A PRIORI BOUNDS FOR SOME
INFINITELY RENORMALIZABLE QUADRATICS: III. MOLECULES.

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CONTENTS

1. INTRODUCTION

The most prominent component of interior of the Mandlebrot set M is the one bounded by the main cardioid. There are infinitely many secondary hyperbolic components of int M attached to it. In turn, infinitely many hyperbolic components are attached to each of the secondary components, etc. Let us take the union of all hyperbolic components of int M obtained this way, close it up and fill it in (i.e., add all bounded components of its complement[1]). We obtain the set called the *molecule* $\mathcal{M}$ of M, see Figure 1.[2] In this paper we consider infinitely primitively renormalizable quadratic polynomials satisfying a *molecule condition*, which means that the combinatorics of the primitive renormalization operators involved stays away from the molecule (see §2.2 for the precise definition in purely combinatorial terms).

An infinitely renormalizable quadratic map f is said to have *a priori bounds* if its renormalizations can be represented by quadratic-like maps $R^n f : U'_n \to U_n$ with $\mathrm{mod}(U_n \smallsetminus U'_n) \geq \mu > 0$, $n = 1, 2 \ldots$.

Our goal is to prove the following result:
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Main Theorem. Infinitely renormalizable quadratic maps satisfying the molecule condition have a priori bounds.

By [L], this implies:

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[1]These bounded components could be only queer components of int M
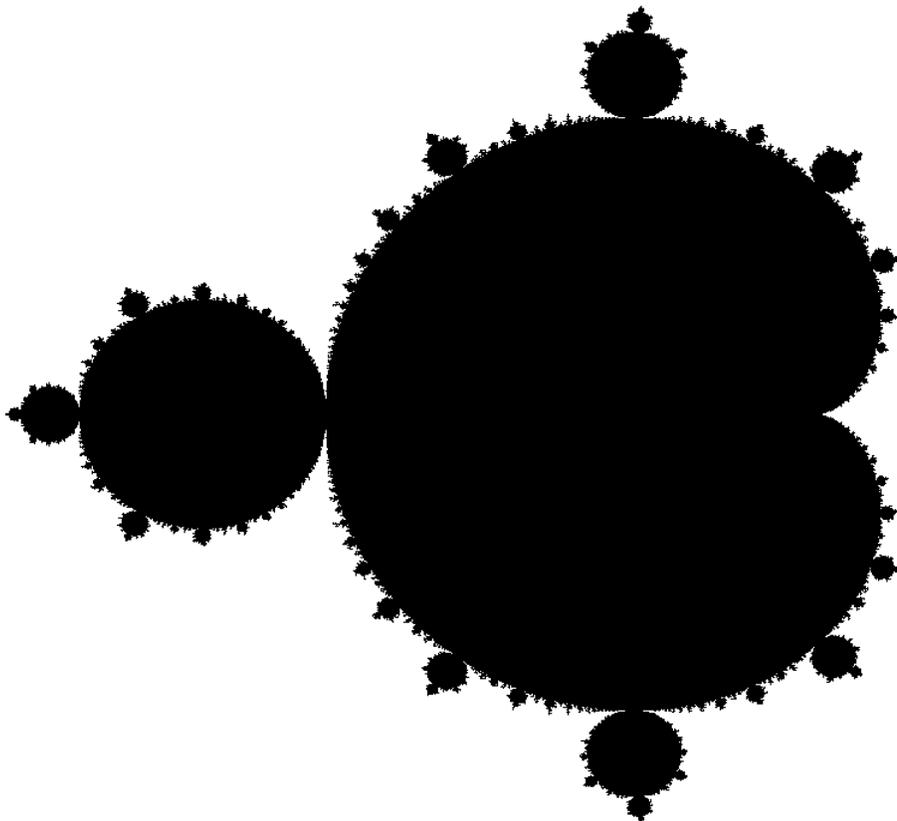
[2]It is also called the *cactus*.

FIGURE 1.1. The central molecule of the Mandelbrot set

Corollary 1.1. *Let $f_c : z \mapsto z^2 + c$ be an infinitely renormalizable quadratic map satisfying the molecule condition. Then the Julia set $J(f_c)$ is locally connected, and the Mandelbrot set M is locally connected at c.*

Given and $\eta > 0$, let us say that an renormalizable quadratic map satisfies the η-*molecule* condition if the combinatorics of the renormalization operators involved stays η-away from the molecule of $\mathcal{M}$.

In this paper we will deal with the case of renormalizations with sufficiently high periods. Roughly speaking, we show that if a quadratic-like map is nearly degenerate then its geometry is improving under such a renormalization. The precise statement requires the notion of "pseudo-quadratic-like map" f defined in §3, and its modulus, $\mathbf{mod}\,(f)$.

Theorem 1.2. *Given $\eta > 0$ and $\rho \in (0, 1)$, there exist $\bar{\mu} > 0$ and $\underline{p} \in \mathbb{N}$ with the following property. Let f be a renormalizable with period p quadratic-like map satisfying the η-molecule condition. If $\mathbf{mod}\,(Rf) < \bar{\mu}$ and $p \geq \underline{p}$ then $\mathbf{mod}\,(f) < \rho\,\mathbf{mod}\,(Rf)$.*

The complementary case of "bounded periods" is dealt in [K].

Remark 1.1. Theorem 1.2 is proved in a similar way as a more special result of [KL3]. The main difference occurs on the top level of the Yoccoz puzzle, which is modified here so that it is associated with an appropriate periodic point rather than with the fixed point of f.[3] We

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[3]It is similar to the difference between "non-renormalizable" and "not infinitely renormalizable" cases in the Yoccoz Theorem.

will focus on explaining these new elements, while only outlining the parts that are similar to [KL3].

Let us now outline the structure of the paper.

In the next section, §2, we lay down the combinatorial framework for our result, the Yoccoz puzzle associated to dividing cycles, and formulate precisely the Molecule Condition.

In §3 we summarize necessary background about *pseudo-quadratic-like maps* introduced in [K], and the *pseudo-puzzle* introduced in [KL3]. From now on, the usual puzzle will serve only as a combinatorial frame, while all the geometric estimates will be made for the pseudo-puzzle. Only at the last moment (§5.6) we return back to the standard quadratic-like context.

In §4 we formulate a Transfer Prinsiple, that will allow us to show that if $\mathbf{mod}\,(Rf)$ is small then the the pseudo-modulus in between appropriate puzzle pieces is even smaller.

In §5 we apply the Transfer Principle to the dynamical context. It implies that the extremal pseuso-distance between two specific parts of the Julia set (obtained by removing from the Julia set the central puzzle piece Y^1) is much bigger than $\mathbf{mod}\,(f)$ (provided the renormalization period is big). On the other hand, we show that under the Molecule Condition, this pseudo-distance is comparable with $\mathbf{mod}\,(f)$. This yields Theorem 1.2.

1.1. **Terminology and Notation.**
$\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers; $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$; $\mathbb{D} = \{z : |z| < 1\}$ is the unit disk, and $\mathbb{T}$ is the unit circle.

A *topological disk* means a simply connected domain in $\mathbb{C}$. A *continuum* K is a connected closed subset in $\mathbb{C}$. It is called *full* if all components of $\mathbb{C} \smallsetminus K$ are unbounded.

For subsets K, Y of a topological space X, notaion $K \Subset Y$ will mean (in a slightly non-standard way) that the closure of K is a compact contained in $\mathrm{int}\,Y$.

We let $\mathrm{orb}(z) \equiv \mathrm{orb}_g(z) = (g^n z)_{n=0}^\infty$ be the *orbit* of z under a map g.

Given a map $g : U \to V$ and an open topological disk $D \subset V$, components of $g^{-1}(D)$ are called *pullbacks* of D under g. If the disk D is closed, we define pullbacks of D as the closures of the pullbacks of $\mathrm{int}\,D$.[4] In either case, given a connected set $X \subset g^{-1}(\mathrm{int}\,D)$, we let $g^{-1}(D)|X$ be the pullback of D containing X.

1.2. **Acknowledgement.**

2. Dividing cycles, Yoccoz puzzle, and renormalization

Let $(f : U' \to U)$ be a quadratic-like map. We assume that the domains U' and U are smooth disks, f is even, and we normalize f so that 0 is its critical point. We let $U^m = f^{-m}(U)$. The boundary of U^m is called the *equipotential of depth m*.

By means of straightening, we can define external rays for f. They form a foliation of $U \smallsetminus K(f)$ transversal to the equipotential ∂U. Each ray is labeled by its *external angle*. These rays will play purely combinatorial role, so particular choice of the straightening is not important.

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[4]Note that the pullbacks of a closed disk D can touch one another, so they are not necessarily connected components of $g^{-1}(D)$.

2.1. Dividing cycles and associated Yoccoz puzzles.

Let us consider a repelling periodic point γ of period $\mathbf{t}$ and the correspoding cycle $\boldsymbol{\gamma} = \{f^k\gamma\}_{k=0}^{\mathbf{t}}$. This point (and the cycle) is called *dividing* if there exist at least two rays landing at it. For instance, the landing point of the zero ray is a non-dividing fixed point, while the other fixed point is dividing (if repelling).

In what follows, we assume that γ is dividing. Let $\mathcal{R}(\gamma)$ (resp., $\mathcal{R}(\boldsymbol{\gamma})$) stand for the family of rays landing at γ (resp., $\boldsymbol{\gamma}$). Let $\mathbf{s} = \#\mathcal{R}(\gamma)$ and let $\mathbf{r} = \mathbf{ts} = \#\mathcal{R}(\boldsymbol{\gamma})$. These rays divide U into $\mathbf{t}(\mathbf{s}-1)+1$ closed topological disks $Y^0(j) \equiv Y^0_\gamma(j)$ called *Yoccoz puzzle pieces of depth* 0.

