A PRESENTATION FOR THE BASELEAF PRESERVING MAPPING CLASS GROUP OF THE PUNCTURED SOLENOID

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ABSTRACT. We give a presentation for the baseleaf preserving mapping class group $Mod(\mathcal{H})$ of the punctured solenoid $\mathcal{H}$. The generators for our presentation were introduced previously, and several relations among them were derived. In addition, we show that $Mod(\mathcal{H})$ has no non-trivial central elements. Our main tool is a new complex of triangulations of the disk upon which $Mod(\mathcal{H})$ acts.

1. INTRODUCTION

This note continues the investigation (begun in [16]) of the baseleaf preserving mapping class group $Mod(\mathcal{H})$ for the punctured solenoid $\mathcal{H}$. Our main result is a presentation for $Mod(\mathcal{H})$. The punctured solenoid $\mathcal{H}$ is an inverse limit of the system of all finite unbranched covers of a punctured surface of negative Euler characteristic, and its baseleaf preserving mapping class group $Mod(\mathcal{H})$ consists of all homotopy classes of appropriate self maps of $\mathcal{H}$ which preserve a distinguished leaf. (See Section 2 or [16] for more details.) One motivation for studying $Mod(\mathcal{H})$ comes from Sullivan's observation [11] that the Ehrenpreis conjecture is equivalent to the statement that $Mod(\mathcal{H})$ has dense orbits in the Teichmüller space of the solenoid $\mathcal{H}$. Another motivation is that the baseleaf preserving mapping class group $Mod(\mathcal{H})$ is a large subgroup of the studied commensurator group $Comm(F_2)$ of the free group F_2 on two generators (for the definition, see Section 2). Namely, if we identify F_2 with the once punctured torus group G, then $Mod(\mathcal{H})$ is the subgroup of $Comm(G)$ which preserves peripheral elements, i.e., preserves parabolic elements.

The path components of the punctured solenoid $\mathcal{H}$ are called *leaves* and the *baseleaf* is a fixed distinguished leaf. Since $Mod(\mathcal{H})$ preserves the baseleaf, which is dense in $\mathcal{H}$, it is enough to analyse the action on the baseleaf. We recall that the baseleaf (and indeed any other leaf) is conformally equivalent to the unit disk **D**.

Given an ideal triangulation of the unit disk (i.e., the baseleaf) which is invariant under a finite-index subgroup K of $PSL_2(\mathbb{Z})$ and a specified edge of the triangulation, there are two adjacent triangles which together form a "neighboring" quadrilateral. We may replace the specified edge of this quadrilateral by its other diagonal, and performing this modification for each edge in the K-orbit of the specified edge, we define the K-equivariant Whitehead move. The resulting ideal triangulation is also invariant under K. A Whitehead homeomorphism of S^1 is obtained by mapping an ideal triangulation of the unit disk onto its image under a Whitehead move. It is shown in [16] that the Whitehead homeomorphisms together with $PSL_2(\mathbb{Z})$ generate the baseleaf preserving mapping class group $Mod(\mathcal{H})$ (see Section 2 or [16] for more details). In fact, $Mod(\mathcal{H})$ consists of quasisymmetric homeomorphisms of S^1 which conjugate one finite-index subgroup of $PSL_2(\mathbb{Z})$ onto another [16] or [13]. In [16], four relations among Whitehead homeomorphisms are identified, and three of them arise in our presentation (see Theorem 4.4(c).)

We first introduce the *triangulation complex* $\mathcal{X}$ for the punctured solenoid $\mathcal{H}$. The vertices of $\mathcal{X}$ are TLC tesselations, i.e., ideal triangulations of **D** invariant under some finite-index subgroup of $PSL_2(\mathbb{Z})$. Two vertices of $\mathcal{X}$ are joined by an edge if they differ by a Whitehead move.

There are several types of two-cells in $\mathcal{X}$: two edges of a triangulation may have disjoint neighboring quadrilaterals, in which case there is a two-cell corresponding to commutativity of their associated Whitehead moves; the two edges may have neighboring quadrilaterals which share a triangle, in which case there is a two-cell corresponding to the pentagon relation; or a single Whitehead move equivariant for a finite-index

subgroup $K < PSL_2(\mathbb{Z})$ may be written as the finite composition of Whitehead moves equivariant for a subgroup of K of finite index. (The two-cells are described more precisely in Section 3).

Theorem 3.1. *The triangulation complex $\mathcal{X}$ is connected and simply connected.*

The action of $Mod(\mathcal{H})$ on the triangulation complex $\mathcal{X}$ is evidently cellular. Furthermore (see [16]), there is only one orbit of vertices in $\mathcal{X}$, and the isotropy group of a vertex v, i.e., its stabilizer $\Gamma(v)$, is a conjugate of $PSL_2(\mathbb{Z})$. Together with the further analysis of the isotropy group $\Gamma(E)$ of an unoriented edge E, standard techniques [2] allow us to derive a presentation of $Mod(\mathcal{H})$. To simplify this presentation, we actually choose a larger set of generators for $Mod(\mathcal{H})$, namely, we take as generators all Whitehead moves starting from the basepoint of $\mathcal{X}$. (This is a smaller set of generators than in [16] but larger than necessary.) We denote by $\mathcal{E}^+$ the set of edges of $\mathcal{X}$ which contain the basepoint and are not inverted by an element of $Mod(\mathcal{H})$, and by $\mathcal{E}^-$ the set of edges which contain the basepoint and are inverted by an element of $Mod(\mathcal{H})$. It is necessary to fix one Whitehead homeomorphism g_E for each edge $E \in \mathcal{E}^+$ in a consistent way. (See Section 4 regarding this choice.) Let $\mathcal{E}^\pm = \mathcal{E}^+ \sqcup \mathcal{E}^-$ denote the set of unoriented edges, and let $\Gamma^+(E)$ denote the subgroup of $\Gamma(E)$ which does not invert the edge $E \in \mathcal{E}^-$.

Theorem 4.3. *The modular group $Mod(\mathcal{H})$ is generated by the isotropy subgroup $PSL_2(\mathbb{Z})$ of the basepoint $\tau_* \in \mathcal{X}$, the isotropy subgroups $\Gamma(E)$ for $E \in \mathcal{E}^\pm$, and by the elements g_E for $E \in \mathcal{E}^+$. The following relations on these generators give a complete presentation of $Mod(\mathcal{H})$:*

a) The inclusions of $\Gamma(E)$ into $PSL_2(\mathbb{Z})$, for $E \in \mathcal{E}^+$, are given by $\Gamma(E) = K'$, where the terminal endpoint of E is invariant under the finite-index subgroup $K' < PSL_2(\mathbb{Z})$;

b) The inclusions of $\Gamma^+(E)$ into $PSL_2(\mathbb{Z})$, for $E \in \mathcal{E}^-$, are given by $\Gamma(E) = K'$, where the terminal endpoint of E is invariant under the finite-index subgroup $K' < PSL_2(\mathbb{Z})$;

c) The relations introduced by the boundary edge-paths of two-cells in $\mathcal{F}$ given by the equations (1), (2), (3), (4), (5), (6) and (7);

d) The redundancy relations: for any two edges E and E' in $\mathcal{E}^\pm$ and for any $\gamma \in PSL_2(\mathbb{Z})$ such that $\gamma(E) = E'$, we get the relation

$$g_{E'} \circ \gamma' = \gamma \circ g_E,$$

where γ' is the unique element of $PSL_2(\mathbb{Z})$ that satisfies $\gamma'(e_0) = e'_1$ with $e'_1 = g_{E'}^{-1}(\gamma(e_0))$.

It is well-known that the mapping class group of a Riemann surface of finite type has trivial center provided the genus is at least three, and we obtain the analogous result for $Mod(\mathcal{H})$.

Theorem 5.1. *The modular group $Mod(\mathcal{H})$ of the punctured solenoid $\mathcal{H}$ has trivial center.*

Define $\mathcal{Y} = \mathcal{X}/Mod(\mathcal{H})$ and let $\mathcal{N}$ be the subgroup of $Mod(\mathcal{H})$ generated by all elements which fix a point in $\mathcal{X}$. By a standard result [1], we get

Theorem 5.4. *The topological fundamental group of $\mathcal{Y} = \mathcal{X}/Mod(\mathcal{H})$ is given by*

$$\pi_1(\mathcal{Y}) = Mod(\mathcal{H})/\mathcal{N}.$$

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2. Preliminaries

Fix a punctured surface S (*the base surface*) with negative Euler characteristic and empty boundary, and consider the system of all finite unbranched covers of S. There is a partial ordering on the covers as follows. If one cover π_1 can be factored as the composition of two covers $\pi_1 = \pi \circ \pi_2$, where π, π_2 are also finite unbranched covers, then $\pi_1 \geq \pi_2$. The system of covers is inverse directed, and there is thus an inverse limit.

