CHAOTIC PERIOD DOUBLING

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Abstract. The period doubling renormalization operator was introduced by M. Feigenbaum and by P. Coullet and C. Tresser in the nineteen-seventieth to study the asymptotic small scale geometry of the attractor of one-dimensional systems which are at the transition from simple to chaotic dynamics. This geometry turns out to not depend on the choice of the map under rather mild smoothness conditions. The existence of a unique renormalization fixed point which is also hyperbolic among generic smooth enough maps plays a crucial role in the corresponding renormalization theory. The uniqueness and hyperbolicity of the renormalization fixed point were first shown in the holomorphic context, by means that generalize to other renormalization operators. It was then proved that in the space of $C^{2+\alpha}$ unimodal maps, for α close to one, the period doubling renormalization fixed point is hyperbolic as well. In this paper we study what happens when one approaches from below the minimal smoothness thresholds for the uniqueness and for the hyperbolicity of the period doubling renormalization generic fixed point. Indeed, our main results states that in the space of C^2 unimodal maps the analytic fixed point is not hyperbolic and that the same remains true when adding enough smoothness to get a priori bounds. In this smoother class, called $C^{2+|\cdot|}$ the failure of hyperbolicity is tamer than in C^2. Things get much worse with just a bit less of smoothness than C^2 as then even the uniqueness is lost and other asymptotic behavior become possible. We show that the period doubling renormalization operator acting on the space of C^{1+Lip} unimodal maps has infinite topological entropy.

Contents

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1. Introduction

The period doubling renormalization operator was introduced by M. Feigenbaum [Fe], [Fe2] and by P. Coullet and C. Tresser [CT], [TC] to study the asymptotic small scale geometry of the attractor of one-dimensional systems which are at the transition from simple to chaotic dynamics. In 1978, they published certain rigidity properties of such systems, the small scale geometry of the invariant Cantor set of generic smooth maps at the boundary of chaos being independent of the particular map being considered. Coullet and Tresser treated this phenomenon as similar to *universality* that has been observed in critical phenomena for long and explained since the early seventieth by Kenneth Wilson (see, *e.g.,* [Ma]). In an attempt to explain universality at the transition to chaos, both groups formulated the following conjectures that are similar to what was conjectured in statistical mechanics.

Renormalization conjectures: *In the proper class of maps, the period doubling* renormalization operator *has a unique fixed point that is hyperbolic with a one-dimensional unstable manifold and a codimension one stable manifold consisting of the systems at the transition to chaos.*

These conjectures were extended to other types of dynamics on the interval and on other manifolds but we will not be concerned here with such generalizations. During the last 30 years many authors have contributed to the development of a rigorous theory proving the renormalization conjectures and explaining the phenomenology. The ultimate goal may still be far since the universality class of smooth maps at the boundary of chaos contains many sorts of dynamical systems, including useful differential models of natural phenomena and there even are predictions about natural phenomena in [CT], which turned out to be experimentally corroborated. A historical review of the mathematics that have been developed can be found in [FMP] so that we recall here only a few milestones that will serve to better understand the contribution to the overall picture brought by the present paper.

The type of differentiability of the systems under consideration has a crucial influence on the actual small scale geometrical behavior (like it is the case in the related problem of smooth conjugacy of circle diffeomorphisms to rotations: compare [He] to [KO] and [KS]). The first result dealt with holomorphic systems and were first local [La], and later global [Su], [McM], [Ly] (a progression similar to what had been seen in the problem of smooth conjugacy to rotations: compare [Ar] to [He] and [Yo]). With global methods came also means to consider other renormalizations. Indeed, the hyperbolicity of the unique renormalization fixed point has been shown in [La] for period doubling, and later in [Ly] by means that generalize to other sorts of dynamics. Then it was showed in [Da] that the renormalization fixed point is also hyperbolic in the space of $C^{2+\alpha}$ unimodal maps with α close to one (using [La]), these results being later extended in [FMP] to more general types of renormalization (using [Ly]). As far as existence of fixed points is concerned, a satisfactory theory could be obtained some time ago, first for period doubling only and then for maps with bounded combinatorics after several subclasses of dynamics had been solved, see [M] for the most general results, assuming the lowest degree of smoothness and references to the prior literature.

We are interested in exploring from below the limit of smoothness that permits hyperbolicity of the fixed point of renormalization. Our main result concern a new smoothness class, $C^{2+|\cdot|}$, which is bigger than $C^{2+\alpha}$ for any positive $\alpha \leq 1$, and is in fact wider than C^2 in ways that are rather technical as we shall describe later (this is the bigger class where the usual method to get a priori bounds for the geometry of the Cantor set work). We are interested here in the part of hyperbolicity that consists in the attraction in the stable manifold made of infinitely renomalizable maps. We show that in the space of $C^{2+|\cdot|}$ unimodal maps the analytic fixed point is not hyperbolic for the action of the period doubling renormalization operator. We also show that nevertheless, the renormalization converges to the analytic generic fixed point (here generic means that the second derivative at the critical point is not zero), proving it to be globally unique, a uniqueness that was formerly known in classes smaller than $C^{2+|\cdot|}$ (hence assuming more smoothness). The convergence might only be polynomial as a concrete sign of non-hyperbolicity. The failure of hyperbolicity happens in a more serious way in the space of C^2 unimodal maps since there the convergence can be arbitrarily slow. The uniqueness of the fixed point in this case, remains an open question. The uniqueness was known to be wrong in a serious way among C^{1+Lip} unimodal maps since a continuum of fixed points of renormalization could be produced [Tr]. Here we show that the period doubling renormalization operator acting on the space of C^{1+Lip} unimodal maps has infinite topological entropy.

After this informal discussion of what will be done here and how it relates to universality theory, we now give some definitions, which allows us next to turn to the precise formulation of our main results.

A *unimodal* map $f:[0,1]\to[0,1]$ is a C^1 mapping with the following properties.

- $f(1)=0$,
- there is a unique point $c\in(0,1)$, the *critical point*, where $Df(c)=0$,
- $f(c)=1$.

A map is a C^r unimodal maps if f is C^r. We will concentrate on unimodal maps of the type C^{1+Lip}, C^2, and $C^{2+|\cdot|}$. This last type of differentiability will be introduced in § 5.

The critical point c of a C^2 unimodal map f is called *non-flat* if $D^2f(c)\neq 0$. A critical point c of a unimodal map f is a *quadratic tip* if there exists a sequence of points $x_n\to c$ and constant $A>0$ such that

$$\lim_{n\to\infty}\frac{f(x_n)-f(c)}{(x_n-c)^2}=-A.$$

The set of C^r unimodal maps with a quadratic tip is denoted by $\mathcal{U}^r$. We will consider different metrics on this set denoted by dist_k with $k=0,1,2$ (in fact the usual C^k metrics).

A unimodal map $f:[0,1]\to[0,1]$ with quadratic tip c is *renormalizable* if

- $c\in[f^2(c),f^4(c)]\equiv I_0^1$,
- $f(I_0^1)=[f^3(c),f(c)]\equiv I_1^1$,
- $I_0^1\cap I_1^1=\emptyset$.

The set of renormalizable C^r unimodal maps is denoted by $\mathcal{U}_0^r\subset\mathcal{U}^r$. Let $f\in\mathcal{U}_0^r$ be a renormalizable map. The renormalization of f is defined by

$$Rf(x)=h^{-1}\circ f^2\circ h(x),$$

where $h:[0,1]\to I_0^1$ is the orientation reversing affine homeomorphism. This map Rf is again a unimodal map. The nonlinear operator $R:\mathcal{U}_0^r\to\mathcal{U}^r$ defined by

$$R:f\mapsto Rf$$

is called the *renormalization operator*. The set of *infinitely* renormalizable maps is denoted by

$$W^r = \bigcap_{n \geq 1} R^{-n}(\mathcal{U}_0^r).$$

There are many fundamental steps needed to reach the following result by Davie, see [Da]. For a brief history see [FMP] and references therein.

Theorem 1.1. *(Davie) Let $\alpha < 1$ close enough to one. There exists a unique renormalization fixed point $f_*^\omega \in \mathcal{U}^{2+\alpha}$. It has the following properties.*

- *f_*^ω is analytic,*
- *f_*^ω is a hyperbolic fixed point of $R : \mathcal{U}_0^{2+\alpha} \to \mathcal{U}^{2+\alpha}$,*
- *the codimension one stable manifold of f_*^ω coincides with $W^{2+\alpha}$.*
- *f_*^ω has a one dimensional unstable manifold which consists of analytic maps.*

In our discussion we only deal with period doubling renormalization. However, there are other renormalization schemes. The hyperbolicity for the corresponding generalized renormalization operator has been established in [FMP].

Our main results deal with $R : \mathcal{U}_0^r \to \mathcal{U}^r$ where $r \in \{1 + Lip, 2, 2 + |\cdot|\}$.

Theorem 1.2. *Let $d_n > 0$ be any sequence with $d_n \to 0$. There exists an infinitely renormalizable C^2 unimodal map f with quadratic tip such that*

$$dist_0\,(R^n f, f_*^\omega) \geq d_n.$$

Corollary 1.3. *The analytic unimodal map f_*^ω is not a hyperbolic fixed point of $R : \mathcal{U}_0^2 \to \mathcal{U}^2$.*

In § 5 we will introduce a type of differentiability of a unimodal map, called $C^{2+|\cdot|}$, which is the minimal needed to be able to apply the classical proofs of a priori bounds for the invariant Cantor sets of infinitely renormalizable maps, see for example [M2],[MMSS],[MS]. This type of differentiability will allow us to represent any $C^{2+|\cdot|}$ unimodal map as

$$f = \phi \circ q,$$

where q is a quadratic polynomial and ϕ has still enough differentiability to control cross-ratio distortion. The precise description of this decomposition is given in Proposition 5.6. For completeness we include the proof of the a priori bounds in § 7.

Theorem 1.4. *If f is an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map then*

$$\lim_{n \to \infty} dist_0\,(R^n f,\ f_*^\omega) = 0.$$

A construction similar to the one provided for C^2 unimodal maps leads to the following result:

Theorem 1.5. *Let $d_n > 0$ be any sequence with $\sum_{n \geq 1} d_n < \infty$. There exists an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map f with a quadratic tip such that*

$$dist_0\,(R^n f, f_*^\omega) \geq d_n.$$

The analytic unimodal map f_*^ω is not a hyperbolic fixed point of $R : \mathcal{U}_0^{2+|\cdot|} \to \mathcal{U}^{2+|\cdot|}$.

Our second set of theorems deals with renormalization of C^{1+Lip} unimodal maps with a quadratic tip.

Theorem 1.6. *There exists an infinitely renormalizable C^{1+Lip} unimodal map f with a quadratic tip which is not C^2 but*

$$Rf = f.$$

The *topological entropy* of a system defined on a noncompact space is defined to be the supremum of the topological entropies contained in compact invariant subsets: we will always mean topological entropy when the type of entropy is not specified. As a consequence of Theorem 1.1 we get that renormalization on $\mathcal{U}_0^{2+\alpha}$ has entropy zero.

Theorem 1.7. *The renormalization operator acting on the space of C^{1+Lip} unimodal maps with quadratic tip has infinite entropy.*

The last theorem illustrates a specific aspect of the chaotic behavior of the renormalization operator on $\mathcal{U}_0^{1+Lip}$:

Theorem 1.8. *There exists an infinitely renormalizable C^{1+Lip} unimodal map f with quadratic tip such that $\{c_n\}_{n\geq 0}$ is dense in a Cantor set. Here c_n is the critical point of $R^n f$.*

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2. Notation

Let $I, J \subset \mathbb{R}^n$, with $n \geq 1$. We will use the following notation.

- $cl(I)$, $int(J)$, ∂I, stands for resp. the closure, the interior, and the boundary of I.
- $|I|$ stands for the Lebesgue measure of I.
- If $n = 1$ then $[I, J]$ is smallest interval which contains I and J.
- $dist\ (x, y)$ is the Euclidean distance between x and y, and

$$dist\ (I, J) = \inf_{x\in I,\, y\in J} dist\ (x, y).$$

- If F is a map between two sets then image(F) stand for the image of F.
- Define $\mathrm{Diff}_+^k\ ([0,1])$, $k \geq 1$, is the set of orientation preserving C^k-diffeomorphisms.
- $|.|_k$, $k \geq 0$, stands for the C^k norm of the functions under consideration.
- $dist_k$, $k \geq 0$, stands for the C^k distance in the function spaces under consideration.
- There is a constant $K > 0$, held fixed throughout the paper, which lets us write $Q_1 \asymp Q_2$ if and only if

$$\frac{1}{K} \leq \frac{Q_1}{Q_2} \leq K.$$

There are two rather independent discussions. One on C^{1+Lip}maps and the other on C^2 maps. There is a slight conflict in the notation used for these two discussions. In particular, the notation I_1^n stands for different intervals in the two parts, but the context will make the meaning of the symbols unambiguous.