Yoccoz puzzle pieces $Y^m(j) \equiv Y^m_\gamma(j)$ *of depth m* are defined as the pullbacks of $Y^0(i)$ under f^m. They tile the neighborhood of $K(f)$ bounded by the equipotential ∂U^m. Each of them is bounded by finitely many arcs of this equipotential and finitely many external rays of $f^{-m}(\mathcal{R}(\boldsymbol{\gamma}))$. We will also use notation $Y^m(z)$ for the puzzle piece $Y^m(j)$ containing z in its interior. If $f^m(0) \notin \boldsymbol{\gamma}$, then there is a well defined *critical* puzzle piece $Y^m \equiv Y^m(0)$. The critical puzzle pieces are nested around the origin:

$$Y^0 \supset Y^1 \supset Y^2 \cdots \ni 0.$$

Notice that all Y^m, $m \geq 1$, are symmetric with respect to the origin.

Let us take a closer look at some puzzle piece $Y = Y^m(i)$. Different arcs of ∂Y meet at the *corners* of Y. The corners where two external rays meet will be called *vertices* of Y; they are f^m-preimages of $\boldsymbol{\gamma}$. Let $K_Y = K(f) \cap Y$. It is a closed connected set that meets the boundary ∂Y at its vertices. Moreover, the external rays meeting at a vertex $v \in \partial Y$ chop off from $K(f)$ a continuum S_Y^v, the component of $K(f) \setminus \operatorname{int} Y$ containing v.

Let $\mathcal{Y}_\gamma$ stand for the family of all puzzle pieces $Y^m_\gamma(j)$.

Let us finish with an obvious observation that will be constantly exploited:

Lemma 2.1. *If a puzzle piece $Y^n(z)$ of $\mathcal{Y}_\gamma$ does not touch the cycle $\boldsymbol{\gamma}$ then $Y^n(z) \Subset Y^0(z)$.*

2.2. Renormalization associated with a dividing cycle.

Lemma 2.2 (see [Th, M2]). *The puzzle piece $X^0 \equiv Y^0(f(0))$ of $\mathcal{Y}_\gamma$ containing the critical value has only one vertex, and thus is bounded by only two external rays (and one equipotential).*

In what follows, γ will denote the point of the cycle $\boldsymbol{\gamma}$ such that $f(\gamma)$ is the vertex of X^0. Notice that $f(0) \in \operatorname{int} X^0$ for otherwise $f(0) = f(\gamma)$, which is impossible since 0 is the only preimage of $f(0)$.

Since the critical puzzle piece Y^1 is the pullback of X^0 under f, it has two vertices, γ and $\gamma' = -\gamma$, and is bounded by four rays, two of them landing at γ and two landing at γ'.[5]

Lemma 2.3. *(see [D]) Let $X^{\mathbf{r}} \equiv Y^{\mathbf{r}}(j) \subset X^0$ be the puzzle piece attached to the boundary of X^0. Then $f^{\mathbf{r}} : X^{\mathbf{r}} \to X^0$ is a double branched covering.*

Proof. Let $\mathcal{C}^0$ be the union of the two rays that bound X^0, and let $\mathcal{C}^{\mathbf{r}}$ be $\mathcal{C}^0$ cut by the equipotential $\partial U^{\mathbf{r}}$. Let us orient $\mathcal{C}^0$ and then induce the orientation to $\mathcal{C}^{\mathbf{r}}$. Since $f^{\mathbf{r}} : \mathcal{C}^{\mathbf{r}} \to \mathcal{C}^0$ is an orientation preserving homeomorphism, it maps $X^{\mathbf{r}}$ onto X^0.

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[5]We will usually say that "a puzzle piece is bounded by several external rays" without mentioning equipotentials that also form part of its boundary.

Since for $m = 1, \ldots, \mathbf{r} - 1$, the arcs $f^m(\mathcal{C}^\mathbf{r}) \subset \partial(f^m X^\mathbf{r})$ are disjoint from int X^0, the puzzle pieces $f^m X^\mathbf{r}$ are not contained in X^0. Since they have a bigger depth than X^0, they are disjoint from int X^0. It follows that all the puzzle pieces $f^m(X^\mathbf{r})$, $m = 0, 1, \ldots, \mathbf{r} - 1$, have pairwise disjoint interiors. (Otherwise $f^m(X^\mathbf{r}) \supset f^n(X^\mathbf{r})$ for some $\mathbf{r} > m > n \geq 0$, and applying $f^{\mathbf{r}-m}$, we would conclude that $X^0 \supset f^{\mathbf{r}-n+m}(X^\mathbf{r})$.)

Moreover, the puzzle piece $f^{\mathbf{r}-1}X^\mathbf{r} = Y^1$ is critical since Y^1 is the only pullback of X^0 under f. Hence the puzzle pieces $f^m(X^\mathbf{r})$, $m = 0, 1, \ldots, \mathbf{r} - 2$, do not contain 0. It follows that $\deg(f^\mathbf{r} : X^\mathbf{r} \to X^0) = 2$. $\qquad\square$
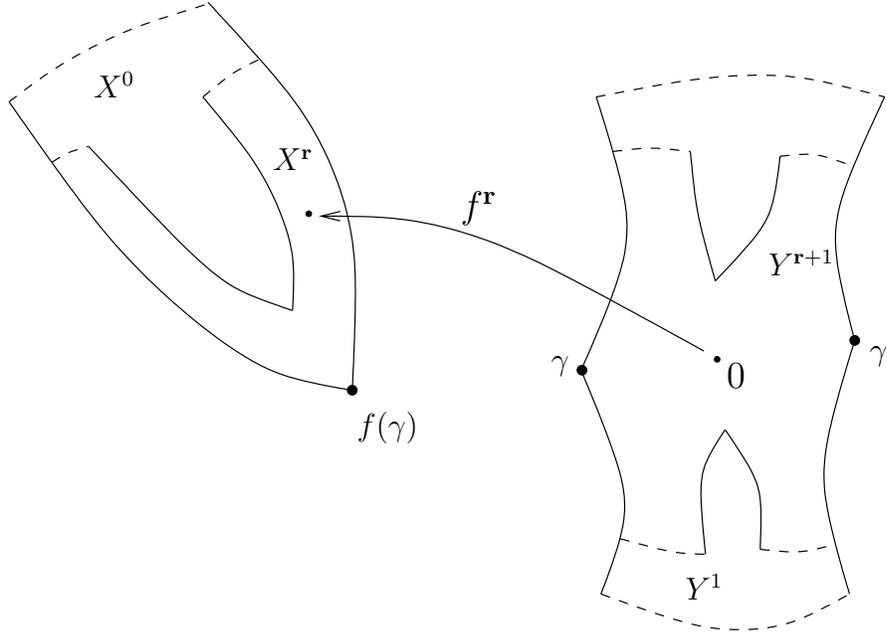

FIGURE 2.1

Corollary 2.4. *If $f(0) \in$ int $X^\mathbf{r}$ then the puzzle piece $Y^{\mathbf{r}+1}$ has four vertices, and the map $f^\mathbf{r} : Y^{\mathbf{r}+1} \to Y^1$ is a double branched covering.*

Let $\Theta(\boldsymbol{\gamma}) \subset \mathbb{T}$ be the set of external angles of the rays of $\mathcal{R}(\boldsymbol{\gamma})$. There is a natural equivalence relation on $\Theta(\boldsymbol{\gamma}) \subset \mathbb{T}$: two angles are equivalent if the corresponding rays land at the same periodic point. Let us consider the hyperbolic convex hulls of these equivalence classes (in the disk $\mathbb{D}$ viewed as the hyperbolic plane). The union of the boundaries of these convex hulls is a finite lamination $\mathcal{P} = \mathcal{P}(\boldsymbol{\gamma})$ in $\mathbb{D}$ which is also called the *periodic ray portrait*. One can characterize all possible ray portraits that appear in this way (see [M2]).

Definition 2.1. A map f is called $\mathcal{P}$-*renormalizable* (or, "renormalizable with combinatorics $\mathcal{P}$") if $f(0) \in$ int $X^\mathbf{r}$ and $f^{\mathbf{r}m}(0) \in Y^{\mathbf{r}+1}$ for all $m = 0, 1, 2, \ldots$. In this case, the double covering $f^\mathbf{r} : Y^{\mathbf{r}+1} \to Y^1$ is called the *renormalization* $R_\mathcal{P} f = R_\gamma f$ of f (associated with the cycle $\boldsymbol{\gamma}$). The corresponding *little (filled) Julia set* $\mathcal{K} = K(Rf)$ is defined as

$$\{z : f^{\mathbf{r}m}z \in Y^{\mathbf{r}+1}, \quad m = 0, 1, 2 \ldots\}$$

If the little Julia sets $f^m \mathcal{K}$, $k = 0, 1, \ldots, \mathbf{r} - 1$, are pairwise disjoint, then the renormalization is called *primitive*; otherwise it is called *satellite*.

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In case $\boldsymbol{\gamma}$ is the dividing fixed point of f, the map is also called *immediately renormalizable.* (This is a particular case of the satellite renormalization.)

Remark 2.1. The above definition of renormalization is not quite standard since the map $R_{\boldsymbol{\gamma}}f$ is not quadratic-like. To obtain the usual notion of renormalization, one should thicken the domain of $R_{\boldsymbol{\gamma}}f$ a bit to make it quadratic-like (see [D, M1]). This thickenning does not change the Julia set, so $K(Rf)$ possesses all the properties of quadratic-like Julia sets. In particular, it has two fixed points, one of which is either non-repelling or dividing.

Note also that in the case when $f^r(0) = \gamma'$ the puzzle piece $Y^{\mathbf{r+1}}$ degenerates (is pinched at 0), but this does not effect any of further considerations.

Given a periodic ray portrait $\mathcal{P}$, the set of parameters $c \in \mathbb{C}$ for which the quadratic polynomial P_c is $\mathcal{P}$-renormalizable form a *little copy $M_{\mathcal{P}}$* of the Mandelbrot set ("*M*-copy"). Thus, there is one-to-one correspondence between the admissible ray portraits and the little M-copies. So, one can encode the combinatorics of the renormalization by the little M-copies themselves.

2.3. Molecule Condition.
The molecule $\mathcal{M}$ defined in the Introduction consists of the quadratic maps which are:
• either finitely many times renormalizable, all these renormalizations are satellite, and the last renormalization has a non-repelling cycle;
• or infinitely many times renormalizable, with all the renormalizations satellite.

