# DYNAMIC RAYS OF BOUNDED-TYPE ENTIRE FUNCTIONS

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ABSTRACT. We construct an entire function in the Eremenko-Lyubich class $\mathcal{B}$ whose Julia set has only bounded path-components. This answers a question of Eremenko from 1989 in the negative.

On the other hand, we show that for many functions in $\mathcal{B}$, in particular those of finite order, every escaping point can be connected to $\infty$ by a curve of escaping points. This gives a partial positive answer to the aforementioned question of Eremenko, and answers a question of Fatou from 1926.

## 1. INTRODUCTION

The dynamical study of transcendental entire functions was initiated by Fatou in 1926 [F]. As well as being a fascinating field in its own right, the topic has recently received increasing interest partly because transcendental phenomena seem to be deeply linked with the behavior of polynomials in cases where the degree gets large. A recent example is provided by the surprising results of Avila and Lyubich [AL], who showed that a constant-type Feigenbaum quadratic polynomial whose Julia set has positive measure would have hyperbolic dimension less than two. This phenomenon occurs naturally in transcendental dynamics, see [UZ]. Other interesting applications of transcendental dynamics include the study of the standard family of circle maps and the use of Newton's method to study zeros of transcendental functions.

In his seminal 1926 article, Fatou observed that the Julia sets of certain explicit entire functions, such as $z \mapsto r\sin(z)$, $r \in \mathbb{R}$, contain curves of points which escape to infinity under iteration. He then remarks

> Il serait intéressant de rechercher si cette propriété n'appartiendrait pas à des substitutions beaucoup plus générales. [1]

Sixty years later, Eremenko [E] was the first to undertake a thorough study of the *escaping set*

$$I(f) := \{z \in \mathbb{C} : |f^{\circ n}(z)| \to \infty\}$$

of an arbitrary entire transcendental function. In particular, he showed that every component of $\overline{I(f)}$ is unbounded, and asks whether in fact each component of $I(f)$ is

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[1] "It would be interesting to investigate whether this property holds for much more general functions."

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unbounded. We will call this problem (the weak form of) *Eremenko's conjecture*. He also states that

> It is plausible that the set $I(f)$ always has the following property: every point $z \in I(f)$ can be joined with $\infty$ by a curve in $I(f)$.

This can be seen as making Fatou's original question more precise, and will be referred to in the following as the *strong form* of Eremenko's conjecture.

These problems are of particular importance since the existence of such curves can be used to study entire functions using combinatorial methods. This is analogous to the notion of "dynamic rays" of polynomials introduced by Douady and Hubbard [DH], which has proved to be one of the fundamental tools for the successful study of polynomial dynamics. Consequently, Fatou's and Eremenko's questions are among the most prominent open problems in the field of transcendental dynamics.

We will show that, in general, the answer to Fatou's question (and thus also to Eremenko's conjecture in its strong form) is negative, even when restricted to the *Eremenko-Lyubich class* $\mathcal{B}$ of entire functions with a bounded set of singular values. (For such functions, all escaping points lie in the Julia set. The class $\mathcal{B}$ appears to be a very natural setting for this type of problem; compare also [R1].)

**1.1. Theorem** (Entire Functions Without Dynamic Rays).
*There exists a hyperbolic entire function $f \in \mathcal{B}$ such that every path-connected component of $J(f)$ is bounded.*

*Remark.* In fact, it is even possible to ensure that $J(f)$ contains no nontrivial curves at all (Theorem 8.4.)

On the other hand, we show that the strong form of Eremenko's conjecture does hold for a large subclass of $\mathcal{B}$. Recall that $f$ has *finite order* if $\log\log|f(z)| = O(\log|z|)$ as $|z| \to \infty$.

**1.2. Theorem** (Entire Functions With Dynamic Rays).
*Let $f \in \mathcal{B}$ be a function of finite order, or more generally a finite composition of such functions. Then every point $z \in I(f)$ can be connected to $\infty$ by a curve $\gamma$ such that $f^{\circ n}|_{\gamma} \to \infty$ uniformly.*

*Remark.* Observe that while $\mathcal{B}$ is invariant under finite compositions, the property of having finite order is not.

Theorem 1.2 applies to a large and natural class of functions, extending considerably beyond those which were previously studied by similar methods. As an example, suppose that $p$ is a polynomial with connected Julia set, and let $\alpha$ be a repelling fixed point of $p$. By a classical theorem of Königs, $p$ is conformally conjugate to the linear map $z \mapsto \mu z$ (where $\mu = p'(\alpha)$ is the multiplier of $\alpha$). The inverse of this conjugacy extends to an entire function $\psi : \mathbb{C} \to \mathbb{C}$ with $\psi(0) = \alpha$ and $\psi(\mu z) = f(\psi(z))$. This map $\psi$, called a *Poincaré function*, has finite order. The set of singular values of $\psi$ is the postcritical set of $p$, which is bounded since $J(p)$ is connected. So $\psi \in \mathcal{B}$. The properties of $\psi$ are generally quite different from e.g. those of the commonly considered exponential and trigonometric functions; for example, if $\mu \notin \mathbb{R}$, then the *tracts* of $\psi$ will spiral near infinity. (Compare Figure 1.)

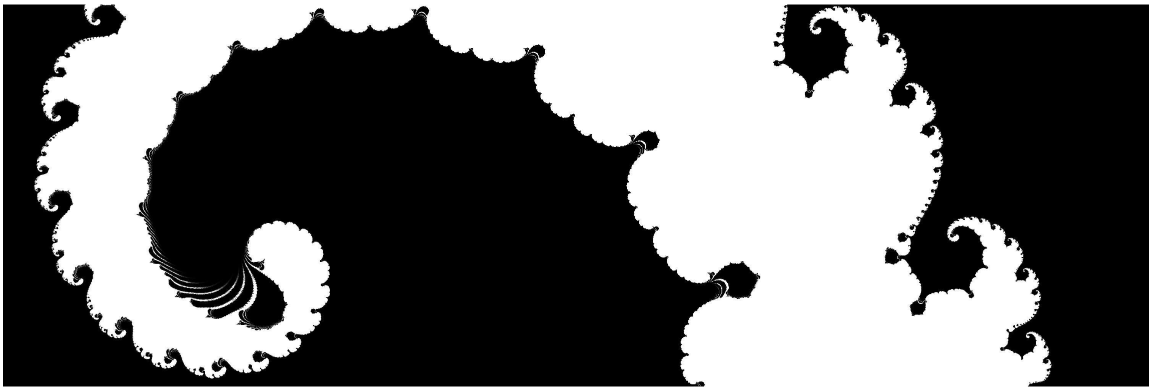

FIGURE 1. Julia set for the Poincaré function around a repelling fixed point of a postcritically-finite quadratic polynomial. This function belongs to $\mathcal{B}$ and has finite order; hence Theorem 1.2 applies to it. (The Julia set, plotted in black, is nowhere dense, but some details are to fine to be visible; this results in the appearance of solid black regions in the figure.)

More generally, given any point $z \in J(p)$, we can find a sequence of rescalings of iterates of $p$ which converges to an entire function $f : \mathbb{C} \to \mathbb{C}$ of finite order by the "Zalcman lemma". Again, since $J(p)$ is connected, we have $f \in \mathcal{B}$. Theorem 1.2 implies that, for all such functions, and their finite compositions, the escaping set consists of rays.

On the other hand, the counterexample from Theorem 1.1 can be constructed such that $\log\log|f(z)| = O(|\log z|^{1+\varepsilon})$ (see Proposition 8.3), so Theorem 1.2 is not far from being optimal.

We remark that our methods are purely local in the sense that they apply to the dynamics of a — not necessarily globally defined — function within any number of logarithmic singularities over $\infty$. Roughly speaking, let $f$ be a function defined on a union of unbounded Jordan domains, such that $f$ is a universal covering of $\{|z| > R\}$ on each of these domains, and extends continuously to the closure of its domain in $\mathbb{C}$. (In fact, our setting is even slightly more general than this; see Section 2 for the class of function we consider.) We will provide sufficient conditions which ensure that every point $z \in I(f)$ eventually maps into a curve in $I(f)$ ending at $\infty$. In particular, we will show that these conditions are satisfied if $f$ has finite order of growth. (Our treatment will also allow us to discuss more generally under which conditions *individual* escaping components, identified by their *external addresses*, are curves to $\infty$.)

For meromorphic functions, we have the following corollary.

**1.3. Corollary** (Meromorphic Functions With Logarithmic Singularities). *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function of finite order.*

(a) *Suppose that $f$ has only finitely many poles and the set of finite singular values of $f$ is bounded. Then every escaping point of $f$ can be connected to $\infty$ or to a pre-pole of $f$ by a curve consisting of escaping points.*

(b) *Suppose that $f$ has a logarithmic singularity over $\infty$. Then $J(f)$ contains uncountably many curves to $\infty$ consisting of escaping points.*

*Remark.* The second part of the corollary applies e.g. to the classical $\Gamma$-function, which has infinitely many poles (at the negative integers), but a logarithmic singularity to the right.

We note finally that our results also apply to the setting of "random iteration" (see [C]). For example, consider a sequence $\mathcal{F} = (f_0, f_1, f_2, \dots)$, where the $f_j$ are finite-order entire functions chosen from some given finite subset of $\mathcal{B}$. If we consider the functions $\mathcal{F}_n := f_n \circ \dots \circ f_0$, and define $I(\mathcal{F}) := \{z \in \mathbb{C} : \mathcal{F}_n(z) \to \infty\}$, then every point of $I(\mathcal{F})$ can be connected to infinity by a curve in $I(\mathcal{F})$.

**Previous results.** In the early 1980s, Devaney gave a complete description of the Julia set of an exponential map with an attracting fixed point; that is, $z \mapsto \lambda \exp(z)$ with $\lambda \in (0, 1/e)$ (see [DK]). This was the first entire function for which it was discovered that the escaping set (and in fact the Julia set) consists of curves to $\infty$. Devaney, Goldberg and Hubbard [DGH] proved the existence of certain curves to $\infty$ in $I(f)$ for arbitrary exponential maps $z \mapsto \lambda \exp(z)$ and championed the idea that these should be thought of as analogs of dynamic rays for polynomials. Devaney and Tangerman [DT] generalized this result to a large subclass of $\mathcal{B}$, namely those functions whose *tracts* (see Section 2) are similar to those of the exponential map. (This includes virtually all functions in the Eremenko-Lyubich class which one can explicitly write down.) It seems that it was partly these developments that led Eremenko to pose the above-mentioned questions in his 1989 paper.

More recently, it was shown in [SZ] that every escaping point of every exponential map can be connected to $\infty$ by a curve consisting of escaping points. This was the first time that a complete classification of all escaping points, and with it a positive answer to both of Eremenko's questions, was given for a complete parameter space of transcendental functions. This result was carried over to the cosine family $z \mapsto a \exp(z) + b \exp(-z)$ in [RoS].

After our proof of Theorem 1.2 was first announced, Baranski [B1] obtained a proof of this result for hyperbolic finite-order functions $f \in \mathcal{B}$ whose Fatou set consists of a single basin of attraction. Together with more recent results [R1] on the escaping dynamics of functions in the Eremenko-Lyubich class, this provides an alternative proof of our theorem when $f$ is of finite order.

A very interesting and surprising case in which the weak form of Eremenko's conjecture can be resolved was discovered by Rippon and Stallard [RiS]. They showed that the escaping set of a function with a multiply-connected wandering domain consists of a single, unbounded, connected component. (Such functions never belong to the Eremenko-Lyubich class $\mathcal{B}$.) In fact, they showed that, for any transcendental entire function, the subset $A(f) \subset I(f)$ introduced by Bergweiler and Hinkkanen [BH] has only unbounded components. Also, recently [R2] the weak form of Eremenko's conjecture was established for functions $f \in \mathcal{B}$ whose postsingular set is bounded (which applies, in particular, to the hyperbolic counterexample constructed in Theorem 1.1).

There has been substantial interest in the set $I(f)$ not only from the point of its topological structure, but also because of its interesting properties from the point of view of Hausdorff dimension. The reasoning is often parallel, and progress on the topology of $I(f)$ has entailed progress on the Hausdorff dimension. For many functions $f$ the set $I(f)$ is an uncountable union of curves, each of which is homeomorphic to either $\mathbb{R}^+$ (a dynamic ray) or $\mathbb{R}_0^+$ (a dynamic ray which lands at an escaping point). Often, the Hausdorff dimension of all the rays is 1, while the endpoints alone have Hausdorff dimension 2, or even infinite planar Lebesgue measure. The first time this "dimension paradox" was discovered was by Karpińska for hyperbolic exponential maps [K] for which the topology of Julia sets was known, using results of McMullen [Mc]. In later extensions for arbitrary exponential maps [SZ] and for the cosine family [RoS], the new parts were the topological classifications; while analogous results on the Hausdorff dimension followed from the methods of Karpińska and McMullen; see also [S1, S2] for extreme results where every point in the complex plane is either on a dynamic ray (whose dimension is still one) or a landing point of those rays (so the landing points of this one-dimensional set of rays is the entire complex plane with only a one-dimensional set of exceptions). Recently, it was shown by Baranski [B2] that the dimension paradox also occurs for finite-order entire transcendental functions which are hyperbolic with a single basin of attraction. In fact, the Hausdorff dimension of $I(f)$ is two for any entire function $f \in \mathcal{B}$ of finite order, which follows from Baranski's result by [R1] and was also shown directly and independently by Schubert. The Hausdorff dimension of the set $I(f)$ in the setting of the present paper is further investigated in [ST].

**Structure of the article.** In Section 2, we define logarithmic coordinates, in which we will perform most of our constructions. Some elementary properties of functions in logarithmic coordinates are proved in Section 3. In Section 4, we show that the escaping set of a function in logarithmic coordinates consists of arcs if the escaping points can be ordered according to their "speed" of escape. We call this property the *head-start condition*. Classes of functions that satisfy this condition, in particular the logarithmic transforms of finite order functions, are discussed in Section 5.

In Section 6, we construct a function in logarithmic coordinates whose escaping set has only bounded path-components and in Section 7, we show how to translate this result into a bounded-type entire function. In an appendix, we recall several important facts from hyperbolic geometry.

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**Notation.** Throughout this article, we denote the Riemann sphere by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the right half plane by $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Also, we write

$$B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\} \quad \text{and} \quad \mathbb{H}_R := \{z \in \mathbb{C} : \operatorname{Re} z > R\} .$$

If $A \subset \mathbb{C}$, the closures of $A$ in $\mathbb{C}$ and $\hat{\mathbb{C}}$ are denoted $\overline{A}$ and $\widehat{A}$, respectively.

Euclidean length and distance are denoted $\ell$ and dist, respectively. If a domain $V \subset \mathbb{C}$ omits at least two points of the plane, we similarly denote hyperbolic length and distance in $V$ by $\ell_V$ and $\operatorname{dist}_V$.

We conclude any proof by the symbol $\blacksquare$. The proofs of separate claims within an argument are concluded by $\triangle$.

## 2. The Eremenko-Lyubich Class $\mathcal{B}$ and the Class $\mathcal{B}_{\log}$

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. We call a point $w \in \mathbb{C}$ a *singular value* of $f$ if for every open neighborhood $U$ of $w$, there exists a component $V$ of $f^{-1}(U)$ such that $f : V \to U$ is not bijective. We denote the set of singular values by $S(f)$. Note that $S(f)$ is the closure of the set $\operatorname{sing}(f^{-1})$ of critical and asymptotic values. We say that $f$ belongs to the *Eremenko-Lyubich class* $\mathcal{B}$ if $S(f)$ is bounded. By $J(f)$ we denote the *Julia set* of $f$, i.e. the set of points at which the sequence of functions $\{f, f \circ f, \dots, f^{\circ n}, \dots\}$ does not form a normal family in the sense of Montel.

**Tracts.** Let $f \in \mathcal{B}$, and let $R_0 > |f(0)| + \max\{|s| : s \in \operatorname{sing}(f^{-1})\}$. Setting $W_{R_0} := \{z \in \mathbb{C} : |z| > R_0\}$, it is easy to see that every component $V$ of

$$\mathcal{V} := f^{-1}(W_{R_0})$$

is an unbounded Jordan domain, and that $f : V \to W_{R_0}$ is a universal covering. (In other words, $f$ has only logarithmic singularities over $\infty$.) The components of $\mathcal{V}$ are called the *tracts* of $f$. Observe that each compact $K \subset \mathbb{C}$ will intersect at most finitely many tracts of $f$.

**Logarithmic coordinates.** To study logarithmic singularities, it is natural to apply a logarithmic change of coordinates (compare [EL, Section 2]). More precisely, let $\varrho_0 := \log R_0$, $\mathcal{T} := \exp^{-1}(\mathcal{V})$ and

$$\mathbb{H}_{\varrho_0} := \exp^{-1}(W_{R_0}) = \{z \in \mathbb{C} : \operatorname{Re} z > \varrho_0\} .$$

Then there is a function $F : \mathcal{T} \to \mathbb{H}_{\varrho_0}$ (the *logarithmic transform* of $f$) such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "F"] \arrow[d, "\exp"'] & \mathbb{H}_{\varrho_0} \arrow[d, "\exp"] \\
\mathcal{V} \arrow[r, "f"'] & W_{R_0}.
\end{tikzcd}$$

The components of $\mathcal{T}$ are called the *tracts* of $F$.