Definition 2.1. The *punctured solenoid* $\mathcal{H}$ is the inverse limit of the system of finite unbranched covers of a punctured surface without boundary and with negative Euler characteristic.

The inverse limit does not depend on the base surface as long as it is of negative Euler characteristic [13], [16]. The punctured solenoid $\mathcal{H}$ is locally homeomorphic to a disk times a Cantor set. Each path component

is called a *leaf*, and each leaf is homeomorphic to the unit disk. The punctured solenoid $\mathcal{H}$ has uncountably many leaves, each of which is dense in $\mathcal{H}$. If we require in the above definition of $\mathcal{H}$ that each punctured surfaces and each covering map is pointed, we obtain a distinguished point, called the *basepoint* of $\mathcal{H}$. The leaf containing the basepoint is called the *baseleaf*. The punctured solenoid $\mathcal{H}$ is a non-compact topological space with one end, which is homeomorphic to a horoball times a Cantor set modulo the continuous action of a countable group. For more details, see [16].

Definition 2.2. The *baseleaf preserving mapping class group* $Mod(\mathcal{H})$ of the punctured solenoid $\mathcal{H}$ is the group of isotopy classes all self homeomorphisms of $\mathcal{H}$ which preserve the baseleaf and which are quasiconformal on leaves.

The restriction of an element of $Mod(\mathcal{H})$ to the baseleaf gives a quasiconformal homeomorphism of the unit disk $\mathbf{D}$ (upon fixing an identification of the baseleaf with $\mathbf{D}$) up to isotopy. Thus, an element of $Mod(\mathcal{H})$ determines a well-defined quasisymmetric homeomorphism of S^1, and we shall thus identify $Mod(\mathcal{H})$ with a appropriate group of quasisymmetric maps. (See Theorem 2.8.)

Definition 2.3. The *commensurator group* $Comm(G)$ of a group G consists of equivalence classes of isomorphisms of finite-index subgroups of G, where two isomorphisms are equivalent if they agree on a finite-index subgroup in the intersection of their domains.

Theorem 2.4. [16] *The modular group $Mod(\mathcal{H})$ is isomorphic to a proper subgroup of the commensurator group $Comm(F_2)$ of the free group F_2 on two generators. Namely, $Mod(\mathcal{H})$ is isomorphic to the subgroup of $Comm(F_2)$ consisting of all elements which preserve the peripheral elements under some fixed identification $F_2 \equiv G$, where $G < PSL_2(\mathbb{Z})$ is the group uniformizing the once-punctured torus.*

In fact, it is convenient in the definition of the punctured solenoid $\mathcal{H}$ to fix the base surface to be the once-punctured torus $\mathbf{D}/G$, where $G < PSL_2(\mathbb{Z})$. Given an isomorphism of two finite-index subgroups K, H of G which preserves peripheral elements, there exists a unique quasisymmetric map of S^1 which conjugates K onto H. Thus, by the previous theorem, we may consider $Mod(\mathcal{H})$ as a group of quasisymmetric maps of S^1 which conjugate one finite-index subgroup of $PSL_2(\mathbb{Z})$ onto another.

We recall that the decorated Teichmüller space $\tilde{T}(\mathcal{H})$ of the punctured solenoid $\mathcal{H}$ is partitioned into sets according to the bending information of the convex hull construction [16]. When the bending locus is a triangulation on each leaf, then it is locally constant in the transverse direction, and the action of the baseleaf preserving mapping class group $Mod(\mathcal{H})$ is transitive on this subspace of $\tilde{T}(\mathcal{H})$. It is convenient to consider the action of $Mod(\mathcal{H})$ on the ideal triangulations of the baseleaf (i.e., the unit disk $\mathbf{D}$) arising by restrictions (to the baseleaf) of the triangulations of $\mathcal{H}$.

Definition 2.5. A *transversely locally constant (TLC) tesselation* τ of the unit disk $\mathbf{D}$ is a lift to $\mathbf{D}$ of an ideal triangulation of some punctured surface $\mathbf{D}/K$ of finite type, i.e., an ideal triangulation of $\mathbf{D}$ invariant under a finite-index subgroup K of $PSL_2(\mathbb{Z})$, where the ideal points of the tesselation agree with $\bar{\mathbb{Q}} \subset S^1$.

A particularly important example of a TLC tesselation is the Farey tesselation τ_* (see, for example, [14] or [16]), which is invariant under the group $PSL_2(\mathbb{Z})$. Let K be a finite-index subgroup of $PSL_2(\mathbb{Z})$ and let τ be a K-invariant TLC tesselation of $\mathbf{D}$. A *characteristic map* for τ is a homeomorphism $h : S^1 \to S^1$ such that $h(\tau_*) = \tau$ (see [14]).

The map $h = h(\tau, e)$ is completely determined by specifying an oriented edge $e \in \tau$, namely, the standard oriented edge $e_0 = (-1, 1)$ in τ_* is mapped onto e, the triangle to the left or right of e_0 in τ_* is mapped to the triangle to the left or right, respectively, of e in τ, and so on. Note that any two characteristic maps for τ differ by pre-composition with an element of $PSL_2(\mathbb{Z})$.

Theorem 2.6. [16] *The characteristic map $h = h(\tau, e)$ for a K-invariant TLC tesselation τ conjugates a finite-index subgroup H of $PSL_2(\mathbb{Z})$ onto K.*

Note that a characteristic map necessarily conjugates peripherals into peripherals because it is a homeomorphisms of S^1, and so a characteristic map for a TLC tesselation lies in $Mod(\mathcal{H})$.

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Let τ be a K-invariant TLC tesselation of $\mathbf{D}$ with a distinguished oriented edge e; we allow for the possibility that τ is invariant under a larger subgroup of $PSL_2(\mathbb{Z})$. Fix an edge $f \in \tau$. We form a new K-invariant TLC tesselation τ' by replacing each $\gamma(f)$, for $\gamma \in K$, by $\gamma(f')$, where f' is the diagonal of the unique ideal quadrilateral in $(\mathbf{D}-\tau) \cup \{f\}$ different from f. We say that τ' is obtained from τ by performing a *Whitehead move* along $K\{f\}$. If $e \notin K\{f\}$ then we let e be the distinguished oriented edge in τ' as well; if $e = \gamma(f)$ for some $\gamma \in K$, then we let $e' = \gamma(f')$ be the distinguished oriented edge for τ', where e' is given the orientation such that the tangent vectors to e and e' at their intersection point comprise a positively oriented basis for the oriented disk $\mathbf{D}$.

Definition 2.7. Let τ be a K-invariant TLC tesselation of $\mathbf{D}$ with a distinguished oriented edge e. The *Whitehead homeomorphism* for τ and e is

$$k(\tau,e) = h(\tau',e') \circ h(\tau,e)^{-1},$$

where (τ',e') arises from (τ,e) under the Whitehead move and $h(\tau',e'), h(\tau,e)$ are the characteristic maps.

A Whitehead homeomorphism lies in $Mod(\mathcal{H})$ since it is the composition of two elements of the group $Mod(\mathcal{H})$ by Theorem 2.6.

Theorem 2.8. [16] *The modular group $Mod(\mathcal{H})$ of the punctured solenoid $\mathcal{H}$ is generated by $PSL_2(\mathbb{Z})$ and by Whitehead homeomorphisms for all TLC tesselations. In addition, $Mod(\mathcal{H})$ acts transitively on the set of all TLC tesselations of the unit disk $\mathbf{D}$.*

3. The triangulation complex

We introduce a two-complex $\mathcal{X}$ associated to TLC tesselations of $\mathbf{D}$. This complex $\mathcal{X}$ is an adaptation to our situation of the two-skeleton of the complex dual to the cell decomposition of the decorated Teichmüller space introduced by Penner [15] and Harer [7]. On the other hand, $\mathcal{X}$ is analogous to the complex of cut systems of Hatcher and Thurston [6] in that $Mod(\mathcal{H})$ acts transitively on its vertices.

We begin the definition of $\mathcal{X}$ by giving its vertices. A *vertex* of the *triangulation complex* $\mathcal{X}$ is a TLC tesselations of the unit disk $\mathbf{D}$. The *basepoint* of $\mathcal{X}$ is Farey tesselation τ_*. A characteristic map between any two TLC tesselations is an element of $Mod(\mathcal{H})$ [16, Lemma 7.5], and so $Mod(\mathcal{H})$ acts transitively on vertices of $\mathcal{X}$.

