{pmatrix} Z \\ s(x',Z) \end{pmatrix} = \begin{pmatrix} Z \\ -s(y',Z) \end{pmatrix} \stackrel{F}{\mapsto} \begin{pmatrix} y' \\ s(Z,y') \end{pmatrix}.$$

According to (Collet *et al* 1981) Λ_F can be chosen to be a linear diagonal transformation:

$$\Lambda_F(x,u) = (\lambda x, \mu u).$$

We, therefore, set $(x',y') = (\lambda x, \lambda y)$, $Z(\lambda x, \lambda y) = z(x,y)$ to obtain:

$$\begin{pmatrix} x \\ -\frac{1}{\mu} s(z,\lambda x) \end{pmatrix} \stackrel{\Lambda_F}{\mapsto} \begin{pmatrix} \lambda x \\ -s(z,\lambda x) \end{pmatrix} \stackrel{F \circ F}{\mapsto} \begin{pmatrix} \lambda y \\ s(z,\lambda y) \end{pmatrix} \stackrel{\Lambda_F^{-1}}{\mapsto} \begin{pmatrix} y \\ \frac{1}{\mu} s(z,\lambda y) \end{pmatrix}, \tag{9}$$

where $z(x,y)$ solves

$$s(\lambda x, z(x,y)) + s(\lambda y, z(x,y)) = 0. \tag{10}$$

If the solution of (10) is unique, then $z(x,y) = z(y,x)$, and it follows from (9) that the generating function of the renormalized F is given by

$$\tilde{s}(x,y) = \mu^{-1} s(z(x,y), \lambda y). \tag{11}$$

One can fix a set of normalization conditions for $\tilde{s}$ and z which serve to determine scalings λ and μ as functions of s. For example, the normalization $s(1,0) = 0$ is reproduced for $\tilde{s}$ as long as $z(1,0) = z(0,1) = 1$. In particular, this implies that

$$s(Z(\lambda,0),0) = 0,$$

which serves as an equation for λ. Furthermore, the condition $\partial_1 s(1,0) = 1$ is reproduced as long as $\mu = \partial_1 z(1,0)$.

We will now summarize the above discussion in the following definition of the renormalization operator acting on generating functions originally due to the authors of (Eckmann *et al* 1982) and (Eckmann *et al* 1984):

Definition 1.1. *Define the prerenormalization of s as*

$$\mathcal{P}_{EKW}[s] = s \circ G[s], \tag{12}$$

where

$$\begin{aligned} 0 &= s(x, Z(x,y)) + s(y, Z(x,y)), && (13)\\ G[s](x,y) &= (Z(x,y), y). && (14) \end{aligned}$$

The renormalization of s will be defined as

$$\mathcal{R}_{EKW}[s] = \frac{1}{\mu} \mathcal{P}_{EKW}[s] \circ \lambda, \tag{15}$$

where

$$\lambda(x,y) = (\lambda x, \lambda y), \quad \mathcal{P}_{EKW}[s](\lambda, 0) = 0 \quad \text{and} \quad \mu = \lambda\ \partial_1 \mathcal{P}_{EKW}[s](\lambda, 0).$$

Definition 1.2. *The Banach space of functions $s(x,y) = \sum_{i,j=0}^{\infty} c_{ij} (x-\beta)^i (y-\beta)^j$, analytic on a bi-disk*

$$\mathcal{D}_\rho(\beta) = \{(x,y) \in \mathbb{C}^2 : |x-\beta| < \rho, |y-\beta| < \rho\},$$

for which the norm

$$\|s\|_\rho = \sum_{i,j=0}^{\infty} |c_{ij}| \rho^{i+j}$$

is finite, will be referred to as $\mathcal{A}^\beta(\rho)$.

$\mathcal{A}_s^\beta(\rho)$ will denote its symmetric subspace $\{s \in \mathcal{A}^\beta(\rho) : s_1(x,y) = s_1(y,x)\}$.

We will use the simplified notation $\mathcal{A}(\rho)$ and $\mathcal{A}_s(\rho)$ for $\mathcal{A}^0(\rho)$ and $\mathcal{A}_s^0(\rho)$, respectively.

As we have already mentioned, the following has been proved with the help of a computer in (Eckmann *et al* 1982) and (Eckmann *et al* 1984):

Theorem 1. *There exist a polynomial $s_{0.5} \in \mathcal{A}_s^{0.5}(\rho)$ and a ball $\mathcal{B}_\varrho(s_{0.5}) \subset \mathcal{A}_s^{0.5}(\rho)$, $\varrho = 6.0 \times 10^{-7}$, $\rho = 1.6$, such that the operator $\mathcal{R}_{EKW}$ is well-defined and analytic on $\mathcal{B}_\varrho(s_{0.5})$.*

Furthermore, its derivative $D\mathcal{R}_{EKW}|_{\mathcal{B}_\varrho(s_{0.5})}$ is a compact linear operator, and has exactly two eigenvalues

$$\delta_1 = 8.721..., \quad \text{and}$$

$$\delta_2 = \frac{1}{\lambda}$$

of modulus larger than 1, while

$$\operatorname{spec}(D\mathcal{R}_{EKW}|_{\mathcal{B}_\varrho(s_{0.5})}) \setminus \{\delta_1, \delta_2\} \subset \{z \in \mathbb{C} : |z| \le \nu\},$$

where

$$\nu < 0.85. \tag{16}$$

Finally, there is an $s^{EKW} \in \mathcal{B}_\varrho(s_{0.5})$ *such that*

$$\mathcal{R}_{EKW}[s^{EKW}] = s^{EKW}.$$

The scalings λ_* *and* μ_* *corresponding to the fixed point* s^{EKW} *satisfy*

$$\lambda_* \in [-0.24887681, -0.24887376], \tag{17}$$

$$\mu_* \in [0.061107811, 0.061112465]. \tag{18}$$

Remark 1.3. *The bound* (16) *is not sharp. In fact, a bound on the largest eigenvalue of* $D\mathcal{R}_{EKW}(s^{EKW})$*, restricted to the tangent space of the stable manifold, is expected to be quite smaller.*

The size of the neighborhood in $\mathcal{A}_s^\beta(\rho)$ where the operator $\mathcal{R}_{EKW}$ is well-defined, analytic and compact has been improved in (Gaidashev 2010). Here, we will cite a somewhat different version of the result of (Gaidashev 2010) which suits the present discussion (in particular, in the Theorem below some parameter, like ρ in $\mathcal{A}_s^\beta(\rho)$, are different from those used in (Gaidashev 2010)). We would like to emphasize that all parameters and bounds used and reported in the Theorem below, and, indeed, throughout the paper, are numbers representable on the computer.

Theorem 2.
There exists a polynomial $s^0 \in \mathcal{A}(\rho)$*,* $\rho = 1.75$*, such that the following holds.*
i) The operator $\mathcal{R}_{EKW}$ *is well-defined and analytic in* $B_R(s^0) \subset \mathcal{A}(\rho)$ *with*

$$R = 0.00426483154296875.$$

ii) For all $s \in B_R(s^0)$ *with real Taylor coefficients, the scalings* $\lambda = \lambda[s]$ *and* $\mu = \mu[s]$ *satisfy*

$$\begin{aligned} 0.0000253506004810333 \leq \quad &\mu \quad \leq 0.121036529541016, \\ -0.27569580078125 \leq \quad &\lambda \quad \leq -0.222587585449219. \end{aligned}$$

iii) The operator $\mathcal{R}_{EKW}$ *is compact in* $B_R(s^0) \subset \mathcal{A}(\rho)$*, with* $\mathcal{R}_{EKW}[s] \in \mathcal{A}(\rho')$*,* $\rho' = 1.0699996948242188\rho$*.*

Definition 1.4. *The set of reversible twist maps* F *of the form (4) with* $s \in \mathcal{B}_\varrho(\tilde{s}) \subset \mathcal{A}_s^\beta(\rho)$ *will be referred to as* $\mathcal{F}_\varrho^{\beta,\rho}(\tilde{s})$*:*

$$\mathcal{F}_\varrho^{\beta,\rho}(\tilde{s}) = \left\{ F : (x, -s(y,x)) \mapsto (y, s(x,y)) | \quad s \in \mathcal{B}_\varrho(\tilde{s}) \subset \mathcal{A}_s^\beta(\rho) \right\}. \tag{19}$$

We will also use the notation

$$\mathcal{F}_\varrho^\rho(\tilde{s}) \equiv \mathcal{F}_\varrho^{0,\rho}(\tilde{s}).$$

We will finish our introduction into period-doubling for area-preserving maps with a summary of properties of the fixed point map. In (Gaidashev and Johnson 2009a) we have described the domain of analyticity of maps in some neighborhood of the fixed point. Additional properties of the domain are studied in (Johnson 2011). Before we state the results of (Gaidashev and Johnson 2009a), we will fix a notation for spaces of functions analytic on a subset of $\mathbb{C}^2$.

Definition 1.5. *Denote $\mathcal{O}_2(\mathcal{D})$ the Banach space of maps $F : \mathcal{D} \mapsto \mathbb{C}^2$, analytic on an open simply connected set $\mathcal{D} \subset \mathbb{C}^2$, continuous on $\partial\mathcal{D}$, equipped with a finite max supremum norm $\|\cdot\|_{\mathcal{D}}$:*

$$\|F\|_{\mathcal{D}} = \max\left\{ \sup_{(x,u)\in\mathcal{D}} |F_1(x,u)|, \sup_{(x,u)\in\mathcal{D}} |F_2(x,u)| \right\}.$$

The Banach space of functions $y : \mathcal{A} \mapsto \mathbb{C}$, analytic on an open simply connected set $\mathcal{A} \subset \mathbb{C}^2$, continuous on $\partial\mathcal{A}$, equipped with a finite supremum norm $\|\cdot\|_{\mathcal{A}}$ will be denoted $\mathcal{O}_1(\mathcal{A})$:

$$\|y\|_{\mathcal{D}} = \sup_{(x,u)\in\mathcal{D}} |y(x,u)|.$$

If $\mathcal{D}$ is a bidisk $\mathcal{D}_\rho \subset \mathbb{C}^2$ for some ρ, then we use the notation

$$\|\cdot\|_\rho \equiv \|\cdot\|_{\mathcal{D}_\rho}.$$

The next Theorem describes the analyticity domains for maps in a neighborhood of the Eckmann-Koch-Wittwer fixed point map, and those for functions in a neighborhood of the Eckmann-Koch-Wittwer fixed point generating function. The Theorem has been proved in two different versions: one for the space $\mathcal{A}_s^{0.5}(1.6)$ (the functional space in the original paper (Eckmann *et al* 1984)), the other for the space $\mathcal{A}_s(1.75)$ — the space in which we will obtain a bound on the renormalization spectral radius in the stable manifold in this paper. To state the Theorem in a compact form, we introduce the following notation:

$$\rho_{0.5} = 1.6, \quad \rho_0 = 1.75,$$

$$\varrho_{0.5} = 6.0 \times 10^{-7}, \quad \varrho_0 = 5.79833984375 \times 10^{-4},$$

while $s_{0.5}$ (as in Theorem 1) and s_0 will denoted the approximate renormalization fixed points in spaces $\mathcal{A}_s^{0.5}(1.6)$ and $\mathcal{A}_s(1.75)$, respectively.

Theorem 3. *There exists a polynomial s_β such that the following holds for all $F \in \mathcal{F}_\varrho^{\beta,\rho}(s_\beta)$, $\beta = 0.5$ or $\beta = 0$.*

i) There exists a simply connected open set $\mathcal{D} = \mathcal{D}(\beta, \varrho_\beta, \rho_\beta) \subset \mathbb{C}^2$ such that the map F is in $\mathcal{O}_2(\mathcal{D})$.

ii) There exist simply connected open sets $\bar{\mathcal{D}} = \bar{\mathcal{D}}(\beta, \varrho_\beta, \rho_\beta) \subset \mathcal{D}$, such that $\bar{\mathcal{D}} \cap \mathbb{R}^2$ is a non-empty simply connected open set, and such that for every $(x,u) \in \bar{\mathcal{D}}$ and $s \in \mathcal{B}_{\varrho_\beta}(s_\beta) \subset \mathcal{A}_s^\beta(\rho_\beta)$, the equation

$$0 = u + s(y, x) \tag{20}$$

has a unique solution $y[s](x,u) \in \mathcal{O}_1(\bar{\mathcal{D}})$. The map

$$\mathcal{S} : s \mapsto y[s]$$

is analytic as a map from $\mathcal{B}_{\varrho_\beta}(s_\beta)$ to $\mathcal{O}_1(\bar{\mathcal{D}})$.

Furthermore, for every $x \in \pi_x\bar{\mathcal{D}}$, there is a function $U \in \mathcal{O}_1(\mathcal{D}_{\rho_\beta}(\beta))$, that satisfies

$$y[s](x, U(x,v)) = v.$$

The map

$$Y : y[s] \mapsto U$$

is analytic as a map from $\mathcal{O}_1(\mathcal{D}_{\rho_\beta}(\beta))$ to $\mathcal{B}_{\varrho_\beta}(s_\beta)$.

Remark 1.6. *It is not too hard to see that the subsets $\mathcal{F}_{\varrho}^{\beta,\rho}(s_\beta)$, $\beta=0$ or 0.5, are Banach submanifolds of the spaces $\mathcal{O}_2(\bar{\mathcal{D}}(\beta,\varrho_\beta,\rho_\beta)$. Indeed, the map*

$$\mathcal{I}: s \mapsto (y[s], s\circ h[s])\,, \tag{21}$$

where $y[s](x,u)$ is the solution of the equation (20), and $h[s](x,u)=(x,y[s](x,u))$, is analytic as a map from $\mathcal{B}_{\varrho_\beta}(s_\beta)$ to $\mathcal{O}_2(\bar{\mathcal{D}}(\beta,\varrho_\beta,\rho_\beta)$ according to Theorem 3, and has an analytic inverse

$$\mathcal{I}^{-1}: F \mapsto \pi_u F\circ g[F], \tag{22}$$

where $g[F](x,y)=(x,U(x,y))$, and U is as in Theorem 3.

We are now ready to give a definition of the Eckmann-Koch-Wittwer renormalization operator for maps of the subset of a plane. Notice, that the condition $\mathcal{P}_{EKW}[s](\lambda,0)=0$ from Definition 1.1 is equivalent to

$$F(F(\lambda,-s(z(\lambda,0),\lambda)))=(0,0),$$

or, using the reversibility

$$\lambda=\pi_x F(F(0,0)).$$

On the other hand,

$$-s(z(y(x,u),x),x)=-\mathcal{P}_{EKW}[s](y(x,u),x)=u,$$

and

$$\begin{aligned}\partial_u\mathcal{P}_{EKW}[s](y(x,u),x) &= \mathcal{P}_{EKW}[s]_1(y(x,u),x)y_2(x,u)\\ &= \mathcal{P}_{EKW}[s]_1(y(x,u),x)\ \pi_x(F\circ F)_2(x,u)=-1,\end{aligned}$$

then

$$\mathcal{P}_{EKW}[s]_1(\lambda,0)\ \pi_x(F\circ F)_2(0,0)=-1,$$

and

$$\mu=\frac{-\lambda}{\pi_x(F\circ F)_2(0,0)}.$$

Definition 1.7. *We will refer to the composition $F\circ F$ as the prerenormalization of F, whenever this composition is defined:*

$$P_{EKW}[F]=F\circ F. \tag{23}$$

Set

$$R_{EKW}[F]=\Lambda^{-1}\circ P_{EKW}[F]\circ\Lambda,$$

where

$$\Lambda(x,u)=(\lambda x,\mu u),\quad \lambda=\pi_x P_{EKW}[F](0,0),\quad \mu=\frac{-\lambda}{\pi_x P_{EKW}[F]_2(0,0)},$$

whenever these operations are defined. $R_{EKW}[F]$ will be called the (EKW-)renormalization of F.

Remark 1.8. *Suppose that for some choice of β, ϱ and ρ, the operator $\mathcal{R}_{EKW}$ and the map $\mathcal{I}$, described in Remark 1.6, are well-defined on some $\mathcal{B}_\varrho(s_\beta)\subset\mathcal{A}_s^\beta(\rho)$. Also, suppose that the inverse of $\mathcal{I}$ exists on $\mathcal{I}(\mathcal{B}_\varrho(s_\beta))$. Then,*

$$R_{EKW}=\mathcal{I}\circ\mathcal{R}_{EKW}\circ\mathcal{I}^{-1}$$

on $\mathcal{F}_{\varrho}^{\beta,\rho}(s_\beta)$.

2. Statement of main results

Consider the dyadic group,

$$\{0,1\}^\infty = \varprojlim \{0,1\}^n, \tag{24}$$

where $\varprojlim$ stands for the inverse limit. An element w of the dyadic group can be represented as a formal power series $w \to \sum_{k=0}^{\infty} w_{k+1} 2^k$. The *odometer*, or the *adding machine*, $p : \{0,1\}^\infty \to \{0,1\}^\infty$ is the operation of adding 1 in this group.

We are now ready to state our main theorems.

Main Theorem 1. *(Existence and Spectral properties) There exists a polynomial* $s_0 : \mathbb{C}^2 \mapsto \mathbb{C}$*, such that*

i) *The operator* $\mathcal{R}_{EKW}$ *is well-defined, analytic and compact in* $\mathcal{B}_{\varrho_0}(s_0) \subset \mathcal{A}_s(\rho)$*, with*

$$\rho = 1.75, \quad \varrho_0 = 5.79833984375 \times 10^{-4}.$$

ii) *There exists a function* $s^* \in \mathcal{B}_r(s_0) \subset \mathcal{A}_s(\rho)$ *with*

$$r = 1.1 \times 10^{-10},$$

such that

$$\mathcal{R}_{EKW}[s^*] = s^*.$$

iii) *The linear operator* $D\mathcal{R}_{EKW}[s^*]$ *has two eigenvalues outside of the unit circle:*

$$8.72021484375 \le \delta_1 \le 8.72216796875, \quad \delta_2 = \frac{1}{\lambda_*},$$

where

$$-0.248875313689 \le \lambda_* \le -0.248886108398438.$$

iv) *The complement of these two eigenvalues in the spectrum is compactly contained in the unit disk. The largest eigenvalue in the unit disk is equal to* λ_**, while*

$$\mathrm{spec}(D\mathcal{R}_{EKW}[s^*]) \setminus \{\delta_1, \delta_2, \lambda_*\} \subset \{z \in \mathbb{C} : |z| \le 0.1258544921875 \equiv \nu\}.$$

The Main Theorem 1 and Theorem 1 imply that there exist codimension 2 local stable manifolds $\mathcal{W}_{\mathcal{R}_{EKW}}(s^*) \subset \mathcal{A}_s(1.75)$ and $\mathcal{W}^{0.5}_{\mathcal{R}_{EKW}}(s^{EKW}) \subset \mathcal{A}^{0.5}_s(1.6)$ of the operator $\mathcal{R}_{EKW}$.

Compactness of the operator $\mathcal{R}_{EKW}$ in neighborhood of s^* implies that there exists a strong "submanifold"

$$\mathcal{W}^s_{\mathcal{R}_{EKW}}(s^*) \subset \mathcal{W}_{\mathcal{R}_{EKW}}(s^*),$$

of codimension 1 in $\mathcal{W}_{\mathcal{R}_{EKW}}(s^*)$, such that the contraction rate in $\mathcal{W}^s_{\mathcal{R}_{EKW}}(s^*)$ is bounded from above by ν:

$$\|\mathcal{R}^n_{EKW}[s] - \mathcal{R}^n_{EKW}[\tilde{s}]\|_\rho = O(\nu^n)$$

for any two s and $\tilde{s}$ in $\mathcal{W}^s_{\mathcal{R}_{EKW}}(s^*)$.

Definition 2.1. *The set of reversible twist maps of the form (4) such that* $s \in \mathcal{W}_{\mathcal{R}_{EKW}}(s^*) \subset \mathcal{A}_s(1.75)$ *will be denoted* W_{EKW}*, and referred to as* **infinitely renormalizable** *maps.*

The set of reversible twist maps of the form (4) such that $s \in \mathcal{W}^s_{\mathcal{R}_{EKW}}(s^*) \subset \mathcal{A}_s(1.75)$ *will be denoted* W^s_{EKW}.

The set of reversible twist maps of the form (4) such that $s \in \mathcal{W}^{0.5}_{\mathcal{R}_{EKW}}(s^{EKW}) \subset \mathcal{A}^{0.5}_s(1.6)$ *will be denoted* $W^{0.5}_{EKW}$.

Recall the Definition 1.4.

Definition 2.2. *Set,*

$$\begin{aligned} W_\varrho(\tilde{s}) &\equiv W_{EKW} \cap \mathcal{F}^{1.75}_\varrho(\tilde{s}), \\ W^s_\varrho(\tilde{s}) &\equiv W^s_{EKW} \cap \mathcal{F}^{1.75}_\varrho(\tilde{s}), \\ W^{0.5}_\varrho(\tilde{s}) &\equiv W^{0.5}_{EKW} \cap \mathcal{F}^{0.5,1.6}_\varrho(\tilde{s}). \end{aligned}$$

Naturally, these sets are invariant under renormalization if ϱ is sufficiently small.

Notice, that, among other things, this Theorem restates the result about existence of the Eckmann-Koch-Wittwer fixed point and renormalization hyperbolicity of Theorem 1 in a setting of a different functional space. We do not prove that the fixed point s^* coincides with s^{EKW} from Theorem 1, although the computer bounds on these two fixed points differ by a tiny amount on any bi-disk contained in the intersection of their domains.

Main Theorem 1 will be proved in Part 2.

Main Theorem 2. *(Stable Set)*

There exists $\varrho > 0$ *such that any* $F \in W_\varrho(s_0)$*, admits a "stable" Cantor set* $\mathcal{C}_F \subset \mathcal{D}$ *with the following properties.*

i) *For all* $x \in \mathcal{C}_F$ *the maximal Lyapunov exponent* $\chi(x; F)$ *exists, is* F*-invariant, is equal to zero:*

$$\chi(x; F) = 0,$$

and

$$\lim_{i \to \pm\infty} \frac{1}{|i|} \log \left\{ \frac{\|DF^i(x)v\|}{\|v\|} \right\} = 0,$$

uniformly for all $v \in \mathbb{R}^2 \setminus \{0\}$ *and* $x \in \mathcal{C}_F$.

ii) *The Hausdorff dimension of* $\mathcal{C}_F$ *satisfies*

$$\dim_H(\mathcal{C}_F) \le 0.794921875.$$

iii) *The restriction of the dynamics* $F|_{\mathcal{C}_F}$ *is topologically conjugate to the adding machine.*

Main Theorem 3. *(Rigidity) Let* s^* *and* $\mathcal{C}_F$ *be as in Main Theorems 1 and 2. There exists* $\varrho > 0$*, such that for all* F *and* $\tilde{F}$ *in* $W_\varrho(s^*)$*,*

$$F|_{\mathcal{C}_F} \underset{h}{\approx} \tilde{F}|_{\mathcal{C}_{\tilde{F}}},$$

where h *extends to a neighborhood of* $\mathcal{C}_F$ *as a differentiable transformation, whose derivative* Dh *is Hölder continuous with the Hölder exponent*

$$\alpha \ge 0.0129241943359375.$$

3. Existence of the invariant stable Cantor sets

Parts *i)*—*iii)* of Main Theorem 2 for $F \in W_{\varrho}^{0.5}(s^{EKW})$ have already been proved in (Gaidashev and Johnson 2009b) with the help of the so called *presentation functions*. We will, however, redo this proof in the setting of the space $\mathcal{A}_s(1.75)$.