By construction, the function $F$ and its domain $\mathcal{T}$ have the following properties, see also Figure 2:

(a) every component $T$ of $\mathcal{T}$ is an unbounded Jordan domain with real parts bounded below, but unbounded from above;

(b) $\overline{\mathcal{T}}$ can be written as the disjoint union

$$\overline{\mathcal{T}} = \bigcup_{T \text{ component of } \mathcal{T}} \overline{T}.$$

(c) for every component $T$ of $\mathcal{T}$, $F : T \to \mathbb{H}_{\varrho_0}$ is a conformal isomorphism. In particular, $F$ extends continuously to the closure $\overline{T}$ of $T$ in $\mathbb{C}$;
(d) for every component $T$ of $\mathcal{T}$, $\exp|_T$ is injective;
(e) $\mathcal{T}$ is invariant under translation by $2\pi i$;
(f) $|F'(z)| \geq 2$, provided that $R_0$ was chosen large enough.

Property (f) is a simple application of Koebe's 1/4-theorem, see [EL, Lemma 2.1]. In the following, we will refer to this property as *expansivity* of $F$. Expansivity and convexity of $\mathbb{H}$ imply that if $z, w \in T$, then $|F(z) - F(w)| \geq 2|z - w|$. Furthermore, applying the change of variable $w = z/R_0$, we may suppose without loss of generality that $R_0 = 1$; i.e., $\varrho_0 = 0$.

Note that for every tract $T$ of $F$, $F|_T$ is unique up to translation by $2\pi i n_T$, where $n_T \in \mathbb{Z}$ can be chosen independently for each tract $T$.

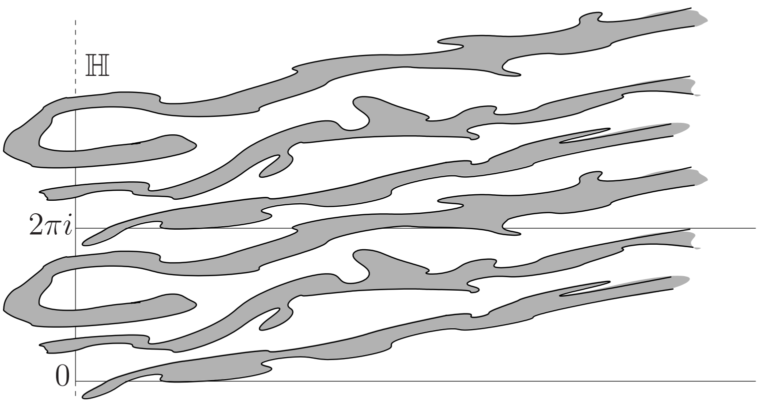


FIGURE 2. The domain of definition for a function $F \in \mathcal{B}_{\log}$ is $2\pi i$-periodic. Note that a tract $T$ need not be contained in $\mathbb{H}$.

We will denote by $\mathcal{B}_{\log}$ the class of all $F : \mathcal{T} \to \mathbb{H}$ such that $\mathcal{T}$ and $F$ satisfy (a) to (f), regardless of whether they arise from an entire function $f \in \mathcal{B}$ or not. In particular, if $f : \mathbb{C} \to \hat{\mathbb{C}}$ is any meromorphic function which has one or several logarithmic singularities over $\infty$, then we can associate to $f$ a function $F \in \mathcal{B}_{\log}$ which encodes the behavior of $f$ near its logarithmic singularities.

If $F \in \mathcal{B}_{\log}$ and $T$ is a tract of $F$, we denote the inverse of the conformal isomorphism $F : T \to \mathbb{H}$ by $F_T^{-1}$.

**Combinatorics in $\mathcal{B}_{\log}$.** Let $F \in \mathcal{B}_{\log}$; we denote the Julia set and the set of escaping points of $F$ by

$$J(F) := \{z \in \overline{\mathcal{T}} : F^{\circ n}(z) \text{ is defined and in } \overline{\mathcal{T}} \text{ for all } n \geq 0\} \quad \text{and}$$
$$I(F) := \{z \in J(F) : \operatorname{Re} F^{\circ n}(z) \to \infty\}.$$

If $f \in \mathcal{B}$ and $F$ is its logarithmic transform, then it is not hard to show that $\exp(J(F)) \subset J(f)$ (the Julia set of $f$) and $\exp(I(F)) \subset I(f)$. Furthermore, every escaping point of $f$ eventually maps to some point in $\exp(I(F))$. For $K > 0$ we also define more generally

$$J^K(F) := \{z \in \overline{\mathcal{T}} : F^{\circ n}(z) \text{ is defined and } \operatorname{Re} F^{\circ n}(z) \geq K \text{ for all } n \geq 0\} .$$

The partition of the domain of $F$ into tracts suggests a natural way to assign symbolic dynamics to points in $J(F)$. More precisely, let $z \in J(F)$ and, for $j \geq 0$, let $T_j$ be the tract of $F$ with $F^{\circ j}(z) \in \overline{T}_j$. Then the sequence

$$\underline{s} := T_0 T_1 T_2 \ldots$$

is called the *external address* of $z$. More generally, we refer to any sequence of tracts of $F$ as an external address (of $F$). If $\underline{s}$ is such an external address, we define the closed set

$$J_{\underline{s}} := \{z \in J(F) : z \text{ has address } \underline{s}\} ;$$

we define $I_{\underline{s}}$ and $J^K_{\underline{s}}$ in a similar fashion (note that $J_{\underline{s}}$, and hence $I_{\underline{s}}$ and $J^K_{\underline{s}}$, may well be empty for some addresses).

We denote the one-sided *shift-operator* on external addresses by $\sigma$; i.e. $\sigma(T_0 T_1 T_2 \ldots) = T_1 T_2 \ldots$.

**2.1. Definition** (Dynamic Rays, Ray Tails)**.**
*Let $F \in \mathcal{B}_{\log}$. A* ray tail *of $F$ is an injective curve*

$$\gamma : [0, \infty) \to I(F)$$

*such that* $\lim_{t\to\infty} \operatorname{Re} F^{\circ n}(\gamma(t)) = +\infty$ *for all* $n \geq 0$ *and such that* $\operatorname{Re} F^{\circ n}(\gamma(t)) \to +\infty$ *uniformly in $t$ as $n \to \infty$.*

*Likewise, we can define ray tails for an entire function $f$. A* dynamic ray *of $f$ is then a maximal injective curve $\gamma : (0, \infty) \to I(f)$ such that $\gamma|_{[t,\infty)}$ is a ray tail for every $t > 0$.*

In Sections 4 and 5, we will construct ray tails for certain functions in class $\mathcal{B}_{\log}$, and in particular for logarithmic transforms of the functions treated in Theorem 1.2 and Corollary 1.3. By the following fact, this will be sufficient to complete our objective.

**2.2. Proposition** (Escaping Points on Rays)**.**
*Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let $z \in I(f)$. Suppose that some iterate $f^{\circ k}(z)$ is on a ray tail $\gamma_k$ of $f$. Then one of the following holds:*

(a) *$z$ is on a dynamic ray,*
(b) *$z$ is the landing point of a dynamic ray, or*
(c) *for some $n \leq k$, $f^{\circ n}(z)$ belongs to a dynamic ray which contains an asymptotic value of $f$.*

*In particular, there is a curve $\gamma_0$ connecting $z$ to $\infty$ such that $f^{\circ j}|_{\gamma_0}$ tends to $\infty$ uniformly (in fact, $f^{\circ k}(\gamma_0) \subset \gamma_k$).*

*Proof.* Let the ray tail be parametrized as $\gamma_k : [0, \infty) \to \mathbb{C}$. Let $T \in (0, \infty]$ be maximal such that there is a curve $\gamma_{k-1} : [0, T) \to \mathbb{C}$ with $\gamma_{k-1}(0) = f^{\circ(k-1)}(z)$ and $f(\gamma_{k-1}(t)) = \gamma_k(t)$. If $T = \infty$, then clearly $\gamma_{k-1}(t) \to \infty$ as $t \to \infty$. Otherwise, $w =$

$\lim_{t\to T}\gamma_{k-1}(t)$ exists in $\hat{\mathbb{C}}$. If $w \neq \infty$, we could extend $\gamma_{k-1}$ further (choosing any one of the possible branches of $f^{-1}$ in the case where $w$ is a critical point), contradicting maximality of $T$. Thus $w = \infty$ (and, in particular, $\gamma_k(T)$ is an asymptotic value of $f$).

In either case, we have found a curve $\gamma_{k-1} \subset f^{-1}(\gamma_k) \subset I(f)$ which connects $f^{\circ(k-1)}(z)$ to infinity. This curve is a ray tail, except possibly if $\gamma_k$ contained an asymptotic value of $f$. Continuing this method inductively, we are done. (Note that, if $z$ is on a ray tail, then $z$ is on a dynamic ray or the landing point of such a ray.) ■

## 3. General Properties of Class $\mathcal{B}_{\log}$

In this section, we prove some general results for functions in class $\mathcal{B}_{\log}$. The first of these strengthens the aforementioned expansion estimate of [EL, Lemma 2.1] by showing that such a function expands distances exponentially.

**3.1. Lemma** (Exponential Separation of Orbits)**.**
*Let $F \in \mathcal{B}_{\log}$ and $T$ be a tract of $F$. If $\omega, \zeta \in T$ are such that $|\omega - \zeta| \geq 2$, then*

$$|F(\omega) - F(\zeta)| \geq \exp(|\omega - \zeta|/8\pi) \cdot \min\{\operatorname{Re} F(\omega), \operatorname{Re} F(\zeta)\} .$$

*Proof.* Suppose without loss of generality that $\operatorname{Re} F(\omega) \geq \operatorname{Re} F(\zeta)$. Since $T$ has height at most $2\pi$, it follows by the standard estimate on hyperbolic distances that

$$|\omega - \zeta|/2\pi \leq \operatorname{dist}_T(\omega, \zeta) = \operatorname{dist}_{\mathbb{H}}(F(\omega), F(\zeta)) .$$

Let $\xi \in \mathbb{H}$ be a point that satisfies $\operatorname{dist}_{\mathbb{H}}(F(\zeta), \xi) = \operatorname{dist}_{\mathbb{H}}(F(\omega), F(\zeta))$ and $\operatorname{Re} F(\zeta) = \operatorname{Re}\xi$, see Figure 3. We will estimate the Euclidean distance $s = |F(\zeta) - \xi|$. Then, $|F(\omega) - F(\zeta)| \geq s$. Let $\gamma$ be the curve consisting of three straight line segments pictured in Figure 3, connecting $F(\zeta)$ to $\xi$ through $F(\zeta) + s$ and $\xi + s$.

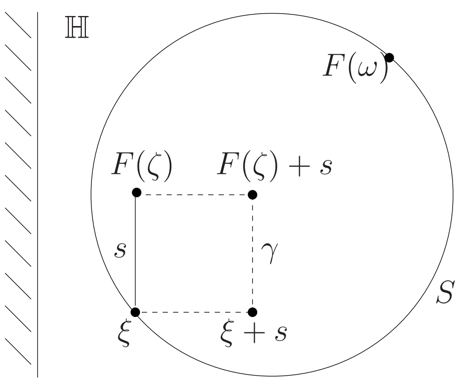


FIGURE 3. The set $S$ of points of equal hyperbolic distance in $\mathbb{H}$ to $F(\zeta)$ is a Euclidean circle. Clearly, $\xi$ is the Euclidean closest point to $F(\zeta)$ on $S$ that satisfies $\operatorname{Re}\xi \geq \operatorname{Re} F(\zeta)$. We use the dotted line $\gamma$ to estimate $s$.

Each of the horizontal parts of $\gamma \subset \mathbb{H}$ has hyperbolic length $\log\left((\operatorname{Re} F(\zeta) + s)/\operatorname{Re} F(\zeta)\right)$, and it is easy to see that the hyperbolic length of the vertical part is less than 1. Hence, we get

$$\frac{|\omega - \zeta|}{2\pi} < 2 \log \left( \frac{\operatorname{Re} F(\zeta) + s}{\operatorname{Re} F(\zeta)} \right) + 1 \ ,$$

and therefore

$$|F(\omega) - F(\zeta)| \geq s \geq \left( \exp \left( \frac{|\omega - \zeta|}{4\pi} - \frac{1}{2} \right) - 1 \right) \cdot \operatorname{Re} F(\zeta) \ .$$

Since $e^{x-1/2} - 1 > e^{x/2}$ for $x \geq 2$, the claim follows. ■

*Remark.* It follows from expansivity of $F$ that for any two distinct points $w, z$ with the same external address, there exists $k \in \mathbb{N}$ such that $|F^{\circ k}(w) - F^{\circ k}(z)| > 2$. Hence Lemma 3.1 will apply eventually.

**3.2. Lemma** (Growth of Real Parts)**.**
*Let $F \in \mathcal{B}_{\log}$. If $\zeta, \omega \in J(F)$ are distinct points with the same external address $\underline{s}$, then*

$$\lim_{k \to \infty} \max(\operatorname{Re} F^{\circ k}(\zeta), \operatorname{Re} F^{\circ k}(\omega)) = \infty \ .$$

*Proof.* Suppose that $\zeta, \omega \in J(F)$ satisfy $\operatorname{Re} F^{\circ k}(\zeta), \operatorname{Re} F^{\circ k}(\omega) < S$ for some $S > 0$ and infinitely many $k \in \mathbb{N}$. For any tract $T$, the set $\overline{T} \cap \{z \in \mathbb{C} : \operatorname{Re} z \leq S\}$ is compact and thus has bounded imaginary parts. Furthermore, up to translations in $2\pi i \mathbb{Z}$ there are only finitely tracts of $F$ which intersect the line $\{\operatorname{Re} z = S\}$ at all (this follows from property (b) in the definition of $\mathcal{B}_{\log}$). We conclude that there is $C > 0$ such that $|F^{\circ k}(\zeta) - F^{\circ k}(\omega)| < C$ whenever $\operatorname{Re} F^{\circ k}(\zeta), \operatorname{Re} F^{\circ k}(\omega) < S$. In particular,

$$|\zeta - \omega| \leq \frac{1}{2^k} \cdot |F^{\circ k}(\zeta) - F^{\circ k}(\omega)| \leq \frac{C}{2^k}$$

by expansivity of $F$ (we have $|(F_T^{-1})'(z)| < 1/2$ for any tract $T$). Since this happens for infinitely many $k$, it follows that $\zeta = \omega$, as required. ■

Note that Lemma 3.2 does not imply that either $\zeta$ or $\omega$ escapes: indeed, it is conceivable that both points have unbounded orbits but return to some bounded real parts infinitely many times. In the next section, we introduce a property, called a *head-start condition*, which is designed precisely so that this does not occur.

As mentioned in the introduction, Rippon and Stallard [RiS] showed that the escaping set of every entire function $f$ contains unbounded connected sets. The following theorem is a version of this result for functions in $\mathcal{B}_{\log}$.

**3.3. Theorem** (Existence of Unbounded Continua in $J_{\underline{s}}$)**.**
*For every $F \in \mathcal{B}_{\log}$ there exists $K \geq 0$ with the following property. If $z_0 \in J^K(F)$ and $\underline{s}$ is the external address of $z_0$, then there exists an unbounded closed connected set $A \subset J_{\underline{s}}$ with $\operatorname{dist}(z_0, A) \leq 2\pi$.*

*Proof.* Choose $K$ large enough so that no bounded component of $\mathbb{H} \cap \overline{T}$ intersects the line $\{\operatorname{Re} z = K\}$ and set $z_k := F^{\circ k}(z_0)$. If $S \subset \mathbb{C}$ is an unbounded set such that $S \setminus B_{2\pi}(z_k)$ has exactly one unbounded component, let us denote this component by $X_k(S)$.

We claim that $X_k(\overline{T}_k)$ is non-empty and contained in $\mathbb{H}$ for all $k \geq 1$. (However, this set is *not* necessarily contained in $\mathbb{H}_K$.) Indeed, this is trivial if $\overline{T}_k \subset \mathbb{H}$. Otherwise, let $\alpha^-$ and $\alpha^+$ denote the two unbounded components of $\mathbb{H} \cap \partial T_k$. We claim that both $\alpha^-$ and $\alpha^+$ must intersect the vertical line segment $L := z_k + i[-2\pi, 2\pi]$. Indeed, otherwise some $2\pi i\mathbb{Z}$-translate of $\alpha^-$ or $\alpha^+$ would separate $z_k$ from $\alpha^\pm$ in $\mathbb{H}$, which is not possible since $z_k$ belongs to the unbounded component of $T_k \cap \mathbb{H}$. Hence it follows that the unbounded component of $T_k \setminus L$, which contains $X_k(\overline{T}_k)$, is contained in $\mathbb{H}$.

