# GEOMETRIC MEASURES FOR HYPERBOLIC SETS ON SURFACES

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ABSTRACT. We present a moduli space for all hyperbolic basic sets of diffeomorphisms on surfaces that have an invariant measure that is absolutely continuous with respect to Hausdorff measure. To do this we introduce two new invariants: the measure solenoid function and the cocycle-gap pair. We extend the eigenvalue formula of A. N. Livšic and Ja. G. Sinai for Anosov diffeomorphisms which preserve an absolutely continuous measure to hyperbolic basic sets on surfaces which possess an invariant measure absolutely continuous with respect to Hausdorff measure. We characterise the Lipschitz conjugacy classes of such hyperbolic systems in a number of ways, for example, in terms of eigenvalues of periodic points and Gibbs measures.

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#### 1. INTRODUCTION

We say that  $(f, \Lambda)$  is a  $C^{1+}$  hyperbolic diffeomorphism if it has the following properties:

- (i)  $f: M \to M$  is a  $C^{1+\alpha}$  diffeomorphism of a compact surface M with respect to a
- $C^{1+\alpha}$  structure  $C_f$  on M, for some  $\alpha > 0$ ; (ii)  $\Lambda$  is a hyperbolic invariant subset of M; and
- (ii) A is a hyperbolic invariant subset of *W*, and
- (iii)  $f|\Lambda$  is topologically transitive and that  $\Lambda$  has a local product structure.

We denote by  $\mathcal{T}(f,\Lambda)$  the set of all  $C^{1+}$  hyperbolic diffeomorphisms  $(g,\Lambda_g)$  such that  $(g,\Lambda_g)$  and  $(f,\Lambda)$  are topologically conjugated by a homeomorphism  $h_{f,g}$  (see Section 2.5). We allow both the case where  $\Lambda = M$  and the case where  $\Lambda$  is a proper subset of M. If  $\Lambda = M$  then f is Anosov and M is a torus [9, 19]. The best known examples where  $\Lambda$  is a proper subset of M are the Smale horseshoes, and the codimension one attractors such as the Plykin attractor and the derived-Anosov diffeomorphisms.

For every  $g \in \mathcal{T}(f, \Lambda)$ , we denote by  $\delta_{g,s}$  (resp.  $\delta_{g,u}$ ) the Hausdorff dimension of the local stable (resp. local unstable) leaves of g intersected with  $\Lambda$ . Let  $\lambda_{g,s}(x)$  and  $\lambda_{g,u}(x)$  denote the stable and unstable eigenvalues of the periodic orbit of g containing a point x. A. N. Livšic and Ja. G. Sinai [13] proved that an Anosov diffeomorphism g has an invariant measure that is absolutely continuous with respect to Lesbegue measure if, and only if,  $\lambda_{g,s}(x)\lambda_{g,u}(x) = 1$  for every periodic point x. We extend the theorem of A. N. Livšic and Ja. G. Sinai to  $C^{1+}$  hyperbolic diffeomorphisms with hyperbolic sets on surfaces such as Smale horseshoes and codimension one attractors.

**Theorem 1.1.** A  $C^{1+}$  hyperbolic diffeomorphism  $g \in \mathcal{T}(f, \Lambda)$  has a g-invariant probability measure which is absolutely continuous to the Hausdorff measure on  $\Lambda_g$  if and only if for every periodic point x of  $g|\Lambda_q$ ,

$$\lambda_{q,s}(x)^{\delta_{g,s}}\lambda_{q,u}(x)^{\delta_{g,u}} = 1 .$$

The proof of all theorems stated in the introduction are given in Section 10.

Since  $(f, \Lambda)$  is a  $C^{1+}$  hyperbolic diffeomorphism it admits a Markov partition  $\mathcal{R} = \{R_1, \ldots, R_k\}$ . This implies the existence of a two-sided subshift  $\tau : \Theta \to \Theta$  of finite type  $\Theta$  in the symbol space  $\{1, \ldots, k\}^{\mathbb{Z}}$ , and an inclusion  $i : \Theta \to \Lambda$  such that (a)  $f \circ i = i \circ \tau$  and (b)  $i(\Theta_j) = R_j$  for every  $j = 1, \ldots, k$ , where  $\Theta_j$  is the cylinder containing all words  $\ldots \varepsilon_{-1}\varepsilon_0\varepsilon_1 \ldots \in \Theta$  with  $\varepsilon_0 = j$ . For every  $g \in \mathcal{T}(f, \Lambda)$ , the inclusion  $i_g = h_{f,g} \circ i : \Theta \to \Lambda_g$  is such that  $g \circ i_g = i_g \circ \tau$ . We call such a map  $i_g : \Theta \to \Lambda_g$  a marking of  $(g, \Lambda_g)$ .

**Definition 1.1.** If  $g \in \mathcal{T}(f,\Lambda)$  is a  $C^{1+}$  hyperbolic diffeomorphism as above and  $\nu$  is a Gibbs measure on  $\Theta$  then we say that  $(g,\Lambda_g,\nu)$  is a Hausdorff realisation of  $\nu$  if  $(i_g)_*\nu$  is absolutely continuous with respect to the Hausdorff measure on  $\Lambda_g$ . If this is the case then we will often just say that  $\nu$  is a Hausdorff realisation for  $(g,\Lambda_g)$ .

We note that if  $g \in \mathcal{T}(f, \Lambda)$  the Hausdorff measure on  $\Lambda_g$  exists and is unique. However, a Hausdorff realisation need not exist for  $(g, \Lambda_g)$ .

Let  $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  be the set of all  $C^{1+}$  hyperbolic diffeomorphisms  $(g, \Lambda_g)$  in  $\mathcal{T}(f, \Lambda)$  such that (i)  $\delta_{g,s} = \delta_s$  and  $\delta_{g,u} = \delta_u$ ; (ii) there is a g-invariant measure  $\mu_g$  on  $\Lambda_g$  which is absolutely continuous with respect to the Hausdorff measure on  $\Lambda_g$ . We denote by  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  the subset of all  $C^{1+}$ -realisations of a Gibbs measure  $\nu$  in  $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ .

De la Llave, Marco and Moriyon [14, 15, 17, 18] have shown that the set of stable and unstable eigenvalues of all periodic points is a complete invariant of the  $C^{1+}$  conjugacy classes of Anosov diffeomorphisms. We extend their result to the sets  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ .

**Theorem 1.2.** (i) Any two elements of  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  have the same set of stable and unstable eigenvalues and these sets are a complete invariant of  $[\nu]$  in the sense that if  $g_1, g_2 \in \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  have the same eigenvalues if, and only if, they are in the same subset  $[\nu]$ .

(ii) The map  $\nu \to [\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  gives a 1-1 correspondence between  $C^{1+}$ -Hausdorff realisable Gibbs measures  $\nu$  and Lipschitz conjugacy classes in  $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ .

In Theorem 9.6, we also prove that the set of stable and unstable eigenvalues of all periodic orbits of a  $C^{1+}$  hyperbolic diffeomorphism  $g \in \mathcal{T}(f, \Lambda)$  is a complete invariant of each Lipschitz conjugacy class. We note that for Anosov diffeomorphisms every Lipschitz conjugacy class is a  $C^{1+}$ conjugacy class. This can be proved by combining Remark 8.1 with Lemmas 4.2 and 8.1.

**Remark 1.3.** We have restricted our discussion to Gibbs measures because it follows from Theorem 1.2 that, if  $g \in \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  has a g-invariant measure  $\mu$  which is absolutely continuous with respect to the Hausdorff measure then  $\mu$  is a  $C^{1+}$ -Hausdorff realisation of a Gibbs measure  $\nu$  so that  $\mu = (i_g)_*\nu$ .

E. Cawley [3] characterised all  $C^{1+}$ -Hausdorff realisable Gibbs measures as Anosov diffeomorphisms using cohomology classes on the torus. While it is possible that her cocycles could give enough information to characterise other hyperbolic systems on surfaces up to lippeomorphism, it is clear that they cannot encode enough for  $C^{1+}$  conjugacy because, for example, they do not encode enough information about gaps and so do not determine the smooth structure of stable leaves in the case where they are Cantor sets. To deal with all these cases in an integrated way we use measure solenoid functions and gap-cocycle pairs to classify  $C^{1+}$ -Hausdorff realisable Gibbs measures of all  $C^{1+}$  hyperbolic diffeomorphisms on surfaces.

The stable and unstable measure solenoid functions are built from the Gibbs measure as we show in Section 5.4. For Anosov diffeomorphisms, the domains  $msol^s$  and  $msol^u$  of the stable and unstable measure solenoid functions are dense subsets of finite disjoint unions of closed intervals. Define a stable leaf segment of a Markov rectangle to be a segment of a stable leaf crossing the Markov rectangle (see Section 2.3, Section 2.6, Section 2.7 and Figure 2). Every point in  $msol^s$ consists of a pair (I, J) of adjacent stable spanning leaf segments of Markov rectangles which are not contained in a stable global leaf containing a stable boundary of a Markov rectangle. The stable measure solenoid function  $\sigma_{\nu,s}$  associates to each pair (I,J) the ratio between the measure of J and the measure of I computed with respect to the conditional measure, determined by the Gibbs measure  $\nu$ , of a stable leaf containing I and J. The construction of the unstable solenoid function  $\sigma_{\nu,\mu}$  is similar. In Section 3.7, we define a boundary condition which consists of a finite set of simple algebraic equalities that the continuous extensions of the stable and unstable measure solenoid functions have to satisfy, for the corresponding Gibbs measures to be  $C^{1+}$ -Hausdorff realisable in the Anosov case. This is necessary because in this case the Markov rectangles have common boundaries along the stable and unstable foliations. We note that in the Anosov case, the Lebesgue measure is the Hausdorff measure.

**Theorem 1.4.** (Anosov diffeomorphisms) Suppose that f is a  $C^{1+}$  Anosov diffeomorphism of the torus  $\Lambda$ . Fix a Gibbs measure  $\nu$  on  $\Theta$ . Then the following statements are equivalent:

- (i) The set  $\nu$ ,  $[\nu] \subset \mathcal{T}_{f,\Lambda}(1,1)$  is non-empty and is precisely the set of  $g \in \mathcal{T}_{f,\Lambda}(1,1)$ such that  $(g, \Lambda_g, \nu)$  is a  $C^{1+}$  Hausdorff realisation. In this case  $\mu = (i_g)_* \nu$  is absolutely continuous with respect to Lesbegue measure.
- (ii) The stable measure solenoid function  $\sigma_{\nu,s} : \operatorname{msol}^s \to \mathbb{R}^+$  has a non-vanishing Hölder continuous extension to the closure of  $\operatorname{msol}^s$  satisfying the boundary condition.
- (iii) The unstable measure solenoid function  $\sigma_{\nu,u} : \operatorname{msol}^u \to \mathbb{R}^+$  has a non-vanishing Hölder continuous extension to the closure of  $\operatorname{msol}^s$  satisfying the boundary condition.

The treatment of codimension one attractors has a number of extra-dificulties due to the fact that the invariant set  $\Lambda$  is locally a Cartesian product of a Cantor set with an interval but the stable

and unstable measure solenoid functions are built in a similar way to the construction for Anosov diffeomorphisms. In the case of codimension one attractors, the continuous extension of the stable measure solenoid functions have to satisfy the *cylinder-cylinder condition* for the corresponding Gibbs measures to be  $C^{1+}$ -Hausdorff realisable (see Section 5.4). The cylinder-cylinder condition, like the boundary condition, consists of a finite set of simple algebraic equalities and is needed because the Markov rectangles have common boundaries along the stable laminations. Hence, the cylinder-cylinder condition just applies to the stable measure function.

**Theorem 1.5.** (Codimension one attractors) Suppose that f is a  $C^{1+}$  surface diffeomorphism and  $\Lambda$  is a codimension one hyperbolic attractor. Fix a Gibbs measure  $\nu$  on  $\Theta$ . Then the following statements are equivalent:

(i) For all  $0 < \delta_s < 1$ ,  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, 1)$  is non-empty and is precisely the set of  $g \in \mathcal{T}_{f,\Lambda}(\delta_s, 1)$  such that  $(g, \Lambda_g, \nu)$  is a  $C^{1+}$  Hausdorff realisation. In this case  $\mu = (i_g)_*\nu$  is absolutely continuous with respect to the Hausdorff measure on  $\Lambda_g$ . (ii) The stable measure solenoid function  $\sigma_{\nu,s} : \operatorname{msol}^s \to \mathbb{R}^+$  has a non-vanishing Hölder continuous extension to the closure of  $\operatorname{msol}^s$  satisfying the cylinder-cylinder condition.

(iii) The unstable measure solenoid function  $\sigma_{\nu,u} : \operatorname{msol}^u \to \mathbb{R}^+$  has a non-vanishing Hölder continuous extension to the closure of  $\operatorname{msol}^u$ .

In the case of Smale horseshoes, there are no extra conditions that the measure solenoid functions have to satisfy for the corresponding Gibbs measures to be  $C^{1+}$ -Hausdorff realisable.

**Theorem 1.6.** (Smale horseshoes) Suppose that  $(f, \Lambda)$  is a Smale horseshoe and  $\nu$  is a Gibbs measure on  $\Theta$ . Then for all  $0 < \delta_s, \delta_u < 1$ ,  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  is non-empty and is precisely the set of  $g \in \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  such that  $(g, \Lambda_g, \nu)$  is a  $C^{1+}$  Hausdorff realisation. In this case  $\mu = (i_g)_* \nu$  is absolutely continuous with respect to the Hausdorff measure on  $\Lambda_q$ .

Using Theorems 1.2-1.5, for  $\iota \in \{s, u\}$ , we prove that the map  $\nu \to \sigma_{\nu,\iota}$  gives a one-to-one correspondence between the subsets  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  and the measure solenoid functions  $\sigma_{g,\iota}$  satisfying the conditions indicated in the above Theorems 1.4 and 1.5. By Theorem 1.2, we have that the sets  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  are precisely the Lipschitz conjugacy classes contained in  $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ .

**Theorem 1.7.** The measure solenoid functions determine a pair of infinite-dimensional metric spaces, that we denote by  $SOL^s$  and  $SOL^u$ , which parametrize all Lipschitz conjugacy classes  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u).$ 

The scaling functions, presented by M. Feigenbaum in [7, 8] and D. Sullivan in [33], and the solenoid functions, presented in [21, 24, 28] (see also Section 1.3 and Section 3), are used to classify all  $C^{1+}$  conjugacy classes of expanding maps on train-tracks and of hyperbolic diffeomorphisms for a given a topological conjugacy class. Both scaling function and solenoid function are complete invariants of the smooth structure but the solenoid functions have the great advantage that, unlike the scaling functions, one knows which solenoid functions occur as the solenoid functions of  $C^{1+}$  expanding maps.

In Section 7, we introduce the stable and unstable cocycle-gap pairs  $(\gamma_s, J_s)$  and  $(\gamma_u, J_u)$  which allow us to parametrize the  $C^{1+}$  conjugacy classes inside of each Lipschitz conjugacy class  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ . The cocycle-gap pair  $(\gamma, J)$  consists of a gap function  $\gamma$  and a measure-length ratio cocycle function J (see Definitions 7.1 and 7.2). The domain of a stable gap function is the set of all pairs  $(\xi_1, \xi_2)$  with the following properties: (a) the stable leaf segments  $\xi_1$  and  $\xi_2$  intersect the invariant set  $\Lambda$  just in their end points; (b) there is a stable leaf K of a Markov rectangle which contains  $f\xi_1$  and  $f\xi_2$ . The stable gap function  $\gamma$  is a Hölder continuous function satisfying the following algebraic equality  $\gamma(\xi_1 : \xi_2) = \gamma(\xi_1 : \xi_3)\gamma(\xi_3 : \xi_2)$ . (We use the notation  $\xi_1 : \xi_2$  rather than  $\xi_1, \xi_2$  just to emphasise that  $\gamma$  will be measuring ratios.) The domain of a stable measure-length ratio cocycle is the set of all stable spanning leaves of Markov rectangles. A *stable* measure-length ratio cocycle is a function  $J = \kappa/(\kappa \circ f)$  where  $\kappa$  is a positive Hölder continuous function satisfying a set of algebraic inequalities given by (7.1). The unstable gap functions and measure-length ratio cocycle are defined similarly. In the case of Smale horseshoes the domains of the gap functions and of the measure-length ratio cocycle functions are Cantor sets. In the case of codimension one attractors the domains of the stable gap functions and of the stable measurelength ratio cocycle functions are a finite union of intervals, and the stable cocycle-gap pairs have to satisfy the cocycle-gap property (see Definiton 7.4 and Lemma 7.9) which is due to the fact that the Markov rectangles have common boundaries along the stable laminations. The stable and unstable cocycle-gap pairs give rise to an infinite dimensional metric space, that we denote respectively by  $\mathcal{J}G^s(\nu, \delta_s)$  and  $\mathcal{J}G^u(\nu, \delta_u)$ .

**Theorem 1.8.** (i) (Smale horseshoes) There is a natural map

$$g \to (\gamma_s(g), J_s(g), \gamma_u(g), J_u(g))$$

which gives a one-to-one correspondence between  $C^{1+}$  conjugacy classes of diffeomorphisms g contained in  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  and stable and unstable cocycle-gap pairs contained in  $\mathcal{J}G^s(\nu, \delta_s) \times \mathcal{J}G^u(\nu, \delta_u)$ .

(ii) (Codimension one attractors) There is a natural map  $g \to (\gamma_s(g), J_s(g))$  which gives a oneto-one correspondence between  $C^{1+}$  conjugacy classes of diffeomorphisms g contained in  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, 1)$  and stable cocycle-gap pairs contained in  $\mathcal{J}G^s(\nu, \delta_s)$ .

1.1. Self-renormalisable structures. In Section 4, we construct  $C^{1+}$  stable and unstable selfrenormalisable structures on train-tracks. The train-tracks are a form of optimal local leaf-quotient space of the stable and unstable laminations of  $\Lambda$ . Locally, these train-tracks are just the quotient space of stable or unstable leaves within a Markov rectangle, but globally the identification of leaves common to two more than one rectangle gives a non-trivial structure and introduces junctions. They are characterised by being the compact quotient on which the Markov map induced by the action of f is continuous with the minimal number of identifications. A smooth structure on the stable or unstable leaves of  $\Lambda$  induces a smooth structure on the corresponding train-tracks and vice-versa. Now, we use that the holonomies of codimension one hyperbolic systems are  $C^{1+}$  (see [25]), and so the holonomies also project in the train-tracks and together with the Markov maps give rise to what we call self-renormalisable  $C^{1+}$  structures (see Figures 1, 8 and 9).

**Remark 1.9.** These structures are called self-renormalisable because the train track **B** has defined on it both a Markov map m and a pseudo group of holonomy maps  $\{h\}$ . For the train tracks arising from Anosov diffeomorphisms of the torus  $\{h\}$  can be identified with a  $C^{1+}$  diffeomorphism g of the circle and the Markov map m defines a renormalisation of g which agrees with the usual one ([29, 30]). The renormalised map Rg is  $C^{1+}$  conjugate to g. Thus these  $C^{1+}$  self-renormalisable structures are in one-to-one correspondance with  $C^{1+}$  fixed points of the the circle map renormalisation operator. It then follows from the results of this paper that these fixed points are in one-to-one correspondance with those Anosov diffeomorphisms of the the torus which preserve an a measure that is absolutely continuous with respect to Lebesgue measure. For train tracks arising from non-Anosov diffeomorphisms one gets other interesting renormalisation structures, for example, for interval exchange maps.

In Section 4.7, we prove the following useful equivalence between 2-dimensional dynamics and 1-dimensional dynamics.

**Theorem 1.10.** There is a natural map  $g \to (S_s(g), S_u(g))$  which gives a one-to-one correspondence between  $C^{1+}$  conjugacy classes in  $\mathcal{T}(f, \Lambda)$  and pairs of stable and unstable  $C^{1+}$  self-renormalisable structures.



FIGURE 1. This figure illustrates a (unstable) train track for the Anosov map  $g : \mathbb{R}^2 \setminus (\mathbb{Z}v \times \mathbb{Z}w) \to \mathbb{R}^2 \setminus (\mathbb{Z}v \times \mathbb{Z}w)$  defined by g(x, y) = (x + y, y). The rectangles A and B are the Markov rectangles and the vertical arrows show paths along unstable manifolds from A to A and from B to A. The train track is represented by the pair of circles and the curves below it show the smooth paths through the junction of the two circles which arise from the smooth paths between the rectangles A and B along unstable manifolds. Note that there is no smooth path from B to B even though in this representation of the train track it looks as though there ought to be. This is because there is no unstable manifold running directly from the rectangle B to itself.

Hence, for a pair  $(S_s, S_u)$  of  $C^{1+}$  self-renormalisable structures to be realisable by a  $C^{1+}$  hyperbolic diffeomorphism in  $\mathcal{T}(f, \Lambda)$ , the unstable  $C^{1+}$  self-renormalisable structure does not impose any restriction in the stable  $C^{1+}$  self-renormalisable structure, and *vice-versa*. The same is no longer true if we ask  $g \in \mathcal{T}(f, \Lambda)$  to be a  $C^{1+}$ -Hausdorff realisation of a Gibbs measure as we describe in the next section.