We next introduce edges of $\mathcal{X}$ with one endpoint at the basepoint τ_*. An unordered pair of vertices $\{\tau, \tau_*\}$ determines an edge in $\mathcal{X}$ if τ can be obtained from τ_* by a single Whitehead move, i.e., τ is obtained from τ_* by replacing an orbit $K\{f\}$ of an edge f in τ_* by the orbit $K\{f'\}$, where K is torsion free and of finite-index in $PSL_2(\mathbb{Z})$, and f' is the diagonal of the quadrilateral in $(\mathbf{D}-\tau_*) \cup \{f\}$ different from f.

More generally and by definition, an unordered pair of vertices $\{\tau_1, \tau_2\}$ determines an edge of $\mathcal{X}$ if $\{\tau_1, \tau_2\}$ is the image by an element of $Mod(\mathcal{H})$ of an edge $\{\tau_*, \tau\}$ defined above. In particular, this implies that if a TLC tesselation τ_2 is obtained by performing a K-invariant Whitehead move on a TLC tesselation τ_1 then $\{\tau_1, \tau_2\}$ is an edge in $\mathcal{X}$. To see this, take a characteristic map h for τ_1 (i.e., h is a homeomorphism of S^1 such that $h(\tau_*) = \tau_1$) and consider $\{\tau_*, h^{-1}(\tau_2)\}$. Thus, τ_1 and τ_2 differ only in the K orbit of the diagonals of an ideal rectangle. The characteristic map h conjugates a finite index subgroup H of $PSL_2(\mathbb{Z})$ onto K [16, Lemma 7.5], so $h^{-1}(\tau_1) = \tau_*$ and $h^{-1}(\tau_2)$ differ only in that they have different diagonals on an H-orbit of a rectangle. It follows that $\{\tau_*, h^{-1}(\tau_2)\}$ is an edge corresponding to an H-invariant Whitehead move, i.e., $\{\tau_1, \tau_2\} = h(\{\tau_*, h^{-1}(\tau_2)\})$ is an edge in $\mathcal{X}$.

However, there are edges which appear away from the basepoint τ_* that do not correspond to Whitehead moves. They can be described as a *generalized* Whitehead move invariant under a conjugate by a characteristic map of a finite-index subgroup of $PSL_2(\mathbb{Z})$. Note that the set of edges in $\mathcal{X}$ is invariant under the action of $Mod(\mathcal{H})$ by construction. This completes the definition of the one-skeleton of $\mathcal{X}$.

We introduce two-cells of $\mathcal{X}$ by first defining those that have one vertex at the basepoint τ_*. Let K be a torsion free finite-index subgroup of $PSL_2(\mathbb{Z})$. There are three types of two-cells:

[Pentagon] Suppose that K is of index at least 9, i.e., τ_*/K is a triangulation of $\mathbf{D}/K$ which has at least three complementary ideal triangles. Any three adjacent complementary triangles form a pentagon

on $\mathbf{D}/K$ whose boundary sides are possibly identified in pairs. Let e_1 and e_2 be two representatives in τ_* of the diagonals of a pentagon on $\mathbf{D}/K$ which share an ideal point. The sequence of five Whitehead homeomorphisms $h(K, e_1)$, $h(K, e_2)$, $h(K, e_1')$, $h(K, e_2')$ and $h(K, e_1'')$ defines a closed edge-path in $\mathcal{X}$ based at τ_*, where e_1' is the new edge corresponding to e_1 under the Whitehead move for τ_* along $K\{e_1\}$, e_2' is the new edge corresponding to e_2 under the Whitehead move for $\tau_1 = h(K, e_1)(\tau_*)$ along $K\{e_2\}$, and e_1'' is the new edge corresponding to e_1' under the Whitehead move for $\tau_2 = h(K, e_2)(\tau_1)$ along $K\{e_1'\}$ (see [16] for more details). We add a two-cell in $\mathcal{X}$ whose boundary is this closed edge-path of length five starting and ending at the basepoint τ_* and call this two-cell a *pentagon* at the basepoint τ_*.

[**Square**] Let K be a torsion free finite-index subgroup of $PSL_2(\mathbb{Z})$ such that the triangulation τ_*/K of $\mathbf{D}/K$ has two edges which do not lie in the boundary of a common complementary triangle. Let e_1 and e_2 be two lifts to τ_* of the two non-adjacent edges. Consider the closed edge-path of length four given by Whitehead homeomorphisms $h(K, e_1)$, $h(K, e_2)$, $h(K, e_1')$ and $h(K, e_2')$, where e_1' corresponds to e_1 under the Whitehead move $h(K, e_1)$ and e_2' corresponds to e_2 under the Whitehead move $h(K, e_2)$. We add a two-cell to $\mathcal{X}$ with boundary equal to the above edge-path of length four and call it a *square* cell at the basepoint τ_*.

[**Coset**] Suppose $e \in \tau_*$ and let H be a finite-index subgroup of K. The orbit $K\{e\}$ is canonically decomposed into finitely many orbits $H\{e_1\}, H\{e_2\}, \ldots, H\{e_k\}$, where $e_1 = e, e_2, \ldots, e_k \in K\{e\}$ and $k = [K : H]$. Let f be the other diagonal in the unique ideal quadrilateral in $(\mathbf{D} - \tau_*) \cup \{e\}$ and let $f_1, f_2, \ldots, f_k \in K\{f\}$ be the altered edges corresponding to $e_1, e_2, \ldots, e_k$. Consider a finite edge-path based at τ_* consisting of the Whitehead homeomorphisms $h(K, e)$, $h(H, f_1)$, $h(H, f_2)$, $\ldots$, $h(H, f_k)$ corresponding to the tesselations τ_*, $\tau_1 = h(K, e)(\tau_*)$, $\tau_2 = h(H, e_1)(\tau_1)$, $\ldots$, $\tau_k = h(H, e_{k-1})(\tau_{k-1})$, $\tau_{k+1} = h(H, e_k) = \tau_*$. We add a two-cell to $\mathcal{X}$ whose boundary is this edge-path and call it the *coset* cell at the basepoint. Note that a different ordering of $f_1, \ldots, f_k$ gives a different edge-path and hence a different coset cell. In fact, there are $k!$ corresponding coset cells when $[K : H] = k$. The edge $\{\tau_*, \tau_1\}$ is called a *long edge*, and all other edges are called *short edges* corresponding to this coset cell.

Note that all two-cells introduced above have their boundaries given by compositions of Whitehead moves invariant under subgroups of $PSL_2(\mathbb{Z})$ as opposed to more general edges in $\mathcal{X}$ where moves are only conjugate to Whitehead moves invariant under subgroups of $PSL_2(\mathbb{Z})$.

To complete the definition of $\mathcal{X}$, an arbitrary two-cell in $\mathcal{X}$ is the image under $Mod(\mathcal{H})$ of a two-cell at the basepoint. If $h \in Mod(\mathcal{H})$ and P is a two-cell based at τ_*, then we say that $h(P)$ is based at $\tau := h(\tau_*)$. Note that closed edge-paths based at τ_* are mapped to closed edge-paths, and hence the boundaries of two-cells are well defined. The boundary of a pentagon or a square two-cell based at a tesselation $\tau \neq \tau_*$ each of whose edges is a Whitehead move invariant with respect to a fixed finite-index subgroup K of $PSL_2(\mathbb{Z})$ (or equivalently, whose vertices are TLC tesselations invariant under K) is likewise the boundary of a two-cell. Furthermore, the boundary of a coset two-cell starting at $\tau \neq \tau_*$ whose initial vertex is invariant under K and whose other vertices are invariant under a subgroup $K_1 < K$ of finite-index is the image of the boundary of a coset two-cell based at τ_* by simply noting that a characteristic map which sends τ_* onto τ conjugates $H_1 < H$ onto $K_1 < K$ where $H < PSL_2(\mathbb{Z})$.

By construction, the set of two-cells in $\mathcal{X}$ is invariant under $Mod(\mathcal{H})$, and $Mod(\mathcal{H})$ consequently acts cellularly on the two-complex $\mathcal{X}$.

