Triviality of fibers for Misiurewicz parameters in the exponential family

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Abstract

We consider the family of holomorphic maps $e^z + c$ and show that fibers of postcritically finite parameters are trivial. This can be considered as the first and simplest class of non-escaping parameters for which we can obtain triviality of fibers in the exponential family.

Introduction

Among transcendental functions, the complex exponential family $e^z + c$ is one of the most widely studied examples, because there is only one singular value at c and because in some respects it can be seen as combinatorial limit of unicritical polynomials of degree d (See e.g.[3]).
The foundational work has been set by Eremenko and Lyubich [6] and Baker and Rippon [1], has been carried over by Devaney and coauthors, to flow eventually into an extensive combinatorial study carried out mainly by Rempe, Schleicher and Zimmer; many other aspects, like ergodic property, have also been investigated by various groups of people; for a review of exponential dynamics and a more complete set of references see [2].
Many results, like the theory of parabolic bifurcations, have been approached in analogy with the more mature theory of quadratic polynomials. A rich class of results which presents difficulties in being generalized are rigidity results; in this class, most results for polynomials involve some version of Yoccoz puzzle and estimates on the modulus of the annuli between puzzle

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pieces, but for the exponential family most dynamically arising objects including puzzle pieces are unbounded, so that the corresponding annuli are degenerate at infinity. An additional difficulty due to the same unboundedness property is that the conjugacy between the original map and the tentative renormalized map does not extend to the boundary of its domain of definition [13].
On the other side, topological properties of Misiurewicz parameters for polynomials have been studied extensively [5]; for triviality of fibers see for example [23] and [20].
A study about general properties of fibers for the exponential family as compared to the polynomial family has been carried out in [17], however Misiurewicz parameters are the first class of non-escaping exponential maps for which it was actually possible to show triviality of fibers. Our main result is stated as follows:

Theorem 1. *Fibers of Misiurewicz parameters in parameter space are trivial, i.e. given any postcritically finite parameter c_0, for any other parameter c which does not belong to one of the finitely many parameter rays landing at c_0 there is a pair of parameter rays with periodic addresses landing together at a parabolic parameter which separate c from c_0.*

We will devote the first section to a collection of relevant results about existence and landing properties of dynamic and parameter rays for the exponential family. The second section will introduce Misiurewicz parameters and their combinatorial properties, followed by a section on ray portraits where we will prove some explicit theorem about the correspondence of ray portraits between exponentials and polynomials. After that we will give a short introduction to fibers and rigidity, and in the last section we will present the statement and the proof of theorem 1.
Remark about notation: we will refer as Π_P to the parameter plane and as Π_c to the dynamical plane for the parameter c.
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1 Dynamic and parameter rays

This section has the purpose of recollecting some of the relevant results about rays and their landing properties.
Rays for the exponential family have been introduced in analogy with the polynomial case in order to construct symbolic dynamics on the set of escaping points

$$I = \{z, |f_c^n(z)| \to \infty\} \subset \Pi_c.$$

In other words we want to subdivide I into connected sets g_s^c (our rays) indicized by a sequence s, called *address*, so that $f_c(g_s^c) = g_{\sigma s}^c$ where σ is the left shift; if we look at e^z as limit of unicritical polynomials of degree d, for which rays are parametrized by angles written in d-adyc expansion, it would be natural to label the exponential rays by sequences over the integers.
In addition to the transversal direction parametrized by addresses we can model the dynamics along the rays (the radial direction) with the function $F : \mathbb{R} \mapsto \mathbb{R}$, $F : t \mapsto e^t - 1$, so that the previous relation among the rays becomes

$$f_c(g_c^c(t)) = g_{\sigma s}^c(F(t)).$$

The natural construction [4] is to divide the plane into strips S_j on which e^z is univalent, for example under the preimages of the positive semiaxis which passes through c; label the strips with integers respecting the vertical order, for example

$$S_j = \{z, \operatorname{Im} z \in ((2j-1)\pi, (2j+1)\pi)\}$$

and consider itineraries of points with respect to this partition, i.e.