Set $\psi_0^F = \Lambda_F$ and $\psi_1^F = F \circ \Lambda_F$, these are the two presentation functions of F. Clearly, for any two $F \in W_r(s_0)$, where r and s_0 are as in Main Theorem 1,

$$D\psi_0^{R^n[F]}(x,u) - D\psi_0^*(x,u) = \left[\begin{array}{cc} C_1|\lambda_*|^n & 0 \\ 0|\lambda_*|^n & C_2|\lambda_*|^n \end{array}\right] = O(|\lambda_*|^n),$$

and, similarly,

$$D\psi_1^{R^n[F]}(x,u) - D\psi_1^*(x,u) = O(|\lambda_*|^n),$$

Furthermore, set

$$\Psi_0^F \equiv \psi_0^F \quad \text{and} \quad \Psi_1^F \equiv \psi_1^F, \tag{25}$$

$$\Psi_\omega^F \equiv \psi_{\omega_1}^F \circ \ldots \circ \psi_{\omega_n}^{R^{n-1}[F]}, \quad \omega = (\omega_1, \ldots, \omega_n) \in \{0,1\}^n. \tag{26}$$

Lemma 3.1. *For every $F \in W_r(s_0)$, $r = 1.1 \times 10^{-10}$, there exists a simply connected closed set $B_F \Subset \bar{\mathcal{D}} \cap \mathbb{R}^2$, where $\bar{\mathcal{D}}$ is as in Theorem 3, such that the following holds.*

1) *$B_0^1(F) \equiv \psi_0^F(B_F) \subset B_F$ and $B_1^1(F) \equiv \psi_1^F(B_F) \subset B_F$ are disjoint, $F(B_1^1(F)) \cap B_0^1 \neq \emptyset$, and*

$$\max\{\|D\psi_0^F\|_{B_F}, \|D\psi_1^F\|_{B_F}\} \leq \theta, \quad \theta = 0.41796875. \tag{27}$$

2) *There exists $\epsilon > 0$ such that*

$$\text{dist}\left\{B_F, \partial\bar{\mathcal{D}}\right\} > \epsilon.$$

Proof. Part 1) First, we verify the following on the computer:

$$\psi_0^F(\tilde{B}) \subset \tilde{B}, \quad \psi_1^F(\tilde{B}) \subset \tilde{B} \quad \text{and} \quad F(\psi_1^F(\tilde{B})) \cap \psi_0^F(\tilde{B}) \neq \emptyset$$

for all $F \in W_r(s_0)$, where

$$\tilde{B} = \{(x,u) \in \mathbb{R}^2 : \frac{(x - 0.469970703125)^2}{0.8199462890625^2} + \frac{(u + 0.0399169921875)^2}{0.301361083984375^2} \leq 1\} \Subset \bar{\mathcal{D}} \cap \mathbb{R}^2.$$

The fact that $\tilde{B} \Subset \bar{\mathcal{D}}$ is proved in Part 2 of this proof. We also check that the sets $\psi_0^F(\tilde{B}) \subset \tilde{B}$ and $\psi_1^F(\tilde{B}) \subset \tilde{B}$ are disjoint.

One can now add another set $\hat{B} \subset \tilde{B}$, so that the set

$$B_F \equiv \psi_0^F(\tilde{B}) \cap \psi_1^F(\tilde{B}) \cap \hat{B}$$

would be simply connected. For example, the ellipse $\hat{B} \subset \tilde{B}$,

$$\hat{B} = \{(x,u) \in \mathbb{R}^2 : \frac{(x - 0.0.469970703125)^2}{0.529998779296875^2} + \frac{u^2}{0.00237011909484863^2} \leq 1\} \Subset \tilde{B},$$

intersects each of $\psi_0^F(\tilde{B})$, $\psi_1^F(\tilde{B})$ along a single arc, and hence, B_F is indeed simply connected (see Figure 3), and satisfies the claim.

Notice, that all numbers used in this Lemma are representable on a computer.

Part 2) A computer bound on $\bar{\mathcal{D}} \cap \mathbb{R}^2$ (see Theorem 3) has been obtained with the help of the *interval Newton operator*. We will now recall the definition of this operator (cf. (Neumaier 1990)).

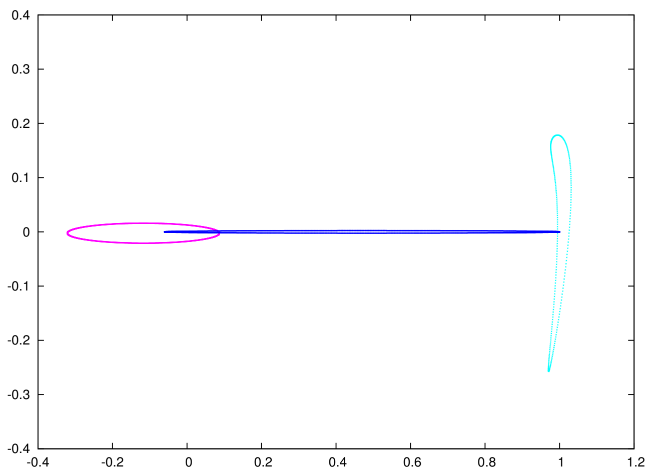

FIGURE 2. Sets $\psi_0^F(\tilde{B})$ (magenta), $\psi_1^F(\tilde{B})$ (cyan) and $\hat{B}$ (blue).

Let $h : \mathcal{A} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and let $D\mathbf{h}$ be an interval matrix valued function such that $[Dh(x)]_{ij} \in [D(\mathbf{h})(x)]_{ij}$, for all $x \in \mathcal{A}$. Let $\mathbf{X} \subset \mathcal{A}$ be a Cartesian product of finite intervals, $\hat{x} \in \mathbf{X}$, and assume that if $A \in D\mathbf{h}(\mathbf{X})$, then A is non-singular. The interval Newton operator is defined as:

$$N(h, \mathbf{X}, \hat{x}) = \hat{x} - (D\mathbf{h})^{-1}(\mathbf{X})\mathbf{h}(\hat{x}).$$

The main properties of N is that if $N(h, \mathbf{X}, \hat{x}) \subset \operatorname{int}\mathbf{X}$, then there exists a unique solution to $h(x) = 0$ in $\mathbf{X}$, which is contained in $N(h, \mathbf{X}, \hat{x})$, and if $N(h, \mathbf{X}, \hat{x}) \cap \mathbf{X} = \emptyset$, then there is no solution to $h(x) = 0$ in $\mathbf{X}$.

To obtain the bound on $\bar{\mathcal{D}} \cap \mathbb{R}^2$, we have verified the containment property for $N(h_{(x,u)}, \mathbf{Y}, \hat{y})$, where

$$\mathbf{Y} = \{y \in \mathbb{R} \, : \, |y| < 1.75\}, \quad \text{and} \quad h_{(x,u)}(y) = u + s(y, x),$$

specifically, we have shown that there exists a non-empty set $\tilde{\mathcal{D}} \subset \mathbb{R}^2$, such that for all $(x, u) \in \tilde{\mathcal{D}}$

$$0 \notin D\mathbf{h}_{x,u}(\mathbf{Y}), \tag{28}$$

and

$$N(h_{(x,u)}, \mathbf{Y}, \hat{y}) \subset \mathbf{Y}.$$

In particular, (28) implies via the Implicit Function Theorem that there exists an open neighborhood $\bar{\mathcal{D}}$ of $\tilde{\mathcal{D}}$ in $\mathbb{C}^2$, such that $h_{x,u}(y) = 0$ has a solution $y(x, u)$ for all $(x, u) \in \bar{\mathcal{D}}$ with y being an analytic function on $\bar{\mathcal{D}}$. We verify that $\tilde{B} \Subset [-0.4, 1.4] \times [-0.6, 0.6] \subset \tilde{D}$. Clearly, the boundary of $\bar{\mathcal{D}}$ is a definite distance away from any set compactly contained in $\bar{\mathcal{D}} \cap \mathbb{R}^2$.

□

Set $B_0^F = \psi_0^F(B_F)$, $B_1^F = \psi_1^F(B_F)$, and define "pieces"

$$B_\omega^F = \Psi_\omega^F(B_F), \quad \omega \in \{0, 1\}^n.$$

One can view $\{0,1\}^n$ as an additive group of residues $mod\, 2^n$ via an identification

$$w \to \sum_{k=0}^{n-1} w_{k+1} 2^k.$$

Let $p : \{0,1\}^n \to \{0,1\}^n$, be the operation of adding 1 in this group. The following Lemma has been proved in (de Carvalho *et al* 2005), and it's proof holds in our case of area-preserving maps word by word:

Lemma 3.2.

1) *The above families of pieces are nested:*
$$B^F_{wv} \subset B^F_\omega, \quad w \in \{0,1\}^{n-1}, \quad v \in \{0,1\}.$$
2) *The pieces* $B^F_\omega, \quad \omega \in \{0,1\}^n$ *are pairwise disjoint.*
3) F *permutes the pieces as follows:* $F(B^F_\omega) = B^F_{p(w)}$ *unless* $p(w) = 0^n$. *If* $p(w) = 0^n$, *then* $F(B^F_\omega) \cap B^F_{0^n} \neq \emptyset$.
4) $\operatorname{diam}(B^F_\omega) \leq \operatorname{const} \theta^n$.
5) $\dim_H(\mathcal{C}_F) \leq -\log(2)/\log(\theta) < 0.794921875$, *where*

$$\mathcal{C}_F \equiv \bigcap_{n=1}^{\infty} \bigcup_{\omega \in \{0,1\}^n} B^n_\omega. \tag{29}$$

We will denote
$$\mathcal{C}_* \equiv \mathcal{C}_{F^*}.$$

Since the set $\tilde{B}$ from Lemma 3.1 contains $(0,0)$, so does each piece $B^F_{0^n}$. It follows from part 3) of Lemma 3.2 that the set $\bigcup_{w \in \{0,1\}^n} B^F_\omega$ contains iterates $F^i((0,0))$ up to order 2^n. Therefore, the Cantor set $\mathcal{C}_F$ is the closure of the orbit of zero.

Recall the definition of the odometer p from Section 2. Lemma 3.2 implies the following:

Corollary 3.3. *The restriction* $F|_{\mathcal{C}_F}$ *is homeomorphic to the odometer* $p : \{0,1\}^\infty \to \{0,1\}^\infty$ *via* $h : \{0,1\}^\infty \to \mathcal{C}_F$ *defined as*

$$h(w) = \bigcap_{n=1}^{\infty} B^F_{w_1 w_2 \ldots w_n}.$$

4. Lyapunov exponents

Recall, the definition of the upper Lyapunov exponent of $(p, v) \in (\mathcal{D} \cap \mathbb{R}^2) \times \mathbb{R}^2$ with respect to F:

$$\chi(p, v; F) \equiv \overline{\lim}_{i \to \infty} \frac{1}{i} \log \left[\| DF^i(p) v \| \right],$$

where $\|\|$ is some norm in $\mathbb{R}^2$. The maximal Lyapunov exponent of $p \in (\mathcal{D} \cap \mathbb{R}^2)$ with respect to F is defined as

$$\chi(p; F) \equiv \sup_{||v||=1} \chi(p, v; F).$$

The following Lemma about the existence of hyperbolic fixed points for maps in a small neighborhood of the renormalization fixed point map F^* is a restatement of a result from (Gaidashev and Johnson 2009a) in the setting of the functional space $\mathcal{A}_s(1.75)$. The proof of the Lemma is computer-assisted (see (Gaidashev and Johnson 2009a)).

Lemma 4.1. *Every map $F \subset \mathcal{F}^\rho_r(s_0)$, with $r = 1.1 \times 10^{-10}$ and $\rho = 1.75$, possesses a hyperbolic fixed point $p^F \in \mathcal{D}$, such that*

1) *$\pi_x p^F \in (0.577606201171875, 0.577629923820496)$, and $\pi_u p^F = 0$, where $\pi_{x,u}$ are projections on the x and u coordinates;*
2) *$DF(p^F)$ has two negative eigenvalues.*

$$\begin{aligned} e^F_+ &\in (-2.0576171875, -2.057373046875), \\ e^F_- &\in (-0.486053466796875, -0.48602294921875), \end{aligned}$$

corresponding to the following two eigenvectors:

$$\mathbf{s}^F = [1.0, -(0.77978515625, 0.779815673828125)], \quad \text{and} \quad \mathbf{u}^F = T(\mathbf{s}^F).$$

This Lemma implies existence of hyperbolic 2^n-th periodic orbits for maps in $W_r(s_0)$. Let $\mathcal{O}_n(F)$ denote such 2^n-th periodic orbit of $F \in W_r(s_0)$, specifically:

$$\mathcal{O}_n(F) = \bigcup_{i=0}^{2^n-1} F^i(\Psi^F_{0^n}(p^{F_n})),$$

where p^{F_n} is the fixed point of $F_n \equiv R^n[F] \in W_r(s_0)$. We will also denote

$$p^F_{0^n} = \Psi^F_{0^n}(p^{F_n}), \quad p^F_\omega = F^{\sum_{i=1}^n \omega_i 2^{i-1}}(p^F_{0^n}).$$

Consider the stable and unstable invariant direction fields on the 2^n-th periodic orbit $\mathcal{O}_n(F)$. At every point p^F_ω, $\omega \in \{0,1\}^n$ of $\mathcal{O}_n(F)$, these directions are given by

$$\mathbf{u}^F_\omega = D\Psi^F_\omega(p^{F_n})\mathbf{u}^{F_n}, \tag{30}$$

$$\mathbf{s}^F_\omega = D\Psi^F_\omega(p^{F_n})\mathbf{s}^{F_n}. \tag{31}$$

The angles between these vectors and the positive real line will be denoted by α^F_ω and β^F_ω.

Lemma 4.2. *The set $\cup_{n=0}^\infty \mathcal{O}_n(F) \cup \mathcal{C}_F$ is in the set of regular points for F, specifically,*

1) *The decomposition*

$$\mathbb{R}^2 = E_-(p^F_\omega) \bigoplus E_+(p^F_\omega) \equiv \text{span}\{\mathbf{s}^F_\omega\} \bigoplus \text{span}\{\mathbf{u}^F_\omega\}$$

is invariant under

$$DF : \mathcal{D} \times \mathbb{R}^2 \mapsto \mathbb{R}^2.$$

The Lyapunov exponents

$$\chi_-(n; F) \equiv -\chi_+(n; F) = \frac{\log |e^{F_n}_-|}{2^n}, \quad x \in \mathcal{O}_n(F),$$

where e^F_- is as in Lemma 4.1, exist, are F-invariant, and

$$\lim_{i \to \infty} \frac{1}{i} \log \left\{ \frac{\|DF^i(x)v\|}{\|v\|} \right\} = \chi_\pm(n; F),$$

uniformly for all $v \in E_\pm(p^F_\omega) \setminus \{0\}$;

2) *The Lyapunov exponent*

$$\chi^\infty = 0, \quad x \in \mathcal{C}_F,$$

exists, is F-invariant, and

$$\lim_{i \to \pm\infty} \frac{1}{|i|} \log \left\{ \frac{\|DF^i(x)v\|}{\|v\|} \right\} = 0,$$

uniformly for all $v \in \mathbb{R}^2 \setminus \{0\}$.

Proof. 1) Let $i = q2^n + k$, $k = 2^{j_1} + 2^{j_2} + \ldots + 2^{j_m} < 2^n$, then

$$\begin{aligned}
DF^i(p^F_{0^n})\mathbf{s}^F_{0^n} &= DF^{k+q2^n}(p^F_{0^n})\mathbf{s}^F_{0^n} = DF^k(F^{q2^n}(p^F_{0^n})) \cdot DF^{q2^n}(p^F_{0^n})\mathbf{s}^F_{0^n} \\
&= DF^k(p^F_{0^n}) \cdot D\left(\Psi^F_{0^n} \circ F^q_n \circ \left(\Psi^F_{0^n}\right)^{-1}\right)(p^F_{0^n})\mathbf{s}^F_{0^n} \\
&= DF^k(p^F_{0^n}) \cdot D\Psi^F_{0^n}\left(F^q_n \circ \left(\Psi^F_{0^n}\right)^{-1}(p^F_{0^n})\right) \\
&\qquad \cdot DF^q_n\left(\left(\Psi^F_{0^n}\right)^{-1}(p^F_{0^n})\right) \cdot D\left(\Psi^F_{0^n}\right)^{-1}(p^F_{0^n})\mathbf{s}^F_{0^n} \\
&= DF^k(p^F_{0^n}) \cdot D\Psi^F_{0^n}(p^{F_n}) \cdot DF^q_n(p^{F_n}) \cdot D\left(\Psi^F_{0^n}\right)^{-1}(p^F_{0^n})\mathbf{s}^F_{0^n} \\
&= DF^k(p^F_{0^n}) \cdot D\Psi^F_{0^n}(p^{F_n}) \cdot DF^q_n(p^{F_n})\mathbf{s}^{F_n} \\
&= DF^k(p^F_{0^n}) \cdot D\Psi^F_{0^n}(p^{F_n}) \cdot (e^{F_n}_-)^q\mathbf{s}^{F_n} \\
&= (e^{F_n}_-)^q DF^k(p^F_{0^n}) \cdot \mathbf{s}^F_{0^n}.
\end{aligned}$$

Denote C_n and c_n - upper and lower bounds on the derivative norm of F on $\mathcal{O}_n(F)$. Then

$$c^k_n |e^{F_n}_-|^q \|\mathbf{s}^F_{0^n}\| \le \|DF^i(p^F_{0^n})\mathbf{s}^F_{0^n}\| \le C^k_n |e^{F_n}_-|^q \|\mathbf{s}^F_{0^n}\|, \tag{32}$$

and,

$$\begin{aligned}
\overline{\lim}_{i\to\infty} \frac{1}{i} \log\left[\frac{\|DF^i(p^F_{0^n})\mathbf{s}^F_{0^n}\|}{\|\mathbf{s}^F_{0^n}\|}\right] &\le \lim_{i\to\infty} \frac{k}{i}\log C_n + \frac{q}{i}\log\{|e^{F_n}_-|\} = \frac{\log\{|e^{F_n}_-|\}}{2^n}, \\
\underline{\lim}_{i\to\infty} \frac{1}{i} \log\left[\frac{\|DF^i(p^F_{0^n})\mathbf{s}^F_{0^n}\|}{\|\mathbf{s}^F_{0^n}\|}\right] &\ge \lim_{i\to\infty} \frac{k}{i}\log c_n + \frac{q}{i}\log\{|e^{F_n}_-|\} = \frac{\log\{|e^{F_n}_-|\}}{2^n},
\end{aligned}$$

therefore, the limit

$$\lim_{i\to\infty} \frac{1}{i} \log\left[\frac{\|DF^i(p^F_{0^n})\mathbf{s}^F_{0^n}\|}{\|\mathbf{s}^F_{0^n}\|}\right]$$

exists, and is equal to

$$\chi_-(n;F) \equiv \frac{\log\{|e^{F_n}_-|\}}{2^n}.$$

A similar computation demonstrates that

$$\lim_{i\to\infty} \frac{1}{i} \log\left[\frac{\|DF^i(p^F_{0^n})\mathbf{u}^F_{0^n}\|}{\|\mathbf{u}^F_{0^n}\|}\right] = \frac{\log\{|e^{F_n}_+|\}}{2^n} = -\chi_-(n;F).$$

2) Clearly, the above implies stability of the accumulation locus of the periodic orbits $\mathcal{O}_n(F)$, that is for any $F \in W_r(s_0)$, and any $x \in \mathcal{C}_F$, $v \in \mathbb{R}^2 \setminus \{0\}$,

$$\chi^\infty = \lim_{i\to\infty} \frac{1}{i} \log\left[\|DF^i(x)v\|\right] = 0.$$

∎

5. Existence of the conjugacy on the stable dynamics for infinitely renormalizable maps

In this Section we will demonstrate the existence of the conjugacy on the Cantor sets for maps on the renormalization stable manifold.

For any $\omega^n = \{\omega_1, \ldots, \omega_n\} \in \{0,1\}^n$, and $F \in W_r(s_0)$, where r and s_0 are as in Main Theorem 1, define $h^F_{\omega^n}$, formally as:

$$h^F_{\omega^n} = \Psi^F_{\omega^n} \circ \left(\Psi^*_{\omega^n}\right)^{-1},$$

and

$$\left(h^F_{\omega^n}\right)^{-1} = \Psi^*_{\omega^n} \circ \left(\Psi^F_{\omega^n}\right)^{-1},$$

Let $\omega \in \{0,1\}^\infty$. We will demonstrate that the map

$$h^F_\omega \equiv \lim_{n \to \infty} h^F_{\omega^n}, \tag{33}$$

and its inverse

$$\left(h^F_\omega\right)^{-1} \equiv \lim_{n \to \infty} \left(h^F_{\omega^n}\right)^{-1}, \tag{34}$$

are well-defined at each point $y^*_\omega \in \mathcal{C}_*$ coded by ω.