In particular, we can pull back the set $X_k(T_k)$ into $T_{k-1}$ using $F^{-1}_{T_{k-1}}$. By expansivity of $F$, $F^{-1}_{T_{k-1}}(X_k(T_k))$ has distance at most $\pi$ from $z_{k-1}$. Continuing inductively, we obtain the sets

$$A_k := X_0(F^{-1}_{T_0}(X_1(F^{-1}_{T_1}(\dots(X_{k-1}(F^{-1}_{T_{k-1}}(X_k(\overline{T}_k))))\dots))))$$

for $k \geq 1$; let $A_0 = X_0(\overline{T}_0)$. Each $\widehat{A}_k \subset \hat{\mathbb{C}}$ is a continuum, has distance at most $2\pi$ from $z_0$ and contains $\widehat{A}_{k+1}$. Hence, the set $A' = \bigcap_{k \geq 0} \widehat{A}_k$ has the same properties and there exists a component $A$ of $A' \setminus \{\infty\}$ with $\operatorname{dist}(A, z_0) \leq 2\pi$. By definition, $A$ is closed and connected, and it is unbounded by the Boundary Bumping theorem [N, Theorem 5.6]. ■

## 4. Functions Satisfying a Head-Start Condition

Throughout most of this and the next section, we will fix some function $F \in \mathcal{B}_{\log}$. Fix an external address $\underline{s}$, and suppose that the set $J_{\underline{s}}$ is a ray tail. Then $J_{\underline{s}} \cup \{\infty\}$ is homeomorphic to $[0, \infty]$, and as such possesses a natural total ordering. In this section, we will use a converse idea: we introduce a "head-start condition", which implies that the points in $J_{\underline{s}}$ are totally ordered by their speed of escape, and deduce from this that $J_{\underline{s}}$ is (essentially) a ray tail. In the next section, we then develop sufficient conditions on $F$ under which a head-start condition is satisfied.

**4.1. Definition** (Head-Start Condition)**.**
*Let $T$ and $T'$ be tracts of $F$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a (not necessarily strictly) monotonically increasing continuous function with $\varphi(x) > x$ for all $x \in \mathbb{R}$. We say that the pair $(T, T')$ satisfies the* head-start condition for $\varphi$ *if, for all $z, w \in T$ with $F(z), F(w) \in T'$,*

$$\operatorname{Re} w > \varphi(\operatorname{Re} z) \Longrightarrow \operatorname{Re} F(w) > \varphi(\operatorname{Re} F(z)) .$$

*An external address $\underline{s}$ satisfies the* head-start condition for $\varphi$ *if all consecutive pairs of tracts $(T_k, T_{k+1})$ satisfy the head-start condition for $\varphi$, and if for all distinct $z, w \in J_{\underline{s}}$, there exists $M \in \mathbb{N}$ such that either $\operatorname{Re} F^{\circ M}(z) > \varphi(\operatorname{Re} F^{\circ M}(w))$ or $\operatorname{Re} F^{\circ M}(w) > \varphi(\operatorname{Re} F^{\circ M}(z))$.*

*We say that $F$ satisfies a* head-start condition *if every external address of $F$ satisfies the head-start condition for some $\varphi$. If the same function $\varphi$ can be chosen for all external addresses, we say that $F$ satisfies the* uniform head-start condition for $\varphi$.

**4.2. Theorem** (Ray Tails)**.**
*Suppose that $F \in \mathcal{B}_{\log}$ satisfies a head-start condition. Then for every escaping point $z$, there exists $k \in \mathbb{N}$ such that $F^{\circ k}(z)$ is on a ray tail $\gamma$. This ray tail is the unique arc in $J(F)$ connecting $F^{\circ k}(z)$ to $\infty$ (up to reparametrization).*

We will devote the remainder of this section to the proof of Theorem 4.2.

If $\underline{s}$ satisfies any head-start condition, the points in $J_{\underline{s}}$ are eventually ordered by their real parts: for any two points $z, w \in J_{\underline{s}}$, $F^{\circ k}(z)$ is to the right of $F^{\circ k}(w)$ for all sufficiently large $k$ (or vice versa).

**4.3. Definition and Lemma** (Speed Ordering)**.**
*Let $\underline{s}$ be an external address satisfying the head-start condition for $\varphi$. For $z, w \in J_{\underline{s}}$, we say that $z \succ w$ if there exists $K \in \mathbb{N}$ such that $\operatorname{Re} F^{\circ K}(z) > \varphi(\operatorname{Re} F^{\circ K}(w))$. We extend this order to $\widehat{J}_{\underline{s}} = J_{\underline{s}} \cup \{\infty\}$ by the convention that $\infty \succ z$ for all $z \in J_{\underline{s}}$.*

*With this definition, $(\widehat{J}_{\underline{s}}, \succ)$ is a totally ordered space. Moreover, the order does not depend on $\varphi$.*

Note that if $z \succ w$, then $\operatorname{Re} F^{\circ k}(z) > \varphi(\operatorname{Re} F^{\circ k}(w))$ for all $k \geq K$.

*Proof.* By definition, $\operatorname{Re} F^{\circ k}(z) < \varphi(\operatorname{Re} F^{\circ k}(z))$ for all $k \in \mathbb{N}$ and $z \in J_{\underline{s}}$. Hence "$\succ$" is non-reflexive.

Let $a, b, c \in J_{\underline{s}}$ such that $a \succ b$ and $b \succ c$. Then, there exist $k, l \in \mathbb{N}$ such that $\operatorname{Re} F^{\circ k}(a) > \varphi(\operatorname{Re} F^{\circ k}(b))$ and $\operatorname{Re} F^{\circ l}(b) > \varphi(\operatorname{Re} F^{\circ l}(c))$. Setting $m := \max\{k, l\}$, we obtain from the head-start condition that $\operatorname{Re} F^{\circ m}(a) > \varphi(\operatorname{Re} F^{\circ m}(b)) > \operatorname{Re} F^{\circ m}(b) > \varphi(\operatorname{Re} F^{\circ m}(c))$. Hence $a \succ c$ and "$\succ$" is transitive.

By assumption, for any distinct $z, w \in J_{\underline{s}}$ there exists $k \in \mathbb{N}$ such that $\operatorname{Re} F^{\circ k}(w) > \varphi(\operatorname{Re} F^{\circ k}(z))$ or $\operatorname{Re} F^{\circ k}(z) > \varphi(\operatorname{Re} F^{\circ k}(w))$. It follows that any two distinct points are comparable under "$\succ$".

If $\underline{s}$ also satisfies the head-start condition for $\tilde{\varphi}$, and $\tilde{\varphi}$ induces an order $\widetilde{\succ}$, then it is easy to see that $w \mathrel{\widetilde{\succ}} z$ if and only if $w \succ z$. This completes the proof. ∎

**4.4. Corollary** (Growth of Real Parts)**.**
*Let $\underline{s}$ be an external address that satisfies the head-start condition for $\varphi$ and let $z, w \in J_{\underline{s}}$. If $w \succ z$, then $w \in I(F)$. In particular, $J_{\underline{s}} \setminus I_{\underline{s}}$ consists of at most one point.*

*Proof.* This is an immediate corollary of Lemmas 3.2 and 4.3. ∎

**4.5. Proposition** (Arcs in $J_{\underline{s}}$)**.**
*Let $\underline{s}$ be an external address satisfying the head-start condition for $\varphi$. Then the topology of $\widehat{J}_{\underline{s}}$ as a subset of the Riemann sphere $\hat{\mathbb{C}}$ agrees with the order topology induced by $\succ$. In particular,*

(a) *every component of $\widehat{J}_{\underline{s}}$ is homeomorphic to a (possibly degenerate) compact interval, and*
(b) *if $J^K_{\underline{s}} \neq \emptyset$ for $K$ as in Theorem 3.3, then $J_{\underline{s}}$ has a unique unbounded component, which is a closed arc to infinity.*

*Proof.* Let us first show that $\mathrm{id} : \widehat{J}_{\underline{s}} \to (\widehat{J}_{\underline{s}}, \succ)$ is continuous. Since $\widehat{J}_{\underline{s}}$ is compact and the order topology on $\widehat{J}_{\underline{s}}$ is Hausdorff, this will imply that id is a homeomorphism and that both topologies agree. It suffices to show that sub-basis elements for the order topology of the form $U_a^- := \{w \in J_{\underline{s}} : a \succ w\}$ and $U_a^+ := \{w \in \widehat{J}_{\underline{s}} : w \succ a\}$ are open in $\widehat{J}_{\underline{s}}$ for any $a \in \widehat{J}_{\underline{s}}$. We will only give a proof for the sets $U_a^-$; the proof for $U_a^+$ is analogous.

Let $w \in U_a^-$ and choose $k \in \mathbb{N}$ minimal such that $\operatorname{Re} F^{\circ k}(a) > \varphi(\operatorname{Re} F^{\circ k}(w))$. Since $\varphi$, Re and $F^{\circ k}$ are continuous, this is true for a neighborhood $V$ of $w$. It follows that $V \cap \widehat{J}_{\underline{s}} \subset U_a^-$, hence $U_a^-$ is a neighborhood of $w$ in $\widehat{J}_{\underline{s}}$.

Thus the topology of $\widehat{J}_{\underline{s}}$ agrees with the order topology. Every connected component $C$ of $\widehat{J}_{\underline{s}}$ is compact; it follows from [N, Theorems 6.16 & 6.17] that $C$ is either a point or an arc. This proves (a). To prove (b), observe that existence follows from Theorem 3.3, while uniqueness follows because $\infty$ is the largest element of $(\widehat{J}_{\underline{s}}, \succ)$. ■

**4.6. Proposition** (Points in the Unbounded Component of $J_{\underline{s}}$).
*Let $\underline{s}$ be an external address that satisfies the head-start condition for $\varphi$. Then there exists $K' \geq 0$ such that $J_{\underline{s}}^{K'}$ is contained in the unbounded component of $J_{\underline{s}}$, which is a closed arc. (The value $K'$ depends on $F$ and $\varphi$, but not on $\underline{s}$.)*

*Proof.* Let $K$ be the constant from Theorem 3.3, set $K' := \max\{\varphi(0) + 1, K\}$ and let $z_0 \in J_{\underline{s}}^{K'}$. For each $k \geq 0$, we let $z_k := F^{\circ k}(z_0)$ and consider the set

$$S_k := \{w \in J_{\sigma^k(\underline{s})} : w \succeq z_k\};.$$

By Proposition 4.5, each $S_k$ has a unique unbounded component $A_k$ which is a closed arc. By Theorem 3.3, $A_k$ satisfies $\operatorname{dist}(z_k, A_k) \leq 2\pi$.

Let us show $A_k \subset \mathbb{H}$, so that we may apply $F^{-1}$ to it. Indeed, if $w \in J_{\sigma^k(\underline{s})}$ with $\operatorname{Re} w \leq 0$, then the choice of $K'$ and monotonicity of $\varphi$ yield that $\operatorname{Re} z_k > \varphi(0) \geq \varphi(\operatorname{Re} w)$, and therefore $z_k \succ w$. Thus, $w \notin S_k$. We conclude that $F_{T_{k-1}}^{-1}(A_k) \subset A_{k-1}$, because it is unbounded and contained in $S_{k-1}$. Since $F$ is expanding, this means that

$$\operatorname{dist}(A_0, z_0) \leq 2^{-k} \operatorname{dist}(z_k, A_k) \leq 2^{-(k-1)}\pi$$

for all $k \geq 0$. Thus $z_0 \in A_0$, as required. That $A_0$ is an arc follows from Proposition 4.5. ■

*Proof of Theorem 4.2.* Let $z$ be an escaping point for $F$ and $\underline{s}$ its external address. By hypothesis, there exists $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\underline{s}$ satisfies the head-start condition for $\varphi$. If $K'$ is the constant from Proposition 4.6, then there exists $k \geq 0$ such that $F^{\circ k}(z) \in J_{\underline{s}}^{K'}$ and $\gamma_k := \{w \in I_{\sigma^k(\underline{s})} : w \succeq F^{\circ k}(z)\}$ is an injective curve connecting $F^{\circ k}(z)$ to $\infty$. Furthermore, since the order topology agrees with the usual topology on $J_{\underline{s}} \cup \infty$, $\gamma_k$ is unique with this property.

To see that $\gamma_k$ is a ray tail, it remains to show that the real parts of all points on $\gamma_k$ grow uniformly. Observe that, for all $w \in \gamma_k$ and $m \in \mathbb{N}$, $\operatorname{Re} F^{\circ m}(w) \geq \inf\{\varphi^{-1}(\operatorname{Re} F^{\circ(k+m)}(z))\}$, because $w \succ z$ or $w = z$ (we have to take the infimum because $\varphi$ need not be invertible). Hence the $\operatorname{Re} F^{\circ m}(w)$ converge uniformly to $\infty$ as $m \to \infty$. ■

**4.7. Theorem** (Existence of Absorbing Brush).
*Suppose that $F$ satisfies a head-start condition. Then there exists a closed subset $X \subset J(F)$ with the following properties:*

(a) *$F(X) \subset X$;*
(b) *each connected component $C$ of $X$ is a closed arc to infinity all of whose points except possibly the finite endpoint escape;*

(c) *every escaping point of $F$ enters $X$ after finitely many iterations. If $F$ satisfies the uniform head-start condition for some function, then there exists $K' > 0$ such that $J^{K'}(F) \subset X$.*

*Proof.* Let $X$ denote the union over all external addresses of all unbounded components of $J(F)$. By the Boundary Bumping theorem [N, Theorem 5.6], $\widehat{X}$ is the unbounded connected component of the compact set $J(F) \cup \{\infty\}$, hence $X \subset \mathbb{C}$ is a closed set. Clearly $X$ is $F$-invariant, and satisfies (b) and (c) by Propositions 4.5 and 4.6 (recall that the choice of $K'$ did not depend on the external address). ■

## 5. Geometry, Growth & Head-Start

This section discusses geometric properties of tracts which imply a head-start condition. Moreover, we show that (compositions of) functions of finite order satisfy these properties, hence completing the proof of Theorem 1.2.

Let $K > 1$ and $M > 0$. We say that $\underline{s}$ satisfies the *linear head-start condition* with constants $K$ and $M$ if it satisfies the head-start condition for

$$\varphi(t) := K \cdot t^+ + M \ ,$$

where $t^+ = \max\{t, 0\}$.

We will restrict our attention to functions whose tracts do not grow too quickly in the imaginary direction.

**5.1. Definition** (Bounded Slope).
*Let $F \in \mathcal{B}_{\log}$. We say that the tracts of $F$ have bounded slope (with constants $\alpha, \beta > 0$) if*

$$|\operatorname{Im} z - \operatorname{Im} w| \leq \alpha \max\{\operatorname{Re} z, \operatorname{Re} w, 0\} + \beta$$

*whenever $z$ and $w$ belong to a common tract of $F$. We denote the class of all functions with this property by $\mathcal{B}_{\log}(\alpha, \beta)$.*

*Remark.* By property (e) in the definition of $\mathcal{B}_{\log}$, this condition is equivalent to the existence of a curve $\gamma : [0, \infty) \to \mathcal{T}$ with $|F(\gamma(t))| \to \infty$ and $\limsup |\operatorname{Im} \gamma(t)| / \operatorname{Re} \gamma(t) < \infty$. Hence if one tract of $F$ has bounded slope, then all tracts do.

The bounded slope condition means that the absolute value of a point is proportional to its real part. As we see in the next lemma, this easily implies that the second requirement of a linear head-start condition (that any two orbits eventually separate far enough for one to have a head-start over the other) is automatically satisfied when the tracts have bounded slope.

**5.2. Lemma** (Linear Separation of Orbits).
*Let $F \in \mathcal{B}_{\log}(\alpha, \beta)$ and $K, M > 0$. Then there exists $\delta = \delta(\alpha, \beta, K, M) > 0$ with the following property: if $\underline{s}$ is an external address and $z, w \in J_{\underline{s}}$ with $|z - w| > \delta$, then*

$$\operatorname{Re} F^{\circ k}(z) > K \operatorname{Re} F^{\circ k}(w) + M \quad \textit{or} \quad \operatorname{Re} F^{\circ k}(w) > K \operatorname{Re} F^{\circ k}(z) + M$$

*for all $k \geq 1$.*

*Proof.* Again, let us denote $z_k = F^{\circ k}(z)$ and $w_k = F^{\circ k}(w)$. Let $\delta' = \alpha + \beta + 2$, $\delta = \max\{\delta', 16\pi \log \delta', M\}$ and fix $k \in \mathbb{N}$. By possibly enlarging $\delta$ further, we may assume that $e^{t/16\pi} > K + t + 1/2$ for $t \geq \delta - 1/2$.

Suppose without loss of generality that $\operatorname{Re} w_{k+1} \geq \operatorname{Re} z_{k+1}$ (note that both points are in $\mathbb{H}$). Using the bounded slope condition, it is easy to estimate $|w_{k+1} - z_{k+1}| \leq (\alpha+1)\operatorname{Re}(w_{k+1}) + \beta$. Then $\operatorname{Re} w_{k+1} \geq 1$, as otherwise we would have $|w_{k+1} - z_{k+1}| \leq \alpha+\beta+1 < \delta'$, contradicting expansivity of $F$. Hence we conclude that $|w_{k+1} - z_{k+1}| \leq \delta' \operatorname{Re} w_{k+1}$. Since $\delta' \geq 2$, Lemma 3.1 now yields

$$(1) \qquad \operatorname{Re} w_{k+1} \geq \frac{|w_{k+1} - z_{k+1}|}{\delta'} \geq \frac{\exp(|w_k - z_k|/8\pi)}{\delta'} \cdot \operatorname{Re} z_{k+1} > e^{\frac{|w_k - z_k|}{16\pi}} \cdot \operatorname{Re} z_{k+1} ,$$

because $\exp(x/8\pi)/\delta' > \exp(x/16\pi)$ for all $x > 16\pi \log \delta'$.