3. Renormalization of C^{1+Lip} unimodal maps

3.1. **Piece-wise affine infinitely renormalizable maps.** Consider the open triangle $\Delta = \{(x, y) \ : \ x, y > 0 \text{ and } x + y < 1\}$. A point $(\sigma_0, \sigma_1) \in \Delta$ is called a *scaling bi-factor*. A scaling bi-factor induces a pair of affine maps

$$\tilde{\sigma}_0 : [0,1] \to [0,1]\,,$$
$$\tilde{\sigma}_1 : [0,1] \to [0,1]\,,$$

defined by

$$\begin{aligned} \tilde{\sigma}_0(t) &= -\sigma_0 t + \sigma_0 = \sigma_0(1-t) \\ \tilde{\sigma}_1(t) &= \sigma_1 t + 1 - \sigma_1 = 1 - \sigma_1(1-t). \end{aligned}$$

A function $\sigma : \mathbb{N} \to \Delta$ is called a *scaling data*. For each $n \in \mathbb{N}$ we set $\sigma(n) = (\sigma_0(n), \sigma_1(n))$, so that the point $(\sigma_0(n), \sigma_1(n) \in \Delta$ induces a pair of maps $(\tilde{\sigma}_0(n), \tilde{\sigma}_1(n)$ as we have just described. For each $n \in \mathbb{N}$ we can now define the pair of intervals:

$$I_0^n = \tilde{\sigma}_0(1) \circ \tilde{\sigma}_0(2) \circ \cdots \circ \tilde{\sigma}_0(n)([0,1])\,,$$
$$I_1^n = \tilde{\sigma}_0(1) \circ \tilde{\sigma}_0(2) \circ \cdots \circ \tilde{\sigma}_0(n-1) \circ \tilde{\sigma}_1(n)([0,1])\,.$$

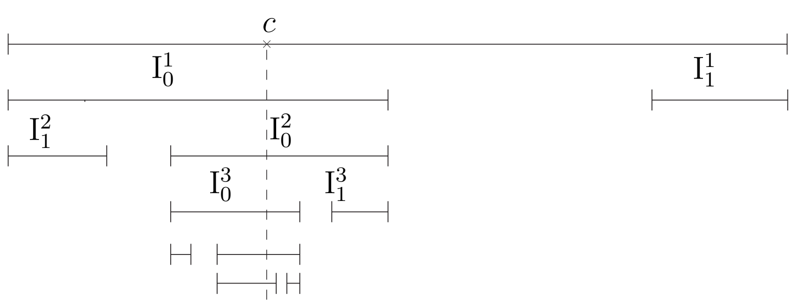

Figure 1

A scaling data with the property

$$dist\,(\sigma(n), \partial\Delta) \geq \epsilon > 0$$

is called $\epsilon-$*proper*, and *proper* if it is $\epsilon-$proper for some $\epsilon > 0$. For $\epsilon-$*proper* scaling data we have

$$|I_j^n| \leq (1-\epsilon)^n$$

with $n \geq 1$ and $j = 0, 1$. Given proper scaling data define

$$\{c\} = \cap_{n\geq 1} I_0^n.$$

The point c, called the *critical* point, is shown in Figure 1. Consider the quadratic map $q_c : [0,1] \to [0,1]$ defined as:

$$q_c(x) = 1 - \left(\frac{x-c}{1-c}\right)^2.$$

Given a proper scaling data $\sigma : \mathbb{N} \to \Delta$ and the set $D_\sigma = \cup_{n\geq 1} I_1^n$ induced by σ, we define a map

$$f_\sigma : D_\sigma \to [0,1]$$

by letting $f_\sigma|_{I_1^n}$ be the affine extension of $q_c|_{\partial I_1^n}$. The graph of f_σ is shown in Figure 2.

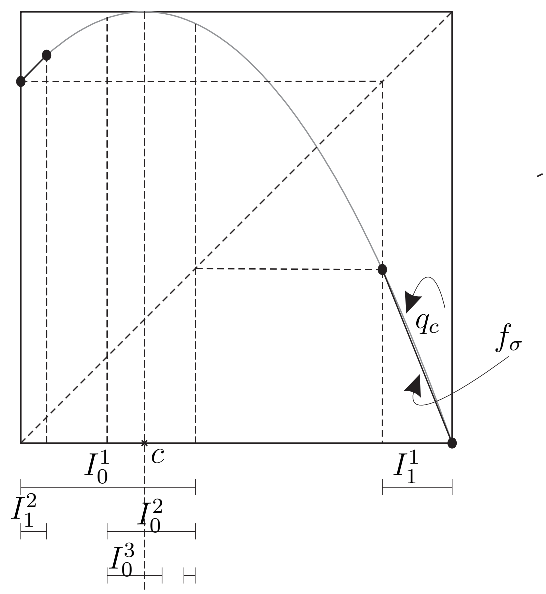

FIGURE 2

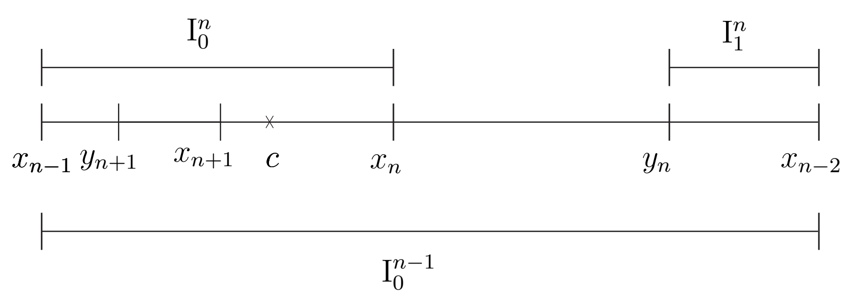

FIGURE 3

Define $x_0 = 0, x_{-1} = 1$ and for $n \geq 1$

$$\begin{aligned} x_n &= \partial I_0^n \setminus \partial I_0^{n-1}, \\ y_n &= \partial I_1^n \setminus \partial I_0^{n-1}. \end{aligned}$$

These points are illustrated in Figure 3.

Definition 1. *A map f_σ corresponding to proper scaling data $\sigma : \mathbb{N} \to \Delta$ is called infinitely renormalizable if for $n \geq 1$*

(i) $[f_\sigma(x_{n-1}), 1]$ is the maximal domain containing 1 on which $f_\sigma^{2^n-1}$ is defined affinely.
(ii) $f_\sigma^{2^n-1}([f_\sigma(x_{n-1}),\ 1]) = I_0^n$.

Define $W = \{f_\sigma \ : f_\sigma$ is infinitely renormalizable$\}$. Let $f \in W$ be given by the proper scaling data $\sigma : \mathbb{N} \to \Delta$ and define

$$\hat{I_0^n} = [q_c(x_{n-1}),\ 1] = [f(x_{n-1}),\ 1].$$

Let

$$h_{\sigma,\, n} : [0,1] \to [0,1]$$

be defined by

$$h_{\sigma,\,n} = \sigma_0(1)\circ\sigma_0(2)\circ\cdots\circ\sigma_0(n).$$

Furthermore let

$$\hat{h}_{\sigma,\,n} : [0,1] \to \hat{I}_0^n$$

be the affine orientation preserving homeomorphism. Then define

$$\mathrm{R}_n f_\sigma : h_{\sigma,n}^{-1}(D_\sigma) \to [0,\ 1]$$

by

$$\mathrm{R}_n f_\sigma = \hat{h}_{\sigma,\,n}^{-1}\circ f_\sigma \circ h_{\sigma,\,n}.$$

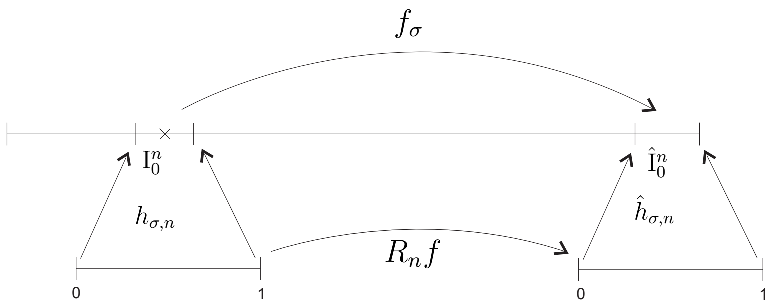

FIGURE 4

It is shown in Figure 4. Let $s : \Delta^{\mathbb{N}} \to \Delta^{\mathbb{N}}$ be the shift

$$s(\sigma)(k) = \sigma(k+1).$$

The construction implies the following result:

Lemma 3.1. *Let $\sigma : \mathbb{N} \to \Delta$ be proper scaling data such that f_σ is infinitely renormalizable. Then*

$$\mathrm{R}_n f_\sigma = f_{s^n(\sigma)}.$$

Let next f_σ be infinitely renormalizable, then for $n \geq 0$ we have

$$f_\sigma^{2^n} : \mathrm{D}_\sigma \cap I_0^n \to I_0^n$$

is well defined. Define the renormalization $\mathrm{R} : W \to W$ by

$$\mathrm{R} f_\sigma = h_{\sigma,\,1}^{-1}\circ f_\sigma^2 \circ h_{\sigma,\,1}.$$

The map $f_\sigma^{2^n-1} : \hat{I}_0^n \to I_0^n$ is an affine homeomorphism whenever $f_\sigma \in W$. This implies immediately the following Lemma.

Lemma 3.2. $\mathrm{R}^n f_\sigma : \mathrm{D}_{s^n(\sigma)} \to [0,1]$ *and* $\mathrm{R}^n f_\sigma = \mathrm{R}_n f_\sigma$.

Proposition 3.3. $W = \{f_{\sigma^*}\}$ *where σ^* is characterized by* $\mathrm{R} f_{\sigma^*} = f_{\sigma^*}$

Proof. Let $\sigma : \mathbb{N} \to \Delta$ be proper scaling data such that f_σ is infinitely renormalizable. Let c_n be the critical point of $f_{s^n(\sigma)}$. Then

$$q_{c_n}(0) = 1 - \sigma_1(n) \tag{1}$$

$$q_{c_n}(1-\sigma_1(n)) = \sigma_0(n) \tag{2}$$

$$c_{n+1} = \frac{\sigma_0(n) - c_n}{\sigma_0(n)}. \tag{3}$$

We also have the conditions

$$\sigma_0(n), \sigma_1(n) > 0 \tag{4}$$

$$\sigma_0(n) + \sigma_1(n) < 1 \tag{5}$$

$$0 < c_n < \frac{1}{2} \tag{6}$$

From conditions (1), (2) and (3) we get

$$\sigma_0(n) = \frac{2c_n^2 - 6c_n^3 + 5c_n^4 - 2c_n^5}{(c_n - 1)^6} \equiv A_0(c_n) \tag{7}$$

$$\sigma_1(n) = \frac{c_n^2}{(c_n - 1)^2} \equiv A_1(c_n) \tag{8}$$

$$c_{n+1} = \frac{c_n^6 - 6c_n^5 + 17c_n^4 - 25c_n^3 + 21c_n^2 - 8c_n + 1}{2c_n^4 - 5c_n^3 + 6c_n^2 - 2c_n} \equiv R(c_n) \tag{9}$$

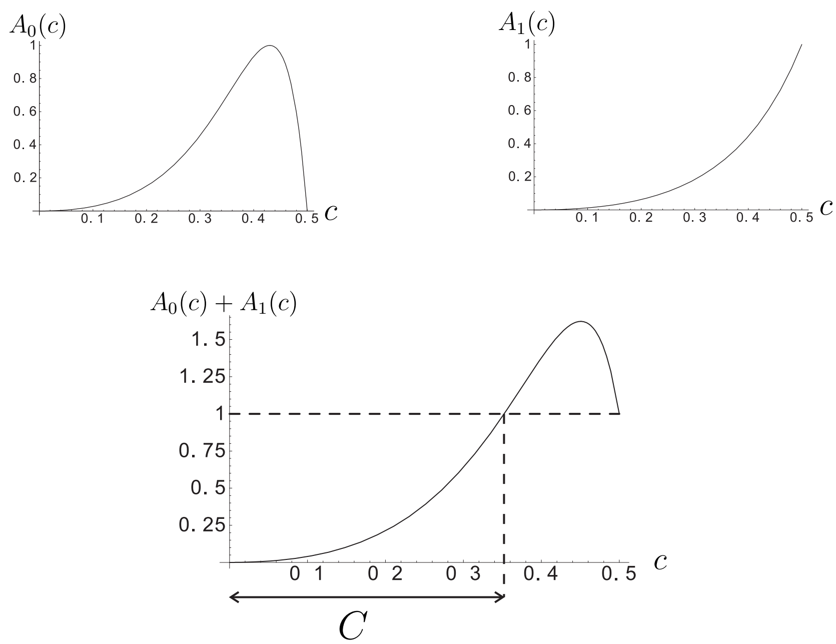

FIGURE 5. The graphs of A_0, A_1 and $A_0 + A_1$

The conditions (4), (5) and (6) reduces to $c \in (0, 1/2)$ and $A_0(c) + A_1(c) < 1$. In particular this lets the feasible domain be:

$$\begin{aligned} C &= \left\{ c \in (0,\ 1/2)\ :\ 0 \leq \frac{c^2(3 - 10c + 11c^2 - 6c^3 + c^4)}{(c-1)^6} < 1 \right\} \\ &= [0,\ 0.35...] \end{aligned}$$