Notice that if $r' \le r$ is sufficiently small, then for any $F \in W_{r'}(s_0)$,

$$\tilde{B}^* = \tilde{B}^F \equiv B^F \cap B^*$$

is again, a zeroth generation piece for the Cantor sets $\mathcal{C}_F$ and $\mathcal{C}_*$. Clearly, the map $h^F_{\omega^n}$ is real-analytic on $\tilde{B}^*_{\omega^n}$, and

$$h^F_{\omega^n}\left(\tilde{B}^*_{\omega^n}\right) = \tilde{B}^F_{\omega^n}.$$

Next, given $x \in \tilde{B}^*_{\omega^n}$, denote $x' = h^F_{\omega^n}(x)$, and

$$x_{\omega^n} = \left(\Psi^*_{\omega^n}\right)^{-1}(x), \quad x'_{\omega^k} = \Psi^{F_k}_{\omega_{k+1}} \circ \ldots \circ \Psi^{F_{n-1}}_{\omega_n}(x_{\omega^n}), \quad k < n, \quad x'_{\omega^n} = x_{\omega^n}, \tag{35}$$

and consider

$$h^F_{\omega^n}(x) - h^F_{\omega^n \omega_{n+1}}(x) = \Psi^F_{\omega^n} \circ \left(\Psi^*_{\omega^n}\right)^{-1}(x) - \Psi^F_{\omega^n \omega_{n+1}} \circ \left(\Psi^*_{\omega^n \omega_{n+1}}\right)^{-1}(x).$$

Since

$$\begin{aligned}
\Psi^{F_n}_{\omega_{n+1}}(x_{\omega^n \omega_{n+1}}) &= \Psi^*_{\omega_{n+1}}(x_{\omega^n \omega_{n+1}}) + \left[\Psi^{F_n}_{\omega_{n+1}}(x_{\omega^n \omega_{n+1}}) - \Psi^*_{\omega_{n+1}}(x_{\omega^n \omega_{n+1}})\right] \\
&= x_{\omega^n} + O(|\lambda_*|^n),
\end{aligned}$$

we get

$$h^F_{\omega^n}(x) - h^F_{\omega^n \omega_{n+1}}(x) = \Psi^F_{\omega^n}(x_{\omega^n}) - \Psi^F_{\omega^n}(x_{\omega^n} + O(|\lambda_*|^n)).$$

Notice, that

$$\begin{aligned}
\Psi^{F_{k-1}}_{\omega_k}(x'_{\omega^k} + c) &= \Psi^{F_{k-1}}_{\omega_k}(x'_{\omega^k}) + \left[\Psi^{F_{k-1}}_{\omega_k}(x'_{\omega^k} + c) - \Psi^{F_{k-1}}_{\omega_k}(x'_{\omega^k})\right] \\
&= x'_{\omega^{k-1}} + D\Psi^{F_{k-1}}_{\omega_k}(\tilde{x}'_{\omega^k}) \cdot c \\
&= x'_{\omega^{k-1}} + O(\theta c),
\end{aligned}$$

where $\tilde{x}'_{\omega^k}$ is a point in the interval $\left[x'_{\omega^k} + c, x'_{\omega^k}\right]$. Therefore,

$$\begin{aligned}
\Psi^F_{\omega^n}(x_{\omega^n}+O(|\lambda_*|^n)) &= \Psi^F_{\omega_1}\circ\Psi^{F_1}_{\omega_2}\circ\ldots\circ\Psi^{F_{n-2}}_{\omega_{n-1}}\circ\Psi^{F_{n-1}}_{\omega_n}(x_{\omega^n}+O(|\lambda_*|^n))\\
&= \Psi^F_{\omega_1}\circ\Psi^{F_1}_{\omega_2}\circ\ldots\circ\Psi^{F_{n-1}}_{\omega_n}(x'_{\omega^{n-1}}+O(\theta|\lambda_*|^n))\\
&= \Psi^F_{\omega_1}(x'_{\omega^1}+O(\theta^{n-1}|\lambda_*|^n)\\
&= x'+O(\theta^n|\lambda_*|^n),
\end{aligned}$$

which implies that the limit

$$h^F_\omega(y^*_\omega)=\lim_{n\to\infty}h^F_{\omega^n}(y^*_\omega)$$

exists, and since $h^F_{\omega^n}(y^*_\omega)\in\tilde{B}^F_{\omega^n}$ for all $n\ge 1$, the point $h^F_\omega(y^*_\omega)$ is the point $y^F_\omega\in\mathcal{C}_F$ coded by ω.

A computation similar to the above shows that the map (34) is also well-defined, and

$$h^F_\omega(y^F_\omega)=y^*_\omega.$$

Finally, we show that the map $h^F:\mathcal{C}_*\mapsto\mathcal{C}_F$,

$$h^F(y_\omega)\equiv h^F_\omega(y_\omega),$$

is a conjugacy of the dynamics of F_* with that of F on their Cantor sets. First, notice that for any $\omega^n=\{\omega_1,\omega_2,\ldots,\omega_n\}\in\{0,1\}^n$

$$\Psi^n_\omega=F^{\sum_{i=1}^n\omega_i 2^{i-1}}\circ\Lambda_{F_0}\circ\ldots\circ\Lambda_{F_{n-2}}\circ\Lambda_{F_{n-1}}.$$

Denote

$$\Lambda_{n,F}\equiv\Lambda_{F_0}\circ\ldots\circ\Lambda_{F_{n-2}}\circ\Lambda_{F_{n-1}},$$

then we have for all $\omega^n\in\{0,1\}^n$, different from 1^n,

$$\begin{aligned}
\left(h^F_{p(\omega^n)}\right)^{-1}\circ F\circ h^F_{\omega^n} &= (F^*)^{\sum_{i=1}^n p(\omega)_i 2^{i-1}}\circ\Lambda_{n,*}\circ\Lambda^{-1}_{n,F}\circ F^{-\sum_{i=1}^n p(\omega)_i 2^{i-1}}\circ F\circ\\
&\quad F^{\sum_{i=1}^n\omega_i 2^{i-1}}\circ\Lambda_{n,F}\circ\Lambda^{-1}_{n,*}\circ(F^*)^{-\sum_{i=1}^n\omega_i 2^{i-1}}\\
&= (F^*)^{\sum_{i=1}^n p(\omega)_i 2^{i-1}}\circ\Lambda_{n,*}\circ\Lambda^{-1}_{n,F}\circ F^{-\sum_{i=1}^n p(\omega)_i 2^{i-1}}\circ\\
&\quad F^{1+\sum_{i=1}^n\omega_i 2^{i-1}}\circ\Lambda_{n,F}\circ\Lambda^{-1}_{n,*}\circ(F^*)^{-\sum_{i=1}^n\omega_i 2^{i-1}}\\
&= (F^*)^{\sum_{i=1}^n p(\omega)_i 2^{i-1}}\circ\Lambda_{n,*}\circ\Lambda^{-1}_{n,F}\circ\Lambda_{n,F}\circ\Lambda^{-1}_{n,*}\circ(F^*)^{-\sum_{i=1}^n\omega_i 2^{i-1}}\\
&= F^*.
\end{aligned}$$

This equality holds on B_{ω^n}, $\omega\neq 1^n$. If $\omega^n=1^n$, then $p(\omega^n)=0^n$, and we have on $B_{1^n}\cap(F^*)^{-1}(B_{0^n})$

$$\begin{aligned}
\left(h^F_{p(\omega^n)}\right)^{-1}\circ F\circ h^F_\omega &= \Lambda_{n,*}\circ\Lambda^{-1}_{n,F}\circ F^{2^n}\circ\Lambda_{n,F}\circ\Lambda^{-1}_{n,*}\circ(F^*)^{-2^n}\circ F^*\\
&= \Lambda_{n,*}\circ F_n\circ\Lambda^{-1}_{n,*}\circ(F^*)^{-2^n}\circ F^*\\
&= \Lambda_{n,*}\circ F_n\circ\Lambda^{-1}_{n,*}\circ(F^*)^{-2^n}\circ\Lambda_{n,*}\circ\Lambda^{-1}_{n,*}\circ F^*\\
&= \Lambda_{n,*}\circ F_n\circ(F^*)^{-1}\circ\Lambda^{-1}_{n,*}\circ F^*\\
&= \Lambda_{n,*}\circ F_n\circ(F^*)^{-1}\circ\Lambda^{-1}_{n,*}\circ F^*.
\end{aligned}$$

Notice, that every $B^F_{0^n}$ contains point $\{(0,0)\}$. This, together with the fact that $\mathrm{diam}(B^F_{0^n})=O(\theta^n)$, implies that

$$y^F_{0^\infty}=\cap_n B^F_{0^n}=\{(0,0)\}.$$

We, therefore, have

$$\begin{aligned}\Lambda_{n,*} \circ F_n \circ (F^*)^{-1} \circ \Lambda_{n,*}^{-1} \circ F^*(y^*_{1\infty}) &= \Lambda_{n,*} \circ F_n \circ (F^*)^{-1} \circ \Lambda_{n,*}^{-1}(0,0)\\ &= \Lambda_{n,*} \circ F_n \circ (F^*)^{-1}(0,0)\\ &= O(|\lambda_*|^n O(|\lambda_*|^n).\end{aligned}$$

In the limit $n \to \infty$:

$$\left(h^F_{p(\omega^n)}\right)^{-1} \circ F \circ h^F_\omega(y^*_{1\infty}) \to (0,0) = y^*_{0\infty},$$

and

$$\left(h^F_{p(\omega)}\right)^{-1} \circ F \circ h^F_\omega(y^*_\omega) = F^*(y^*_\omega).$$

6. Existence of the derivative of the conjugacy for maps on the strong stable manifold

We will now demonstrate existence of derivatives of the conjugacy. To be more precise, given $F \in W^s_{r'}(s_0)$ (with $r' \le r$ as in the previous Section), and $\omega \in \{0,1\}^\infty$, we will show that the map

$$Dh^F_\omega \equiv \lim_{n\to\infty} Dh^F_{\omega^n} \tag{36}$$

exists at the point $y^*_\omega \in \mathcal{C}_*$ with coding ω, and that

$$h^F_\omega(y^*_\omega) - h^F_{\hat\omega}(y^*_{\hat\omega}) = Dh^F_{\hat\omega}(y^*_{\hat\omega})(y^*_\omega - y^*_{\hat\omega}) + o(|y^*_\omega - y^*_{\hat\omega}|) \tag{37}$$

(in particular, h is a homeomorphism on $\mathcal{C}_*$).

Given $v \in \mathbb{R}^2$ and $x \in B_{\omega^n\omega_{n+1}}$, consider

$$\begin{aligned}Dh^F_{\omega^n}(x)v \quad &- \quad Dh^F_{\omega^n\omega_{n+1}}(x)v = \Big[D\Psi^F_{\omega^n}(x_{\omega^n}) \cdot D\left(\Psi^*_{\omega^n}\right)^{-1}(x)\\ &- \quad D\Psi^F_{\omega^n\omega_{n+1}}(x_{\omega^n\omega_{n+1}}) \cdot D\left(\Psi^*_{\omega^n\omega_{n+1}}\right)^{-1}(x)\Big]v,\end{aligned}$$

where we have used the notation (35).

Since

$$\begin{aligned}\Psi^{F_n}_{\omega_{n+1}}(x_{\omega^n\omega_{n+1}}) &= \Psi^*_{\omega_{n+1}}(x_{\omega^n\omega_{n+1}}) + \left[\Psi^{F_n}_{\omega_{n+1}}(x_{\omega^n\omega_{n+1}}) - \Psi^*_{\omega_{n+1}}(x_{\omega^n\omega_{n+1}})\right]\\ &= x_{\omega^n} + O(\nu^n),\end{aligned}$$

we get

$$\begin{aligned}Dh^F_{\omega^n}(x)v \quad &- \quad Dh^F_{\omega^n\omega_{n+1}}(x)v = D\Psi^F_{\omega^n}(x_{\omega^n}) \cdot D\left(\Psi^*_{\omega^n}\right)^{-1}(x)v\\ &- \quad D\Psi^F_{\omega^n}(x_{\omega^n} + O(\nu^n)) \cdot D\Psi^{F_n}_{\omega_{n+1}}(x_{\omega^n\omega_{n+1}}) \cdot D\left(\Psi^*_{\omega_{n+1}}\right)^{-1}(x_{\omega^n})\\ &\qquad \cdot D\left(\Psi^*_{\omega^n}\right)^{-1}(x)v.\end{aligned}$$

Let $N \ge 1$ be fixed, and $n = mN$. Notice, that

$$\begin{aligned}\Psi^{F_{k-1}}_{\omega_k}(x'_{\omega^k} + c) &= \Psi^{F_{k-1}}_{\omega_k}(x'_{\omega^k}) + \left[\Psi^{F_{k-1}}_{\omega_k}(x'_{\omega^k} + c) - \Psi^{F_{k-1}}_{\omega_k}(x'_{\omega^k})\right]\\ &= x'_{\omega^{k-1}} + D\Psi^{F_{k-1}}_{\omega_k}(\tilde{x}'_{\omega^k}) \cdot c\\ &= x'_{\omega^{k-1}} + O(\theta c),\end{aligned}$$

where $\tilde{x}'_{\omega^k}$ is a point in the interval $\left[x'_{\omega^k}+c, x'_{\omega^k}\right]$. Therefore,

$$\begin{aligned}
&D\Psi^F_{\omega^n}(x_{\omega^n}+O(\nu^n)) = \\
&= D\Psi^F_{\omega_1}(x'_{\omega_1}+O(\nu^n)\theta^{n-1})\cdot D\Psi^{F_1}_{\omega_2}(x'_{\omega^2}+O(\nu^n)\theta^{n-2}) \\
&\qquad\qquad \cdot\ldots\cdot D\Psi^{F_{n-1}}_{\omega_n}(x_{\omega^n}+O(\nu^n)) \\
&= D\Psi^F_{\omega_1}(x'_{\omega_1})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-1}\right)\cdot D\Psi^{F_1}_{\omega_2}(x'_{\omega^2})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-2}\right) \\
&\qquad\qquad \cdot\ldots\cdot D\Psi^{F_{n-1}}_{\omega_n}(x_{\omega^n})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\right) \\
&= D\Psi^F_{\omega_1}(x'_{\omega_1})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-1}\right)\cdot \\
&\qquad \ldots D\Psi^{F_{N-1}}_{\omega_N}(x'_{\omega^N})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-N}\right)\cdot \\
&\quad D\Psi^{F_N}_{\omega_{N+1}}(x'_{\omega_{N+1}})\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-N-1}\right)\cdot \\
&\qquad \ldots D\Psi^{F_{2N-1}}_{\omega_{2N}}(x'_{\omega^{2N}})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-2N}\right)\cdot \\
&\qquad \ldots \\
&\quad D\Psi^{F_{(m-1)N}}_{\omega_{(m-1)N+1}}(x'_{\omega_{(m-1)N+1}})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-(m-1)N-1}\right)\cdot \\
&\qquad \ldots D\Psi^{F_{mN-1}}_{\omega_{mN}}(x'_{\omega^{mN}})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-mN}\right) \\
&= D\Psi^F_{\omega_1}(x'_{\omega_1})\cdot\ldots\cdot D\Psi^{F_{N-1}}_{\omega_N}(x'_{\omega^N})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-N}\right)\cdot \\
&\quad D\Psi^{F_N}_{\omega_{N+1}}(x'_{\omega^{N+1}})\cdot\ldots\cdot D\Psi^{F_{2N-1}}_{\omega_{2N}}(x'_{\omega^{2N}})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-2N}\right) \\
&\qquad\qquad \ldots \\
&\quad D\Psi^{F_{(m-1)N}}_{\omega_{(m-1)N+1}}(x'_{\omega^{(m-1)N+1}})\cdot\ldots\cdot D\Psi^{F_{mN-1}}_{\omega_{mN}}(x'_{\omega^{mN}})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-mN}\right) \\
&= \prod_{i=1}^{m} D\Psi^{F_{Ni-N}}_{\omega^{Ni-N+1}}(x'_{\omega^{Ni-N+1}})\cdot\ldots\cdot D\Psi^{F_{Ni-1}}_{\omega^{Ni}}(x'_{\omega^{Ni}})\cdot\left(\mathbb{I}+\mathbb{O}(\nu^n)\theta^{n-iN}\right). \qquad (38)
\end{aligned}$$

We estimate the norm of the above expression using the following Lemma

Lemma 6.1. *For any $\omega^4\in\{0,1\}^\infty$, and all $F\in W_r(s_0)$, where r and s_0 are as in the Main Theorem 1,*

$$\|D\Psi^F_{\omega^4}\|_{B_0} \le a_0 = 0.00383651256561279, \qquad (39)$$

$$\|D\Psi^F_{\omega^4}\|_{B_1} \le a_1 = 0.00383651256561279, \qquad (40)$$

$$\min_{x\in B_0,\ \|v\|=1}\|D\Psi^F_{\omega^4}(x)v\| \ge b_0 = 0.000013950102593, \qquad (41)$$

$$\min_{x\in B_1,\ \|v\|=1}\|D\Psi^F_{\omega^4}(x)v\| \ge b_1 = 0.000013958149969. \qquad (42)$$

Proof. The proof is done by brute force computer aided estimates of the above norms. □

We will denote

$$\kappa_4 = \max\left\{\frac{a_0}{b_0},\frac{a_1}{b_1}\right\}.$$

More, generally, similar maxima for a general N, not necessarily equal to 4, will be denoted κ_N.

Notice, that according to Lemma 6.1,

$$\kappa_4^{\frac{1}{4}} \le 4.072601318359375.$$

We will now fix $N = 4$ in (38), and permute the factors

$$\left(\mathbb{I} + \mathbb{O}(\nu^n)\theta^{n-iN}\right) \tag{43}$$

in (38) to the end of the expression. Notice, that every time (43) exchanges places with the matrix $D\Psi^{F_{Ni-N}}_{\omega^{Ni-N+1}}(x'_{\omega^{Ni-N+1}}) \cdot \ldots \cdot D\Psi^{F_{Ni-1}}_{\omega^{Ni}}(x'_{\omega^{Ni}})$, which is of the form $D\Psi^F_{\omega^4}$ as in the Lemma above, the term proportional to $\mathbb{O}(\nu^n)$ in (43) at most acquires a multiplicative factor of κ_4:

$$\begin{aligned} D\Psi^F_{\omega^n}(x_{\omega^n} + O(\nu^n)) &= \prod_{i=1}^{m} D\Psi^{F_{Ni-N}}_{\omega^{Ni-N+1}}(x'_{\omega^{Ni-N+1}}) \cdot \ldots \cdot D\Psi^{F_{Ni-1}}_{\omega^{Ni}}(x'_{\omega^{Ni}}) \cdot \\ & \prod_{i=1}^{m} \left(\mathbb{I} + \kappa_N^{m-i}\theta^{n-iN}\mathbb{O}(\nu^n)\right). \end{aligned}$$

The difference of

$$\mathcal{N}_n \equiv \prod_{i=1}^{m} \left(\mathbb{I} + \kappa_N^{m-i}\theta^{n-iN}\mathbb{O}(\nu^n)\right),$$

from identity satisfies

$$\begin{aligned} \|\mathcal{N}_n - \mathbb{I}\| &\le \exp\left[\sum_{i=1}^{m} \log\left(1 + C\left(\kappa_N\theta^N\right)^{m-i}\nu^n\right)\right] - 1 \\ &\le \exp\left[C'\sum_{i=1}^{m}\left(\kappa_N\theta^N\right)^{m-i}\nu^n\right] - 1 \\ &\le \exp\left[C'\nu^n\theta^n\kappa_N^m\sum_{i=1}^{m}\left(\kappa_N\theta^N\right)^{-i}\right] - 1 \\ &\le \exp\left[C'\nu^n\theta^n\kappa_N^m\frac{\kappa_N\theta^N}{\kappa_N\theta^N - 1}\right] - 1 \\ &\le C''\left(\nu^N\theta^N\kappa_N\right)^m, \end{aligned}$$

where we have used that $\kappa_N\theta^N > 1$ and $\kappa_N\theta^N\nu^N < 1$. Therefore,

$$\begin{aligned} \|Dh^F_{\omega^n\omega_{n+1}}(x) &- Dh^F_{\omega^n}(x)\| = \|D\Psi^F_{\omega^n}(x'_{\omega^n}) \cdot \\ &\cdot \left[\mathcal{N}_n \cdot D\Psi^{F_{n+1}}_{\omega_{n+1}}(x_{\omega^{n+1}}) \cdot D\left(\Psi^*_{\omega_{n+1}}\right)^{-1}(x_{\omega^n}) - \mathbb{I}\right] \\ &\cdot D\left(\Psi^*_{\omega^n}\right)^{-1}(x)\| \\ &= \|D\Psi^F_{\omega^n}(x'_{\omega^n}) \cdot [\mathcal{N}_n \cdot (\mathbb{I} + \mathbb{O}(\nu^n)) - \mathbb{I}] \cdot D\left(\Psi^*_{\omega^n}\right)^{-1}(x)\| \\ &= \|D\Psi^F_{\omega^n}(x'_{\omega^n}) \cdot [\mathcal{N}_n - \mathbb{I} + \mathcal{N}_n \cdot \mathbb{O}(\nu^n)] \cdot D\left(\Psi^*_{\omega^n}\right)^{-1}(x)\| \\ &= \|D\Psi^F_{\omega^n}(x'_{\omega^n}) \cdot [\mathbb{O}(\nu^n\theta^n\kappa_N^m) + \mathbb{O}(\nu^n)] \cdot D\left(\Psi^*_{\omega^n}\right)^{-1}(x)\| \\ &\le C\kappa_N^{2m}\nu^n\theta^n \\ &= C\left(\kappa_N^{\frac{2}{N}}\nu\,\theta\right)^n. \end{aligned} \tag{44}$$

and since

$$\gamma \equiv \kappa_4^{\frac{2}{4}}\ \nu\,\theta < 0.86541748046875, \tag{45}$$

clearly, the limit (36) exists.

We will now demonstrate (37). We have for any two y^*_ω and $y^*_{\hat\omega}$ in $\mathcal{C}_*$, such that ω and $\hat\omega$ coincide in the first n positions, that is $\omega^n = \hat\omega^n$:

$$\begin{aligned} h^F_\omega(y^*_\omega) - h^F_{\hat\omega}(y^*_{\hat\omega}) &= h^F_{\omega^n}(y^*_\omega) - h^F_{\omega^n}(y^*_{\hat\omega}) + \left(h^F_\omega(y^*_\omega) - h^F_{\omega^n}(y^*_\omega)\right) + \left(h^F_{\omega^n}(y^*_{\hat\omega}) - h^F_{\hat\omega}(y^*_{\hat\omega})\right) \\ &= Dh^F_{\omega^n}(y^*_{\hat\omega})\,(y^*_\omega - y^*_{\hat\omega}) + o(y^*_\omega - y^*_{\hat\omega}) + O(\theta^n\nu^n) \\ &= \left(Dh^F_{\hat\omega}(y^*_{\hat\omega}) + O(\gamma^n)\right)(y^*_\omega - y^*_{\hat\omega}) + o(y^*_\omega - y^*_{\hat\omega}) + O(\theta^n\nu^n). \end{aligned}$$

Therefore,

$$\begin{aligned} h^F_\omega(y^*_\omega) - h^F_{\hat\omega}(y^*_{\hat\omega}) - Dh^F_{\hat\omega}(y^*_{\hat\omega})\,(y^*_\omega - y^*_{\hat\omega}) &= O(\gamma^n)\,(y^*_\omega - y^*_{\hat\omega}) + o(y^*_\omega - y^*_{\hat\omega}) + O(\theta^n\nu^n) \\ &\equiv I. \end{aligned}$$

As $|y^*_\omega - y^*_{\hat\omega}| \to 0$ (necessarily, $n \to \infty$), the right hand side I of (46) satisfies

$$\frac{|I|}{|y^*_\omega - y^*_{\hat\omega}|} \le \left| O(\gamma^n) + O(y^*_\omega - y^*_{\hat\omega}) + O\left(\frac{\theta^n\nu^n}{b^{\frac{n}{4}}}\right) \right|.$$

where

$$b \equiv \min\{b_0, b_1\}, \tag{46}$$

with b_i as in Lemma 6.1.

Since γ, $\theta\nu$ and $\theta^n\nu^n b^{-\frac{n}{4}} < 0.88$ are all less than 1, we have that

$$\frac{|I|}{|y^*_\omega - y^*_{\hat\omega}|} \to 0$$

as $y^*_\omega \to y^*_{\hat\omega}$, and (37) is verified.