$$\operatorname{itin}(z) = s_1 s_2 \ldots \text{ iff } f^j(z) \in S_{s_j}.$$

For points whose jth iterate belongs to the boundary separating two strips S_{s_j} and $S_{s_{j+1}}$ the corresponding entry in the address will be the boundary symbol $\binom{s_j}{s_{j+1}}$.
It would be tempting to define the ray of address $s = s_1 s_2 \ldots$ as the set of points whose itinerary is s, but this would introduce the unnatural condition that rays can't cross the boundaries of the partition creating disconnected sets. The actual construction uses this guideline to first define rays for large real part and then use inverse dynamics to extend them to their maximal length choosing branches of f^{-1} in order to preserve continuity of the curve.

Note that by the above construction we cannot realize as a ray any sequence whose entries grow faster than iterates of the exponential function, leading to the following notion:

Definition 1. A sequence $s = s_1 s_2 \dots$ is said *exponentially bounded* if $\exists\, x \in \mathbb{R},\ |2\pi s_j| < F^j(x)\ \forall j$.

This condition turns out to be not only necessary but also sufficient [21], so that we can consider all sequences s contained in the set

$$\mathcal{S} = \{s = s_1 s_2 ... \in \mathbb{Z}^{\mathbb{N}} \text{ such that } s \text{ is exponentially bounded}\}$$

and state the following theorem ([21]):

Theorem 2. Existence of dynamic rays *Given a non-escaping parameter c, for any $s \in \mathcal{S}$ there exists a real number t_s and a curve $g_s^c : (t_s, \infty) \to \mathbb{C}$ such that*

- *$g_s^c(t)$ has external address s for sufficiently large t*
- *$f_c(g_s^c(t)) = g_{\sigma s}^c(F(t))$*
- *We have the asymptotics $g_s^c(t) = 2\pi i s_1 + t + o(e^{-t})$*

The question whether periodic rays land for the exponential family remained open for some time, and was finally solved by Rempe using the previously known fact that periodic rays land for hyperbolic parameters and an argument about persistence of landing inside wakes. This led to the following theorem ([14]):

Theorem 3. Landing theorem for periodic dynamic rays *Let c be such that the singular value c of f_c does not escape to infinity. Then every periodic dynamic ray g_s^c lands at a repelling or parabolic periodic point.*

Note that as preperiodic points are preimages of periodic points this automatically implies that preperiodic rays also land and that, like for polynomials, when c is escaping the only exceptions are rays which are preimages of the ray containing the singular value.
The construction of parameter rays is also done keeping in mind the fundamental property of parameter rays that we have for polynomials: a point c belongs to some parameter ray G_s in Π_P if and only if it belongs to the dynamic ray g_s^c in Π_c.
It is carried out by Forster and Schleicher and is summarized in the following theorem about existence of parameter rays:

Theorem 4. Existence of parameter rays*([8],prop 2.2). Let $s \in \mathcal{S}$.Then for any t bigger than some t_s, there exist a unique paramter $c = G_s(t)$ such that $c = g_s^c(t)$.*
The map $G_s : (t_s, \infty) \to \mathbb{C}$ is continuous, and $|G_s(t) - (t + 2\pi i s_1)| \to 0$ as $t \to \infty$.

Not much is known about the landing properties of parameter rays, however the basic result that we will need, proved in [19], is that periodic and preperiodic parameter rays land at parabolic or Misiurewicz parameters respectively [19].
Now that we recollected the essential general results about rays and their landing properties we will move to the specific class of parameters that we are interested in, Misiurewicz parameters.