We claim that a pentagon or a square two-cell P based at $\tau \neq \tau_*$ with one vertex at the basepoint τ_* has all edges given by Whitehead moves invariant under a fixed finite-index subgroup K of $PSL_2(\mathbb{Z})$. Since any characteristic map h of τ_* onto the base vertex τ of the two-cell P conjugates a finite-index subgroup $H < PSL_2(\mathbb{Z})$ onto a finite-index subgroup $K < PSL_2(\mathbb{Z})$ (under which τ is invariant), it follows that a two-cell (pentagon or square) P' invariant under H and based at τ_* is mapped by h onto the above two-cell P based at τ whose vertices are invariant under K. Moreover, a coset two-cell P which is based at $\tau \neq \tau_*$ whose initial vertex is τ_* does not necessarily have edges arising from Whitehead moves invariant under $K_1 < K$. Let P' be a coset cell based at τ_* such that $h(P') = P$. In fact, if the image under h of the long edge of P' is not incident on τ_*, then it is represented by a generalized Whitehead move (invariant under a conjugate by h of a subgroup of $PSL_2(\mathbb{Z})$ which is not itself a group of Möbius transformations).

Theorem 3.1. *The triangulation complex $\mathcal{X}$ is connected and simply connected.*

Proof. We first prove that $\mathcal{X}$ is connected by showing that any vertex τ can be connected to the basepoint τ_* by a finite edge-path. Let K be a finite-index subgroup of $PSL_2(\mathbb{Z})$ under which τ is invariant. Thus, τ/K and τ_*/K are two tesselations of a punctured surface $\mathbf{D}/K$. By results of Penner [15] (or Harer [7] or Hatcher [8]), there is a sequence of Whitehead moves on $\mathbf{D}/K$ which transforms τ_*/K into τ/K. The lifts of the Whitehead moves on $\mathbf{D}/K$ to $\mathbf{D}$ are TLC Whitehead moves on $\mathbf{D}$ and they provide an edge-path from τ_* to τ in $\mathcal{X}$. This establishes that $\mathcal{X}$ is connected.

It remains to show that $\mathcal{X}$ is simply connected. We recall a result of Harer [7, Theorem 1.3] or Penner [15] for triangulations of punctured surfaces. The set of top-dimensional simplices of the triangulation complex of a finite punctured surface consists of ideal triangulations, the codimension-one simplices are ideal triangulations with one ideal geodesic erased, the codimension-two simplices are ideal triangulations with two ideal geodesics erased, etc.. The main fact is that the triangulation complex of the finite surface minus simplices which are given by decompositions of the surface, where at least one complementary component is not topologically a disc is homeomorphic to the decorated Teichmüller space of the punctured surface. In particular, the triangulation complex for a punctured surface is contractible.

Consider a closed edge-path α in the triangulation complex $\mathcal{X}$ for the punctured solenoid. It is possible that an edge E in the path α is given by a generalized Whitehead move, i.e., the two tesselations at the endpoints of E are invariant under hKh^{-1}, where K is a finite-index subgroup of $PSL_2(\mathbb{Z})$ and $h \in Mod(\mathcal{H})$. Since $h \in Mod(\mathcal{H})$, there exists $H_1, H_2 < PSL_2(\mathbb{Z})$ of finite index such that $H_2 = hH_1h^{-1}$. Thus, $H_1 \cap K =: K_1$ is of finite-index in K, and we consider a coset two-cell corresponding to the groups $K_1 < K$ with long edge $h^{-1}(E)$. The edge $h^{-1}(E)$ is homotopic modulo its endpoints to the path of short edges in the coset two-cell, where each vertex is invariant under K_1. The image under h is a coset two-cell with long edge corresponding to hKh^{-1} and short edges corresponding to $hK_1h^{-1} < PSL_2(\mathbb{Z})$. Thus, we can replace the long edge invariant under hKh^{-1}, which is not a subgroup of $PSL_2(\mathbb{Z})$, by the homotopic edge-path invariant under $hK_1h^{-1} < H_2 < PSL_2(\mathbb{Z})$. We may therefore replace α by an edge-path α' each of whose edges corresponds to a Whitehead move invariant under a finite-index subgroup of $PSL_2(\mathbb{Z})$ using only coset two-cells.

Let $K_1, K_2, \ldots, K_n$ be finite-index subgroups of $PSL_2(\mathbb{Z})$ which correspond to invariant Whitehead moves defining the edges of α'. Using coset two-cells corresponding to each K_i, we first homotope the above edge-path α' into a closed edge-path α'' where each edge corresponds to an invariant Whitehead move with respect to a single finite-index subgroup $K := K_1 \cap K_2 \cap \ldots \cap K_n$. The new edge-path α'' invariant under K in the triangulation complex $\mathcal{X}$ of the punctured solenoid can be represented by a closed path γ in the above triangulation complex of a finite surface $\mathbf{D}/K$. (Recall that $\mathcal{X}$ is an extension of the dual of the triangulation complex of finite surface $\mathbf{D}/K$.) The path γ starts and ends in the top-dimensional simplex which corresponds to the triangulation of $\mathbf{D}/K$ obtained by projecting the TLC tesselation of $\mathbf{D}$ defining the basepoint of α'' onto $\mathbf{D}/K$. Furthermore, γ crosses transversely codimension-one simplices of the triangulation complex of $\mathbf{D}/K$ corresponding to each edge in α'', and it enters each top-dimensional simplex which corresponds to a vertex of α'' in the given order. Since the triangulation complex for punctured surface $\mathbf{D}/K$ is simply connected [7] or [15], there exists a homotopy of γ into the trivial path which transversely crosses codimension-two cells. The number of times the homotopy crosses codimension-two cells is finite, and it is possible to choose a homotopy which does not intersect simplices of codimension greater than two. For each intersection point of the homotopy with a codimension-two simplex, there is a corresponding two-cell in $\mathcal{X}$ because two-cells corresponding to ideal triangulations of the surface with two edges erased lift to ideal triangulations of $\mathbf{D}$ with orbits of two edges erased such that each complementary region is finite sided. This exactly correspond to two-cells (either pentagon or square) in $\mathcal{X}$. Thus, the homotopy for γ gives a homotopy between α'' and the trivial path in $\mathcal{X}$, and $\mathcal{X}$ is therefore simply connected. $\square$

In the spirit of Ivanov's work [9], we may ask:

Question. Is the group of automorphisms $Aut(\mathcal{X})$ of the triangulation complex $\mathcal{X}$ isomorphic to the (extended) baseleaf preserving modular group?

4. Presentation of $Mod(\mathcal{H})$

Applying a general theorem of Brown [2, Theorem 1] to the action of $Mod(\mathcal{H})$ on $\mathcal{X}$, we give a presentation for the modular group $Mod(\mathcal{H})$ of the punctured solenoid $\mathcal{H}$.

First recall [16] that $PSL_2(\mathbb{Z})$ is the isotropy group of the basepoint $\tau_* \in \mathcal{X}$, that $Mod(\mathcal{H})$ acts transitively on the vertices of $\mathcal{X}$, that an arbitrary vertex $\tau \in \mathcal{X}$ has isotropy group $hPSL_2(\mathbb{Z})h^{-1}$, where the characteristic map $h : \tau_* \mapsto \tau$ lies in $Mod(\mathcal{H})$, and that $hPSL_2(\mathbb{Z})h^{-1}$ contains a finite-index subgroup of $PSL_2(\mathbb{Z})$.

Consider the isotropy group of an edge in $\mathcal{X}$. Since each vertex is mapped to the basepoint τ_*, it is enough to consider edges with an endpoint at τ_*. The isotropy group of any other edge is the conjugate of the isotropy group of such an edge.

Let $E = \{\tau_*, \tau\}$ be an arbitrary edge of $\mathcal{X}$ with one endpoint at the basepoint τ_* of $\mathcal{X}$. There are two possibilities: either the isotropy group $\Gamma(E)$ of E contains elements which reverse the orientation of E (i.e., interchanges τ_* and τ), or each element of $\Gamma(E)$ fixes each endpoint of E.

Let τ be obtained by a Whitehead move on τ_* invariant under a torsion free finite-index subgroup K of $PSL_2(\mathbb{Z})$, and let us choose a characteristic map $g : \tau_* \mapsto \tau$, where $g \in Mod(\mathcal{H})$. Denote by K' the maximal extension of K in $PSL_2(\mathbb{Z})$ which fixes τ. If $h \in \Gamma(E)$ preserves the orientation of E as above, then h fixes both τ_* and τ. By [16, Lemma 7.3], $h \in PSL_2(\mathbb{Z})$ and similarly $h \in gPSL_2(\mathbb{Z})g^{-1}$, and so $h \in PSL_2(\mathbb{Z}) \cap gPSL_2(\mathbb{Z})g^{-1} = K'$. It follows that the subgroup $\Gamma^+(E)$ of the isotropy group $\Gamma(E)$ of an edge E which consists of elements which do not reverse orientation on E is equal to $K' < PSL_2(\mathbb{Z})$.