7. Hölder property of the derivative on W^s_{EKW}

Proposition 7.1. *Let $\omega, \omega' \in \{0,1\}^\infty$, $\omega \ne \omega'$, and let x^*_ω and $x^*_{\omega'}$ be two points in the Cantor set $\mathcal{C}_*$ whose codings are ω and ω'. Then,*

$$\|Dh^F_\omega(x^*_\omega) - Dh^F_{\omega'}(x^*_{\omega'})\| \le C|x^*_\omega - x^*_{\omega'}|^\alpha, \tag{47}$$

where C is some constant independent of ω and ω',

$$\alpha = \min\left\{\frac{N\log\gamma}{L\log b}, \frac{N\log\theta}{\log b} - \frac{1}{L}\right\} \ge 0.0129241943359375, \tag{48}$$

γ is as in (45), θ is as in (27), $N = 4$,

$$L = \left[\frac{1}{N}\frac{\log b}{\log\theta}\right] + 1 > 1, \tag{49}$$

and b is as in (46).

Proof. Suppose that ω and ω' coincide in the first n positions, and differ in $n+1$-st, for some $n > 2L$ (if $n \le 2L$ then $|x^*_\omega - x^*_{\omega'}|$ is bounded from below by a constant depending on L only, and (47) can be satisfied by a choice of C). Set

$$k = \left[\frac{n}{L}\right] - 1 \ge 1.$$

We have that such x^*_ω and $x^*_{\omega'}$ both lie in the piece $B_{\omega^n} \subset B_{\omega^k}$, and therefore, on one hand,

$$c_1 b^{\frac{n}{N}} \le |x^*_\omega - x^*_{\omega'}| \le c_2\theta^n, \tag{50}$$

and, on the other hand,

$$\begin{aligned}\|Dh^F_\omega(x^*_\omega)-Dh^F_{\omega'}(x^*_{\omega'})\| &\le \|Dh^F_\omega(x^*_\omega)-Dh^F_{\omega^k}(x^*_\omega)\|+\|Dh^F_\omega(x^*_{\omega'})-Dh^F_{\omega^k}(x^*_{\omega'})\| \\ &\quad +\|Dh^F_{\omega^k}(x^*_\omega)-Dh^F_{\omega^k}(x^*_{\omega'})\| \\ &\le C\gamma^k+\|Dh^F_{\omega^k}(x^*_\omega)-Dh^F_{\omega^k}(x^*_{\omega'})\|.\end{aligned}$$

To estimate $\|Dh^F_{\omega^k}(x^*_\omega)-Dh^F_{\omega^k}(x^*_{\omega'})\|$, we first recall that, according to Lemma 3.1, the zero generation piece B is compactly contained in the domain $\bar{\mathcal{D}}$ of definition of F^*, while B_0 and B_1 are compactly contained in B. Therefore, the piece B can be made smaller: there exist $\delta>0$, a compact set $B'\Subset B$ and a complex δ-neighborhood B^δ of B, $B\Subset B^\delta\Subset\bar{\mathcal{D}}\subset\mathbb{C}^2$, such that

$$\operatorname{dist}\left\{B',\partial B^\delta\right\}>\delta,$$

and such that B' is a zero-generation piece for the hierarchy of covers of the Cantor set $\mathcal{C}_F$, and $B'_{\omega^k}\Subset B_{\omega^k}$.

Recall, that $h^F_{\omega^k}$ is real-analytic on the piece B_{ω^k}, and therefore, on B'_{ω^k}. We have

$$\|Dh^F_{\omega^k}(x^*_\omega)-Dh^F_{\omega^k}(x^*_{\omega'})\|\le D_k|x^*_\omega-x^*_{\omega'}|,$$

where D_k is a bound on the second derivative of $h^F_{\omega^k}$ on the piece B'_{ω^k}:

$$D^2h^F_{\omega^k}:B'_{\omega^k}\times\mathbb{R}^2\times\mathbb{R}^2\mapsto\mathbb{R}^2.$$

Now, notice that $b^{\frac{k}{N}}$ is a lower bound on the contraction rate of distances by $D\Psi^F_{\omega^k}$ on B. We denote $b_\delta^{\frac{k}{N}}$,

$$b_\delta^{\frac{k}{N}}\underset{\delta\to 0}{\longrightarrow}b^{\frac{k}{N}},$$

the contraction rate of distances by $D\Psi^F_{\omega^k}$ on B^δ. Then

$$\operatorname{dist}\left(\partial B^\delta_{\omega^k},B'_{\omega^k}\right)\ge\delta b_\delta^{\frac{k}{N}}.$$

According to the previous Section, $\|Dh^F_{\omega^k}\|_{B^\delta_{\omega^k}}$ is bounded. We can now use Cauchy estimates to obtain that the operator norm of the second derivative satisfies

$$\|D^2h^F_{\omega^k}\|_{B'_{\omega^k}}\le\frac{C}{\delta b_\delta^{\frac{k}{N}}}.$$

We get

$$\|Dh^F_\omega(x^*_\omega)-Dh^F_{\omega'}(x^*_{\omega'})\|\le C_1\gamma^k+\frac{C_2}{\delta b_\delta^{\frac{k}{N}}}|x^*_\omega-x^*_{\omega'}|.$$

Notice, that by our choice of n and k, if δ is sufficiently small, then

$$\frac{1}{L}\ge\frac{k\log b_\delta}{n\log b},$$

and

$$\alpha\le\frac{N\log\theta}{\log b}-\frac{1}{L}\le\frac{N\log\theta}{\log b}-\frac{k\log b_\delta}{n\log b},$$

which implies that

$$b^{\alpha\frac{n}{N}}\ge\frac{\theta^n}{b_\delta^{\frac{k}{N}}}. \tag{51}$$

We can now use (51), (50) and the definition (48) of α to finish the demonstration of the claim of the Proposition:

$$\begin{aligned}\|Dh^F_\omega(x^*_\omega) - Dh^F_{\omega'}(x^*_{\omega'})\| &\le C_1\gamma^k + \frac{C_2}{\delta b_\delta^{\frac{k}{N}}}|x^*_\omega - x^*_{\omega'}| \\ &\le C_1 b^{\alpha\frac{n}{N}} + \frac{C_2}{\delta b_\delta^{\frac{k}{N}}}\theta^n \\ &\le C_1 b^{\alpha\frac{n}{N}} + \frac{C_2}{\delta} b^{\alpha\frac{n}{N}} \\ &\le C|x^*_\omega - x^*_{\omega'}|^\alpha.\end{aligned}$$

□

8. Rigidity of infinitely renormalizable maps

Existence of the well-defined maps h and Dh on the Cantor set $\mathcal{C}_*$, together with the condition (37), implies via the Whitney Extension Theorem, that for all $F \in W_{r'}(s_0)$ the map h extends to a C^1 map on a neighborhood $\mathcal{O}$ of $\mathcal{C}_*$. Such h is analytic on $\mathcal{O} \setminus \mathcal{C}_*$, and according to the previous Section, Dh is Hölder of exponent α on all of $\mathcal{O}$.

Part 2. Spectral Properties of Renormalization

1. Coordinate changes and renormalization eigenvalues

Let $\mathcal{D}$ and $\bar{\mathcal{D}}$ be as in the Theorem 3. Consider the action of the operator

$$R_0[F] = \Lambda_*^{-1} \circ F \circ F \circ \Lambda_* \tag{52}$$

on $\mathcal{O}_2(\mathcal{D})$, where

$$\Lambda_*(x,u) = (\lambda_* x, \mu_* u),$$

with λ_* and μ_* being the fixed scaling parameters corresponding to the Collet-Eckmann-Koch as in Theorem 1.

According to Theorem 1 this operator is analytic and compact on the subset $\mathcal{F}_\varrho^{0.5,1.6}(s_{0.5})$, $\varrho = 6.0 \times 10^{-7}$, of $\mathcal{O}_2(\mathcal{D})$, and has a fixed point F_{EKW}. In this paper, we will prove the existence of a fixed point s^* of the operator $\mathcal{R}_{EKW}$ in a Banach space different from that in Theorem 1. Therefore, we will state most of our results concerning the spectra of renormalization operators for general spaces $\mathcal{A}_s^\beta(\rho)$ and sets $\mathcal{F}_\varrho^{\beta,\rho}(s^*)$, under the hypotheses of existence of a fixed point s^*, and analyticity and compactness of the operators in some neighborhood of the fixed point. Later, a specific choice of parameters β, ρ and ϱ will be made, and the hypotheses - verified.

Let $S = id + \sigma$ be a coordinate transformation of the domain $\mathcal{D}$ of maps F, satisfying

$$DS \circ F = DS.$$

In particular, these transformations preserve the subset of area-preserving maps.

Notice, that

$$\begin{aligned}(id+\epsilon\sigma)^{-1}\circ F\circ(id+\epsilon\sigma) &= F+\epsilon\left(-\sigma\circ F+DF\cdot\sigma\right)+O(\epsilon^2)\\ &\equiv F+\epsilon h_{F,\sigma}+O(\epsilon^2).\end{aligned}$$

Suppose that the operator R_0 has a fixed point F^* in some neighborhood $\mathcal{B}\subset\mathcal{O}_2(\mathcal{D})$, on which R_0 is analytic and compact. Consider the action $DR_0[F]h_{F,\sigma}$ of the derivative of this operator.

$$\begin{aligned}DR_0[F]h_{F,\sigma} &= \partial_\epsilon\left(\Lambda_*^{-1}\circ(F+\epsilon h_\sigma)\circ(F+\epsilon h_\sigma)\circ\Lambda_*\right)|_{\epsilon=0}\\ &= \partial_\epsilon\left(\Lambda_*^{-1}\circ(id+\epsilon\sigma)^{-1}\circ F\circ F\circ(id+\epsilon\sigma\circ\Lambda_*\right)|_{\epsilon=0}\\ &= \Lambda_*^{-1}\cdot[-\sigma\circ F\circ F+D(F\circ F)\cdot\sigma]\circ\Lambda_*\\ &= \Lambda_*^{-1}\cdot h_{F\circ F,\sigma}\circ\Lambda_*.\end{aligned}\tag{53}$$

Specifically, if $F=F^*$, one gets

$$DR_0[F^*]h_{F^*,\sigma}=h_{F^*,\tau},\quad \tau=\Lambda_*^{-1}\cdot\sigma\circ\Lambda_*,$$

and clearly, $h_{F^*,\sigma}$ is an eigenvector, if $\tau=\kappa\sigma$, of eigenvalue κ. In particular,

$$\kappa=\lambda_*^i\mu_*^j,\quad i\geq 0,\ j\geq 0$$

is an eigenvalue of multiplicity (at least) 2 with eigenvectors $h_{F^*,\sigma}$ generated by

$$\sigma^1_{i,j}(x,u)=(x^{i+1}u^j,0),\quad \sigma^2_{i,j}(x,u)=(0,x^iu^{j+1}),\tag{54}$$

while

$$\kappa=\mu_*^j\lambda_*^{-1}, j\geq 0,\quad\text{and}\quad \kappa=\lambda_*^i\mu_*^{-1}, i\geq 0,$$

are each eigenvalues of multiplicity (at least) 1, generated by

$$\sigma^1_{-1,j}(x,u)=(u^j,0),\quad\text{and}\quad \sigma^2_{i,-1}(x,u)=(0,x^i),\tag{55}$$

respectively.

Next, suppose S^σ_t, $S^\sigma_0=Id$, is a transformation of coordinates generated by a function σ as in (54)-(55), associated with an eigenvalue κ of $DR_0[F^*]$. In addition to the operator (52), consider

$$R_\sigma[F]=\Lambda_*^{-1}\circ\left(S^\sigma_{t_\sigma[F]}\right)^{-1}\circ F\circ F\circ S^\sigma_{t_\sigma[F]}\circ\Lambda_*.\tag{56}$$

where the parameter $t_\sigma[F]$ is chosen as

$$t_\sigma[F]=-\frac{1}{\kappa\|h_{F^*,\sigma}\|_{\mathcal{D}}}\|E(\kappa)(R_0[F]-F^*)\|_{\mathcal{D}},\tag{57}$$

$E(\kappa)$ being the Riesz spectral projection associated with κ:

$$E(\kappa)=\frac{1}{2\pi i}\int_\gamma(z-DR_0[F^*])^{-1}dz$$

(γ - a Jordan contour that enclose only κ in the spectrum of $DR_0[F^*]$).

We will now compare the spectra of the operators R_0 and R_σ. The result below should be interpreted as follows: if $h_{F^*,\sigma}$ is an eigenvector of $DR_0[F^*]$ generated by a coordinate change $id+\epsilon\sigma$, and associated with some eigenvalue κ, then this eigenvalue is eliminated from the spectrum of $DR_\sigma[F^*]$, *if its multiplicity is* 1.

Lemma 1.1. *Suppose, there exists a map F^* in some $\mathcal{O}_2(\mathcal{D})$, and a neighborhood $\mathcal{B}(F^*) \subset \mathcal{O}_2(\mathcal{D})$, such that the operators R_0 and R_σ are analytic and compact as maps from $\mathcal{B}(F^*)$ to $\mathcal{O}_2(\mathcal{D})$, and $R_0[F^*] = R_\sigma[F^*] = F^*$.*

Then,

$$\mathrm{spec}(DR_0[F^*]) = \mathrm{spec}(DR_\sigma[F^*]) \cup \{\kappa\}.$$

Moreover, if the multiplicity of κ is 1, then

$$\mathrm{spec}(DR_0[F^*]) \setminus \mathrm{spec}(DR_\sigma[F^*]) = \{\kappa\}.$$

Proof. Since $DR_\sigma[F^*]$ and $DR_0[F^*]$ are both compact operators acting on an infinite-dimensional space, their spectra contain $\{0\}$.

Suppose h is a eigenvector of $DR_0[F^*]$ corresponding to some eigenvalue δ, then

$$\begin{aligned}
DR_\sigma[F^*]h &= DR_0[F^*]h \\
&+ \Lambda_*^{-1} \cdot \left(D_F \left(S^\sigma_{t_\sigma[F^*]} \right)^{-1} h \right) \circ F^* \circ F^* \circ S^\sigma_{t_\sigma[F^*]} \circ \Lambda_* \\
&+ \Lambda_*^{-1} \cdot \left[D \left(\left(S^\sigma_{t_\sigma[F^*]} \right)^{-1} \circ F^* \circ F^* \right) \circ S^\sigma_{t_\sigma[F^*]} \cdot \right. \\
&\qquad \left. \cdot \left(D_F S^\sigma_{t_\sigma[F^*]} h \right) \right] \circ \Lambda_* \\
&= \delta h + \Lambda_*^{-1} \cdot \left(D_F \left(S^\sigma_{t_\sigma[F^*]} \right)^{-1} h \right) \circ \Lambda_* \circ F^* \\
&\quad + \left[DF^* \cdot \Lambda_*^{-1} \cdot \left(D_F S^\sigma_{t_\sigma[F^*]} h \right) \right] \circ \Lambda_*
\end{aligned} \tag{58}$$

(we have used the fact that F^* satisfies the fixed point equation), where

$$t_\sigma[F^*] \equiv 0 \quad \text{and} \quad D_F S^\sigma_{t_\sigma[F^*]} h \equiv \partial_\epsilon \left[S^\sigma_{t_\sigma[F^* + \epsilon h]} \right]_{\epsilon=0} = (D_F t_\sigma[F^*] h)\, \sigma.$$

More specifically,

$$\begin{aligned}
t_\sigma[F^* + \epsilon h] &= -\kappa^{-1} \|h_{F^*,\sigma}\|_{\mathcal{D}}^{-1} \|E(\kappa) \left(R_0(F^* + \epsilon h) - F^* \right) \|_{\mathcal{D}} \\
&= -\epsilon \kappa^{-1} \|h_{F^*,\sigma}\|_{\mathcal{D}}^{-1} \|E(\kappa) \left(DR_0[F^*] h \right) \|_{\mathcal{D}} + O(\epsilon^2) \\
&= -\epsilon \|h_{F^*,\sigma}\|_{\mathcal{D}}^{-1} \kappa^{-1} \delta \| \left(E(\kappa) h \right) \|_{\mathcal{D}} + O(\epsilon^2), \\
&= -\epsilon \|h_{F^*,\sigma}\|_{\mathcal{D}}^{-1} \kappa^{-1} \delta \| \left(E(\kappa) \left(E(\delta) h \right) \right) \|_{\mathcal{D}} + O(\epsilon^2),
\end{aligned}$$

and

$$D_F t_\sigma[F^*] h = \partial_\epsilon \left[t_\sigma[F^* + \epsilon h] \right]_{\epsilon=0} = -\|h_{F^*,\sigma}\|_{\mathcal{D}}^{-1} \kappa^{-1} \delta \| \left(E(\kappa) \left(E(\delta) h \right) \right) \|_{\mathcal{D}}. \tag{59}$$

If $\delta = \kappa$ and $h = h_{F^*,\sigma}$ then

$$D_F t_\sigma[F^*] h = -1$$

(recall, that $E(\delta)^2 = E(\delta)$) and

$$\begin{aligned}
\Lambda_*^{-1} \quad &\cdot \quad \left(D_F \left(S^\sigma_{t_\sigma[F^*]} \right)^{-1} h \right) \circ \Lambda_* \circ F^* + DF^* \cdot \Lambda_*^{-1} \cdot \left(D_F S^\sigma_{t_\sigma[F^*]} h \right) \circ \Lambda_* \\
&= -\left[-\Lambda_*^{-1} \cdot \sigma \circ \Lambda_* \circ F^* + DF^* \cdot \Lambda_*^{-1} \cdot \sigma \circ \Lambda_* \right] \\
&= -\kappa \left[-\sigma \circ F^* + DF^* \cdot \sigma \right] \\
&= -\kappa h_{F^*,\sigma},
\end{aligned}$$

therefore

$$DR_\sigma[F^*] h_{F^*,\sigma} = 0.$$

Now, suppose h is an eigenvector of $DR_0[F^*]$ corresponding to the eigenvalue $\delta \neq \kappa$, hence, $h \neq h_{F^*,\sigma}$, then, since $E(\kappa)E(\delta) = 0$, so is $D_F t_\sigma[F^*]h$, and $D_F S^\sigma_{t_\sigma[F^*]}h$. It follows from (58) that

$$DR_\sigma[F^*]h = \delta h.$$

Vice verse, suppose h is an eigenvector of $DR_\sigma[F^*]$ corresponding to an eigenvalue $\delta \neq \kappa$, then,

$$D_F t_\sigma[F^*]h = -\kappa^{-1} \|h_{F^*,\sigma}\|^{-1}_{\mathcal{D}} \|E(\kappa) DR_0[F^*]h\|_{\mathcal{D}},$$

and by (58) and a similar computation as above, for $a \in \mathbb{R}$,

$$\begin{aligned} DR_0[F^*](h + a h_{F^*,\sigma}) &= a\kappa h_{F^*,\sigma} + DR_0[F^*]h \\ &= a\kappa h_{F^*,\sigma} + \delta h - \left(\Lambda_*^{-1} \cdot \left(D_F \left(S^\sigma_{t[F^*]}\right)^{-1} h\right) \circ \Lambda_* \circ F^* \right. \\ &\qquad \left. + \left[DF^* \cdot \Lambda_*^{-1} \cdot \left(D_F S^\sigma_{t[F^*]} h\right)\right] \circ \Lambda_*\right) \\ &= a\kappa h_{F^*,\sigma} + \delta h + \|h_{F^*,\sigma}\|^{-1}_{\mathcal{D}} \|E(\kappa) DR_0[F^*]h\|_{\mathcal{D}} h_{F^*,\sigma}. \end{aligned}$$

Let,

$$a = \frac{\|E(\kappa) DR_0[F^*]h\|_{\mathcal{D}}}{\|h_{F^*,\sigma}\|_{\mathcal{D}} (\delta - \kappa)},$$

then $h + a h_{F^*,\sigma}$ is an eigenvector of $DR_0[F^*]$ with eigenvalue δ.

$\square$

Lemma 1.2. *Suppose that there are β, ϱ and ρ, and a function $s^* \in \mathcal{A}^\beta_s(\rho)$ such that the operator R_{EKW} is analytic and compact as maps from $\mathcal{F}^{\beta,\rho}_\varrho(s^*)$ to $\mathcal{O}_2(\mathcal{D})$, and*

$$R_{EKW}[F^*] = R_0[F^*] = F^*,$$

where F^* is generated by s^*.

Then, there exists a neighborhood $\mathcal{B}(F^*) \subset \mathcal{F}^{\beta,\rho}_\varrho(s^*)$, in which R_0 is analytic and compact, and

$$\operatorname{spec}(DR_0[F^*]|_{T_{F^*}\mathcal{B}(F^*)}) = \operatorname{spec}(DR_{EKW}[F^*]|_{T_{F^*}\mathcal{F}^{\beta,\rho}_\varrho(s^*)}) \cup \{1\}.$$

Proof. Let $\sigma^1_{0,0}$ and $\sigma^2_{0,0}$ be as in (54), then

$$\begin{aligned} S^{\sigma^1_{0,0}}_\epsilon(x,u) &= ((1+\epsilon)x, u), \quad h_{F,\sigma^1_{0,0}} = \pi_x F + DF \cdot (\pi_x, 0), \\ S^{\sigma^2_{0,0}}_\epsilon(x,u) &= (x, (1+\epsilon)u), \quad h_{F,\sigma^2_{0,0}} = \pi_u F + DF \cdot (0, \pi_u). \end{aligned}$$

Now, notice, that the operator $R_{EKW}[F]$ can be written as

$$R_{EKW}[F] = \Lambda_*^{-1} \circ \left(S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\right)^{-1} \circ \left(S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\right)^{-1} \circ F \circ F \circ S^{\sigma^2_{0,0}}_{r_{EKW}[F]} \circ S^{\sigma^1_{0,0}}_{t_{EKW}[F]} \circ \Lambda_*,$$

where

$$t_{EKW}[F] = \frac{\pi_x F(F(0,0))}{\lambda_*} - 1, \quad r_{EKW}[F] = \frac{\pi_x F(F(0,0))}{\mu_* \pi_x (F \circ F)_2(0,0)} - 1 = \frac{\lambda_*(1 + t_{EKW}[F])}{\mu_* \pi_x (F \circ F)_2(0,0)} - 1,$$

Notice, that that $t_{EKW}[F]$, $r_{EKW}[F]$, and therefore the transformations $S^{\sigma^1_{0,0}}_{t_{EKW}[F]}$ and $S^{\sigma^2_{0,0}}_{r_{EKW}[F]}$, depend only on $P_{EKW}[F]$. Therefore, the maps $F \mapsto S^{\sigma^1_{0,0}}_{t_{EKW}[F]}$ and $F \mapsto S^{\sigma^2_{0,0}}_{r_{EKW}[F]}$ are analytic (differentiable). In particular, by the continuity of

$F \mapsto S^{\sigma^1_{0,0}}_{t_{EKW}[F]}$ and $F \mapsto S^{\sigma^2_{0,0}}_{r_{EKW}[F]}$, there exists a neighborhood $\mathcal{B}(F^*) \subset \mathcal{F}^{\beta,\rho}_{\varrho}(s^*)$, such that R_0 is compact in $\mathcal{B}(F^*)$. In particular, both $DR_0[F^*]$ and $DR_{EKW}[F^*]$ exist, and are compact linear operators.