2 Misiurewicz parameters

We call a parameter c_0 *Misiurewicz* if it is postsingularly finite, so that the singular value c_0 lands at some repelling orbit $\{z_i\}$ of period m after k steps. From the definition above and the discreteness of solutions of the equation $f_c^{k+m}(c) = f_c^k(c)$ it follows immediately that Misiurewicz parameters belong to the bifurcation locus; it has been also proved that for such parameters the Julia set is equal to $\mathbb{C}$.
There are a few reasons why proving rigidity for Misiurewicz parameters is easier than the other cases. Among them, Schleicher and Zimmer have proved in [21] that such a parameter c_0 is the landing point of exactly q parameter rays whose addresses $s_1 < ... < s_q$ are preperiodic of period mq and preperiod k, and that the dynamic rays with the corresponding addresses land at c_0 in Π_{c_0}:

Theorem 5. Dynamical-parameter plane correspondence at Misiurewicz points *A preperiodic parameter ray lands at some Misiurewicz parameter c_0 iff the dynamic ray with the same address lands at c_0 in Π_{c_0}.*

This theorem by itself expresses a form of combinatorial similarity between parameter and dynamical plane at Misurewicz points, and together with the generalization of Thurston's rigidity theorem for exponentials ([10]) and a subsequent work ([11]), gives a combinatorial classification of postsingularly finite exponential maps expressed by the following theorem (thm 2.6 in [11]):

Theorem 6. Classification of Misiurewicz exponential maps *For every preperiodic external address s, there is a unique postsingularly finite exponential map such that the dynamic ray at external address s lands at the singular value. Every postsingularly finite exponential map is associated in this way to a positive finite number of preperiodic external addresses.*

This classification mimics the corresponding classification of postsingularly finite unicritical polynomials, so that the theorem above offers a natural correspondence between exponential Misiurewicz parameters and polynomial Misiurewicz parameters which will be exploited later.
Before exploring further the consequences of the combinatorial classification of Misiurewicz exponential maps, let us mention that the second main ingredient in proving triviality of fibers is offered by the linearizing coordinates which give contraction under the inverse map in a neighborhood of the postsingular periodic orbit.

A combinatorial property of Misiurewicz parameters One of the features of Misurewicz parameters that we are going to use in the proof of our main theorem is a lemma connecting topology to combinatorics, proven in [21] for exponentials and probably well known for unicritical poynomials of degree D; for completeness we will include a proof following the outline of [22].
Let $f = e^z + c_0$ or $f = z^D + c_0$ where c_0 is a Misiurewicz parameter, and choose one of the finitely many dynamic rays landing at c_0, say g_{s_1}, where s_1 is the address/angle respectively. Then topologically the preimage of g_{s_1} under f is a set of countably many curves going to $-\infty$ in the case of exponentials, and a set of D curves connecting at 0 for polynomials of degree D; in both cases, those first preimages of g_{s_1} partition the plane into open regions W_j. Similarly the preimages of s_1 under the shift map partition into the same number of sectors the combinatorial space, which for exponentials is given by all exponentially bounded sequences over the integers, and for polynomials of degree D is given by sequences over D symbols.
Label with the entry 0 the dynamical and the combinatorial sector containing c_0 and s_1 respectively.
Any ray g_s which is not a preimage of g_{s_1} has a well defined itinerary whose entries keep track of the sectors visited by iterates of s under the shift map.
In order to use a hyperbolic contraction argument we need a picture which is forward invariant, so we also need to remove from $\mathbb{C}$ the finitely many

forward images of g_{s_1}, obtaining a subpartition of the plane into regions $\hat{W}_{i,j}$ where each region $\hat{W}_{i,j} \subset W_i$.