1.2. Realisation of Gibbs measures on train-tracks. We are going to study the  $C^{1+}$ -Hausdorff realisations of Gibbs measures as self-renormalisable structures. Then we use this information to study the  $C^{1+}$ -Hausdorff realisations of Gibbs measures as  $C^{1+}$  hyperbolic diffeomorphisms, going like this from one-dimensional dynamics to two-dimensional dynamics.

Let  $\pi_u: \Theta \to \Theta^u$  and  $\pi_s: \Theta \to \Theta^s$  be respectively the natural projections on the left and right infinite words. Let  $\tau_u: \Theta^u \to \Theta^u$  and  $\tau_s: \Theta^s \to \Theta^s$  be the corresponding right and left shifts. Let  $\iota \in \{s, u\}$  where s denotes stable and u denotes unstable. Let us denote by  $\mathbf{B}^\iota$  the  $\iota$  train-track and by  $i_\iota: \Theta^\iota \to \mathbf{B}^\iota$  the natural marking induced by  $i: \Theta \to \Lambda$  (see Section 4.5). A  $\tau$ -invariant measure  $\nu$  on  $\Theta$  determines a unique  $\tau_\iota$ -invariant measure  $\nu_\iota = (\pi_\iota)_*\nu$  on  $\Theta^\iota$ . Conversely, a  $\tau_\iota$ -invariant measure  $\nu_\iota$  on  $\Theta^\iota$  has an unique natural extension to a  $\tau$ -invariant measure  $\nu$  on  $\Theta$ . We say that a  $\tau_\iota$ -invariant measure  $\nu_\iota$  is a *Gibbs measure* if its natural extension  $\nu$  is a *Gibbs measure* on  $\Theta$ . A  $C^{1+}$  $\iota$  self-renormalisable structure  $\mathcal{S}_\iota$  is a  $C^{1+}$ -Hausdorff realisation of a *Gibbs measure*  $\nu$  on  $\Theta$  if, for every chart (e, U) of the self-renormalisable structure  $\mathcal{S}_\iota$ , the pusforward  $(e \circ i_\iota)_*\nu_\iota$  of the measure  $\nu_\iota$  is a measure absolutely continuous with respect to the Hausdorff measure of the set e(U). Let us denote by  $\delta(\mathcal{S}_\iota)$  the Hausdorff dimension of the set e(U) for every chart (e, U) of the smooth structure of  $\mathcal{S}_\iota$ . Clearly, this is independent of the chart (e, U). We denote by  $\mathcal{D}^\iota(\nu, \delta)$  the set of all  $C^{1+}$   $\iota$  self-renormalisable structures  $\mathcal{S}_\iota$  with  $\delta(\mathcal{S}_\iota) = \delta$  and which are  $C^{1+}$ -Hausdorff realisations of the Gibbs measure  $\nu$  on  $\Theta$ . By Theorem 6.1, every  $C^{1+} \iota$  self-renormalisable structure  $\mathcal{S}_{\iota}$  is a  $C^{1+}$ -Hausdorff realisation of an unique Gibbs measure  $\nu_{\mathcal{S}_{\iota}}$  on  $\Theta$ .

**Theorem 1.11.** The map  $g \to (\mathcal{S}_s(g), \mathcal{S}_u(g))$  gives a 1-1 correspondence between  $C^{1+}$  conjugacy classes in  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  and pairs in  $\mathcal{D}^s(\nu, \delta_s) \times \mathcal{D}^u(\nu, \delta_u)$ .

Hence, if  $g \in \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  then  $\delta(\mathcal{S}_s(g)) = \delta_s$  and  $\delta(\mathcal{S}_u(g)) = \delta_u$ . Let  $\mathcal{S}_\iota$  be a  $C^{1+} \iota$  selfrenormalisable structure. If  $\delta(\mathcal{S}_\iota) = 1$  we call  $\mathbf{B}^\iota$  a no-gap train-track. If  $0 < \delta(\mathcal{S}_\iota) < 1$  we call  $\mathbf{B}^\iota$ a gap train-track. Let  $\iota'$  denote the element of  $\{s, u\}$  which is not  $\iota \in \{s, u\}$ .

**Theorem 1.12.** Let  $\mathbf{B}^s$  and  $\mathbf{B}^u$  be the stable and unstable train-tracks determined by a  $C^{1+}$  hyperbolic diffeomorphism  $(f, \Lambda)$ . The set  $\mathcal{D}^\iota(\nu, \delta_\iota)$  is non-empty if, and ony if, the  $\iota$ -measure solenoid function  $\sigma_{\nu} : \mathrm{msol}^\iota \to \mathbb{R}^+$  of the Gibbs measure  $\nu$  has the following properties:

(i) If  $\mathbf{B}^{\iota}$  and  $\mathbf{B}^{\iota'}$  are no-gap train-tracks then  $\sigma_{\nu}$  has a non-vanishing Hölder continuous extension to the closure of msol<sup> $\iota$ </sup> satisfying the boundary condition.

(ii) If  $\mathbf{B}^{\iota}$  is a no-gap train-track and  $\mathbf{B}^{\iota'}$  is a gap train-track then  $\sigma_{\nu}$  has a non-vanishing Hölder continuous extension to the closure of  $\mathrm{msol}^{\iota}$ .

(iii) If  $\mathbf{B}^{\iota}$  is a gap train-track and  $\mathbf{B}^{\iota'}$  is a no-gap train-track then  $\sigma_{\nu}$  has a non-vanishing Hölder continuous extension to the closure of msol<sup> $\iota$ </sup> satisfying the cylinder-cylinder condition.

(iv) If  $\mathbf{B}^{\iota}$  and  $\mathbf{B}^{\iota'}$  are gap train-tracks then  $\sigma_{\nu}$  does not have to satisfy any extracondition.

Furthermore,  $\mathcal{D}^{\iota}(\nu, \delta_{\iota}) \neq \emptyset$  if, and only if,  $\mathcal{D}^{\iota'}(\nu, \delta_{\iota'}) \neq \emptyset$ 

By Theorem 10.1, the set of all  $\iota$ -measure solenoid functions  $\sigma_{\nu}$  with the properties indicated in Theorem 1.12 determine an infinite dimensional metric space  $SOL^{\iota}$  which gives a nice parametrization of all Lipschitz conjugacy classes  $\mathcal{D}^{\iota}(\nu, \delta)$  of  $C^{1+}$  self-renormalisable structures  $\mathcal{S}_{\iota}$  with a given Hausdorff dimension  $\delta(\mathcal{S}_{\iota}) = \delta$ . We have used the same notation  $SOL^{\iota}$  as in Section 1 above because we will show that they are effectively the same sets.

**Theorem 1.13.** Let us suppose that  $\mathcal{D}^{\iota}(\nu, \delta) \neq \emptyset$ .

(i) (Flexibility) If  $\mathbf{B}^{\iota}$  is a gap train-track then  $\mathcal{D}^{\iota}(\nu, \delta)$  is an infinite dimensional space parametrized by cocycle-gap pairs contained in  $\mathcal{J}G^{\iota}(\nu, \delta)$ . (ii) (Rigidity) If  $\mathbf{B}^{\iota}$  is a no-gap train-track then  $\mathcal{D}^{\iota}(\nu, 1)$  consists of a single  $C^{1+}$  self-renormalisable structure.

By Lemma 9.1, each set  $\mathcal{D}^{\iota}(\nu, \delta)$  is either empty or a Lipschitz conjugacy class. Hence, if  $\mathbf{B}^{\iota}$  is a no-gap train-track then the Lipschitz conjugacy class consists of a single  $C^{1+}$  self-renormalisable structure. Furthermore, by Lemma 9.5, the set of eigenvalues of all periodic orbits of  $\mathcal{S}_{\iota}$  is a complete invariant of each set  $\mathcal{D}^{\iota}(\nu, \delta)$  (see also [34]).

**Theorem 1.14.** (Rigidity) If  $\delta_{\iota} = 1$ , the mapping  $g \to S_{\iota'}(g)$  gives a 1-1 correspondence between  $C^{1+}$  conjugacy classes in  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  and  $C^{1+}$  self-renormalisable structures in  $\mathcal{D}^{\iota'}(\nu, \delta_{\iota'})$ .

Hence, the map  $g \to S_{\iota'}(g)$  gives rise to the following one-to-one correspondences. For Anosov diffeomorphisms  $(f, \Lambda)$ , there is an one-to-one correspondence between (i)  $C^{1+}$  conjugacy classes of hyperbolic diffeomorphisms  $g \in \mathcal{T}(f, \Lambda)$  which are  $C^{1+}$ -Hausdorff realisations of a Gibbs measure; (ii)  $C^{1+}$  unstable self-renormalisable structures; and (iii)  $C^{1+}$  stable self-renormalisable structures. For codimension one attractors  $(f, \Lambda)$ , there is a one-to-one correspondence between (i)  $C^{1+}$  conjugacy classes of hyperbolic diffeomorphisms  $g \in \mathcal{T}(f, \Lambda)$  which are  $C^{1+}$ -Hausdorff realisations of a Gibbs measure; and (ii)  $C^{1+}$  stable self-renormalisable structures. 1.3. Solenoid functions and Gibbs measures. In Section 3, we introduce HR-structures (HR for Hölder-ratios). These associate an affine structure to each stable and unstable leaf segment in such a way that these vary Hölder continuously with the leaf and are kept invariant by the dynamics of f. As we will describe the HR-structures are in one-to-one correspondence with the  $C^{1+}$  conjugacy classes of  $C^{1+}$  hyperbolic diffeomorphisms in  $\mathcal{T}(f, \Lambda)$ . The HR-structures are characterized by a pair  $(r_s, r_u)$  of ratio functions. The ratio functions  $r_s$  and  $r_u$  are independent one from another. However, if we ask that the  $C^{1+}$  hyperbolic diffeomorphism have an associated geometric measure then we will see later that the ratio function  $r_u$  imposes restrictions on the ratio function  $r_s$  and vice-versa.

For  $\iota \in \{s, u\}$ , we construct a topological set  $\mathbf{S}^{\iota}$  which is either isomorphic to a finite union of closed intervals on the real line or to an embedded Cantor set on the real line. The ratio functions  $r_{\iota}$  when restricted to the set  $\mathbf{S}^{\iota}$  are Hölder continuous functions  $\sigma_{\iota} = r_{\iota} | \mathbf{S}^{\iota}$  completely characterised by a finite set of properties that we explain also in Section 3. We call the functions  $\sigma_{\iota}$  solenoid functions. They form an infinite dimensional metric space which parametrizes the  $C^{1+}$  conjugacy classes of  $C^{1+}$  hyperbolic diffeomorphisms contained in  $\mathcal{T}(f, \Lambda)$  (see Corollary 3.6).

We say that D is a stable leaf n-cylinder segment of  $\Lambda$  if  $f^{-n}D$  is a stable spanning leaf segment of a Markov rectangle. Let  $E_D$  be a stable spanning leaf segment of a Markov rectangle such that  $D \subset E_D$ . Let  $\nu$  be a Gibbs measure and  $\mu = i_*\nu$  the corresponding invariant measure in  $\Lambda$ . Let  $\rho(D \cap \Lambda, E_D \cap \Lambda)$  be the ratio between the measure of  $D \cap \Lambda$  and the measure of  $E_D \cap \Lambda$  with respect to the conditional measure of  $\mu$  in  $E_D$ . The  $\delta$ -stable bounded solenoid class of a Gibbs measure  $\nu$  is the set of all stable solenoid functions  $\sigma$  with the following property: There is  $C = C(\sigma) > 0$  such that for every stable leaf n-cylinder segment of  $\Lambda$  we have

$$\left|\delta_{\iota} \log r(D \cap \Lambda, E_D \cap \Lambda) - \log \rho(D \cap \Lambda, E_D \cap \Lambda)\right| < C ,$$

where r is the stable ratio function determined by the stable solenoid function  $\sigma$  (see also Definition 8.2). The  $\delta$ -unstable bounded solenoid class of a Gibbs measure  $\nu$  is defined in a similar way.

- **Theorem 1.15.** (i) There is a natural map  $g \to (\sigma_s(g), \sigma_u(g))$  which gives a oneto-one correspondence between  $C^{1+}$  conjugacy classes of  $C^{1+}$  hyperbolic diffeomorphisms  $g \in \mathcal{T}(\nu, \delta_s, \delta_u)$  and pairs  $(\sigma_s(g), \sigma_u(g))$  of stable and unstable solenoid functions such that, for  $\iota$  equal to s and  $u, \sigma_\iota(g)$  is contained in the  $\delta_\iota$ -bounded solenoid equivalence class of  $\nu$ .
  - (ii) There is a natural map  $S_{\iota} \to \sigma_{S_{\iota}}$  which gives a one-to-one correspondence between  $C^{1+}$  self-renormalisable structures  $S_{\iota}$  contained in  $\mathcal{D}^{\iota}(\nu, \delta_{\iota})$  and  $\iota$ -solenoid functions  $\sigma_{S_{\iota}}$  contained in the  $\delta_{\iota}$ -bounded equivalence class of  $\nu$ .
  - (iii) Let us suppose that  $\mathcal{D}^{\iota}(\nu, \delta_{\iota}) \neq \emptyset$ .
    - (a) (Rigidity) If  $\delta_{\iota} = 1$  then the  $\delta_{\iota}$ -bounded solenoid equivalence class of  $\nu$  is a singleton consisting in the continuous extension of the  $\iota$  measure solenoid function  $\sigma_{\nu,\iota}$  to  $\mathbf{S}^{\iota}$ .
    - (b) (Flexibility) If  $0 < \delta_{\iota} < 1$  then the  $\delta_{\iota}$ -bounded solenoid equivalence class of  $\nu$  is an infinite dimensional space of solenoid functions.

In Section 7, we use the cocycle-gap pairs to construct explicitly the solenoid functions in the  $\delta_{\iota}$ -bounded equivalence classes of the  $C^{1+}$ -Hausdorff realisable Gibbs measures  $\nu$ .

By Lemma 4.1, given an  $\iota$ -solenoid function  $\sigma_{\iota}$  there is a unique  $C^{1+}$  self-renormalisable structure  $S_{\iota}$  such that  $\sigma_{\iota} = \sigma_{S_{\iota}}$  and, by Theorem 6.1 and Lemma 6.3, there is a unique  $C^{1+}$ -Hausdorff realisable Gibbs measure  $\nu = \nu_{\sigma_{\iota}}$  such that  $S_{\iota} \in \mathcal{D}^{\iota}(\nu, \delta_{\iota})$  with  $\delta_{\iota} = \delta(S_{\iota})$ . Hence, by Theorem 1.15 (ii), given an  $\iota$ -solenoid function  $\sigma_{\iota}$  there is an unique  $C^{1+}$ -Hausdorff realisable Gibbs measure  $\nu$  such that  $\sigma_{\iota}$  belongs to the  $\delta_{\iota}$ -bounded solenoid equivalence class of  $\nu$ .

**Theorem 1.16.** Given an  $\iota$ -solenoid function  $\sigma_{\iota}$  and  $0 < \delta_{\iota'} \leq 1$ , there is a unique Gibbs measure  $\nu$  and a unique  $\delta_{\iota'}$ -bounded equivalence class of  $\nu$  consisting of  $\iota'$ -solenoid functions  $\sigma_{\iota'}$  such that the  $C^{1+}$  conjugacy class of hyperbolic diffeomorphisms  $g \in \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  determined by the pair  $(\sigma_s, \sigma_u)$  have an invariant measure  $\mu = (i_g)_*\nu$  absolutely continuous with respect to the Hausdorff measure.

Putting together Theorem 1.15 and Theorem 1.16, we obtain the following implications:

- (i) (Flexibility for Smale horseshoes) For  $\iota = s$  and u, given a  $\iota$ -solenoid function  $\sigma_{\iota}$  there is an infinite dimensional space of solenoid functions  $\sigma_{\iota'}$  such that the  $C^{1+}$  hyperbolic Smale horseshoes determined by the pairs  $(\sigma_s, \sigma_u)$  have an invariant measure  $\mu$  absolutely continuous with respect to the Hausdorff measure.
- (ii) (Rigidity for Anosov diffeomorphisms) For  $\iota = s$  and u, given an  $\iota$ -solenoid function  $\sigma_{\iota}$  there is an unique  $\iota'$ -solenoid function such that the  $C^{1+}$  Anosov diffeomorphisms determined by the pair  $(\sigma_s, \sigma_u)$  has an invariant measure  $\mu$  absolutely continuous with respect to Lebesgue.
- (iii) (Flexibility for codimension one attractors) Given an unstable solenoid function  $\sigma_u$ there is an infinite dimensional space of stable solenoid functions  $\sigma_s$  such that the  $C^{1+}$  hyperbolic codimension one attractors determined by the pairs ( $\sigma_s, \sigma_u$ ) have an invariant measure  $\mu$  absolutely continuous with respect to the Hausdorff measure.
- (iv) (Rigidity for codimension one attractors) Given an s-solenoid function  $\sigma_s$  there is an unique unstable solenoid function  $\sigma_u$  such that the  $C^{1+}$  hyperbolic codimension one attractors determined by the pair ( $\sigma_s, \sigma_u$ ) have an invariant measure  $\mu$  absolutely continuous with respect to the Hausdorff measure using non-zero stable and unstable pressures.

In this paper we prove, in fact, a generalized version of the results presented in the introduction. The theory that we develop here studies the properties and classifies all the Gibbs measures which are  $C^{1+}$ -realisable as natural geometric measures of  $C^{1+}$  self-renormalisable structures (see Definition 6.1) and of  $C^{1+}$  hyperbolic diffeomorphisms (see Definition 8.3). The set of natural geometric measures which are absolutely continuous with respect to Hausdorff measure.

The proof of all theorems stated in the introduction are given in Section 10.

#### 2. Hyperbolic diffeomorphisms

In this section, we present some basic facts on hyperbolic dynamics, that we include for clarity of the exposition.

2.1. Interval notation. We also use the notation of interval arithmetic for some inequalities where:

- (i) if I and J are intervals then I + J, I.J and I/J have the obvious meaning as intervals,
- (ii) if  $I = \{x\}$  then we often denote I by x, and
- (iii)  $I \pm \varepsilon$  denotes the interval consisting of those x such that  $|x y| < \varepsilon$  for all  $y \in I$ .

By  $\phi(n) \in 1 \pm \mathcal{O}(\nu^n)$  we mean that there exists a constant c > 0 depending only upon explicitly mentioned quantities such that for all  $n \ge 0$ ,  $1 - c\nu^n < \phi(n) < 1 + c\nu^n$ . By  $\phi(n) = \mathcal{O}(\nu^n)$  we mean that there exists a constant  $c \ge 1$  depending only upon explicitly mentioned quantities such that for all  $n \ge 0$ ,  $c^{-1}\nu^n \le \phi(n) \le c\nu^n$ .

2.2. Stable and unstable superscripts. Throughout the paper we will use the following notation: we use  $\iota$  to denote an element of the set  $\{s, u\}$  of the stable and unstable superscripts and  $\iota'$ to denote the element of  $\{s, u\}$  that is not  $\iota$ . In the main discussion we will often refer to objects which are qualified by  $\iota$  such as, for example, an  $\iota$ -leaf: This means a leaf which is a leaf of the stable lamination if  $\iota = s$ , or a leaf of the unstable lamination if  $\iota = u$ . In general the meaning should be quite clear.

We define the map  $f_{\iota} = f$  if  $\iota = u$  or  $f_{\iota} = f^{-1}$  if  $\iota = s$ .

2.3. Leaf segments. Let d be a metric on M. For  $\iota \in \{s, u\}$ , if  $x \in \Lambda$  we denote the local  $\iota$ -manifolds through x by

$$W^{\iota}(x,\varepsilon) = \left\{ y \in M : d(f_{\iota}^{-n}(x), f_{\iota}^{-n}(y)) \le \varepsilon, \text{ for all } n \ge 0 \right\}.$$

By the Stable Manifold Theorem (see [11]), these sets are respectively contained in the stable and unstable immersed manifolds

$$W^{\iota}(x) = \bigcup_{n \ge 0} f^{n}_{\iota} \left( W^{\iota} \left( f^{-n}_{\iota}(x), \varepsilon_{0} \right) \right)$$

which are the image of a  $C^{1+\gamma}$  immersion  $\kappa_{\iota,x} : \mathbb{R} \to M$ . An open (resp. closed) full  $\iota$ -leaf segment I is defined as a subset of  $W^{\iota}(x)$  of the form  $\kappa_{\iota,x}(I_1)$  where  $I_1$  is a non-empty open (resp. closed) subinterval in  $\mathbb{R}$ . An open (resp. closed)  $\iota$ -leaf segment is the intersection with  $\Lambda$  of an open (resp. closed) full  $\iota$ -leaf segment such that the intersection contains at least two distinct points. If the intersection is exactly two points we call this closed  $\iota$ -leaf segment an  $\iota$ -leaf gap. A full  $\iota$ -leaf segment is either an open or closed full  $\iota$ -leaf segment are the points  $\kappa_{\iota,x}(u)$  and  $\kappa_{\iota,x}(v)$  where u and v are the endpoints of a full  $\iota$ -leaf segment I are the points of the minimal closed full  $\iota$ -leaf segment I has empty interior if, and only if, it is an  $\iota$ -leaf gap. A map  $c: I \to \mathbb{R}$  is an  $\iota$ -leaf chart of an  $\iota$ -leaf segment I if has an extension  $c_E: I_E \to \mathbb{R}$  to a full  $\iota$ -leaf segment  $I_E$  with the following properties:  $I \subset I_E$  and  $c_E$  is a homeomorphism onto its image.