The molecule condition that we are about to introduce will ensure that our map f has frequent "qualified" primitive renormalizations. Though f is allowed to be satellite renormalizable once in a while, *we will record only the primitive renormalizations.* They are naturlly ordered according to their periods, $1 = p_0 < p_1 < p_2 < \ldots$, where p_i is a multiple of p_{i-1}.

Along with these "absolute" periods of the primitive renormalizations, we will consider *relative periods $\tilde{p}_i = p_i/p_{i-1}$* and the corresponding M-copies $\tilde{M}_i$ that encode the combinatorics of $R^i f$ as the renormalization of $R^{i-1} f$.

Given an $\eta > 0$, we say that a sequence of primitively renormalizable quadratic-like maps f_i satisfies the *η-molecule condition* if the corresponding M-copies M_i stay η-away from the molecule $\mathcal{M}$ (the latter is defined in the Introduction). We say that $\{f_i\}$ satisfies the *molecule condition* if it does it for some $\eta > 0$.

An infinitely primitively renormalizable map f satisfies the η-molecule condition if the sequence of its primitive renormalizations $R^i f$ does (i.e., the corresponding relative copies $\tilde{M}_i$ stay η-away from the molecule $\mathcal{M}$). (And similarly, for the non-quantified molecule condition.)

There is, however, a more specific combinatorial way to describe the molecule condition.

Let us consider a quadratic-like map f with straightening P_c, $c \in M$. We are going to associate to f (in some combinatorial region, and with some choice involved) three combinatorial parameters, $(\mathbf{r}, \mathbf{q}, \mathbf{n})$ ("period, valence, and escaping time") whose boundedness will be equivalent to the molecule condition.

Assume first that f admits a dividing cycle $\boldsymbol{\gamma}$ with the ray portrait $\mathcal{P}$ with $\mathbf{r}$ rays. This happens if and only if c belongs to the parabolic limb of M cut off by two external rays landing at an appropriate parabolic point.

On the central domain of $\mathbb{C} \setminus (\mathcal{R}(\boldsymbol{\gamma}) \cup \mathcal{R}(\gamma'))$, $f^{\mathbf{r}}$ has a unique fixed poin α. *Next, we assume that α is repelling and there are $\mathbf{q}$ rays landing at it.*

Assume next that the finite orbit $f^{\mathbf{r}j}(0)$, $j = 1, \ldots, \mathbf{qn} - 1$, does not escape the central domain of $\mathbb{C} \setminus (\mathcal{R}(\gamma) \cup \mathcal{R}(\gamma'))$. This happens if and only if c lies outside certain *decorations* (see [KL3]) of the above parabolic limb. In particular, this happens if f is *satellite $\mathcal{P}(\gamma)$-renormalizable.*

Finally, assume that $\mathbf{n}$ is the first moment n such that $f^{\mathbf{rq}n}(0)$ escapes the central domain of $\mathbb{C} \setminus (\mathcal{R}(\alpha) \cup \mathcal{R}(\alpha'))$. This happens if and only if c belongs to the union of $2^{\mathbf{n}}$ decorations inside the above parabolic limb. In particular, the map *f is not $\mathcal{P}(\boldsymbol{\alpha})$-renormalizable.*

Under the above assumptions, we say that f satisfies the $(\bar{\mathbf{r}}, \bar{\mathbf{q}}, \bar{\mathbf{n}})$-*molecule condition* if there is a choice of $(\mathbf{r}, \mathbf{q}, \mathbf{n})$ with $\mathbf{r} \leq \bar{\mathbf{r}}$, $\mathbf{q} \leq \bar{\mathbf{q}}$ and $\mathbf{n} \leq \bar{\mathbf{n}}$.

Lemma 2.5. *The $(\mathbf{r}, \mathbf{q}, \mathbf{n})$-molecule condition is equivalent to the η-molecule condition.*[6]

Proof. If f satisfies $(\mathbf{r}, \mathbf{q}, \mathbf{n})$-molecule condition then c belongs to the finite union of decorations. Each of them does not intersect the molecule $\mathcal{M}$, so c stays some distance η away from $\mathcal{M}$.

Vice versa, assume there is a sequence of maps f_i satisfying the η-molecule condition, but with $(\mathbf{r}, \mathbf{q}, \mathbf{n}) \to \infty$ for any choice of $(\mathbf{r}, \mathbf{q}, \mathbf{n})$. Let us select a convergent subsequence $c_i \to c$. Since $c \notin \mathcal{M}$, there can be only finitely many hyperbolic components $H_0, H_1, \ldots, H_m$ of $\mathcal{M}$ such that H_0 is bounded by the main cardioid, H_{k+1} bifurcates from H_k, and f is m times immediately renormalizable with the corresponding combinatorics. Let us consider the two rays landing at the last bifurcation point (where H_m is attached to H_{m-1}), and the corresponding parabolic limb of the Mandelbrot set.[7] This parabolic point has certain period $\mathbf{r}$.

Since the quadratic polynomial P_c is not immediately renormalizable any more, the corresponding cycle $\boldsymbol{\alpha}$ is repelling with $\mathbf{q}$ rays landing at each of its periodic points, and there is some escaping time $\mathbf{n}$. So, P_c satisfies $(\mathbf{r}, \mathbf{q}, \mathbf{n})$-molecule condition. Since this condition is stable under perturbations, P_{c_i} satisfy it as well – contradiction. $\square$

In what follows we assume that parameters $\mathbf{r}, \mathbf{q}, \mathbf{n}$ are well defined for a map f under consideration, so in particular, we have two dividing cycles, $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$. We let $\mathbf{k} = \mathbf{rqn}$. Let us state for the record the following well-known combinatorial property:

Lemma 2.6. *The point $\zeta = f^{\mathbf{k}}(0)$ is separated from α and 0 by the rays landing at α'.*[8]

2.4. Combinatorial separation between γ and α. Along with the puzzle $\mathcal{Y}_{\boldsymbol{\gamma}}$ associated with $\boldsymbol{\gamma}$, let us consider the puzzle $\mathcal{Y}_{\boldsymbol{\alpha}}$ associated with $\boldsymbol{\alpha}$. The critical puzzle piece $Y_{\boldsymbol{\alpha}}^1$ has two vertices, α and α', and is bounded by four external rays landing at these vertices. Let $\mathcal{C}$ be the union of the two external rays of $\partial Y_{\boldsymbol{\alpha}}^1$ landing at α, and let $\mathcal{C}'$ be the symmetric pair of external rays landing at α'.

Recall that $\mathbf{t}$ stands for the period of $\boldsymbol{\gamma}$ and $\mathbf{s}$ stands for $\#\mathcal{R}(\boldsymbol{\gamma})$.

Lemma 2.7. *There exist inverse branches $f^{-\mathbf{t}m}|\mathcal{C}'$, $m = 0, 1, \ldots, \mathbf{s} - 1$, such that the union of the arcs $f^{-\mathbf{t}m}(\mathcal{C}')$ separates γ from the cycle $\boldsymbol{\alpha}$ and the co-cycle $\boldsymbol{\alpha}'$ (except that $\alpha' \in \mathcal{C}'$).*

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[6]In the sense that if f satisfies $(\mathbf{r}, \mathbf{q}, \mathbf{n})$-molecule condition then it satisfies η-molecule condition with some $\eta = \eta(\mathbf{r}, \mathbf{q}, \mathbf{n})$, and the other way around.

[7]If $m = 0$ then we consider the whole Mandelbrot set.

[8]while the latter two points are not separated.

Proof. Let us pull the puzzle piece Y_γ^1 along the orbit $\boldsymbol{\gamma}$ (or equivalently, along the orbit $\boldsymbol{\alpha}$). By Corollary 2.4, the corresponding inverse branches

$$f^{-m} : Y_\gamma^1 \to f^{-m}(Y_\gamma^1) = f^{\mathbf{r}-m}Y_\gamma^{\mathbf{r}+1}, \quad m = 0, 1, \ldots, \mathbf{r} - 1,$$

have disjoint interiors. Hence each of these puzzle pieces contains exactly one point of $\boldsymbol{\alpha}$. Moreover, non of these puzzle pieces except Y_γ^1 may intersect $\boldsymbol{\alpha}'$ (for otherwise, its image would contain two points of $\boldsymbol{\alpha}$).

It follows from standard properties of quadratic maps that the arc $\mathcal{C}'$ *separates* γ (which is the non-dividing fixed point of $R_\gamma f$) from γ' and α (which is the dividing fixed point of $R_\gamma f$). Hence the arcs $f^{-\mathbf{t}m}(\mathcal{C}')$, separate γ from $f^{-\mathbf{t}m}(\gamma')$ and $f^{-\mathbf{t}m}(\alpha)$.

Since each of the puzzle pieces $f^{-\mathbf{t}m}(Y_\gamma^1)$, $m = 0, 1, \ldots, \mathbf{s} - 1$, has two vertices ($\gamma$ and $f^{-\mathbf{t}m}(\gamma')$) and their union forms a neighborhood of γ, the rest of the Julia set is separated from γ by the union of arcs $f^{-\mathbf{t}m}(\mathcal{C}')$, $m = 0, 1, \ldots, \mathbf{s}-1$. It follows that this union separates γ from the whole cycle $\boldsymbol{\alpha}$, and from the co-cycle $\boldsymbol{\alpha}'$. $\square$

Together with Lemma 2.1 this yields:

Corollary 2.8. *We have:* $Y_{\boldsymbol{\alpha}}^{\mathbf{r}}(\gamma) \Subset Y_{\boldsymbol{\alpha}}^0(\gamma)$

Proof. The puzzle piece $Y_{\boldsymbol{\alpha}}^{\mathbf{r}}(\gamma)$ contains γ and does not cross the arcs $f^{-\mathbf{t}m}(\mathcal{C}')$, $m = 0, 1, \ldots, \mathbf{s} - 1$, from Lemma 2.7. Hence it does not intersect $\boldsymbol{\alpha}$, and the the conclusion follows from Lemma 2.1. $\square$

2.5. **A non-degenerate annulus.** *In what follows we will be dealing only with the puzzle* $\mathcal{Y}_{\boldsymbol{\alpha}}$, *so we will skip the label* $\boldsymbol{\alpha}$ *in notation.*

Since by Lemma 2.6, the point $\zeta = f^{\mathbf{k}}(0)$ is separated from α by $\mathcal{C}'$, the union of arcs $f^{-\mathbf{t}m}(\mathcal{C}')$, $m = 0, 1, \ldots, \mathbf{s} - 1$, from Lemma 2.7 separates ζ from the whole cycle $\boldsymbol{\alpha}$. By Lemma 2.1, $Y^{\mathbf{r}}(\zeta) \Subset Y^0(\zeta)$. Pulling this back by $f^{\mathbf{k}}$, we conclude:

Lemma 2.9. *We have:* $Y^{\mathbf{k}+\mathbf{r}} \Subset Y^{\mathbf{k}}$.