Lemma 7. Significance of dynamical partition for Misiurewicz parameters *Two rational rays which are not preimages of the rays landing at the singular value land together iff they have the same itinerary with respect to the dynamical partition described above*

Proof. Given a region W_i, we can choose a branch of f_i^{-1} mapping $\mathbb{C}$ into W_i, so that the same branch restricted to any $W_{j'}$ contracts the hyperbolic metric from $W_{j'}$ to W_i. Similarly each $\hat{W}_{i,j}$ carries its own hyperbolic metric which is bigger than the metric of W_i, and the restriction of f_i^{-1} to any $\hat{W}_{i,j}$ contracts the hyperbolic metric of that $\hat{W}_{i,j}$.
Let us start by considering any two periodic rays which have the same itinerary; they land at two points w_1 and w_2 which are periodic, so that up to selecting branches they are both fixed under some M-th iterate f^{-M} of the inverse of f.
The periodic points w_1 and w_2 have the same itinerary under $f^k,\ k = 1...m$ so at each step we can select the same branch of f_{-1} and get hyperbolic contraction along the backward orbits until we get back to w_1 and w_2 decreasing hyperbolic distance, which is a contradiction unless $w_1 = w_2$ to start with.
This proves the theorem for periodic rays unless the iterates of w_1 and w_2 always belong to different connected components of $W_j - \cup_n f^n(g_{s_1})$ (i.e to different $\hat{W}_{i,j}$). So suppose that w_1 and w_2 belong to the same W_i but to different $\hat{W}_{i,j}$; then at least one of them, say w_1, belongs to one of the internal sectors defined in the section about orbit portraits and originating at some point z of the postsingular periodic orbit: the dynamics permutes those sectors transitively, so each image $f^n(w_1)$ belongs to the same W_i as $f^n(z)$, hence w_1 has the same itinerary as z and, as w_2 has the same itinerary as w_1, it also has the same itinerary as z.
Remains to prove that no periodic point w_i can have the same itinerary as some postcritical periodic point z. The family of inverse iterates is normal in a neighborhood of z, and is defined in a connected set containing w_i because the two points have the same itinerary. As the iterates converge to the constant map contracting everything to z in a linearizing neighborhood for z, they converge to the same map in the entire domain where they are defined, contradicting the fact that w_i is fixed under some appropriate iterate of the

inverse function.
If the rays are preperiodic and have the same itinerary their periodic images also have the same itineray, hence land together by previous part; and since our preperiodic rays are not preimages of the rays landing at the singular value, and they have the same itinerary, we can take pullbacks using the same branch for both, so that they keep landing together.
On the other side if two rays land together they form a connected set, which never intersects the original partition under iterates of f, so they always belong to the same region of the partition.

$\square$

3 Combinatorics and Ray Portraits

This section introduces ray portraits (in analogy with [12] and with [16]) and studies the correspondence between rational parameter rays for polynomials and rational parameter rays for exponentials. The theorems about this correspondence are probably known or not surprising to people in the field but we could not find a precise reference in the literature.

Definition 2. We call a *ray pair* any couple of rational rays landing together. When we refer to a ray pair as a couple of addresses or angles, we mean the ray pair corresponding to that couple of angles. A *rational ray pair* is a ray pair whose addresses or angles are either periodic or preperiodic.

We will define the following distance between two sequences $l = l_1 l_2 \dots$ and $s = s_1 s_2 \dots$:

$$\operatorname{dist}(l, s) = \sum_{s_k \neq l_k} \frac{1}{2^k}$$

Definition 3. Let $\{z_i\}_{i=1 \cdots n}$ be a periodic repelling or parabolic orbit of period n in Π_c, and $A_i = \{r \in \mathcal{S}, r$ is periodic and g^c_s lands at $z_i\}$. Then $\mathcal{P} = \{A_1, \cdots A_n\}$ is said to be the *orbit portrait* for $\{z_i\}$.
$\mathcal{P}$ is called *essential* if each A_i contains at least two addresses, *satellite* if the addresses form only one cycle and *primitive* otherwise.