2.4. Smoothness. In this paper, when we say that a map, atlas or structure is  $C^r$  we include the case  $C^{k+}$  where k is a positive integer. For maps f this means that f is  $C^{k+\alpha}$  for some  $0 < \alpha < 1$ , i.e.  $C^k$  with  $\alpha$ -Hölder continuous kth-order derivatives. For an atlas or structure by  $C^{k+}$  we mean that each pair of charts in the atlas or structure are  $C^{k+\alpha}$  compatible for some  $0 < \alpha < 1$  where the  $\alpha$  might depend upon the charts. In the case of an atlas, we suppose that (i) one can choose  $\alpha$  to be independent of the charts and (ii) the overlap maps have  $C^{k+\alpha}$  norm bounded independent of the charts considered. This is immediately verified if the number of charts contained in the  $C^{k+}$  atlas is finite. Thus a  $C^{k+}$  atlas is  $C^{k+\alpha}$ , for some  $0 < \alpha < 1$ . This is not the case for  $C^{k+}$  structures.

2.5. Topological and smooth conjugacies. Let  $(f, \Lambda)$  be a  $C^{1+}$  hyperbolic diffeomorphism. Somewhat unusually we also desire to highlight the  $C^{1+}$  structure on M in which f is a diffeomorphism. By a  $C^{1+}$  structure on M we mean a maximal set of charts with open domains in M such that the union of their domains cover M and whenever U is an open subset contained in the domains of any two of these charts i and j then the overlap map  $j \circ i^{-1} : i(U) \to j(U)$  is  $C^{1+\alpha}$ , where  $\alpha > 0$  depends on i, j and U. We note that by compactness of M, given such a  $C^{1+}$  structure on M, there is an atlas consisting of a finite set of these charts which cover M and for which the overlap maps are  $C^{1+\alpha}$  compatible and uniformly bounded in the  $C^{1+\alpha}$  norm, where  $\alpha > 0$  just depends upon the atlas. We denote by  $C_f$  the  $C^{1+}$  structure on M in which f is a diffeomorphism. Usually one is not concerned with this as, given two such structures, there is a homeomorphism of M sending one onto the other and thus, from this point of view, all such structures can be identified. For our discussion it will be important to maintain the identity of the different smooth structures on M. We say that a map  $h : \Lambda_f \to \Lambda_g$  is a *topological conjugacy* between two  $C^{1+}$  hyperbolic diffeomorphisms  $(f, \Lambda_f)$  and  $(g, \Lambda_g)$  if there is a homeomorphism  $h : \Lambda_f \to \Lambda_g$  with the following properties:

(i)  $g \circ h(x) = h \circ f(x)$  for every  $x \in \Lambda_f$ .

(ii) The pull-back of the  $\iota$ -leaf segments of g by h are  $\iota$ -leaf segments of f.

We say that a topological conjugacy  $h : \Lambda_f \to \Lambda_g$  is a *Lipschitz conjugacy* if h has a bi-Lipschitz homeomorphic extension to an open neighbourhood of  $\Lambda_f$  in the surface M (with respect to the  $C^{1+}$  structures  $C_f$  and  $C_g$ , respectively).

Similarly, we say that a topological conjugacy  $h : \Lambda_f \to \Lambda_g$  is a  $C^{1+}$  conjugacy if h has a  $C^{1+\alpha}$  diffeomorphic extension to an open neighbourhood of  $\Lambda_f$  in the surface M, for some  $\alpha > 0$ .

Our approach is to fix a  $C^{1+}$  hyperbolic diffeomorphism  $(f, \Lambda)$  and consider  $C^{1+}$  hyperbolic diffeomorphism  $(g_1, \Lambda_{g_1})$  topologically conjugate to  $(f, \Lambda)$ . The topological conjugacy  $h : \Lambda \to \Lambda_{g_1}$ between f and  $g_1$  extends to a homeomorphism H defined on a neighbourhood of  $\Lambda$ . Then, we obtain the new  $C^{1+}$ -realization  $(g_2, \Lambda_{g_2})$  of f defined as follows: (i) the map  $g_2 = H^{-1} \circ g_1 \circ H$ ; (ii) the basic set is  $\Lambda_{g_2} = H^{-1} | \Lambda_{g_1}$ ; (iii) the  $C^{1+}$  structure  $C_{g_2}$  is given by the pull-back  $(H)_* C_{g_1}$  of the  $C^{1+}$  structure  $C_{g_1}$ . From (i) and (ii), we get that  $\Lambda_{g_2} = \Lambda$  and  $g_2 | \Lambda = f$ . From (iii), we get that  $g_2$ is  $C^{1+}$  conjugated to  $g_1$ . Hence, to study the conjugacy classes of  $C^{1+}$  hyperbolic diffeomorphisms  $(f, \Lambda)$  of f, we can just consider the  $C^{1+}$  hyperbolic diffeomorphisms  $(g, \Lambda_g)$  with  $\Lambda_g = \Lambda$  and  $g | \Lambda = f | \Lambda$ , which we will do from now on for simplicity of our exposition.

2.6. Rectangles. Since  $\Lambda$  is a hyperbolic invariant set of a diffeomorphism  $f : M \to M$ , for  $0 < \varepsilon < \varepsilon_0$  there is  $\delta = \delta(\varepsilon) > 0$  such that, for all points  $w, z \in \Lambda$  with  $d(w, z) < \delta$ ,  $W^u(w, \varepsilon)$  and  $W^s(z, \varepsilon)$  intersect in a unique point that we denote by [w, z]. Since we assume that the hyperbolic set has a *local product structure*, we have that  $[w, z] \in \Lambda$ . Furthermore, the following properties are satisfied: (i) [w, z] varies continuously with  $w, z \in \Lambda$ ; (ii) the bracket map is continuous on a  $\delta$ -uniform neighbourhood of the diagonal in  $\Lambda \times \Lambda$ ; and (iii) whenever both sides are defined f([w, z]) = [f(w), f(z)]. Note that the bracket map does not really depend on  $\delta$  provided it is sufficiently small.

Let us underline that it is a standing hypothesis that all the hyperbolic sets considered here have such a local product structure.

A rectangle R is a subset of  $\Lambda$  which is (i) closed under the bracket i.e.  $x, y \in R \implies [x, y] \in R$ , and (ii) proper i.e. is the closure of its interior in  $\Lambda$ . This definition imposes that a rectangle has always to be proper which is more restrictive than the usual one which only insists on the closure condition.

If  $\ell^s$  and  $\ell^u$  are respectively stable and unstable leaf segments intersecting in a single point then we denote by  $[\ell^s, \ell^u]$  the set consisting of all points of the form [w, z] with  $w \in \ell^s$  and  $z \in \ell^u$ . We note that if the stable and unstable leaf segments  $\ell$  and  $\ell'$  are closed then the set  $[\ell, \ell']$  is a rectangle. Conversely in this 2-dimensional situations, any rectangle R has a product structure in the following sense: for each  $x \in R$  there are closed stable and unstable leaf segments of  $\Lambda$ ,  $\ell^s(x, R) \subset W^s(x)$ and  $\ell^u(x, R) \subset W^u(x)$  such that  $R = [\ell^s(x, R), \ell^u(x, R)]$ . The leaf segments  $\ell^s(x, R)$  and  $\ell^u(x, R)$ are called stable and unstable spanning leaf segments for R (see Figure 2). For  $\iota \in \{s, u\}$ , we denote by  $\partial \ell^\iota(x, R)$  the set consisting of the endpoints of  $\ell^\iota(x, R)$ , and we denote by  $\operatorname{int} \ell^\iota(x, R)$ the set  $\ell^\iota(x, R) \setminus \partial \ell^\iota(x, R)$ . The interior of R is given by  $\operatorname{int} R = [\operatorname{int} \ell^s(x, R), \operatorname{int} \ell^u(x, R)]$ , and the boundary of R is given by  $\partial R = [\partial \ell^s(x, R), \ell^u(x, R)] \bigcup [\ell^s(x, R), \partial \ell^u(x, R)]$ .

2.7. Markov partitions. A Markov partition of f is a collection  $\mathcal{R} = \{R_1, \ldots, R_k\}$  of rectangles such that (i)  $\Lambda \subset \bigcup_{i=1}^k R_i$ ; (ii)  $R_i \cap R_j = \partial R_i \cap \partial R_j$  for all i and j; (iii) if  $x \in int R_i$  and  $fx \in int R_j$  then

(a) 
$$f(\ell^s(x, R_i)) \subset \ell^s(fx, R_j)$$
 and  $f^{-1}(\ell^u(fx, R_j)) \subset \ell^u(x, R_i)$ 



FIGURE 2. A rectangle.

(b)  $f(\ell^u(x, R_i)) \cap R_j = \ell^u(fx, R_j)$  and  $f^{-1}(\ell^s(fx, R_j)) \cap R_i = \ell^s(x, R_i)$ .

The last condition means that  $f(R_i)$  goes across  $R_j$  just once. In fact, it follows from condition (a) providing the rectangles  $R_j$  are chosen sufficiently small (see [16]). The rectangles making up the Markov partition are called *Markov rectangles*.

We note that there is a Markov partition  $\mathcal{R}$  of f with the following *disjointness property* (see [2, 20, 32]):

- (i) if  $0 < \delta_{f,s} < 1$  and  $0 < \delta_{f,u} < 1$  then the stable and unstable leaf boundaries of any two Markov rectangles do not intesect.
- (ii) if  $0 < \delta_{f,\iota} < 1$  and  $\delta_{f,\iota'} = 1$  then the  $\iota'$ -leaf boundaries of any two Markov rectangles do not intersect except, possibly, at their endpoints.

If  $\delta_{f,s} = \delta_{f,u} = 1$ , the disjointness property does not apply and so we consider that it is trivially satisfied for every Markov partition. For simplicity of our exposition, we will just consider Markov partitions satisfying the disjointness property.

2.8. Marking the invariant set  $\Lambda$ . The properties of the Markov partition  $\mathcal{R} = \{R_1, \ldots, R_k\}$  of f imply the existence of an unique two-sided subshift  $\tau$  of finite type  $\Theta = \Theta_A$  and a continuous surjection  $i: \Theta \to \Lambda$  such that (a)  $f \circ i = i \circ \tau$  and (b)  $i(\Theta_j) = R_j$  for every  $j = 1, \ldots, k$ . We call such a map  $i: \Theta \to \Lambda$  a marking of a  $C^{1+}$  hyperbolic diffeomorphism  $(f, \Lambda)$ .

As we have explained before a  $C^{1+}$  hyperbolic diffeomorphism  $(f, \Lambda)$  admits always a marking which is not necessarily unique.

2.9. Leaf *n*-cylinders and leaf *n*-gaps. For  $\iota = s$  or u, an  $\iota$ -leaf primary cylinder of a Markov rectangle R is a spanning  $\iota$ -leaf segment of R. For  $n \ge 1$ , an  $\iota$ -leaf *n*-cylinder of R is an  $\iota$ -leaf segment I such that

- (i)  $f_{\iota}^{n}I$  is an  $\iota$ -leaf primary cylinder of a Markov rectangle M;
- (ii)  $f_{\iota}^{n}\left(\ell^{\iota'}(x,R)\right) \subset M$  for every  $x \in I$ .

For  $n \geq 2$ , an  $\iota$ -leaf n-gap G of R is an  $\iota$ -leaf gap  $\{x, y\}$  in a Markov rectangle R such that n is the smallest integer such that both leaves  $f_{\iota}^{n-1}\ell^{\iota'}(x, R)$  and  $f_{\iota}^{n-1}\ell^{\iota'}(y, R)$  are contained in  $\iota'$ -boundaries of Markov rectangles; An  $\iota$ -leaf primary gap G is the image  $f_{\iota}G'$  by  $f_{\iota}$  of an  $\iota$ -leaf 2-gap G'.

We note that an  $\iota$ -leaf segment I of a Markov rectangle R can be simultaneously an  $n_1$ -cylinder,  $(n_1 + 1)$ -cylinder, ...,  $n_2$ -cylinder of R if  $f^{n_1}(I)$ ,  $f^{n_1+1}(I)$ , ...,  $f^{n_2}(I)$  are all spanning  $\iota$ -leaf segments. Furthermore, if I is an  $\iota$ -leaf segment contained in the common boundary of two Markov rectangles  $R_i$  and  $R_j$  then I can be an  $n_1$ -cylinder of  $R_i$  and an  $n_2$ -cylinder of  $R_j$  with  $n_1$  distinct of  $n_2$ . If  $G = \{x, y\}$  is an  $\iota$ -gap of R contained in the interior of R then there is a unique n such that G is an n-gap. However, if  $G = \{x, y\}$  is contained in the common boundary of two Markov



FIGURE 3. A basic stable holonomy from I to J.

rectangles  $R_i$  and  $R_j$  then G can be an  $n_1$ -gap of  $R_i$  and an  $n_2$ -gap of  $R_j$  with  $n_1$  distinct of  $n_2$ . Since the number of Markov rectangles  $R_1, \ldots, R_k$  is finite, there is  $C \ge 1$  such that, in all the above cases for cylinders and gaps we have  $|n_2 - n_1| \le C$ .

We say that a leaf segment K is the *i*-th mother of an n-cylinder or an n-gap J of R if  $J \subset K$ and K is a leaf (n-i)-cylinder of R. We denote K by  $m^i J$ .

By the properties of a Markov partition, the smallest full  $\iota$ -leaf  $\hat{K}$  containing a leaf *n*-cylinder K of a Markov rectangle R is equal to the union of all smallest full  $\iota$ -leaves containing either a leaf (n + j)-cylinder or a leaf (n + i)-gap of R, with  $i \in \{1, \ldots, j\}$ , contained in K.

2.10. Metric on  $\Lambda$ . We say that a rectangle R is an  $(n_s, n_u)$ -rectangle if there is  $x \in R$  such that, for  $\iota = s$  and u, the spanning leaf segments  $\ell^{\iota}(x, R)$  are either an  $\iota$ -leaf  $n_{\iota}$ -cylinder or the union of two such cylinders with a common endpoint.

The reason for allowing the possibility of the spanning leaf segments being inside two touching cylinders is to allow us to regard geometrically very small rectangles intersecting a common boundary of two Markov rectangles to be small in the sense of having  $n_s$  and  $n_u$  large.

If  $x, y \in \Lambda$  and  $x \neq y$  then  $d_{\Lambda}(x, y) = 2^{-n}$  where *n* is the biggest integer such that both *x* and *y* are contained in an  $(n_s, n_u)$ -rectangle with  $n_s \geq n$  and  $n_u \geq n$ . Similarly if *I* and *J* are *ι*-leaf segments then  $d_{\Lambda}(I, J) = 2^{-n_{\iota'}}$  where  $n_{\iota} = 1$  and  $n_{\iota'}$  is the biggest integer such that both *I* and *J* are contained in an  $(n_s, n_u)$ -rectangle.

2.11. **Basic holonomies.** Suppose that x and y are two points inside any rectangle R of  $\Lambda$ . Let  $\ell(x, R)$  and  $\ell(y, R)$  be two stable leaf segments respectively containing x and y and inside R. Then we define  $\theta : \ell(x, R) \to \ell(y, R)$  by  $\theta(w) = [w, y]$ . Such maps are called the *basic stable holonomies* (see Figure 3). They generate the pseudo-group of all stable holonomies. Similarly we define the basic unstable holonomies.

By Theorem 2.1 in [25], the holonomy  $\theta : \ell^{\iota}(x, R) \to \ell^{\iota}(y, R)$  has a  $C^{1+\alpha}$  extension to the leaves containing  $\ell^{\iota}(x, R)$  and  $\ell^{\iota}(y, R)$ , for some  $\alpha > 0$ .

2.12. Foliated lamination atlas. In this section when we refer to a  $C^r$  object r is allowed to take the values  $k + \alpha$  where k is a positive integer and  $0 < \alpha \leq 1$ . Two  $\iota$ -leaf charts i and j are  $C^r$  compatible if whenever U is an open subset of an  $\iota$ -leaf segment contained in the domains of i and j then  $j \circ i^{-1} : i(U) \to j(U)$  extends to a  $C^r$  diffeomorphism of the real line. Such maps are called *chart overlap maps*. A *bounded*  $C^r \iota$ -*lamination atlas*  $\mathcal{A}^{\iota}$  is a set of such charts which (a) cover  $\Lambda$ , (b) are pairwise  $C^r$  compatible, and (c) the chart overlap maps are uniformly bounded in the  $C^r$  norm.

Let  $\mathcal{A}^{\iota}$  be a bounded  $C^{1+\alpha}$   $\iota$ -lamination atlas, with  $0 < \alpha \leq 1$ . If  $i: I \to \mathbb{R}$  is a chart in  $\mathcal{A}^{\iota}$  defined on the leaf segment I and K is a leaf segment in I then we define  $|K|_i$  to be the length of the minimal closed interval containing i(K). Since the atlas is bounded, if  $j: J \to \mathbb{R}$  is another chart in  $\mathcal{A}^{\iota}$  defined on the leaf segment J which contains K then the ratio between the lengths  $|K|_i$  and  $|K|_j$  is universally bounded away from 0 and  $\infty$ . If  $K' \subset I \bigcap J$  is another such segment then

we can define the ratio  $r_i(K : K') = |K|_i/|K'|_i$ . Although this ratio depends upon *i*, the ratio is exponentially determined in the sense that if *T* is the smallest segment containing both *K* and *K'* then

$$r_j(K:K') \in (1 \pm \mathcal{O}(|T|_i^{\alpha})) r_i(K:K')$$
.

This follows from the  $C^{1+\alpha}$  smoothness of the overlap maps and Taylor's Theorem.

A  $C^r$  lamination atlas  $\mathcal{A}^{\iota}$  has bounded geometry (i) if f is a  $C^r$  diffeomorphism with  $C^r$  norm uniformly bounded in this atlas; (ii) if for all pairs  $I_1, I_2$  of  $\iota$ -leaf n-cylinders or  $\iota$ -leaf n-gaps with a common point, we have that  $r_i(I_1 : I_2)$  is uniformly bounded away from 0 and  $\infty$  with the bounds being independent of i,  $I_1, I_2$  and n; and (iii) for all endpoints x and y of an  $\iota$ -leaf n-cylinder or  $\iota$ -leaf n-gap I, we have that  $|I|_i \leq \mathcal{O}\left((d_{\Lambda}(x,y))^{\beta}\right)$  and  $d_{\Lambda}(x,y) \leq \mathcal{O}\left(|I|_i^{\beta}\right)$ , for some  $0 < \beta < 1$ , independent of i, I and n.

A  $C^r$  bounded lamination atlas  $\mathcal{A}^{\iota}$  is  $C^r$  foliated (i) if  $\mathcal{A}^{\iota}$  has bounded geometry; and (ii) if the basic holonomies are  $C^r$  and have a  $C^r$  norm uniformly bounded in this atlas, except possibly for the dependence upon the rectangles defining the basic holonomy. A bounded lamination atlas  $\mathcal{A}^{\iota}$  is  $C^{1+}$  foliated if  $\mathcal{A}^{\iota}$  is  $C^r$  foliated for some r > 1.

2.13. Foliated atlas  $\mathcal{A}^{\iota}(g,\rho)$ . Let  $g \in \mathcal{T}(f,\Lambda)$  and  $\rho = \rho_g$  be a  $C^{1+}$  Riemannian metric on the manifold containing  $\Lambda$ . The  $\iota$ -lamination atlas  $\mathcal{A}^{\iota}(g,\rho)$  determined by  $\rho$  is the set of all maps  $e: I \to \mathbb{R}$  where  $I = \Lambda \cap \hat{I}$  with  $\hat{I}$  a full  $\iota$ -leaf segment, such that e extends to an isometry between the induced Riemannian metric on  $\hat{I}$  and the Euclidean metric on the reals. We call the maps  $e \in \mathcal{A}^{\iota}(\rho)$  the  $\iota$ -lamination charts. If I is an  $\iota$ -leaf segment (or a full  $\iota$ -leaf segment) then by  $|I|_{\rho}$  we mean the length in the Riemannian metric  $\rho$  of the minimal full  $\iota$ -leaf containing I. By Theorem 2.2 in [25], the lamination atlas  $\mathcal{A}^{\iota}(g,\rho)$  is  $C^{1+}$  foliated for  $\iota = \{s, u\}$ .

# 3. Solenoid functions

In this section, we construct the stable and unstable solenoid functions, and we prove an equivalence between  $C^{1+}$  hyperbolic diffeomorphisms and pairs of stable and unstable solenoid functions.