We let $E^0 = Y^{\mathbf{k}+\mathbf{r}}$.

2.6. **Buffers attached to the vertices of** $P = Y^{\mathbf{r}\mathbf{q}(\mathbf{n}-1)+1}$. Let us consider a nest of critical puzzle pieces

$$Y^1 \supset Y^{\mathbf{r}\mathbf{q}+1} \supset Y^{2\mathbf{r}\mathbf{q}+1} \supset \cdots \supset Y^{\mathbf{r}\mathbf{q}(\mathbf{n}-1)+1} = P.$$

Since $f^{\mathbf{r}\mathbf{q}} : Y^{\mathbf{r}\mathbf{q}+1} \to Y^1$ is a double branched covering such that

$$f^{\mathbf{r}\mathbf{q}m}(0) \in Y^1, \quad m = 0, 1, \ldots, \mathbf{n} - 1,$$

the puzzle piece $Y^{\mathbf{r}\mathbf{q}k+1}$ is mapped by $f^{\mathbf{r}\mathbf{q}}$ onto $Y^{\mathbf{r}\mathbf{q}(k-1)+1}$ as a double branched covering, $k = 1 \ldots, \mathbf{n} - 1$. However, since $f^{\mathbf{k}}(0) = f^{\mathbf{r}\mathbf{q}\mathbf{n}}(0) \notin Y^1$, there are two non-critical puzzle pieces of depth $\mathbf{k}+1$ mapped univalently onto P under $g^{\mathbf{q}}$. One of these puzzle pieces, called Q_L, is attached to the point α, another one, called Q_R, is attached to α'. The following Lemma is similar to Lemma 2.1 of [KL3]:

Lemma 2.10. *For any vertex of P, there exists a puzzle piece $Q^v \subset P$ of depth $\mathbf{r}(2\mathbf{n}-1)\mathbf{q}+1$ attached to the boundary rays of P landing at v which is a univalent $f^{\mathbf{k}}$-pullback of P. Moreover, these puzzle pieces are pairwise disjoint.*

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2.7. **Modified principle nest.** Until now, the combinatorics of the puzzle depended only on the parameters $(\mathbf{r}, \mathbf{q}, \mathbf{n})$. Now we will dive into the deeper waters.

Let l be the first return time of 0 to int E^0 and let $E^1 = Y^{\mathbf{k}+\mathbf{r}+l}$ be the pullback of E^0 along the orbit $\{f^m(0)\}_{m=0}^l$. Then $f^l : E^1 \to E^0$ is a double branched covering.

Corollary 2.11. *We have: $E^1 \Subset E^0$.*

Proof. Since $\{f^m(0)\}_{m=1}^{\mathbf{r}-1}$ is disjoint from $Y_\gamma^1 \supset Y_\alpha^1 \supset E^0$, we have: $l \geq \mathbf{r}$. Hence

$$f^l(E^0) \supset f^{\mathbf{r}}(E^0) \ni E^0 = f^l(E^1),$$

and the conclusion follows. □

Given two critical puzzle pieces $E^1 \subset$ int E^0, we can construct the *(Modified) Principle Nest* of critical puzzle pieces

$$E^0 \ni E^1 \ni E^1 \ni \ldots \ni E^{\chi-1} \ni E^{\chi}$$

as described in in [KL2]. It comes together with quadratic-like maps $g_n : E^n \to E^{n-1}$.

If the map f is renormalizable then the Principle Nest terminates at some level χ. In this case, the last quadratic-like map $g_\chi : E^\chi \to E^{\chi-1}$ has connected Julia set that coincides with the Julia set of the renormalization $R_\beta f$, where β is the f-orbit of the non-dividing fixed point β of g_χ. The renormalization level χ is also called the *height* of the nest.

2.8. **Stars.** Given a vertex v of some puzzle piece of depts n, let $S^n(v)$ stand for the union of the puzzle pieces of depth n attached to v (the "star" of v). Given a finite set $\mathbf{v} = \{v_j\}$ of vertices v_j, we let

$$S^n(\mathbf{v}) = \bigcup_j S^n(v_j).$$

Let us begin with an obvious observation that follows from Lemma 2.1:

Lemma 2.12. *If a puzzle piece $Y^n(z)$ is not contained in $S^n(\boldsymbol{\alpha})$ then $Y^n(z) \Subset Y^0(z)$.*

Lemma 2.13. *For $\boldsymbol{\lambda} = \mathbf{k} + 1 + 2\mathbf{r}$, the stars $S^{\boldsymbol{\lambda}}(\alpha_j)$ do not overlap and do not contain the critical point.*

Proof. Let us consider the curves $\mathcal{C}$ and $f^{-\mathbf{t}m}\mathcal{C}'$, $m = 1, \ldots, \mathbf{s} - 1$, from Lemma 2.7. They separate α' from all points of $\boldsymbol{\alpha} \cup \boldsymbol{\alpha}' \smallsetminus \{\alpha\}$. Furthermore, since $f^{\mathbf{k}}(0)$ is separated from 0 by $\mathcal{C}'$, there is a lift Γ of $\mathcal{C}'$ under $f^{\mathbf{k}}$ that separates α' from 0 and hence from α. It follows that the curves Γ and $f^{-\mathbf{t}m}\mathcal{C}'$, $m = 1, \ldots, \mathbf{s} - 1$, separate α' from all points of $\boldsymbol{\alpha} \cup \boldsymbol{\alpha}'$. Since the maximal depth of these curves is $\boldsymbol{\lambda} = \mathbf{k} + 1$ (which is the depth of Γ), the star $S^{\mathbf{k}+1}(\alpha')$ does not overlap with the interior of the stars $S^{\mathbf{k}+1}(a)$ for all other $a \in \boldsymbol{\alpha} \cup \boldsymbol{\alpha}'$.

By symmetry, the same is true for the star $S^{\mathbf{k}+1}(\alpha)$. Since these stars do not contain 0, the pullback of $S^{\mathbf{k}+1+\mathbf{r}}(\alpha)$ under $f^{\mathbf{r}}$ (along $\boldsymbol{\alpha}$) is compactly contained in its interior, int $S^{\mathbf{k}+1+\mathbf{r}}(\alpha)$. It follows that $S^{\mathbf{k}+1+\mathbf{r}}(\alpha)$ does overlap with the stars $S^{\mathbf{k}+1}(a)$ for all other $a \in \boldsymbol{\alpha} \cup \boldsymbol{\alpha}'$.

Pulling this star once more around $\boldsymbol{\alpha}$, we obtain a disjoint family of stars. Hence all the stars $S^{\mathbf{k}+1+2\mathbf{r}}(a)$, $a \in \boldsymbol{\alpha} \cup \boldsymbol{\alpha}'$, are pairwise disjoint. □

2.9. **Geometric puzzle pieces.** In what follows we will deal with more general puzzle pieces.

Given a puzzle piece Y, of depth m, let $Y[l]$ stand for a Jordan disk bounded by the same external rays as Y and arcs of equipotentials of level l (so $Y[m] = Y$). Such a disk will be called a puzzle piece of *bidepth* (m, l).

A *geometric puzzle piece* of bidepth (m, l) is a closed Jordan domain which is the union of several puzzle pieces of the same bidepth. As for ordinary pieces, a pullback of a geometric puzzle piece of bidepth (m, l) under some iterate f^k is a geometric puzzle piece of bidepth $(m + k, l + k)$. Note also that if P and P' are geometric puzzle pieces with[9] bidepth $P \geq$ bidepth P' and $K_P \subset K_{P'}$ then $P \subset P'$.

The family of geometric puzzle pieces of bidepth (m, l) will be called $\mathcal{Y}^m[l]$.

Stars give examples of geometric puzzle pieces. Note that pullback of a star is a geometric puzzle piece as well but it is a star only if the pullback is univalent.

Lemma 2.14. *Given a point $z \in \operatorname{int} S^1(\alpha) \cup \operatorname{int} S^1(\alpha')$ such that $f^k z \in \operatorname{int} S^1(\alpha)$, let $P = f^{-k}(S^1(\alpha))|z$. Then $P \subset S^1(\alpha)$ or $P \subset S^1(\alpha')$.*

Proof. Notice that α and α' are the only points of $\boldsymbol{\alpha} \cup \boldsymbol{\alpha}'$ contained in $\operatorname{int} S^1(\alpha) \cup \operatorname{int} S^1(\alpha')$. Since f^k maps $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \cup \boldsymbol{\alpha}' \smallsetminus \{\alpha, \alpha'\}$ to $\boldsymbol{\alpha} \smallsetminus \{\alpha\}$, no point of $\tilde{\boldsymbol{\alpha}}$ is contained in $\operatorname{int} P$. Hence P is contained in $S^1(\alpha) \cup S^1(\alpha')$.

But by construction of Y^{k+1}, the interior of $f^k(Y^{k+1})$ does not overlap with $S^1(\alpha)$. Hence $\operatorname{int} Y^{k+1}$ does not overlap with P. But $S^1(\alpha) \cup S^1(\alpha') \smallsetminus \operatorname{int} Y^{k+1}$ consists of two components, one inside $S^1(\alpha)$ and the other inside $S^1(\alpha')$. Since P is connected, it is contained in one of them. $\qquad\square$

3. Pseudo-quadratic-like maps and pseudo-puzzle

In this section, we will summarize the needed background on pseudo-quadratic-like maps and pseudo-puzzle. The details can be found in [K, KL3].