Theorem 8. Basic properties of orbit portraits [16] *Given an orbit portrait $\mathcal{P}$, all A_i's contain a finite number of addresses, and the shift map sends A_i bijectively onto A_{i+1}. All addresses share the same period qn.*

Theorem 9. Misiurewicz addresses for exponentials and polynomials *The parameter rays $G_{s_1} \dots G_{s_q}$ land together at some exponential Misiurewicz parameter iff they land together at some polynomial Misiurewicz parameter in the parameter plane for unicritical polynomials of sufficiently high degree D.*

Proof. All the rays landing together at some Misiurewicz parameter c_0 in exponential parameter plane have the same itineraries with respect to the dynamical partition induced by g_{s_1} in Π_{c_0}. The address s_1 is preperiodic, so it is a sequence over finitely many values, so for polynomials of sufficiently high degree there's a Misiurewicz parameter c_1 which is the landing point of the corresponding parameter ray.
All the polynomial dynamic rays $g_{s_2} \dots g_{s_q}$ also have the same itinerary with respect to the partition induced by g_{s_1} so they all land together in the dynamical plane for the polynomial whose critical value is c_1, hence the corresponding parameter rays land together at c_1 in the polynomial parameter plane. □

Definition 4. Given a ray portrait, the *characteristic rays* are the rays which separate the singular value from all other rays in the portrait

Definition 5. A *characteristic ray pair* is a pair of periodic rays G_{s_1}, G_{s_2} landing together in parameter plane. By [16], the dynamic rays of addresses s_1, s_2 also land together for parameters in the wake defined by G_{s_1}, G_{s_2}. We will call s_1, s_2 *characteristic addresses.*

By [16], the two definitions above coincide.

Remark 1. It follows from the theory of parabolic bifurcation worked out in [16] that for any parameter c there's a 1-to-1 correspondence between characteristic ray pairs in Π_c and the periodic parameter ray pairs separating the parameter c from the period one component.

Theorem 10. Correspondence of characteristic rays *A pair of rays is characteristic for exponentials iff it is characteristic for some unicritical polynomial of some degree D.*

Proof. Given a combinatorial ray portrait, by vertical/cyclic order of rays the correct topological picture is already encoded in the combinatorial portrait. For polynomials we know that the singular value always belongs to the sector with smallest width; this implies the corresponding result for exponentials,

because due to branching only the preimage of the singular sector is bounded by rays whose addresses differ in the first entry by one, and it is encoded in the combinatorics which sector maps to which other sector.
We want to show that a ray pair which is characteristic for polynomials also is characteristic for exponentials. We only need to show that all polynomial portraits are realized for exponentials as well and vice versa, because once we know that the portrait is realized this determines the characteristic sector. Any portrait for polynomials persists in the whole wake bounded by its characteristic adresses so in particular it is realized for some polynomial Misiurewicz parameter as well.
Then by lemma 7 the correspondent Misiurewicz parameter in the exponential family also realizes that orbit portrait, because the rays exist and the pattern in which they land together is encoded in the itineraries with respect to the Misiurewicz partition described in the previous section. The proof that Misiurewicz portraits are realized for polynomials is symmetric once you observe that we have only finitely many addresses in any portrait hence finitely many entries. □

4 Fibers and Rigidity

One of the main problems in one dymensional complex dynamics is to show that the set of structurally stable parameters consists only of hyperbolic components. If there was a non hyperbolic component all maps in a neighborhood would be conjugated, so that any two maps in the component would have exactly the same set of ray portraits; by the theory of parabolic bifurcation in [16], this means that two parameters in the same non hyperbolic components could not be separated by a parameter ray pair, or otherwise one of the two would have an additional ray portrait. This leads to the following definitions:

Definition 6. The *parameter fiber* of a parameter c_0 is the set of parameters which cannot be separated from c_0 by some pair of (pre)periodic parameter rays landing together at some parabolic or Misiurewicz parameter, or by two periodic parameter rays landing at the boundary of the same hyperbolic component.
By analogy, the *dynamical fiber* of a point c_0 is the set of points which cannot be separated from c_0 by some pair of (pre)periodic rays landing together at some (pre)periodic point.

Definition 7. We will say that the fiber of a point c_0 in dynamical/parameter space is *trivial*, if all other points can be separated from c_0 via a pair of rational rays landing together, except for the rays which might land at the point c_0 itself.