3.1. **HR-Hölder ratios.** A *HR-structure* associates an affine structure to each stable and unstable leaf segment in such a way that these vary Hölder continuously with the leaf and are invariant under f.

An affine structure on a stable or unstable leaf is equivalent to a ratio function r(I:J) which can be thought of as prescribing the ratio of the size of two leaf segments I and J in the same stable or unstable leaf. A ratio function r(I:J) is positive (we recall that each leaf segment has at least two distinct points) and continuous in the endpoints of I and J. Moreover,

(3.1) 
$$r(I:J) = r(J:I)^{-1} \text{ and } r(I_1 \cup I_2:K) = r(I_1:K) + r(I_2:K)$$

provided  $I_1$  and  $I_2$  intersect at most in one of their endpoints.

We say that r is a  $\iota$ -ratio function if (i) for all  $\iota$ -leaf segments K, r(I:J)  $(I, J \subset K)$  defines a ratio function on K; (ii) r is invariant under f, i.e. r(I:J) = r(fI:fJ) for all  $\iota$ -leaf segments; and (iii) for every basic  $\iota$ -holonomy map  $\theta: I \to J$  between the leaf segment I and the leaf segment J defined with respect to a rectangle R and for every  $\iota$ -leaf segment  $I_0 \subset I$  and every  $\iota$ -leaf segment or gap  $I_1 \subset I$ ,

(3.2) 
$$\left|\log\frac{r(\theta I_0:\theta I_1)}{r(I_0:I_1)}\right| \le \mathcal{O}\left((d_{\Lambda}(I,J))^{\varepsilon}\right)$$

where  $\varepsilon \in (0, 1)$  depends upon r and the constant of proportionality also depends upon R, but not on the segments considered.

A *HR-structure* is a pair  $(r_s, r_u)$  consisting of a stable and an unstable ratio function.



FIGURE 4. The embedding  $e: I \to \mathbb{R}$ .

3.2. Realised ratio functions. Let  $(g, \Lambda) \in \mathcal{T}(f, \Lambda)$  and let  $\mathcal{A}(g, \rho)$  be an  $\iota$ -lamination atlas which is  $C^{1+}$  foliated. Let  $|I| = |I|_{\rho}$  for every  $\iota$ -leaf segment I. By hyperbolicity of g in  $\Lambda$ , there are  $0 < \nu < 1$  and C > 0 such that for all  $\iota$ -leaf segments I and all  $m \ge 0$  we get  $|g_{\iota'}^m I| \le C\nu^m |I|$ . Thus, using the mean value theorem and the fact that  $g_{\iota}$  is  $C^r$ , for all short leaf segments K and all leaf segments I and J contained in it, the  $\iota$ -realised ratio function  $r_{g,\iota}$  given by

$$\begin{aligned} r_{g,\iota}(I:J) &= \lim_{n \to \infty} \frac{|g_{\iota'}^n I|}{|g_{\iota'}^n J|} \\ &= \frac{|g_{\iota'}^m I|}{|g_{\iota'}^m J|} \prod_{n=m}^{\infty} \left( \frac{|g_{\iota'}^{n+1} I|}{|g_{\iota'}^{n+1} J|} \frac{|g_{\iota'}^n J|}{|g_{\iota'}^n I|} \right) \\ &\in \frac{|g_{\iota'}^m I|}{|g_{\iota'}^m J|} \prod_{n=m}^{\infty} \left( 1 \pm \mathcal{O}\left(\nu^n |K|^{\alpha}\right) \right) \\ &\subset \frac{|g_{\iota'}^m I|}{|g_{\iota'}^m J|} \left( 1 \pm \mathcal{O}\left(\nu^n |K|^{\alpha}\right) \right) \end{aligned}$$

is well-defined, where  $\alpha = \min\{1, r-1\}$ . This construction gives the HR-structure on  $\Lambda$  determined by g.By [24], we get the following equivalence:

**Theorem 3.1.** The map  $g \to (r_{g,s}, r_{g,u})$  determines a one-to-one correspondence between  $C^{1+}$  conjugacy classes in  $\mathcal{T}(f, \Lambda)$  and HR-structures.

3.3. Lamination atlas. Given an  $\iota$ -ratio function r, we define the embeddings  $e: I \to \mathbb{R}$  by

(3.3) 
$$e(x) = r(\ell(\xi, x), \ell(\xi, R))$$

where  $\xi$  is an endpoint of the  $\iota$ -leaf segment I and R is a Markov rectangle containing  $\xi$  (see Figure 4). For this definition it is not necessary that R contains I. We denote the set of all these embeddings e by  $\mathcal{A}(r)$ .

The embeddings e of  $\mathcal{A}(r)$  have overlap maps with affine extensions, therefore the atlas  $\mathcal{A}(r)$  extends to a  $C^{1+\alpha}$  lamination structure  $\mathcal{L}(r)$ . By Proposition 4.2 in [25], we obtain that  $\mathcal{A}(r)$  is a  $C^{1+\alpha}$  foliated atlas.

Let  $g \in \mathcal{T}(f, \Lambda)$  and  $\mathcal{A}(g, \rho)$  a  $C^{1+}$  foliated  $\iota$ -lamination atlas determined by a Riemmanian metric  $\rho$ . Putting together Proposition 2.5 and Proposition 3.5 of [24], we get that the overlap map  $e_1 \circ e_2^{-1}$  between a chart  $e_1 \in \mathcal{A}(g, \rho)$  and a chart  $e_2 \in \mathcal{A}(r_{g,\iota})$  has a  $C^{1+}$  diffeomorphic extension to the reals. Therefore, the atlasses  $\mathcal{A}(g, \rho)$  and  $\mathcal{A}(r_{g,\iota})$  determine the same  $C^{1+}$  foliated  $\iota$ -lamination.



FIGURE 5. The *f*-matching condition for  $\iota$ -leaf segments.

In particular, for all short leaf segments K and all leaf segments I and J contained in it, we obtain that

$$r_{g,\iota}(I:J) = \lim_{n \to \infty} \frac{|g_{\iota'}^n I|_{\rho}}{|g_{\iota'}^n J|_{\rho}} = \lim_{n \to \infty} \frac{|g_{\iota'}^n I|_{i_n}}{|g_{\iota'}^n J|_{i_n}}$$

where  $i_n$  is any chart in  $\mathcal{A}(r_{q,\iota})$  containing the segment  $g_{\iota'}^n K$  in its domain.

3.4. Realised solenoid functions. For  $\iota = s$  and u, let  $\mathbf{S}^{\iota}$  denote the set of all ordered pairs (I, J) of  $\iota$ -leaf segments with the following properties:

- (i) The intersection of I and J consists of a single endpoint.
- (ii) if  $\delta_{f,\iota} = 1$  then I and J are primary  $\iota$ -leaf cylinders.
- (iii) if  $0 < \delta_{f,\iota} < 1$  then  $f_{\iota'}I$  is an  $\iota$ -leaf 2-cylinder of a Markov rectangle R and  $f_{\iota'}J$  is an  $\iota$ -leaf 2-gap also of the same Markov rectangle R.

(See Section 2.6 for the definitions of leaf cylinders and gaps). Pairs (I, J) where both are primary cylinders are called *leaf-leaf pairs*. Pairs (I, J) where J is a gap are called *leaf-gap pairs* and in this case we refer to J as a *primary gap*. The set  $\mathbf{S}^{\iota}$  has a very nice topological structure. If  $\delta_{f,\iota'} = 1$  then the set  $\mathbf{S}^{\iota}$  is isomorphic to a finite union of intervals, and if  $\delta_{f,\iota'} < 1$  then the set  $\mathbf{S}^{\iota}$ is isomorphic to an embedded Cantor set.

We define a pseudo-metric  $d_{\mathbf{S}^{\iota}}: \mathbf{S}^{\iota} \times \mathbf{S}^{\iota} \to \mathbb{R}^+$  on the set  $\mathbf{S}^{\iota}$  by

$$d_{\mathbf{S}^{\iota}}\left(\left(I,J\right),\left(I',J'\right)\right) = \max\left\{d_{\Lambda}\left(I,I'\right),d_{\Lambda}\left(J,J'\right)\right\} \ .$$

Let  $g \in \mathcal{T}(f, \Lambda)$ . For  $\iota = s$  and u, we call the restriction of an  $\iota$ -ratio function  $r_{g,\iota}$  to  $\mathbf{S}^i$  a realised solenoid function  $\sigma_{g,\iota}$ . By construction, for  $\iota = s$  and u, the restriction of an  $\iota$ -ratio function to  $\mathbf{S}^i$  gives an Hölder continuous function satisfying the matching condition, the boundary condition and the cylinder-gap condition as we now proceed to describe.

3.5. Hölder continuity of solenoid functions. This means that for t = (I, J) and t' = (I', J')in  $\mathbf{S}^{\iota}$ ,  $|\sigma_{\iota}(t) - \sigma_{\iota}(t')| \leq \mathcal{O}\left((d_{\mathbf{S}^{\iota}}(t, t'))^{\alpha}\right)$ . The Hölder continuity of  $\sigma_{g,\iota}$  and the compactness of its domain imply that  $\sigma_{g,\iota}$  is bounded away from zero and infinity.

3.6. Matching condition. Let  $(I, J) \in \mathbf{S}^{\iota}$  be a pair of primary cylinders and suppose that we have pairs

 $(I_0, I_1), (I_1, I_2), \dots, (I_{n-2}, I_{n-1}) \in \mathbf{S}^{\iota}$ 

of primary cylinders such that  $f_{\iota}I = \bigcup_{j=0}^{k-1} I_j$  and  $f_{\iota}J = \bigcup_{j=k}^{n-1} I_j$ . Then

$$\frac{|f_{\iota}I|}{|f_{\iota}J|} = \frac{\sum_{j=0}^{k-1} |I_j|}{\sum_{j=k}^{n-1} |I_j|} = \frac{1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} |I_i|/|I_{i-1}|}{\sum_{j=k}^{n-1} \prod_{i=1}^{j} |I_i|/|I_{i-1}|}$$



FIGURE 6. The boundary condition for  $\iota$ -leaf segments.

Hence, noting that  $g|\Lambda = f|\Lambda$ , the realised solenoid function  $\sigma_{g,\iota}$  must satisfy the matching condition (see Figure 5) for all such leaf segments:

(3.4) 
$$\sigma_{g,\iota}(I:J) = \frac{1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} \sigma_{g,\iota}(I_i:I_{i-1})}{\sum_{j=k}^{n-1} \prod_{i=1}^{j} \sigma_{g,\iota}(I_i:I_{i-1})}$$

3.7. Boundary condition. If the stable and unstable leaf segments have Hausdorff dimension equal to 1, then leaf segments I in the boundaries of Markov rectangles can sometimes be written as the union of primary cylinders in more than one way. This gives rise to the existence of a boundary condition that the realised solenoid functions have to satisfy as we pass to explain.

If J is another leaf segment adjacent to the leaf segment I then the value of |I|/|J| must be the same whichever decomposition we use. If we write  $J = I_0 = K_0$  and I as  $\bigcup_{i=1}^m I_i$  and  $\bigcup_{j=1}^n K_j$ where the  $I_i$  and  $K_j$  are primary cylinders with  $I_i \neq K_j$  for all i and j, then the above two ratios are

$$\sum_{i=1}^{m} \prod_{j=1}^{i} \frac{|I_j|}{|I_{j-1}|} = \frac{|I|}{|J|} = \sum_{i=1}^{n} \prod_{j=1}^{i} \frac{|K_j|}{|K_{j-1}|}$$

Thus, noting that  $g|\Lambda = f|\Lambda$ , a realised solenoid function  $\sigma_{g,\iota}$  must satisfy the following boundary condition (see Figure 6) for all such leaf segments:

(3.5) 
$$\sum_{i=1}^{m} \prod_{j=1}^{i} \sigma_{g,\iota} \left( I_j : I_{j-1} \right) = \sum_{i=1}^{n} \prod_{j=1}^{i} \sigma_{g,\iota} \left( K_j : K_{j-1} \right)$$

3.8. Scaling function. If the  $\iota$ -leaf segments have Hausdorff dimension less than one and the  $\iota'$ -leaf segments have Hausdorff dimension equal to 1, then a primary cylinder I in the  $\iota$ -boundary of a Markov rectangle can also be written as the union of gaps and cylinders of other Markov rectangles. This gives rise to the existence of a cylinder-gap condition that the  $\iota$ -realised solenoid functions have to satisfy.

Before defining the cylinder-gap condition, we will introduce the scaling function that will be useful to express the cylinder-gap condition, and also, in Definitions 3.2, the bounded equivalence classes of solenoid functions and, in Definition 8.2, the  $(\delta, P)$ -bounded solenoid equivalence classes of a Gibbs measure.

Let  $scl^{\iota}$  be the set of all pairs (K, J) of  $\iota$ -leaf segments with the following properties:

- (i) K is a leaf  $n_1$ -cylinder or an  $n_1$ -gap segment for some  $n_1 > 1$ ;
- (ii) J is a leaf  $n_2$ -cylinder or an  $n_2$ -gap segment for some  $n_2 > 1$ ; (iii)  $m^{n_1-1}K$  and  $m^{n_2-1}J$  are the same primary cylinder.

**Lemma 3.2.** Every function  $\sigma_{\iota} : \mathbf{S}^{\iota} \to \mathbb{R}^+$  has a canonical extension  $s_{\iota}$  to scl<sup> $\iota$ </sup>. Furthermore, if  $\sigma_{\iota}$  is the restriction of a ratio function  $r_{\iota}|\mathbf{S}^{\iota}$  to  $\mathbf{S}^{\iota}$  then  $s_{\iota} = r_{\iota}|\mathrm{scl}^{\iota}$ .

**Remark 3.3.** The above map  $s_{\iota} : \operatorname{scl}^{\iota} \to \mathbb{R}^+$  is the scaling function determined by the solenoid function  $\sigma_{\iota} : \mathbf{S}^{\iota} \to \mathbb{R}^+$ .

Proof of Lemma 3.2. We are going to give an explicit construction of a realised scaling function  $s_{g,\iota}$  from a realised solenoid function  $\sigma_{g,\iota}$  with the property that  $s_{g,\iota} = r_{g,\iota} |\mathrm{scl}^i$  where  $r_{g,\iota}$  is a ratio function, i.e for every  $(K, J) \in \mathrm{scl}^{\iota}$  we have

$$s_{g,\iota}(K,J) = \lim_{k \to \infty} \frac{|g_{\iota'}^k K|_{\rho}}{|g_{\iota'}^k J|_{\rho}}$$

where  $\mathcal{A}(g,\rho)$  is an *i*-lamination atlas.

This construction is canonical and applies to every function  $\sigma_{\iota} : \mathbf{S}^{\iota} \to \mathbb{R}^+$  determining a canonical extension  $s_{\iota} : \operatorname{scl}^{\iota} \to \mathbb{R}^+$  of  $\sigma_{\iota}$ .

Let us proceed to construct the  $\iota$ -scaling function  $s : S^{\iota} \to \mathbb{R}^+$  from an  $\iota$ -solenoid function  $\sigma$ . Suppose that J is an  $\iota$ -leaf n-cylinder or n-gap. Then there are pairs

$$(I_0, I_1), (I_1, I_2), \dots, (I_{l-1}, I_l) \in \mathbf{S}^d$$

such that  $mJ = \bigcup_{j=0}^{l} f_{\iota'}^{n-1}I_j$  and  $J = f_{\iota'}^{n-1}I_s$  for some  $0 \le s \le l$ . Let us denote  $f_{\iota'}^{n-1}I_j$  by  $I'_j$  for every  $0 \le s \le l$ . Then

$$\frac{|mJ|}{|J|} = \sum_{j=0}^{l} \frac{|I'_j|}{|I'_s|} = 1 + \sum_{j=0}^{s-1} \prod_{i=s}^{j+1} \frac{|I'_{i-1}|}{|I'_i|} + \sum_{j=s+1}^{l} \prod_{i=s}^{j-1} \frac{|I'_{i+1}|}{|I'_i|} ,$$

where the first sum above is empty if s = 0, and the second sum above is empty if s = 1. Therefore, we define the extension s from  $\sigma$  to the pairs (mJ, J) by

$$s(mJ,J) = 1 + \sum_{j=0}^{s-1} \prod_{i=s}^{j+1} \sigma(I_{i-1},I_i) + \sum_{j=s+1}^{l} \prod_{i=s}^{j-1} \sigma(I_{i+1},I_i) + \sum_{j=s+1}^{l} \prod_{i=s}^{l} \sigma(I_{i+1},I_i) + \sum_{j=s+1}^{l} \prod_{i=s+1}^{l} \prod_{i=s$$

where the first sum above is empty if s = 0, and the second sum above is empty if s = 1. For every  $(K, J) \in \operatorname{scl}^{\iota}$  there is a primary leaf segment I such that  $m^{m_1}K = I = m^{m_2}J$  for some  $m_1 \ge 1$  and  $m_2 \ge 1$ . Then,

$$\frac{|K|}{|J|} = \prod_{j=1}^{m_1} \frac{|m^j J|}{|m^{j-1}J|} \prod_{j=1}^{m_2} \frac{|m^{j-1}K|}{|m^j K|}$$

Therefore, we define the extension s to (K, J) by

$$s(K,J) = \prod_{j=1}^{m_1} s(m^j J, m^{j-1} J) \prod_{j=1}^{m_2} s(m^{j-1} K, m^j K)$$

Hence, we have constructed a scaling function s from  $\sigma$  on the set scl<sup> $\iota$ </sup> such that if  $\sigma$  is the restriction of a ratio function  $r_{g,\iota}|\mathbf{S}^{\iota}$  to  $\mathbf{S}^{\iota}$  then  $s = r_{g,\iota}|\mathrm{scl}^{\iota}.\square$ 

3.9. Cylinder-gap condition. Let (I, K) be a leaf-gap pair such that the primary cylinder I is the  $\iota$ -boundary of a Markov rectangle  $R_1$ . Then the primary cylinder I intersects another Markov rectangle  $R_2$  giving rise to the existence of a cylinder-gap condition that the realised solenoid functions have to satisfy as we proceed to explain. Take the smallest  $l \ge 0$  such that  $f_{\iota'}^l I \cup f_{\iota'}^l K$  is contained in the intersection of the boundaries of two Markov rectangles  $M_1$  and  $M_2$ . Let  $M_1$  be the Markov rectangle with the property that  $M_1 \cap f_{\iota'}^l R_1$  is a rectangle with non-empty interior (and so  $M_2 \cap f_{\iota'}^l R_2$  also has non-empty interior). Then, for some positive n, there are distinct n-cylinder and gap leaf segments  $J_1, \ldots, J_m$  contained in a primary cylinder of  $M_2$  such that  $f_{\iota'}^l K = J_m$  and



FIGURE 7. The cylinder-gap condition for  $\iota$ -leaf segments.

the smallest full  $\iota$ -leaf segment containing  $f_{\iota'}^l I$  is equal to the union  $\bigcup_{i=1}^{m-1} \hat{J}_i$ , where  $\hat{J}_i$  is the smallest full  $\iota$ -leaf segment containing  $J_i$ . Hence,

$$\frac{|f_{\iota'}^l I|}{|f_{\iota'}^l K|} = \sum_{i=1}^{m-1} \frac{|J_i|}{|J_m|} \; .$$

Hence, noting that  $g|\Lambda = f|\Lambda$ , a realised solenoid function  $\sigma_{g,\iota}$  must satisfy the cylinder-gap condition (see Figure 7) for all such leaf segments:

$$\sigma_{g,\iota}(I,K) = \sum_{i=1}^{m-1} s_{g,\iota}(J_i,J_m)$$

where  $s_{q,\iota}$  is the scaling function determined by the solenoid function  $\sigma_{q,\iota}$ .

3.10. Solenoid functions. Now, we are ready to present the definition of an  $\iota$ -solenoid function.

**Definition 3.1.** An Hölder continuous function  $\sigma_{\iota} : \mathbf{S}^{\iota} \to \mathbb{R}^+$  is an  $\iota$ -solenoid function if  $\sigma_{\iota}$ satisfies the matching condition, the boundary condition and the cylinder-gap condition.

We denote by  $\mathcal{P}S(f)$  the set of pairs  $(\sigma_s, \sigma_u)$  of stable and unstable solenoid functions.

**Remark 3.4.** Let  $\sigma_{\iota} : \mathbf{S}^{\iota} \to \mathbb{R}^+$  be an  $\iota$ -solenoid function. The matching, the boundary and the cylinder-gap conditions are trivially satisfied except in the following cases:

- (i) The matching condition if  $\delta_{f,\iota} = 1$ .
- (ii) The boundary condition if  $\delta_{f,s} = \delta_{f,u} = 1$ . (iii) The cylinder-gap condition if  $\delta_{f,\iota} < 1$  and  $\delta_{f,\iota'} = 1$ .

**Lemma 3.5.** The map  $r_{\iota} \to r_{\iota} | \mathbf{S}^{\iota}$  gives a one-to-one correspondence between  $\iota$ -ratio functions and *i*-solenoid functions.