For any $F \in \mathcal{B}(F^*)$ and $h \in T_{F^*}\mathcal{F}^{\beta,\rho}_{\varrho}(s^*)$,

$$
\begin{aligned}
&DR_{EKW}[F]h = DR_0[F]h \\
+\quad & \Lambda_*^{-1}\left(D_F\left(S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\right)^{-1}h\right)\circ\left(S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\right)^{-1}\circ F\circ F\circ S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\circ S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\circ\Lambda_* \\
+\quad & \Lambda_*^{-1}\left[D\left(\left(S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\right)^{-1}\circ\left(S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\right)^{-1}\circ F\circ F\circ S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\right)\cdot\left(D_F S^{\sigma^1_{0,0}}_{t_{EKW}[F]}h\right)\right]\circ\Lambda_* \\
+\quad & \Lambda_*^{-1}\cdot D\left(S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\right)^{-1}\cdot\left(D_F\left(S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\right)^{-1}h\right)\circ F\circ F\circ S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\circ S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\circ\Lambda_* \\
+\quad & \Lambda_*^{-1}\left[D\left(\left(S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\right)^{-1}\circ\left(S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\right)^{-1}\circ F\circ F\right)\cdot\left(D_F S^{\sigma^2_{0,0}}_{r_{EKW}[F]}h\right)\right]\circ S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\circ\Lambda_* \\
=\quad & DR_0[F]h-\left(D_F t_{EKW}[F]h\right)\Lambda_*^{-1}\circ\sigma^1_{0,0}\circ\left(S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\right)^{-1}\circ F\circ F\circ S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\circ S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\circ\Lambda_* \\
+\quad & \left(D_F t_{EKW}[F]h\right)\Lambda_*^{-1}\left[D\left(\left(S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\right)^{-1}\circ\left(S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\right)^{-1}\circ F\circ F\circ S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\right)\circ\sigma^1_{0,0}\right]\circ\Lambda_* \\
-\quad & \left(D_F r_{EKW}[F]h\right)\Lambda_*^{-1}\cdot D\left(S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\right)^{-1}\circ\sigma^2_{0,0}\circ F\circ F\circ S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\circ S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\circ\Lambda_* \\
+\quad & \left(D_F r_{EKW}[F]h\right)\Lambda_*^{-1}\left[D\left(\left(S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\right)^{-1}\circ\left(S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\right)^{-1}\circ F\circ F\right)\circ\sigma^2_{0,0}\right]\circ S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\circ\Lambda_*.
\end{aligned}
$$

Specifically, if $F = F^*$, then (cf. (53))

$$
\begin{aligned}
DR_{EKW}[F^*]h &= DR_0[F^*]h + \left(D_F t_{EKW}[F^*]h\right) h_{F^*,\sigma^1_{0,0}} \\
&\quad + \left(D_F r_{EKW}[F^*]h\right) h_{F^*,\sigma^2_{0,0}}.
\end{aligned}
\tag{60}
$$

Next,

$$
\begin{aligned}
D_F S^{\sigma^1_{0,0}}_{t_{EKW}[F]}h &= (D_F t_{EKW}[F]h\pi_x, 0), \\
D_F S^{\sigma^2_{0,0}}_{r_{EKW}[F]}h &= (0, D_F r_{EKW}[F]h\pi_u), \\
D_F t_{EKW}[F]h &= \frac{\pi_x DP_{EKW}[F]h(0,0)}{\lambda_*}, \\
D_F r_{EKW}[F]h &= \frac{\lambda_* D_F t_{EKW}[F]h}{\mu_*\pi_x(F\circ F)_2(0,0)} - \frac{\lambda_*\pi_x\left(DP_{EKW}[F]h\right)_2(0,0)}{\mu_*\left(\pi_x(F\circ F)_2(0,0)\right)^2}.
\end{aligned}
$$

If $h = h_{F^*,\sigma^1_{0,0}}$, then

$$\begin{aligned}
DP_{EKW}[F]h(x,u) &= \left(-\pi_x P_{EKW}[F](x,u) + \pi_x P_{EKW}[F]_1(x,u)x,\right.\\
&\qquad \left.\pi_u P_{EKW}[F]_1(x,u)x\right),\\
\pi_x DP_{EKW}[F]h(0,0) &= -\pi_x P_{EKW}[F](0,0) = -\lambda_*,\\
D_F t_{EKW}[F]h &= -1,\\
D_F r_{EKW}[F]h &= -\frac{\lambda_*}{\mu_* \pi_x (F\circ F)_2(0,0)}\\
&\quad -\frac{\lambda_*\left(-\pi_x P_{EKW}[F]_2(0,0) + \pi_x P_{EKW}[F]_{1,2}(0,0)0\right)}{\mu_*\left(\pi_x (F\circ F)_2(0,0)\right)^2} = 0,\\
D_F S^{\sigma^1_{0,0}}_{t_{EKW}[F]} h &= (-\pi_x, 0),\\
D_F \left(S^{\sigma^1_{0,0}}_{t_{EKW}[F]}\right)^{-1} h &= (\pi_x, 0).
\end{aligned}$$

Similarly, if $h = h_{F^*,\sigma^2_{0,0}}$, then

$$\begin{aligned}
DP_{EKW}[F]h(x,u) &= \left(\pi_x P_{EKW}[F]_2(x,u)u,\right.\\
&\qquad \left.-\pi_u P_{EKW}[F](x,u) + \pi_u P_{EKW}[F]_2(x,u)u\right),\\
\pi_x DP_{EKW}[F]h(0,0) &= 0,\\
D_F t_{EKW}[F]h &= 0,\\
D_F r_{EKW}[F]h &= -1,\\
D_F S^{\sigma^2_{0,0}}_{r_{EKW}[F]} h &= (0, -\pi_u),\\
D_F \left(S^{\sigma^2_{0,0}}_{r_{EKW}[F]}\right)^{-1} h &= (0, \pi_u).
\end{aligned}$$

Therefore, if $h = h_{F^*,\sigma^1_{0,0}}$, we get

$$\begin{aligned}
DR_{EKW}[F^*]h &= \Lambda_*^{-1} DP_{EKW}[F^*]h\circ\Lambda_* + \Lambda_*^{-1}\pi_x F\circ F\circ\Lambda_*\\
&\qquad + \Lambda_*^{-1}\left[D\left(F\circ F\right)\cdot(-\pi_x,0)\right]\circ\Lambda_*\\
&\qquad + \left(D_F r_{EKW}[F^*]h\right) h_{F^*,\sigma^2_{0,0}}.\\
&= \Lambda_*^{-1}\left[DP_{EKW}[F^*]h + \pi_x P_{EKW}[F^*]\right.\\
&\qquad \left.-\left(\pi_x P_{EKW}[F]_1\pi_x, \pi_u P_{EKW}[F]_1\pi_x\right)\right]\circ\Lambda_*\\
&\qquad +0\\
&= 0.
\end{aligned}$$

If $h = h_{F^*,\sigma^2_{0,0}}$, then

$$\begin{aligned}
DR_{EKW}[F^*]h &= \Lambda_*^{-1} DP_{EKW}[F^*]h\circ\Lambda_* + \Lambda_*^{-1}\pi_u F\circ F\circ\Lambda_*\\
&\qquad + \Lambda_*^{-1}\left[D\left(F\circ F\right)\cdot(0,-\pi_u)\right]\circ\Lambda_*\\
&\qquad + \left(D_F t_{EKW}[F^*]h\right) h_{F^*,\sigma^1_{0,0}}\\
&= \Lambda_*^{-1}\left[DP_{EKW}[F^*]h + \pi_u P_{EKW}[F^*]\right.\\
&\qquad \left.-\left(\pi_x P_{EKW}[F]_2\pi_u, \pi_u P_{EKW}[F]_2\pi_u\right)\right]\circ\Lambda_*\\
&\qquad +0\\
&= 0.
\end{aligned}$$

If h is an eigenvector of $DR_0[F^*]$ associated with a non-zero eigenvalue κ, $h \neq h_{F^*,\sigma^1_{0,0}}$, and $h \neq h_{F^*,\sigma^2_{0,0}}$, then for any constant a and b

$$\begin{aligned}
& DR_{EKW}[F^*](h + ah_{F^*,\sigma^1_{0,0}} + bh_{F^*,\sigma^2_{0,0}}) = \\
& = DR_0[F^*]h + ah_{F^*,\sigma^1_{0,0}} + bh_{F^*,\sigma^2_{0,0}} + \\
& \qquad\qquad + \left(D_F t_{EKW}[F^*]\left(h + ah_{F^*,\sigma^1_{0,0}} + bh_{F^*,\sigma^2_{0,0}}\right)\right) h_{F^*,\sigma^1_{0,0}} \\
& \qquad\qquad + \left(D_F r_{EKW}[F^*]\left(h + ah_{F^*,\sigma^1_{0,0}} + bh_{F^*,\sigma^2_{0,0}}\right)\right) h_{F^*,\sigma^2_{0,0}} \\
& = \kappa h + ah_{F^*,\sigma^1_{0,0}} + bh_{F^*,\sigma^2_{0,0}} + \\
& \qquad + \left(D_F t_{EKW}[F^*]\left(h + bh_{F^*,\sigma^2_{0,0}}\right)\right) h_{F^*,\sigma^1_{0,0}} - ah_{F^*,\sigma^1_{0,0}} \\
& \qquad + \left(D_F r_{EKW}[F^*]\left(h + ah_{F^*,\sigma^1_{0,0}}\right)\right) h_{F^*,\sigma^2_{0,0}} - bh_{F^*,\sigma^2_{0,0}} \\
& = \kappa h + \kappa_1 h_{F^*,\sigma^1_{0,0}} + \kappa_2 h_{F^*,\sigma^2_{0,0}},
\end{aligned}$$

where

$$\kappa_1[h] = D_F t_{EKW}[F^*]h, \quad \kappa_2[h] = D_F r_{EKW}[F^*]h,$$

and we see, that if $a[h] = \kappa_1/\kappa$ and $b[h] = \kappa_2/\kappa$, then

$$h + ah_{F^*,\sigma^1_{0,0}} + bh_{F^*,\sigma^2_{0,0}}$$

is an eigenvector for $DR_{EKW}[F^*]$ with the eigenvalues κ.

On the other hand, if h is en eigenvector of $DR_{EKW}[F^*]$ associated with the eigenvalue $\kappa \neq 1$, then

$$h - ah_{F^*,\sigma^1_{0,0}} - bh_{F^*,\sigma^2_{0,0}}$$

is an eigenvector of $DR_0[F^*]$ associated with κ.

□

2. Strong contraction on the stable manifold

Lemma 2.1. *Suppose that β, ϱ and ρ are such that the operator*

$$\mathcal{R}_0[s] = \frac{1}{\mu_*}\mathcal{P}_{EKW}[s] \circ \lambda_*$$

has a fixed point $s^* \in \mathcal{B}_\varrho \subset \mathcal{A}^\beta_s(\rho)$, and $\mathcal{R}_0$ is analytic and compact as a map from $\mathcal{B}_\varrho$ to $\mathcal{A}^\beta_s(\rho)$.

Then, the number λ_* is an eigenvalue of $D\mathcal{R}_0[s^*]$, and the eigenspace of λ_* contains the eigenvector

$$\psi_{s^*}(x,y) = s^*_1(x,y)x^2 + s^*_2(x,y)y^2 + 2s^*(x,y)y. \tag{61}$$

Proof. Consider the coordinate transformation

$$\begin{aligned}
S_\epsilon(x,u) &= \left(x + \epsilon x^2, \frac{u}{1+2\epsilon x}\right) \\
&= (x,u) + \epsilon\sigma^1_{1,0}(x,u) - 2\epsilon\sigma^2_{1,0}(x,u) + O(\epsilon^2), && (62)\\
S^{-1}_\epsilon(y,v) &= \left(\frac{\sqrt{1+4\epsilon y}-1}{2\epsilon}, v\sqrt{1+4\epsilon y}\right), && (63)
\end{aligned}$$

for real ϵ, $|\epsilon| < 4/(\rho + |\beta|)$ (recall Definition 1.2).

Let $s \in \mathcal{A}_s^\beta(\rho)$ be the generating function for some F, then the following demonstrates that $S_\epsilon^{-1} \circ F \circ S_\epsilon$ is reversible, area-preserving and generated by

$$\hat{s}(x,y) = s(x+\epsilon x^2, y+\epsilon y^2)(1+2\epsilon y):$$

$$\begin{aligned}
\begin{pmatrix} x \\ -s(y+\epsilon y^2, x+\epsilon x^2)(1+2\epsilon x) \end{pmatrix} & \overset{S_\epsilon}{\mapsto} & \begin{pmatrix} x+\epsilon x^2 \\ -s(y+\epsilon y^2, x+\epsilon x^2) \end{pmatrix} \\
= \begin{pmatrix} x' \\ -s(y', x') \end{pmatrix} & \overset{F}{\mapsto} & \begin{pmatrix} y' \\ s(x', y') \end{pmatrix} \\
= \begin{pmatrix} y+\epsilon y^2 \\ s(x+\epsilon x^2, y+\epsilon y^2) \end{pmatrix} & \overset{S_\epsilon^{-1}}{\mapsto} & \begin{pmatrix} y \\ s(x+\epsilon x^2, y+\epsilon y^2)(1+2\epsilon y) \end{pmatrix}.
\end{aligned}$$

Next,

$$\hat{s}(x,y) = s(x,y) + \epsilon s_1(x,y)x^2 + \epsilon s_2(x,y)y^2 + \epsilon 2 s(x,y) y + O(\epsilon^2).$$

We will demonstrate that

$$\psi_{s^*}(x,y) = s_1^*(x,y)x^2 + s_2^*(x,y)y^2 + 2s^*(x,y)y.$$

is an eigenvector of $D\mathcal{R}_0[s^*]$ of the eigenvalue λ_*. Notice, that

$$\partial_1 \psi_s = \partial_1 \psi_s \circ I, \quad I(x,y) = (y,x),$$

and therefore, the function $s + \epsilon \psi_s \in \mathcal{A}_s^\beta(\rho)$.

Consider the midpoint equation

$$\begin{aligned}
0 = O(\epsilon^2) & + & s(x, Z(x,y) + \epsilon DZ[s]\psi_s(x,y)) + s(y, Z(x,y) + \epsilon DZ[s]\psi_s(x,y)) \\
& + & \epsilon \psi_s(x, Z(x,y)) + \epsilon \psi_s(y, Z(x,y))
\end{aligned}$$

for the generating function $s + \epsilon \psi_s$. We get that

$$DZ[s]\psi_s(x,y) = -\frac{\psi_s(x, Z(x,y)) + \psi_s(y, Z(x,y))}{s_2(x, Z(x,y)) + s_2(y, Z(x,y))},$$

and

$$\begin{aligned}
D\mathcal{P}_{EKW}\psi_s(x,y) &= s_1(Z(x,y),y) DZ[s]\psi_s(x,y) + \psi_s(Z(x,y),y) \\
&= -2s_1(Z(x,y),y)\frac{s(x,Z(x,y))Z + s(y,Z(x,y))Z}{s_2(x,Z(x,y)) + s_2(y,Z(x,y))} \\
&\quad -s_1(Z(x,y),y)\frac{s_2(x,Z(x,y))Z(x,y)^2 + s_2(y,Z(x,y))Z(x,y)^2}{s_2(x,Z(x,y)) + s_2(y,Z(x,y))} \\
&\quad +s_1(Z(x,y),y)Z(x,y)^2 \\
&\quad -s_1(Z(x,y),y)\frac{s_1(y,Z(x,y))y^2}{s_2(x,Z(x,y)) + s_2(y,Z(x,y))} + s_2(Z(x,y),y)y^2 \\
&\quad -s_1(Z(x,y),y)\frac{s_1(x,Z(x,y))x^2}{s_2(x,Z(x,y)) + s_2(y,Z(x,y))} \\
&\quad +2s(Z(x,y),y)y
\end{aligned}$$

The terms on line 2 add up to zero (the numerator is equal to zero because of the midpoint equation), and so do those on lines 3 and 4. We can also use the equalities

$$\begin{aligned} s_2(x,Z(x,y))+s_2(y,Z(x,y)) &= -\frac{s_1(y,Z(x,y))}{Z_2(x,y)} \\ \partial_2\mathcal{P}_{EKW}[s](x,y) &= s_2(Z(x,y),y)+s_1(Z(x,y),y)Z_2(x,y) \end{aligned}$$

(the first one being the midpoint equation differentiated with respect to y) to reduce the 5-th line to

$$\partial_2\mathcal{P}_{EKW}[s](x,y)y^2.$$

The 6-th line reduces to

$$\partial_1\mathcal{P}_{EKW}[s](x,y)x^2$$

after we use the midpoint equation differentiated with respect to x:

$$s_2(x,Z(x,y)+s_2(y,Z(x,y)=-\frac{s_1(x,Z(x,y))}{Z_1(x,y)}.$$

To summarize,

$$\begin{aligned} D\mathcal{P}_{EKW}\psi_s(x,y) &= \partial_1\mathcal{P}_{EKW}[s](x,y)x^2+\partial_2\mathcal{P}_{EKW}[s](x,y)y^2+2\mathcal{P}_{EKW}[s](x,y)y \\ &= \psi_{\mathcal{P}_{EKW}[s]}(x,y). \end{aligned}$$

Finally, we use the fact that

$$\lambda_*\partial_i\mathcal{P}_{EKW}[s](\lambda_*x,\lambda_*y)=\partial_i\left(\mathcal{P}[s](\lambda_*x,\lambda_*y)\right)$$

to get

$$D\mathcal{R}_0[s^*]\psi_{s^*}=\lambda_*\psi_{s^*}.$$

□

The Lemma below, whose elementary proof we will omit, shows that λ_* is also in the spectrum of $DR_0[F_*]$:

Lemma 2.2. *Suppose that β, ϱ and ρ are such that $s^*\in\mathcal{A}_s^\beta(\rho)$ is a fixed point of $\mathcal{R}_0$, and the operator $\mathcal{R}_0$ is analytic and compact as a map from $\mathcal{B}_\varrho(s^*)$ to $\mathcal{A}_s^\beta(\rho)$. Also, suppose that the map $\mathcal{I}$, described in Remark* 1.6, *is well-defined and analytic on $\mathcal{B}_\varrho(s^*)$, and that it has an analytic inverse $\mathcal{I}^{-1}$ on $\mathcal{I}(\mathcal{B}_\varrho(s^*))$. Then,*

$$\mathrm{spec}\left(\left(DR_0[F^*]\right)|_{T_{F^*}\mathcal{F}_\varrho^{\beta,\rho}(s^*)}\right)=\mathrm{spec}\left(D\mathcal{R}_0[s^*]\right).$$

in particular,

$$\lambda_*\in\mathrm{spec}\left(DR_0[F_*]\right).$$

At the same time, it is straightforward to see that the spectra of $DR_{EKW}[F_{EKW}]|_{T_{F_{EKW}}\mathcal{F}_\varrho^{\beta,\rho}(s^*)}$ and $D\mathcal{R}_{EKW}[s^{EKW}]$ are identical.

Lemma 2.3. *Suppose that β, ϱ and ρ are such that $s^*\in\mathcal{A}_s^\beta(\rho)$, and the operator $\mathcal{R}_{EKW}$ is analytic and compact as a map from $\mathcal{B}_\varrho(s^*)$ to $\mathcal{A}_s^\beta(\rho)$. Also, suppose that the map $\mathcal{I}$, described in Remark* 1.6, *is well-defined and analytic on $\mathcal{B}_\varrho(s^*)$, and that it has an analytic inverse $\mathcal{I}^{-1}$ on $\mathcal{I}(\mathcal{B}_\varrho(s^*))$. Then,*

$$\mathrm{spec}\left(\left(DR_{EKW}[F^*]\right)|_{T_{F^*}\mathcal{F}_\varrho^{\beta,\rho}(s^*)}\right)=\mathrm{spec}\left(D\mathcal{R}_{EKW}[s^*]\right),$$

in particular,

$$\lambda_*\in\mathrm{spec}\left(D\mathcal{R}_{EKW}[s^*]\right).$$

In Part 1 we saw that the convergence rate in the stable manifold of the renormalization operator plays a crucial role in demonstrating rigidity. It turns out that *the eigenvalue* λ_* *is the largest eigenvalues in the stable subspace of* $DR_{EKW}[F_*]$, or equivalently $D\mathcal{R}_{EKW}[s^*]$. However, it's value $|\lambda_*| \approx 0.2488$ is not small enough to ensure rigidity. At the same time, the eigenspace of the eigenvalue λ_* is, in the terminology of the renormalization theory, *irrelevant* to dynamics (the associated eigenvector is generated by a coordinate transformation). We, therefore, would like to eliminate this eigenvalue via an appropriate coordinate change, as described above.

However, first we would like to identify the eigenvector corresponding to the eigenvalue λ_* for the operator $\mathcal{R}_{EKW}$. This vector turns out to be different from ψ_{s^*}.

Lemma 2.4. *Suppose that* β*,* ϱ *and* ρ *are such that the operator* $\mathcal{R}_{EKW}$ *has a fixed point* $s^* \in \mathcal{A}^\beta_s(\rho)$*, and* $\mathcal{R}_{EKW}$ *is analytic and compact as a map from* $\mathcal{B}_\varrho(s^*)$ *to* $\mathcal{A}^\beta_s(\rho)$*. Also, suppose that the map* $\mathcal{I}$*, described in Remark* 1.6*, is well-defined and analytic on* $\mathcal{B}_\varrho(s^*)$*, and that it has an analytic inverse* $\mathcal{I}^{-1}$ *on* $\mathcal{I}(\mathcal{B}_\varrho(s^*))$*.*

Then, the number λ_* *is an eigenvalue of* $D\mathcal{R}_{EKW}[s^*]$*, and the eigenspace of* λ_* *contains the eigenvector*

$$\psi^{EKW}_{s^*}(x,y) = \psi_{s^*} + \tilde{\psi}, \tag{64}$$

where

$$\tilde{\psi} = s^* - (s^*_1(x,y)x + s^*_2(x,y)y).$$

Proof. Notice, that $\tilde{\psi}$ is of the form

$$\tilde{\psi}(x,y) = \psi_u - \psi_x,$$

where

$$\psi_x(x,y) = s^*_1(x,y)x + s^*_2(x,y)y$$

is the eigenvector of $D\mathcal{R}_0[s^*]$ corresponding to the rescaling of the variables x and y, while

$$\psi_u(x,y) = s^*(x,y)$$

is the eigenvector corresponding to the rescaling of s. $\psi_x(x,y)$ and $\psi_u(x,y)$ correspond to the eigenvectors $h_{F^*,\sigma^1_{0,0}}$ and $h_{F^*,\sigma^2_{0,0}}$, respectively, of $DR_0[F^*]$.

Recall, that $h_{F^*,\sigma^1_{0,0}}$ and $h_{F^*,\sigma^2_{0,0}}$ are eigenvectors of $DR_0[F^*]$, with eigenvalue 1, and eigenvectors of $DR_{EKW}[F^*]$ with eigenvalue 0.