Definition 8. We will call any result about triviality of fibers a *rigidity result*. This comes from the fact that any map whose singular value does not escape and with trivial fiber can not be conjugated to any other map in a neighborhood because two maps with different ray portraits can not be topologically conjugated.

From the above definitions it follows immediately the implication below (see again [17] for a slightly different formulation of this discussion)

Theorem 11. *If the fiber of every non hyperbolic parameter with non escaping singular value is trivial, then there are no non hyperbolic components in the set of structurally stable parameters*

There are two main points in considering fibers to study density of hyperbolicity: for the exponential case, periodic parameter rays for exponentials are closely related to parameter rays for unicritical polynomials (see thm 10), so that it is possible to infer results about exponentials using known results about polynomials; the second one, and more general one, is that fibers are a way to "localize" the global conjecture, and select specific classes of parameters which are easier to study.
Our combinatorial rigidity statement 1 deals exactly with the easiest class of parameters, we restate it here for convenience:

Theorem 12. *Fibers of Misiurewicz parameters in parameter space are trivial, i.e. given any postcritically finite parameter c_0, for any other parameter c which does not belong to one of the finitely many parameter rays landing at c_0 there is a pair of parameter rays with periodic addresses landing together at a parabolic parameter which separate c from c_0.*

5 Triviality of Misiurewicz fibers

In this section we prove theorem 12. The proof follows the outline of the corresponding result for polynomials (lemma 7.1 and thm 7.3 in [20]), using theorem 9 to establish a bridge between the combinatorics for polynomials

and the combinatorics for exponentials.
In particular we will prove the exponential version of lemma 7.1 in [20], whose statement is exactly the same except for replacing polynomial maps with exponential maps:

Proposition 13. Combinatorial approximation of parameter rays *Let $G_{s_1}...G_{s_q}$ be the parameter rays landing at c_0. We will approximate each sector by ray pairs arbitrarily close to the rays landing at c_0. This means that $\forall \epsilon > 0$ there exist parameter ray pairs α_i, α'_i depending on ϵ such that $s_i < \alpha_i < \alpha'_i < s_{i+1}$ for $i = 1...q-1$ and $dist(\alpha_i, s_i) < \epsilon$, $dist\ (\alpha'_i, s_{i+1}) < \epsilon$; we also want a parameter ray pair α_0, α'_0 such that $\alpha_0 < s_1 < s_q < \alpha'_0$ and $dist(\alpha_0, s_1) < \epsilon$, $dist\ (\alpha'_0, s_q) < \epsilon$*

There's a crucial point here: at first sight it might seem that this proposition would solve our problem, but the relation between the "combinatorial topology" and the topology on $\mathbb{C}$ are far from clear, so we still have to show that those rays which approximate the Misiurewicz rays combinatorially actually converge to them in $\mathbb{C}$ topology in a neighborhood of c_0. We will derive this from the following propositions:

Proposition 14. Triviality in dynamical plane *Dynamical fibers of the postsingular periodic orbit are trivial*

Proposition 15. Persistence of dynamical triviality *Dynamical fibers of the analytic continuation of the postsingular periodic orbit are trivial for parameters belonging to some neighborhood of c_0*

At this point we will be able to prove our final theorem (equivalent to 12)

Theorem 16. Triviality of Misiurewicz fibers *Any parameter c can be separated from c_0 by a parameter ray pair, except for those parameters lying on the rays G_{s_i} landing at c_0*

Now let us prove the propositions above and the main theorem.
Proof of proposition 13: Combinatorial approximation of parameter rays

Proof. The core of the proof relies on the correspondence between combinatorics for polynomials and for exponential parameters: the idea is that the angles labeling rays for polynomials of degree D are written in D-adic expansion as sequences with D symbols, which can be seen as a subset of the exponentially bounded sequences encoding the combinatorics for exponential maps.

Consider the dynamic rays of addresses $s_1 \dots s_q$ landing at our Misiurewicz parameter c_0. As noted in lemma 7, each one of them defines a partition with respect to which other rays have the same itinerary if and only if they land together in dynamical plane.