*Proof.* Every  $\iota$ -ratio function restricted to the set  $\mathbf{S}^{\iota}$  determines an  $\iota$ -solenoid function  $r_{\iota}|\mathbf{S}^{\iota}$ . Now we prove the converse. Since the solenoid functions are continuous and their domains are compact they are bounded away from 0 and  $\infty$ . By this boundedness and the f-matching condition of the solenoid functions and by iterating the domains  $\mathbf{S}^s$  and  $\mathbf{S}^u$  of the solenoid functions backward and forward by f, we determine the ratio functions  $r^s$  and  $r^u$  at very small (and large) scales, such that f leaves the ratios invariant. Then, using the boundedness again, we extend the ratio functions to all pairs of small adjacent leaf segments by continuity. By the boundary condition and the cylinder-gap condition of the solenoid functions, the ratio functions are well determined at the boundaries of the Markov rectangles. Using the Hölder continuity of the solenoid function, we deduce inequality (3.2).

The set  $\mathcal{P}S(f)$  of all pairs  $(\sigma_s, \sigma_u)$  has a natural metric. Combining Theorem 3.1 with Theorem 3.5, we obtain that the set  $\mathcal{P}S(f)$  forms a moduli space for the  $C^{1+}$  conjugacy classes of  $C^{1+}$ hyperbolic diffeomorphisms  $q \in \mathcal{T}(f, \Lambda)$ :

**Corollary 3.6.** The map  $g \to (r_{g,s}|\mathbf{S}^s, r_{g,u}|\mathbf{S}^u)$  determines a one-to-one correspondence between  $C^{1+}$  conjugacy classes of  $g \in \mathcal{T}(f, \Lambda)$  and pairs of solenoid functions in  $\mathcal{P}S(f)$ .

**Definition 3.2.** We say that any two  $\iota$ -solenoid functions  $\sigma_1 : \mathbf{S}^{\iota} \to \mathbb{R}^+$  and  $\sigma_2 : \mathbf{S}^{\iota} \to \mathbb{R}^+$  are in the same bounded equivalence class if the corresponding scaling functions  $s_1 : \operatorname{scl}^{\iota} \to \mathbb{R}^+$  and  $s_2 : \operatorname{scl}^{\iota} \to \mathbb{R}^+$  satisfy the following property: There is C > 0 such that for every  $\iota$ -leaf (i + 1)cylinder or (i + 1)-gap J

(3.6) 
$$\left|\log s_1(J, m^i J) - \log s_2(J, m^i J)\right| < C$$
.

Later, in Lemma 9.2, we prove that two  $C^{1+}$  hyperbolic diffeomorphisms  $g_1$  and  $g_2$  are Lipschitz conjugate if, and only if, the solenoid functions  $s_{g_1,\iota}$  and  $s_{g_2,\iota}$  are in the same bounded equivalence class for  $\iota$  equal to s and u.

#### 4. Self-renormalisable structures

In this section, we construct the stable and unstable self-renormalisable structures living in 1dimensional spaces, and we prove an equivalence between  $C^{1+}$  hyperbolic diffeomorphisms and pairs of stable and unstable self-renormalisable structures.

4.1. **Train-tracks.** Roughly speaking train-tracks are the optimal leaf-quotient spaces on which the stable and unstable Markov maps induced by the action of f on leaf segments are local home-omorphisms.

For each Markov rectangle R let  $t_R^{\iota}$  be the set of  $\iota'$ -segments of R. Thus by the local product structure one can identify  $t_R^{\iota}$  with any spanning  $\iota$ -leaf segment  $\ell^{\iota}(x, R)$  of R.

We form the space  $\mathbf{B}^{\iota}$  by taking the disjoint union  $\bigsqcup_{R \in \mathcal{R}} t_R^{\iota}$  (union over all Markov rectangles R of the Markov partition  $\mathcal{R}$ ) and identifying two points  $I \in t_R^{\iota}$  and  $J \in t_{R'}^{\iota}$  if either (i) the  $\iota'$ -leaf segments I and J are  $\iota'$ -boundaries of Markov rectangles and their intersection contains at least a point which is not an endpoint of I or J or (ii) there is a sequence  $I = I_1, \ldots, I_n = J$  such that all  $I_i, I_{i+1}$  are both identified in the sense of (i). This space is called the  $\iota$ -train-track and is denoted  $\mathbf{B}^{\iota}$ .

Let  $\pi_{\mathbf{B}^{\iota}} : \bigsqcup_{R \in \mathcal{R}} R \to \mathbf{B}^{\iota}$  be the natural projection sending  $x \in R$  to the point in  $\mathbf{B}^{\iota}$  represented by  $\ell^{\iota'}(x, R)$ . A topologically regular point I in  $\mathbf{B}^{\iota}$  is a point with an unique preimage under  $\pi_{\mathbf{B}^{\iota}}$ (i.e. the pre-image of I is not a union of distinct  $\iota'$ -boundaries of Markov rectangles). If a point has more than one preimage by  $\pi_{\mathbf{B}^{\iota}}$  then we call it a *junction*. Since there are only a finite number of  $\iota'$ -boundaries of Markov rectangles there are only finitely many junctions (see Figures 1).

Let  $d_{\iota}$  be the metric on  $\mathbf{B}^{\iota}$  defined as follows: if  $\xi, \eta \in \mathbf{B}^{\iota}, d_{\mathbf{B}^{\iota}}(\xi, \eta) = d_{\Lambda}(\xi, \eta)$ .

4.2. Train-track segments and charts. We say that  $I_T$  is a train-track segment if there is an  $\iota$ -leaf segment I, not intersecting  $\iota$ -boundaries of Markov rectangles, such that  $\pi_{\iota}|I$  is an injection and  $\pi_{\iota}(I) = I_T$ . Let  $\mathcal{A}$  be an  $\iota$ -lamination atlas (take for instance  $\mathcal{A}$  equal to  $\mathcal{A}^{\iota}(f, \rho)$  or to  $\mathcal{A}(r_{f,\iota})$ ). The chart  $i: I \to \mathbb{R}$  in  $\mathcal{A}$  determines a train-track chart  $i_T: I_T \to \mathbb{R}$  for  $I_T$  given by  $i_T = i \circ \pi_{\iota}^{-1}$ . We denote by  $\mathcal{B}$  the set of all train-track charts for all train-track segments determined by  $\mathcal{A}$ .

Two train-track charts  $(i_T, I_T)$  and  $(j_T, J_T)$  on the train-track  $\mathbf{B}^i$  are  $C^{1+}$ -compatible if the overlap map  $j_T \circ i_T^{-1} : i_T(I_T \cap J_T) \to j_T(I_T \cap J_T)$  has a  $C^{1+}$  extension. A  $C^{1+}$  atlas  $\mathcal{B}$  is a set of  $C^{1+}$ -compatible charts with the following property: For every short train-track segment  $K_T$  there is a chart  $(i_T, I_T) \in \mathcal{B}$  such that  $K_T \subset I_T$ . A  $C^{1+}$  structure  $\mathcal{S}$  on  $\mathbf{B}^i$  is a maximal set of  $C^{1+}$ -compatible charts with a given atlas  $\mathcal{B}$  on  $\mathbf{B}^i$ . We say that two  $C^{1+}$  structures  $\mathcal{S}$  and  $\mathcal{S}'$  are in the same Lipschitz equivalence class if for every chart in  $\mathcal{S}$  and every chart in  $\mathcal{S}'$  the overlap map  $e_1 \circ e_2^{-1}$  has a bi-Lipschitz extension.

Given any train-track charts  $i_T : I_T \to \mathbb{R}$  and  $j_T : J_T \to \mathbb{R}$  in  $\mathcal{B}$ , the overlap map  $j_T \circ i_T^{-1} : i_T(I_T \cap J_T) \to j_T(I_T \cap J_T)$  is equal to  $j_T \circ i_T^{-1} = j \circ h \circ i^{-1}$  where  $i = i_T \circ \pi_\iota : I \to \mathbb{R}$  and  $j = j_T \circ \pi_\iota : J \to \mathbb{R}$  are charts in  $\mathcal{A}$ , and

$$h: i^{-1}(i_T(I_T \cap J_T)) \to j^{-1}(j_T(I_T \cap J_T))$$

is a basic  $\iota$ -holonomy. Let us denote by  $\mathcal{B}^{\iota}(g,\rho)$  and  $\mathcal{B}(r_{g,\iota})$  the train-track atlasses determined respectively by  $\mathcal{A}^{\iota}(g,\rho)$  and  $\mathcal{A}(r_{g,\iota})$  with  $g \in \mathcal{T}(f,\Lambda)$ . Since  $\mathcal{A}^{\iota}(g,\rho)$  and  $\mathcal{A}(r_{g,\iota})$  are  $C^{1+}$  foliated atlases, there is  $\eta > 0$  such that, for all train-track charts  $i_T$  and  $j_T$  in  $\mathcal{B}^{\iota}(g,\rho)$  (or in  $\mathcal{B}(r_{g,\iota})$ ), the overlap maps  $j_T \circ i_T^{-1} = j \circ h \circ i^{-1}$  have  $C^{1+\eta}$  diffeomorphic extensions with a uniform bound for the  $C^{1+\eta}$  norm. Hence,  $\mathcal{B}^{\iota}(g,\rho)$  and  $\mathcal{B}(r_{g,\iota})$  are  $C^{1+\eta}$  atlas.

4.3. Markov maps. The Markov map  $\tau_{\iota} : \mathbf{B}^{\iota} \to \mathbf{B}^{\iota}$  is the mapping induced by the action of f on leaf segments i.e. it is defined as follows: if  $I \in \mathbf{B}^{\iota}$ ,  $\tau_{\iota}I = \pi_{\mathbf{B}^{\iota}}(f_{\iota}I)$  is the  $\iota'$ -leaf segment containing the  $f_{\iota}$ -image of the  $\iota'$ -leaf segment I (for simplicity of notation we use the same symbols for the Markov maps as for the shift maps). This map  $\tau_{\iota}$  is a local homeomorphism because  $f_{\iota}$  sends a short  $\iota$ -leaf segment homeomorphically onto a short  $\iota$ -leaf segment. For simplicity of notation, we will denote  $\tau_{\iota}$  by  $f_{\iota}$  through the paper.

Given a topological chart (e, U) on the train-track  $\mathbf{B}^{\iota}$  and a train-track segment  $C \subset U$ , we denote by  $|C|_e$  the lenght of the smallest interval containing e(C). We say that  $f_{\iota}$  has bounded geometry in a  $C^{1+}$  atlas  $\mathcal{B}$  if there is  $\kappa_1 > 0$  such that, for every *n*-cylinder  $C_1$  and *n*-cylinder or *n*-gap  $C_2$  with a common endpoint with  $C_1$ , we have  $\kappa_1^{-1} < |C_1|_e/|C_2|_e < \kappa_1$ , where the lengths are measured in any chart (e, U) of the atlas such that  $C_1 \cup C_2 \subset U$ . Hence there is  $\kappa_2 > 0$  and  $0 < \nu < 1$  such that  $|C|_e \le \kappa_2 \nu^n$  for every *n*-cylinder or *n*-gap *C*. This property is equivalent to the Markov map  $f_{\iota}$  being uniformly expanding in  $\mathbf{B}^{\iota}$ .

Since f on  $\Lambda$  along leaves has affine extensions with respect to the charts in  $\mathcal{A}(r^{\iota})$  and the basic  $\iota$ -bolonomies have  $C^{1+\eta}$  extensions we get that the Markov maps  $\tau_{\iota}$  also have  $C^{1+\eta}$  extensions with respect to the charts in  $\mathcal{B}(r^{\iota})$  for some  $\eta > 0$ . Since  $\mathcal{A}(r^{\iota})$  has bounded geometry, we obtain that  $f_{\iota}$  also has *bounded geometry* in  $\mathcal{B}(r^{\iota})$ . Since, for every  $g \in \mathcal{T}(f, \Lambda)$ , the  $C^{1+\eta}$  lamination atlas  $\mathcal{A}(g, \rho_g)$  has bounded geometry we obtain that the Markov map  $f_{\iota}$  has  $C^{1+\eta}$  extensions with respect to the charts in  $\mathcal{B}(g, \rho_g)$ , for some  $\eta > 0$ , and has bounded geometry.

4.4. Holonomy pseudo-groups on  $\mathbf{B}^{\iota}$ . The elements  $\theta_{\iota} = \theta_{f,\iota}$  of the holonomy pseudo-group on  $\mathbf{B}^{\iota}$  are the mappings defined as follows. Suppose that I and J are  $\iota$ -leaf segments and  $h: I \to J$  a holonomy. Then it follows from the definition of the train-track  $\mathbf{B}^{\iota}$  that the map  $\theta: \pi_{\mathbf{B}^{\iota}}(I) \to \pi_{\mathbf{B}^{\iota}}(J)$  given by  $\theta(\pi_{\mathbf{B}^{\iota}}(x)) = \pi_{\mathbf{B}^{\iota}}(h(x))$  is well-defined. The collection of all such local mappings forms the basic holonomy pseudo-group of  $\mathbf{B}^{\iota}$ . Note that if x is a junction of  $\mathbf{B}^{\iota}$  then there may be segments I and J containing x such that  $I \cap J = \{x\}$ . In this case the image of I and J under the holonomies will not agree in that they will map x differently.

4.5. Markings on train-tracks. For  $\iota = s$  and u, the Markov partition  $\mathcal{R} = \{R_1, \ldots, R_m\}$  for  $(f, \Lambda)$  induces a Markov partition  $\mathcal{R}^{\iota} = \{R_1^{\iota}, \ldots, R_m^{\iota}\}$  for the Markov map  $\tau = \tau_{\iota}$  on the train-track  $\mathbf{B}^{\iota}$ . The marking  $i: \Theta \to \Lambda$  determines unique markings  $i_u: \Theta^u \to \mathbf{B}^u$  and  $i_s: \Theta^s \to \mathbf{B}^s$  such that  $i_u(w_0w_1\ldots) = \bigcap_{i\geq 0} R_{w_i}^u$  and  $i_s(\ldots w_{-1}w_0) = \bigcap_{i\geq 0} R_{w_i}^u$ . We note that  $\pi_{\mathbf{B}^{\iota}} \circ i = i_{\iota} \circ \pi_{\iota}$ . The map  $i_{\iota}$  is continuous, onto  $\mathbf{B}^{\iota}$  and semiconjugates the shift map on  $\Theta^{\iota}$  to the Markov map on  $\mathbf{B}^{\iota}$ . Defining  $\varepsilon, \varepsilon' \in \Theta^{\iota}$  to be equivalent ( $\varepsilon \sim \varepsilon'$ ) if the point  $i^{\iota}(\varepsilon) = i^{\iota}(\varepsilon')$ , we get that the space  $\Theta^{\iota}/\sim$  is homeomorphic to the train-track  $\mathbf{B}^{\iota}$ .

Consider the Markov map  $f_{\iota}$  on  $\mathbf{B}^{\iota}$  induced by the action of f on  $\iota'$ -leaves and described above. For  $n \geq 1$ , an *n*-cylinder is the projection into  $\mathbf{B}^{\iota}$  of an  $\iota$ -leaf *n*-cylinder segment in  $\Lambda$ . Thus, each Markov rectangle in  $\Lambda$  projects in an unique primary  $\iota$ -leaf segment in  $\mathbf{B}^{\iota}$ . For  $n \geq 1$ , an *n*-gap of  $f_{\iota}$  is the projection into  $\mathbf{B}^{\iota}$  of a  $\iota$ -leaf *n*-gap in  $\Lambda$ .



FIGURE 8. A Markov partition for the Smale-shoe f into two rectangles A and B. A representation of the Markov maps  $m_s: \Theta^s \to \Theta^s$  and  $m_u: \Theta^u \to \Theta^u$  for Smale horseshoes.

We say that  $\mathbf{B}^{\iota}$  is a *no-gap train-track* if  $\mathbf{B}^{\iota}$  does not have gaps. Otherwise, we call  $\mathbf{B}^{\iota}$  a *gap train-track*.

4.6. Self-renormalisable structures. The  $C^{1+}$  structure  $S_{\iota}$  on  $\mathbf{B}^{\iota}$  is an  $\iota$  self-renormalisable if it has the following properties:

- (i) in this structure the Markov mapping  $m_{\iota}$  is a local diffeomorphism and has bounded geometry in some  $C^{1+}$  atlas of this structure; and
- (ii) the elements of the basic holonomy pseudo-group are local diffeomorphisms in  $S_{\iota}$ .

We say that  $\mathcal{B}$  is a  $C^{1+}$  self-renormalisable atlas if  $\mathcal{B}$  has bounded geometry and extends to a  $C^{1+}$  self-renormalisable structure. By definition, a  $C^{1+}$  self-renormalisable structure contains a  $C^{1+}$  self-renormalisable atlas.

A  $C^{1+}$  foliated  $\iota$ -lamination atlas  $\mathcal{A}$  induces a  $C^{1+}$   $\iota$  self-renormalisable atlas  $\mathcal{B}$  on  $\mathbf{B}^{\iota}$  (and vice-versa) as follows: The holonomies are  $C^{1+}$  with respect to the atlas  $\mathcal{A}$  and so the charts in  $\mathcal{B}$  are  $C^{1+}$  compatible, and the basic holonomy pseudo-group of  $\mathbf{B}^{\iota}$  are local diffeomorphisms. Since  $\mathcal{A}$  has bounded geometry the Markov mapping  $\tau_{\iota}$  is a local diffeomorphism and also has



FIGURE 9. A representation of the Markov maps  $m_s: \Theta^s \to \Theta^s$  and  $m_u: \Theta^u \to \Theta^u$  as maps of the interval for Anosov diffeomorphisms.

bounded geometry in  $\mathcal{B}$ . Therefore,  $\mathcal{B}$  is a  $C^{1+}$  self-renormalisable atlas and extends to a  $C^{1+}$  self-renormalisable structure  $\mathcal{S}(\mathcal{B})$  on  $\mathbf{B}^{\iota}$ . Since  $\mathcal{A}(r_{\iota})$  and  $\mathcal{A}^{\iota}(g, \rho_g)$  are  $C^{1+}$  foliated  $\iota$ -lamination atlas we obtain that the atlases  $\mathcal{B}(r_{\iota})$  and  $\mathcal{B}^{\iota}(g, \rho_g)$  determine respectively  $C^{1+}$  self-renormalisable structures  $\mathcal{S}(r_{\iota})$  and  $\mathcal{S}(g, \iota)$ .

**Lemma 4.1.** The map  $r_{\iota} \to S(r_{\iota})$  determines a one-to-one correspondence between  $\iota$ -ratio functions (or equivalently,  $\iota$ -solenoid functions  $r_{\iota}|\mathbf{S}^{\iota}$ ) and  $C^{1+}$  self-renormalisable structures on  $\mathbf{B}^{\iota}$ . *Proof.* Every ratio function  $r_{\iota}$  determines an unique  $C^{1+}$  self-renormalisable S. Conversely, let us prove that a given  $C^{1+}$  self-renormalisable structure S on  $\mathbf{B}^{\iota}$  also determines an unique ratio function  $r_{S,\iota}$ . Let  $\mathcal{B}$  be a bounded atlas for S. Consider a small leaf segment K and two leaf segments I and J contained in K. Since the elements of the basic holonomy pseudo-group on  $\mathbf{B}^{\iota}$  are  $C^{1+}$  and the Markov map is also  $C^{1+}$  and has bounded geometry we obtain by Taylor's Theorem that the following limit exists

(4.1) 
$$r_{\mathcal{S},\iota}(I:J) = \lim_{n \to \infty} \frac{|\pi_{\iota} f_{\iota'}^n I|_{i_n}}{|\pi_{\iota} f_{\iota'}^n J|_{i_n}} \\ \in \frac{|\pi_{\iota} I|_{i_0}}{|\pi_{\iota} J|_{i_0}} \left(1 \pm \mathcal{O}(|\pi_{\iota} K|_{i_0}^{\gamma})\right) ,$$

where the size of the leaf segments are measured in charts of the bounded atlas  $\mathcal{B}$ . Furthermore, by [24] and (4.1), the charts in  $\mathcal{B}(r_{\iota})$  and the charts in  $\mathcal{B}$  are  $C^{1+}$  equivalent and so determine the same  $C^{1+}$  self-renormalisable structure.

4.7. Hyperbolic diffeomorphisms. Let  $g \in \mathcal{T}(f, \Lambda)$  and  $\mathcal{A}(g, \rho_g)$  be the  $C^{1+}$  foliated  $\iota$ -lamination atlas determined by the Riemannian metric  $\rho_g$ . As shown in Section 4.6, the atlas  $\mathcal{A}(g, \rho_g)$  induces a  $C^{1+}$  self-renormalisable atlas  $\mathcal{B}(g, \rho_g)$  on  $\mathbf{B}^{\iota}$  which generates a  $C^{1+}$  self-renormalisable structure  $\mathcal{S}(g, \iota)$ .