If $k \in \Gamma(E)$ reverses orientation of $E = \{\tau_*, \tau\}$, i.e., $k(\tau_*) = \tau$ and $k(\tau) = \tau_*$, then $k^2 \in K' = \Gamma^+(E)$. In particular, k^2 is a lift of a self map of the Riemann surface $\mathbf{D}/K$. By [12, Proposition 1.3.6], k is a lift of a self map of a Riemann surface which finitely covers $\mathbf{D}/K$. We show that k is actually a lift of a self map of $\mathbf{D}/K$ itself.

Lemma 4.1. *Let τ be the image of τ_* under a K-invariant Whitehead move, let $E = \{\tau_*, \tau\}$ be the corresponding edge and let $k \in \Gamma(E) - \Gamma^+(E)$. Then k conjugates K onto itself.*

Proof. The proof proceeds in several steps.

Simplification of the homeomorphism k. Since k preserves the union of the two tesselations τ, τ_*, it therefore sends a pair of intersecting edges to a pair of intersecting edges. Fix such an intersecting pair $e \in \tau_*$ and $f \in \tau$, and consider the corresponding Whitehead move. Since $k(e) \in K\{f\}$, there exists $\gamma \in K$ such that $(k \circ \gamma)(e) = f$. It is enough to prove the lemma for $k \circ \gamma$ and we continue to denote it by k.

Orientation of edges. Let us choose an orientation of e and assign an orientation to f such that $k : e \mapsto f$ is orientation preserving. Assign an orientation to each edge in the orbits of e and f under K as follows. Let $e' = \gamma'(e)$ for some $\gamma' \in G$. Let α be a differentiable arc connecting e to e' which transversely crosses the minimal number of edges of τ_*. Give the orientation to the curve α such that the tangent vector to α and the tangent vector to e at their point of intersection form a positively oriented basis for the tangent space of $\mathbf{D}$ at the intersection point, and assign an orientation on e' such that the tangent vector to α and the tangent vector to e' at the intersection point $\alpha \cap e'$ form a positively oriented basis to the tangent space. We may assign an orientation to any $f' = \gamma'(f)$ in a similar fashion.

k preserves the orientation. We noted above that k maps the orbit $K\{e\}$ onto $K\{f\}$ without specifying an orientation, and we noted that we may assume that $k(e) = f$ preserving orientation. It is a standard fact that $k : S^1 \to S^1$ extends to a differentiable self map $\tilde{k}$ of $\mathbf{D}$ which sends complementary triangles of τ_* onto complementary triangles of τ [14]. If α is a differentiable path between e and e' as above, then $\tilde{k}(\alpha)$ is a differentiable path between f and $f'' := k(e')$ which satisfies the required properties. Note that it is not necessarily true that $f' = \gamma'(f)$ and f'' are equal. However, the inductive definition of the characteristic map k immediately implies that $k : e' \mapsto f''$ is orientation preserving.

G is orientation preserving. Elements of G are covering transformations for the surface $\mathbf{D}/K$. We show that $\gamma : e \mapsto e' := \gamma(e)$ is orientation preserving, and a similar statement for f follows immediately. Denote

by $\pi_K : \mathbf{D} \to \mathbf{D}/K$ the universal covering map. Let α' be a differentiable curve on $\mathbf{D}/K$ representing the covering transformation γ which is transverse to $\pi_K(e)$ and crosses the minimal number of edges of τ_*/K. We denote by α a part of the lift of α' to $\mathbf{D}$ which connects e and e', so $\gamma(e \cap \alpha) = e' \cap \alpha$. Since γ preserves the orientation of α, it follows that $\gamma : e \mapsto e'$ is orientation preserving.

k conjugates K onto itself. Recall that k conjugates a finite-index subgroup H of $PSL_2(\mathbb{Z})$ onto K. Since G preserves the orientation of the orbits $K\{e\}$ and $K\{f\}$ in the sense of the previous paragraph and k maps $K\{e\}$ onto $K\{f\}$, it follows that k conjugates the action of K on the orbit $K\{e\}$ onto the action of K on the orbit $K\{f\}$. Since H and K have the same index in $PSL_2(\mathbb{Z})$ it follows that $H = K$. $\square$

Thus, k descends to a self map $\bar{k}$ of $\mathbf{D}/K$ sending the tesselation τ_*/K onto the tesselation τ/K and vice versa. Let $\bar{e} \in \tau_*/K$ and $\bar{f} \in \tau/K$ be the corresponding edges on $\mathbf{D}/K$ of the orbit $K\{e\} \subset \tau_*$ and its corresponding orbit $K\{f\} \in \tau - \tau_*$ under the Whitehead move defining the edge E.

It follows from the proof above that $\bar{e}$ is necessarily mapped onto $\bar{f}$ by $\bar{k}$, whence $\bar{k}^2(\bar{e}) = \bar{e}$ and $\bar{k}^2(\bar{f}) = \bar{f}$ with the orientations of $\bar{e}$ and $\bar{f}$ reversed. This implies that $\bar{k}^4(\bar{e}) = \bar{e}$ with an orientation of $\bar{e}$ preserved. Since in addition $\bar{k}^4(\tau_*/K) = \tau_*/K$, we conclude that $\bar{k}^4 = id$. This implies that k (after possibly pre-composing by an element of K and for simplicity renaming the composition again by k) maps e onto f sending τ_* onto τ, and vice versa. Since $k^2 \in \Gamma(E)$ maps e onto itself by reversing its orientation, we conclude that $k^2 \in K'$ is an involution with fixed point on e. Thus,

$$\Gamma(E) \ = \ <K', k>$$

where $K' < PSL_2(\mathbb{Z})$, and $k \in Mod(\mathcal{H})$ with $k^2 \in K' - K$ and $k^4 = id$. In particular, k^2 is an elliptic involution whose fixed point lies on e and $K' \neq K$ if k^2 is non trivial. Note that any Whitehead move on the once punctured torus can be obtained as a homeomorphism of the torus which interchanges the two tesselations, and hence the corresponding edge is inverted. In the following example we show that edges admit orientation reversing isotropy also for higher genus.

Example 4.2. We give in Figure 1 just one illustrative example of a surface $\mathbf{D}/K$ with a distinguished quadrilateral, together with a self homeomorphism performing a Whitehead move on the quadrilateral. In this figure, the homeomorphism h is a rotation by $\pi/4$ along the horizontal axis. The dots represent the punctures of the surface.
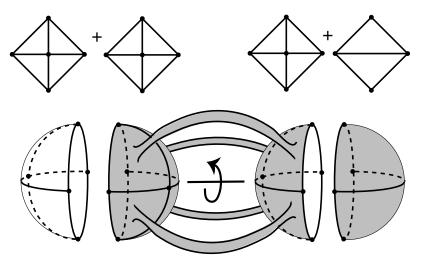

FIGURE 1. Inverting an edge

We also note that there are infinitely many edges $E \in \mathcal{X}$ with $\Gamma(E) - \Gamma^+(E) = \emptyset$. This follows from the fact that there are infinitely many Whitehead moves on finite surfaces (whose Euler characteristics are increasing without bound) such that there is no homeomorphism of the surface which maps the starting tesselation onto the ending tesselation and by Lemma 4.1.

Consider a two-cell of $\mathcal{X}$ with one vertex at the basepoint τ_*. Recall that for pentagon and square two-cells, each vertex is invariant under a finite-index subgroup K of $PSL_2(\mathbb{Z})$, and edges correspond to Whitehead moves invariant under this group K. For coset two-cells, either the long edge has τ_* as endpoint, in which case all edges are Whitehead moves invariant under K, or the long edge does have τ_* as endpoint, in which case the long edge is given by a generalized Whitehead move.

We may now apply Brown's theorem [2, Theorem 1] to obtain a presentation of the modular group $Mod(\mathcal{H})$ since it acts cellularly on the connected and simply connected triangulation complex $\mathcal{X}$ with a single vertex orbit. In fact, we shall introduce a somewhat larger set of generators than necessary for the application of Brown's theorem in order to obtain a simpler presentation.

It is a standard fact (which follows from Tietze's Theorem for instance) that for a given presentation, if one adds extra generators, then an equivalent presentation arises by expressing the new generators in terms of the old as new relations. One can in effect replace any occurrence of a subsequence of old generators in the old relations by new generators, in order to presumably simplify the presentation. We shall ultimately give the presentation of an abstract group $\mathcal{G}$, which is equivalent in this sense to the presentation of the group in Brown's Theorem.