By the 2π vertical periodicity of parameter plane we can restrict ourselves to Misiurewicz parameters whose addresses only have nonnegative entries. So consider the Misiurewicz polynomial $z^D + c$ of degree D, where D is the maximal entry in $\{s_1 \dots s_q\}$ for which the rays at angles $s_1 \dots s_q$ lands at c , and go to the corresponding dynamical plane. Note that the address s_1 is preperiodic, so by classification of Misiurewicz polynomials there is a polynomial such that the dynamic ray at angle s_1 lands at the singular value, and that all other angles $s_2 \dots s_q$ have the same itinerary with respect to s_1. This means that by lemma 7 the rays corresponding to those addresses all land together, and no other ray can land together with them otherwise its angle would be an admissible sequence for exponentials and would have the same itinerary, so the corresponding ray would land together with $g_{s_1} \dots g_{s_q}$ in the exponential dynamical plane as well.

There by lemma 7.1 in [20] we have characteristic periodic dynamic ray pairs approximating each sector arbitrarily close, and the two rays in each ray pair have the same itinerary again by lemma 7; this is a purely combinatorial notion, so that it carries over to exponentials and the ray pairs with the same addresses keep landing together in the dynamical plane for $e^z + c_0$, giving the wanted approximating couples of rays in the dynamical plane.

Now we want to transfer those approximating ray pairs in the parameter plane for exponential maps. By remark 1 we can transfer the couples of characteristic rays , so the ray pairs approximating the 'sector' between s_1 and s_q carry over straightforwardly to the parameter plane.

To approximate the other parameter sectors as well, fix a sector, say the sector betweem G_{s_1} and G_{s_2}, call it $s_1 \hat{} s_2$

Let $V \subset \Pi_P$ be a neighborhood of c_0 such that there is an analytic continuation $\tilde{z}(c)$ of c_0 which keeps all the rays landing at c_0, and pick a Misiurewicz parametes c in $V \cap s_1 \hat{} s_2$. In Π_c we will have the same relative position between $\tilde{z}$ and c as we have in parameter plane between c_0 and c, in the sense that c in Π_c belongs to the sector defined by the rays of addresses s_1 and s_2: this follows from the fact that rays respect the vertical order induced by their addresses both in dynamical and in parameter plane.

Lemma 7.1 in [20] gives us characteristic dynamic ray pairs approximating $g^c_{s_1}$ and $g^c_{s_2}$ for polynomials (now $g^c_{s_1}$ and $g^c_{s_2}$ are landing at the repelling point $\tilde{z}(c)$, not at the singular value c); the corresponding rays can be obtained in the exponential dynamical plane by the same technique described above, and they can be transfered in parameter plane by remark 1.

□

Note that this proposition proves that we can separate a Misiurewicz parameter from all other Misiurewicz parameters, and from any parameter which is described combinatorially, for example parabolic and escaping parameters and landing points of parameter rays.

Remark 2. By the correspondence of characteristic ray pairs between polynomials and exponentials as stated in theorem 10, we could have obtained the combinatorial approximation directly in the parameter plane, but we need it also in dynamical plane in order to prove that dynamical fibers of the postsingular orbit are trivial and to proceed with the topological part of the proof.

Proof of prop 14: Triviality in dynamical plane

Proof. Let z be the first periodic point in the postsingular orbit, and L its linearizing neighborhood. Taking the kth image of the approximating ray pairs found in the proof of proposition 13 we obtain dynamic ray pairs which approximate combinatorially the q rays $g_{s'_1} \dots g_{s'_q}$ landing at z. We want to show that this combinatorial separation corresponds to an actual separation of all points in L from z.
So for each sector defined by the $g_{s'_i}$ consider an approximating ray pair which enters V. Note that such a ray pairs must exist, because it is known that $J = \overline{\{\text{Set of escaping points}\}} = \mathbb{C}$, which means there are no open sets containing no escaping points, so at least one ray must enter each sector, and once we have a ray inside we can surround it by one of the combinatorially approximating ray pairs.
So each sector contains a ray pair, and the region between that ray pair and the boundaries of the sector is uniformly contracted under f^{-m}, where m was the period of the orbit, so that the region left out by the approximating ray pairs shrinks to $\{z\}$.