Now suppose first that $\operatorname{Re} z_{k+1} \geq 1$; then equation (1) implies

$$\operatorname{Re} w_{k+1} > e^{\frac{|w_k - z_k|}{16\pi}} \operatorname{Re} z_{k+1} > (K + |w_k - z_k| + \tfrac{1}{2}) \operatorname{Re} z_{k+1} > K \operatorname{Re} z_{k+1} + |w_k - z_k| + 1/2 .$$

On the other hand, if $\operatorname{Re} z_{k+1} < 1$, then we can replace $z_{k+1}$ by $z' = F_T^{-1}(1 + \operatorname{Im} z_{k+1})$ in the previous argument. Since $|z_k - z'| < 1/2$, and hence $|z' - w_k| \geq |w_k - z_k| - 1/2$, we have

$$\operatorname{Re} w_{k+1} > K \operatorname{Re} F(z') + |z' - w_k| + 1/2 \geq K \operatorname{Re} z_{k+1} + |w_k - z_k| .$$

The claim follows because $|w_k - z_k| > \delta \geq M$. ■

*Remark.* Thus, to show that a function whose tracts have bounded slope satisfies a linear head-start condition, we need only verify the first requirement: if $w_k$ is ahead of $z_k$ in terms of real parts, the same should be true for $w_{k+1}$ and $z_{k+1}$. Note that this condition — in contrast to the second one — is not dynamical in nature; rather, it concerns the mapping behavior of the conformal map $F : T \to \mathbb{H}$. As such, it is not too difficult to translate this requirement into a geometrical condition. Roughly speaking, the tract should not "wiggle" in the sense of first growing in real parts to reach the larger point $w_k$, then turning around to return to $z_k$, until finally starting to grow again. (We exploit this idea in Section 6 to construct a counterexample; compare also Figure 4). The precise geometric condition is as follows.

**5.3. Definition** (Bounded Wiggling)**.**
*Let $F \in \mathcal{B}_{\log}$, and let $T$ be a tract of $F$. We say that $T$ has* bounded wiggling *if there exist $K > 1$ and $\mu > 0$ such that for every $z_0 \in T$, every point $z$ on the hyperbolic geodesic of $T$ that connects $z_0$ to $\infty$ satisfies*

$$\operatorname{Re} z > \tfrac{1}{K} \operatorname{Re} z_0 - \mu .$$

*We say that $F \in \mathcal{B}_{\log}$ has* uniformly bounded wiggling *if the wiggling of all tracts of $F$ is bounded by the same constants $K, \mu$.*

**5.4. Proposition** (Head-Start and Wiggling for Bounded Slope)**.**
*Let $F \in \mathcal{B}_{\log}(\alpha, \beta)$, $\underline{s} = T_0 T_1 T_2 \ldots$ be an external address, and let $K > 1$. Then the following are equivalent:*

(a) *For some $\mu > 0$, the tracts $T_k$ have bounded wiggling with constants $K$ and $\mu$;*

(b) *for some $M > 0$, $\underline{s}$ satisfies the linear head-start condition with constants $K$ and $M$.*

*The relation between $\mu$ and $M$ depends on $K$, $\alpha$ and $\beta$, but not on $\underline{s}$.*

Observe that if $F \in \mathcal{B}_{\log}(\alpha, \beta)$ has uniformly bounded wiggling, then Proposition 5.4 implies that $F$ satisfies a uniform linear head-start condition.

*Proof.* Suppose that (a) holds. Setting $\widetilde{M} := K(\mu + 2\pi(\alpha+\beta))$, we will define $M := \max(\delta, 1)$, where $\delta = \delta(\alpha, \beta, K, \widetilde{M})$ is the constant from Lemma 5.2. Let $k \in \mathbb{N}$ and choose $z, w \in T_k$ such that $F(z), F(w) \in T_{k+1}$ and $\operatorname{Re} w > K(\operatorname{Re} z)^+ + M$. Then $|z - w| > M$ and, by Lemma 5.2, it suffices to show that $\operatorname{Re} F(w) \geq \operatorname{Re} F(z)$.

So suppose by way of contradiction that $\operatorname{Re} F(z) > \operatorname{Re} F(w)$. Since $M \geq 1$, we see from Lemma 5.2 that $\operatorname{Re} F(z) \geq M \geq 1$. Set $\Gamma := \{F(w) + t : t \geq 0\}$ and $\gamma := F_{T_k}^{-1}(\Gamma)$; in other words, $\gamma$ is the geodesic of $T_k$ connecting $w$ to $\infty$. The bounded slope condition ensures that

$$\operatorname{dist}_{T_k}(z, \gamma) = \operatorname{dist}_{\mathbb{H}}(F(z), \Gamma) \leq \frac{|\operatorname{Im} F(z) - \operatorname{Im} F(w)|}{\operatorname{Re} F(z)} \leq \alpha + \beta \,.$$

Therefore, $\operatorname{dist}(z, \gamma) \leq 2\pi(\alpha+\beta)$ and consequently, $\operatorname{Re} z \geq \min_{\zeta\in\gamma}\{\operatorname{Re}\zeta\} - 2\pi(\alpha+\beta)$. By the bounded wiggling condition, we have $\operatorname{Re}\zeta \geq \frac{1}{K}\operatorname{Re} w - \mu$ for all $\zeta \in \gamma$. Thus

$$\operatorname{Re} w \leq K(\operatorname{Re} z + \mu + 2\pi(\alpha+\beta)) \leq K \operatorname{Re} z + M \,,$$

a contradiction.

For the converse direction, suppose thta (b) holds. Let $T$ be a tract of $F$ and $z \in T$. Observe that there exists a constant $\kappa > 0$ — independent of $T$ and $z$ — such that $\operatorname{dist}(F(z), I(F)) < \kappa$ (this is because $I(F)$ contains a curve with real parts tending to $+\infty$, as well as its $2\pi i\mathbb{Z}$-translates). Pulling back, we find that there exists an escaping point $\zeta \in T$ such that $z$ and $\zeta$ can be joined in $T$ with a curve of Euclidean length at most $\kappa/2$. Let $\underline{s}$ be the external address of $\zeta$ and $\gamma \subset \{w \in I_{\underline{s}} : w \succ \zeta\}$ be a curve of escaping points that joins $\zeta$ to $\infty$. By the head-start condition, every $w \in \gamma$ satisfies

$$\operatorname{Re} w \geq \frac{\operatorname{Re}\zeta}{K} - \frac{M}{K} \,.$$

Hence there exists a curve $\gamma' \subset T$ that connects $z$ to $\infty$ such that for every $w \in \gamma'$,

$$\operatorname{Re} w \geq \frac{\operatorname{Re}\zeta}{K} - \frac{M}{K} - \kappa \geq \frac{\operatorname{Re} z}{K} - \frac{\kappa}{K} - \frac{M}{K} - \kappa \,.$$

Now the claim follows easily by general principles of hyperbolic geometry (see Lemma A.2 in the appendix). ■

Finally, let us consider functions of finite order.

**5.5. Definition** (Finite Order)**.**
*We say that $F \in \mathcal{B}_{\log}$ has* finite order *if*

$$\log \operatorname{Re} F(w) = O(\operatorname{Re} w)$$

*as $\operatorname{Re} w \to \infty$ in $\mathcal{T}$.*

Note that this definition ensures that $f \in \mathcal{B}$ has finite order (i.e.

$$\lim_{r\to\infty} \sup_{|z|=r} \frac{\log\log|f(z)|}{\log|z|} < \infty)$$

if and only if its logarithmic transform $F \in \mathcal{B}_{\log}$ has finite order in the sense of Definition 5.5.

**5.6. Theorem** (Finite Order Functions have Good Geometry)**.**
*Suppose that $F \in \mathcal{B}_{\log}$ has finite order. Then the tracts of $F$ have bounded slope and (uniformly) bounded wiggling.*

*Proof.* By the Ahlfors non-spiralling theorem (Theorem A.1), $F \in \mathcal{B}_{\log}(\alpha,\beta)$ for some constants $\alpha, \beta$. By the finite-order condition, there are $\varrho$ and $M$ such that $\log \operatorname{Re} F(z) \leq \varrho \operatorname{Re} z + M$ for all $z \in \mathcal{T}$. Let $T$ be a tract of $F$ and $z \in T$.

Suppose first that $\operatorname{Re} F(z) \geq 1$. Consider the geodesic $\gamma(t) := F_T^{-1}(F(z)+t)$ (for $t \geq 0$). Since the hyperbolic distance between $z$ and $\gamma(t)$ is at most $\log t$, we have

$$\operatorname{Re} z - \operatorname{Re}\gamma(t) \leq 2\pi \log t \leq 2\pi \log \operatorname{Re} F(\gamma(t)) \leq 2\pi(\varrho \operatorname{Re}\gamma(t) + M) .$$

In other words, $\operatorname{Re} z \leq (1+2\pi\varrho)\operatorname{Re}\gamma(t) + 2\pi M$, i.e.

$$\operatorname{Re}\gamma(t) \geq \frac{1}{1+2\pi\varrho}\operatorname{Re} z - \frac{2\pi M}{1+2\pi\varrho} .$$

Since $z$ was chosen arbitrarily, $F$ has uniformly bounded wiggling with constants $1/(1+2\pi\varrho)$ and $2\pi M/(1+2\pi\varrho)$.

If $\operatorname{Re} F(z) < 1$, we can connect $z$ to a point $w \in T$ with $\operatorname{Re} F(w) \geq 1$ by a curve of bounded diameter. ■

To complete the proof of Theorem 1.2, it only remains to show that linear head-start conditions are preserved under composition. In logarithmic coordinates, this is given by the following statement; let $\tau_a(z) = z - a$ for $a \geq 0$ and $\mathbb{H}_a := \{z \in \mathbb{C} \colon \operatorname{Re}(z) > a\}$.

**5.7. Lemma** (Linear Head-Start is Preserved by Composition)**.**
*Let $F_i \colon \mathcal{T}_{F_i} \to \mathbb{H}$ be in $\mathcal{B}_{\log}$, for $i = 1, 2, \ldots, n$. Then there is an $a \geq 0$ so that $G_a := \tau_a \circ F_n \circ \cdots \circ F_1 \in \mathcal{B}_{\log}$ on appropriate tracts $\mathcal{T}_a \subset \mathcal{T}_{F_1}$, so that $G_a$ is a conformal isomorphism from each component of $\mathcal{T}_a$ onto $\mathbb{H}$. If $F_1$ has bounded slope and all $F_i$ satisfy linear head-start conditions, then $G_a$ also has bounded slope and satisfies a linear head-start condition.*

*Proof.* There is an $a_2 \geq 0$ so that $F_2^{-1}(\mathbb{H}_{a_2}) \subset \mathbb{H}$; there is an $a_3 \geq 0$ so that $F_3^{-1}(\mathbb{H}_{a_3}) \subset \mathbb{H}_{a_2}$, etc.. Finally, there is an $a = a_n \geq 0$ so that $(F_n \circ \cdots \circ F_1)^{-1}$ is defined on all of $\mathbb{H}_a$. Let $\mathcal{T}_a := (F_n \circ \cdots \circ F_1)^{-1}(\mathbb{H}_a) \subset \mathcal{T}_{F_1}$. Then $F_n \circ \cdots \circ F_1$ is a conformal isomorphism from each component of $\mathcal{T}_a$ onto $\mathbb{H}_a$, and the first claim follows. In particular, the tracts of $G_a$ have bounded slope.

For $i = 1, \ldots, n$, let $K_i$ and $M_i$ be the constants for the linear head-start condition of $F_i$, and set $K := \max_i\{K_i\}$ and $M := \delta$, where $\delta = \delta(\alpha, \beta, K, \max_i\{M_i\})$ is the constant from Lemma 5.2. Let $T$ be a tract of $F_i$ and $w, z \in T$, such that $\operatorname{Re} w > K \operatorname{Re} z + M$. Then, $|w - z| \geq \operatorname{Re} w - \operatorname{Re} z > M = \delta$, and Lemma 5.2 gives that

$$\operatorname{Re} F_i(w) > K \operatorname{Re} F_i(z) + M \quad \text{or} \quad \operatorname{Re} F_i(z) > K \operatorname{Re} F_i(w) + M \ .$$

But since $F_i$ satisfies a head-start condition, the first possibility holds. Hence, all $F_i$ satisfy a linear head-start condition with constants $K, M$, and it is now easy to see that $G_a$ does, too. ■

*Proof of Theorem 1.2 and Corollary 1.3.* Let $f_1, \dots, f_n \in \mathcal{B}$ be functions of finite order, and let $F_j \in \mathcal{B}_{\log}$ be their logarithmic transforms. By Theorem 5.6, each $F_j$ satisfies a linear head-start condition, and by Lemma 5.7, $G_a := \tau_a \circ F_n \circ \dots \circ F_1 \in \mathcal{B}_{\log}$ satisfies a linear head-start condition. (The purpose of $\tau_a$ is only to arrange the maps so that their image is all of $\mathbb{H}$.) Now, on a sufficiently restricted domain, $F := G_a \circ \tau_a^{-1}$ is a logarithmic transform of $f = f_n \circ \dots \circ f_1$ and satisfies a linear head-start condition. Hence every escaping point of $F$, and hence of $f$, is eventually mapped into some ray tail. By Proposition 2.2, this completes the proof of Theorem 1.2.

The proof of Corollary 1.3 is analogous. (Recall that the order of a meromorphic function is defined in terms of its Nevanlinna characteristic. However, if $f$ has finite order, then it is well-known that the restriction of $f$ to its logarithmic tracts will also have finite order in the previously defined sense.) ■

*Remark.* If our goal was only to prove Theorem 1.2 and Corollary 1.3, a somewhat faster route would be possible (compare [Ro, Chapter 3]). For example, the linear head-start condition can be verified directly for functions of finite order, without explicitly considering the geometry of their tracts. Also, the bounded slope condition can be used to simplify the proof of Theorem 4.2 in this context, eliminating e.g. the need for Theorem 3.3. We have chosen the current approach because it provides both additional information and a clear conceptual picture of the proof.

Finally, let us collect together some of the results obtained in this and the previous section for future reference.

**5.8. Corollary** (Linear Head-Start Conditions)**.**
*Let $\mathcal{H}_{\log}$ consist of all functions $F \in \mathcal{B}_{\log}$ which satisfy a linear head-start condition and have tracts of bounded slope.*

(a) *The class $\mathcal{H}_{\log}$ contains all functions $F \in \mathcal{B}_{\log}$ of finite order.*
(b) *The class $\mathcal{H}_{\log}$ is closed under composition.*
(c) *If $F \in \mathcal{H}_{\log}$, then there is some $K > 0$ such that every point of $J^K(F)$ can be connected to infinity by a curve in $I(F)$.*

*Remark 1.* Here closure under composition should be understood in the sense of Lemma 5.7. I.e., given functions $F_1, \dots, F_n \in \mathcal{H}_{\log}$, the function $F_1 \circ \dots \circ F_n$ belongs to $\mathcal{B}_{\log}$ after a suitable restriction and conjugation with a translation; this map then also belongs to $\mathcal{H}_{\log}$.

*Remark 2.* It is easy to see that the class $\mathcal{H}_{\log}$ is also closed under *quasiconformal equivalence near infinity* in the sense of [R1].

*Proof.* The first claim is a combination of Theorem 5.6 and Proposition 5.4. The second follows from Lemma 5.7, and the final statement follows from Proposition 4.6. ■

To conclude the section, let us comment on the issue of "random iteration", where we are considering a sequence $\mathcal{F} = (F_0, F_1, F_2, \dots)$ of functions, and study the corresponding "escaping set" set $I(\mathcal{F}) = \{z \in \mathbb{C} : \mathcal{F}_n(z) \to \infty\}$, where $\mathcal{F}_n = F_n \circ F_{n-1} \circ \dots \circ F_0$. (Now the pairs $(T, T')$ will consist of a tract $T$ of $F_k$ and a tract $T'$ of $F_{k+1}$, etc.) Our proofs carry through analogously in this setting. In particular, if all tracts of all $F_j$ have uniformly bounded wiggling and uniformly bounded slope, then again for every $z \in I(\mathcal{F})$, there is some iterate $\mathcal{F}_n(z)$ which can be connected to infinity by a curve in the escaping set $I(F_n, F_{n+1}, \dots)$.

## 6. Counterexamples

This section is devoted to the proof of Theorem 1.1; that is, the construction of a counterexample to the strong form of Eremenko's Conjecture. As mentioned in the previous section, such an example will be provided by a function with a tract which has sufficiently large "wiggles".

We begin by formulating the exact properties our counterexample should have. Then we construct a tract (and hence a function $F \in \mathcal{B}_{\log}$) with the required properties. Finally, we show how to realize such a tract as that of an entire function $f \in \mathcal{B}$, using a function-theoretic principle.

To facilitate discussion in this and the next section, let us call an unbounded Jordan domain $T$ a *tract* if the real parts of $T$ are unbounded from above and all translates $T + 2\pi i n$ (for $n \in \mathbb{Z}$) have disjoint closure in $\mathbb{C}$.