By Lemma 2.1 ψ_{s^*} is an eigenvector of $D\mathcal{R}_0$, the corresponding eigenvector of DR_0 is $h_{F^*,\sigma^1_{1,0}-2\sigma^2_{1,0}}$. Thus, $\psi_{s^*} + \tilde{\psi}$ corresponds to the vector

$$h^{EKW}_{\lambda_*} := h_{F^*,\sigma^1_{1,0}-2\sigma^2_{1,0}} - h_{F^*,\sigma^1_{0,0}} + h_{F^*,\sigma^2_{0,0}}. \tag{65}$$

To finish the proof, it suffices to prove that

$$DR_{EKW} h^{EKW}_{\lambda_*} = \lambda_* h^{EKW}_{\lambda_*}.$$

By (60)

$$\begin{aligned}
DR_{EKW}[F^*]h^{EKW}_{\lambda_*} &= DR_{EKW}[F^*]h_{F^*,\sigma^1_{1,0}-2\sigma^2_{1,0}} \\
&= DR_0[F^*]h_{F^*,\sigma^1_{1,0}-2\sigma^2_{1,0}} \\
&+ \left(D_F t_{EKW}[F^*]h_{F^*,\sigma^1_{1,0}-2\sigma^2_{1,0}}\right) h_{F^*,\sigma^1_{0,0}} \\
&+ \left(D_F r_{EKW}[F^*]h_{F^*,\sigma^1_{1,0}-2\sigma^2_{1,0}}\right) h_{F^*,\sigma^2_{0,0}} \\
&= \lambda_* h_{F^*,\sigma^1_{1,0}-2\sigma^2_{1,0}} \\
&+ \left(D_F t_{EKW}[F^*]h_{F^*,\sigma^1_{1,0}-2\sigma^2_{1,0}}\right) h_{F^*,\sigma^1_{0,0}} \\
&+ \left(D_F r_{EKW}[F^*]h_{F^*,\sigma^1_{1,0}-2\sigma^2_{1,0}}\right) h_{F^*,\sigma^2_{0,0}}
\end{aligned}$$

The result follows if

$$D_F t_{EKW}[F^*]h_{F^*,\sigma^1_{1,0}-2\sigma^2_{1,0}} = -\lambda_*$$

and

$$D_F r_{EKW}[F^*]h_{F^*,\sigma^1_{1,0}-2\sigma^2_{1,0}} = \lambda_*.$$

Indeed, as in the proof of Lemma 1.2. If $h = h_{F^*,\sigma^1_{1,0}}$, then

$$\begin{aligned}
DP_{EKW}[F^*]h(x,u) &= \big(-(\pi_x P_{EKW}[F^*](x,u))^2 + \pi_x P_{EKW}[F^*]_1(x,u)x^2, \\
&\qquad \pi_u P_{EKW}[F^*]_1(x,u)x^2\big), \\
\pi_x DP_{EKW}[F^*]h(0,0) &= -(\pi_x P_{EKW}[F^*](0,0))^2 = -\lambda_*^2, \\
D_F t_{EKW}[F^*]h &= -\lambda_* \\
D_F r_{EKW}[F^*]h &= \frac{\lambda_*^2}{\mu_* \pi_x (F^*\circ F^*)_2(0,0)} \\
&+ \lambda_* \,(-2\pi_x P_{EKW}[F^*](0,0)\pi_x P_{EKW}[F^*]_2(0,0) \\
&\qquad + \frac{\pi_x P_{EKW}[F^*]_{1,2}(0,0)0^2)}{\mu_* \left(\pi_x (F^*\circ F^*)_2(0,0)\right)^2} \\
&= -\lambda_* + 2\pi_x P_{EKW}[F^*](0,0) = \lambda_*
\end{aligned}$$

If $h = h_{F^*,\sigma^2_{1,0}}$, then

$$\begin{aligned}
DP_{EKW}[F^*]h(x,u) &= (\pi_x P_{EKW}[F^*]_2(x,u)xu, \\
&\qquad -\pi_x P_{EKW}[F^*](x,u)\pi_u P_{EKW}[F^*](x,u) \\
&\qquad +\pi_u P_{EKW}[F^*]_2(x,u)xu)\,, \\
\pi_x DP_{EKW}[F^*]h(0,0) &= 0 \\
D_F t_{EKW}[F^*]h &= 0 \\
D_F r_{EKW}[F^*]h &= 0 + \frac{\lambda_* \left(\pi_x P_{EKW}[F^*]_{2,2}(0,0)0 + \pi_x P_{EKW}[F^*]_2(0,0)0\right)}{\mu_* \left(\pi_x (F^*\circ F^*)_2(0,0)\right)^2} = 0.
\end{aligned}$$

□

Definition 2.5. *Suppose s^* is a fixed point of the operator $\mathcal{R}_0$ (or, equivalently, $\mathcal{R}_{EKW}$). Set, formally,*

$$\mathcal{P}[s](x,y) = (1+2ty)s(G(\xi_t(x,y))), \quad \text{and} \quad \mathcal{R}[s] = \mu^{-1}\mathcal{P}[s]\circ\lambda,$$

where

$$
\begin{aligned}
0 &= s(x, Z(x,y)) + s(y, Z(x,y)), \\
t &= -\frac{1}{\lambda_* \|\psi_{s^*}^{EKW}\|_\rho} \|E(\lambda_*)(\mathcal{R}_{EKW}[s] - s^*)\|, \\
0 &= \mathcal{P}[s](\lambda, 0), \quad (66)\\
\mu &= \lambda \partial_1 \mathcal{P}[s](\lambda, 0), \quad (67)\\
\xi_t(x,y) &= (x + tx^2, y + ty^2), \quad (68)
\end{aligned}
$$

$\psi_{s^*}^{EKW}$ *is as in* (64)*, G as in (14), and E is the Riesz projection for the operator* $D\mathcal{R}_{EKW}[s^*]$.

We will quote a version of a lemma from (*Gaidashev* 2010) which we will require to demonstrate analyticity and compactness of the operator $\mathcal{R}$. The proof of the Lemma is computer-assisted. Notice, the parameters that enter the Lemma are different from those used in (*Gaidashev* 2010). As before, the reported numbers are representable on a computer.

Lemma 2.6. *For all $s \in \mathcal{B}_R(s^0)$, where*

$$R = 5.47321968732772541 \times 10^{-3},$$

and s^0 is as in Theorem 2*, the prerenormalization $\mathcal{P}_{EKW}[s]$ is well-defined and analytic function on the set*

$$\mathcal{D}_r \equiv \mathcal{D}_r(0) = \{(x,y) \in \mathbb{C}^2 : |x| < r, |y| < r\}, \quad r = 0.51853174082497335,$$

with

$$\|Z\|_r \leq 1.63160151494042404.$$

We will now demonstrate analyticity and compactness of the modified renormalization operator in a functional space, different from that used in (Eckmann *et al* 1984), specifically, in the space $\mathcal{A}_s(1.75)$. It is in this space that we will eventually compute a bound on the spectral radius of the action of the modified renormalization operator on infinitely renormalizable maps.

Proposition 2.7. *There exists a polynomial $s_0 \subset \mathcal{B}_R(s^0) \subset \mathcal{A}_s(1.75)$, where R and s^0 are as in Lemma* 2.6*, such that the operator $\mathcal{R}$ is well-defined, analytic and compact as a map from $\mathcal{B}_{\varrho_0}(s_0)$, $\varrho_0 = 5.79833984375 \times 10^{-4}$, to $\mathcal{A}_s(1.75)$, if $\mathcal{B}_{\varrho_0}(s_0) \subset \mathcal{B}_R(s^0)$ contains the fixed point s^*.*

Proof. The polynomial s_0 has been computed as a high order numerical approximation of a fixed point s^* of $\mathcal{R}$.

First, we get a bound on t for all $s \in \mathcal{B}_\delta(s_0)$:

$$
\begin{aligned}
|t| &= \frac{1}{|\lambda_*| \|\psi_{s^*}^{EKW}\|_\rho} \|E(\lambda_*)(\mathcal{R}_{EKW}[s] - s^*)\|_\rho \\
&\leq \frac{1}{|\lambda_+| \psi_{s^*}^{EKW}\|_\rho} \|\mathcal{R}_{EKW}[s] - s^*\|_\rho.
\end{aligned}
$$

We estimate the right hand side rigorously on the computer and obtain

$$|t| \leq 2.1095979213715 \times 10^{-6}. \quad (69)$$

The condition of the hypothesis that $s^* \in \mathcal{B}_\delta(s_0)$ is specifically required to be able to compute this estimate.

Notice that according to Definition 2.5 and Theorem 2, the maps $s \mapsto t$ and, hence, $s \mapsto \xi_t$ are analytic on a larger neighborhood $\mathcal{B}_R(s^0)$ of analyticity of $\mathcal{R}_{EKW}$. According to Theorem 2 and Lemma 2.6, the prerenormalization $\mathcal{P}_{EKW}$ is also analytic as a map from $\mathcal{B}_R(s^0)$ to $\mathcal{A}_s(r)$, $r = 0.516235055482147608$. We verify that for all $s \in \mathcal{B}_\delta(s_0)$ and t as in (69) the following holds:

$$\{\xi_t(x,y) : (x,y) \in \mathcal{D}_{r'}\} \Subset \mathcal{D}_r, \quad r' = |\lambda_-|\rho, \tag{70}$$

where $\lambda_- = -0.27569580078125$ is the lower bound from Theorem 2. Furthermore,

$$1 > 2|t|\rho$$

with t as in (69). Therefore, the map $s \mapsto \mathcal{P}[s]$ is analytic on $\mathcal{B}_\delta(s_0)$.

Since the inclusion of sets (70) is compact, $\mathcal{R}[s]$ has an analytic extension to a neighborhood of $\mathcal{D}_{1.75}$, $\mathcal{R}[s] \in \mathcal{A}_s(\rho')$, $\rho' > 1.75$. Compactness of the map $s \mapsto \mathcal{R}[s]$ now follows from the fact that the inclusions of spaces $\mathcal{A}_s(\rho') \subset \mathcal{A}_s(\rho)$ is compact. □

Recall, that according to Lemma 2.2, λ_* is an eigenvalue of $DR_0[F^*]$ of multiplicity at least 1. According to Lemma 1.2, λ_* is in the spectrum of $DR_{EKW}[F_*]$, and according to Lemma 2.3, $\lambda_* \in D\mathcal{R}_{EKW}[s^*]$.

Proposition 2.8. *Suppose that β, ρ, ϱ and the neighborhood $\mathcal{B}_\varrho(s^*) \subset \mathcal{A}_s^\beta(\rho)$ satisfy the hypothesis of Lemma 2.2. Furthermore, suppose that the operator $\mathcal{R}$ is analytic and compact in $\mathcal{B}_\varrho(s^*)$.*

Then

$$\mathrm{spec}(D\mathcal{R}_{EKW}[s^*]) \setminus \{\lambda_*\} \subset \mathrm{spec}(D\mathcal{R}[s^*]),$$

and $\psi_{s^*}^{EKW}$ is an eigenvector of $D\mathcal{R}[s^*]$ associated with the eigenvalue 0.

In addition,

$$\mathrm{spec}(D\mathcal{R}[s^*]) \subset \mathrm{spec}(D\mathcal{R}_{EKW}[s^*]),$$

and if $\lambda_* \notin \mathrm{spec}(D\mathcal{R}[s^*])$, then λ_* has multiplicity 1 in $\mathrm{spec}(D\mathcal{R}_{EKW}[s^*])$.

Proof. First, notice the difference between the definition of λ in (1.1)

$$s(G(\lambda,0)) = 0,$$

and in Definition (2.5)

$$s(G(\lambda + t\lambda^2, 0)) = 0$$

(we will use the notation λ_{EKW} below to emphasize the difference). This implies that if $D_s\lambda_{EKW}[s]\psi$ is an action of the derivative of $\lambda_{EKW}[s]$ on a vector ψ, then

$$D_s\lambda[s^*]\psi = D_s\lambda_{EKW}[s^*]\psi - \lambda_*^2 D_s t[s^*]\psi$$

is that of the derivative of $\lambda[s]$.

Similarly,

$$\begin{aligned}
D_s\mu_{EKW}[s^*]\psi &= \left[\partial_1(s^* \circ G)(\lambda_*,0) + \lambda_*\partial_1^2(s^* \circ G)(\lambda_*,0)\right] D_s\lambda_{EKW}[s^*]\psi \\
&\qquad + \lambda_*\partial_1(D_s\mathcal{P}_{EKW}[s^*]\psi)(\lambda_*,0), \\
D_s\mu[s^*]\psi &= \left[\partial_1(s^* \circ G)(\lambda_*,0) + \lambda_*\partial_1^2(s^* \circ G)(\lambda_*,0)\right] D_s\lambda[s^*]\psi \\
&\qquad + \lambda_*\partial_1(D_s\mathcal{P}_{EKW}[s^*]\psi)(\lambda_*,0) \\
&\qquad + \lambda_*^3\partial_1^2(s^* \circ G)(\lambda_*,0) D_s t[s^*]\psi \\
&= D_s\mu_{EKW}[s^*]\psi - \partial_1\mathcal{P}_{EKW}[s^*](\lambda_*,0)\lambda_*^2 D_s t[s^*]\psi \\
&= D_s\mu_{EKW}[s^*]\psi - \lambda_*\mu_* D_s t[s^*]\psi.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
D\mathcal{R}[s^*]\psi = D\mathcal{R}_{EKW}[s^*]\psi &+ 2\lambda_* \left(D_s t[s^*]\psi\right) s^* \pi_y + \frac{1}{\mu_*}\left(D\mathcal{P}_{EKW}[s^*]\cdot (D_s\xi_t\psi)\right)\circ \lambda_* \\
&- D_s t[s^*]\psi \frac{\lambda_*^2}{\mu_*} D\mathcal{P}_{EKW}[s^*]\circ\lambda_*\cdot(\pi_x,\pi_y) \\
&+ \lambda_* D_s t[s^*]\psi s^* \\
= D\mathcal{R}_{EKW}[s^*]\psi &- \lambda_*\left(D_s t[s^*]\psi\right) Ds^*\cdot(\pi_x,\pi_y) + \lambda_*\left(D_s t[s^*]\psi\right) s^* \\
&+ \lambda_*\left(D_s t[s^*]\psi\right)\psi_{s^*} \\
= D\mathcal{R}_{EKW}[s^*]\psi &+ \lambda_*\left(D_s t[s^*]\psi\right)\psi^{EKW}_{s^*} \qquad (71)
\end{aligned}
$$

where

$$
\begin{aligned}
D_s t[s^*]\psi &= -\lambda_*^{-1}\|\psi^{EKW}_{s^*}\|_\rho^{-1}\|E(\lambda_*)\left(D\mathcal{R}_{EKW}[s^*]\psi\right)\|_\rho, \\
D_s\xi_t[s^*]\psi(x,y) &= \left(D_s t\psi\right)(x^2,y^2) \\
&= -\lambda_*^{-1}\|\psi^{EKW}_{s^*}\|_\rho^{-1}\|E(\lambda_*)\left(D\mathcal{R}_{EKW}[s^*]\psi\right)\|_\rho(x^2,y^2).
\end{aligned}
$$

Similarly to Lemma (1.1), we get that if ψ is an eigenvector of $D\mathcal{R}_{EKW}[s^*]$ associated with the eigenvalue $\delta \neq \lambda_*$, then $\psi \neq \psi^{EKW}_{s^*}$, and

$$E(\lambda_*)\left(D\mathcal{R}_{EKW}[s^*]\psi\right) = \delta E(\lambda_*)\psi = 0,$$

so is $D_s t[s^*]\psi$, and

$$D\mathcal{R}[s^*]\psi = D\mathcal{R}_{EKW}[s^*]\psi = \delta\psi.$$

If $\delta = \lambda_*$ and $\psi = \psi^{EKW}_{s^*}$, then

$$D_s t[s^*]\psi = -1, \quad D_s\xi_t[s^*]\psi(x,y) = -(x^2,y^2),$$

and therefore,

$$D\mathcal{R}[s^*]\psi^{EKW}_{s^*} = \lambda_*\psi^{EKW}_{s^*} - \lambda_*\psi^{EKW}_{s^*} = 0,$$

and $\psi^{EKW}_{s^*}$ is an eigenvector of $D\mathcal{R}[s^*]$ associated with the eigenvalue 0.

Vice verse, by (71), if ψ is an eigenvector of $D\mathcal{R}[s^*]$ associated with the eigenvalue $\delta \neq \lambda_*$, then

$$
\begin{aligned}
D\mathcal{R}_{EKW}[s^*](\psi + a\psi^{EKW}_{s^*}) &= D\mathcal{R}[s^*]\psi - \lambda_*(D_s t[s^*](\psi + a\psi^{EKW}_{s^*})\psi^{EKW}_{s^*} \\
&= \delta\psi - \lambda_*(D_s t[s^*]\psi - a)\psi^{EKW}_{s^*}
\end{aligned}
$$

Hence, $\psi + \frac{\lambda_* D_s t[s^*]\psi}{\lambda_* - \delta}\psi^{EKW}_{s^*}$ is an eigenvector of $D\mathcal{R}_{EKW}[s^*]$ with the eigenvalue δ.

Finally, assume that $\lambda_* \notin \operatorname{spec}(D\mathcal{R}[s^*])$, but that there exists an eigenvector $\varphi \neq \psi^{EKW}_{s^*}$ of $D\mathcal{R}_{EKW}[s^*]$ with eigenvalue λ_*. Then

$$D_s t[s^*]\varphi = -\frac{\|\varphi\|_\rho}{\|\psi^{EKW}_{s^*}\|_\rho},$$

and, by (71),

$$
\begin{aligned}
D\mathcal{R}[s^*]\left(\varphi - \frac{\|\varphi\|_\rho}{\|\psi^{EKW}_{s^*}\|_\rho}\psi^{EKW}_{s^*}\right) &= D\mathcal{R}[s^*]\varphi \\
&= \lambda_*\varphi + \lambda_*\left(-\frac{\|\varphi\|_\rho}{\|\psi^{EKW}_{s^*}\|_\rho}\right)\psi^{EKW}_{s^*} \\
&= \lambda_*\left(\varphi - \frac{\|\varphi\|_\rho}{\|\psi^{EKW}_{s^*}\|_\rho}\psi^{EKW}_{s^*}\right).
\end{aligned}
$$

This contradiction finishes the proof. □

So far we were not able to make any claims about the multiplicity of the eigenvalue λ_* in the spectrum of $D\mathcal{R}_{EKW}[s^*]$. However, we will demonstrate in Section 3 that it is indeed equal to 1.

Definition 2.9. *Set, formally,*

$$\begin{aligned} R[F] &= \Lambda_F^{-1} \circ P[F] \circ \Lambda_F, \\ P[F] &= S_{t[F]}^{-1} \circ F \circ F \circ S_{t[F]}, \end{aligned} \tag{72}$$

where $S_{t[F]}$ *is as in* (62), $\Lambda_F(x,u) = (\lambda[F]x, \mu[F]u)$,

$$t[F] = -\frac{1}{\lambda_* \|h_{F_*,\sigma}\|_{\mathcal{D}}} \|E(\lambda_*)(R_{EKW}[F] - F_*)\|_{\mathcal{D}},$$

where

$$\sigma = \sigma^1_{1,0} - 2\sigma^2_{1,0} - \sigma^1_{0,0} + \sigma^2_{0,0},$$

and, furthermore,

$$\begin{aligned} \lambda[F] &= \pi_x P[F](0,0), \\ \mu[F] &= \frac{-\lambda[F]}{\pi_x P[F]_2(0,0)}. \end{aligned}$$

The above is a formal definition. As usual, one would have to verify the properties of being well-defined, analytic and compact, in a setting of a specific functional space.

3. Spectral properties of $\mathcal{R}$. Proof of Main Theorem 1

We will now describe our computer-assisted proof of Main Theorem 1.

To implement the operator $D\mathcal{R}[s^*]$ on the computer, we would have to implement the Riesz projection as well. Unfortunately, this is not easy, therefore, we will do it only approximately. Specifically, the component $(0,3)$ of the composition $s \circ G$ will be consistently normalized to be

$$c_0 = (s_0 \circ G[s_0])_{(0,3)},$$

where s_0 is our polynomial approximation for the fixed point.

Definition 3.1. *Set, formally,*

$$\mathcal{P}_c[s](x,y) = (1 + 2t_c y) s(G(\xi_{t_c}(x,y))), \quad \text{and} \quad \mathcal{R}_c[s] = \mu^{-1}\mathcal{P}_c[s] \circ \lambda,$$

where

$$\begin{aligned} G(x,y) &= (Z(x,y), y), \\ 0 &= s(x, Z(x,y)) + s(y, Z(x,y)), \\ t_c[s] &= \frac{1}{4}\frac{c - (s \circ G)_{(0,3)}}{(s \circ G)_{(0,2)}}, \quad c \in \mathbb{R} \\ 0 &= \mathcal{P}_c[s](\lambda[s], 0), \\ \mu[s] &= \lambda \partial_1 \mathcal{P}_c[s](\lambda, 0). \end{aligned}$$

The operator $\mathcal{R}_c$ differs from $\mathcal{R}$ (cf.2.5) only in the "amount" by which the eigendirection $\psi_{s^*}^{EKW}$ is "eliminated". In particular, as the next proposition demonstrates, R_c is still analytic and compact in the same neighborhood of s_0.

Proposition 3.2. *There exists a polynomial $s_0 \subset \mathcal{B}_R(s^0) \subset \mathcal{A}_s(1.75)$, where R and s^0 are as in Theorem 2, such that the operators $\mathcal{R}_c$, $c \in [c_0 - \delta, c_0 + \delta]$,*

$$c_0 = (s_0 \circ G[s_0])_{(0,3)} \quad \text{and} \quad \delta = 1.068115234375 \times 10^{-4},$$

are well-defined and analytic as maps from $\mathcal{B}_{\varrho_0}(s_0)$, $\varrho_0 = 5.79833984375 \times 10^{-4}$, to $\mathcal{A}_s(1.75)$.

Furthermore, the operators $\mathcal{R}_c$ are compact in $B_R(s^0) \subset \mathcal{A}(\rho)$, with $\mathcal{R}_c[s] \in \mathcal{A}(\rho')$, $\rho' = 1.0699996948242188\rho$.

Proof. The proof is almost identical to that of Proposition 2.7, with a different (but still sufficiently small) bound on $|t_c[s]|$. □

Definition 3.3. *Set, formally,*

$$R_c[F] = \Lambda_F^{-1} \circ P_c[F] \circ \Lambda_F, \quad P_c[F] = S_{t_c}^{-1} \circ F \circ F \circ S_{t_c}, \tag{73}$$

where S_{t_c} is as in (62), $\Lambda_F(x,u) = (\lambda[F]x, \mu[F]u)$, and

$$t_c[F] = \frac{1}{4}\frac{c - (\pi_u(F \circ F))_{(0,3)}}{(\pi_u(F \circ F))_{(0,2)}}, \; c \in \mathbb{R}, \; \lambda[F] = \pi_x P_c[F](0,0), \; \mu[F] = \frac{-\lambda[F]}{\pi_x P_c[F]_2(0,0)}.$$

Lemma 3.4. *Suppose that the neighborhood $\mathcal{B}_{\varrho_0}(s_0)$, with ϱ_0 as in Propositions 2.7 and 3.2, contains a fixed point s^* of $\mathcal{R}$, and of $\mathcal{R}_{c^*}$ for*

$$c^* = (s^* \circ G[s^*])_{(0,3)}.$$

Set

$$\delta = 0.00124359130859375,$$

then

$$\operatorname{spec}(D\mathcal{R}[s^*]) \setminus \{z \in \mathbb{C} : |z| \le \delta\} \subset \operatorname{spec}(D\mathcal{R}_{c^*}[s^*]) \setminus \{z \in \mathbb{C} : |z| \le \delta\}.$$

Proof. According to Propositions 2.7 and 3.2, under the hypothesis of the Lemma, $\mathcal{R}$ and $\mathcal{R}_{c^*}$ are analytic and compact as operators from $\mathcal{B}_\delta(s_0)$ to $\mathcal{A}_s(1.75)$.