Notice that the map $R : C \to \mathbb{R}$ is expanding. It follows readily that only the fixed point $c^* \in C$ and $R(c^*) = c^*$ corresponds to an infinitely renormalizable f_{σ^*}. Otherwise speaking, consider the scaling data $\sigma^* : \mathbb{N} \to \Delta$ with

$$\sigma^*(n) = \left(q_{c^*}^2(0),\ 1 - q_{c^*}(0) \right),\ n \geq 1.$$

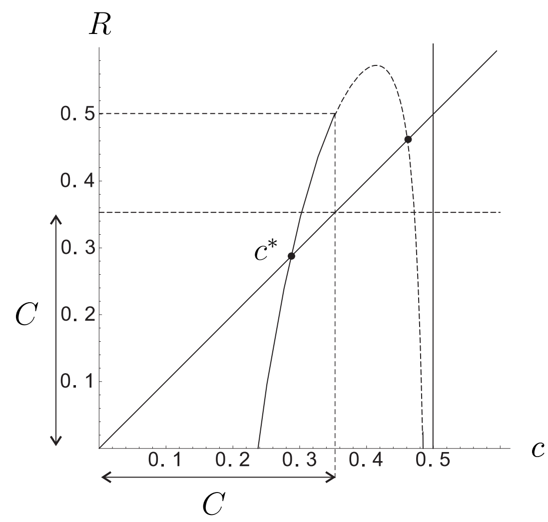

FIGURE 6. $R : C \to \mathbb{R}$

Then $s(\sigma^*) = \sigma^*$ and Lemma 3.1 implies

$$\mathrm{R} f_{\sigma^*} = f_{\sigma^*}.$$

$\square$

Remark 3.4. *Let $I_0^n = [x_{n-1}, x_n]$ be the interval corresponding to σ^* then*

$$f_{\sigma^*}(x_{n-1}) = q_{c^*}(x_{n-1}).$$

Hence f_{σ^*} has a quadratic tip.

Remark 3.5. *The invariant Cantor set of the map f_{σ^*} is next in complexity to the well known middle third Cantor set in the following sense:*

- like in the middle third Cantor set, on each scale and everywhere the same scaling ratios are used,

- but unlike in the middle third Cantor set, there are now two ratios (a small one and a bigger one) at each scale .

This situation of rather extreme tameness of the scaling data is very different from the geometry of the Cantor attractor of the analytic renormalization fixed point in which there are no two places where the same scaling ratios are used at all scales, and where the closure of the set of ratios is itself a Cantor set [BMT].

Lemma 3.6. *Let $f_* = f_{\sigma^*}$ where $\sigma^* : \mathbb{N} \to \Delta$ is the scaling data with $\sigma^*(n)(\sigma_0^*, \sigma_1^*)$. Then*

$$(\sigma_0^*)^2 = \sigma_1^*.$$

Proof. Let $\hat{I}_0^n = f_*(I_0^n) = [f_*(x_{n-1}),\ 1]$ and $\hat{I}_1^{n+1} = f_*(I_1^{n+1})$. Then $f_*^{2^n-1} : \hat{I}_0^n \to I_0^n$ is affine, monotone and onto. Further, by construction

$$f^{2^n-1}(\hat{I}_0^{n+1}) = I_1^{n+1}.$$

Hence,

$$\frac{|\hat{I}_0^{n+1}|}{|\hat{I}_0^n|} = \sigma_1^*.$$

So $|I_0^n| = (\sigma_0^*)^n$ and $|\hat{I}_0^n| = (\sigma_1^*)^n$. Now f_{σ^*} has a quadratic tip with

$$f_{\sigma^*}(x_n) = q_{c_*}(x_n).$$

Hence,

$$\sigma_1^* = \frac{|\hat{I}_0^{n+1}|}{|\hat{I}_0^n|} = \left(\frac{x_n - c}{x_{n-1} - c}\right)^2 = \left(\frac{|I_0^{n+1}|}{|I_0^n|}\right)^2 = (\sigma_0^*)^2.$$

This completes the proof. □

3.2. C^{1+Lip} **extension.** In this sub-section we will extend the piece-wise affine map f_* to a C^{1+Lip} unimodal map. Let $S : [0,1]^2 \to [0,1]^2$ be the scaling function defined by

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\sigma_0^* x + \sigma_0^* \\ \sigma_1^* y + 1 - \sigma_1^* \end{pmatrix} \equiv \begin{pmatrix} S_1(x) \\ S_2(y) \end{pmatrix}$$

and let F be the graph of $f_* = f_{\sigma^*}$, where $f_{\sigma^*} : D_{\sigma^*} \to [0,1]$, $D_{\sigma^*} = \cup_{n\geq 1} I_1^n$. Then the idea of how to construct an extension g of f_* is contained in the following lemma:

Lemma 3.7. *$F \cap$ image$(S) = S(F)$.*

Proof. Let $\hat{h} = \hat{h}_{\sigma^*,1}$ and $h = h_{\sigma^*,1}$. Let $(x,y) \in$ *graph*(f_*) $\cap$ *image*(S). Say $(x,y) = (S_1(x'), S_2(y'))$ with $S_2(y') = f_*(S_1(x'))$. Since $S_1(x') = h(x')$ and $S_2(y') = \hat{h}(y')$, we can write $y' = \hat{h}^{-1} \circ f_* \circ h(x')$. By Lemma 3.1

$$y' = R_1 f_*(x') = f_*(x'),$$

which gives $(x', y') \in$ *graph*(f_*). This in turn implies $(x,y) \in S($*graph*$f_*)$. By reading the previous argument backward we prove $S(\text{graph } f_*) \subset F \cap$ *image*(S). □

Lemma 3.8. *$S($graph $q_{c^*}) \subset$ graph(q_{c^*}).*

Proof. Let $S($*graph*$(q_{c^*}))$ be the graph of the function q. Since S is linear and q_c is quadratic we get that q is also a quadratic function. Then both $q_{c^*}(c^*) = 1$ and $q(c^*) = 1$, because of $S(c^*, 1) = (c^*, 1)$. Furthermore, by construction

$$S(1,0) = (0, q_{c^*}(0)) = (0, q(0)).$$

Hence $q_{c^*}(0) = q(0)$. Differentiate twice $S_2(y) = q(S_1(x))$ and use $(\sigma_0^*)^2 = \sigma_1^*$ from Lemma 3.6, which proves $q''(c^*) = q''_{c^*}(c^*)$. Now we conclude that the quadratic maps q and q_{c^*} are equal. □

Let F_0 be the graph of $f_*|_{I_1^1}$. Then by Lemma 3.7, $F = \cup_{k\geq 0} S^k(F_0)$. Let g be a C^{1+Lip} extension of f_* on $D_{\sigma_*} \cup [x_1, 1]$ and $G_0 =$ *graph* $(g|_{[x_1,\,1]})$. Then $G = \cup_{k\geq 0} S^k(G_0)$ is the graph of an extension of f_*. We prove that g is C^{1+Lip} and also has a quadratic tip. Let $B^k = S^k([0,1]^2)$, where

$$\begin{aligned} B^k &= [x_{k-1},\ x_k] \times [\hat{x}_{k-1},\ 1] \qquad \text{for } k = 1,3,5,\ldots \\ B^k &= [x_k,\ x_{k-1}] \times [\hat{x}_{k-1},\ 1] \qquad \text{for } k = 2,4,\ldots \end{aligned}$$

where $\hat{x}_{k-1} = q_c(x_{k-1}) = 1 - (\sigma_1^*)^k$. Let $b_n = (x_{n-1}, \hat{x}_{n-1}) = S^n(1,0)$.

Remark 3.9. *Notice that the points b_n lie on the graph of q_{c^*}. This follows from Lemma 3.8.*

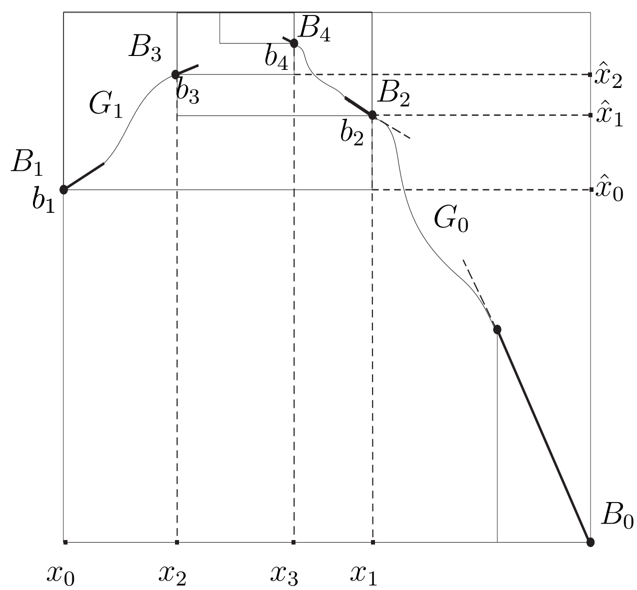

FIGURE 7. extension of f_{σ_*}

Lemma 3.10. *G is the graph of a C^1 extension of f_*.*

Proof. Note that $G_k = S^k(G_0)$ is the graph of a C^1 function on $[x_{k-1}, x_{k+1}]$ for k odd and on $[x_{k+1}, x_{k-1}]$ for k is even. To prove the Lemma we need to show continuous differentiability at the points b_n, where these graphs intersect. By construction G_0 is C^1 at b_2. Namely, consider a small interval $(x_1 - \delta, x_1 + \delta)$. Then on the interval $(x_1 - \delta, x_1)$, the slope is given by an affine piece of f_* and on $(x_1, x_1 + \delta)$ the slope is given by the chosen C^{1+Lip} extension. Let $\Gamma \subset G$ be the graph over this interval $(x_1 - \delta, x_1 + \delta)$. Then locally around b_n the graph G equals $S^{n-1}(\Gamma)$. Hence G is C^1 on $[0, 1] \setminus \{c^*\}$. From Lemma 3.6, notice that the vertical contraction of S is stronger than the horizontal contraction. This implies that the slope of G_n tends to zero. Indeed, G is the graph of a C^1 function on $[0, 1]$. $\square$

Proposition 3.11. *Let g be the function whose graph is G then g is C^{1+Lip} with a quadratic tip.*

Proof. Since $f_*|_{D_\sigma}$ has a quadratic tip, the extension g has a quadratic tip. Because g is C^1 we only need to show that G_n is the graph of a C^{1+Lip} function

$$g_n : [x_{n-1}, x_{n+1}] \to [0, 1]$$

with an uniform Lipschitz bound. That is, for $n \geq 1$

$$Lip(g'_{n+1}) \leq Lip(g'_n).$$

Assume that g_n is C^{1+Lip} with Lipschitz constant Lip_n for its derivative. We prove that $Lip_{n+1} \leq Lip_n$, and in particular $Lip_n \leq Lip_0$. For, given (x, y) on the graph of g_n there is $(x', y') = S(x, y)$, on the graph of g_{n+1}. Therefore, we can write

$$g_{n+1}(x') = \sigma_1^* \, g_n(x) + 1 - \sigma_1^*.$$

Since $x = 1 - \dfrac{x'}{\sigma_0^*}$, we have

$$g_{n+1}(x') = \sigma_1^* \, g_n \left(1 - \frac{x'}{\sigma_0^*}\right) + 1 - \sigma_1^*.$$

Differentiate,

$$g'_{n+1}(x') = \frac{-\sigma_1^*}{\sigma_0^*} \, g'_n \left(1 - \frac{x'}{\sigma_0^*}\right).$$

Therefore,

$$\begin{aligned} \left|g'_{n+1}(x'_1) - g'_{n+1}(x'_2)\right| &= \left|\frac{-\sigma_1^*}{\sigma_0^*}\right| \cdot \left|g'_n \left(1 - \frac{x'_1}{\sigma_0^*}\right) - g'_n \left(1 - \frac{x'_2}{\sigma_0^*}\right)\right| \\ &\leq \frac{\sigma_1^*}{(\sigma_0^*)^2} \, Lip(g'_n) \, |x'_1 - x'_2| \end{aligned}$$

From Lemma 3.6 we have $\frac{\sigma_1^*}{(\sigma_0^*)^2} = 1$. Hence

$$Lip(g'_{n+1}) \leq Lip(g'_n) \leq Lip(g'_1).$$

which completes the proof. $\square$

Remark 3.12. *Notice that if f_σ is infinitely renormalizable then every extension g is renormalizable in the classical sense.*

Theorem 3.13. *There exists an infinitely renormalizable C^{1+Lip} unimodal map f with a quadratic tip which is not C^2 but*

$$Rf = f.$$

3.3. **Entropy of renormalization.** For all $\phi \in C^{1+Lip}$, $\phi : [x_1, 1] \to [0, 1]$, which extends f_* we constructed $f_\phi \in C^{1+Lip}$ in such a way that

(i) $\mathrm{R} f_\phi = f_\phi$
(ii) f_ϕ has a quadratic tip.