**6.1. Theorem** (No Curve To Infinity)**.**
*Let $T \subset \mathbb{H}$ be a tract, and let $F_0 \colon T \to \mathbb{H}$ be a Riemann map, with continuous extension $F_0 \colon \widehat{T} \to \widehat{\mathbb{H}}$ given by Carathéodory's Theorem. Suppose that the following hold:*

(a) *$F_0(\infty) = \infty$;*
(b) *$|\operatorname{Im} z - \operatorname{Im} z'| < H$ for some $H < 2\pi$ and all $z, z' \in T$;*
(c) *there are countably many disjoint hyperbolic geodesics $C_k, \dot{C}_k \subset T$, for $k = 0, 1, \dots$, so that all $F_0(C_k)$ and $F_0(\dot{C}_k)$ are semi-circles in $\mathbb{H}$ centered at $0$ with radii $\varrho_{k+1}$ and $\dot{\varrho}_{k+1}$ so that $\varrho_1 < \dot{\varrho}_1 < \varrho_2 < \dots$;*
(d) *all $\varrho_k + H < \dot{\varrho}_k/2$ and all $\dot{\varrho}_k + H < \varrho_{k+1}/2$;*
(e) *all $C_k$ and $\dot{C}_k$ have real parts strictly between $\dot{\varrho}_k + H$ and $\varrho_{k+1}/2$;*
(f) *all points in the unbounded component of $T \setminus \dot{C}_k$ have real parts greater than $\dot{\varrho}_k$;*
(g) *every curve in $T$ which connects $C_k$ to $\dot{C}_k$ intersects the line $\{z \in \mathbb{C} \colon \operatorname{Re} z = \varrho_k/2\}$.*

*Define $T_n := T + 2\pi i n$ for $n \in \mathbb{Z}$ and $\mathcal{T} := \bigcup_n T_n$, define Riemann maps $F_n \colon T_n \to \mathbb{H}$ via $F_n(z) := F_0(z - 2\pi i n)$, and define a map $F \colon \mathcal{T} \to \mathbb{H}$ which coincides on $T_n$ with $F_n$ for each $n$.*

*Then the set $J := J(F) = \{z \in T \colon F^{\circ k}(z) \in \mathcal{T}$ for all $k\}$ contains no curve to $\infty$.*

*Proof.* Since the $T_n$ have disjoint closures, $F$ extends continuously to $\overline{\mathcal{T}}$. Let $R_k$ and $\dot{R}_k$ be semicircles in $\mathbb{H}$ centered at 0 with radii $\varrho_k$ and $\dot{\varrho}_k$, respectively.

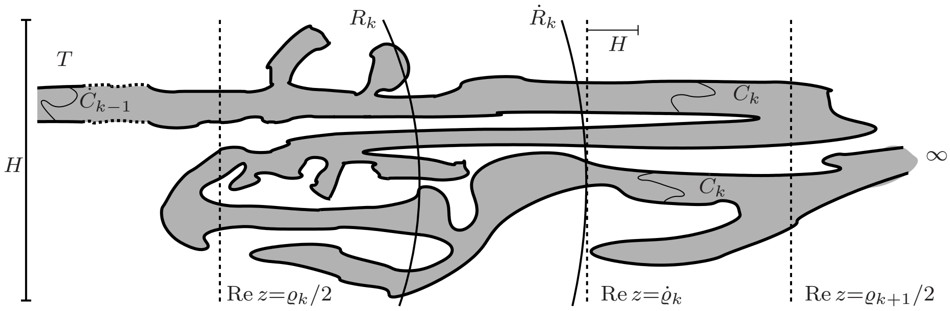


FIGURE 4. A tract that satisfies the conditions of Theorem 6.1. The figure is not to scale, as can be seen from the horizontal and vertical dimensions of the length $H$.

Every $z \in J$ has an external address $\underline{s} = T_{s_0}T_{s_1}T_{s_2}\dots$ so that $F^{\circ k}(z) \in T_{s_k}$ for all $k$. Clearly, all points within any connected component of $J$ have the same external address, so we may fix an external address $\underline{s}$ and show that the set $J_{\underline{s}}$ (i.e., the points in $J$ with address $\underline{s}$) contains no curve to $\infty$. We may assume that there is an orbit $(w_k)$ with external address $\underline{s}$ (if not, then we have nothing to show).

For simplicity, we write $C^k_m$ for $C_m + 2\pi i s_k$ and $\dot C^k_m$ for $\dot C_m + 2\pi i s_k$.

*Claim 1.* There is an $m \geq 0$ so that for all $k \geq 0$, $\dot C^k_{m+k}$ separates $w_k$ from $\infty$ within $T_{s_k}$, and $|w_{k+1}| < \dot\varrho_{m+k+1}$.

*Proof.* We prove this claim by induction, based on Condition (f): some $\dot C^0_m$ separates $w_0$ from $\infty$. For the inductive step, suppose that $\dot C^k_{m+k}$ separates $w_k$ from $\infty$ within $T_{s_k}$. Then $\dot R_{m+k+1}$ separates $w_{k+1}$ from $\infty$ within $\mathbb{H}$, i.e., $\operatorname{Re} w_{k+1} \leq |w_{k+1}| < \dot\varrho_{m+k+1}$ (Condition (c)). By Condition (f), it follows that $w_{k+1}$ is in the bounded component of $T_{s_{k+1}} \setminus \dot C^{k+1}_{m+k+1}$, so $\dot C^{k+1}_{m+k+1}$ separates $w_{k+1}$ from $\infty$ within $T_{s_{k+1}}$, and this keeps the induction going and proves the claim. $\triangle$

*Claim 2.* For all $k \geq 1$, the semicircle $R_{m+k+1}$ surrounds $C^k_{m+k}$, $\dot C^k_{m+k}$ and all points in $T_{s_k}$ with real parts at most $\varrho_{m+k+1}/2$.

*Proof.* Recall that $C^k_{m+k}$ and $\dot C^k_{m+k}$ have real parts at most $\varrho_{m+k+1}/2$ by Condition (e).

So suppose that $z \in T_{s_k}$ has $\operatorname{Re} z \leq \varrho_{m+k+1}/2$. We have $|\operatorname{Im} w_k| \leq |w_k| < \dot\varrho_{m+k}$ by the first claim, and since $T_{s_k}$ contains $w_k$ as well as $z$ and has height at most $H$ (Condition (b)), it follows that $|\operatorname{Im} z| < \dot\varrho_{m+k} + H$. So, by Condition (d),

$$|z| \leq \operatorname{Re} z + |\operatorname{Im} z| < \varrho_{m+k+1}/2 + \dot\varrho_{m+k} + H < \varrho_{m+k+1} \qquad \triangle$$

Now suppose there is a curve $\gamma \subset J_{\underline{s}}$ that converges to $\infty$, and suppose that $w_0$ was chosen with $w_0 \in \gamma$. For every $k \geq 0$, the curve $F^{\circ k}(\gamma)$ connects $w_k$ to $\infty$ (Condition (a)). The point $w_k$ is surrounded by both $R_{m+k+1}$ and $\dot R_{m+k+1}$: by the first claim, we have $|w_k| < \dot\varrho_{m+k} < \varrho_{m+k+1} < \dot\varrho_{m+k+1}$. As a result, $F^{\circ k}(\gamma)$ must contain a subcurve $\gamma_k$ connecting $R_{m+k+1}$ with $\dot R_{m+k+1}$. But this implies that $F^{\circ(k-1)}(\gamma)$ contains a subcurve $\gamma_{k-1}$ connecting $C^{k-1}_{m+k}$ with $\dot C^{k-1}_{m+k}$ (Condition c). Since $C^{k-1}_{m+k}$ and $\dot C^{k-1}_{m+k}$ have real parts greater than $\dot\varrho_{m+k} + H$ by Condition (e), it follows that both endpoints of $\gamma_{k-1}$ are outside of $\dot{R}_{m+k}$. But $\gamma_{k-1}$ must be contained within $T_{s_{k-1}}$, so Condition (g) implies that $\gamma_{k-1}$ must contain a point $z_{k-1} \in T_{s_{k-1}}$ with real part $\varrho_{m+k}/2$. Now the last claim shows that $z_{k-1}$ is surrounded by $R_{m+k}$. As a result, $\gamma_{k-1}$ must contain two disjoint subcurves that connect $R_{m+k}$ with $\dot{R}_{m+k}$.

Continuing the argument inductively, it follows that $\gamma$ contains $2^k$ disjoint subcurves that connect $C^0_{m+1}$ with $\dot{C}^0_{m+1}$. Since this is true for every $k \geq 0$, this is a contradiction. ■

Now we give conditions under which the set $J$ not only contains no curve to $\infty$, but in fact no unbounded curve at all. In many cases these conditions are satisfied automatically, such as in the example that we construct below (see Theorem 6.3).

**6.2. Corollary** (Bounded Path Components)**.**
*Suppose that, in addition to the conditions of Theorem 6.1, there are countably many disjoint hyperbolic geodesics $\ddot{C}_k \subset T$ so that all $F_0(\ddot{C}_k)$ are semi-circles in $\mathbb{H}$ centered at 0 with radii $\ddot{\varrho}_{k+1} > \dot{\varrho}_{k+1}$ so that the bounded component of $T \setminus \ddot{C}_{k+1}$ has real parts at most $\ddot{\varrho}_{k+1}/2$.*

*Then every path component of $J$ is bounded.*

*Proof.* We continue the proof of the previous theorem. Suppose there is an unbounded curve $\gamma \subset J_{\underline{s}}$ with $w_0 \in \gamma$. As before, for every $k \geq 0$ the curve $F^{\circ k}(\gamma)$ connects $w_k$ with $\infty$. We will show that then there must be a point $z_0 \in \gamma$ so that for every $k$ the subcurve of $\gamma$ between $w_0$ and $z_0$ contains $2^k$ disjoint subcurves that connect $C^0_{m+1}$ with $\dot{C}^0_{m+1}$, and this is a contradiction.

Let $\ddot{R}_k$ be semi-circles centered at 0 with radii $\ddot{\varrho}_k$. Since $\ddot{\varrho}_{k+1} > \dot{\varrho}_{k+1}$, it follows that every $\ddot{C}_k$ is in the unbounded component of $T \setminus \dot{C}_k$. Define vertical translates $\ddot{C}^k_m = \ddot{C}_m + 2\pi i s_k$ in analogy to the $C^k_m$ and $\dot{C}^k_m$. As in the second claim in the proof above, it follows that the bounded component of $T \setminus \ddot{C}_{k+1}$ is surrounded by $\ddot{R}_{m+k}$.

By the first claim in the proof above, $\dot{C}^0_m$ separates $w_0$ from $\infty$ within $T_{s_0}$, so $\ddot{C}^0_m$ and also $\ddot{C}^0_{m+1}$ must do the same. Let $z_0$ be a point in the intersection of $\gamma$ with $\ddot{C}^0_{m+1}$ and denote by $[w_0, z_0]_\gamma$ the subcurve of $\gamma$ connecting $w_0$ with $z_0$. Then $F([w_0, z_0]_\gamma)$ connects $w_1$ with $F(z_0) \in \ddot{R}_{m+2}$. So $F(z_0)$ belongs to the unbounded component of $T \setminus \ddot{C}^1_{m+2}$, and there is thus a point $z_1 \in [w_0, z_0]_\gamma$ with $F(z_1) \in \ddot{C}^1_{m+2}$, and $F^{\circ 2}([w_0, z_1]_\gamma)$ connects $w_2$ with $\ddot{R}_{m+3}$. By induction, for any $k \geq 0$, the curve $F^{\circ k}([w_0, z_{k-1}]_\gamma)$ connects $w_k$ with $\ddot{R}_{m+k+1}$ and hence it connects $R_{m+k+1}$ with $\dot{R}_{m+k+1}$.

The same arguments from the proof of the theorem now show that in fact $[w_0, z_k]_\gamma \subset [w_0, z_0]_\gamma$ must contain $2^k$ subcurves connecting $C^0_{m+1}$ with $\dot{C}^0_{m+1}$ for every $k \geq 0$, and this is the desired contradiction. ■

*Remark.* We stated the results in the form above in order to minimize the order of growth of the resulting entire function, and to show that the entire functions we construct can be rather close to finite order; see Section 8. If we were only interested in the non-existence of unbounded path components in $I$, we could have formulated conditions that are somewhat simpler than those in the preceding theorem and its corollary. For instance, the necessity for introducing a third geodesic $\ddot{C}_k$ would have been removed if we had placed $C_k$ at real parts at most $\varrho_k/2$, and required that the entire bounded component of $T \setminus C_k$ has real parts less than $\varrho_k/2$ (the image of any curve in $T$ connecting $\dot{C}^k_{m+k}$ with $C^k_{m+k+2}$ would then connect $\dot{C}^{k+1}_{m+k+1}$ and $C^{k+1}_{m+k+3}$; this keeps the induction going as before.)

**6.3. Theorem** (Tract with Bounded Path Components)**.**
*There exist a tract $T$ with $\overline{T} \subset \mathbb{H}$ and a conformal isomorphism $F : T \to \mathbb{H}$ fixing $\infty$ which satisfies the conditions of Theorem 6.1 and Corollary 6.2, so every path component of $J$ is bounded.*

*In fact, $T$ and $F$ can be chosen so as to satisfy the following conditions for an arbitrary $M \in (1, 1.75)$ (with the same notation as in Theorem 6.1):*

(d') $\varrho_k^M < \dot{\varrho}_k$ *and* $\dot{\varrho}_k^M < \varrho_{k+1}$*;*
(e') *the geodesics $C_k$ and $\dot{C}_k$ have real parts strictly between $\dot{\varrho}_k^M$ and $\varrho_{k+1}/3$;*
(f') *all points in the unbounded component of $T \setminus \dot{C}_k$ have real parts greater than $\dot{\varrho}_k^M$;*
(g') *every curve in $T$ which connects $C_k$ to $\dot{C}_k$ intersects the line $\{z \in \mathbb{C} : \mathrm{Re}(z) = \varrho_k^{1/M}\}$;*
(h') *the geodesics $\ddot{C}_{k+1}$ from Corollary 6.2 have the property that the bounded component of $T \setminus \ddot{C}_{k+1}$ has real parts at most $(\ddot{\varrho})^{1/M}$.*

*Remark.* It is easy to see that the modified conditions indeed imply the original conditions at least for sufficiently large $\varrho_k$, and this can be assured by a sufficiently large choice of $x_1$ in the proof below. The modified conditions as written in this theorem are needed in order to show that this tract is "approximately" realized by an entire function: they are adapted to the quality of the approximation that we get later in this section.

*Proof.* Our domain $T$ will be a countable union of long horizontal tubes of unit thickness, together with countably many vertical tubes and countably many turns made of quarter and half annuli, all of unit thickness as well (see Figure 5). The domain $T$ terminates at the far left with a semidisk at center $P$. The lengths of the various tubes are labelled as in Figure 6.

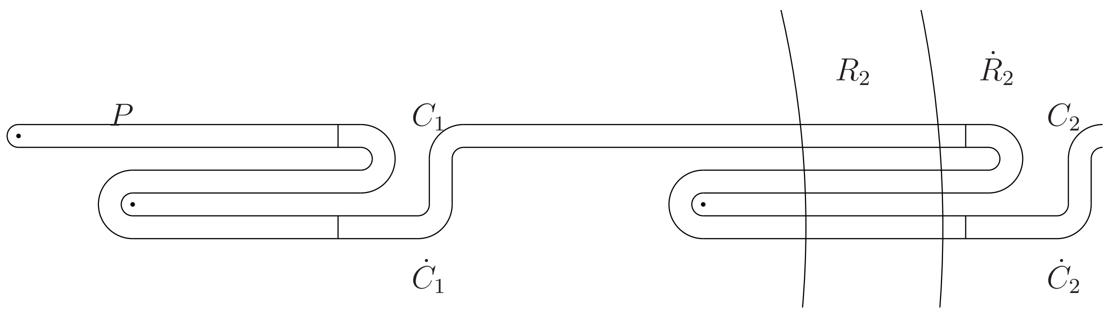


FIGURE 5. *Construction of an example for Theorem 6.3.*

The tract $T$ is specified completely by the following data:

- the position of the base point $P$;

- the universal lengths $h$, $h'$ and $h''$;
- the exponent $M > 1$;
- the horizontal lengths $x_k$ and $\dot{x}_k$.

The Riemann map $F\colon T \to \mathbb{H}$ is then specified completely by the choice of $F(P)$.

We choose $h = 1$, so that all tubes have width 1, and choose $h' = 3/2$ and $h'' = 2$. Choose $P = 1$ and $F(P) = P$. Choose initial values $x_0$ and $\dot{x}_0$ so that both $x_0$ and $\dot{x}_0 - x_0$ are sufficiently large (see below). Then define recursively

$$x_{k+1} := \exp\left(\dot{x}_k^M\right) \qquad \text{and} \qquad \dot{x}_{k+1} := \exp\left(12\dot{x}_k^M\right) \ .$$

In order for the geometry to be feasible (so that the left end of the new wiggle is disjoint from the previous wiggles), we need

$$x_{k+1}^{1/M} - \dot{x}_k^M > h' + h'' + 4h \tag{2}$$

But as soon as $\dot{x}_k$ is sufficiently large, this follows from the definition $x_{k+1} = \exp\left(\dot{x}_k^M\right)$. We will have a number of inequalities below that hold only if $x_0$ (and hence all $x_k$ and $\dot{x}_k$) are sufficiently large; we use the symbol "$\star$" to mark such inequalities (e.g. "$\overset{\star}{<}$").