3.1. **Pseudo-quadratic-like maps.** Suppose that $\mathbf{U}'$, $\mathbf{U}$ are disks, $i : \mathbf{U}' \to \mathbf{U}$ is a holomorphic immersion, and $f : \mathbf{U}' \to \mathbf{U}$ is a degree d holomorphic branched cover. Suppose further that there exist full continua $K \Subset \mathbf{U}$ and $K' \Subset \mathbf{U}'$ such that $K' = i^{-1}(K) = f^{-1}(K)$. Then we say that $F = (i, f) : \mathbf{U}' \to (\mathbf{U}, \mathbf{U})$ is a *ψ-quadratic-like (ψ-ql)* map with filled Julia set K. We let

$$\mathbf{mod}\,(F) = \mathbf{mod}\,(f) = \operatorname{mod}(\mathbf{U} \smallsetminus K).$$

Lemma 3.1. *Let $F = (i, f) \colon \mathbf{U}' \to \mathbf{U}$ be a ψ-ql map of degree d with filled Julia set K. Then i is an embedding in a neighborhood of $K' \equiv f^{-1}(K)$, and the map $g \equiv f \circ i^{-1} \colon U' \to U$ near K is quadratic-like.*

Moreover, the domains U and U' can be selected in such a way that

$$\operatorname{mod}(U \smallsetminus i(U')) \geq \mu(\mathbf{mod}\,(F) > 0.$$

There is a natural ψ-ql map $\mathbf{U}^n \to \mathbf{U}^{n-1}$, the "restriction" of (i, f) to $\mathbf{U}^n$. Somewhat loosely, we will use the same notation $F = (i, f)$ for this restriction.

Let us normalize the ψ-quadratic-like maps under consideration so that $\operatorname{diam} K' = \operatorname{diam} K = 1$, both K and K' contain 0 and 1, 0 is the critical point of f, and $i(0) = 0$. Let us endow

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[9] the inequality between bidepths is understood componentwise

the space of ψ-quadratic-like maps (considered up to independent rescalings in the domain and the range) with the Carathéodory topology. In this topology, a sequence of normalized maps $(i_n, f_n) : \mathbf{U}'_n \to \mathbf{U}_n$ converges to $(i, f) : \mathbf{U}' \to \mathbf{U}$ if the pointed domains $(\mathbf{U}'_n, 0)$ and $(\mathbf{U}_n, 0)$ converge to $\mathbf{U}'$ and $\mathbf{U}$ respectively, and the maps i_n, f_n converge respectively to i, f, uniformly on compact subsets of $\mathbf{U}'$.

Lemma 3.2. *Let $\mu > 0$. Then the space of ψ-PL maps F with connected Julia set and* $\mathbf{mod}\,(F) \geq \mu$ *is compact.*

To simplify notation, we will often refer to f as a "ψ-ql map" keeping i in mind implicitly.

3.2. **Pseudo-puzzle.**

3.2.1. *Definitions.* Let $(i, f) : \mathbf{U}' \to \mathbf{U}$ be a ψ-ql map. By Lemma 3.1, it admits a quadratic-like restriction $U' \to U$ to a neighborhood of its (filled) Julia set $K = K_{\mathbf{U}}$. Here U' is embedded into U, so we can identify U' with $i(U')$ and $f : U' \to U$ with $f \circ i^{-1}$.

Assume that K is connected and both fixed points of f are repelling. Then we can cut U by external rays landing at the α-fixed point and consider the corresponding Yoccoz puzzle.

Given a (geometric) puzzle piece Y of bidepth (m, l), recall that K_Y stands for $Y \cap K(f)$. Let us consider the topological annulus $A = \mathbf{U}^l \smallsetminus K(f)$ and its universal covering $\hat{A}$. Let Y_i be the components of $Y \smallsetminus K_Y$. There are finitely many of them, and each Y_i is simply connected. Hence they can be embedded into $\hat{A}$. Select such an embedding $e_i : Y_i \to \hat{A}_i$ where $\hat{A}_i$ stands for a copy of $\hat{A}$. Then glue the A_i to Y by means of e_i, i.e., let $\mathbf{Y} = Y \sqcup_{e_i} \hat{A}_i$. This is the *pseudo-piece* ("ψ-piece") associated with Y. Noter that the Julia piece K_Y naturally embeds into $\mathbf{Y}$.

Lemma 3.3.   (i) *Consider two puzzle pieces Y and Z such that the map $f : Y \to Z$ is a branched covering of degree k (where $k = 1$ or $k = 2$ depending on whether Y is off-critical or not). Then there exists an induced map $\mathbf{f} : \mathbf{Y} \to \mathbf{Z}$ which is a branched covering of the same degree k.*

  (ii) *Given two puzzle pieces $Y \subset Z$, the inclusion $i : Y \to Z$ extends to an immersion $\mathbf{i} : \mathbf{Y} \to \mathbf{Z}$.*

3.2.2. *Boundary of puzzle pieces.* The ideal boundary of a ψ-puzzle piece $\mathbf{Y}$ is tiled by (finitely many) arcs $\lambda_i \subset \partial \hat{A}_i$ that cover the ideal boundary of $\mathbf{U}^m$ (where $m = \operatorname{depth} Y$) and arcs $\xi_i, \eta_i \subset \partial \hat{A}_i$ mapped onto the Julia set $J(f)$. The arc λ_i meets each ξ_i, η_i at a single boundary point corresponding to a path $\delta : [0, 1) \mapsto A$ that wraps around $K(f)$ infinitely many times, while η_i meets ξ_{i+1} at a vertex $v_i \in Y \cap K(f)$. We say that the arcs λ_i form the *outer boundary* (or "O-boundary") $\partial_O \mathbf{Y}$ of the puzzle piece $\mathbf{Y}$, while the arcs ξ_i and η_i form its *J-boundary* $\partial_J \mathbf{Y}$. Given a vertex $v = v_i$ of a puzzle piece Y, let $\partial^v \mathbf{Y} = \eta_i \cup \xi_{i+1}$ stand for the part of the J-boundary of $\mathbf{Y}$ attached to v.

4. Transfer Principle

Let us now formulate two analytic results which will play a crucial role in what follows. The first one appeares in §2.10.3 of [KL1]:

Quasi-Additivity Law. Fix some $\eta \in (0, 1)$. Let $\mathbf{V}$ be a topological disk, let $K_i \Subset \mathbf{V}$, $i = 1, \ldots, m$, be pairwise disjoint full compact continua, and let $\phi_i : \mathbb{A}(1, r_i) \to \mathbf{V} \smallsetminus \cup K_j$ be holomorphic annuli such that each ϕ_i is an embedding of some proper collar of $\mathbb{T}$ to a proper

collar of ∂K_i. Then there exists a $\delta_0 > 0$ (depending on η and m) such that: If for some $\delta \in (0, \delta_0)$, $\mathrm{mod}(\mathbf{V}, K_i) < \delta$ while $\log r_i > 2\pi\eta\delta$ for all i, then

$$\mathrm{mod}(\mathbf{V}, \cup K_i) < \frac{2\eta^{-1}\delta}{m}.$$

The next result appears in §3.1.5 of [KL1]:

Covering Lemma. Fix some $\eta \in (0, 1)$. Let us consider two topological disks $\mathbf{U}$ and $\mathbf{V}$, two full continua $A' \subset \mathbf{U}$ and $B' \subset \mathbf{V}$, and two full compact continua $A \Subset A'$ and $B \Subset B'$.

Let $f : \mathbf{U} \to \mathbf{V}$ be a branched covering of degree D such that A' is a component of $f^{-1}(B')$, and A is the union of some components of $f^{-1}(B)$. Let $d = \deg(f : A' \to B')$.

Let B' be also embedded into another topological disk $\mathbf{B}'$. Assume $\mathbf{B}'$ is immersed into $\mathbf{V}$ by a map i in such a way that $i|\, B' = \mathrm{id}$, $i^{-1}(B') = B'$, and $i(\mathbf{B}') \smallsetminus B'$ does not contain the critical values of f.

Under the following "Collar Assumption":

$$\mathrm{mod}(\mathbf{B}', B) > \eta\, \mathrm{mod}(\mathbf{U}, A),$$

if

$$\mathrm{mod}(\mathbf{U}, A) < \varepsilon(\eta, D)$$

then

$$\mathrm{mod}(\mathbf{V}, B) < 2\eta^{-1}d^2\, \mathrm{mod}(\mathbf{U}, A).$$

We will now apply these two geometric results to a dynamical situation. Recall from §2.7 that χ stands for the height of the Principal Nest, so that the quadratic-like map $g_\chi : E^\chi \to E^{\chi-1}$ represents the renormalization of f with the filled Julia set $\mathcal{K}$.
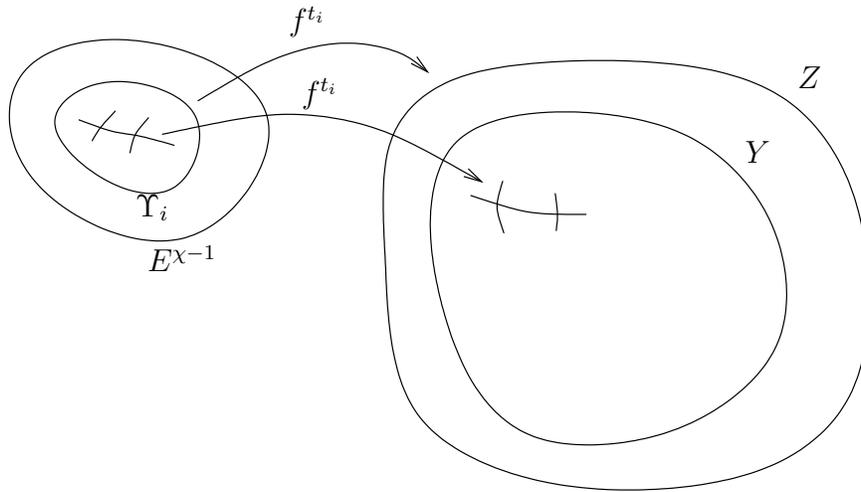

FIGURE 4.1. The Transfer Principle

Transfer Principle. Suppose there are two geometric puzzle pieces $Y \Subset Z$ with depth $Z <$ depth $E^{\chi-1}$, and a sequence of moments of time $0 < t_1 < t_2 < \cdots < t_m$ such that:
- $t_m - t_1 < p$ and $t_m < 2p$;
- $f^{t_i}(\mathcal{K}) \subset Y$;

- $\Upsilon_i = f^{-t_i}(Z)|\mathcal{K} \subset E^{\chi-1}$;
- $\deg(f^{t_i} : \Upsilon_i \to Z) \leq D$.