Recall, that $\psi_{s^*}^{EKW}$ is an eigenvector of $D\mathcal{R}_{EKW}[s^*]$ corresponding to the eigenvalue λ_*.

We consider the action of $D\mathcal{R}_{c^*}[s^*]$ on a vector ψ. Similarly to (71),

$$\begin{aligned} D\mathcal{R}_{c^*}[s^*]\psi &= D\mathcal{R}_{EKW}[s^*]\psi + \lambda_* (D_s t_c[s^*]\psi)\,\psi_{s^*} + \lambda_* (D_s t_c[s^*]\psi)\,\tilde{\psi} \\ &= D\mathcal{R}[s^*]\psi + \lambda_* ((D_s t_c[s^*] - D_s t[s])\,\psi)\,\psi_{s^*}^{EKW}. \end{aligned}$$

Now, let ψ be an eigenvector of $D\mathcal{R}[s^*]$ of eigenvalue $\kappa \neq 0$ (that is, $\psi \neq \psi_{s^*}^{EKW}$). Consider the action of $D\mathcal{R}_{c^*}[s^*]$ on $\psi + a\psi_{s^*}^{EKW}$.

$$D\mathcal{R}_{c^*}[s^*](\psi + a\psi_{s^*}^{EKW}) = \kappa\psi + \lambda_* (D_s t_c[s^*] - D_s t[s^*])\,(\psi + a\psi_{s^*}^{EKW})\psi_{s^*}^{EKW}.$$

Notice,

$$
\begin{aligned}
D_s t_c[s^*]\psi^{EKW}_{s^*} &= D_s t_c[s^*](\psi_{s^*}+\psi_u-\psi_x)\\
&= -\frac{1}{4}\frac{(D\mathcal{P}_{EKW}[s^*](\psi_{s^*}+\psi_u-\psi_x))_{0,3}}{\mathcal{P}_{EKW}[s^*]_{0,2}}\\
&\quad -\frac{1}{4}\frac{(D\mathcal{P}_{EKW}[s^*](\psi_{s^*}+\psi_u-\psi_x))_{0,2}\,(c-\mathcal{P}_{EKW}[s^*]_{0,3})}{\left(\mathcal{P}_{EKW}[s^*]_{0,2}\right)^2}\\
&= -\frac{1}{4}\frac{\left(\psi_{\mathcal{P}_{EKW}[s^*]}+\mathcal{P}_{EKW}[s^*]-D\mathcal{P}_{EKW}[s^*]\cdot(\pi_x,\pi_y)\right)_{0,3}}{\mathcal{P}_{EKW}[s^*]_{0,2}}\\
&\quad -\frac{1}{4}\frac{\left(\psi_{\mathcal{P}_{EKW}[s^*]}+\mathcal{P}_{EKW}[s^*]-D\mathcal{P}_{EKW}[s^*]\cdot(\pi_x,\pi_y)\right)_{0,2}(c-\mathcal{P}_{EKW}[s^*]_{0,3})}{\left(\mathcal{P}_{EKW}[s^*]_{0,2}\right)^2}\\
&= -\frac{1}{4}\frac{(\partial_2\mathcal{P}_{EKW}[s^*])_{0,1}+2\left(\mathcal{P}_{EKW}[s^*]\right)_{0,2}}{\mathcal{P}_{EKW}[s^*]_{0,2}}\\
&\quad -\frac{1}{4}\frac{(\mathcal{P}_{EKW}[s^*])_{0,3}-(\partial_2\mathcal{P}_{EKW}[s^*])_{0,2}}{\mathcal{P}_{EKW}[s^*]_{0,2}}\\
&\quad -\frac{1}{4}\frac{\left((\partial_2\mathcal{P}_{EKW}[s^*])_{0,0}+2\left(\mathcal{P}_{EKW}[s^*]\right)_{0,1}\right)(c-\mathcal{P}_{EKW}[s^*]_{0,3})}{\left(\mathcal{P}_{EKW}[s^*]_{0,2}\right)^2}\\
&\quad -\frac{1}{4}\frac{\left((\mathcal{P}_{EKW}[s^*])_{0,2}-(\partial_2\mathcal{P}_{EKW}[s^*])_{0,1}\right)(c-\mathcal{P}_{EKW}[s^*]_{0,3})}{\left(\mathcal{P}_{EKW}[s^*]_{0,2}\right)^2}\\
&= -1+\frac{1}{2}\frac{c^*}{\mathcal{P}_{EKW}[s^*]_{0,2}}-\frac{1}{4}\left(\frac{3\mathcal{P}_{EKW}[s^*]_{0,1}}{\mathcal{P}_{EKW}[s^*]_{0,2}}-1\right)\frac{c-c^*}{\mathcal{P}_{EKW}[s^*]_{0,2}}\\
&= -1+C,\\
D_s t[s^*]\psi^{EKW}_{s^*} &= -1
\end{aligned}
$$

Denote $d_1 \equiv D_s t_c[s^*]\psi$ and $d_2 \equiv D_s t[s^*]\psi$, then

$$
\begin{aligned}
D\mathcal{R}_{c^*}[s^*](\psi + a\psi^{EKW}_{s^*}) &= \kappa\psi + \lambda_*(d_1 - d_2 + a(-1+C)+a)\psi^{EKW}_{s^*}\\
&= \kappa\left(\psi + \frac{\lambda_*}{\kappa}(d_1-d_2+aC)\psi^{EKW}_{s^*}\right),
\end{aligned}
$$

and we see that the equation

$$a = \frac{\lambda_*}{\kappa}(d_1 - d_2 + aC)$$

has a unique solution a if

$$\kappa \neq \lambda_* C. \tag{74}$$

For such κ, the vector

$$\psi + \frac{\lambda_*(d_1-d_2)}{\kappa-\lambda_* C}\psi^{EKW}_{s^*}$$

is an eigenvector of $D\mathcal{R}_{c^*}[s^*]$ associated with the eigenvalue κ.

The eigenvalues κ as in (74) satisfy

$$|\kappa| > 0.00124359130859375$$

□

We will now describe a rigorous computer upper bound on the spectrum of the operator $DR_c[s^*]$. Since the bound itself is an intermediate results, here, we will not give a thorough introduction into rigorous computations in Banach space, and, in fact, will skip many technicalities of the proof. For a thorough treatise of computations in Banach spaces, an interested reader is referred to (Koch *et al* 1996).

Proof of part ii) of Main Theorem 1.

Step 1). Recall the Definition 1.2 of the Banach subspace $\mathcal{A}_s(\rho)$ of $\mathcal{A}(\rho)$. We will now choose a new bases $\{\psi_{i,j}\}$ in $\mathcal{A}_s(\rho)$. Given $s \in \mathcal{A}_s(\rho)$ we write its Taylor expansion in the form

$$s(x,y) = \sum_{(i,j)\in I} s_{i,j}\psi_{i,j}(x,y),$$

where $\psi_{i,j} \in \mathcal{A}_s(\rho)$:

$$\begin{aligned}
\tilde{\psi}_{i,j}(x,y) &= x^{i+1}y^j, \quad i=-1, \quad j\geq 0,\\
\tilde{\psi}_{i,j}(x,y) &= x^{i+1}y^j + \frac{i+1}{j+1}x^{j+1}y^i, \quad i>-1, \quad j\geq i,\\
\psi_{i,j} &= \frac{\tilde{\psi}_{i,j}}{\|\tilde{\psi}_{i,j}\|_\rho}, \quad i\geq -1, \quad j\geq \max\{0,i\},
\end{aligned}$$

and the index set I of these basis vectors is defined as

$$I = \{(i,j)\in \mathbb{Z}^2: \quad i\geq -1, \quad j \geq \max\{0,i\}\}.$$

Denote $\tilde{\mathcal{A}}_s(\rho)$ the set of all sequences

$$\tilde{s} = \left\{ s_{i,j} : s_{i,j}\in\mathbb{C}, \sum_{(i,j)\in I} |s_{i,j}| < \infty \right\}.$$

Equipped with the l_1-norm

$$|s|_1 = \sum_{(i,j)\in I} |s_{i,j}|, \tag{75}$$

$\tilde{\mathcal{A}}_s(\rho)$ is a Banach space, which is isomorphic to $\mathcal{A}_s(\rho)$. Clearly, the isomorphism $J : \mathcal{A}_s(\rho) \mapsto \tilde{\mathcal{A}}_s(\rho)$ is an isometry:

$$\|\cdot\|_\rho = |\cdot|_1.$$

We divide the set I in three disjoint parts:

$$\begin{aligned}
I_1 &= \{(i,j)\in I : i+j<N\},\\
I_2 &= \{(i,j)\in I : N\leq i+j<M\},\\
I_3 &= \{(i,j)\in I : i+j\geq M\},
\end{aligned}$$

with

$$N=22, \quad M=60.$$

We will denote the cardinality of the first set as $D(N)$, the cardinality of $I_1\cup I_2$ as $D(M)$.

We assign a single index to vectors $\psi_{i,j}$, $(i,j)\in I_1\cup I_2$, as follows:

$$k(-1,0)=1, \quad k(-1,1)=2, \quad \ldots, \quad k(-1,M)=M+1, \quad k(0,0)=M+2,$$
$$k(0,1)=M+3, \quad \ldots, \quad k\left(\left[\frac{M-1}{2}\right], M-1-\left[\frac{M-1}{2}\right]\right) = D(M).$$

This correspondence $(i,j) \mapsto k$ is one-to-one, we will, therefore, also use the notation $(i(k), j(k))$.

For any $s \in \mathcal{A}_s(\rho)$, we define the following projections on the subspaces of the linear subspace $E_{D(N)}$ spanned by $\{\psi_k\}_{k=1}^{D(N)}$.

$$\Pi_k s = s_{i(k),j(k)} \psi_k, \quad \Pi_{E_{D(N)}} s = \sum_{m \le D(N)} \Pi_m s.$$

Fix

$$c_0 = (s_0 \circ G[s_0])_{0,3},$$

where s_0 is some good numerical approximation of the fixed point. Denote for brevity $\mathcal{L}_c^s \equiv D\mathcal{R}_c[s]$. We can now write a matrix representation of the finite-dimensional linear operator

$$\Pi_{E_{D(N)}} \mathcal{L}_{c_0}^{s_0} \Pi_{E_{D(N)}}$$

as

$$D_{n,m} = \Pi_m \mathcal{L}_{c_0}^{s_0} \psi_n.$$

Step 2). We compute the unit eigenvectors e_k of the matrix D *numerically*, and form a $D(N) \times D(N)$ matrix A whose columns are the approximate eigenvectors e_k. We would now like to find a rigorous bound $\mathbf{B}$ on the inverse B of A.

Let B_0 be an approximate inverse of A. Consider the operator C in the Banach space of all $D(N) \times D(N)$ matrices (isomorphic to $\mathbb{R}^{D(N)^2}$) equipped with the l_1-norm, given by

$$C[B] = (A + \mathbb{I})B - \mathbb{I}.$$

Notice, that if B is a fixed point of C then $AB = \mathbb{I}$. Consider a "Newton map" for C:

$$N[z] = z + C[B_0 - B_0 z] - B_0 + B_0 z.$$

If z is a fixed point of N, then $B_0 - B_0 z$ is a fixed point of C. Furthermore,

$$DN[z] = \mathbb{I} - AB_0$$

is constant. We therefore, estimate l_∞ matrix norms

$$\|N[0]\|_1 \le \equiv \epsilon, \quad \|\mathbb{I} - AB_0\|_1 \le \equiv \mathcal{D},$$

and obtain via the Contraction Mapping Principle, that the inverse of A is contained in the l_1 δ-neighborhood of B_0, with

$$\delta = \|B_0\|_1 \frac{\epsilon}{1 - \mathcal{D}}.$$

Step 3). Define the linear operator

$$\mathcal{A} = A \Pi_{E_{D(N)}} \bigoplus \left(\mathbb{I} - \Pi_{E_{D(N)}}\right),$$

and its inverse

$$\mathcal{B} = B \Pi_{E_{D(N)}} \bigoplus \left(\mathbb{I} - \Pi_{E_{D(N)}}\right).$$

Consider the action of the operator $\mathcal{L}_{c_0}^s$ in the new basis

$$e_k = \frac{\tilde{e}_k}{\|\tilde{e}_k\|_\rho}, 1 \le k \le D(N), \quad e_k \equiv \psi_k, \quad k > D(N),$$

where

$$[e_1, e_2, \ldots, e_{D(N)}] \equiv [\psi_1, \psi_2, \ldots, \psi_{D(N)}] A, \tag{76}$$

in $\mathcal{A}_s(\rho)$. To be specific, we consider a new Banach space $\hat{\mathcal{A}}_s(\rho)$: the space of all functions

$$s=\sum_k c_k e_k,$$

analytic on a bi-disk $\mathcal{D}_\rho$, for which the norm

$$\|s\|_1=\sum_k |c_k|$$

is finite.

For any $s\in\hat{\mathcal{A}}_s(\rho)$, we define the following projections on the basis vectors.

$$P_i s=c_i e_i,\quad P_{>k}s=\left(\mathbb{I}-\sum_{i=1}^{k}P_i\right)s.$$

Clearly, the Banach spaces $\mathcal{A}_s(\rho)$ and $\hat{\mathcal{A}}_s(\rho)$ are isomorphic, while the norms $\|\cdot\|_\rho$ and $\|\cdot\|_1$ are equivalent. We can use (76) to compute the equivalence constant α in

$$\alpha\|\cdot\|_1\geq\|\cdot\|_\rho=|\cdot|_1$$

(recall, norms $\|\cdot\|_\rho$ and $|\cdot|_1$, defined in (75) are equal). Notice, that

$$\begin{aligned}
s &= \sum_k c_k e_k=\sum_{1\leq k\leq D(N)} c_k\left(\sum_{1\leq i\leq D(N)} A^i_k\psi_i\right)+\sum_{k>D(N)} c_k\psi_k\\
&= \sum_{1\leq i\leq D(N)}\left(\sum_{1\leq k\leq D(N)} c_k A^i_k\right)\psi_i+\sum_{i>D(N)} c_i\psi_i,
\end{aligned}$$

therefore, if A^i is the i-th row of the matrix A, then

$$\begin{aligned}
|s|_1 &= \sum_{1\leq i\leq D(N)}\left|\sum_{1\leq k\leq D(N)} c_k A^i_k\right|+\sum_{i>D(N)}|c_i|\\
&\leq \sum_{1\leq i\leq D(N)}\left(\|A^i\|_\infty\sum_{1\leq k\leq D(N)}|c_k|\right)+\sum_{i>D(N)}|c_i|\\
&= \left[\sum_{1\leq i\leq D(N)}\|A^i\|_\infty\right]\sum_{1\leq k\leq D(N)}|c_k|+\sum_{i>D(N)}|c_i|\\
&\leq \max\left\{\sum_{1\leq i\leq D(N)}\|A^i\|_\infty,1\right\}\|s\|_1
\end{aligned}$$

and

$$\alpha=\max\left\{\sum_{1\leq i\leq D(N)}\|A^i\|_\infty,1\right\}.$$

The constant has been rigorously evaluated on the computer:

$$\alpha\leq 49.435546875. \tag{77}$$

The operator $\mathcal{L}^s_{c_0}$ is "almost" diagonal in this new basis for all $s \in \mathcal{B}_\varrho(s_0) \subset \mathcal{A}_s(\rho)$,

$$\varrho = 6.0 \times 10^{-12}.$$

We proceed to quantify this claim.

$$\begin{array}{ll}
\|P_2\mathcal{L}^s_{c_0}e_1\|_1 \ \ \leq 5.19007444381714 \times 10^{-4}, & \|P_1\mathcal{L}^s_{c_0}e_2\|_1 \ \ \leq 1.76560133695602 \times 10^{-4}, \\
\|P_{>2}\mathcal{L}^s_{c_0}e_1\|_1 \leq 3.5819411277771 \times 10^{-3}, & \|P_{>2}\mathcal{L}^s_{c_0}e_2\|_1 \leq 1.49521231651306 \times 10^{-3}, \\
\|P_1\mathcal{L}^s_{c_0}P_{>2}\|_1 \leq 1.22539699077606 \times 10^{-4}, & \|P_2\mathcal{L}^s_{c_0}P_{>2}\|_1 \leq 8.2328915596008310^{-5},
\end{array}$$

for all $h \in \mathcal{B}_\varrho(s_0) \subset \mathcal{A}_s(\rho)$.

Step 4). We will now demonstrate existence of a fixed point $s^*_{c_0}$ in $\mathcal{B}_\varrho \in \mathcal{A}_s(\rho)$, of the operator $\mathcal{R}_{c_0}$, where

$$c = (s_0 \circ G[s_0])_{0,3}.$$

We will use the Contraction Mapping Principle in the following form. Define the following linear operator on $\mathcal{A}_s(\rho)$

$$M \equiv [\mathbb{I} - K]^{-1},$$

where

$$Kh \equiv \hat{\delta}_1 P_1 h + \hat{\delta}_2 P_2 h,$$

and $\hat{\delta}_1$ and $\hat{\delta}_2$ are defined via

$$P_1\mathcal{L}^{s_0}_{c_0}e_1 = \hat{\delta}_1 e_1, \quad P_2\mathcal{L}^{s_0}_{c_0}e_2 = \hat{\delta}_2 e_2.$$

Consider the operator

$$\mathcal{N}[h] = h + \mathcal{R}_{c_0}[s_0 + Mh] - (s_0 + Mh)$$

on $\hat{\mathcal{A}}_s(\rho)$ and for all z.

The operator $\mathcal{N}$ is analytic and compact on $\mathcal{B}_{\|M\|_1^{-1}\alpha^{-1}\varrho}(0)$, where c is the norm equivalence constant (77), and

$$\|M\|_1 = \max\left\{\left|\frac{1}{1-\hat{\delta}_1}\right|, \left|\frac{1}{1-\hat{\delta}_2}\right|, 1\right\} = 1.$$

Notice, that if h^* is a fixed point of $\mathcal{N}$, then $s_0 + Mh^*$ is a fixed point of R_{c_0}.

The derivative norm of the operator $\mathcal{N}$ is "small", indeed,

$$\begin{aligned}
D\mathcal{N}[h] &= \mathbb{I} + D\mathcal{R}_{c_0}[s_0 + Mh] \cdot M - M \\
&= \left[M^{-1} + D\mathcal{R}_{c_0}[s_0 + Mh] - \mathbb{I}\right] \cdot M \\
&= [\mathbb{I} - K + D\mathcal{R}_{c_0}[s_0 + Mh] - \mathbb{I}] \cdot M \\
&= [D\mathcal{R}_{c_0}[s_0 + Mh] - K] \cdot M.
\end{aligned}$$

We have evaluated the operator norm of this derivative for all $h \in \mathcal{B}_{\alpha^{-1}\varrho}(0)$:

$$\|D\mathcal{N}[h]\|_1 \equiv \mathcal{D} \leq 0.1258544921875$$

At the same time

$$\|\mathcal{N}[0]\|_1 = \|\mathcal{R}_{c_0}[s_0] - s_0\|_1 \equiv \epsilon \leq 4.9560546875 \times 10^{-16}.$$

We can now see that the hypothesis of the Contraction Mapping Principle is indeed verified:

$$\epsilon < 4.9560546875 \times 10^{-14} < 1.058349609375 \times 10^{-13} < (1 - \mathcal{D})\alpha^{-1}\varrho,$$

and therefore, the neighborhood $\mathcal{B}_{\epsilon/(1-\mathcal{D})}(0) \subset \mathcal{B}_{0.5\alpha^{-1}\varrho}(0)$ contains a fixed point h^* of $\mathcal{N}$, i.e. the neighborhood $\mathcal{B}_{\varrho/2}(s_0) \subset \mathcal{B}_{\varrho}(s_0) \subset \mathcal{A}_s(\rho)$ contains a fixed point $s^*_{c_0} = s_0 + Mh^*$ of $\mathcal{R}_{c_0}$.