Now choose two C^{1+Lip} functions which extend f_*, say $\phi_0 : [x_1, 1] \to [0, 1]$ and $\phi_1 : [x_1, 1] \to [0, 1]$. For $\omega = (\omega_k)_{k \geq 1} \in \{0, 1\}^{\mathbb{N}}$, define

$$F_n(\omega) = S^n \left(graph \ \phi_{\omega_n}\right)$$

and

$$F(\omega) = \cup_{k \geq 1} F_k(\omega).$$

Then $F(\omega)$ is the graph of C^{1+Lip} with a quadratic tip f_ω, by an argument similar to what is given above. Let now

$$\tau : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$$

be the shift map defined by

$$\tau(\omega)_n = \omega_{n+1},$$

(so that the map τ acting on the set $\{0, 1\}^{\mathbb{N}}$ is the full 2-shift).

Proposition 3.14. *For all* $\omega \in \{0,1\}^{\mathbb{N}}$

$$f_\omega^2 : [0, x_1] \to [0, x_1]$$

is a unimodal map. In particular f_ω *is renormalizable and*

$$Rf_\omega = f_{\tau(\omega)}.$$

Proof. Note that $f_\omega : [0, x_1] \to I_1^1$ is unimodal and onto. Furthermore, $f_\omega : I_1^1 \to [0, x_1]$ is affine and onto. Hence f_ω is renormalizable. The construction also gives

$$Rf_\omega = f_{\tau(\omega)}.$$

□

Theorem 3.15. *Renormalization acting on the space of* C^{1+Lip} *unimodal maps has positive entropy.*

Proof. Note that $\omega \to f_\omega \in C^{1+Lip}$ is injective. Hence the domain of R contains a copy of the full 2-shift (i.e., contains a subset on which the restriction of R is topologically conjugate to the full 2-shift). □

Remark 3.16. *We can also embedded a full* k*-shift in the domain of* R *by choosing* $\phi_0, \phi_1, \dots, \phi_{k-1}$ *and repeat the construction. The entropy of* R *on* C^{1+Lip} *is actually unbounded.*

4. Chaotic scaling data

In this section we will use a variation on the construction of scaling data as presented in § 3 to obtain the following

Theorem 4.1. *There exists an infinitely renormalizable* C^{1+Lip} *unimodal map* g *with quadratic tip such that* $\{c_n\}_{n\geq 0}$*, where* c_n *is the critical point of* $R^n g$*, is dense in a Cantor set.*

The proof needs some preparation. For $\epsilon > 0$ we will modify the construction as described in § 3. This modification is illustrated in Figure 8. For $c \in (0, \frac{1}{2})$ let

$$\begin{aligned} \sigma_1(c,\epsilon) &= 1 - q_c(0), \\ \sigma_0(c,\epsilon) &= \epsilon\, q_c^2(0), \end{aligned}$$

where $\epsilon > 0$ and close to 1. Also let

$$R(c,\epsilon) = \frac{\sigma_0(c,\epsilon) - c}{\sigma_0(c,\epsilon)} = 1 - \frac{c}{q_c^2(0)} \cdot \frac{1}{\epsilon}.$$

In § 3 we observed that $R(c,1)$ has a unique fixed point $c^* \in (0, \frac{1}{2})$ with feasible $\sigma_0(c^*,1)$ and $\sigma_1(c^*,1)$. This fixed point is expanding. Although we will not use this, a numerical computation gives

$$\frac{\partial R}{\partial c}(c^*, 1) > 2.$$

Now choose $\epsilon_0 > \epsilon_1$ close to 1. Then $R(\cdot, \epsilon_0)$ will have an expanding fixed point c_0^* and $R(\cdot, \epsilon_1)$ a fixed point c_1^*. In particular, by choosing $\epsilon_0 > \epsilon_1$ close enough to 1 we will get the following horseshoe as shown in Figure 9; more precisely there exists an interval $A_0 = [c_0^*, a_0]$ and $A_1 = [a_1, c_1^*]$ such that

$$R_0 : A_0 \to [c_0^*, c_1^*] \supset A_0$$

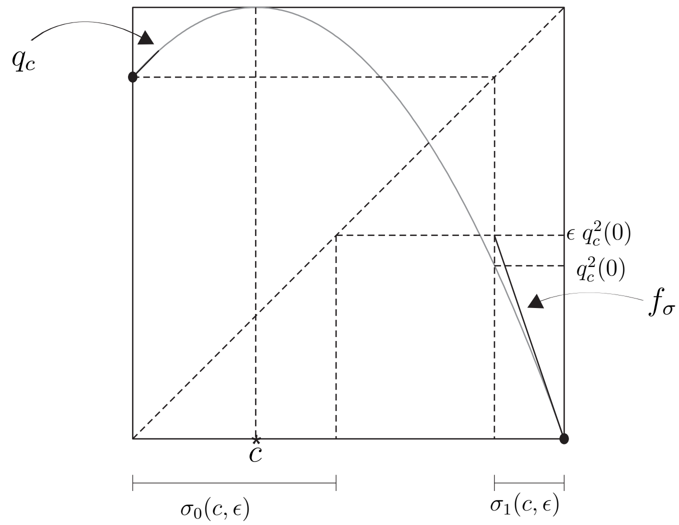

FIGURE 8

and

$$R_1 : A_1 \to [c_0^*, c_1^*] \supset A_1$$

are expanding diffeomorphisms (with derivative larger than 2, but larger than one would suffice to get a horseshoe). Here

$$R_0(c) = R(c, \epsilon_0)$$

and

$$R_1(c) = R(c, \epsilon_1).$$

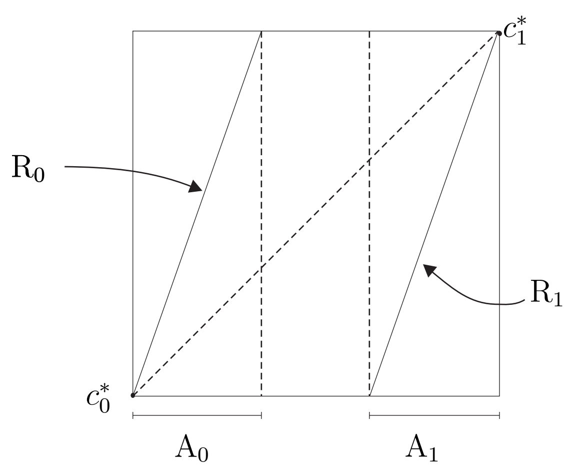

FIGURE 9

Use the following coding for the invariant Cantor set of the horseshoe map

$$c : \{0,1\}^{\mathbb{N}} \to [c_0^*, c_1^*]$$

with

$$c(\tau\omega) = R\left(c(\omega), \epsilon_{\omega_0}\right)$$

where $\tau : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is the shift. Given $\omega \in \{0,1\}^{\mathbb{N}}$ define the following scaling data $\sigma : \mathbb{N} \to \Delta$.

$$\sigma(n) = \left(\sigma_0\left(c(\tau^n\omega), \epsilon_{\omega_n}\right), \sigma_1\left(c(\tau^n\omega), \epsilon_{\omega_n}\right)\right).$$

Again, by taking ϵ_0, ϵ_1, close enough to 1, we can assume that $\sigma(n)$ is proper scaling data for any chosen $\omega \in \{0,1\}^{\mathbb{N}}$. As in § 3 we will define a piece wise affine map

$$f_\omega : D_\omega = \cup_{n\geq 1} I_1^n \to [0,1].$$

The precise definition needs some preparation. Use the notation as illustrated in Figure 10. For $n \geq 0$ let

$$I_0^n = [x_n,\ x_{n-1}]$$

where $x_n = \partial I_0^n \setminus \partial I_0^{n-1},\ \ n \geq 1$ and

$$I_1^n = [y_n,\ x_{n-2}]$$

where $y_n = \partial I_1^n \setminus \partial I_0^{n-1},\ \ n \geq 1$.

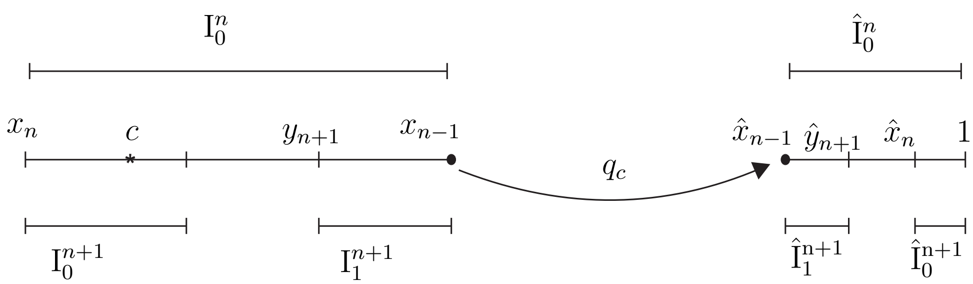

FIGURE 10

Let

$$\hat{I}_0^n = q_c([x_{n-1},\ 1]) = q_c(I_0^n) = [\hat{x}_{n-1},\ 1]$$

where $\hat{x}_{n-1} = q_c(x_{n-1})$. Finally, let $\hat{I}_1^{n+1} = [\hat{x}_{n-1},\ \hat{y}_{n+1}] \subset \hat{I}_0^n$ such that

$$|\hat{I}_1^{n+1}| = \sigma_0(n) \cdot |\hat{I}_0^n|.$$

Now define $f_\omega : I_1^{n+1} \to \hat{I}_1^{n+1}$ to be the affine homeomorphism such that

$$f_\omega(x_{n-1}) = q_c(x_{n-1}) = \hat{x}_{n-1}.$$

Lemma 4.2. *There exists $K > 0$ such that*

$$\frac{1}{K} \leq \frac{|\hat{I}_0^n|}{|I_0^n|^2} \leq K.$$

Proof. Observe, $c(n) = c(\tau^n\omega) \in [c_0^*, c_1^*]$ which is a small interval around c^*. This implies that for some $K > 0$

$$\frac{1}{K} \leq \frac{|c - x_{n-1}|}{|I_0^n|} \leq K.$$

Then

$$\frac{|\hat{I}_0^n|}{|I_0^n|^2} = \frac{|q_c([c, x_{n-1}])|}{|I_0^n|^2} = \frac{(c - x_{n-1})^2}{(1-c)^2} \cdot \frac{1}{(I_0^n)^2}$$

which implies the bound. □

Let $S_2^n : [0,1] \to \hat{I}_0^n$ be the affine orientation preserving homeomorphism and $S_1^n : [0,1] \to I_0^n$ be the affine homeomorphism with $S_1^n(1) = x_{n-1}$. Define

$$S^n : [0,1]^2 \to [0,1]^2$$

by

$$S^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_1^n(x) \\ S_2^n(y) \end{pmatrix}.$$

The image of S^n is B_n.