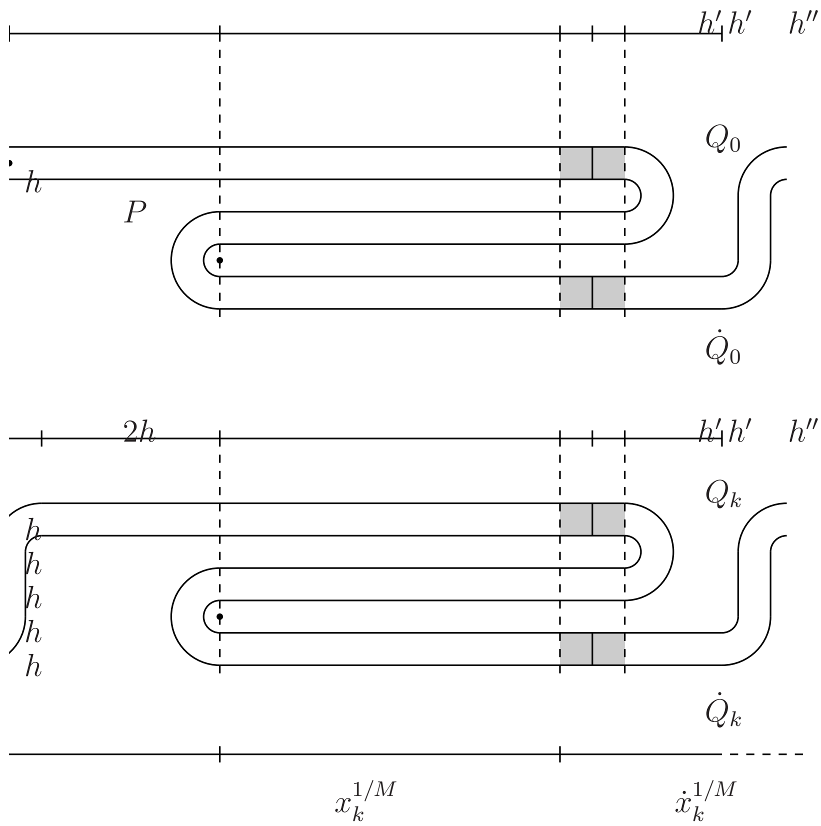


Figure 6. *The length scales in the construction. Top: the first wiggle; bottom: the general wiggle. The lengths $h$, $h'$ and $h''$ are universal for all $k$. Shaded are the boxes $Q_k$ and $\dot{Q}_k$.*

We thus have a well-defined tract $T$ (depending on $x_1$ and $\dot{x}_1$) and need to show that it satisfies all conditions of of Theorem 6.1. Conditions (a) and (b) are satisfied to begin with, and $H = 5 < 2\pi$. Define points $P_k := \dot{x}_k^M + h'$ and $\dot{P}_k := P_k - 4hi$ and set $\varrho_{k+1} := |F(P_k)|$ and $\dot{\varrho}_{k+1} := |F(\dot{P}_k)|$. Let $R_{k+1}$ and $\dot{R}_{k+1}$ be the semicircles around 0 with radii $\varrho_{k+1}$ and $\dot{\varrho}_{k+1}$ and let $C_k := F^{-1}(R_{k+1})$ and $\dot{C}_k := F^{-1}(\dot{R}_{k+1})$.

Then $C_k$ and $\dot{C}_k$ are hyperbolic geodesics of $T$, and by Lemma A.3, they are contained in the boxes $Q_k := \{z \in \mathbb{C}\colon \operatorname{Re}(z) \in (x_k^M, x_k^M + 2h'), |\operatorname{Im} z| < 1/2\}$ and $\dot{Q}_k := Q_k - 4hi$, and they connect the upper with the lower boundaries of their boxes. As a result, Condition (c) is satisfied automatically.

Now we prove the remaining conditions inductively; we will maintain the additional inductive hypothesis $x_k < \varrho_k < \dot{\varrho}_k < \dot{x}_k$ (the initial values $\varrho_0$ and $\dot{\varrho}_0$ have no meaning; rather than modify the statements for $k = 0$, simply choose arbitrary values $\varrho_0 < \dot{\varrho}_0$ in $(x_0, \dot{x}_0)$). First we need to estimate $\varrho_{k+1}$ and $\dot{\varrho}_{k+1}$; it is clear that $\varrho_{k+1} < \dot{\varrho}_{k+1}$. We have

$$\log(\varrho_{k+1}) = d_{\mathbb{H}}(P, R_{k+1}) = d_T(P, C_k) < 4\left(\dot{x}_K^M + 2\dot{x}_{k-1}^M\right) \overset{\star}{<} (4\pi/3)\,\dot{x}_k^M$$

(the first inequality follows from the fact that every smooth curve in $T$ connecting $P$ with $C_k$ must connect real part 1 to real part $\dot{x}_k^M$, and the wiggles force it to traverse certain real parts in $(1, \dot{x}_{k-1}^M)$ two more times. The extra vertical bits contribute less than the horizontal bits that are counted three times but traversed only once. We can thus connect $P$ to $C_k$ by a smooth curve in $T$ that has Euclidean length at most $\dot{x}_k^M + 2\dot{x}_{k-1}^M$. If we choose the curve so as to run always in the center of the tract, then it maintains distance 1/2 from the boundary of $T$, and the hyperbolic length exceeds the Euclidean length at most by a factor of 4. We will use this kind of reasoning several times in the sequel.) We rewrite this as

$$\varrho_{k+1} < \exp\left((4\pi/3)\,\dot{x}_k^M\right) \ . \tag{3}$$

A lower bound follows in a similar way:

$$\log \varrho_{k+1} = d_{\mathbb{H}}(P, R_{k+1}) = d_T(P, C_k) > \dot{x}_k^M$$

and thus

$$\varrho_{k+1} > \exp(\dot{x}_k^M) = x_{k+1} \ . \tag{4}$$

An upper bound for $\dot{\varrho}_{k+1}$ is as follows:

$$\log \dot{\varrho}_{k+1} = d_{\mathbb{H}}(P, \dot{R}_{k+1}) = d_T(P, \dot{C}_k) < 12\dot{x}_k^M$$

and thus

$$\dot{\varrho}_{k+1} < \exp\left(12\dot{x}_k^M\right) = \dot{x}_{k+1} \ . \tag{5}$$

This establishes the additional inductive hypothesis $x_{k+1} < \varrho_{k+1} < \dot{\varrho}_{k+1} < \dot{x}_{k+1}$.

Now we show that $\varrho_{k+1}$ and $\dot{\varrho}_{k+1}$ satisfy condition (d'). The subdomain of $T$ bounded by $C_k$ and $\dot{C}_k$ maps under $F$ conformally onto the semi-annulus in $\mathbb{H}$ between radii $R_{k+1}$ and $\dot{R}_{k+1}$, so their moduli are equal, and we see by the Grötzsch inequality that

$$\frac{1}{\pi}\log(\dot{\varrho}_{k+1}/\varrho_{k+1}) > 2\left(\dot{x}_k^M - x_k^{1/M}\right) > 2\left(\dot{x}_k^M - \dot{x}_k\right) \overset{\star}{>} \dot{x}_k^M$$

and so, using (3)

$$\dot{\varrho}_{k+1} > \varrho_{k+1} \exp\left(\pi \dot{x}_k^M\right) = \varrho_{k+1} \left(\exp\left((4\pi/3)\dot{x}_k^M\right)\right)^{3/4} > \varrho_{k+1}^M \ . \tag{6}$$

For the second inequality, we use the subdomain of $T$ bounded by $C_k$ and $\dot{C}_{k-1}$; it maps conformally onto the semi-annulus between $R_{k+1}$ and $\dot{R}_k$. We estimate

$$\frac{1}{\pi}\log(\varrho_{k+1}/\dot{\varrho}_k) > \dot{x}_k^M - \dot{x}_k \overset{\star}{>} \dot{x}_k^M/2 > \dot{\varrho}_k/2$$

and thus

$$\varrho_{k+1} > \dot{\varrho}_k \exp\left(\pi \dot{\varrho}_k/2\right) > \dot{\varrho}_k^M \ . \tag{7}$$

For Condition (e'), the construction of $C_{k+1}$ and $\dot{C}_{k+1}$ is such that their real parts are at least $\dot{x}_{k+1}^M > \dot{\varrho}_{k+1}^M$ and at most $\dot{x}_{k+1}^M + 2h' < x_{k+2}^{1/M} \overset{\star}{<} x_{k+2}/3 < \varrho_{k+2}/3$ (see (2)), so this condition is satisfied.

Condition (f') is obvious: the construction assures that all points in the unbounded component of $T \setminus \dot{C}_{k+1}$ have real parts at least $\dot{x}_{k+1}^M > \dot{\varrho}_{k+1}^M$.

Finally, every curve in $T$ which connects $C_{k+1}$ with $\dot{C}_{k+1}$ must reach real parts less than $x_k^{1/M} < \varrho_k^{1/M}$, and this is the final Condition (g').

Now we show that the conditions of Corollary 6.2 are satisfied, even in the stronger form (h'), so that all path components of $J$ are bounded. Define $\ddot{\varrho}_k := \exp\left(M(12M+1)\dot{x}_k^M\right)$, let $\ddot{R}_k$ be the semicircles in $\mathbb{H}$ centered at 0 with radii $\ddot{\varrho}_k$, and let $\ddot{C}_k$ be the $F_0$-preimage of $\ddot{R}_{k+1}$ within $T$. We show that the real parts of $\ddot{C}_k$ are at most $(\ddot{\varrho}_k)^{1/M}$.

Let $s := d_T(P, \ddot{C}_k) = d_{\mathbb{H}}(P, \ddot{R}_{k+1}) = \log \ddot{\varrho}_{k+1}$. Within $T$, we can estimate $s > \operatorname{Re}(\ddot{C}_k)$, so we obtain

$$\begin{aligned} \operatorname{Re}(\ddot{C}_k) \ &< \ s = \log \ddot{\varrho}_{k+1} = M(12M+1)\dot{x}_{k+1}^M = M(12M+1)\left(\exp\left(12\dot{x}_k^M\right)\right)^M \\ &= M(12M+1)\exp\left(12M\dot{x}_k^M\right) \overset{\star}{<} \exp\left((12M+1)\dot{x}_k^M\right) = (\ddot{\varrho}_k)^{1/M} \end{aligned}$$

as claimed. Since $\ddot{C}_k$ is between $\dot{C}_k$ and $C_{k+1}$, the construction is such that all points in the bounded component of $T \setminus \ddot{C}_k$ have real parts bounded by the real parts of $\ddot{C}_k$. ■

In order to complete the proof of Theorem 1.1, we need to show that there is an entire function which suitably approximates the previously constructed map. To this end, we will use the following fact on the existence of entire functions with a prescribed tract; a proof can be found in the next section. For simplicity, let us refer to geodesics $C$ as in the previous results, which are mapped to semi-circles centered at 0 under the map $F : T \to \mathbb{H}$, as *vertical geodesics* (of $F$).

**6.4. Proposition** (Approximation by entire functions).
*Let $T$ be a tract, and let $F : T \to \mathbb{H}$ be a conformal isomorphism fixing $\infty$. Let $\theta > 1$.*

*Then there is an entire function $g$ with $\overline{\operatorname{sing}(f^{-1})} \subset \mathbb{D}$ and a single tract $W = g^{-1}(\{|z| > 1\})$ such that the logarithmic transform $G : \log W \to \mathbb{H}$ of $g$ has the following properties:*

(a) *$\log W$ has a component $\widetilde{T}$ satisfying $\widetilde{T} \subset T$;*
(b) *the vertical geodesics of $G$ have uniformly bounded diameters;*

(c) $|F(z)| \leq |G(z)| \leq |F(z)|^\theta$ *when* $z \in \widetilde{T}$ *with* $\operatorname{Re} z$ *sufficiently large.*

*Remark.* If we apply the above proposition to a tract $T$ with $\overline{T} \subset \mathbb{H}$ (such as the one from Theorem 6.3), then the resulting function $g$ satisfies $f(\mathbb{D}) \Subset \mathbb{D}$. It follows that the postsingular set is compactly contained in the Fatou set of $g$, and hence that $g$ is hyperbolic.

*Proof of Theorem 1.1 (using Proposition 6.4).* Let $F \in \mathcal{B}_{\log}$ be the function constructed in Theorem 6.3, and let $T$ be its single tract. Choose $1 < \theta < M$ (where $M$ is the constant from Theorem 6.3). Let $g$ be a function as in Proposition 6.4, with logarithmic transform $G : \widetilde{T} \to \mathbb{H}$. (Recall that $G$ extends continuously to the closure $\operatorname{cl}(\widetilde{T})$.)

The vertical geodesics $C_k$ and $\dot{C}_k$ of $T$ intersect $\widetilde{T}$ for sufficiently large $k$. Let $\dot{\sigma}_{k+1}$ be maximal with the property that the geodesic $\dot{D}_k := \{z \in \operatorname{cl}(\widetilde{T}) : |G(z)| = \dot{\sigma}_{k+1}\}$ intersects $\dot{C}_k$, and define $\sigma_{k+1}$ and $D_k$ similarly. We claim that, with this choice of geodesics, the function $G$ also satisfies the conclusions of Theorem 6.3 (for a constant $M' < M/\theta$).

Indeed, by (c) of Proposition 6.4, we have $\varrho_k \leq \sigma_k \leq \varrho_k^\theta$ and $\dot{\varrho}_k \leq \dot{\sigma}_k \leq \dot{\varrho}_k^\theta$. Thus

$$\sigma_k^{M'} \leq \varrho_k^M < \dot{\varrho}_k \leq \dot{\sigma}_k \quad \text{and} \quad \dot{\sigma}_k^{M'} \leq \dot{\varrho}_k^M < \varrho_{k+1} \leq \sigma_k.$$

Thus (d') holds. (e') and (g') follow similarly, using the fact that the geodesics $D_k$ and $\dot{D}_k$ have uniformly bounded diameters.

To prove (f'), note that the unbounded component of $\widetilde{T} \setminus \dot{D}_k$ does not intersect $\dot{C}_k$ since we chose $\dot{\sigma}_{k+1}$ to be maximal. Hence it follows from condition (f') for $F$ that this component has real parts at least $\dot{\varrho}_k^M \geq \dot{\sigma}_k^{M'}$. Property (h') follows analogously. ■

## 7. Entire functions with prescribed tracts

We will now prove Proposition 6.4, using the method of approximating a given tract using Cauchy integrals, which is well-established. (Compare e.g. [GE] for a similar construction.) However, there appears to be no result stated in the literature which is immediately applicable to our situation. Hence we will first provide a proof of the following, more classical-looking statement, and then proceed to indicate how it implies Proposition 6.4.

**7.1. Proposition** (Existence of functions with prescribed tracts)**.**
*Let $V \subset \mathbb{C}$ be an unbounded Jordan domain and let $\Psi : V \to \mathbb{H}$ be a conformal isomorphism with $\Psi(\infty) = \infty$. Let $\varrho$ be arbitrary with $1 < \varrho < 2$ and define*

$$f : V \to \mathbb{C}; z \mapsto \exp\big((\Psi(z))^\varrho\big).$$

*Then there exists an entire function $g \in \mathcal{B}$ and a number $K > 0$ such that the following hold:*

(a) *$W := \{z : |g(z)| > K\}$ is a simply connected domain which is contained in $V$, and $g|_W$ is a universal covering;*
(b) *$|g(z) - f(z)| = O(1)$ on $W$, and $g(z) = O(1)$ outside $W$.*

*Remark.* In particular, the tract $W$ of $g$ satisfies

$$V \supset W \supset \{z : \operatorname{Re}\Psi(z) > C \text{ and } |\arg \Psi(z)| < \theta\}$$

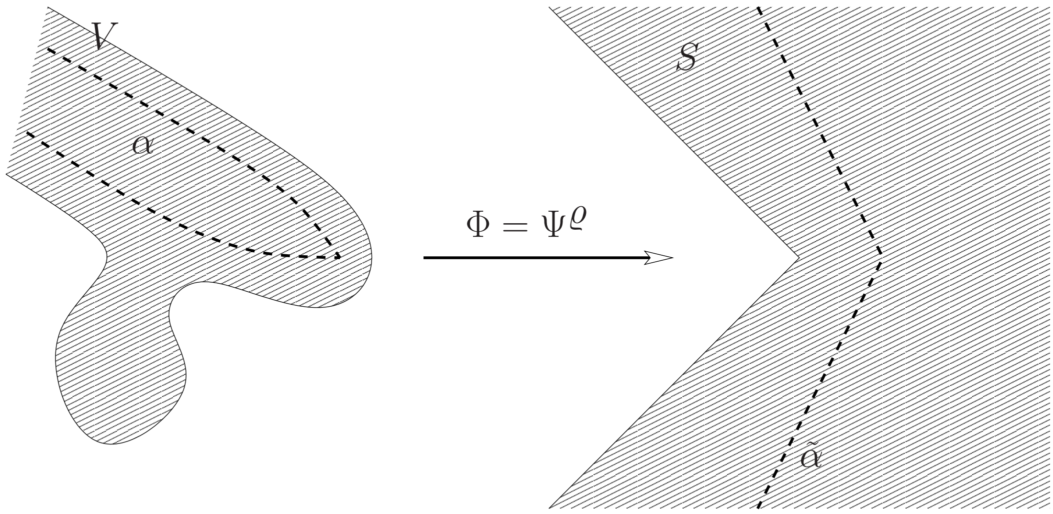


FIGURE 7. Definition of $\alpha$ and $\tilde{\alpha}$ in the proof of Proposition 7.1

(where $\theta$ can be chosen arbitrarily close to $\pi/2\varrho$ if $C$ is sufficiently large). So this proposition really does present a result on the realization of a prescribed tract (up to a certain "pruning" of the edges) by an entire function.