We quote here for reference purposes the bounds on the values of the scalings $\lambda[s^*_c]$ and $\mu[s^*_c]$:

$$\lambda[s^*_c] = [-0.248875288734817765, -0.248875288702286711], \tag{78}$$

$$\mu[s^*_c] = [0.0611101382055370338, 0.0611101382190655586]. \tag{79}$$

Step 5). Notice, that in general,

$$\left(s^*_{c_0} \circ G[s^*_{c_0}]\right)_{0,3} \neq c,$$

therefore

$$t_{c_0}[s^*_{c_0}] \neq 0.$$

However, $t_{c_0}[s^*_{c_0}]$ is a small number which we have estimated to be

$$|t_{c_0}[s^*_{c_0}]| < 7.89560771750566329 \times 10^{-12}. \tag{80}$$

Consider the map $F^*_{c_0}$ generated by $s^*_{c_0}$. Recall that by Theorem 3, there exists a simply connected open set $\mathcal{D}$ such that $F^*_{c_0} \in \mathcal{O}_2(\mathcal{D})$. The fixed point equation for the map $F^*_{c_0}$ is as follows:

$$\Lambda^{-1}_{F^*_{c_0}} \circ S^{-1}_{t_{c_0}[s^*_{c_0}]} \circ F^*_{c_0} \circ F^*_{c_0} \circ S_{t_{c_0}[s^*_{c_0}]} \circ \Lambda_{F^*_{c_0}} = F^*_{c_0}.$$

Suppose that there exists an invertable transformation T_{c_0} such that

$$S_{t_{c_0}[s^*_{c_0}]} \circ \Lambda_{F^*_{c_0}} \circ T_{c_0} = T_{c_0} \circ \Lambda_{F^*_{c_0}} \tag{81}$$

(we will skip the issue of domains for a moment). Then

$$\Lambda^{-1}_{F^*_{c_0}} \circ \tilde{F}^* \circ \tilde{F}^* \circ \Lambda_{F^*_{c_0}} = \tilde{F}^*, \tag{82}$$

on the domain on T_{c_0}, where

$$\tilde{F}^* = T^{-1}_{c_0} \circ F^*_{c_0} \circ T_{c_0}.$$

This $\tilde{F}^*$ is close to a fixed point of the operator $R_{\tilde{c}^*}$ with

$$\tilde{c}^* = \left(\pi_u(\tilde{F}^* \circ \tilde{F}^*)\right)_{(0,3)},$$

the only thing missing is that the rescaling $\Lambda^{-1}_{F^*_{c_0}}$ in the doubling equation (82) is not the one corresponding to $\tilde{F}^*$. To amend this, we rescale

$$F^* \equiv J \circ \tilde{F}^* \circ J^{-1}$$

by a near-identity diagonal transformation

$$J(x,y) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} \frac{\lambda[F^*_{c_0}]}{\lambda[\tilde{F}^*]} & 0 \\ 0 & \frac{\mu[F^*_{c_0}]}{\mu[\tilde{F}^*]} \end{bmatrix}, \tag{83}$$

so that

$$\lambda[F^*_{c_0}] = \pi_x P_{c^*}[F^*](0,0), \quad \mu[F^*_{c_0}] = \frac{-\lambda[F^*_{c_0}]}{\pi_x P_{c^*}[F^*]_2(0,0)},$$

where

$$c^* = \tilde{c}^* b^{-2}.$$

Indeed, then

$$
\begin{aligned}
t_{c^*}[F^*] &= \frac{1}{4}\frac{c^*-\left(\pi_u(J\circ\tilde F^*\circ\tilde F^*\circ J^{-1})\right)_{(0,3)}}{\left(\pi_u(J\circ\tilde F^*\circ\tilde F^*\circ J^{-1})\right)_{(0,2)}}\\
&= \frac{1}{4}\frac{\tilde c^* b^{-2}-b\left(\pi_u(\tilde F^*\circ\tilde F^*)\right)_{(0,3)}b^{-3}}{b\left(\pi_u(\tilde F^*\circ\tilde F^*)\right)_{(0,2)}b^{-2}}\\
&= b^{-1}t_{\tilde c^*}[\tilde F^*]\\
&= 0,
\end{aligned}
$$

$$
\lambda[F^*]=\lambda[F^*_{c_0}],\quad \mu[F^*]=\mu[F^*_{c_0}],
$$

and

$$
\Lambda^{-1}_{F^*}\circ F^*\circ F^*\circ\Lambda_{F^*}=F^*. \tag{84}
$$

Notice, that T_{c_0} in (81) conjugates the transformation $\tilde\Lambda_{F^*_{c_0}}\equiv S_{t_{c_0}[s^*_{c_0}]}\circ\Lambda_{F^*_{c_0}}$ to its linear part $\Lambda_{F^*_{c_0}}$. Since the eigenvalues of $\Lambda_{F^*_{c_0}}$, $\lambda[s^*_{c_0}]$ and $\mu[s^*_{c_0}]$, as given in (78)—(79), are not in resonance, the Sternberg Linearization Theorem guarantees existence of such $T^{-1}_{c_0}$ on a neighborhood of zero, normalized as

$$
T^{-1}_{c_0}(0,0)=(0,0),\quad DT^{-1}_{c_0}(0,0)=\mathbb{I},
$$

and given by

$$
T^{-1}_{c_0}=\lim_{n\to\infty}\Lambda^{-n}_{F^*_{c_0}}\circ\tilde\Lambda^n_{F^*_{c_0}},
$$

while

$$
T_{c_0}=\lim_{n\to\infty}\tilde\Lambda^{-n}_{F^*_{c_0}}\circ\Lambda^n_{F^*_{c_0}}.
$$

We will now look at the domains of convergence of the above transformations. Consider

$$
T_{c,n}=\tilde\Lambda^{-n}_{F^*_{c_0}}\circ\Lambda^n_{F^*_{c_0}},\quad\text{and}\quad T^{-1}_{c,n}=\Lambda^{-n}_{F^*_{c_0}}\circ\tilde\Lambda^n_{F^*_{c_0}}.
$$

Notice, that

$$
S_t\circ\Lambda=\Lambda\circ S_{\lambda t},
$$

therefore,

$$
\begin{aligned}
T^{-1}_{c,n} &= S_{\lambda[F^*_{c_0}]^n t_{c_0}[s^*_{c_0}]}\circ\ldots\circ S_{\lambda[F^*_{c_0}]t_{c_0}[s^*_{c_0}]}\\
T_{c,n} &= S^{-1}_{\lambda[F^*_{c_0}]t_{c_0}[s^*_{c_0}]}\circ\ldots\circ S^{-1}_{\lambda[F^*_{c_0}]^n t_{c_0}[s^*_{c_0}]}.
\end{aligned}
$$

Consider the map

$$
\tilde F^*_n=T^{-1}_{c,n}\circ F^*_{c_0}\circ T_{c,n}.
$$

We use λ for $\lambda[F^*_{c_0}]$ and t for $t_{c_0}[s^*_{c_0}]$ for brevity, and denote

$$
\begin{aligned}
\zeta_t(x) &= x+tx^2,\\
\zeta_t^{-1}(x) &= \frac{\sqrt{1+4tx}-1}{2t},\\
\xi_{t,n} &= \xi_{\lambda^n t}\circ\ldots\circ\xi_{\lambda t},\\
\xi^{-1}_{t,n} &= \xi^{-1}_{\lambda t}-\circ\ldots\circ\xi^{-1}_{\lambda^n t},\\
\xi_{t,0} &\equiv \xi^{-1}_{t,0} id.
\end{aligned}
$$

Notice, $\zeta_t(x) = \pi_x S_t(x,u)$ and $\zeta_t^{-1}(x) = \pi_x S_t^{-1}(x,u)$.

The transformation $\xi_{t,n}$ is analytic on $\mathcal{D}_{\rho'}$, where ρ' is as in Proposition 3.2, and

$$\xi_{t,n}(\mathcal{D}_{\rho'}) \supset \mathcal{D}_{\hat{\rho}},$$

for all $n \geq 0$ where

$$\hat{\rho} = \rho' \prod_{i=0}^{\infty} (1 - |\lambda|^i \kappa) > 1.87249946593321017, \quad \kappa = |\lambda||t|\rho' < 3.67950199077131340 \times 10^{-12}.$$

Therefore,

$$\mathcal{D}_\rho \subset \mathcal{D}_{\hat{\rho}}.$$

Furthermore,

$$\xi_{t,n}(\mathcal{D}_{\rho'}) \subset \mathcal{D}_{\tilde{\rho}},$$

for all $n \geq 0$ where

$$\tilde{\rho} = \rho' \prod_{i=0}^{\infty} (1 + \nu^i \kappa) < 1.87249946595155563, \quad \nu = |\lambda|(1+\kappa) < 0.248875288735733502.$$

Next,

$$\begin{aligned} \|\xi_{t,n+1} - \xi_{t,n}\|_{\rho'} &= \|\xi_{\lambda^{n+1}t} \circ \ldots \circ \xi_{\lambda t} - \xi_{\lambda^n t} \circ \ldots \circ \xi_{\lambda t}\|_{\rho'} \\ &= |\lambda|^{n+1} |t| \|\xi_{\lambda^n t} \circ \ldots \circ \xi_{\lambda t}\|_{\rho'}^2 \\ &\leq |\lambda|^{n+1} |t| \tilde{\rho}^2, \end{aligned}$$

where the norm $\|\cdot\|_{\rho'}$ has been defined in the Definition 1.5. Therefore the uniform limit $\xi_{t,\infty} = \lim_{n\to\infty} \xi_{t,n}$ exists, and is analytic on $\mathcal{D}_{\rho'}$.

We will now turn to the bounds on the parameter a in (83). Notice, that since $T_{c,n}^{-1}$ is of the form

$$T_{c,n}^{-1}(x,u) = (\pi_x \xi_{t,n}(x,0), \ldots),$$

we have

$$\lambda[\tilde{F}^*] = \pi_x T_{c_0}^{-1} \circ F_{c_0}^* \circ F_{c_0}^*(0,0) = \pi_x \xi_{t,\infty}(\lambda[F_{c_0}^*], 0),$$

and

$$\begin{aligned} |\lambda[F_{c_0}^*] - \lambda[\tilde{F}^*]| &\leq \|\xi_{t,\infty} - id\|_{\rho'} \leq \sum_0^\infty \|\xi_{t,n+1} - \xi_{t,n}\|_{\rho'} \leq \sum_0^\infty |\lambda|^{n+1} |t| \tilde{\rho}^2 \\ &= \frac{|\lambda|}{1 - |\lambda|} |t| \tilde{\rho}^2 \leq = \iota. \end{aligned}$$

Therefore,

$$|a - 1| = \left| \frac{\lambda[F_{c_0}^*] - \lambda[\tilde{F}^*]}{\lambda[\tilde{F}^*]} \right| \leq \frac{\iota}{|\lambda[F_{c_0}^*]| - \iota} \leq = \chi.$$

Next, we consider the domains of analyticity of the inverse transformations. First, for all t as in (80), and ρ' as in Proposition 3.2,

$$\zeta_t^{-1}(\mathcal{D}_{\rho'}) \subset \mathcal{D}_{\rho' + |t|\rho'^2}, \tag{85}$$

therefore,

$$\|\xi_{t,n}^{-1}\|_{\rho'} \leq \bar{\rho},$$

where

$$\bar{\rho} = \rho' \prod_{i=1}^{\infty} (1 + |\bar{\lambda}|^i |t| \rho'), \quad \bar{\lambda} = (1 + |t|\rho))\lambda.$$

At the same time

$$\|\xi_{t,n}^{-1}\|_\rho < \rho'',$$

where

$$\rho'' = \rho \prod_{i=1}^{\infty} (1 + |\bar{\lambda}|^i |t| \rho) < 1.75000000000801182 < \rho',$$

therefore

$$\xi_{t,n}^{-1}(\mathcal{D}_\rho) \subset \mathcal{D}_{\rho'}.$$

We, therefore, have

$$\begin{aligned} \|\xi_{t,n+1}^{-1} - \xi_{t,n}^{-1}\|_\rho &= \|\xi_{\lambda t}^{-1} \circ \ldots \circ \xi_{\lambda^{n+1} t}^{-1} - \xi_{\lambda t}^{-1} \circ \ldots \circ \xi_{\lambda^n t}^{-1}\|_\rho \\ &\leq \|D\xi_{t,n}^{-1}\|_{\rho + \lambda^{n+1}|t|\rho} \lambda^{n+1} |t| \rho^2 \\ &\leq \frac{C}{\rho' - \rho - \lambda^{n+1}|t|\rho} \|\xi_{t,n}^{-1}\|_{\rho'} \lambda^{n+1} |t| \rho^2 \\ &\leq \frac{C}{\rho' - \rho - \lambda^{n+1}|t|\rho} \rho' \lambda^{n+1} |t| \rho^2, \end{aligned}$$

and $\xi_{t,n}^{-1}$ converges uniformly on $\mathcal{D}_\rho$ to a limit transformation $\xi_{t,\infty}^{-1}$, analytic on $\mathcal{D}_\rho$:

$$\xi_{t,\infty}^{-1}(\mathcal{D}_\rho) \subset \mathcal{D}_{\rho'}. \tag{86}$$

Next, we obtain a bound on the rescaling parameter b in (83):

$$\begin{aligned} \left| \frac{1}{\mu[F_{c_0}^*]} - \frac{1}{\mu[\tilde{F}^*]} \right| &= \left| \frac{\pi_x \left(F_{c_0}^* \circ F_{c_0}^*\right)_2 (0,0)}{\lambda[F_{c_0}^*]} - \frac{\pi_x \left(\tilde{F}^* \circ \tilde{F}^*\right)_2 (0,0)}{\lambda[\tilde{F}^*]} \right| \\ &= \left| \frac{1}{\mu[F_{c_0}^*]} - \frac{\pi_x D T_{c_0}^{-1}(\lambda[F_{c_0}^*], 0) \cdot D\left(F_{c_0}^* \circ F_{c_0}^*\right)(0,0) \cdot \begin{bmatrix} (\partial_2 \pi_x T_{c_0})(0,0) \\ (\partial_2 \pi_u T_{c_0})(0,0) \end{bmatrix}}{\lambda[\tilde{F}^*]} \right| \\ &= \left| \frac{1}{\mu[F_{c_0}^*]} - \frac{\pi_x \left(\xi_{t,\infty}\right)_1 (\lambda[F_{c_0}^*], 0) \pi_x \left(F_{c_0}^* \circ F_{c_0}^*\right)_2 (0,0)}{\lambda[\tilde{F}^*]} \right| \\ &= \left| \frac{1}{\mu[F_{c_0}^*]} - \frac{a \pi_x \left(\xi_{t,\infty}\right)_1 (\lambda[F_{c_0}^*], 0) \pi_x \left(F_{c_0}^* \circ F_{c_0}^*\right)_2 (0,0)}{\lambda[F_{c_0}^*]} \right| \\ &= \left| \frac{1}{\mu[F_{c_0}^*]} \right| \left| 1 - a \pi_x \left(\xi_{t,\infty}\right)_1 (\lambda[F_{c_0}^*], 0) \right| \\ &\leq \left| \frac{1}{\mu[F_{c_0}^*]} \right| \left\{ \left| 1 - \pi_x \left(\xi_{t,\infty}\right)_1 (\lambda[F_{c_0}^*], 0) \right| + |1 - a| \left| \pi_x \left(\xi_{t,\infty}\right)_1 (\lambda[F_{c_0}^*], 0) \right| \right\}, \end{aligned}$$

where we have used $DT_{c_0}(0,0) = \mathbb{I}$. Now, write

$$\pi_x \xi_{t,\infty}(x, 0) = x + \sum_{n=2}^{\infty} c_n x^n,$$

then

$$
\begin{aligned}
|1-\pi_x\left(\xi_{t,\infty}\right)_1\left(\lambda[F^*_{c_0}],0\right)| &= \left|\sum_{n=2}^{\infty} n c_n \lambda[F^*_{c_0}]^{n-1}\right| \le \frac{1}{|\lambda[F^*_{c_0}]|}\max_{n\ge 2}\left\{ n\frac{|\lambda[F^*_{c_0}]|^n}{\rho'^n}\right\}\|\xi_{t,\infty}-id\|_{\rho'} \\
&\le \frac{1}{|\lambda[F^*_{c_0}]|} N \frac{|\lambda[F^*_{c_0}]|^N}{\rho'^N}\|\xi_{t,\infty}-id\|_{\rho'} \le = \upsilon,
\end{aligned}
$$

where

$$
N=\max\left\{\frac{1}{\ln\frac{\rho'}{|\lambda[F^*_{c_0}]|}},2\right\}=2.
$$

Therefore

$$
|1-b|\le \upsilon+|1-a|(1+\upsilon)\le \upsilon+\chi(1+\upsilon)\le=\omega.
$$

Finally, we revisit the inclusion (86). Denote $a(x)=ax$, then

$$
\|\xi^{-1}_{t,n}\circ a^{-1}\|_\rho=\|\xi^{-1}_{t,n}\circ\|_{|a^{-1}|\rho}<\rho'',
$$

where

$$
\rho''=\rho\prod_{i=1}^{\infty}(1+|\bar\lambda|^i|t||a^{-1}|\rho)<1.750001<\rho',
$$

and, the transformation $\xi^{-1}_{t,\infty}\circ a^{-1}$, again, maps $\mathcal{D}_\rho$ into $\mathcal{D}_{\rho'}$:

$$
\xi^{-1}_{t,\infty}\left(a^{-1}\left(\mathcal{D}_\rho\right)\right)\subset\mathcal{D}_{\rho'}.
$$

We will now write the composition $T^{-1}_{c,n}\circ F^*_{c_0}\circ T_{c,n}$ above via generating functions:

$$
\begin{aligned}
&\begin{pmatrix} \zeta_{t,n}(x') \\ \frac{-s(y',x')}{\prod_{i=1}^n(1+2\lambda^i t\zeta_{\lambda^{i-1}t}(x'))}\end{pmatrix} = \\
&= \begin{pmatrix} \zeta_{\lambda^n t}\circ\ldots\circ\zeta_{\lambda t}(x') \\ \frac{-s(y',x')}{\prod_{i=1}^n(1+2\lambda^i t\zeta_{\lambda^{i-1}t}(x'))}\end{pmatrix} \overset{S^{-1}_{\lambda^n t}}{\mapsto} \ldots \overset{S^{-1}_{\lambda^3 t}}{\mapsto} \begin{pmatrix} \zeta_{\lambda^2 t}(\zeta_{\lambda t}(x')) \\ \frac{-s(y',x')}{(1+2\lambda t x')(1+2\lambda^2 t\zeta_{\lambda t}(x'))}\end{pmatrix} \\
&\overset{S^{-1}_{\lambda^2 t}}{\mapsto} \begin{pmatrix} \zeta_{\lambda t}(x') \\ \frac{-s(y',x')}{(1+2\lambda t x')}\end{pmatrix} \overset{S^{-1}_{\lambda t}}{\mapsto} \begin{pmatrix} x' \\ -s(y',x')\end{pmatrix} \overset{F}{\mapsto} \begin{pmatrix} y' \\ s(x',y')\end{pmatrix} \overset{S_{\lambda t}}{\mapsto} \begin{pmatrix} \zeta_{\lambda t}(y') \\ \frac{s(x',y')}{(1+2\lambda t y')}\end{pmatrix} \overset{S_{\lambda^2 t}}{\mapsto} \\
&\begin{pmatrix} \zeta_{\lambda^2 t}(\zeta_{\lambda t}(y')) \\ \frac{s(x',y')}{(1+2\lambda t y')(1+2\lambda^2 t\zeta_{\lambda t}(y'))}\end{pmatrix} \overset{S_{\lambda^3 t}}{\mapsto} \ldots \overset{S_{\lambda^n t}}{\mapsto} \begin{pmatrix} \zeta_{\lambda^n t}\circ\ldots\circ\zeta_{\lambda t}(y') \\ \frac{s(x',y')}{\prod_{i=1}^n(1+2\lambda^i t\zeta_{\lambda^{i-1}t}(y'))}\end{pmatrix} = \\
&= \begin{pmatrix} \zeta_{t,n}(y') \\ \frac{s(x',y')}{\prod_{i=1}^n(1+2\lambda^i t\zeta_{\lambda^{i-1}t}(y'))}\end{pmatrix}
\end{aligned}
$$

Now, let $x'=\zeta^{-1}_{t,n}(x)$, then

$$
\begin{pmatrix} x \\ \frac{-s(\xi^{-1}_{t,n}(y,x))}{\prod_{i=1}^n(1+2\lambda^i t\zeta^{-1}_{\lambda^{i-1}t}(x))}\end{pmatrix} \overset{T_{c,n}}{\mapsto} \begin{pmatrix} \zeta^{-1}_{t,n}(x) \\ -s(\xi^{-1}_{t,n}(y,x))\end{pmatrix} \overset{F}{\mapsto} \begin{pmatrix} \zeta^{-1}_{t,n}(y) \\ s(\xi^{-1}_{t,n}(x,y))\end{pmatrix} \overset{T^{-1}_{c,n}}{\mapsto} \begin{pmatrix} y \\ \frac{s(\xi^{-1}_{t,n}(x,y))}{\prod_{i=1}^n(1+2\lambda^i t\zeta^{-1}_{\lambda^{i-1}t}(y))}\end{pmatrix},
$$

and

$$
\frac{s^*_{c_0}(\xi^{-1}_{t,n}(x,y))}{\prod_{i=1}^n(1+2\lambda^i t\zeta^{-1}_{\lambda^{i-1}t}(y))}
$$

is the generating function of $\tilde F^*_n$ if if $s^*_{c_0}$ is that for $F^*_{c_0}$.

By Proposition 3.2, $s^*_{c_0}$ is in $\mathcal{A}(\rho')$. Since $\xi^{-1}_{t,\infty}(a^{-1}(\mathcal{D}_{1.75})) \subset \mathcal{D}_{\rho'}$, the function $s^*_{c_0} \circ \xi^{-1}_{t,\infty} \circ a^{-1}$ is analytic on $\mathcal{D}_{1.75}$.

Furthermore, since

$$2|\lambda||t|(|a^{-1}|\rho + |t||a^{-2}|\rho^2) < 1$$

for all t as in (80) (cf. (85)), the generating function of F^*,

$$s^* = \frac{b s^*_{c_0}(\xi^{-1}_{t,\infty}(a^{-1}x, a^{-1}y))}{\prod_{i=1}^{\infty}(1 + 2\lambda^i t \zeta^{-1}_{\lambda^{i-1}t}(a^{-1}y))},$$

is in $\mathcal{A}_s(1.75)$.

In particular,

$$\begin{aligned} \|s^*_{c_0} - s^*\|_\rho &\le \max\left\{ \left\| s^*_{c_0} - \frac{b s^*_{c_0}}{\prod_{i=1}^{\infty}(1+2|\lambda^i||t||a^{-1}|\rho')} \right\|_\rho, \left\| s^*_{c_0} - \frac{b s^*_{c_0}}{\prod_{i=1}^{\infty}(1-2|\lambda^i||t||a^{-1}|\rho')} \right\|_\rho \right\} \\ &+ \frac{|b| \left\| D s^*_{c_0} \right\|_{\rho'}}{\prod_{i=1}^{\infty}(1-2|\lambda^i||t||a^{-1}|\rho')} \max\left\{ \prod_{i=1}^{\infty}(1+2|\lambda^i||t||a^{-1}|\rho') - 1, 1 - \prod_{i=1}^{\infty}(1-2|\lambda^i||t||a^{-1}|\rho') \right\} \\ &\le 1.04748302248271977 \times 10^{-10} \equiv \beta', \end{aligned}$$

and

and since $0.5\varrho + \beta < r$, where r is as in the Main Theorem 1,

$$s^* \in \mathcal{B}_r(s_0).$$

Step 6). At the last step we repeat the calculations of the bound on the operator $\mathcal{L}^s_c$ in Step 3) for all $c \in I$,where the interval I,

$$I = \left[c_0 - \frac{r}{\rho^3}, c_0 + \frac{r}{\rho^3} \right],$$

contains c^*.

An almost-diagonal linear operator P transforms the operator $\mathcal{L}^s_c$ in a block-diagonal form

$$P^{-1}\mathcal{L}^s_c P = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \mathcal{L}^s_{>2} \end{bmatrix},$$

with

$$\begin{aligned} \delta_1 &\in (8.72021484375, 8.72216796875), \\ \|\mathcal{L}^s_{>2}\| &\le 0.1258544921875, \end{aligned}$$

for all $s \in \mathcal{B}_r(s_0) \subset \mathcal{A}_s(\rho)$ and $c \in I$.

Now, it follows immediately that

$$R_{\mathrm{spec}}(D\mathcal{R}_{c^*}[s^*]|_{T_{s^*}\mathcal{W}_{\mathcal{R}_{c^*}}(s^*)}) \le 0.1258544921875, \tag{87}$$

where $\mathcal{W}_{\mathcal{R}_{c^*}}(s^*)$ is the local stable manifold of $\mathcal{R}_{c^*}$ at s^*.

This, together with Lemma 3.4 implies that the same is true for the spectral radius of $D\mathcal{R}[s^*]$

□

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Programs available at `http://math.uu.se/~gaidash`
Programs available at `http://www.math.cornell.edu/~tjohnson`

Department of Mathematics, Uppsala University, Uppsala, Sweden, `gaidash@math.uu.se`

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA, `tomas.johnson@cornell.edu`

The Mathematics Department, Stony Brook University, Stony Brook NY, USA, `marco@math.sunysb.edu`