The set of edges of $\mathcal{X}$ that are not inverted by the action of $Mod(\mathcal{H})$ can be oriented consistently for the action of $Mod(\mathcal{H})$, and we fix one such orientation on each such edge. Let $\mathcal{E}^+$ be the set of edges which are not inverted by the action of $Mod(\mathcal{H})$ that have initial point τ_*. If $E = (\tau_*, \tau) \in \mathcal{E}^+$ then τ is invariant under a finite-index subgroup K of $PSL_2(\mathbb{Z})$. By Lemma 4.1 and the subsequent discussion, there is no homeomorphism $f : \mathbf{D}/K \to \mathbf{D}/K$ such that $f(\tau_*/K) = \tau/K$ and $f(\tau/K) = \tau_*/K$. The elements of $\mathcal{E}^+$ therefore are obtained by taking all finite-index torsion free subgroups of $PSL_2(\mathbb{Z})$ and performing all possible Whitehead moves on τ_* invariant under the chosen groups, where the Farey tesselation τ_* and the image tesselation satisfy the additional property of not being mapped onto each other by a single map conjugating the group onto itself. The images of τ_* under the Whitehead moves are the terminal vertices of edges in $\mathcal{E}^+$. For any such $E = (\tau_*, \tau) \in \mathcal{E}^+$, we fix the characteristic map $g_E \in Mod(\mathcal{H})$ such that $g_E(\tau_*) = \tau$ and the standard distinguished oriented edge $e_0 = (-1, 1)$ of τ_* is mapped to either itself if the Whitehead move is not along an orbit of e_0, or it is mapped onto $f_0 = (-i, i)$ if the Whitehead move is along an orbit of e_0. The characteristic map is uniquely determined by these conditions, and we fix this choice g_E. (Notice that $\mathcal{E}^+$ is larger than necessary, since it is enough to take only the edges corresponding to representatives of *conjugacy classes* in $PSL_2(\mathbb{Z})$ of finite-index subgroups. However, this larger set simplifies the presentation, and not much is lost because both sets are infinite.)

Let $\mathcal{E}^-$ denote the set of inverted edges with initial point τ_*. By Lemma 4.1, an edge $E = (\tau_*, \tau)$ is inverted if there exists $k : S^1 \to S^1$ such that $k(\tau_*) = \tau$, $k(\tau) = \tau_*$ and $kHk^{-1} = H$, where τ is invariant under a torsion free finite-index subgroup H of $PSL_2(\mathbb{Z})$. The isotropy group $\Gamma(E)$ of the cell underlying $E \in \mathcal{E}^-$ is the subgroup of $Mod(\mathcal{H})$ generated by H' and k, where $H' > H$ is the maximal subgroup of $PSL_2(\mathbb{Z})$ under which τ is invariant, where $k^2 \in PSL_2(\mathbb{Z})$ and where $k^4 = id$. Fix some choice g_E of characteristic map associated to E and take $k = g_E$. (Again, we take $\mathcal{E}^-$ larger then necessary for ease in writing down the relations.)

Denote by $\mathcal{F}$ the set of two-cells of $\mathcal{X}$ based at τ_*. This condition implies that for each coset two-cell in $\mathcal{F}$ the initial point of the long edge is τ_* and each edge of the cell is consequently obtained by a geometric Whitehead move, i.e., one invariant under a subgroup of $PSL_2(\mathbb{Z})$ as opposed to a conjugate of a subgroup of $PSL_2(\mathbb{Z})$. This property also holds for pentagon and square cells in $\mathcal{F}$. The set $\mathcal{F}$ is obtained by taking all torsion free finite-index subgroups of $PSL_2(\mathbb{Z})$ and taking all possible pentagon, square and coset edge-paths in $\mathcal{X}$ corresponding to the chosen groups.

Note that a single choice of group for coset two-cells gives countably many coset cells because there are countably many finite-index subgroups and each finite-index subgroup yields finitely many coset cells. In the case of a square or a pentagon cell, a choice of a finite-index group determines finitely many cells because there are finitely many edge orbits in τ_* under the group. (Again, we could have taken only representatives of orbits of two-cells based at τ_* together with subsets of $\mathcal{E}^\pm$, but for simplicity later, we have expanded these sets.)

Let us assume for a moment that we had instead chosen for $\mathcal{E}^{\pm}$ and $\mathcal{F}$ only representatives of classes under the action of $Mod(\mathcal{H})$. We describe the assignment of a composition of elements in $Mod(\mathcal{H})$ (depending upon the above choices) to any closed boundary edge-path of a two-cell in $\mathcal{F}$ in order to give a relation corresponding to the two-cell as in Brown's theorem [2]. Given an oriented edge $E = (\tau_*, \tau)$ starting at τ_*, we assign to it a chosen element of $g \in Mod(\mathcal{H})$ such that $\tau = g(\tau_*)$. If $E \in \mathcal{E}^+$ then set $g := g_E$. If $E \notin \mathcal{E}^+$ is not inverted by $Mod(\mathcal{H})$ then $g := \gamma \circ g_{E'}$, where $E' = (\tau_*, \tau') \in \mathcal{E}^+$, and $\gamma \in PSL_2(\mathbb{Z})$ satisfies $\gamma(E') = E$; γ is well-defined up to pre-composition by an element of $\Gamma(E')$. The two Whitehead moves from τ_* to τ and τ' determine distinguished oriented edges e and e' of τ and τ', and we choose unique $\gamma \in PSL_2(\mathbb{Z})$ mapping e' to e. Our choice of $\gamma \circ g_E$ is in this case unique. If $E \in \mathcal{E}^-$, then set $g := g_E \in \Gamma(E) - \Gamma^+(E)$. If $E \notin \mathcal{E}^-$ is inverted by the action of $Mod(\mathcal{H})$ then $g := \gamma \circ g_{E'}$, where $E' \in \mathcal{E}^-$ and $\gamma \in PSL_2(\mathbb{Z})$ with $\gamma(\tau') = \tau$ and $\gamma(e') = e$. The edge E therefore ends at $g(\tau_*)$, but it seems complicated to explicitly determine E' and γ.

Continuing to assume that we had chosen for $\mathcal{E}^{\pm}$ and $\mathcal{F}$ only representatives of classes under the action of $Mod(\mathcal{H})$, consider a closed path α of edges in $\mathcal{X}$ based at τ_*. Let $(E_1, E_2, \ldots, E_n)$ be the sequential edges of α. Denote by g_1 the unique element of $Mod(\mathcal{H})$ chosen for the edge $E_1 = (\tau_*, \tau_1)$ starting at τ_* as above, so $g_1(\tau_*) = \tau_1$. The edge $E_2 = (\tau_1, \tau_2)$ is therefore of the form $g_1(E_2')$ for an edge $E_2' = (\tau_*, \tau_2')$ based at τ_*. Denote by g_2 the unique element of $Mod(\mathcal{H})$ associated to E_2' as above, so $g_1 \circ g_2(\tau_*) = \tau_2$. This implies that $E_3 = (\tau_2, \tau_3)$ is given by $g_1 \circ g_2(E_3')$, where E_3' starts at τ_*. Take $g_3 \in Mod(\mathcal{H})$ associated to the edge E_3', and continue in this manner until we exhaust all edges of α. This yields a composition $g_1 \circ g_2 \circ \ldots \circ g_n$ in terms of generators such that $g_1 \circ g_2 \circ \ldots \circ g_n(\tau_*) = \tau_*$. Thus, there is $\gamma \in PSL_2(\mathbb{Z})$ such that $g_1 \circ g_2 \circ \ldots \circ g_n = \gamma$, and this is the relation associated with a closed edge path α based at τ_*. It seems complicated to determine the maps g_i from the given description or to decide which elements $\gamma \in PSL_2(\mathbb{Z})$ arise. However, the choice of g_i simplifies if we allow all edges with initial point τ_*, and this will require additional relations as discussed before.

¿From this point on, we go back to our choice of $\mathcal{E}^{\pm}$ to consist of all edges with initial point τ_* and of $\mathcal{F}$ to consists of all two-cells based at τ_*.