**Lemma 4.2.** The mapping  $g \to (\mathcal{S}(g,s), \mathcal{S}(g,u))$  gives a 1-1 correspondence between  $C^{1+}$  conjugacy classes in  $\mathcal{T}(f,\Lambda)$  and pairs  $(\mathcal{S}(g,s), \mathcal{S}(g,u))$  of  $C^{1+}$  self-renormalisable structures. Furthermore,  $r_{g,s} = r_{\mathcal{S}(g,s),s}$  and  $r_{g,u} = r_{\mathcal{S}(g,u),u}$ .

Proof. By Lemma 4.1, the pair  $(\mathcal{S}_s, \mathcal{S}_u)$  determines a pair  $(r_{s,\mathcal{S}}|\mathbf{S}^s, r_{u,\mathcal{S}}|\mathbf{S}^u)$  of solenoid functions and vice-versa. By Corollary 3.6, the pair  $(r_{s,\mathcal{S}}|\mathbf{S}^s, r_{u,\mathcal{S}}|\mathbf{S}^u)$  determines an unique  $C^{1+}$  conjugacy class of diffeomorphisms  $g \in \mathcal{T}(f, \Lambda)$  which realise the pair  $(r_{s,\mathcal{S}}|\mathbf{S}^s, r_{u,\mathcal{S}}|\mathbf{S}^u)$  and vice-versa (and so  $(\mathcal{S}(g, s), \mathcal{S}(g, u)) = (\mathcal{S}_s, \mathcal{S}_u)$ ). Furthermore, by Lemma 3.5, we get  $r_{g,s} = r_{\mathcal{S}(g,s),s}$  and  $r_{g,u} = r_{\mathcal{S}(g,u),u}$ .

#### 5. Measure solenoid functions

In this section, we introduce the following new concepts: stable and unstable measure solenoid functions and stable and unstable measure ratio functions. Later, we will use the measure solenoid functions and the measure ratio functions to determine which Gibbs measures are  $C^{1+}$ -realisable by  $C^{1+}$  hyperbolic diffeomorphisms and by  $C^{1+}$  self-renormalisable structures.

5.1. **Gibbs measures.** Let us give the definition of an infinite two-sided subshift of finite type  $\Theta = \Theta(A)$ . The elements of  $\Theta$  are all infinite two-sided words  $w = \ldots w_{-1}w_0w_1\ldots$  in the symbols  $1, \ldots, k$  such that  $A_{w_iw_{i+1}} = 1$ , for all  $i \in \mathbb{Z}$ . Here  $A = (A_{ij})$  is any matrix with entries 0 and 1 such that  $A^n$  has all entries positive for some  $n \geq 1$ . We write  $w \overset{n_1,n_2}{\sim} w'$  if  $w_j = w'_j$  for every  $j = -n_1, \ldots, n_2$ . The metric d on  $\Theta$  is given by  $d(w, w') = 2^{-n}$  if  $n \geq 0$  is the largest such that  $w \overset{n,n}{\sim} w'$ . Together with this metric  $\Theta$  is a compact metric space. The two-sided shift map  $\tau : \Theta \to \Theta$  is the mapping which sends  $w = \ldots w_{-1}w_0w_1\ldots$  to  $v = \ldots v_{-1}v_0v_1\ldots$  where  $v_j = w_{j+1}$  for every  $j \in \mathbb{Z}$ . An  $(n_1, n_2)$ -cylinder  $\Theta_{w_{-n_1}\ldots w_{n_2}}$ , where  $w \in \Theta$ , consists of all those words w' in  $\Theta$  such that  $w \overset{n_1,n_2}{\sim} w'$ . Let  $\Theta^u$  be the set of all right-handed words  $w_0w_1\ldots$  which extend to words  $\ldots w_{-1}w_0\ldots$  in  $\Theta$ . Then  $\pi_u : \Theta \to \Theta^u$  and  $\pi_s : \Theta \to \Theta^s$  are the natural projection given, respectively, by

$$\pi_u(\dots w_{-1}w_0w_1\dots) = w_0w_1\dots$$
 and  $\pi_s(\dots w_{-1}w_0w_1\dots) = \dots w_{-1}w_0$ 



FIGURE 10. An  $\iota$ -admissible pair  $(\xi, C)$  where  $\xi_{\Lambda} = i(\pi_{\iota'}^{-1}\xi)$  and  $C_{\Lambda} = i(\pi_{\iota}^{-1}C)$ .

An *n*-cylinder  $\Theta_{w_0...w_{n-1}}^u$  is equal to  $\pi_u(\Theta_{w_0...w_{n-1}})$  and an *n*-cylinder  $\Theta_{w_{-(n-1)}...w_0}^s$  is equal to  $\pi_s(\Theta_{w_{-(n-1)}...w_0})$ . Let  $\tau_u: \Theta^u \to \Theta^u$  and  $\tau_s: \Theta^s \to \Theta^s$  be the corresponding one-sided shifts.

**Definition 5.1.** For  $\iota = s$  and u, we say that  $s_{\iota} : \Theta^{\iota} \to \mathbb{R}^+$  is an  $\iota$ -measure scaling function if  $s_{\iota}$  is a Hölder continuous function, and for every  $\xi \in \Theta^{\iota}$ 

$$\sum_{\tau_{\iota}\eta=\xi}s_{\iota}(\eta)=1\;,$$

where the sum is upon all  $\xi \in \Theta^{\iota}$  such that  $\tau_{\iota}\eta = \xi$ .

For  $\iota \in \{s, u\}$ , a  $\tau$ -invariant measure  $\nu$  on  $\Theta$  determines a unique  $\tau_{\iota}$ -invariant measure  $\nu_{\iota} = (\pi_{\iota})_* \nu$ on  $\Theta^{\iota}$ . We note that a  $\tau_{\iota}$ -invariant measure  $\nu_{\iota}$  on  $\Theta^{\iota}$  has an unique  $\tau$ -invariant natural extension to an invariant measure  $\nu$  on  $\Theta$  such that  $\nu(\Theta_{w_0...w_{n_2}}) = \nu_{\iota}(\Theta^{\iota}_{w_0...w_{n_2}})$ .

**Definition 5.2.** A  $\tau$ -invariant measure  $\nu$  on  $\Theta$  is a Gibbs measure:

(i) if the function  $s_{\nu,u}: \Theta^u \to \mathbb{R}^+$  given by

$$s_{\nu,u}(w_0w_1\ldots) = \lim_{n \to \infty} \frac{\nu(\Theta_{w_0\ldots w_n})}{\nu(\Theta_{w_1\ldots w_n})} ,$$

is well-defined and it is an u-measure scaling function; and (ii) if the function  $s_{\nu,s}: \Theta^s \to \mathbb{R}^+$  given by

$$s_{\nu,s}(\ldots w_1w_0) = \lim_{n \to \infty} \frac{\nu(\Theta_{w_n \ldots w_0})}{\nu(\Theta_{w_n \ldots w_1})} ,$$

is well-defined and it is a s-measure scaling function.

The following theorem follows from the results proved in [23]. It can also be deduced from standard results about Gibbs states such as those in [2].

**Theorem 5.1.** (Moduli space for Gibbs measures) Let  $s_{\iota} : \Theta^{\iota} \to \mathbb{R}^+$  be an  $\iota$ -measure scaling function for  $\iota = s$  or u. Then there is an unique  $\tau$ -invariant Gibbs measure  $\nu$  such that  $s_{\nu,\iota} = s_{\iota}$ .

5.2. Extended measure scaling function. To present a classification of Gibbs measures  $C^{1+}$ -Hausdorff which are realisable by codimension one attractors, we have to define the cylinder-cylinder condition. We will express the cylinder-cylinder condition, in Section 5.3, using the extended measure scaling functions. These extended measure scaling functions are also used to present, in Section 5.2, the  $\delta_{\iota}$ -bounded solenoid equivalence class of a Gibbs measure.

Throughout the paper, if  $\xi \in \Theta^{\iota'}$ , we denote by  $\xi_{\Lambda}$  the leaf primary cylinder segment  $i(\pi_{\iota'}^{-1}\xi) \subset \Lambda$ . Similarly, if C is an  $n_{\iota}$ -cylinder of  $\Theta^{\iota}$  then we denote by  $C_{\Lambda}$  the  $(1, n_{\iota})$ -rectangle  $i(\pi_{\iota}^{-1}C) \subset \Lambda$ .



FIGURE 11. The (n-j+1)-cylinder leaf segment  $I = \xi_{\Lambda} \cap D_{\Lambda}$  and the primary leaf segment  $f^{n-j}(I) = i(\pi_{\iota'}\tau_{\iota}^{n-j}(\xi.D))$ , where  $D = m_{\iota}^{j-1}C$ .

We say that a  $(n_1, n_2)$ -cylinder  $\theta_{w_{-n_1}...w_{n_2}}$  of  $\Theta$  is an *u*-symbolic leaf  $n_2$ -cylinder if  $n_1 = -\infty$ . Similarly, we say that a  $(n_1, n_2)$ -cylinder  $\theta_{w_{-n_1}...w_{n_2}}$  is a *s*-symbolic leaf  $n_1$ -cylinder if  $n_2 = +\infty$ . Let  $\xi \in \Theta^{\iota'}$  and *C* be a *n*-cylinder of  $\Theta^{\iota}$ . We say that the pair  $(\xi, C)$  is *ι*-admissible if the set

$$\xi.C = \pi_{\iota}^{-1}C \cap \pi_{\iota'}^{-1}\xi$$

is non-empty (see Figure 10). We note that if the pair  $(\xi, C)$  is  $\iota$ -admissible then  $\xi.C$  is an  $\iota$ -symbolic leaf *n*-cylinder, and, conversely, any  $\iota$ -symbolic leaf *n*-cylinder can be expressed as  $\xi.C$  where the pair  $(\xi, C)$  is  $\iota$ -admissible. The set of all  $\iota$ -admissible pairs  $(\xi, C)$  is the  $\iota$ -measure scaling set msc<sup> $\iota$ </sup>.

Let  $C^u = \Theta^u_{w_0...w_{n-1}}$  and  $C^s = \Theta^s_{w_{-(n-1)}...w_0}$  be the *n*-cylinders of  $\Theta^u$  and of  $\Theta^s$ , respectively. For i < n, we denote by  $m^i C^u$  the *i*-th mother  $\Theta^u_{w_0...w_{(n-i-1)}}$  of  $C^u$ . Similarly, we denote by  $m^i C^s$  the *i*-th mother  $\Theta^s_{w_{-(n-i-1)}...w_0}$  of  $C^s$ .

Given an  $\iota$ -measure scaling function s, we construct the extended  $\iota$ -measure scaling function  $\rho : \operatorname{msc}_{\iota} \to \mathbb{R}^+$  of s as follows: If C is a 1-cylinder then we define  $\rho_{\xi}(C) = 1$ . If C is a n-cylinder with  $n \geq 2$ , then we define

$$\rho_{\xi}(C) = \prod_{j=1}^{n-1} s(\pi_{\iota'} \tau_{\iota}^{n-j}(\xi . m_{\iota}^{j-1}C))$$

(see Figure 11). We note that, if  $s_{\nu,s}$  is the stable measure scaling function of a Gibbs measure  $\nu$ , then

$$\rho_{\xi}(C) = \lim_{m \to \infty} \frac{\nu(\pi_s \circ \tau^m(\xi.C))}{\nu(\pi_s \circ \tau^m(\xi))}$$

The unstable case is similar to the one above by taking -m instead of m. Hence,  $\rho_{\xi}(C)$  is the ratio between the measure of  $\xi.C$  and the measure of  $\xi$  with respect to the conditional measure determined by the Gibbs measure  $\nu$  in  $\xi$ .

Recall that a  $\tau$ -invariant measure  $\nu$  on  $\Theta$  determines a unique  $\tau_u$ -invariant measure  $\nu_u = (\pi_u)_* \nu$ on  $\Theta^u$  and a unique  $\tau_s$ -invariant measure  $\nu_s = (\pi_s)_* \nu$  on  $\Theta^s$ . The following theorem follows from [23].

**Theorem 5.2.** (Ratio decomposition of a Gibbs measure) Let  $\rho : \operatorname{msc}_{\iota} \to \mathbb{R}^+$  the an extended *ι*-measure scaling function and  $\nu$  the corresponding  $\tau$ -invariant Gibbs measure. If C is an n-cylinder of  $\Theta^{\iota}$  then

$$\nu_{\iota}(C) = \int_{\xi \in M} \rho_{\xi}(C) \nu_{\iota'}(d\xi).$$

where  $M = \pi_{\iota'} \circ \pi_{\iota}^{-1}C$  a 1-cylinder of  $\Theta^{\iota'}$ . Furthermore, the ratios  $\nu_{\iota}(C)/\rho_{\xi}(C)$  are uniformly bounded away from 0 and  $\infty$ .



FIGURE 12. The cylinder-cylinder condition for  $\iota$ -leaf segments.

5.3. Cylinder-cylinder condition. We introduce the cylinder-cylinder condition that we will use to classify all Gibbs measures that are  $C^{1+}$ -Hausdorff realizable by codimension one attractors.

Similarly to the cylinder-gap condition given in Section 3.9 for a given solenoid function, we are going to construct the cylinder-cylinder condition for a given measure solenoid function  $\sigma_{\nu,\iota}$ . Let  $\delta_{\iota} < 1$  and  $\delta_{\iota'} = 1$ . Let  $(I, J) \in \text{Msol}^{\iota}$  be such that the  $\iota$ -leaf segment  $f_{\iota'}I \cup f_{\iota'}J$  is contained in an  $\iota$ -boundary K of a Markov rectangle  $R_1$ . Then  $f_{\iota'}I \cup f_{\iota'}J$  intersects another Markov rectangle  $R_2$ . Take the smallest  $k \geq 0$  such that  $f_{\iota'}^kI \cup f_{\iota'}^kJ$  is contained in the intersection of the boundaries of two Markov rectangles  $M_1$  and  $M_2$ . Let  $M_1$  be the Markov rectangle with the property that  $M_1 \cap f_{\iota'}^kR_1$  is a rectangle with non empty interior, and so  $M_2 \cap f_{\iota'}^kR_2$  has also non-empty interior. Then, for some positive n, there are distinct  $\iota$ -leaf n-cylinders  $J_1, \ldots, J_m$  contained in a primary cylinder L of  $M_2$  such that  $f_{\iota'}^kI = \bigcup_{i=1}^{p-1}J_i$  and  $f_{\iota'}^kJ = \bigcup_{i=p}^m J_i$ . Let  $\eta \in \Theta^{\iota'}$  be such that  $\eta_{\Lambda} = L$  and, for every  $i = 1, \ldots, m$ , let  $D_i$  be a cylinder of  $\Theta^{\iota}$  such that  $i(\eta.D_i) = J_i$ . Let  $\xi \in \Theta^{\iota'}$  be such that  $\iota$ -extended scaling function  $\rho$  satisfies the cylinder-cylinder condition (see Figure 12) if for all such leaf segments:

$$\frac{\rho_{\xi}(C_2)}{\rho_{\xi}(C_1)} = \frac{\sum_{i=p}^m \rho_{\eta}(D_i)}{\sum_{i=1}^{p-1} \rho_{\eta}(D_i)}$$

5.4. Measure solenoid functions. Let  $\operatorname{Msol}^{\iota}$  be the set of all pairs (I, J) with the following properties: (a) If  $\delta_{\iota} = 1$  then  $\operatorname{Msol}^{\iota} = \mathbf{S}^{\iota}$ . (b) If  $\delta_{\iota} < 1$  then  $f_{\iota'}I$  and  $f_{\iota'}J$  are  $\iota$ -leaf 2-cylinders of a Markov rectangle R such that  $f_{\iota'}I \cup f_{\iota'}J$  is an  $\iota$ -leaf segment, i.e. there is an unique  $\iota$ -leaf 2-gap between them. Let  $\operatorname{msol}^{\iota}$  be the set of all pairs  $(I, J) \in \operatorname{Msol}^{\iota}$  such that the leaf segments I and J are not contained in an  $\iota$ -global leaf containing an  $\iota$ -boundary of a Markov rectangle. By construction, the set  $\operatorname{msol}^{\iota}$  is dense in  $\operatorname{Msol}^{\iota}$ , and for every pair  $(C, D) \subset \operatorname{msol}^{\iota}$  there is an unique  $\psi \in \Theta^{\iota'}$  and an unique  $\xi \in \Theta^{\iota'}$  such that  $i(\pi_{\iota'}^{-1}(\psi)) = C$  and  $i(\pi_{\iota'}^{-1}(\xi)) = D$ . We will denote, in what follows,  $i(\pi_{\iota'}^{-1}(\psi))$  by  $\psi_{\Lambda}$  and  $i(\pi_{\iota'}^{-1}(\xi))$  by  $\xi_{\Lambda}$ .

**Lemma 5.3.** Let  $\nu$  be a Gibbs measure on  $\Theta$ . The s-measure solenoid function  $\sigma_{\nu,s} : \operatorname{msol}^s \to \mathbb{R}^+$ of  $\nu$  and the u-measure solenoid function  $\sigma_{\nu,u} : \operatorname{msol}^u \to \mathbb{R}^+$  of  $\nu$  given by

$$\sigma_{\nu,s}(\psi_{\Lambda},\xi_{\Lambda}) = \lim_{n \to \infty} \frac{\nu(\Theta_{\psi_0\dots\psi_n})}{\nu(\Theta_{\xi_0\dots\xi_n})}$$

and

$$\sigma_{\nu,u}(\psi_{\Lambda},\xi_{\Lambda}) = \lim_{n \to \infty} \frac{\nu(\Theta_{\psi_n...\psi_0})}{\nu(\Theta_{\xi_n...\xi_0})}$$

are both well-defined.

Proof. Let  $(I, J) \in \text{msol}^{\iota}$ . By Property (iii) of  $\text{msol}^{\iota}$ , there is k = k(I, J) such that  $f_{\iota'}^{k}I$  and  $f_{\iota'}^{k}J$  are cylinders with the same mother  $mf_{\iota'}^{k}I = mf_{\iota'}^{k}J$ . Let  $(\xi : C)$  and  $(\xi : D)$  be the admissible pairs in  $\text{msc}_{\iota}$  such that  $i(\xi.C) = f_{\iota'}^{k}I$  and  $i(\xi.D) = f_{\iota'}^{k}J$ . Since the measure  $\nu$  is  $\tau$ -invariant, we obtain that

$$\sigma_{\nu,\iota}(I,J) = \rho_{\xi}(C)\rho_{\xi}(D)^{-1} ,$$
<sup>27</sup>

where  $\rho$  is the extended scaling function determined by the Gibbs measure  $\nu$ . Therefore, the  $\iota$ -measure solenoid function  $\sigma_{\nu,\iota}$  is well-defined for  $\iota \in \{s, u\}$ .

**Lemma 5.4.** Suppose  $\delta_{f,\iota} = 1$ . If an  $\iota$ -measure solenoid function  $\sigma_{\nu,\iota} : \operatorname{msol}^{\iota} \to \mathbb{R}^+$  has a continuous extension to  $\mathbf{S}^{\iota}$  then its extension satisfies the matching condition.

*Proof.* Let  $(J_0, J_1) \in \mathbf{S}^{\iota}$  be a pair of primary cylinders and suppose that we have pairs

$$(I_0, I_1), (I_1, I_2), \dots, (I_{n-2}, I_{n-1}) \in \mathbf{S}$$

of primary cylinders such that  $f_{\iota}J_0 = \bigcup_{j=0}^{k-1} I_j$  and  $f_{\iota}J_1 = \bigcup_{j=k}^{n-1} I_j$ . Since the set msol<sup> $\iota$ </sup> is dense in  $\mathbf{S}^{\iota}$  there are pairs  $(J_0^l, J_1^l) \in \mathrm{msol}^{\iota}$  and pairs  $(I_i^l, I_{i+1}^l)$  with the following properties:

(i)  $f_{\iota}J_0^l = \bigcup_{j=0}^{k-1} I_j^l$  and  $f_{\iota}J_1^l = \bigcup_{j=k}^{n-1} I_j^i$ . (ii) The pair  $(J_0^l, J_1^l)$  converges to  $(J_0, J_1)$  when *i* tends to infinity.

Therefore, for every  $j = 0, \ldots, n-2$  the pair  $(I_j^l, I_{j+1}^l)$  converges to  $(I_j, I_{j+1})$  when i tends to infinity. Since  $\nu$  is a  $\tau$ -invariant measure, we get that the matching condition

$$\sigma_{\nu,\iota}(J_0^l:J_1^l) = \frac{1 + \sum_{j=1}^{k-1} \prod_{i=1}^j \sigma_{\nu,\iota}(I_j^l:I_{i-1}^l)}{\sum_{j=k}^{n-1} \prod_{i=1}^j \sigma_{\nu,\iota}(I_j^l:I_{i-1}^l)}$$

is satisfied for every  $l \geq 1$ . Since the extension of  $\sigma_{\nu,\iota} : \mathrm{msol}^{\iota} \to \mathbb{R}^+$  to the set  $\mathbf{S}^{\iota}$  is continuous, we get that the matching condition also holds for the pairs  $(J_0, J_1)$  and  $(I_0, I_1), \ldots, (I_{n-2}, I_{n-1})$ .