*Proof.* The idea of the proof is simple: we define a function $h$, using an integral along the boundary $\alpha$ of the desired tract, which changes by $f(z)$ as $z$ crosses the curve $\alpha$. That is, we set

$$h(z) := \int_\alpha \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We are using $\Psi^\varrho$, rather than $\Psi$ itself, in the definition of $f$ to ensure that this integral converges uniformly and that $h$ is bounded. Then the function $g$ which agrees with $h$ on the outside of $\alpha$ and with $h + f$ on the inside will be entire, and it follows easily that it is in class $\mathcal{B}$.

Let us now provide the details of this argument. We define $\Phi := \Psi^\varrho$ and let $S$ denote the sector $S := \Phi(V) = \left\{z : |\arg z| < \frac{\varrho\pi}{2}\right\}$. Also fix some $\eta \in (\pi/2, \varrho\pi/2)$ and set $\lambda := \exp(i\eta)$. We define

$$\tilde{\alpha} : (-\infty, \infty) \to S; t \mapsto \begin{cases} 1 + \lambda t & t \geq 0 \\ 1 + \overline{\lambda}|t| & t < 0 \end{cases} \quad \text{and} \quad \alpha := \Phi^{-1} \circ \tilde{\alpha}.$$

Let $V'$ denote the component of $\mathbb{C} \setminus \alpha$ with $V' \subset V$.

*Claim 1.* The integral $\int_\alpha f(\zeta) d\zeta$ converges absolutely. In particular,

$$h(z) := \int_\alpha \frac{f(\zeta)}{\zeta - z} d\zeta.$$

defines a holomorphic function $h : \mathbb{C} \setminus \alpha \to \mathbb{C}$.

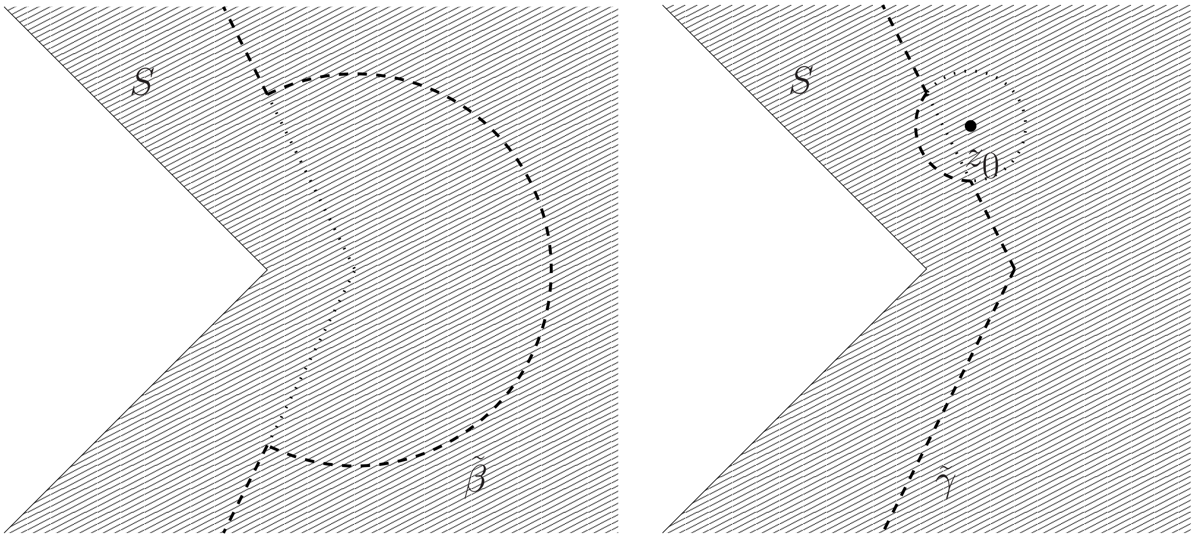


FIGURE 8. Definition of $\tilde{\beta}$ and $\tilde{\gamma}$ as in Claims 2 and 3

*Proof.* Note that $|\alpha'(t)| = |1/\Phi'(\alpha(t))|$ for $t \neq 0$. By the Schwarz lemma and Koebe's theorem, we have

$$|\Phi'(\alpha(t))| \geq \frac{\operatorname{dist}(\tilde{\alpha}(t), \partial S)}{4\operatorname{dist}(\alpha(t), \partial V)}.$$

Clearly $\operatorname{dist}(\tilde{\alpha}(t), \partial S) \geq C_1(1+|t|)$ for some $C_1 > 0$. Furthermore, using the standard estimates on the hyperbolic metric of $V$, it is easy to see that $\alpha(t)$ grows at most polynomially in $t$; in particular, $\operatorname{dist}(\alpha(t), \partial V) \leq C_2(1+|t|)^c$ for some $c, C_2 > 0$.

Together, these estimates imply by the Koebe theorem that $|\Phi'(\alpha(t))| \geq C/4(1+|t|)^{c-1}$ for $C := C_1/C_2$. In particular,

$$\begin{aligned}\int_\alpha |f(\zeta)d\zeta| &= \int_{-\infty}^{+\infty} \exp(\operatorname{Re}\tilde{\alpha}(t))|\alpha'(t)|dt \\ &= \int_{-\infty}^{+\infty} \frac{\exp(1-|\operatorname{Re}(\lambda)t|)}{|\Phi'(\alpha(t))|}dt \leq \frac{e}{C}\int_{-\infty}^{+\infty}(1+|t|)^{c-1}e^{-|\operatorname{Re}(\lambda)t|}dt < \infty.\end{aligned}$$

This completes the proof. △

*Claim 2.* The function

$$g(z) := \begin{cases} h(z) & z \notin \overline{V'} \\ h(z) + f(z) & z \in V' \end{cases}$$

extends to an entire function $g : \mathbb{C} \to \mathbb{C}$.

*Proof.* Let $R \gg 1$ be arbitrary, and modify $\tilde{\alpha}$ to obtain a curve

$$\tilde{\beta} := (\tilde{\alpha} \cap \{|\zeta| > R\}) \cup \{1 + Re^{2\pi i\theta} : \theta \in [-\eta, \eta]\}.$$

Set $\beta := \Phi^{-1} \circ \tilde{\beta}$ and let $W$ be the unbounded component of $\mathbb{C} \setminus \beta$ which contains $\mathbb{C} \setminus \overline{V'}$. Then

$$\widetilde{g} : W \to \mathbb{C}; z \mapsto \int_\beta \frac{f(\zeta)}{\zeta - z}d\zeta$$

defines an analytic function on $W$. By the Cauchy integral theorem, $\widetilde{g}$ agrees with $g$ on $\mathbb{C} \setminus \overline{V'}$.

Furthermore, for $z \in W \cap V'$, we have by the residue theorem that

$$\widetilde{g}(z) - h(z) = \operatorname{res}_z\Big(\frac{f(\zeta)}{\zeta - z}\Big) = f(z).$$

In particular, $\widetilde{g} = g|_W$. Since $R$ was arbitrary, the claim follows. △

*Claim 3.* The function $h$ is uniformly bounded.

*Proof.* Let $z_0 \in \mathbb{C}\backslash\alpha$. We set $\delta := \sin(\eta) = \operatorname{dist}(\tilde{\alpha}, \partial S)$ and define a curve $\tilde{\gamma}$ (depending on $z_0$) as follows. If $z_0 \notin V$ or if $z_0 \in V$ and $\operatorname{dist}(\Phi(z_0), \tilde{\alpha}) \geq \delta/2$, we simply set $\tilde{\gamma} := \tilde{\alpha}$. Otherwise, we set

$$\tilde{\gamma} := (\tilde{\alpha} \setminus \{z : |z - \Phi(z_0)| < \delta/2\}) \cup C,$$

where $C$ is the arc of the circle $\{|z - \Phi(z_0)| = \delta/2\}$ for which $\tilde{\alpha} \cup C$ does not separate $\Phi(z_0)$ from $\infty$.

We also set $\gamma := \Phi^{-1} \circ \tilde{\gamma}$. By Cauchy's integral theorem, we have

$$h(z_0) = \int_\alpha \frac{\exp(\Phi(\zeta))}{\zeta - z} d\zeta = \int_\gamma \frac{\exp(\Phi(\zeta))}{\zeta - z} d\zeta.$$

Thus it is sufficient to show that the second integral is bounded independently of $z_0$. By the Koebe 1/4-theorem and the definition of $\gamma$, we have $|\gamma(t) - z_0| \geq \delta/8|\Phi'(\gamma(t))|$ for all $t$. If we parametrize $\tilde{\gamma}$ by arclength, then clearly

$$\operatorname{Re} \tilde{\gamma}(t) \leq C - K|t|,$$

where the constants $K = |\operatorname{Re} \lambda|$ and $C$ are independent of $z_0$. We thus have

$$\begin{aligned} |h(z_0)| &= \left|\int_\gamma \frac{\exp(\Phi(\zeta))}{\zeta - z_0} d\zeta\right| \leq \int_{-\infty}^{+\infty} \frac{|\exp(\tilde{\gamma}(t))|}{|\gamma(t) - z_0|} |\tilde{\beta}'(t)| dt = \int_{-\infty}^{+\infty} \frac{\exp(\operatorname{Re} \tilde{\gamma}(t))}{|\Phi'(\gamma(t))| \cdot |\gamma(t) - z_0|} dt \\ &\leq \int_{-\infty}^{+\infty} \exp(C - K|t|) \frac{8|\Phi'(\gamma(t))|}{\delta|\Phi'(\gamma(t))|} dt = \frac{8}{\delta} \int_{-\infty}^{+\infty} \exp(C - K|t|) dt < \infty. \end{aligned}$$

So $h(z_0)$ is uniformly bounded, as required. △

To complete the proof, let $M > 0$ such that $|h(z)| < M$ for all $z$. Set $K := 2M$ and $W := \{z \in \mathbb{C} : |g(z)| > K\}$. If $\beta$ is a simple closed curve in $W$, then $|\Phi(z)| > M$ on $\beta$. By the minimum principle, we also have $|\Phi(z)| > M$ on the region $U$ surrounded by $\beta$. It follows that $g$ has no zeros in $U$, and by the minimum principle $U \subset W$. Thus $W$ is simply connected.

We can therefore define a function

$$G := \log g : W \to \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta > \log(2M)\}.$$

It is easy to see that $G$ is proper. Since there is exactly one homotopy class of curves in $W$ along which $G(z) \to \infty$, the degree of $G$ is 1. In other words, $G$ is a conformal isomorphism, and $f|_W = \exp \circ G$ is a universal cover, as required. ■

*Proof of Proposition 6.4.* Let $V := \exp(T)$, and let $\Psi : V \to \mathbb{H}$ be the conformal isomorphism with $\Psi \circ \exp = F$. Let $1 < \varrho < \min(\theta, 2)$, let $f$ be as in Proposition 7.1, and let $\widetilde{g}$ be the entire function constructed there. Recall that this function satisfies $|\widetilde{g}(z) - f(z)| = O(1)$ on its tract $W = \widetilde{g}^{-1}(\{|z| > K\})$. It easily follows that the logarithmic transform $\widetilde{G}$ also satisfies $|\widetilde{G}(z) - F(z)^\varrho| \leq C_1$ for some $C_1 > 0$.

Now set $g(z) := \widetilde{g}(z)/K$, and let $G : \widetilde{T} \to \mathbb{H}$ be its logarithmic transform; i.e. $G(z) = \widetilde{G}(z) - \log K$. We claim that $g$ is the desired entire function. Indeed, by choice of $\widetilde{g}$, we have

$$|(F(z))^\varrho - G(z)| \leq C_1 + \log K =: C,$$

which proves (c).

To complete the proof, let $\gamma = \{z \in \widetilde{T} : |G(z)| = R\}$ be a vertical geodesic (where $R$ is sufficiently large; say $R \geq C + 1$). We need to prove that the diameter of $\gamma$ is bounded independently of $R$. So let $z \in \gamma$. Then $|F(z)^\varrho - G(z)| \leq C$, which implies that

$$\big||F(z)| - R^{1/\varrho}\big| \leq C \quad \text{and} \quad |\arg F(z)| < \pi/(1+\varepsilon)$$

(where $\varrho = 1 + 2\varepsilon$), provided $R$ was chosen large enough.

The hyperbolic diameter of the subset of $\mathbb{H}$ described by these inequalities — and hence that of $F(\gamma)$ — is uniformly bounded. Since $F : T \to \mathbb{H}$ is a conformal isomorphism, the usual estimate on the hyperbolic metric of $T$ implies that the euclidean diameter of $\gamma$ is uniformly bounded as well. ■

## 8. Properties of the counterexample

The goal of this section is to indicate how the counterexample $f$ from Theorem 1.1 (constructed in Section 6) can be strengthened in various ways. We begin by discussing the growth behavior of the function $f$, and how to modify the construction to reduce this growth further. The section concludes with a sketch of the construction of a hyperbolic entire function whose Julia set contains no nontrivial curves at all.

**Order of growth.** By Theorem 1.2, we know that the counterexample $f$ from Theorem 1.1 cannot have finite order; that is, we cannot have $\log\log|f(z)| = O(\log|z|)$. We now see that its growth is not all that much faster than this.

**8.1. Proposition** (Growth of counterexample)**.**
*The function $f$ constructed in the proof of Theorem 1.1 satisfies*

$$\log\log|f(z)| = O\left((\log|z|)^{12M^2}\right) .$$

*Proof.* We verify that the function $F : T \to \mathbb{H}$ from Theorem 6.3 satisfies

$$\log \operatorname{Re} F(z) = O\left((\operatorname{Re} z)^{12M^2}\right). \tag{8}$$

(The claim then follows immediately from the fact that $f$ is obtained from $F$ by applying Proposition 6.4).

We use the notation of the proof of Theorem 6.3 (recall Figure 6). Pick points $p_k$ with real parts $\operatorname{Re}(p_k) = x_k^{1/M}$ and satisfying $F(p_k) \in (\varrho_{k+1}, \infty)$ (that is, $p_k$ lies half-way between the geodesics $C_k$ and $\dot{C}_k$, in the place where $T$ "turns around": it is here that the values of $\operatorname{Re} F(z)$ is largest in terms of $\operatorname{Re} z$). It is not difficult to see that it is sufficient to verify (8) when $z = p_k$. (In other parts of the tract, $\log \operatorname{Re} F(z)$ increases at most linearly with $\operatorname{Re} z$.)

We have

$$\log \operatorname{Re} F(p_k) \leq \log \dot{\varrho}_{k+1} < \log \dot{x}_{k+1} = 12\dot{x}_k^M.$$

It remains to estimate $\dot{x}_k$ in terms of $\operatorname{Re}(p_k) = x_k^{1/M}$, which we can do from the definition:

$$\dot{x}_k = \exp(12\dot{x}_{k-1}^M) = x_k^{12}, \tag{9}$$

so

$$\log \operatorname{Re} F(p_k) \leq 12\dot{x}_k^M \leq 12 x_k^{12M} = 12 \operatorname{Re}(p_k)^{12M^2},$$

as required. ■

We are now going to discuss how to improve the growth behavior of $f$. Recall that $M > 1$ was arbitrary; we will show how to reduce the constant 12 in the growth estimate to any number greater than 1. Note that the main estimate which influenced the growth of $f$ in the previous proof was (9), which estimates $\dot{x}_k$ in terms of $x_k$. We can improve the growth behavior of our counterexample by making thinner the part of the tract leading up to $C_{k-1}$: this will increase $R_k$ and hence $x_k$, while keeping $\dot{x}_k/x_k$ essentially the same.

More precisely, consider a tract described by a variation of Figure 6, where the upper of the three horizontal tubes connecting real parts $x_k^{1/M}$ and $\dot{x}_k^M$ has small height $\delta > 0$, while the other two tubes remain at unit height. Then $x_{k+1}$ will be roughly of size $\exp(\dot{x}_k^M/\delta)$, while $\dot{x}_{k+1}$ is still of size

$$\dot{x}_{k+1} \sim x_{k+1} \cdot \exp(C\dot{x}_k^M).$$

In other words, we will have

$$\dot{x}_{k+1} \lesssim x_{k+1}^{1+\delta C}.$$

We can then estimate $\log \operatorname{Re} F(p_k)$ as in Proposition 8.1 to see that

$$\log \operatorname{Re} F(p_k) \leq C \operatorname{Re}(p_k)^{(1+\delta C)M^2}.$$

By letting $\delta > 0$ and $M > 1$ be sufficiently small, we have obtained the following result.