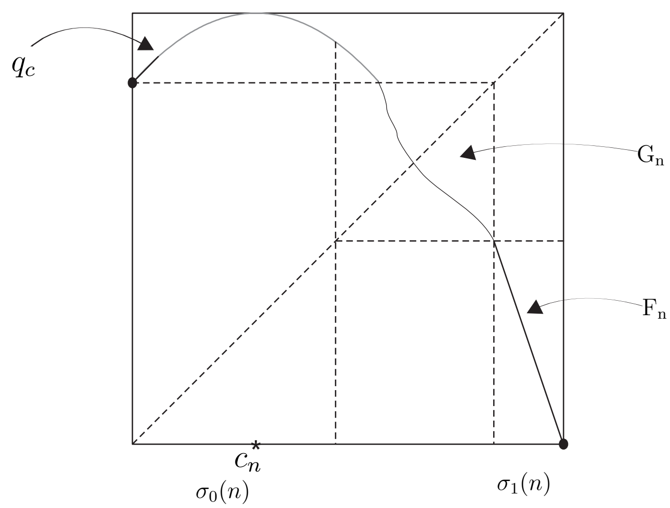

FIGURE 11

Let $F_n = (S^n)^{-1}(graph\ f_\omega)$. This is the graph of a function f_n. We will extend this function (and its graph) on the gap $[\sigma_0(n),\ 1-\sigma_1(n)]$. Notice, that $\sigma_0(n), 1-\sigma_1(n), Df_n(\sigma_0(n)),$ and $Df_n(1-\sigma_1(n))$ vary within a compact family. This allows us to choose from a compact family of C^{1+Lip} diffeomorphisms an extension

$$g_n : [\sigma_0(n), 1] \to [0, f_n(\sigma_0(n))]$$

of the map f_n. The Lipschitz constant of Dg_n is bounded by $K_0 > 0$. Let G_n be the graph of g_n and

$$G = \cup_{n \geq 0}\ S^n(G_n).$$

Then G is the graph of a unimodal map

$$g : [0,1] \to [0,1]$$

which extends f_ω. Notice, g is C^1. It has a quadratic tip because f_ω has a quadratic tip. Also notice that $S^n(G_n)$ is the graph of a C^{1+Lip} diffeomorphism. The Lipschitz bound L_n of its derivative satisfies, for a similar reason as in § 3,

$$L_n \leq \frac{|\hat{I}_0^n|}{|(I_0^n)|^2} \cdot K_0.$$

This is bounded by Lemma 4.2. Thus g_ω is a C^{1+Lip} unimodal map with quadratic tip. The construction implies that g is infinitely renormalizable and

$$graph\ (R^n g_\omega) \supset F_n.$$

One can prove Theorem 4.1 by choosing $\omega \in \{0,1\}^{\mathbb{N}}$ such that the orbit under the shift τ is dense in the invariant Cantor set of the horseshoe map.

Remark 4.3. *Let $\omega = \{0, 0, \dots\}$, then we will get another renormalization fixed point which is a modification of the one constructed in* § 3.

5. $C^{2+|\cdot|}$ unimodal maps

Let $f : [0,1] \to [0,1]$ be a C^2 unimodal map with critical point $c \in (0,1)$. Say, $D^2 f(x) = E(1 + \varepsilon(x))$, where

$$\varepsilon : [0,1] \to \mathbb{R}$$

is continuous with $\varepsilon(c) = 0$ and $E = D^2 f(c) \neq 0$. Let then

$$\bar{\varepsilon} : [0,1] \to \mathbb{R}$$

be defined by

$$\bar{\varepsilon}(x) = \frac{1}{x-c} \int_c^x \varepsilon(t) dt.$$

Notice, $\bar{\varepsilon}$ is continuous with $\bar{\varepsilon}(c) = 0$. Furthermore, $1 + \bar{\varepsilon}(x) \neq 0$ for all $x \in [0,1]$. Since

$$Df(x) = E(x-c)(1 + \bar{\varepsilon}(x))$$

and $Df(x)$ equals zero only when $x = c$. Let the map

$$\delta : [0,1] \to \mathbb{R}$$

defined by

$$\delta(x) = \varepsilon(x) - \bar{\varepsilon}(x).$$

Notice that δ is continuous and $\delta(c) = 0$. Finally, define

$$\beta : [0,1] \to \mathbb{R}$$

by

$$\beta(x) = \int_c^x \frac{1}{t-c} \delta(t) dt.$$

Lemma 5.1. *The function β is continuous and $\varepsilon = \delta + \beta$.*

Proof. The definition of δ gives $\bar{\varepsilon} = \varepsilon - \delta$, which is differentiable on $[0,1] \setminus \{c\}$, and

$$\begin{aligned} \varepsilon(x) &= ((x-c)(\varepsilon - \delta)(x))' \\ &= \varepsilon(x) - \delta(x) + (x-c)(\varepsilon - \delta)'(x). \end{aligned}$$

Hence,

$$\delta(x) = (x-c)(\varepsilon - \delta)'(x).$$

This implies

$$\varepsilon(x) = \delta(x) + \int_c^x \frac{1}{t-c} \delta(t) dt = \delta(x) + \beta(x).$$

$\square$

Definition 2. *Let $f : [0,1] \to [0,1]$ be unimodal map with critical point $c \in (0,1)$. We say f is $C^{2+|\cdot|}$ if and only if*

$$\hat{\beta} : x \longmapsto \int_c^x \frac{1}{|t-c|} \, |\delta(t)| dt$$

is continuous.

Remark 5.2. *Every $C^{2+\alpha}$ Hölder unimodal map, $\alpha > 0$, is $C^{2+|\cdot|}$.*

Remark 5.3. *The very weak condition of local monotonicity of $D^2 f$ is sufficient for f to be $C^{2+|\cdot|}$.*

Remark 5.4. *$C^{2+|\cdot|}$ unimodal maps are dense in C^2.*

Remark 5.5. *There exists C^2 unimodal maps which are not $C^{2+|\cdot|}$. See also remark 11.2.*

The *non-linearity* $\eta_\phi : [0,1] \to \mathbb{R}$ of a C^1 diffeomorphism $\phi : [0,1] \to [0,1]$ is given by

$$\eta_\phi(x) = D \; ln D\phi(x),$$

wherever it is defined.

Proposition 5.6. *Let f be a $C^{2+|\cdot|}$ unimodal map with critical point $c \in (0,1)$. There exist diffeomorphisms*

$$\phi_\pm : [0,1] \to [0,1]$$

such that

$$f(x) = \begin{cases} \phi_+ \left(q_c(x)\right) & x \in [c, \; 1] \\ \phi_- \left(q_c(x)\right) & x \in [0, \; c] \end{cases}$$

with

$$\eta_{\phi_\pm} \in L^1([0,1]).$$

Proof. It is plain that there exists a C^1 diffeomorphism

$$\phi_+ : [0, \; 1] \to [0, \; 1]$$

such that for $x \in [c, 1]$

$$f(x) = \phi_+ \left(q_c(x)\right).$$

We will analyze the nonlinearity of ϕ_+. Observe that:

$$Df(x) = -2 \, \frac{(x-c)}{(1-c)^2} \; \cdot \; D\phi_+ \left(q_c(x)\right)$$

and

$$\begin{aligned} D^2 f(x) \; &= \; 4 \, \frac{(x-c)^2}{(1-c)^4} \; \cdot \; D^2\phi_+ \left(q_c(x)\right) - 2 \, \frac{1}{(1-c)^2} \; \cdot D\phi_+ \left(q_c(x)\right) \\ &= \; E \; (1 + \varepsilon(x)). \end{aligned} \tag{10}$$

As we have seen before, we also have

$$Df(x) = E \; (x-c) \cdot (1 + \bar{\varepsilon}(x)).$$

This implies that

$$\eta_{\phi_+} \left(q_c(x)\right) = \frac{-(1-c)^2}{2} \; \cdot \; \frac{\varepsilon(x) - \bar{\varepsilon}(x)}{1 + \bar{\varepsilon}(x)} \; \cdot \frac{1}{(x-c)^2}. \tag{11}$$

Therefore, by performing the substitution $u = q_c(x)$, we get:

$$\int_0^1 |\eta_\phi(u)|\, du = \int_1^c -2\, |\eta_{\phi_+}(q_c(x))|\, \frac{x-c}{(1-c)^2}\, dx \tag{12}$$

$$= \int_c^1 \frac{|\varepsilon(x)-\bar{\varepsilon}(x)|}{1+\bar{\varepsilon}(x)}\, \frac{1}{x-c}\, dx \tag{13}$$

$$\leq \frac{1}{\min(1+\bar{\varepsilon})} \int_c^1 \frac{|\delta(x)|}{|x-c|}\, dx < \infty \tag{14}$$

We have proved $\eta_{\phi_+} \in L^1([0,1])$. Similarly one can prove the existence of a C^1 diffeomorphism

$$\phi_- : [0,\ 1] \to [0,\ 1]$$

such that for $x \in [0,c]$

$$f(x) = \phi_-(q_c(x))$$

and

$$\eta_{\phi_-} \in L^1([0,1]).$$

$\square$

6. Distortion of cross ratios

Definition 3. *Let $J \subset T \subset [0,1]$ be open and bounded intervals such that $T \setminus J$ consists of two components L and R. Define the cross ratios of these intervals as*

$$D(T,J) = \frac{|J||T|}{|L||R|}.$$

If f is continuous and monotone on T then define the cross ratio distortion of f as

$$B(f,T,J) = \frac{D(f(T),f(J))}{D(T,J)}.$$

If $f^n|_T$ is monotone and continuous then

$$B(f^n,T,J) = \prod_{i=0}^{n-1} B\left(f, f^i(T), f^i(J)\right).$$

Definition 4. *Let $f : [0,1] \to [0,1]$ be a unimodal map and $T \subset [0,1]$. We say that*

$$\left\{f^i(T) : 0 \leq i \leq n\right\}$$

has intersection multiplicity $m \in \mathbb{N}$ if and only if for every $x \in [0,1]$

$$\#\left\{i \leq n \mid x \in f^i(T)\right\} \leq m$$

and m is minimal with this property.

Theorem 6.1. *Let $f : [0,1] \to [0,1]$ be a $C^{2+|\cdot|}$ unimodal map with critical point $c \in (0,1)$. Then there exists $K > 0$, such that the following holds. If T is an interval such that $f^n|_T$ is a diffeomorphism then for any interval $J \subset T$ with $cl(J) \subset int(T)$ we have,*

$$B(f^n,T,J) \geq \exp\{-K \cdot m\}$$

where m is the intersection multiplicity of $\{f^i(T) : 0 \leq i \leq n\}$.

Proof. Observe that q_c expands cross-ratios. Then Proposition 5.6 implies

$$B\left(f, f^i(T), f^i(J)\right) > \frac{D\phi_i(j_i) \cdot D\phi_i(t_i)}{D\phi_i(l_i) \cdot D\phi_i(r_i)}$$

where $\phi_i = \phi_+$ or ϕ_- depending whether $f^i(T) \subset [c,\ 1]$ or $[0,\ c]$ and

$$\begin{array}{rcl} j_i & \in & q_c\left(f^i(J)\right), \\ t_i & \in & q_c\left(f^i(T)\right), \\ l_i & \in & q_c\left(f^i(L)\right), \\ r_i & \in & q_c\left(f^i(R)\right). \end{array}$$

Thus

$$\begin{array}{l} ln\ B(f^n, T, J) = \displaystyle\sum_{i=0}^{n-1}\ ln\ B\left(f, f^i(T), f^i(J)\right) \geq \\ \displaystyle\sum_{i=0}^{n-1} (ln\ D\phi_i(j_i) - ln\ D\phi_i(l_i)) + (ln\ D\phi_i(t_i) - ln\ D\phi_i(r_i)) \geq \\ -\displaystyle\sum_{i=0}^{n-1} |\eta_{\phi_i}(\xi_i^1)|\ |j_i - l_i| + |\eta_{\phi_i}(\xi_i^2)|\ |t_i - r_i| \geq \\ -2\ m \left(\displaystyle\int |\eta_{\phi_+}| + \int |\eta_{\phi_-}|\right) = -K \cdot m. \end{array}$$

Therefore

$$B(f^n, T, J) \geq \exp\{-K \cdot m\}.$$

□

The previous Theorem allows us to apply the Koebe Lemma. See [MS] for a proof.

Lemma 6.2. *(Koebe Lemma) For each $K_1 > 0$, $0 < \tau < 1/4$, there exists $K < \infty$ with the following property:*
Let $g : T \to g(T) \subset [0, 1]$ be a C^1 diffeomorphism on some interval T. Assume that for any intervals J^* and T^* with $J^* \subset T^* \subset T$ one has

$$B(g, T^*, J^*) \geq K_1 > 0,$$

for an interval $M \subset T$ such that $cl(M) \subset int(T)$. Let L, R be the components of $T \setminus M$. Then, if:

$$\frac{|g(L)|}{|g(M)|} \geq \tau \ \ and\ \ \frac{|g(R)|}{|g(M)|} \geq \tau$$

we have:

$$\forall x, y \in M, \qquad \frac{1}{K} \leq \frac{|g'(x)|}{|g'(y)|} \leq K.$$

Remark 6.3. *The conclusion of the Koebe-Lemma is summarized by saying that $g|_M$ has bounded distortion.*

7. A priori bounds

Let f be an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map with quadratic tip at $c \in (0,1)$. Let $I_0^n = [f^{2^n}(c), f^{2^{n+1}}(c)]$ be the central interval whose first return map corresponds to the n^{th}-renormalization. Here, we study the geometry of the cycle consisting of the intervals

$$I_j^n = f^j(I_0^n), \quad j = 0, 1, \dots, 2^n - 1.$$

Notice that

$$I_j^{n+1}, I_{j+2^n}^{n+1} \subset I_j^n, \quad j = 0, 1, \dots, 2^n - 1.$$

Let I_l^n and I_r^n be the direct neighbors of I_j^n for $3 \le j \le 2^n$.