□

Proof of proposition 15: Persistence of dynamical triviality

Proof. Let $\{p_i\}_{i=1\ldots q}$ be the landing points of the ray pairs which enter the linearizing neighborhood in the proof of propositon 14; then we can find a parameter neighborhood V of c_0 in which we can continue analytically both c_0, the p_i's and the postsingular periodic orbit $\{z_i\}$ with the same rays landing at them.
Up to shrinking V, we can also assume that the rays enter the new linearizing neighborhood, and by contraction under the inverse map the neighborhoods between the approximating rays and the actual rays landing at the analytic continuation of the z_i shrink to points. By carrying the approximating rays forward we obtain that the dynamical fiber of every $z_i(c)$ is trivial. □

Proof of theorem 16: Triviality of Misiurewicz fibers

Proof. We want to find a parameter neighborhood V of c_0 so that every $c \in V$ can be separated from c_0 by some parameter ray pair.
Note that it is enough to separate from c_0 any parameter c in the bifurcation locus, as rays cannot cross non-hyperbolic components.
Essentially we want to use propositions 14 and 15 to show that the combinatorially approximating ray pairs found in proposition 13 do converge on the rays landing at c_0 in the complex plane, so that the regions which we can separate combinatorially actually fill in the whole neighborhood $V - \cup G_{s_i}$.
Like we did before in dynamical plane, let us distinguish the cases in which the parameter c that we want to separate from c_0 is in the external sector which contains $-\infty$ (the one bounded by G_{s_1} and G_{s_q}) and the case in which c belongs to some of the other internal sector.

If c belongs to the external sector, consider the dynamical plane for c_0, and separate c from c_0 there by a preperiodic dynamic ray pair as from proposition 14 follows directly that the dynamical fiber of c_0 is trivial. Now separate this preperiodic ray pair by by one of the approximating characteristic ray pairs found in proposition 13 and then transfer this characteristic ray pair into parameter plane by remark 1. Note that c in general does not have a ray landing at it. However the parameter ray pair and c keep the same relative position in parameter plane that had c and the corresponding dynamic rays in Π_{c_0}: c crosses a dynamic ray pair iff it crosses the corresponding parameter ray pair, because inside the wake things can be moved holomorphically. Also note that in this case we have not restricted c to any neighborhood of c_0.

If c belongs to one of the internal sectors, say $s_1 \hat{} s_2$, and also belongs to the neighborhood V as in proposition 15 then consider the dynamical plane Π_c. There, c belongs to the corresponding dynamical sector $s_1 \hat{} s_2$ defined at the analytic continuation $\tilde{z}(c)$. The dynamical fiber of $\tilde{z}$ is trivial by proposition 15, so we can separate c and $\tilde{z}(c)$ by some periodic ray pair (α, α'). This ray pair is persistent over a parameter neighborhood U of c. This means that, for the parameters in this neighborhood, in dynamical plane the singular value will be inside the sector bounded by the dynamical rays (α, α'). In particular, by vertical order, escaping parameters in this neighborhood lie on a dynamic ray of address between α and α' in dynamical plane, so they lie on a parameter ray of address between α and α'. By the combinatorial approximation given by proposition 13, such a parameter is separated from c_0 by any of the ray pairs whose address are closer to s_1 and s_2 than α and α'. This means that we can separate all those escaping parameters simultaneously from c_0 using the same ray pair (β, β'). By density of escaping points in the bifurcation locus, we can approximate c by escaping parameters, so the ray pair (β, β') also separates c_0 from c unless c lies on β or β' in which case it has a well defined address and can be separated from c_0 by any ray pair closer than β or β'. $\square$

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