**8.2. Proposition** (Counterexamples of mild growth)**.**
*For every $\varepsilon > 0$, there is a hyperbolic function $f \in \mathcal{B}$ such that $J(f)$ has no unbounded path-connected components, and such that*

$$\log \log |f(z)| = O((\log |z|)^{1+\varepsilon}). \quad ■$$

Finally, we do not need to fix the height $\delta$, but rather can let it tend to 0 in a controlled fashion, so that wiggles at large real parts have values of $\delta$ close to 0.

Also note that, in all our examples, $\log \operatorname{Re} F(z)$ grows at most linearly with $\operatorname{Re}(z)$ within the long horizontal tubes *between* two consecutive "wiggles" (i.e., between $\dot{x}_k^M$

and $x_{k+1}^{1/M}$ in Figure 6). We claim that this means that the *lower order* of $F$; i.e. the number

$$\liminf_{r\to\infty} \sup_{\operatorname{Re} z = r} \frac{\log \operatorname{Re} F(z)}{r}$$

is finite.

Indeed, let $w_n$ and $\dot{w}_n$ be points at the beginning and the end of this tube, respectively. That is, $w_n$ is at real parts slightly larger than $\dot{x}_k^M$ and $\dot{w}_n$ is at real parts slightly below $x_{k+1}^{1/M}$. We then have

$$|F(\dot{w}_n)| \leq |F(w_n)| \cdot \exp(C \cdot \operatorname{Re} \dot{w}_n),$$

where essentially $C = \pi/h$. It follows from the construction that $|F(w_n)|$ grows at most like $\dot{x}_{k+1}^M$, and hence by (9) is bounded by $x_{k+1}^A$ for some $A > 1$. So overall

$$\log \operatorname{Re} F(\dot{w}_n) \leq A \log x_{k+1} + C \operatorname{Re} \dot{w}_n \leq A \log \operatorname{Re} \dot{w}_n + C \operatorname{Re} \dot{w}_n,$$

and the lower order is at most $C$.

Since there are no other parts of the tract $T$ between the real parts of $w_n$ and $\dot{w}_n$, we can actually modify $T$ so that these tubes have the maximal possible height $h = 2\pi$. Then the lower order of the resulting function $F$ will be $C = 1/2$, which is the minimal possible value for a function in $\mathcal{B}_{\log}$.

Altogether, this yields the following.

**8.3. Proposition** (More counterexamples of mild growth)**.**
*There exists a function $F \in \mathcal{B}_{\log}$ such that*

(a) $\log \operatorname{Re} F(z) = (\operatorname{Re} z)^{1+o(1)}$ *as* $\operatorname{Re} z \to \infty$*, and*
(b) $F$ *has lower order* $1/2$*.*
(c) $J(F)$ *has no unbounded path-connected components.*

Note that this function will not satisfy the stronger requirements in Theorem 6.3 for a fixed $M$ (we will need to let $M$ tend to 1 as $k \to \infty$). So we will not be able to use Proposition 6.4 to obtain an entire function from $F$. (Also, an application of Proposition 6.4 would slightly increase the lower order.) We believe that it should be possible to modify Proposition 6.4 so as to construct an entire function with these properties.

**No nontrivial path components.** To conclude, we would like to note that our construction can also be adapted to yield a topologically stronger form of the counterexample. We content ourselves with giving a sketch of the proof, which involves a non-trivial amount of bookkeeping but is not conceptionally more involved than the previous arguments.

**8.4. Theorem** (No Nontrivial Paths in the Julia Set)**.**
*There exists a (hyperbolic) function $f \in \mathcal{B}$ such that $J(f) \cup \{\infty\}$ is a compact connected set which contains no nontrivial curve.*

*Sketch of proof.* Again, the result will be established by designing a function $F \in \mathcal{B}_{\log}$ with a single tract $T \subset \mathbb{H}$ whose Julia set contains no nontrivial curve; the existence of an entire function with the same property is easily obtained using Proposition 6.4.

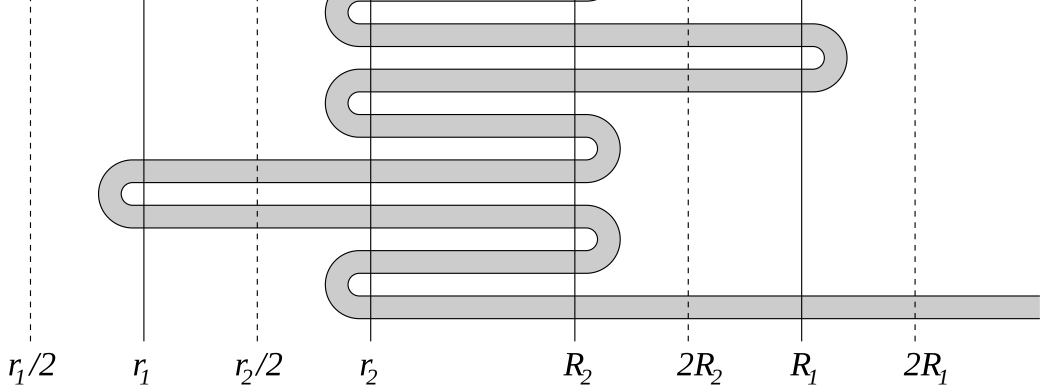


FIGURE 9. Illustration of the proof of Theorem 8.4. The tract pictured here has a wiggle over $(r_1, R_1)$ and over $(r_2, R_2)$.

(Recall that $J(f) \cup \{\infty\}$ is always a compact connected set when $f \in \mathcal{B}$, so only the second part of the claim needs to be established.)

Let us say that a tract $T$ has a *wiggle over* $(r, R)$ if any curve in $T$ which connects a point at real part $r/2$ to one at real part at least $2R$ contains at least three disjoint subcurves connecting the real parts $r$ and $R$.

Our aim is now to construct a tract $T$, a conformal map $F : T \to \mathbb{H}$, and an associated set $\mathcal{W}$ of wiggles $(r, R)$ such that

1. $T$ has a wiggle over $(r, R)$ for every $(r, R) \in \mathcal{W}$.
2. For every $\eta \in [1, \infty)$, there is some wiggle $(r, R) \in \mathcal{W}$ with $\eta \leq r \leq R \leq 3\eta$.
3. Every wiggle $(r, R) \in \mathcal{W}$ "propagates", roughly in the sense that curves connecting real parts $r$ and $R$ are going to map to an "image wiggle" $(r', R') \in \mathcal{W}$.

   More precisely, suppose that $\gamma : [0, 1] \to T$ connects real parts $r/2$ and $2R$, and $\mathrm{Re}(\gamma(t)) \in (r/2, 2R)$ for all $t \in (0, 1)$. Let us suppose without loss of generality that $|F(\gamma(0))| < |F(\gamma(1))|$. Then there should be $(r', R') \in \mathcal{W}$ such that, for every $t \in (0, 1)$ with $\mathrm{Re}\,\gamma(t) \in (r, R)$, we have

$$|F(\gamma(0))| < r'/2 < r' < |F(\gamma(t))| < R' < 2R' < |F(\gamma(1))|.$$

By linear separation of real parts (Lemma 3.2), for any two points $z, w \in J(F)$ with the same external address, there is an iterate $F^k$ so that $\mathrm{Re}\,F^k(z)/\,\mathrm{Re}\,F^k(w) > 12$ (assuming without loss of generality that $\mathrm{Re}\,F^k(z) > \mathrm{Re}\,F^k(w)$). So, by 2., there is a wiggle $(r, R) \in \mathcal{W}$ such that

$$\mathrm{Re}\,F^k(z) < r/2 < 2R < \mathrm{Re}\,F^k(w).$$

The condition in 3. will then guarantee, by an inductive argument as in Theorem 6.1, that any curve in $J(F)$ connecting $F^k(z)$ and $F^k(w)$ would need to connect real parts $r$ and $R$ at least $3^n$ times for every $n$, which is impossible.

To complete our sketch, we now indicate how to construct such a tract $T$, which will be a winding strip contained in $\{|\,\mathrm{Im}\,z| < \pi\}$, similarly as before. However, the number of times that $T$ crosses the line $\{\operatorname{Re} z = R\}$ will tend to infinity as $R$ does, so the width of $T$ will necessarily tend to 0 as real parts increase. Similarly as in Theorem 6.3, the tract will be constructed by inductively defining pieces $T_1, T_2, \dots$, in the following fashion:

(a) $T_j$ is the piece of $T$ between real parts $\eta_{j-1}$ and $\eta_j$, where $\eta_0 < \eta_1 < \eta_2 < \dots$ is a sequence tending to infinity.
(b) At each step in the construction, there is a set $\mathcal{W}_k = \{(r_1^k, R_1^k), \dots, (r_{m_k}^k, R_{m_k}^k)\}$ of "wiggles", with $r_j^k/2 \geq \eta_{k-1}$ and $2R_j^k \leq \eta_k$. $T_k$ is constructed to have a wiggle over each $(r, R) \in \mathcal{W}_k$ (see Figure 9).
(c) The next set of wiggles $\mathcal{W}_{k+1}$ is determined by the construction of $T_k$.

More precisely, we begin by setting $\eta_0 := 1$, $\mathcal{W}_1 := \{(r_1, r_1 + A)\}$, where $A$ is a sufficiently large number (fixed for the whole construction), and $r_1$ is large enough. We also set $\eta_1 := 2(r_1 + A) > 3\eta_0$.

Given $\mathcal{W}_k$, we construct a piece $T_k$, connecting real parts $\eta_{k-1}$ and $\eta_k$, by first constructing a "central curve" which has a wiggle over every $(r, R) \in \mathcal{W}_k$ (this is easy to achieve, compare Figure 9), and then thickening this curve slightly (see below) to obtain $T_k$.

We then construct the set $\mathcal{W}_{k+1}$ as follows. Suppose that $(r, R) \in \mathcal{W}_k$, and that $\gamma : [0, 1] \to T_k$ is a minimal piece of the central curve of $T_k$ which connects real parts $r/2$ and $2R$. (Note that there may be several such pieces; we will add a wiggle to $\mathcal{W}_{k+1}$ for each of them.)

Let $z$ be the first point on $\gamma$ which has real part $r$, and let $Z$ be the last point on $\gamma$ which has real part $R$. Using the semi-hyperbolic metric, i.e. the reciprocal of the distance to $\partial T_k$, we can estimate $|F(z)|$ and $|F(Z)|$ (up to an exponent of 2), independently of the construction of $T_K$ for $K > k$. Hence we can add a new wiggle $(r_j^{k+1}, R_j^{k+1})$ to $\mathcal{W}_{k+1}$ such that $|F(z)| \geq r_j^{k+1} + A$ and $|F(Z)| \leq R_j^{k+1} - A$.

If the width of $T_j$ along $\gamma$ was chosen sufficiently thin, we can easily ensure that $r_j^{k+1}/2 > |F(\gamma(0))| > \eta_k$, and that $2R_j^{k+1} < |F(\gamma(1))|$.

Having added these (finitely many) wiggles to $\mathcal{W}_{k+1}$, we set $\eta_{k+1} := \max_{(r,R)\in\mathcal{W}_{k+1}} 2R$. Finally, we add sufficiently many wiggles of the form $(t, t+A)$ to $\mathcal{W}_{k+1}$ to ensure that, for every $\eta \in [\eta_k, \eta_{k+1}/3]$, there is some wiggle between real parts $\eta$ and $3\eta$. This completes the description of the inductive construction. ■

## Appendix A. Some Geometric Facts

In this section, we collect some of the simple results from hyperbolic geometry which we required in the course of the article. The first is a version of the *Ahlfors spiral theorem* [H, Theorem 8.21] (which states that any entire function of finite order has controlled spiralling). We give a simple proof of this fact for functions in class $\mathcal{B}_{\log}$ below. In Section 5, we also required a characterization of domains with bounded wiggling, which we prove here for completeness. Finally, Lemma A.3 below was used in Theorem 6.3.

**A.1. Theorem** (Spiral Theorem).
*Suppose that $F \in \mathcal{B}_{\log}$ has finite order. Then the tracts of $F$ have bounded slope.*

*Proof.* Let $T$ be a tract of $F$, set $\varrho := \sup\{\frac{\log \operatorname{Re} F(z)}{\operatorname{Re} z} : z \in \mathbb{H} \cap \mathcal{T}\} < \infty$, and consider the central geodesic $\gamma : [1,\infty) \to T; t \mapsto F_T^{-1}(t)$. Then for every $t \geq 1$,

$$|\gamma(t)| - |\gamma(1)| \leq |\gamma(t) - \gamma(1)| \leq 2\pi \ell_T\big(\gamma([1,t])\big) = 2\pi \log t \leq 2\pi\varrho \operatorname{Re} \gamma(t)\ .$$

Thus we have proved the existence of an asymptotic curve $\gamma$ satisfying $|\operatorname{Im}\gamma(t)| \leq |\gamma(t)| \leq K \operatorname{Re}\gamma(t) + M$, for $K = 2\pi\varrho$ and $M = |\gamma(1)|$, which is equivalent to the bounded slope condition. ■

**A.2. Lemma** (Domains with bounded wiggling).
*Let $V$ be an unbounded Jordan domain such that $\exp|_V$ is injective. Suppose that there are $K, M > 0$ such that every $z_0 \in V$ can be connected to $\infty$ by a curve $\gamma \subset V$ satisfying*

$$\operatorname{Re} z \geq \frac{\operatorname{Re} z_0}{K} - M$$

*for all $z \in \gamma$. Then there is $M' > 0$ which depends only on $M$ such that, for every $z_0 \in V$,*

$$\operatorname{Re} z \geq \frac{\operatorname{Re} z_0}{K} - M'$$

*for all $z$ on the geodesic connecting $z_0$ to $\infty$.*

*Proof.* Let $z_0 \in V$, let $\gamma$ be a curve as in the statement of the theorem, and let $F : V \to \mathbb{H}$ be a conformal isomorphism with $F(\infty) = \infty$ and $F(z_0) = 1$. Then

$$\gamma' := F^{-1}\big([1,\infty)\big)$$

is the horizontal geodesic connecting $z_0$ to $\infty$.

Let $z \in \gamma'$. By [P, Corollary 4.18], we can find geodesics $\alpha^+$ and $\alpha^-$ of $\mathbb{H}$, connecting $F(z)$ to the positive resp. negative imaginary axis, such that the geodesics $F^{-1}(\alpha^\pm)$ of $V$ have diameter at most $C \operatorname{dist}(z, \partial V)$. (Here $C$ is a universal constant.) Hence the crosscut $\alpha := F^{-1}(\alpha^+) \cup F^{-1}(\alpha^-)$, which separates $z_0$ from $\infty$ in $V$, has diameter at most $2C \operatorname{dist}(z, \partial V) \leq 4C\pi$.

The curve $\gamma$ must intersect $\alpha$ in some point $w$. We thus have

$$\operatorname{Re} z \geq \operatorname{Re} w - 4C\pi \geq \operatorname{Re} z_0/K - M - 4C\pi\ .$$ ■

**A.3. Lemma** (Geometry of Geodesics).
*Consider the rectangle $Q = \{z \in \mathbb{C} \colon |\operatorname{Re} z| < 3, |\operatorname{Im} z| < 1\}$ and let $U \subset \mathbb{C}$ be a simply connected domain with Jordan boundary and $U \supsetneq Q$ so that $\partial Q \cap \partial U$ consists exactly of the two horizontal boundary sides of $Q$. Let $P, R, P', R' \in \partial U$ be four distinct boundary points in this cyclic order, subject to the condition that $P$ and $P'$ are in the boundary of different components of $U \setminus Q$, and so that the quadrilateral $U$ with the marked points $P, R, P', R'$ has modulus 1.*

*Let $\gamma$ be the hyperbolic geodesic in $U$ connecting $R$ with $R'$. If $0 \in \gamma$, then the two endpoints of $\gamma$ are on the vertical boundaries of $Q$, one endpoint each on the upper and lower boundary.*

*Sketch of proof.* If $R$ and $R'$ are in the same connected component of $\partial U \setminus \partial Q$, then $\gamma$ cannot contain 0, so either $R$ and $R'$ are in different connected components of $\partial U \setminus \partial Q$, or at least one of the points is in $\partial U \cap \partial Q$.

Consider the special case $R = x - 1 - i$ and $R' = x + 1 + i$, for $x \in [-2, 2]$. Using extremal length arguments and a constant density function on $\{z \in Q\colon |\operatorname{Re} z| < 1\}$, it is easy to show that the modulus of $U$ is different from 1. Monotonicity then rules out the cases when $R$ and $R'$ are both outside of $\partial Q$.

At least one point $R$ or $R'$ must thus be on $\partial Q$; without loss of generality, suppose that $R = x - 1 - i$ for $x \in [-2, 2]$. But then either $R' = x' + 1 + i$ for $x' \in [-2, x]$, or $R'$ is in the component of $\partial U \setminus \partial Q$ adjacent to the left boundary of $Q$. The first case is what we claim, and in the second case $\gamma$ cannot contain 0 (here the extremal case is that $R' = -3 + i$ is on a vertex of $Q$, and $R = -1 - i$ is furthest possible to the right). ■

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