**Remark 5.5.** We say that an  $\iota$ -measure solenoid function  $\sigma_{\nu,\iota}$  of a Gibbs measure  $\nu$  satisfies the cylinder-cylinder condition if the extended scaling function of the Gibbs measure  $\nu$  satisfies the cylinder-cylinder condition.

#### 5.5. Measure ratio functions. We say that $\rho$ is a *i*-measure ratio function if

- (i)  $\rho(I:J)$  is well-defined for every pair of  $\iota$ -leaf segments I and J such that (a) there is an  $\iota$ -leaf segment K such that  $I, J \subset K$ , and (b) I or J has non-empty interior;
- (ii) if I is an  $\iota$ -leaf gap then  $\rho(I:J) = 0$  (and  $\rho(J:I) = +\infty$ );
- (iii) if I and J have non-empty interiors then  $\rho(I:J)$  is strictly positive;
- (iv)  $\rho(I:J) = \rho(J:I)^{-1};$
- (v) if  $I_1$  and  $I_2$  intersect at most in one of their endpoints then  $\rho(I_1 \cup I_2 : K) = \rho(I_1 : K)$  $K) + \rho(I_2:K);$
- (vi)  $\rho$  is invariant under f, i.e.  $\rho(I:J) = r(fI:fJ)$  for all  $\iota$ -leaf segments;
- (vii) for every basic *i*-holonomy map  $\theta: I \to J$  between the leaf segment I and the leaf segment J defined with respect to a rectangle R and for every  $\iota$ -leaf segment  $I_0 \subset I$ and every  $\iota$ -leaf segment or gap  $I_1 \subset I$ ,

(5.1) 
$$\left|\log\frac{\rho(\theta I_0:\theta I_1)}{\rho(I_0:I_1)}\right| \le \mathcal{O}\left((d_{\Lambda}(I,J))^{\varepsilon}\right)$$

where  $\varepsilon \in (0,1)$  depends upon  $\rho$  and the constant of proportionality also depends upon R, but not on the segments considered.

We note that if  $\mathbf{B}^{\iota}$  is a no-gap train-track then an  $\iota$ -measure ratio function is an  $\iota$ -ratio function.

**Remark 5.6.** A function  $\sigma$  : msol<sup> $\iota$ </sup>  $\to \mathbb{R}^+$  that has an Hölder continuous extension to Msol<sup> $\iota$ </sup> determines a unique extended scaling function  $\rho$ , and so we say that  $\sigma$  satisfies the cylinder-cylinder condition if the extended scaling function  $\rho$  satisfies the cylinder-cylinder condition.

Let  $\mathcal{SOL}^{\iota}$  be the space of all Hölder continuous functions  $\sigma_{\iota} : \mathrm{Msol}^{\iota} \to \mathbb{R}^+$  with the following properties:

- (i) If  $\mathbf{B}^{\iota}$  is a no-gap train-track then  $\sigma_{\iota}$  is an  $\iota$ -solenoid function.
- (ii) If  $\mathbf{B}^{\iota}$  is a gap train-track and  $\mathbf{B}^{\iota'}$  is a no-gap train-track then  $\sigma_{\iota}$  satisfies the cylindercylinder condition.
- (iii) If  $\mathbf{B}^{\iota}$  and  $\mathbf{B}^{\iota'}$  are no-gap train-tracks then  $\sigma_{\iota}$  does not have to satisfy any extra property.

**Lemma 5.7.** The map  $\rho \to \rho | \text{Msol}^{\iota}$  determines an one-to-one correspondence between  $\iota$ -measure ratio functions and functions contained in  $SOL^{\iota}$ .

*Proof.* The proof follows similarly to the proof of Lemma 3.5.

- **Remark 5.8.** (i) By Lemma 5.7, a Gibbs measure  $\nu$  with an  $\iota$ -measure solenoid function with an extension  $\hat{\sigma}$  to  $\text{Msol}^{\iota}$  such that  $\hat{\sigma} \in SOL^{\iota}$  determines an unique  $\iota$ -measure ratio function  $\rho_{\nu}$ .
  - (ii) A measure ratio function  $\rho$  determines naturally a measure scaling function, and so, by Lemma 5.1, a Gibbs measure  $\nu_{\rho}$ .
  - (iii) By Lemma 5.7, a function  $\sigma : \text{msol}^{\iota} \to \mathbb{R}^+$  with an extension  $\hat{\sigma}$  to  $\text{Msol}^{\iota}$  such that  $\hat{\sigma} \in SOL^{\iota}$  determines an  $\iota$ -measure ratio function, and, by (ii), a unique Gibbs measure  $\nu$  such that  $\sigma = \sigma_{\nu}$ .

#### 6. NATURAL GEOMETRIC MEASURES

In this section, we define the natural geometric measures  $\mu_{S,\delta}$  associated with a self-renormalisable structure S and  $\delta > 0$ . The natural geometric measures are measures determined by the length scaling structure of the cylinders. We will prove that every natural geometric measure is a pushforward of a Gibbs measure with the property that the measure solenoid function determines a measure ratio function. In Section 8, we will show that a Gibbs measure with the property that its measure solenoid function determines a measure ratio function determines a measure ratio function is  $C^{1+}$ -realisable by a self-renormalisable structure.

**Definition 6.1.** Let S be a  $C^{1+}$  self-renormalisable structure on  $\mathbf{B}^{\iota}$ . If  $\mathbf{B}^{\iota}$  is a gap train-track let  $0 < \delta < 1$ , and if  $\mathbf{B}^{\iota}$  is a no-gap train-track let  $\delta = 1$ .

(i) We say that S has a natural geometric measure  $\mu_{\iota} = \mu_{S,\delta}$  with pressure  $P = P(S,\delta)$  if (a)  $\mu_{\iota}$  is a  $f_{\iota}$ -invariant measure; (b), there exists  $\kappa > 1$  such that for all  $n \ge 1$  and all n-cylinders I of  $\mathbf{B}^{\iota}$ , we have

(6.1) 
$$\kappa^{-1} < \frac{\mu_{\iota}(I)}{|I|_{i}^{\delta} e^{-nP}} < \kappa ,$$

where i is a chart containing I of a bounded atlas  $\mathcal{B}$  of  $\mathcal{S}$ ;

(ii) We say that S is a  $C^{1+}$  realisation of a Gibbs measure  $\nu = \nu_{S,\delta}$  if  $\mu_{\iota} = (i_{\iota})_*\nu_{\iota}$ where  $\nu_{\iota} = (\pi_{\iota})_*\nu$  and  $\mu_{\iota} = \mu_{S,\delta}$  is a natural geometric measure of S.

Suppose that we have a  $C^{1+}$  self-renormalisable structure S on  $\mathbf{B}^{\iota}$  and that  $\mathcal{B}$  is a bounded atlas for it. Let  $\delta > 0$ . If I is a segment in  $\mathbf{B}^{\iota}$ , let  $|I| = |I|_i$  be its length in any chart i of this atlas which contains it. If C is a *m*-cylinder, let us denote m by n(C) and  $i_{\iota}(C)$  by  $I_C$ . For  $m_1 \ge 1$  and  $m_2 \ge 1$ , let C be an  $m_1$ -cylinder and D an  $m_2$ -cylinder contained in the same 1-cylinder. Let

(6.2) 
$$L_{\delta,s}(C:D) = \frac{\sum_{C' \subset C} |I_{C'}|^{\delta} e^{-n(C')s}}{\sum_{D' \subset D} |I_{D'}|^{\delta} e^{-n(D')s}}$$

where the sums are respectively over all cylinders contained in C and D and the values  $|I_{C'}|$  and  $|I_{D'}|$  are determined using the same chart in  $\mathcal{B}$ . Let the pressure  $P = P(\mathcal{S}, \delta)$  be the infimum value of s for which the numerator (and the denominator) are finite.

ξ,		$C^1_{\Lambda}$		$C_{\Lambda}^{2}$		
	$\eta_{\Lambda}$	$D^1_{\Lambda}$	 $D^p_{\Lambda}$	$D^{p+1}_{\Lambda}$	 $D^q_{\Lambda}$	

FIGURE 13. The rectangles  $C^1_{\Lambda}, C^2_{\Lambda}$  and  $D^1_{\Lambda}, \ldots, D^q_{\Lambda}$ 

If  $\xi \in \Theta^{\iota'}$ , then the leaf 1-cylinder segment  $\xi_{\Lambda} = i(\pi_{\iota'}^{-1}\xi) \subset \Lambda$  is also regarded, without ambiguity, as a point in the train-track  $\mathbf{B}^{\iota'}$ . Similarly, if *C* is an *n*-cylinder of  $\Theta^{\iota}$  then the (1, n)-rectangle  $C_{\Lambda} = i(\pi_{\iota}^{-1}\xi) \subset \Lambda$  is also regarded, without ambiguity, as an *n*-cylinder of the train-track  $\mathbf{B}^{\iota}$ .

The following theorem follows from the results proved in [23]. It can also be deduced from standard results about Gibbs states such as those in [2].

**Theorem 6.1.** Let S be a  $C^{1+}$  self-renormalisable structure on  $\mathbf{B}^{\iota}$ . For every  $\delta > 0$ , there is a unique geometric natural measure  $\mu_{\iota} = \mu_{S,\delta}$  with pressure  $P = P(S,\delta) \in \mathbb{R}$ , and there is an unique  $\tau$ -invariant Gibbs measure  $\nu = \nu_{S,\delta}$  on  $\Theta$  such that  $\mu_{\iota} = (i_{\iota})_*\nu_{\iota}$  where  $\nu_{\iota} = (\pi_{\iota})_*\nu$ . Furthermore, the measure  $\mu_{\iota}$  has the following properties:

(i) There is  $0 < \alpha < 1$  such that if C and D are any two n-cylinders in  $\Theta^{\iota}$  such that  $I_C$  and  $I_D$  are contained in a common segment K then

$$\frac{\mu_{\iota}(I_C)}{\mu_{\iota}(I_D)} \in (1 \pm \mathcal{O}(|K|^{\alpha})) L_{\delta,P}(C:D) .$$

(ii) If  $\rho : \operatorname{msol}^{\iota} \to \mathbb{R}$  is the extended measure scaling function of  $\nu_{\iota}$ , then

$$\rho_{\xi}(C) = \lim_{m \to \infty} L_{\delta, P}(C_m : \xi_m)$$

where  $C_m$  and  $\xi_m$  are the cylinders given by  $I_{C_m} = f_{\iota'}^m(C_\Lambda \cap \xi_\Lambda)$  and  $I_{\xi_m} = f_{\iota'}^{m-1}\xi_\Lambda$ . (iii) (ratio decomposition) if C is an n-cylinder in  $\Theta^{\iota}$  and  $C_p$  is the primary cylinder containing C then

(6.3) 
$$\mu_{\iota}(I_C) = \int_{\xi \in \pi_{\iota'}(C)} \rho_{\xi}(C) \nu_{\iota'}(d\xi)$$

**Lemma 6.2.** Let S be a  $C^{1+}$  self-renormalisable structure on  $\mathbf{B}^{\iota}$  and let  $\rho$  be the extended measure scaling function of the Gibbs measure  $\nu_{S,\delta}$ .

(i) If C and D are two cylinders contained in an n-cylinder E of  $\Theta^{\iota}$  then, for all  $\xi, \eta$  contained in the 1-cylinder  $\pi_{\iota'}(\pi_{\iota}^{-1}E)$  of  $\Theta^{\iota'}$ ,

(6.4) 
$$\frac{\rho_{\eta}(C)}{\rho_{\eta}(D)} \in (1 \pm \mathcal{O}(\theta^n)) \frac{\rho_{\xi}(C)}{\rho_{\xi}(D)}$$

(ii) Let  $\mathbf{B}^{\iota'}$  be a no-gap train-track. Let  $\xi, \eta \in \Theta^{\iota'}$  be such that the corresponding leaf segments in  $\Lambda$  have a common intersection K (or coincide). Let  $(\xi : C^1), (\xi : C^2), (\eta : D^1), \ldots, (\eta : D^q)$  be admissible pairwise distinct pairs in  $\mathrm{msc}^{\iota}$  such that (a)  $\xi_{\Lambda} \cap C^1_{\Lambda} = \xi_{\Lambda} \cap (\cup_{i=1}^p D^i_{\Lambda}) \subset K$ , and (b)  $\xi_{\Lambda} \cap C^2_{\Lambda} = \xi_{\Lambda} \cap (\cup_{i=p+1}^q D^i_{\Lambda}) \subset K$  (see Figure 13). Then

(6.5) 
$$\frac{\rho_{\xi}(C^1)}{\rho_{\xi}(C^2)} = \frac{\sum_{i=1}^p \rho_{\eta}(D^i)}{\sum_{i=p+1}^q \rho_{\eta}(D^i)}$$

(iii) Let  $\mathbf{B}^{\iota}$  be a no-gap train-track (and  $\delta = 1$ ). Then for every admissible pair  $(C:\xi) \in \operatorname{msc}^{\iota}$  we get

(6.6) 
$$\rho_{\xi}(C) = r^{\iota}(C_{\Lambda} \cap \xi_{\Lambda} : \xi_{\Lambda})$$

where  $r^{\iota}$  is the  $\iota$ -ratio function determined by the  $C^{1+}$  self-renormalisable structure.

*Proof.* Proof of (i) and (ii). Suppose that C and D are two cylinders contained in an n-cylinder E. Let  $E_1$  be a (n + 1)-cylinder whose image under the shift map  $\tau$  is E and let  $C_1$  and  $D_1$  be the cylinders in  $E_1$  such that  $\tau C_1 = C$  and  $\tau D_1 = D$ . Then

$$L_{\delta,P}(C_1:D_1) \in (1 \pm \mathcal{O}(\theta^n)) L_{\delta,P}(C:D)$$

where (i)  $0 < \theta < 1$  is independent of C, D, E and  $E_1$ , and  $P = P(S, \delta)$  is the pressure. This follows directly from the definition of  $L_{\delta,P}$  together with the fact that for all cylinders C', D' in  $E_1$ ,

$$\frac{|I_{D'}|}{|I_{C'}|} \in (1 \pm \mathcal{O}(\theta^n)) \frac{|I_{\tau D'}|}{|I_{\tau C'}|}.$$

As a corollary of this we deduce (6.4). Then, equality (6.5) follows from using that the local holonomies are local diffeomorphisms in the self-renormalisable structure of  $\mathbf{B}^{\iota}$ .

Proof of (iii). In this case the self-renormalisable structure S is a local manifold structure as defined in Section 4.6 (i.e. the charts are homeomorphisms onto a subinterval of  $\mathbb{R}$ ), and  $\delta = 1$ . Using (6.2), we get  $P(S, \delta) = 0$  and so the ratios  $\mu(I)/|I|$  are uniformly bounded away from 0 and  $\infty$  for all segments I in  $\mathbf{B}^{\iota}$ . Moreover, in this case, the length system  $\ell$  matches in the sense that, if C is an n-cylinder then  $\sum_{C'} |I_{C'}| = |I_C|$  where the sum is over all m-cylinders C' contained in Cand  $|I_C|$  and  $|I_{C'}|$  are obtained using the same chart in  $\mathcal{B}$ . Thus, if C and D are n-cylinders and  $I_C \cup I_D$  is a segment of  $\mathbf{B}^{\iota}$  then

$$\frac{\mu_{\iota}(I_C)}{\mu_{\iota}(I_D)} \in (1 \pm \mathcal{O}(\theta^n)) \frac{|I_C|}{|I_D|}.$$

Hence,

$$\rho_{\xi}(C) = \lim_{m \to \infty} \frac{|f_{\iota'}^m(C_{\Lambda} \cap \xi_{\Lambda})|}{|f_{\iota'}^m \xi_{\Lambda}|}$$

which implies (6.6).

**Lemma 6.3.** If  $\delta$  is the Hausdorff dimension of  $\mathbf{B}^{\iota}$  then the ratios  $\mu_{\iota}(I_C)/|I_C|^{\delta}$  are uniformly bounded away from 0 and  $\infty$ . It follows from this that the Hausdorff  $\delta$ -measure  $\mathcal{H}^{\delta}$  is finite and positive on  $\mathbf{B}^{\iota}$  and such that  $\mu_{\iota}$  is absolutely continuous with respect to  $\mathcal{H}^{\delta}$ .

Proof. Since in this case  $\delta$  is the Hausdorff dimension of  $\mathbf{B}^{\iota}$ ,  $P_{\delta} = 0$ . Now, let us prove that the ratios  $\mu_{\iota}(I)/|I|^{\delta}$  are uniformly bounded away from 0 and  $\infty$  for all segments I. Suppose that I is any segment in  $\mathbf{B}^{\iota}$ . Then either there exists a cylinder C with  $I_C \subset I$  such that  $I \subset I_{mC}$  or there exist cylinders  $I_C, I_D \subset I$  with a common endpoint such that  $I \subset I_{mC} \cup I_{mD}$ . In the first case let  $\hat{I} = I_C$  and  $m\hat{I} = I_{mC}$ , otherwise let  $\hat{I} = I_C \cup I_D$  and  $m\hat{I} = I_{mC} \cup I_{mD}$ . Since by bounded geometry there exists  $\sigma > 0$  such that for all cylinders  $|I_C|/|I_{mC}| \in [\sigma, \sigma^{-1}]$ ,

$$\sigma|m\hat{I}| \le |\hat{I}| \le |I| \le |m\hat{I}| \le \sigma^{-1}|\hat{I}|.$$

By Theorem 6.1, the ratios  $\mu_{\iota}(I_C)/|I_C|^{\delta}$  are uniformly bounded away from 0 and  $\infty$  for cylinders C, and so the ratios  $\mu_{\iota}(I)/|I|^{\delta}$  for segments I are also uniformly bounded away from 0 and  $\infty$ .

Suppose that A is a subset of  $\mathbf{B}^{\iota}$ . Recall the definition of  $\mathcal{H}^{\delta}_{\varepsilon}(A)$  as the infimum of the sums  $\sum r_{i}^{\delta}$  where the  $r_{i}$  are the lengths of the segments of an  $\varepsilon$ -cover of A. Then the Hausdorff  $\delta$ -dimensional

outer measure of A is  $\mathcal{H}^{\delta}(A) = \lim_{\varepsilon \to 0} \mathcal{H}^{\delta}_{\varepsilon}(A) = \sup_{\varepsilon > 0} \mathcal{H}^{\delta}_{\varepsilon}(A)$ . Now suppose that A is a subset of  $\mathbf{B}^{\iota}$  and  $\{B_i\}$  is a cover of A by segments of length  $r_i$ . Then,  $\mathcal{H}^{\delta}(A) \geq \mathcal{O}(\mu_{\iota}(A))$  because

$$\mathcal{H}^{\delta}(A) \geq \sum r_{i}^{\delta} \geq \mathcal{O}\left(\sum \mu_{\iota}(B_{i})\right) \geq \mathcal{O}\left(\mu_{\iota}\left(\bigcup B_{i}\right)\right) \geq \mathcal{O}(\mu_{\iota}(A)).$$

Now suppose that A is a Borel subset of  $\mathbf{B}^i$  with  $\mu_{\iota}(A) > 0$ . The set of segments in  $\mathbf{B}^{\iota}$  is a Vitali class for A. Therefore, by the Vitali Covering Theorem (e.g. [4, 6]), given  $\varepsilon > 0$ , there is a countable disjoint sequence  $B_j$  of segments such that either  $\sum_j |B_j|^{\delta} = \infty$  or  $\mathcal{H}^{\delta}(A) \leq \sum_j |B_j|^{\delta} + \varepsilon$ . But since the  $B_j$ 's are disjoint we get

$$\sum_{j} |B_{j}|^{\delta} \leq \mathcal{O}\left(\sum_{j} \mu_{\iota}(B_{j})\right) \leq \mathcal{O}(\mu_{\iota}(A)) .$$

Thus  $\mathcal{H}^{\delta}(A) \leq \mathcal{O}(\mu_{\iota}(A)) + \varepsilon$ . Letting  $\varepsilon \to 0$  gives  $\mathcal{H}^{\delta}(A) \leq \mathcal{O}(\mu_{\iota}(A))$ . This proves that the measure  $\mu_{\iota}$  is proportional to  $\mathcal{H}^{\delta}$ . It follows immediately that  $\mathcal{H}^{\delta}$  is positive and finite.

6.1. Measure ratio functions. In this subsection, we prove that, for every  $\delta > 0$ , a given  $C^{1+}$  self-renormalisable structure S on  $\mathbf{B}^{\iota'}$  determines an  $\iota$ -measure ratio function  $\rho_{S,\delta}$  such that the Gibbs measure  $\nu_{\rho}$  determined by  $\rho_{S,\delta}$  (see Remark 5.8) is the same as the Gibbs measure  $\nu_{S,\delta}$  which is  $C^{1+}$  realisable by the self-renormalisable structure S.