We describe the relations associated to boundaries of two-cells in $\mathcal{F}$. Let us start with a pentagon two-cell P based at τ_* whose boundary edges are $\{E_1 = (\tau_*, \tau_1), E_2 = (\tau_1, \tau_2), \ldots, E_5 = (\tau_4, \tau_*)\}$. The pentagon two-cell P is given by changing an orbit of two adjacent edges e_1, e_2 of τ_* under a torsion free finite-index subgroup K of $PSL_2(\mathbb{Z})$ of index at least 9. Assume first that the distinguished oriented edge $e_0 = (-1, 1)$ of τ_* is not an element of the orbit $K\{e_1, e_2\}$ and apply the algorithm of Brown to get the edge-path relation, but using our extended set of generators to simplify it. We denote by g_i the element of $Mod(\mathcal{H})$ which corresponds to the edge E_i. The first edge E_1 gives $g_1 := g_{E_1}$, so $g_1(e_0) = e_0$. We find $g_2 := \gamma \circ g_{E_2''}$, where E_2'' is a representative of the orbit of $(\tau_*, g_1^{-1}(\tau_2))$ and $\gamma(E_2'') = (\tau_*, g_1^{-1}(\tau_2))$ is chosen from $PSL_2(\mathbb{Z})$ in a unique way as above (i.e., $\gamma(e_0) = e_0$). However, since $\mathcal{E}^+$ consists of all edge starting at τ_* we immediately obtain that $g_2 := g_{E_2'}$, where $E_2' := (\tau_*, g_1^{-1}(\tau_2))$. We likewise obtain $g_i := g_{E_i'}$, for $i = 3, 4, 5$, where $E_3' := (\tau_*, (g_1 \circ g_2)^{-1}(\tau_3))$, $E_4' := (\tau_*, (g_1 \circ g_2 \circ g_3)^{-1}(\tau_4))$, and $E_5' := (\tau_*, (g_1 \circ g_2 \circ g_3 \circ g_4)^{-1}(\tau_*))$. Under our assumption that $e_0 \notin K\{e_1, e_2\}$, we find $g_i(e_0) = e_0$ for $i = 1, 2, \ldots, 5$. The relation associated to P is therefore

$$(1) \qquad g_1 \circ g_2 \circ \cdots \circ g_5 = id.$$

On the other hand, now assume $e_0 \in K\{e_1\}$ and without loss of generality we can assume that $e_1 = e_0$. We choose g_i as above and note that $g_1 : (\tau_*, e_0) \mapsto (\tau_1, e_0')$, where $e_0' = (-i, i)$ is the image of $e_0 = (-1, 1)$ under the Whitehead move corresponding to $E_1 = (\tau_*, \tau_1)$, $g_1 \circ g_2 : (\tau_*, e_0) \mapsto (\tau_2, e_0')$, $g_1 \circ g_2 \circ g_3 : (\tau_*, e_0) \mapsto (\tau_3, e_0'')$ where e_0'' is the image of e_0' under the Whitehead move corresponding to E_3, $g_1 \circ g_2 \circ g_3 \circ g_4 : (\tau_*, e_0) \mapsto (\tau_4, e_0'')$ and $g_1 \circ g_2 \circ \cdots \circ g_5 : (\tau_*, e_0) \mapsto (\tau_*, \bar{e}_2)$ where $\bar{e}_2$ is the oriented edge e_2 with orientation given such that the terminal point of e_0 is the initial point of $\bar{e}_2$. Denote by $\gamma_{e_0, \bar{e}_2} \in PSL_2(\mathbb{Z})$ the unique element which maps e_0 onto $\bar{e}_2$ with the given orientation. Thus, $\gamma_{e_0, \bar{e}_2}$ is the composition of the primitive parabolic element with fixed point at the terminal point of e_0 which maps e_0 onto e_2 and the involution which reverses e_2. We obtain the following relation

$$(2) \qquad g_1 \circ g_2 \circ \cdots \circ g_5 = \gamma_{e_0, \bar{e}_2}.$$

When $e_2 = e_0$ the relation is similarly

$$g_1 \circ g_2 \circ \cdots \circ g_5 = \gamma_{e_0, \bar{e}_1}. \tag{3}$$

Let P be a square cell in $\mathcal{F}$. Assume that P is obtained by Whitehead moves along the nonadjacent orbits $K\{e_1\}$ and $K\{e_2\}$ of edges e_1, e_2 in τ_*, where K is a torsion free finite-index subgroup of $PSL_2(\mathbb{Z})$. If $e_0 \notin K\{e_1, e_2\}$ then

$$g_1 \circ \cdots \circ g_4 = id, \tag{4}$$

where g_i are chosen as above. If $e_i = e_0$ then we obtain a relation

$$g_1 \circ \cdots \circ g_4 = s_{e_0}, \tag{5}$$

where $s_{e_0} \in PSL_2(\mathbb{Z})$ is the involution which reverses e_0. The proofs of both relations for the square cell P depend upon keeping track of where e_0 is mapped, and it is sufficiently similar to the pentagon two-cell that we do not repeat it.

Let $P \in \mathcal{F}$ be a coset two-cell for the edge $e \in \tau_*$ and for the groups $K_1 < K < PSL_2(\mathbb{Z})$. If $e \neq e_0$ then we obtain a relation

$$g_1 \circ \cdots \circ g_k = id, \tag{6}$$

where $k = [K : K_1]$ and g_i are uniquely chosen as above. Note that a single choice of $K_1 < K$ gives a decomposition of the orbit $K\{e\}$ into k disjoint coset orbits $K_1\{e_1\}, K_1\{e_2\}, \ldots, K_1\{e_k\}$, where $e_i \in K\{e\}$. This gives $k!$ possible permutations on $K_1\{e_1\}, K_1\{e_2\}, \ldots, K_1\{e_k\}$ which in turn produce $k!$ coset two-cells with the long edge given by the Whitehead move on $K\{e\}$. Note that $g_1 = g_E$ where $E = (\tau_*, \tau)$ and τ is the image of the Whitehead move along $K\{e\}$. The other g_i, for $i = 2, 3, \ldots, k$, are given by the translation to τ_* of the short edges. If $e = e_0$ then we obtain a relation

$$g_1 \circ \cdots \circ g_k = s_{e_0}. \tag{7}$$

The desired group $\mathcal{G}$ is by definition the free product of $PSL_2(\mathbb{Z})$, $\Gamma(E) = \Gamma^+(E)$ for $E \in \mathcal{E}^+$, $\Gamma(E)$ for $E \in \mathcal{E}^-$ and a free group generated by g_E for $E \in \mathcal{E}^+$. The modular group $Mod(\mathcal{H})$ is the quotient of $\mathcal{G}$ by a set of relations as follows.

Theorem 4.3. *The modular group $Mod(\mathcal{H})$ is generated by the isotropy subgroup $PSL_2(\mathbb{Z})$ of the basepoint $\tau_* \in \mathcal{X}$, the isotropy subgroups $\Gamma(E)$ for $E \in \mathcal{E}^{\pm}$, and by the elements g_E for $E \in \mathcal{E}^+$. The following relations on these generators give a complete presentation of $Mod(\mathcal{H})$:*

a) The inclusions of $\Gamma(E)$ into $PSL_2(\mathbb{Z})$, for $E \in \mathcal{E}^+$, are given by $\Gamma(E) = K'$, where the terminal endpoint of E is invariant under the finite-index subgroup $K' < PSL_2(\mathbb{Z})$;

b) The inclusions of $\Gamma^+(E)$ into $PSL_2(\mathbb{Z})$, for $E \in \mathcal{E}^-$, are given by $\Gamma(E) = K'$, where the terminal endpoint of E is invariant under the finite-index subgroup $K' < PSL_2(\mathbb{Z})$;

c) The relations introduced by the boundary edge-paths of two-cells in $\mathcal{F}$ given by the equations (1), (2), (3), (4), (5), (6) and (7);

d) The redundancy relations: for any two edges E and E' in $\mathcal{E}^{\pm}$ and for any $\gamma \in PSL_2(\mathbb{Z})$ such that $\gamma(E) = E'$, we get the relation

$$g_{E'} \circ \gamma' = \gamma \circ g_E,$$

where γ' is the unique element of $PSL_2(\mathbb{Z})$ that satisfies $\gamma'(e_0) = e_1'$ with $e_1' = g_{E'}^{-1}(\gamma(e_0))$.

Proof. The fact about the generators of $Mod(\mathcal{H})$ follows directly from Brown's theorem [2] and from our choice of $\mathcal{E}^{\pm}$ even larger than necessary. The relations from [2, Theorem 1] are included in our theorem as follows. The relations (i) are empty in our case. The relations (ii), (iii) and (iv) translate easily to relations a), b) and c) in our theorem, respectively. The relations d) are extra relations needed because we have taken a larger set of generators than in Brown's presentation. If $g_E(e_0) = e_0$, then the relation d) is immediate. If $g_E(e_0) \neq e_0$, then $g_{E'}(e_0) = e_0$ (since $\gamma \notin K$), and d) follows by reversing the roles of g_E and $g_{E'}$. $\square$

5. No central elements

In analogy to the case of surfaces of finite type, we have:

Theorem 5.1. *The modular group $Mod(\mathcal{H})$ of the punctured solenoid $\mathcal{H}$ has trivial center.*

Proof. Let $h \in Mod(\mathcal{H})$ be a central element, so h is a word in the generators of Theorem 4.3. If $h \in PSL_2(\mathbb{Z}) < Mod(\mathcal{H})$, then since $PSL_2(\mathbb{Z})$ has trivial center, it follows that $h = id$.