Lemma 7.1. *For each $1 \le i < j$, There exists an interval T which contains I_i^n, such that $f^{j-i} : T \to [I_l^n, I_r^n]$ is monotone and onto.*

Proof. Let $T \subset [0,1]$ be the maximal interval which contains I_i^n such that $f^{j-i}|_T$ is monotone. Such interval exists because of monotonicity of $f^{j-i}|_{I_i^n}$. The boundary points of T are $a, b \in [0,1]$. Suppose $f^{j-i}(b)$ is to the right of I_j^n. The maximality of T ensures the existence of k, $k < j - i$ such that $f^k(b) = c$. Because $i + k < j \le 2^n$, we have $c \notin I_{i+k}^n$ and so $f^{k+1}(T) \supset I_1^n$. Moreover, $f^{j-i-(k+1)}|_{f^{k+1}(T)}$ is monotone. Hence $f^{j-i-(k+1)}|_{I_1^n}$ is monotone. So $1 + j - i - (k+1) \le 2^n$. This implies that $f^{j-i}(T)$ contains $I_{1+j-i-(k+1)}^n$. In particular $f^{j-i}(T)$ contains I_r^n. Similarly we can prove $f^{j-i}(T)$ contains I_l^n. $\square$

Lemma 7.2. *(Intersection multiplicity) Let $f^{j-i} : T \to [I_l^n, I_r^n]$ be monotone and onto with $T \supset I_i^n$. Then for all $x \in [0,1]$*

$$\#\{k < j - i \mid f^k(T) \ni x\} \le 7.$$

Proof. Without loss of generality we may restrict ourselves to estimate the intersection multiplicity at a point $x \in U$, where

$$U = [I_l^n, I_r^n] = [u_l, u_r].$$

Let $c_l \in I_l^n$ such that $f^{2^n - l}(c_l) = c$ and

$$C_l = [u_l, c_l] \subset I_l^n.$$

Similarly, define

$$C_r = [c_r, u_r] \subset I_r^n.$$

Let $T_k = f^k(T), \qquad k = 0, 1, \dots. j - i.$
Claim: If $i + k \notin \{l, j, r\}$ and $T_k \cap U \ne \emptyset$ then

(i) $I_{i+k}^n \cap U = \emptyset$
(ii) $U \cap T_k = I_l^n$ *or* C_l *or* I_r^n *or* C_r.

Let $T \setminus I_i^n = L \cup R$ and then we may assume $U \cap T_k = U \cap L_k$ where $L_k = f^k(L)$. This holds because $I_{i+k}^n \cap U = \emptyset$. Consider the situation where

$$I_r^n \cap L_k \ne \emptyset.$$

The other possibilities can be treated similarly. Notice that I_r^n cannot be strictly contained in L_k. Otherwise there would be a third "neighbor" of I_j^n in U. Let $a = \partial L \cap \partial T$. Notice that

$$f^k(a) \in \partial L_k \cap I_r^n.$$

Furthermore,

$$f^{j-k}(f^k(a)) \in \partial U.$$

This means $f^{j-k}(f^k(a))$ is a point in the orbit of c. This holds because all boundary points of the interval I_j^n are in the orbit of c. Hence, $f^k(a)$ is a point in the orbit of c or $f^k(a)$ is a preimage of c. The first possibility implies $f^k(a) \in \partial I_r^n$. This implies

$$U \cap T_k = U \cap L_k = I_r^n.$$

The second possibility implies $f^k(a) = c_r$ which means

$$U \cap T_k = U \cap L_k = C_r.$$

This finishes the proof of claim. This claim gives 7 as bound for the intersection multiplicity.

□

Proposition 7.3. *For $j < 2^n$, $f^{2^n-j} : I_j^n \to I_0^n$ has uniform bounded distortion.*

Proof. Step1 : Choose $j_0 < 2^n$, such that for all $j \leq 2^n$, we have $|I_{j_0}^n| \leq |I_j^n|$. By Lemma 7.1 there exists an interval neighborhood $T_n = L_n^0 \cup I_1^n \cup R_n^0$ such that $f^{j-1} : T_n \to [I_l^n, I_r^n] \supset I_{j_0}^n$ is monotone and onto. Lemma 7.2 together with Theorem 6.1 allow us to apply the Koebe Lemma 6.2. So, there exists $\tau_0 > 0$ such that

$$|L_n^0|, |R_n^0| \geq \tau_0 \, |I_1^n|.$$

Let $U_n = I_0^n$, $V_n = f^{-1}\left(L_n^0 \cup I_1^n \cup R_n^0\right)$ and let L_n^1, R_n^1 be the components of $V_n \setminus U_n$. From Proposition 5.6 we get $\tau_1 > 0$ such that

$$|L_n^1|, |R_n^1| \geq \tau_1 \, |U_n|.$$

Step2 : Suppose $W_n = [I_{l_n}^n, I_{r_n}^n]$, where $I_{l_n}^n, I_{r_n}^n$ are the direct neighbors of U_n. We claim that $V_n \subset W_n$. Suppose it is not. Then, say $I_{r_n}^n \subset int(V_n)$ implies that $f(I_{r_n}^n) \subset int(L_n^1)$. So, $f^{j_0-1}|_{f(I_{r_n}^n)}$ is monotone, implies that $r_n + j_0 \leq 2^n$ and $f^{j_0}(I_{r_n}^n) \subset int([I_l^n, I_r^n])$. This contradiction concludes that $V_n \subset W_n$.
Step3 : Let L_n, R_n be the components of $W_n \setminus U_n$. Then

$$|L_n|, |R_n| \geq \tau_1 \, |U_n|.$$

Step4 : For all $j < 2^n$, there exists an interval neighborhood T_j which contains I_j^n such that $f^{2^n-j} : T_j \to W_n$ is monotone and onto. Now Proposition 7.3 follows from the Lemma 7.2 together with Theorem 6.1 and the Koebe Lemma 6.2. □

Corollary 7.4. *There exists a constant K such that*

$$\left|Df^{2^n}|_{I_0^n}\right| \leq K.$$

Proof. Let $x \in I_1^n$. Then from Proposition 7.3 we get $K_1 > 0$ such that for some $x_0 \in I_1^n$

$$\begin{aligned} |Df^{2^n-1}(x)| &= \frac{|I_0^n|}{|I_1^n|} \cdot \left\{ \frac{Df^{2^n-1}(x)}{Df^{2^n-1}(x_0)} \right\} \\ &\leq \frac{|I_0^n|}{|I_1^n|} \cdot K_1. \end{aligned}$$

Proposition 5.6 implies that there exists $K_2 > 0$ such that for $x \in I_0^n$

$$|Df(x)| \leq K_2 \cdot |x - c|$$

and

$$|I_1^n| \geq \frac{1}{K_2} \cdot |I_0^n|^2.$$

Now for $x \in I_0^n$

$$\begin{aligned} |Df^{2^n}(x)| &\leq K_2 \cdot |x-c| \cdot \frac{|I_0^n|}{|I_1^n|} \cdot K_1 \\ &\leq K_2 \cdot K_1 \cdot \frac{|I_0^n|^2}{|I_1^n|} \leq K_2^2 \cdot K_1 = K \end{aligned}$$

Therefore, we conclude that $\left|Df^{2^n}|_{I_0^n}\right| \leq K$. ☐

Definition 5. *(A priori bounds) Let f be infinitely renormalizable. We say f has a priori bounds if there exists $\tau > 0$ such that for all $n \geq 1$ and $j \leq 2^n$ we have*

$$\tau < \frac{|I_j^{n+1}|}{|I_j^n|}, \frac{|I_{j+2^n}^{n+1}|}{|I_j^n|} \tag{15}$$

$$\tau < \frac{|I_j^n \setminus \left(I_j^{n+1} \cup I_{j+2^n}^{n+1}\right)|}{|I_j^n|} \tag{16}$$

where, $I_j^{n+1}, I_{j+2^n}^{n+1}$ are the intervals of next generation contained in I_j^n.

Proposition 7.5. *Every infinitely renormalizable $C^{2+|\cdot|}$ map has a priori bounds.*

Proof. Step1. There exists $\tau_1 > 0$ such that $\dfrac{|I_0^{n+1}|}{|I_0^n|} > \tau_1$.
Let $I_0^n = [a_n, a_{n-1}]$ be the central interval, and so $a_n = f^{2^n}(c)$. A similar argument as in the proof of Corollary 7.4 gives $K_1 > 0$ such that

$$|f^{2^n}([a_n, c])| \leq \left(\frac{|a_n - c|}{|I_0^n|}\right)^2 \cdot |I_0^n| \cdot K_1.$$

Notice that

$$f^{2^n}([a_n, c]) = I_{2^n}^{n+1}.$$

Thus

$$|I_{2^n}^{n+1}| \leq \frac{|a_n - c|^2}{|I_0^n|} \cdot K_1.$$

Note

$$f^{2^n}(I_{2^n}^{n+1}) = I_0^{n+1} \supset [a_n, c].$$

Therefore, by Corollary 7.4

$$|a_n - c| \leq |f^{2^n}(I_{2^n}^{n+1})| \leq K \cdot |I_{2^n}^{n+1}| \leq K \cdot \frac{|a_n - c|^2}{|I_0^n|} \cdot K_1.$$

This implies

$$|a_n - c| \geq \frac{1}{K} \cdot |I_0^n|.$$

Which proves $\dfrac{|I_0^{n+1}|}{|I_0^n|} > \tau_1$.

Step2. There exists $\tau_2 > 0$ such that $\dfrac{|I_{2^n}^{n+1}|}{|I_0^n|} \geq \tau_2$.

From above we get

$$\tau_1 |I_0^n| \leq |I_0^{n+1}| = |f^{2^n}(I_{2^n}^{n+1})| \leq K \cdot |I_{2^n}^{n+1}|$$

This proves

$$\frac{|I_{2^n}^{n+1}|}{|I_0^n|} \geq \tau_2.$$

Step3. There exists $\tau_3 > 0$ such that the following holds.

$$\frac{|I_j^{n+1}|}{|I_j^n|}, \quad \frac{|I_{j+2^n}^{n+1}|}{|I_j^n|} \geq \tau_3.$$

Because

$$f^{2^n - j}(I_j^{n+1}) = I_0^{n+1}, \quad f^{2^n - j}(I_j^n) = I_0^n$$

and from Proposition 7.3 we get a $K > 0$ such that

$$\frac{|I_j^{n+1}|}{|I_j^n|} \geq \frac{1}{K} \cdot \frac{|I_0^{n+1}|}{|I_0^n|} \geq \frac{\tau_1}{K}.$$

Hence, $\dfrac{|I_j^{n+1}|}{|I_j^n|} \geq \tau_3$. Similarly we prove $\dfrac{|I_{j+2^n}^{n+1}|}{|I_j^n|} \geq \tau_3$. Which completes the proof of (15).

Step4. To complete the proof of the Proposition, it remains to show that the gap between the intervals $I_0^{n+1}, I_{2^n}^{n+1}$ and as well as $I_j^{n+1}, I_{j+2^n}^{n+1}$ are not too small. Let

$$G_n = I_0^n \setminus \left(I_0^{n+1} \cup I_{2^n}^{n+1}\right).$$

We claim that there exists $\tau_4 > 0$ such that

$$\frac{|G_n|}{|I_0^n|} \geq \tau_4.$$

Let H_n be the image of G_n under f^{2^n}. Then $H_n = f^{2^n}(G_n) \supset I_{3 \cdot 2^n}^{n+2}$. The claim follows by using Corollary 7.4 and the bounds we have so far. Namely,

$$K \cdot |G_n| \geq |H_n| \geq |I_{3 \cdot 2^n}^{n+2}| \geq \tau_3 \cdot |I_{2^n}^{n+1}| \geq \tau_3 \cdot \tau_2 \cdot |I_0^n|.$$

This implies

$$|G_n| \geq \tau_4 \cdot |I_0^n|.$$

Step5. Let $G_j^n = I_j^n \setminus \left(I_j^{n+1} \cup I_{j+2^n}^{n+1}\right)$, then there exists $\tau_5 > 0$ such that

$$\frac{|G_j^n|}{|I_j^n|} \geq \tau_5.$$

We have $f^{2^n - j}(G_j^n) = G_n$ and $f^{2^n - j}(I_j^n) = I_0^n$. Since $f^{2^n - j}$ has bounded distortion, we immediately get a constant $K > 0$ such that

$$\frac{|G_j^n|}{|I_j^n|} \geq \frac{1}{K} \cdot \frac{|G_n|}{|I_0^n|} \geq \frac{\tau_4}{K}.$$

This implies

$$|G_j^n| \geq \tau_5 \cdot |I_j^n|.$$

This completes the proof of (16). □

8. Approximation of $f|_{I_j^n}$ by a quadratic map

Let $\phi : [0,1] \to [0,1]$ be an orientation preserving C^2 diffeomorphism with non-linearity $\eta_\phi : [0,1] \to \mathbb{R}$. The norm we consider is

$$|\phi| = |\eta_\phi|_0.$$

Let $[a,b] \subset [0,1]$ and $f : [a,b] \to f([a,b])$ be a diffeomorphism. Let

$$1_{[a\ b]} : [0,1] \to [a,b]$$

and

$$1_{f([a,b])} : [0,1] \to f([a,b])$$

be the affine homeomorphisms with $1_{[a,b]}(0) = a$ and $1_{f([a,b])}(0) = f(a)$. The rescaling $f_{[a,b]} : [0,1] \to [0,1]$ is the diffeomorphism

$$f_{[a,b]} = \left(1_{f([a,b])}\right)^{-1} \circ f \circ 1_{[a,b]}.$$

We say that $0 \in [0,1]$ corresponds to $a \in [a,b]$.