Then there exist an absolute constant C and $\varepsilon = \varepsilon(D)$ such that

$$\mathbf{mod}\,(Z, Y) < \frac{C}{m}\,\mathbf{mod}\,(E^\chi, \mathcal{K}),$$

provided $\mathbf{mod}\,(E^\chi, \mathcal{K}) < \varepsilon$.

Proof. Let $\mathcal{K}_j = f^{t_j}(\mathcal{K})$. We want to apply the Covering Lemma to the maps $f^{t_j} : (\Upsilon_j, \mathcal{K}) \to (Z, \mathcal{K}_j)$. As the buffer around $\mathcal{K}$ we take $E^{\chi+1}$. Since

$$\operatorname{depth} E^{\chi+1} = \operatorname{depth} E^{\chi-1} + 2p > \operatorname{depth} Z + t_j = \operatorname{depth} \Upsilon_j,$$

we have $E^{\chi+1} \subset \Upsilon_j$ for any $j = 1, \ldots, m$.

We let $\Omega_j = f^{t_j}(E^{\chi+1})$ be the corresponding buffer around $\mathcal{K}_j$. Then $\deg(f^{t_j} : E^{\chi+1} \to \Omega_j) \leq 4$ since $t_j < 2p$. Moreover,

(4.1) $\qquad \mathbf{mod}\,(\Omega_j, K_j) \geq 2\,\mathbf{mod}\,(E^{\chi+1}, \mathcal{K}) = \frac{1}{2}\mathbf{mod}\,(E^{\chi-1}, \mathcal{K}) \geq \frac{1}{2}\mathbf{mod}\,(\Upsilon_j, \mathcal{K}),$

which puts us in the position to apply the Covering Lemma with $\eta = 1/2$ and $d = 4$. It yields:

(4.2) $\qquad \mathbf{mod}\,(Z, \mathcal{K}_j) \leq 2^6\,\mathbf{mod}\,(E^{\chi-1}, \mathcal{K}) = 2^7\,\mathbf{mod}\,(E^\chi, \mathcal{K}),$

provided $\mathbf{mod}\,(E^\chi, \mathcal{K}) < \varepsilon(D)$.

Let us define k_j as t_j if $t_j < p$, and as $t_j - p$ otherwise. Then k_j's are pairwise different numbers in between 0 and p, and hence the sets $f^{k_j}(E^\chi)$ are pairwise disjoint. Since $\Omega_j \subset f^{k_j}(E^\chi)$, the buffers Ω_j are pairwise disjoint as well. Moreover, by (4.1)

$$\mathbf{mod}\,(\Omega_j, \mathcal{K}_j) \geq \mathbf{mod}\,(E^\chi, \mathcal{K}),$$

which, together with (4.2), puts us into a position to apply the Quasi-Additivity Law with $\eta = 2^{-7}$. It yields

$$\mathbf{mod}\,(Z, Y) \leq \frac{2^{15}}{m}\mathbf{mod}\,(E^\chi, \mathcal{K}),$$

provided $\mathbf{mod}\,(E^\chi, \mathcal{K}) < \varepsilon(D)$. $\qquad\square$

5. Improving the moduli

In this section we will prove Theorem 1.2 for ψ-ql maps.

Let $f_i : \mathbf{U}_i' \to \mathbf{U}_i$ be a sequence of renormalizable ψ-ql maps satisfying the η-molecule condition. Let $p_i \to \infty$ stand for the renormalization periods of the f_i, and let $\mathbf{mod}\,(Rf_i) \to 0$. We need to show that

$$\mathbf{mod}\,(Rf_i)/\mathbf{mod}\,(f_i) \to \infty.$$

Let $P_{c_i} : z \mapsto z^2 + c_i$ be the straigtenings of the f_i. Without loss of generality we can assume that $c_i \to c$. Then the η-molecule condition implies that the quadratic polynomial P_c satisfies the $(\bar{\mathbf{r}}, \bar{\mathbf{q}}, \bar{\mathbf{n}})$-condition, with $\bar{\mathbf{r}}$, $\bar{\mathbf{q}}$ and $\bar{\mathbf{n}}$ depending only on η. Hence all nearby maps satisfy the $(\bar{\mathbf{r}}, \bar{\mathbf{q}}, \bar{\mathbf{n}})$-condition as well. In what follows, we will fix one of these maps, $f = f_i$, with parameters $(\mathbf{r}, \mathbf{q}, \mathbf{n}) \leq (\bar{\mathbf{r}}, \bar{\mathbf{q}}, \bar{\mathbf{n}})$, and consider its puzzle as described in §2. All the objects under consideration (e.g., the principal nest $E^0 \supset E^1 \supset \ldots$) will be assoicated with f without making it notationally explicit.

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5.1. **From the bottom to the top of the Principal Nest.** The following result proved [KL3] (Lemma 5.3) shows that if χ is big while the modulus $\mathbf{mod}\,(E^{\chi-1}, E^{\chi})$ is small then $\mathbf{mod}\,(E^0, E^1)$ is even smaller:

Lemma 5.1. *For any* $\mathbf{k} \in \mathbb{N}$ *and* $\rho \in (0,1)$*, there exist* $\varepsilon > 0$ *and* $\underline{\chi} \in \mathbb{N}$ *such that if* $\chi \geq \underline{\chi}$ *and* $\mathbf{mod}\,(E^{\chi-1}, E^{\chi}) < \varepsilon$*, then*

$$\mathbf{mod}\,(E^0, E^1) < \rho\,\mathbf{mod}\,(E^{\chi-1}, E^{\chi}) \leq \rho\,\mathrm{mod}(\mathbf{E}^{\chi}, \mathcal{K}).$$

Corollary 5.2. *For any* $\mathbf{k} \in \mathbb{N}$ *and* $\rho \in (0,1)$*, there exist* $\varepsilon > 0$ *and* $\underline{\chi} \in \mathbb{N}$ *such that if* $\chi \geq \underline{\chi}$*, and* $\mathrm{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}) < \varepsilon$*, then for some puzzle piece* $Y^m(z)$ *we have:*

$$\mathbf{mod}\,(Y^0(z), Y^m(z)) < \rho\,\mathrm{mod}(\mathbf{E}^{\chi}, \mathcal{K}).$$

Proof. Let us apply $f^{\mathbf{r}(\mathbf{qn}+1)}$ to the pair (E^0, E^1). It maps E^1 onto some puzzle piece $Y^m(z)$, and maps E^0 onto $Y^0(z)$ with degree at most $2^{\mathbf{r}(\mathbf{qn}+1)}$. By Lemma 3.3,

$$\mathbf{mod}\,(Y^0(z), Y^m(z)) \leq 2^{\mathbf{r}(\mathbf{qn}+1)}\mathbf{mod}\,(E^0, E^1).$$

Together with Lemma 5.1 this yields the assertion. ☐

5.2. **Around the stars.** We will now go back to the original map f. Recall that p stands for its renormalization period, $\mathbf{k} = \mathbf{rqn}$, and $\boldsymbol{\lambda}$ is introduced in Lemma 2.13.

Lemma 5.3. *For any* $\mathbf{k} \in \mathbb{N}$ *and* $\rho \in (0,1)$*, there exist* $\varepsilon > 0$ *and* $\underline{p} \in \mathbb{N}$ *such that if* $p \geq \underline{p}$*, and* $\mathrm{mod}(\mathbf{E}^{\chi-1}, K) < \varepsilon$*, then either for some puzzle piece* $Y^{\boldsymbol{\lambda}}(z)$*,*

$$0 < \mathbf{mod}\,(Y^0(z), Y^{\boldsymbol{\lambda}}(z)) < \rho\,\mathrm{mod}(\mathbf{E}^{\chi}, \mathcal{K}),$$

or for some periodic point $\alpha_{\mu} \in \boldsymbol{\alpha}$*,*

$$0 < \mathbf{mod}\,(S^1(\alpha_{\mu}), S^{\boldsymbol{\lambda}}(\alpha_{\mu})) < \rho\,\mathrm{mod}(\mathbf{E}^{\chi}, \mathcal{K}).$$

Proof. By Corollary 5.2, it is true when $\chi \geq \underline{\chi}$, so assume $\chi \leq \underline{\chi}$. It will follow from the Trnasfer Principle of §4. Let $E^{\chi-1} = Y^{\tau_0}$. Note that

$$\deg(f^{\tau_0} | E^{\chi-1}) \leq 2^{\underline{\chi}+\mathbf{r}(\mathbf{qn}+1)},$$

so it is bounded it terms of $\mathbf{k}$ and ρ.

Let us then select the first moment $\tau \geq \tau_0$ such that $f^{\tau}(K) \not\subset S^{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$. It is bounded by $p + \mathbf{k}$ (since $f^{p+\mathbf{k}}(K) = f^{\mathbf{k}}(K) \not\subset S^{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$).

Let $m = [\frac{C}{\rho}] + 1$, where C is the constant from the Transfer Prinsiple. Let $s = s(\mathbf{r}, \mathbf{q})$ be the number of puzzle pieces $Y^{\boldsymbol{\lambda}}(z)$ in the complement of the star $S^{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$ (see §2.8), and let $N = \mathbf{k}sm^2$. Let us consider the piece of orb $\mathcal{K}$ of length N,

(5.1) $$f^t(\mathcal{K}), \quad t = \tau, \dots, \tau + N.$$

Then one of the following options takes place:

(i) m sets $\mathcal{K}_j = f^{t_j}\mathcal{K}$, $j = 0, \dots, m-1$, in the orbit (5.1) belong to some puzzle piece $Y^{\boldsymbol{\lambda}}(z)$ in the complement of the star $S^{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$;

(ii) $\mathbf{k}m$ consecutive sets $f^t(\mathcal{K})$, $t = i+1, \dots i + \mathbf{k}m$, in the orbit (5.1) belong to the star $S^{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$. Here i is selected so that that $f^i(\mathcal{K})$ does not belong to the star $S^{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$. Note that $i \geq \tau \geq \tau_0$ by definition of τ.