Assuming that $h \notin PSL_2(\mathbb{Z})$, we have $h(\tau_*) = \tau \neq \tau_*$. Since h is supposed to be central, we have $g \circ h \circ g^{-1} = h$ for all $g \in Mod(\mathcal{H})$. Taking $g \in PSL_2(\mathbb{Z})$, we find $g \circ h(\tau_*) = h(\tau_*)$, i.e., $g(\tau) = \tau$ for all $g \in PSL_2(\mathbb{Z})$. By Proposition 5.2 below, we conclude $\tau = \tau_*$, i.e., $h(\tau_*) = \tau_*$, which implies $h \in PSL_2(\mathbb{Z})$. This gives a contradiction, and so again $h = id$. $\square$

Proposition 5.2. *A TLC tesselation of $\mathbf{D}$ is invariant under $PSL_2(\mathbb{Z})$ if and only if it is the Farey tesselation τ_*.*

Proof. The implication that τ_* is the unique tesselation invariant under $PSL_2(\mathbb{Z})$ is given in [16, Lemma 7.3]. For the converse, recall that $PSL_2(\mathbb{Z})$ contains order two elliptic elements with fixed points on each edge of τ_* and order three elliptic elements with fixed points at the center of each ideal complementary triangle of τ_*. Let T be a complementary triangle of τ containing the fixed point a of an elliptic element $\gamma \in PSL_2(\mathbb{Z})$ of order three. It is an exercise in elementary hyperbolic geometry to show that a is the center of T.

Let b be a fixed point of an elliptic involution $\gamma \in PSL_2(\mathbb{Z})$. If b is in the interior of a complementary triangle T of τ then $\gamma(T) \neq T$ and $\gamma(T) \cap T \neq \emptyset$. Thus, the image of the boundary of T under γ intersects transversely the boundary of T. This is in contradiction to the assumption that γ fixes τ. It follows that b must lie on an edge of τ.

Let a be the fixed point of an elliptic element of $PSL_2(\mathbb{Z})$ of order three and let b_1, b_2, b_3 be fixed points of three elliptic involutions of $PSL_2(\mathbb{Z})$ that are shortest distance to a among all such involutions. Thus, b_1, b_2, b_3 lie on a hyperbolic circle centered at a. If T is the ideal triangle in the complement of τ whose center is a, then the boundary sides of T are tangent to this circle. Since b_1, b_2, b_3 must lie on edges of τ, this implies that the boundary sides of T are tangent at the points b_1, b_2, b_3. It follows that T is a complementary triangle of τ_* as well. Since this is true for an arbitrary T, it follows that indeed $\tau = \tau_*$. $\square$

We consider the action of $Mod(\mathcal{H})$ on the first barycentric subdivision $\mathcal{X}'$ of $\mathcal{X}$.

Proposition 5.3. *The first barycentric subdivision $\mathcal{X}'$ of $\mathcal{X}$ is a simplicial complex on which $Mod(\mathcal{H})$ acts simplicially.*

Proof. Note that $Mod(\mathcal{H})$ preserves cells of $\mathcal{X}$. An isotropy group of a vertex of $\mathcal{X}$ is a conjugate of $PSL_2(\mathbb{Z})$. We showed that the isotropy group of an edge is either a finite-index subgroup of $PSL_2(\mathbb{Z})$ which preserves the orientation of the edge or it is generated by an element of $Mod(\mathcal{H})$ which reverses the orientation of the edge and by a finite-index subgroup of $PSL_2(\mathbb{Z})$ which preserves the orientation of the edge. In the first case, each element of the isotropy group fixes each point on the edge. In the second case, an element either fixes each point of the edge or fixes the midpoint and reflects the endpoints of the edge.

Let C be a coset two-cell with long edge given by a Whitehead move on TLC tesselation τ_* invariant under $K < PSL_2(\mathbb{Z})$ along the orbit of $e \in \tau$ and with the short edges given by Whitehead moves invariant under a subgroup $H < K$. The isotropy subgroup of C is a finite extension in $PSL_2(\mathbb{Z})$ of H. To prove the claim, it is enough to show that the long edge cannot be mapped onto a short edge, and this is true because the rectangles in which change of diagonals for the Whitehead move occur must be mapped onto the rectangles on which change of diagonals occur. However, the two groups have different indexes in $PSL_2(\mathbb{Z})$ which gives a contradiction, and the isotropy group $\Gamma(C)$ therefore acts by fixing each point in C. Since an arbitrary coset two-cell is the image of some C as above, the same statement holds for an arbitrary coset two-cell.

Let P be a pentagon two-cell based at τ_* obtained by Whitehead moves along $K\{e_1, e_2\}$. Thus, P has a subgroup K', where $PSL_2(\mathbb{Z}) > K' > K$, of its isotropy group $\Gamma(P)$ fixing each point of P. If $\Gamma(P) \neq K'$, then it is generated by K' and a single element of $h \in Mod(\mathcal{H})$ which maps the first edge onto the second. Since $h^5(\tau_*) = \tau_*$, we conclude $h^5 \in PSL_2(\mathbb{Z})$. Thus, h^5 is mapping class like and therefore h is mapping class like. The map h fixes the center of P and rotates by the angle $2\pi/5$ the pentagon P. The situation for a pentagon not based at τ_* is the same.

Finally, suppose Q is a square two-cell obtained by Whitehead moves along $K\{e_1, e_2\}$ then $\Gamma(Q) > K'$, where $K < K' < PSL_2(\mathbb{Z})$. It is possible a priori that $\Gamma(Q) \neq K'$, in which case the elements $h \in \Gamma(Q) - K'$ permute edges of Q and fix the center of Q. $\square$

We finally investigate the topological fundamental group $\pi_1(\mathcal{Y})$ of the quotient space $\mathcal{Y} = \mathcal{X}/Mod(\mathcal{H})$. To begin, we describe a natural surjection $\phi : Mod(\mathcal{H}) \to \pi_1(\mathcal{Y})$ as follows. Denote by $\Pi : \mathcal{X} \to \mathcal{Y}$ the quotient map, let $h \in Mod(\mathcal{H})$ be arbitrary, and define $\tau = h(\tau_*)$. Let γ be an edge-path between τ_* and τ in $\mathcal{X}$. and define
$$\phi(h) := [\Pi(\gamma)],$$
where $[\Pi(\gamma)]$ is the homotopy class of the closed curve $\Pi(\gamma)$ based at $\Pi(\tau_*)$, i.e., $[\Pi(\gamma)] \in \pi_1(\mathcal{Y}, \Pi(\tau_*)) = \pi_1(\mathcal{Y})$. It is a standard fact that ϕ is a well-defined and surjective homomorphism.

Let $\mathcal{N}$ be the group generated by the isotropy subgroups of all vertices of $\mathcal{X}'$, where $\mathcal{X}'$ denotes the first barycentric subdivision of $\mathcal{X}$, so $\mathcal{N}$ is normal in $Mod(\mathcal{H})$. In fact, $\mathcal{N}$ is generated by all conjugates of $PSL_2(\mathbb{Z})$ and by the isotropy groups of edges and two-cells of $\mathcal{X}$. The isotropy group of an edge in $\mathcal{X}$ fixes the center of the edge and therefore belongs to the isotropy group of a vertex in $\mathcal{X}'$, and likewise the isotropy group of a two-cell in $\mathcal{X}$ fixes a vertex of $\mathcal{X}'$. Moreover, any element of $Mod(\mathcal{H})$ which fixes a point in $\mathcal{X}'$ fixes a point in $\mathcal{X}$. By Proposition 5.3 and by a standard result [1] we get:

Theorem 5.4. *The topological fundamental group of $\mathcal{Y} = \mathcal{X}/Mod(\mathcal{H})$ satisfies*

$$\pi_1(\mathcal{Y}) = Mod(\mathcal{H})/\mathcal{N},$$

where $\mathcal{N} < Mod(\mathcal{H})$ is generated by the isotropy groups of vertices, edges and two-cells of $\mathcal{X}$. $\square$

By our discussion above, each element of $Mod(\mathcal{H})$ which fixes a cell in $\mathcal{X}$ is mapping class like, i.e. it conjugates a finite index subgroup of G onto itself. Therefore $\mathcal{N}$ is generated by some mapping class like elements. We pose the following question:

Question. Is $\mathcal{N}$ equal to the normal subgroup of $Mod(\mathcal{H})$ generated by all mapping class like elements?

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