Proposition 8.1. *Let f be an infinitely renormalizable $C^{2+|\cdot|}$ map with critical point $c \in (0,1)$. For $n \geq 1$ and $1 \leq j < 2^n$ we have*

$$f_{I_j^n} = \phi_j^n \circ q_j^n$$

where

$$q_j^n = (q_c)_{I_j^n} : [0,1] \to [0,1]$$

such that 0 corresponds to $f^j(c) \in I_j^n$ and $\phi_j^n : [0,1] \to [0,1]$ a C^2 diffeomorphism. Moreover

$$\lim_{n\to\infty} \sum_{j=1}^{2^n-1} |\phi_j^n| = 0$$

Proof. If $I_j^n \subset [c,1]$ then use Proposition 5.6 and define

$$\phi_j^n = (\phi_+)_{q_c(I_j^n)} : [0,1] \to [0,1]$$

such that $0 \in [0,1]$ corresponds to $q_c\left(f^j(c)\right) \in q_c(I_j^n)$. In case $I_j^n \in [0,c]$ then let

$$\phi_j^n = (\phi_-)_{q_c(I_j^n)} : [0,1] \to [0,1]$$

where again $0 \in [0,1]$ corresponds to $q_c\left(f^j(c)\right) \in q_c(I_j^n)$. Let η_j^n be the non-linearity of ϕ_j^n. Then the chain rule for non-linearities [M] gives

$$|\eta_j^n(x)| = |q_c(I_j^n)| \cdot |\eta_{\phi_\pm}(1_j^n(x))|$$

where $1^n_j : [0,1] \to q_c(I^n_j)$ is the affine homeomorphism such that $1^n_j(0) = q_c(f^j(c))$. Now use (11) to get

$$\begin{aligned}
|\eta^n_j|_0 &\leq |q_c(I^n_j)| \cdot \frac{(1-c)^2}{2} \cdot \frac{1}{\min_{x\in I^n_j}(1+\bar{\epsilon}(x))} \cdot \sup_{x\in I^n_j} \frac{|\delta(x)|}{(x-c)^2} \\
&\leq \frac{1}{\min_{x\in[0,1]}(1+\bar{\epsilon}(x))} \cdot |\zeta^n_j - c| \cdot |I^n_j| \cdot \sup_{x\in I^n_j} \frac{|\delta(x)|}{|x-c|^2}
\end{aligned}$$

where

$$|Dq_c(\xi^n_j)| = \frac{|q_c(I^n_j)|}{|I^n_j|}$$

and $\xi^n_j \in I^n_j$. The a priori bounds gives $K_1 > 0$ such that

$$dist(c, I^n_j) \geq \frac{1}{K_1} \cdot |I^n_j|.$$

This implies that for some $K > 0$

$$|\eta^n_j| \leq K \cdot \sup_{x\in I^n_j} \frac{|\delta(x)|}{|x-c|} \cdot |I^n_j|.$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^{2^n-1} |\phi_{I^n_j}| &\leq K \cdot \sum_{j=1}^{2^n-1} \sup_{x\in I^n_j} \frac{|\delta(x)|}{|x-c|} \cdot |I^n_j| \\
&= K \cdot Z_n
\end{aligned}$$

Let $\Lambda_n = \cup_{j=0}^{2^n-1} I^n_j$. The a priori bounds imply that there exists $\tau > 0$ such that

$$|\Lambda_n| \leq (1-\tau)\, |\Lambda_{n-1}|.$$

In particular $|\Lambda| = 0$ where $\Lambda \cap\, \Lambda_n$ is the Cantor attractor. Now we go back to our estimate and notice that Z_n is a Riemann sum for

$$\int_{\Lambda_n} \frac{|\delta(x)|}{|x-c|}\, dx.$$

Suppose that $\limsup\ Z_n = Z > 0$. Let $n \geq 1$ and $m > n$. Then we can find a Riemann sum $\Sigma_{m,n}$ for

$$\int_{\Lambda_n} \frac{|\delta(x)|}{|x-c|}\, dx$$

by adding positive terms to Z_m. Then

$$\int_{\Lambda_n} \frac{|\delta(x)|}{|x-c|}\, dx = \limsup_{m\to\infty}\ \Sigma_{m,n} \geq \limsup_{m\to\infty}\ Z_m \geq Z > 0.$$

Hence,

$$\int_{\Lambda} \frac{|\delta(x)|}{|x-c|}\, dx\ \geq Z > 0.$$

This is impossible because $|\Lambda| = 0$. Thus we proved

$$\sum_{j=1}^{2^n-1} |\phi_{I_j^n}| \longrightarrow 0.$$

$\square$

9. Approximation of $R^n f$ by a polynomial map

The following Lemma is a variation on Sandwich Lemma from [M].

Lemma 9.1. *(Sandwich) For every $K > 0$ there exists constant $B > 0$ such that the following holds. Let ψ_1, ψ_2 be the compositions of finitely many $\phi, \phi_j \in \mathrm{Diff}^2_+([0,1]), 1 \le j \le n$;*

$$\psi_1 = \phi_n \circ \cdots \circ \phi_t \circ \ldots \phi_1$$

and

$$\psi_2 = \phi_n \circ \cdots \circ \phi_{t+1} \circ \phi \circ \phi_t \circ \ldots \phi_1.$$

If

$$\sum_j |\phi_j| + |\phi| \le K$$

then

$$|\psi_1 - \psi_2|_1 \le B\ |\phi|.$$

Proof. Let $x \in [0,1]$. For $1 \le j \le n$ let

$$x_j = \phi_{j-1} \circ \cdots \circ \phi_2 \circ \phi_1(x)$$

and

$$D_j = \left(\phi_{j-1} \circ \cdots \circ \phi_2 \circ \phi_1\right)'(x).$$

Furthermore, for $t+1 \le j \le n$, let

$$x'_j = \phi_{j-1} \circ \cdots \circ \phi_{t+1}(\phi(x_{t+1}))$$

and

$$D'_j = \left(\phi_{j-1} \circ \cdots \circ \phi_{t+1}\right)'(x'_{t+1})\ \phi'(x_{t+1})\ D_{t+1}.$$

Now we estimate the difference of the derivatives of ψ_1, ψ_2. Namely,

$$\frac{D\psi_2(x)}{D\psi_1(x)} \quad = \quad D\phi(x_{t+1}) \cdot \prod_{j \ge t+1} \frac{D\phi_j(x'_j)}{D\phi_j(x_j)}.$$

In the following estimates we will repeatedly apply Lemma 10.3 from [M] which says,

$$e^{-|\psi|} \le |D\psi|_0 \le e^{|\psi|}.$$

This allows us to get an estimate on $|D\psi_1 - D\psi_2|_0$ in terms of $\dfrac{D\psi_2}{D\psi_1}$. Now

$$D\phi_j(x'_j) = D\phi_j(x_j) + D^2\phi_j(\zeta_j)\ (x'_j - x_j).$$

Therefore,

$$\begin{aligned}\frac{D\phi_j(x_j')}{D\phi_j(x_j)} &\le 1+\frac{|D^2\phi_j|_0}{D\phi_j(x_j)}\cdot|x_j'-x_j|\\ &= 1+O(\phi_j)\cdot|x_j'-x_j|\end{aligned}$$

To continue, we have to estimate $|x_j'-x_j|$. Apply Lemma 10.2 from [M] to get

$$\begin{aligned}|x_j'-x_j| &= O\left(|x_{t+1}'-x_{t+1}|\right)\\ &= O(|\phi|).\end{aligned}$$

Because $\sum|\phi_j|+|\phi|\le K$ there exists $K_1>0$ such that

$$\begin{aligned}\frac{D\psi_2(x)}{D\psi_1(x)} &\le e^{|\phi|}\prod_{j\ge t+1}\left(1+O(|\phi_j|\;|\phi|)\right)\\ &\le e^{|\phi|}\;e^{K_1\cdot\sum|\phi_j|\;|\phi|}\end{aligned}$$

Hence,

$$\frac{D\psi_2}{D\psi_1}\le e^{|\phi|(1+K_1\cdot K)}.$$

We get a lower bound in similar way. So there exists $K_2>0$ such that

$$e^{-K_2\cdot|\phi|}\le\frac{|D\psi_2|}{|D\psi_1|}\le e^{K_2\cdot|\phi|}.$$

Finally, there exists $B>0$ such that

$$|D\psi_2(x)-D\psi_1(x)|\le B\;|\phi|.$$

$\square$

Let f be an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map.

Lemma 9.2. *There exists $K>0$ such that for all $n\ge1$ the following holds*

$$\sum_{1\,\le\,j\,\le 2^n-1}|q_j^n|\le K.$$

Proof. The non-linearity norm of q_j^n, $j=1,\dots,2^n-1$, is

$$|q_j^n|=\frac{|I_j^n|}{dist\;(I_j^n,c)}.$$

Let

$$Q_n=\sum_{j=1}^{2^n-1}|q_j^n|.$$

Observe that there exists $\tau > 0$ such that for $j = 1, 2, \ldots, 2^n - 1$

$$\begin{aligned} |q_j^{n+1}| + |q_{j+2^n}^{n+1}| &\leq \frac{|I_j^{n+1}| + |I_{j+2^n}^{n+1}|}{dist\ (I_j^n, c)} \\ &= |q_j^n| \, \frac{|I_j^{n+1}| + |I_{j+2^n}^{n+1}|}{|I_j^n|} \\ &= |q_j^n| \, \frac{|I_j^n - G_j^n|}{|I_j^n|} \leq |q_j^n|(1-\tau). \end{aligned}$$

Therefore

$$Q_{n+1} \leq (1-\tau)\ Q_n + |q_{2^n}^{n+1}|.$$

From the a priori bounds we get a constant $K_1 > 0$ such that

$$|q_{2^n}^{n+1}| \leq \frac{|I_{2^n}^{n+1}|}{|G_{2^n}^n|} \leq K_1.$$

Thus

$$Q_{n+1} \leq (1-\tau)Q_n + K_1.$$

This implies the Lemma. $\square$

Consider the map $f : I_0^n \to I_1^n$, and rescaled affinely range and domain to obtain the unimodal map

$$\hat{f}_n : [0,1] \to [0,1].$$

Apply Proposition 5.6 to obtain the following representation of $\hat{f}_n$. There exists $c_n \in (0,1)$ and diffeomorphisms $\phi_\pm^n : [0,1] \to [0,1]$ such that

$$\hat{f}_n(x) = \phi_+^n \circ q_{c_n}(x), \qquad x \in [c_n, 1]$$

and

$$\hat{f}_n(x) = \phi_-^n \circ q_{c_n}(x), \qquad x \in [0, c_n].$$

Furthermore

$$|\phi_\pm^n| \to 0$$

when $n \to \infty$. Let $q_0^n = q_{c_n}$. Use Proposition 8.1 to obtain the following representation for the n^{th} renormalization of f.

$$R^n f = (\phi_{2^n-1}^n \circ q_{2^n-1}^n) \circ \cdots \circ (\phi_j^n \circ q_j^n) \circ \cdots \circ (\phi_1^n \circ q_1^n) \circ \phi_\pm^n \circ q_0^n.$$

Inspired by [AMM] we introduce the unimodal map

$$f_n = q_{2^n-1}^n \circ \cdots \circ q_j^n \circ \cdots \circ q_1^n \circ q_0^n.$$

Proposition 9.3. *If f is an infinitely renormalizable $C^{2+|\cdot|}$ map then*

$$\lim_{n\to\infty} |R^n f - f_n|_1 = 0.$$

Proof. Define the diffeomorphisms

$$\psi_j^\pm = q_{2^n-1}^n \circ \cdots \circ q_j^n \circ (\phi_{j-1}^n \circ q_{j-1}^n) \circ \cdots \circ (\phi_1^n \circ q_1^n) \circ \phi_\pm^n$$

with $j = 0, 1, 2, \ldots 2^n$. Notice that

$$R^n f(x) = \psi_{2^n}^\pm \circ q_0^n(x)$$

and that

$$f_n(x) = \psi_0^\pm \circ q_0^n(x).$$

where we use again the $\pm$ distinction for points $x \in [0, c_n]$ and $x \in [c_n, 1]$. Apply the Sandwich Lemma 9.1 to get a constant $B > 0$ such that

$$|\psi_{j+1}^\pm - \psi_j^\pm|_1 \leq B \cdot |\phi_j^n|$$

for $j \geq 1$, and also notice that

$$|\psi_1^\pm - \psi_0^\pm|_1 \leq B \cdot |\phi_\pm^n| \longrightarrow 0.$$

We can now apply Proposition 8.1 to get

$$\lim_{n\to\infty} |\psi_{2^n}^\pm - \psi_0^\pm|_1 \leq \lim_{n\to\infty} B \cdot \sum_{1\,\leq\, j\, \leq 2^n - 1} |\phi_j^n| + |\phi_\pm^n| = 0,$$

which implies that:

$$\lim_{n\to\infty} |R^n f - f_n|_1 = 0.$$

□

10. Convergence

Fix an infinitely renormalizable $C^{2+|\cdot|}$ map f.