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Assume the first option occured. Then let us consider the puzzle piece $Y^0(z)$ of depth 0 containing $Y^{\boldsymbol{\lambda}}(z)$. By Lemma 2.12, $Y^{\boldsymbol{\lambda}}(z) \Subset Y^0(z)$. Let us consider if pullback $\Upsilon_j = f^{-t_j}(Y^0(z))|\mathcal{K}$ containing $\mathcal{K}$. Then $\Upsilon_j \subset E^{\chi-1}$ (since $t_j \geq \tau_0$)).

Moreover, the map $g^{t_j} : \Upsilon_j \to Y^0(z)$ has degree bounded in terms of $\mathbf{k}$ and ρ. Indeed, $\deg(f^{\tau_0}| E^\chi)$ and N, $\boldsymbol{\lambda}$, $\mathbf{k}$ are bounded in these terms. Hence it is enough to show that the trajectory

$$f^m(\Upsilon_j), \quad \tau_0 < m < \tau - \boldsymbol{\lambda} - \mathbf{k},$$

does not hit the critical point. But if $f^m(\Upsilon_j) \ni 0$ then $f^{m+\mathbf{k}}(\Upsilon_j)$ would land outside $S^{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$ (since $f^{\mathbf{k}}(0) \notin S^{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$ and depth $f^{m+\mathbf{k}}(\Upsilon_j) \geq \boldsymbol{\lambda}$). Then $f^{m+\mathbf{k}}(\mathcal{K})$ would land outside $S^{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$ as well contradicting the defintion of τ as the first landing moment of orb K in $S^{\boldsymbol{\lambda}}(\boldsymbol{\alpha})$ after τ_0.

Now, selecting p bigger than $\mathbf{k}+N$, we bring ourselves in the position to apply the Transfer Principle with $Y = Y^{\boldsymbol{\lambda}}(z)$, $Z = Y^0(z)$. It yields:

$$\mathbf{mod}\,(Y^0(z), Y^{\boldsymbol{\lambda}}(z)) \leq \frac{C}{m}\mathbf{mod}\,(E^\chi, \mathcal{K}) \leq \rho\,\mathbf{mod}\,(E^\chi, \mathcal{K}).$$

Assume now the second option occured. Then there is a point $\alpha_\mu \in \boldsymbol{\alpha}$ such that $f^{i+\mathbf{k}j}(\mathcal{K}) \subset S^{\boldsymbol{\lambda}}(\alpha_\mu)$ for $j = 1, \ldots, m$, while $f^i(\mathcal{K}) \subset S^{\boldsymbol{\lambda}}(\alpha'_\mu)$. Let us pull the star $S^1(\alpha_\mu)$ back by $f^{i+\mathbf{k}j}$:

$$\Upsilon_j = f^{-(i+\mathbf{k}j)}(S^1(\alpha_\mu))|\,\mathcal{K}, \quad j = 0, \ldots, m-1.$$

Let us show that

(5.2) $$\Upsilon_j \subset E^{\chi-1}.$$

We fix some j and let $\Upsilon = \Upsilon_j$. We claim that int $f^i(\Upsilon)$ does not contain any points of $\boldsymbol{\alpha}$. Since α_μ is the only point of $\boldsymbol{\alpha}$ inside int $S^1(\alpha_\mu)$, it is the only point of $\boldsymbol{\alpha}$ that can be inside int $f^i(\Upsilon)$. If $\mu = 0$ then $f^i(\mathcal{K}) \subset S^{\boldsymbol{\lambda}}(\alpha')$, so by Lemma 2.14, $f^i(\Upsilon) \subset S^1(\alpha')$, which does not contain α. If $\mu \neq 0$ then the points α_μ and α'_μ are separated by α in the filled Julia set. Since $f^i(\Upsilon)$ is a geometric puzzle piece containing both α_μ and α'_μ, it must contain α as well – contradiction. The claim follows.

Thus, $f^i(\Upsilon)$ is a geometric puzzle piece whose interior does not contain any points of $\boldsymbol{\alpha}$. Hence it is contained in some puzzle piece $Y^0(z)$ of zero depth. Then $\Upsilon \subset f^{-i}(Y^0(z))| 0 = Y^i$. But since $i \geq \tau_0$, $Y^i \subset Y^{\tau_0} = E^{\chi-1}$, and (5.2) follows.

Other assumptions of the Transfer Principle (with $Y = S^{\boldsymbol{\lambda}}(\alpha_\mu)$ and $Z = S^1(\alpha_\mu)$) are valid for the same reason as in the first case. The lemma follows. □

5.3. **Bigons.** A geometric puzzle piece with two vertices is called a *bigon*, and the corresponding pseudo-puzzle piece is called a ψ-bigon. Given a bigon Y with vertices v and w, let S_Y^v and S_Y^w stand for the components of $K(f) \smallsetminus \text{int}\, Y$ containing v and w respectively. We let $S_Y = S_Y^v \cup S_Y^w$.

Recall from §3.2.2 that the ideal boundary of the corresponding ψ-puzzle piece $\mathbf{Y}$ comprises the outer boundary $\partial_O \mathbf{Y}$ (in the case of bigon consisting of two arcs) and the J-boundary $\partial_J \mathbf{Y} = \partial^v \mathbf{Y} \cup \partial^w \mathbf{Y}$ attached to the vertices. Let $\mathcal{G}_{\mathbf{Y}} = \mathcal{G}_{\mathbf{Y}}(v, w)$ stand (in the case of bigon) for the family of horizontal curves in $\mathbf{Y}$ connecting $\partial^v \mathbf{Y}$ to $\partial^w \mathbf{Y}$, and let $\mathbf{d}_{\mathbf{Y}}(v, w)$ stand for its extremal length.

More generally, let us consider a puzzle piece Y whose vertices are bi-colored, i.e., they are partitioned into two non-empty subsets, B and W. This induces a natural bi-coloring

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of $K(f) \setminus \text{int} \, Y$ and of $\partial_J \mathbf{Y}$: namely, a component of these sets attached to a black/white vertex inherits the corresponding color. Let $\mathcal{G}_\mathbf{Y}$ stands for the family of horizontal curves in $\mathbf{Y}$ connecting boundary components with different colors.

For a geometric puzzle piece Y, let $v(Y) \subset K$ denote the set of vertices of Y. Suppose that $Y \subset Z$ are (geometric) puzzle pieces with the same equipotential depth; we say that Y is *cut out of* Z if $v(Z) \subset v(Y)$, so that we have produced Y by cutting out pieces of Z. If the vertices of Y are bicolored, then the vertices of Z are as well.

Lemma 5.4. *If Y is cut out of Z, and $v(Y)$ is bicolored, then the family of curves $\mathcal{G}_Z$ overflows $\mathcal{G}_Y$.*

Proof. We prove the Lemma by induction on the cardinality of $v(Y) \setminus v(Z)$. First suppose that $v(Y) = v(Z) \cup \{w\}$, where $w \notin v(Z)$. Let $\gamma \in \mathcal{G}_Z$; the two endpoints of γ lie in differently-colored components $\partial^x \mathbf{Z}$, $\partial^y \mathbf{Z}$ of $\partial_J \mathbf{Z}$. If γ lifts to $\mathcal{G}_Y$, then we are finished. Otherwise, we can start lifting γ from the endpoint (of γ) that lies in the component (say $\partial^x Z$) of $\partial_J Z$ whose color is different from that of w. Then that partial lift of γ will connect $\partial^x Y$ and $\partial^w Y$ and hence will belong to $\mathcal{G}_Y$.

In general, if $|v(Y)| > |v(Z)| + 1$, we can let Y' be such that $v(Y) = v(Y') \cup \{w\}$, and $v(Y') \supset v(W)$. Then given $\gamma \in \mathcal{G}_W$, we can lift part of γ to $\mathcal{G}_{Y'}$ by induction, and then to $\mathcal{G}_Y$ as before. $\qquad\square$

Given a puzzle piece Y with $v(Y)$ bicolored and a bigon P, we let $P \succ Y$ if the vertices of P belong to components of $K(f) \setminus \text{int} \, Y$ with different colors, while the equipotential depths of P and Y are the same.

Lemma 5.5. *If $P \succ Y$ then the family $\mathcal{G}_P$ overflows $\mathcal{G}_Y$.*

Proof. Let P' be the bigon whose vertices are the vertices of Y that separate $\text{int} \, Y$ from the $v(P)$. Then $\mathcal{G}_P$ overflows $\mathcal{G}_{P'}$, and $\mathcal{G}_{P'}$ overflows $\mathcal{G}_Y$ by Lemma 5.5. $\qquad\square$

Let $\mathcal{W}_Y$ stand for the width of $\mathcal{G}_\mathbf{Y}$.