Lemma 10.1. *For every $N_0 \geq 1$, there exists $n_1 \geq 1$ such that f_n is N_0 times renormalizable whenever $n \geq n_1$.*

Proof. The a priori bounds from Proposition 7.5 gives $d > 0$ such that for $n \geq 1$

$$|(R^n f)^i(c) - (R^n f)^j(c)| \geq d$$

for all $i, j \leq 2^{N_0+1}$ and $i \neq j$. Now by taking n large enough and using Proposition 9.3 we find

$$|f_n^i(c) - f_n^j(c)| \geq \frac{1}{2} d$$

for $i \neq j$ and $i, j \leq 2^{N_0+1}$. The *kneading sequence* of f_n (i.e., the sequence of signs of the derivatives of that function) coincides with the kneading sequence of $R^n f$ for at least 2^{N_0+1} positions. We proved that f_n is N_0 times renormalizable because $R^n f$ is N_0 times renormalizable. □

The polynomial unimodal maps f_n are in a compact family of quadratic like maps. This follows from Lemma 9.2. The unimodal renormalization theory presented in [Ly] gives us the following.

Proposition 10.2. *There exists $N_0 \geq 1$ and $n_0 \geq 1$ such that f_n is N_0 renormalizable and*

$$dist_1\ (R^{N_0} f_n,\ W^u) \leq \frac{1}{3} \cdot dist_1\ (f_n,\ W^u).$$

Here, W^u is the unstable manifold of the renormalization fixed point contained in the space of quadratic like maps. Recall that $dist_1$ stands for the C^1 distance.

Lemma 10.3. *There exists $K > 0$ such that for $n \geq 1$*

$$dist_1\ (R^n f,\ W^u) \leq K.$$

Proof. This follows from Lemma 9.2 and Proposition 9.3. □

Let $f_*^\omega \in W^u$ be the analytic renormalization fixed point.

Theorem 10.4. *If f is an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map. Then*

$$\lim_{n\to\infty} dist_0\left(R^n f,\ f_*^\omega\right) = 0.$$

Proof. For every $K > 0$, there exists $A > 0$ such that the following holds. Let f, g be renormalizable unimodal maps with

$$|Df|_0,\ |Dg|_0 \le K$$

then

$$dist_0(Rf,\ Rg)\ \le\ A \cdot dist_0(f, g). \tag{17}$$

Let $N_0 \ge 1$ be as in Proposition 10.2. Now

$$\begin{aligned} dist_0(R^{n+N_0} f, W^u) &\le dist_0\left(R^{N_0}(R^n f),\ R^{N_0} f_n\right) + dist_0\left(R^{N_0} f_n,\ W^u\right) \\ &\le A^{N_0} \cdot dist_0\left(R^n f,\ f_n\right) + \frac{1}{3}\, dist_0\left(f_n,\ W^u\right) \end{aligned}$$

Notice,

$$dist_0(f_n,\ W^u) \le dist_0(f_n,\ R^n f) + dist_0(R^n f,\ W^u).$$

Thus there exists $K > 0$,

$$dist_0(R^{n+N_0} f,\ W^u) \le \frac{1}{3}\, dist_0(R^n f,\ W^u) + K \cdot dist_0(R^n f,\ f_n).$$

Let

$$z_n = dist_0(R^{n\cdot N_0} f,\ W^u)$$

and

$$\delta_n = dist_0(R^n f,\ f_n).$$

Then

$$z_{n+1} \le \frac{1}{3} z_n + K \cdot \delta_{n\cdot N_0}.$$

This implies

$$z_n \le \sum_{j<n} K \cdot \delta_{j\cdot N_0} \cdot (\frac{1}{3})^{n-j}.$$

Now we use that $\delta_n \to 0$, see Proposition 9.3, to get $z_n \to 0$. So we proved that $R^{n\cdot N_0} f$ converges to W^u. Use (17) and $R(W^u) \subset W^u$ to get that $R^n f$ converges to W^u in C^0 sense. Notice that any limit of $R^n f$ is infinitely renormalizable. The only infinitely renormalizable map in W^u is the fixed point f_*^ω. Thus

$$\lim_{n\to\infty} dist_0\left(R^n f,\ f_*^\omega\right) = 0.$$

□

11. Slow convergence

Theorem 11.1. *Let $d_n > 0$ be any sequence with $d_n \to 0$. There exists an infinitely renormalizable C^2 map f with quadratic tip such that*

$$dist_0\left(R^n f, f_*^\omega\right) \geq d_n.$$

The proof needs some preparation. Use the representation

$$f_*^\omega = \phi \circ q_c$$

where ϕ is an analytic diffeomorphism. The renormalization domains are denoted by I_0^n with

$$c = \cap_{n\geq 1} I_0^n.$$

Each I_0^n contains two intervals of the $(n+1)^{th}$ generation. Namely I_0^{n+1} and $I_{2^n}^{n+1}$. Let

$$G_n = I_0^n \setminus \left(I_0^{n+1} \cup I_{2^n}^{n+1}\right),$$

$$\hat{G}_n = q_c(G_n) \subset \hat{I}_0^n = q_c(I_0^n)$$

and $\hat{I}_{2^n}^{n+1} = q_c(I_{2^n}^{n+1})$. The invariant Cantor set of f_*^ω is denoted by Λ. Notice,

$$q_c(\Lambda) \cap \hat{I}_0^n \subset \left(\hat{I}_0^{n+1} \cup \hat{I}_{2^n}^{n+1}\right).$$

The gap $\hat{G}_n$ in $\hat{I}_0^n$ does not intersect with Λ. Choose a family of C^2 diffeomorphisms

$$\phi_t : [0,1] \to [0,1]$$

with

(i) $D\phi_t(0) = D\phi_t(1) = 1$.
(ii) $D^2\phi_t(0) = D^2\phi(1) = 0$.
(iii) For some $C_1 > 0$

$$dist_0\ (\phi_t, id) \geq C_1 \cdot t.$$

(iv) For some $C_2 > 0$

$$|\eta_{\phi_t}|_0 \leq C_2 \cdot t.$$

Let $m = min\ D\phi$ and $t_n = \frac{1}{m\ C_1\ |\hat{G}_1|} d_n$. Now we will introduce a perturbation $\tilde{\phi}$ of ϕ. Let

$$1_n : [0,1] \to \hat{G}_n$$

be the affine orientation preserving homeomorphism. Define

$$\psi : [0,1] \to [0,1]$$

as follows

$$\psi(x) = \begin{cases} x & x \notin \cup_{n\geq 0} \hat{G}_n \\ 1_n \circ \phi_{t_n} \circ 1_n^{-1}(x) & x \in \hat{G}_n. \end{cases}$$

Let

$$f = \phi \circ \psi \circ q_c = \tilde{\phi} \circ q_c.$$

Then f is unimodal map with quadratic tip which is infinitely renormalizable and still has Λ as its invariant Cantor set. This follows from the fact that the perturbation did not affect the critical orbit and it is located in the complement of the Cantor set. In particular the invariant Cantor set of $R^n f$ is again $\Lambda \subset I_0^1 \cup I_1^1$ and G_1 is the gap of $R^n f$. Notice, by using that f_*^ω is the fixed point of renormalization that for $x \in G_1$

$$R^n f(x) = \phi \circ 1_1 \circ \phi_{t_n} \circ 1_1^{-1} \circ q_c(x)$$

Hence,

$$\begin{array}{rcl} |R^n f - f_*^\omega|_0 & \geq & \max_{x \in \hat{G}_1} |R^n f(x) - f_*^\omega(x)| \\ & \geq & \max_{x \in \hat{G}_1} m \cdot |\left(1_1 \circ \phi_{t_n} \circ 1_1^{-1}\right) q_c(x) - q_c(x)| \\ & \geq & m \cdot \max_{x \in \hat{G}_1} |\left(1_1 \circ \phi_{t_n} \circ 1_1^{-1}\right)(x) - x| \\ & = & m \cdot |\hat{G}_1| \cdot |\phi_{t_n} - id|_0 \\ & \geq & m \cdot |\hat{G}_1| \cdot C_1 \cdot t_n = d_n. \end{array}$$

It remains to prove that f is C^2. The map f is C^2 on $[0,1] \setminus \{c\}$ because $f = \tilde{\phi} \circ q_c$ with $\tilde{\phi} = \phi \circ \psi$. Where ϕ is analytic diffeomorphism and ψ is by construction C^2 on $[0,1)$. Notice that, from (10) we have,

$$\begin{array}{rcl} D^2 f(x) & = & 4 \cdot \dfrac{(x-c)^2}{(1-c)^4} \cdot D^2\tilde{\phi}\left(q_c(x)\right) \\ & - & 2 \cdot \dfrac{1}{(1-c)^2} \cdot D\tilde{\phi}\left(q_c(x)\right). \end{array} \tag{18}$$

We will analyze the above two terms separately. Observe

$$D\psi(x) = \begin{cases} 1, & x \notin \cup_{n \geq 0} \hat{G}_n \\ |D\phi_{t_n}\left(1_n^{-1}(x)\right)|, & x \in \hat{G}_n. \end{cases}$$

This implies for $x \in G_n$

$$\begin{array}{rcl} D\tilde{\phi}\left(q_c(x)\right) & = & D\phi\left(\psi \circ q_c\right) \cdot D\psi(q_c(x)) \\ & = & D\phi(1) \cdot \left(1 + O(\hat{I}_0^n)\right) \cdot \left(1 + O(t_n)\right) \end{array}$$

For $x \notin \cup_{n \geq 1} G_n$ we have

$$D\tilde{\phi}(q_c(x)) = D\phi(q_c(x))$$

This implies that the term

$$x \longmapsto -2 \cdot \frac{1}{(1-c)^2} \cdot D\tilde{\phi}(q_c(x))$$

extends continuously to the whole domain. The first term in (18) needs more care. Observe, for $u \in \hat{G}_n$,

$$\begin{aligned}
D^2\tilde{\phi}(u) &= D^2\phi(\psi(u)) \cdot (D\psi(u))^2 + D\phi(\psi(u)) \cdot D^2\psi(u) \\
&= D^2\phi(1) \cdot \left(1 + O(\hat{I}_0^n)\right) \cdot (1 + O(t_n)) + \\
&\quad D\phi(1) \cdot \left(1 + O(\hat{I}_0^n)\right) \cdot (1 + O(t_n)) \cdot D^2\psi(u) \\
&= D^2\phi(1) \cdot \left(1 + O(\hat{I}_0^n)\right) \cdot (1 + O(t_n)) + \\
&\quad D\phi(1) \cdot \left(1 + O(\hat{I}_0^n)\right) \cdot (1 + O(t_n)) \cdot \frac{1}{|\hat{G}_n|} \cdot O(t_n).
\end{aligned}$$

This implies that

$$4\,\frac{(x-c)^2}{(1-c)^4} \cdot D^2\tilde{\phi}(q_c(x)) = \begin{cases} O\left((x-c)^2\right) + O(t_n), & x \in \hat{G}_n \\ O\left((x-c)^2\right), & x \notin \cup_{n\geq 0}\hat{G}_n \end{cases}$$

In particular, the first term of $D^2 f$

$$x \longmapsto 4\,\frac{(x-c)^2}{(1-c)^4} \cdot D^2\tilde{\phi}(q_c(x))$$

also extends to a continuous function on $[0,1]$. Indeed, f is C^2.

Remark 11.2. *If the sequence d_n is not summable (and in particular not exponential decaying) then the example constructed above is not $C^{2+|\cdot|}$. This follows from*

$$\int_{\hat{G}_n} |\eta_{\tilde{\phi}}(x)|dx \asymp t_n.$$

Thus

$$\int |\eta_{\tilde{\phi}}| \asymp \sum d_n = \infty.$$

Now, equation 12 implies that f is not $C^{2+|\cdot|}$. However, this construction show that in the space of $C^{2+|\cdot|}$ unimodal maps there are examples whose renormalizations converges only polynomially. The renormalization fixed point is not hyperbolic in the space of $C^{2+|\cdot|}$ unimodal maps.

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