Lemma 5.6. *Let Y and P be two bigons such that the vertices of $f^n(P)$ are separated by Y for some n, and the equipotential depth of Y is 2^n times bigger than the equipotential depth of P. Then $\mathcal{W}_Y \geq 2^{-n}\mathcal{W}_P$.*

Proof. Since the vertices of $f^n(P)$ are separted by Y, there is a component Z of $f^{-n}(Y)$ such that $P \succ Z$. By Lemma 5.5, $\mathcal{W}_P \leq \mathcal{W}_Z$. On the other hand, the map $f^n : Z \to Y$ has degree at most 2^n, and maps horizontal curves in Z to horizontal curves in Y. Hence $\mathcal{W}_Z \leq 2^n \mathcal{W}_Y$. $\qquad\square$

Lemma 5.7. *Let Y be a bigon with vertices u and v of depths l and m satisfying the following property: If $l = m$ then $f^l u \neq f^l v$. Then there exists an $n \leq \max(l, m) + \mathbf{r}$ such that $f^n u$ and $f^n v$ are separated by the puzzle piece Y^1.*

Proof. By symmetry, we can assume that $l \leq m$. Suppose that $f^m u \neq f^m v$; then we can find $0 \leq t < r$ such that $f^{m+t} u$ and $f^{m+t} v$ are on opposite sides of the critical point, so they are separated by Y^1. Otherwise, we must have $l < m$, and then $f^{m-1} u = -f^{m-1} v$; hence they are separated by Y^1. $\qquad\square$

5.4. **Amplification.** We can now put together all the above results of this section as follows:

Lemma 5.8. *For any* $\mathbf{k} \in \mathbb{N}$ *and* $\rho \in (0,1)$, *there exist* $\varepsilon > 0$ *and* $\underline{p} \in \mathbb{N}$ *such that if* $p \geq \underline{p}$, *and* $\mathrm{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}) < \varepsilon$, *then*

$$\mathbf{d}_{Y^1}(\alpha, \alpha') \leq \rho \, \mathrm{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}).$$

Proof. Under our circumstances, Corollary 5.2 and Lemma 5.3 imply that there exist geometric puzzle pieces $Y \Subset Z$ with the bidepth of Z bounded by $(1,1)$ while the bidepth of Y bounded in terms of $\mathbf{k}$, such that

$$\mathbf{mod}\,(Z,Y) \leq \rho \, \mathrm{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}).$$

For any vertex v of Z, there exists a vertex v' of Y such that the rays of ∂Y landing at v' separate $\mathrm{int}\, Y$ from v. These two rays together with the two rays landing at v (trancated by the equipotential of Z) form a bigon B^v. By the Parallel Law, there exists a vertex v of Z such that

$$\mathbf{d}_{B^v}(v, v') \leq N \mathbf{mod}\,(Z,Y),$$

where $N \leq N(\mathbf{r})$ is the number of vertices of Z. By Lemma 5.7, there is an iterate $f^n(B^v)$ such that the vertices $f^n(v)$ and $f^n(v')$ are separated by $\mathrm{int}\, Y^1$. By Lemma 5.6,

$$\mathbf{d}_{Y^1}(\alpha, \alpha') \leq 2^n \mathbf{d}_{B^v}(v, v').$$

Putting the above three estimates together, we obtain the assertion. $\qquad\square$

Remark 5.1. The name "amplification" alludes to the extremal *width* which is amplified under the push-forward procedure described above.

5.5. **Separation.** The final step of our argument is to show that the vertices α and α' are well separated in the bigon Y^1.

Lemma 5.9. *There exists* $\kappa = \kappa(\mathbf{r}, \mathbf{q}, \mathbf{n}) > 0$ *such that*

$$\mathbf{d}_{Y^1}(\alpha, \alpha') \geq \kappa \, \mathrm{mod}(\mathbf{U}, K).$$

Idea of the proof. The proof is the same as the one of Proposition 5.12 in [KL3], so we will only give an idea here.

Let $\mathbf{Y}$ be a ψ-puzzle piece, and let v and w be two vertices of it. A *multicurve in* $\mathbf{Y}$ *connecting* $\partial^v \mathbf{Y}$ *to* $\partial^w \mathbf{Y}$ is a sequence of proper paths γ_i, $i = 1, \ldots, n$, in $\mathbf{Y}$ connecting $\partial^{v_{i-1}} \mathbf{Y}$ to $\partial^{v_i} \mathbf{Y}$, where $v = v_0, v_1, \ldots, v_n = w$ is a sequence of vertices in $\mathbf{Y}$. Let $\mathbf{W_Y}(v, w)$ stand for the extremal width of the family of multicurves in $\mathbf{Y}$ connecting $\partial^v \mathbf{Y}$ to $\partial^w \mathbf{Y}$. Let

$$\mathbf{W}_Y = \sup_{v,w} \mathbf{W}_Y(v, w).$$

Let us estimate this width for the puzzle piece P introduced in §2.6. To this end let us consider puzzle pieces Q^v from Lemma 2.10. Let r be the depth of these puzzle pieces, $T^{vw} = \mathrm{cl}(K_P \smallsetminus (Q^v \cup Q^w))$, and let $v' = Q^v \cap T^{vw}$, $w' = Q^w \cap T^{vw}$. For any multicurve γ in $\mathbf{P}$ connecting $\partial^v \mathbf{P}$ to $\partial^w \mathbf{P}$, one of the following events can happen:

(i) γ skips over T^{vw};

(ii) γ contains an arc γ' connecting an equipotential of depth r to T^{vw};

(iii) γ contains two disjoint multicurves, δ^v and δ^w, that do not cross this equipotential and such that δ^v connects $\partial^v \mathbf{Q}^v$ to $\partial^{v'} \mathbf{Q}^v$, while δ^w connects $\partial^{w'} \mathbf{Q}^w$ to $\partial^w \mathbf{Q}^w$.

It is not hard to show that the width of the first two families of multicurves is $O(\operatorname{mod}(\mathbf{U}, K))$ (see §§5.5-5.6 of [KL3]). Concerning each family of multicurves δ^v or δ^w that appear in (iii), it is conformally equivalent to a family of multicurves connecting appropriate two vertices of P (since Q^v and Q^w are conformal copies of P). By the Series and Parallel Laws,

$$\mathbf{W}_P \leq \frac{1}{2}\mathbf{W}_P + O(\operatorname{mod}(\mathbf{U}, K)),$$

which implies the desired estimate.

5.6. **Conclusion.** Everything is now prepared for the main results. Lemmas 5.8 and 5.9 imply:

Theorem 5.10 (Improving of the moduli). *For any parameters $\bar{\mathbf{r}}, \bar{\mathbf{q}}, \bar{\mathbf{n}}$ and any $\rho > 0$, there exist $\underline{p} \in \mathbb{N}$ and $\varepsilon > 0$ with the following property. Let f be a renormalizable ψ-quadratic-like map with renormalization period p satisfying the $(\bar{\mathbf{r}}, \bar{\mathbf{q}}, \bar{\mathbf{n}})$-molecule condition, and let g be its first renormalization. Then*

$$\{p \geq \underline{p} \ \text{ and } \mathbf{mod}\,(g) < \varepsilon\} \Rightarrow \mathbf{mod}\,(f) < \rho\operatorname{mod}(g).$$

Theorem 5.10, together with Lemma 3.1, implies Theorem 1.2 from the Introduction. The Main Theorem follows from Theorem 5.10 combined with the following result (Theorem 9.1 from [K]):

Theorem 5.11 (Improving of the moduli: bounded period). *For any $\rho \in (0, 1)$, there exists $\underline{p} = \underline{p}(\rho)$ such that for any $\bar{p} \geq \underline{p}$, there exists $\varepsilon = \varepsilon(\bar{p}) > 0$ with the following property. Let f be primitively renormalizable ψ-quadratic-like map, and let g be the corresponding renormalization. Then*

$$\{\underline{p} \leq p \leq \bar{p} \ \text{ and } \mathbf{mod}\,(g) < \varepsilon\} \Rightarrow \mathbf{mod}\,(f) < \rho\,\mathbf{mod}\,(g).$$

Putting the above two theorems together, we obtain:

Corollary 5.12. *For any $(\bar{\mathbf{r}}, \bar{\mathbf{q}}, \bar{\mathbf{n}})$, there exist an $\varepsilon > 0$ and $l \in \mathbb{N}$ with the following property. For any infinitely renormalizable ψ-ql map f satisfying the $(\bar{\mathbf{r}}, \bar{\mathbf{q}}, \bar{\mathbf{n}})$-molecule condition with renormalizations $g_n = R^n f$, if $\mathbf{mod}\,(g_n) < \varepsilon$, $n \geq l$, then $\mathbf{mod}\,(g_{n-l}) < \frac{1}{2}\mathbf{mod}\,(g_n)$.*

This implies the Main Theorem, in an important refined version. We say that a family $\mathcal{M}$ of little Mandelbrot copies (and the corresponding renormalization combinatorics) has *beau*[10] *a priori* bounds if there exists an $\varepsilon = \varepsilon(\mathcal{M}) > 0$ and a function $N : \mathbb{R}_+ \to \mathbb{N}$ with the following property. Let $f : U \to V$ be a quadratic-like map with $\operatorname{mod}(V \smallsetminus U) \geq \delta > 0$ that is at least $N = N(\delta)$ times renormalizable. Then for any $n \geq N$, the n-fold renormalization of f can be represented by a quadratic-like map $R^n f : U_n \to V_n$ with $\operatorname{mod}(V_n \smallsetminus U_n) \geq \varepsilon$.

Beau Bounds. For any parameters $(\bar{\mathbf{r}}, \bar{\mathbf{q}}, \bar{\mathbf{n}})$, the family of renormalization combinatorics satisfying the $(\bar{\mathbf{r}}, \bar{\mathbf{q}}, \bar{\mathbf{n}})$-molecule condition has beau a priori bounds.

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[10]According to Dennis Sullivan, "beau" stands for "bounded and eventually universal".

5.7. **Table of notations.** p is the renormalization period of f;

γ is a dividing periodic point of period $\mathbf{t}$,

$\boldsymbol{\gamma}$ is its cycle;

$\mathbf{s}$ is the numner of rays landing at γ;

α is a dividing periodic point of periods $\mathbf{r} = \mathbf{ts}$;

$\alpha' = -\alpha$, $\alpha_j = f^j \alpha$;

$\boldsymbol{\alpha} = \{\alpha_j\}_{j=0}^{p-1}$ is the cycle of α;

$\mathbf{q}$ is the number of rays landing at α;

$\mathbf{n}$ is the first moment such that $f^{\mathbf{rqn}}(0)$ is separated from 0 by the rays landing at α',

$\mathbf{k} = \mathbf{rqn}$;

$\boldsymbol{\lambda} = \mathbf{k} + 1 + \mathbf{r}^2$ is such a depth that the stars $S^{\boldsymbol{\lambda}}(\alpha_j)$, $j = 0, 1, \ldots, \mathbf{r} - 1$, are all disjoint;

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