Let  $\xi \in \mathbf{B}^{\iota'}$  be an  $\iota$ -leaf segment spanning of a Markov rectangle M. Let R be a rectangle inside M. There are cylinders  $C_j \in \Theta^{\iota}$  such that  $\pi_{\iota}R$  is the countable (or finite) union  $\cup_{j \in \text{Ind}} I_{C_j}$  of cylinders  $I_{C_j} = i_{\iota}(C_j)$  of  $\mathbf{B}^{\iota}$ , and any two of which intersect at most in a point of their boundary (see Figure 14). Suppose that  $\xi \cap R \neq \emptyset$ . Let  $\xi' \in \Theta^{\iota'}$  be such that  $i_{\iota}(\xi') = \xi$  (we note that  $\xi'$  might not be uniquely determined). Using (6.5), the following ratio is well-defined

(6.7) 
$$\rho_{\iota,\xi}(R:M) = \sum_{j \in \text{Ind}} \rho_{\xi'}(C_j) ,$$

where  $\rho_{\xi'}(C_j)$  is the *i*-extended measure scaling function of  $\nu_{S,\delta}$ . If  $\xi \cap R = \emptyset$  then we define  $\rho_{\xi'}(R:M) = 0$ . More generally, suppose that  $R_0$  and  $R_1$  are *i*'-spanning rectangles contained in R. Then we define

$$\rho_{\iota,\xi}(R_0:R_1) = \rho_{\iota,\xi}(R_0:M)\rho_{\iota,\xi}(R_1:M)^{-1} .$$

**Theorem 6.4.** (2-dimensional ratio decomposition) Let S be a  $C^{1+}$  self-renormalisable structure and  $\mu_t = \mu_{S,\delta}$  a natural geometric measure for some  $\delta > 0$ . Suppose that R is a rectangle contained in a Markov rectangle M. Then

(6.8) 
$$\mu(R) = \int_{\pi_{\mathbf{B}^{t'}}(R)} \rho_{\iota,\xi}(R:M) \mu_{\iota'}(d\xi) \ .$$

We now consider the case where  $\mathbf{B}^{\iota}$  is a no-gap train-track. Let  $\mathcal{S}$  be a  $C^{1+}$  self-renormalisable structure and  $\mu_{\iota} = \mu_{\mathcal{S},1}$  the natural measure (with pressure P = 0). Recall the definition of  $t_R^{\prime}$  as the set of spanning  $\iota$ -leaf segments of the rectangle R (not necessarily a Markov rectangle). By the local product structure, one can identify  $t_R^{\prime}$  with any spanning  $\iota'$ -leaf segment  $\ell^{\iota'}(x, R)$  of R. Suppose that R is a rectangle and M is a Markov rectangle and that  $\theta : \ell = \ell^{\iota'}(x, R) \to \ell' \subset \ell^{\iota'}(x', M)$  is a basic holonomy defined on the spanning  $\iota'$ -leaf segment  $\ell$ . This defines an injection  $t_{\theta} : t_R^{\prime} \to t_M^{\prime}$ which we call the holonomy injection induced by  $\theta$  (see Figure 14). The measure  $\mu_{\iota'}$  on  $\mathbf{B}^{\iota'}$ induces a measure on  $t_M^{\prime'}$  which we can pull back to  $t_R^{\prime'}$  using  $t_{\theta}$  to obtain a measure  $\mu_{R,M}^{\theta}$  i.e.  $\mu_{R,M}^{\theta}(E) = \mu_{\iota'}(\pi_{\mathbf{B}^{\iota'}}(t_{\theta}(E))).$ 



FIGURE 14. The holonomy injection  $t_{\theta}$ .

**Theorem 6.5.** (2-dimensional ratio decomposition for SRB measures) Let  $\mathbf{B}^{\iota}$  be a no-gap traintrack. Let S be a  $C^{1+}$  self-renormalisable structure and  $\mu_{\iota} = \mu_{S,1}$  the natural measure (with pressure P = 0). If  $t_{\theta} : t'_R \to t'_M$  is a holonomy injection as above with P a Markov rectangle then

(6.9) 
$$\mu(R) = \int_{t'_R} r_{\iota}(\xi : t_{\theta}(\xi)) \mu_{R,M}^{\theta}(d\xi) ,$$

where  $r_{\iota}$  is the  $\iota$ -ratio function determined by S.

**Remark 6.6.** Note that if  $R \subset M$  then  $t_{\theta}(\xi)$  is just the M-spanning  $\iota$ -leaf containing  $\xi$  and  $\mu_{R,M}^{\theta} = \mu_{\iota'}$ .

Since any rectangle can be written as the union of rectangles R with the property hypothesised in the theorem for some Markov rectangle, the above theorem gives an explicit formula for the measure of any rectangle in terms of a ratio decomposition using the ratio function which characterises the smooth structure of the train-track.

Proof of Theorems 6.4 and 6.5. Suppose that R is any rectangle, M is a Markov rectangle and  $t_{\theta}: t_{R}^{t'} \to t_{M}^{t'}$  is a holonomy injection as above (in the case of Theorem 6.4  $t_{\theta}$  is the identity map). Then we note that there is  $0 < \nu < 1$  such that for all n > 0 we can write  $R = R_0 \cup \cdots \cup R_{N(n)}$  where

- (i)  $R_0, \ldots, R_{N(n)}$  are rectangles which intersect at most in their boundary leaves and their spanning  $\iota$ -leaf segments are also *R*-spanning  $\iota$ -leaf segments;
- (ii)  $P_i = t_{\theta} R_i$  and  $\pi_{\iota'}(P_i)$  is an *n*-cylinder of  $\mathbf{B}^{\iota'}$  for every  $0 \le i \le N(n)$ ;
- (iii)  $R_0$  is the empty set, or  $\pi_{\iota'}(P_0)$  is strictly contained in an *n*-cylinder of  $\mathbf{B}^{\iota'}$ , and so, using the bounded geometry of the Markov map (see Section 4.3) and (6.1),  $\mu(R_0) < \mathcal{O}(\varepsilon_0^n)$  for some  $0 < \varepsilon_0 < 1$ ;

Let  $S_i = f_{\iota'}^n R_i$  and  $Q_i = f_{\iota'}^n P_i$  for  $1 \le i \le N(n)$ , and note that the rectangles  $Q_i$  are  $\iota$ -spanning (1, n)-rectangles of some Markov rectangle  $M_i$ . We note that if  $t_{\theta}$  is not the identity there might be a non-empty set  $V_n$  of values of i such that  $S_i$  is not be contained in the Markov rectangle  $M_i$ . However, since there are a finite number of Markov rectangles, the cardinality of the set  $V_n$  is bounded away from infinity, independently of  $n \ge 0$ . Hence, we desregard in what follows these

values of  $i \in V_n$ , since the measure of the corresponding sets  $S_i$  converges to 0 when n tends to infinity. To prove the theorems we firstly note that by Lemma 6.2 and by (6.7) we obtain that, if  $Q_i, P_i$  and  $M_i$  are as above, for all  $\xi, \eta \in M_i$ ,

$$\rho_{\xi}(S_i:Q_i) \in (1 \pm \mathcal{O}(\varepsilon^n)) \rho_{\eta}(S_i:Q_i)$$

for some  $0 < \varepsilon < 1$ . Thus, since  $\mu(S_i) = \mu(\pi_\iota(S_i))$  and  $\mu(Q_i) = \mu(\pi_\iota(Q_i))$  and by (6.3), if  $\xi \in t_{M_i}^{\iota'}$ ,

$$\mu(S_i) \in (1 \pm \mathcal{O}(\varepsilon^n)) \,\rho_{\xi}(S_i : Q_i) \mu(Q_i).$$

Now consider the case of Theorem 6.4. Then, since  $R_i$  and  $P_i$  are contained in the same Markov rectangle,  $\rho_{\xi}(S_i : Q_i)$  equals  $\rho_{\xi_i}(R : M)$  for some  $\xi_i \in t_{R_i}^{\iota'}$  and  $\rho(Q_i) = \rho(P_i)$  which is equal to  $\mu_{\iota'}P_i$  since  $P_i$  is an  $\iota$ -spanning rectangle of the Markov rectangle M. Thus we have deduced that up to addition of a term that is  $\mathcal{O}(\nu^n)$ ,

$$\mu(R) \in (1 \pm \mathcal{O}(\varepsilon^n)) \sum_{i=1}^{N(n)} \rho_{\xi_i}(R:M) \mu_{\iota'}(P_i).$$

Equation (6.8) follows on taking the limit  $n \to \infty$ .

Now consider the case of Theorem 6.5. Under its hypotheses we have that  $\rho_{\xi}(S_i : Q_i) = r^{\iota}(\xi \cap S_i : \xi \cap Q_i)$  by (6.6) and (6.7). By the *f*-invariance of  $r^{\iota}$  there is  $\xi_i \in t_{R_i}^{\iota'}$  such that  $r^{\iota}(\xi_i : t_{\theta}(\xi_i)) = r^{\iota}(\xi \cap S_i : \xi \cap Q_i)$ . Thus, as above, we deduce that

$$\mu(R) \in (1 \pm \mathcal{O}(\varepsilon^n)) \sum_{i=1}^{N(n)} r^{\iota}(\xi_i : t_{\theta}(\xi_i)) \mu_{\iota'}(t_{\theta}R_i) .$$

Equation (6.9) follows on taking the limit  $n \to \infty$ .  $\Box$ 

**Lemma 6.7.** Let S be a  $C^{1+}$  self-renormalisable structure on  $\mathbf{B}^{\iota}$  with natural measure  $\mu_{\iota} = \mu_{S,\delta}$  for some  $\delta > 0$ . Suppose that R is contained in a  $(n_s, n_u)$ -rectangle and that R' and R'' are  $\iota'$ -spanning rectangles contained in R. Suppose in addition that either (i) R is contained in a Markov rectangle or (ii)  $\mathbf{B}^{\iota}$  does not have gaps and there is a holonomy injection of R into a Markov rectangle (in this case  $\delta = 1$  and P = 0). Then for every  $\iota$ -leaf segment  $\xi \in t_R^{\iota'}$  we have that

(6.10) 
$$\frac{\mu(R')}{\mu(R'')} \in \left(1 \pm \mathcal{O}(\varepsilon^{n_s + n_u})\right) \rho_{\xi}(R':R'')$$

for some constant  $0 < \varepsilon < 1$  independent of R, R', R'',  $n_s$  and  $n_u$ , (and in case (ii)  $\rho_{\xi}(R:R') = r^{\iota}(\xi \cap R:\xi \cap R'))$ .

*Proof.* We give the proof for the second case since that for the first is similar. By Theorem 6.5, we have that

$$\frac{\mu(R)}{\mu(R')} = \frac{\int_{t_R'} r^{\iota}(\xi : t_{\theta}(\xi)) \mu_{R,M}^{\theta}(d\xi)}{\int_{t_{R'}'} r^{\iota}(\xi : t_{\theta}(\xi)) \mu_{R',M}^{\theta}(d\xi)} = \frac{\int_{t_R'} r_{\xi}(R : R') r^{\iota}(\xi : t_{\theta}(\xi)) \mu_{R,M}^{\theta}(d\xi)}{\int_{t_{R'}'} r^{\iota}(\xi : t_{\theta}(\xi)) \mu_{R',M}^{\theta}(d\xi)}$$

where  $r_{\xi}^{\iota}(R:R') = r^{\iota}(R \cap \xi: R' \cap \xi)$ . Let  $F = f_{\iota'}^{n_s+n_u}$ . By inequality (3.2) (or inequality (6.4) in case (i)), there is  $0 < \varepsilon < 1$  such that

$$r_{F\eta}^{\iota}(FR:FR') \in (1 \pm \mathcal{O}(\varepsilon^{n_s+n_u}))r_{F\xi}^{\iota}(FR:FR')$$

for all  $\xi, \eta \in t_R^{\iota'}$ . Thus,

$$r_{\eta}^{\iota}(R:R') \in (1 \pm \mathcal{O}(\varepsilon^{n_s+n_u}))r_{\xi}^{\iota}(R:R')$$

and so

$$\frac{\mu(R)}{\iota(R')} \in (1 \pm \mathcal{O}(\varepsilon^{n_s + n_u})) r_{\xi}^{\iota}(R:R') \; .$$

Similarly,

$$\frac{\mu(R)}{\mu(R'')} \in (1 \pm \mathcal{O}(\varepsilon^{n_s + n_u})) r_{\xi}^{\iota}(R:R') \ .$$

Putting together the previous two equations we obtain (6.10).

**Lemma 6.8.** Let S be a  $C^{1+}$  self-renormalisable structure on  $\mathbf{B}^{\iota}$  with natural measure  $\mu_{\iota} = \mu_{S,\delta}$  for some  $\delta > 0$ . The values

$$\rho_{\mathcal{S},\delta}(\xi \cap R' : \xi \cap R'') = \rho_{\mathcal{E}}(R' : R'')$$

(as in Lemma 6.7) determine an  $\iota$ -measure ratio function  $\rho_{\mathcal{S},\delta}$  with the following properties: (i) The Gibbs measure  $\nu_{\rho}$  determined by the  $\iota$ -measure ratio function  $\rho_{\mathcal{S},\delta}$  (see Remark 5.8) is the same as the Gibbs measure  $\nu_{\mathcal{S},\delta}$  which is  $C^{1+}$  realisable by the self-renormalisable structure  $\mathcal{S}$ ; (ii) If  $\mathbf{B}^{\iota}$  is a no-gap train-track then  $\rho_{\mathcal{S},1} = r$ , where r is the ratio function determined by the  $C^{1+}$ self-renormalisable structure  $\mathcal{S}$ .

*Proof.* Let us prove this lemma first in the case where the train-track  $\mathbf{B}^{\iota}$  does not have gaps and then in the case where the train-track  $\mathbf{B}^{\iota}$  has gaps.

(i)  $\mathbf{B}^{\iota}$  does not have gaps. Then  $\delta = 1$  and, by Lemma 6.2 (iii), we have  $\rho_{\xi}(R': R'') = r(\xi \cap R': \xi \cap R'')$  where r is the ratio function determined by the  $C^{1+}$  self-renormalisable structure  $\mathcal{S}$ . Hence  $\rho_{\mathcal{S},\delta} = r$  is an  $\iota$ -measure ratio function. Using (6.10), we get that the Gibbs measure, which is a  $C^{1+}$  realisation of the natural geometric measure  $\mu_{\mathcal{S},\delta}$ , determines an  $\iota$ -measure solenoid function which induces the  $\iota$ -measure ratio function  $\rho_{\mathcal{S},\delta}$ .

(ii)  $\mathbf{B}^{\iota}$  has gaps. Let  $\xi, \eta \in \mathbf{B}^{\iota'}$  be boundaries of Markov rectangles such that  $\xi \cap \eta \neq \emptyset$ . Let M, M', R and R' be rectangles such that  $\xi \cap R = \eta \cap M$  and  $\xi \cap R' = \eta \cap M'$ . Using (6.5) and (6.7), we get that

$$\rho_{\xi}(R:R') = \rho_{\eta}(M:M') .$$

Therefore, the  $\iota$ -measure ratio function  $\rho_{S,\delta}$  is well-defined for every  $\iota$ -leaf segments I and J contained in a spanning leaf segment of a Markov rectangle. By f-invariance of  $\mu$  and (6.10), we get that

(6.11) 
$$\rho_{\xi}(R:R') = \rho_{f_{\iota'}\xi}(f_{\iota'}R:f_{\iota'}R')$$

is invariant under f. Let I and J be  $\iota$ -leaf segments such that (a) there is an  $\iota$ -leaf segment K such that  $I, J \subset K$ , and (b) I or J has non-empty interior. Then there is  $n > 0, \xi \in \mathbf{B}^{\iota'}, R$  and R' such that  $f_{\iota'}^n I = \xi \cap R$  and  $f_{\iota'}^n J = \xi \cap R'$ . Hence, using (6.11), the ratio

$$\rho_{\mathcal{S},\delta}(I:J) = \rho_{\xi}(R:R')$$

is well defined independently of n. Using (6.10), we get that (5.1) is satisfied and the Gibbs measure, which is a  $C^{1+}$  realisation of the natural geometric measure  $\mu_{S,\delta}$ , determines an  $\iota$ -measure solenoid function which induces the  $\iota$ -measure ratio function  $\rho_{S,\delta}$ .

6.2. Dual measure ratio function. We will show that an  $\iota$ -measure ratio function  $\rho_{\iota}$  determines an unique dual function  $\rho_{\iota'}$  which is an  $\iota'$ -measure ratio function.

**Definition 6.2.** We say that the  $\iota$ -measure ratio function  $\rho_{\iota}$  and the  $\iota'$ -measure ratio function  $\rho_{\iota'}$  are dual if both determine the same Gibbs measure  $\nu = \nu_{\rho_{\iota}} = \nu_{\rho_{\iota'}}$  on  $\Theta$  (see Remark 5.8).

**Lemma 6.9.** Let S be a  $C^{1+}$  self-renormalisable structure on  $\mathbf{B}^{\iota}$  with  $\iota$ -measure ratio function  $\rho_{\iota} = \rho_{S,\delta}$  corresponding to the Gibbs measure  $\nu = \nu_{S,\delta}$ . Then there is an unique  $\iota'$ -measure ratio function  $\rho_{\iota'}$  determining the same Gibbs measure  $\nu_{\rho,\iota} = \nu$  on  $\Theta$ .



FIGURE 15. The rectangles  $A_m = [I, a_m]$  and  $B_m = [K, b_m]$ .

Proof. Let  $\mu = i_*\nu$ . The dual  $\rho_{\iota'}$  of  $\rho_{\iota}$  is constructed as follows: Let I and K be (i) two  $\iota'$ -leaf segments contained in a common *n*-cylinder  $\iota'$ -leaf, or also (ii) two  $\iota'$ -leaf segments contained in a union of two *n*-cylinders with a common endpoint in the case of a local manifold structure. Choose  $p \in I$  and  $p' \in K$ . Let  $a_m$  be the  $\iota$ -leaf N-cylinders containing p, and  $b_m$  the  $\iota$ -leaf containing p' and holonomic to  $a_m$ . Let  $A_m = [I, a_m]$  and  $B_m = [K, b_m]$  (see Figure 15). Now, let us prove that (i)

(6.12) 
$$\frac{\mu(A_{m+1})}{\mu(B_{m+1})} \in (1 \pm \mathcal{O}(\varepsilon^{n+m})) \frac{\mu(A_m)}{\mu(B_m)}$$

for some  $0 < \varepsilon < 1$ ;

(ii) the dual measure raio function is given by

(6.13) 
$$\rho_{\iota'}(I:K) = \lim_{m \to \infty} \frac{\mu(A_m)}{\mu(B_m)}$$

By Lemma 6.7, there is  $0 < \varepsilon < 1$  such that

$$\mu(A_{m+1})/\mu(A_m) \in (1 \pm \mathcal{O}(\varepsilon^{n+m}))\rho_{\iota}(a_{m+1}:a_m) ,$$

and, similarly,

$$\mu(B_{m+1})/\mu(B_m) \in (1 \pm \mathcal{O}(\varepsilon^{n+m}))\rho_\iota(b_{m+1}:b_m)$$

Since  $\rho_{\iota}$  is an  $\iota$ -ratio function,

$$\rho_{\iota}(a_{m+1}:a_m) \in \mathcal{O}(\varepsilon^{n+m}))\rho_{\iota}(b_{m+1}:b_m) .$$

Therefore, (6.12) follows. Furthermore, (6.12) implies (6.13).

Using (6.12), we obtain that  $\rho_{\iota'}$  is an  $\iota'$ -measure ratio function:  $\rho_{\iota'}$  is f-invariant,  $\rho_{\iota'}(I:K) = \rho_{\iota'}(K:I)^{-1}$  and

$$\rho_{\iota'}(I:K) = \rho_{\iota'}(I_1:K) + \rho_{\iota'}(I_2:K)$$

for  $\iota$ -leaf segments  $I_1$  and  $I_2$  with at most one common point and such that  $I = I_1 \cup I_2$ . Again using (6.12),  $\rho_{\iota'}$  satisfies inequality (5.1).

#### 7. Cocycle-gap pairs

In this section, we are going to introduce the gap-cocyle pairs  $(\gamma, J)$  consisting of a gap ratio function  $\gamma$  and of a measure-length ratio cocycle J satisfying a gap-cocyle property. In Section 8, we will apply this description to give an explicit geometric construction of all  $C^{1+}$  self-renormalisable structures and all  $C^{1+}$  hyperbolic diffeomorphisms which have natural geometric measures. 7.1. Measure-length ratio cocycle. Let  $\mathbf{B}^{\iota}$  be a gap train-track. For each Markov rectangle R let  $t_{R}^{\iota'}$  be the set of  $\iota$ -segments of R. Let us denote by  $\mathbf{B}_{o}^{\iota'}$  the disjoint union  $\sqcup_{i=1}^{m} t_{R_{i}}^{\iota'}$  over all Markov rectangles  $R_{1}, \ldots, R_{m}$  (without doing any extra-identification). In this section, for every  $\xi \in \mathbf{B}_{o}^{\iota'}$  and  $n \geq 1$ , we denote by  $\xi_{n}$  the *n*-cylinder  $\pi_{\iota} f_{\iota'}^{n-1} \xi$  of  $\mathbf{B}^{\iota}$ .

**Definition 7.1.** Let  $\mathbf{B}^{\iota}$  be a gap train-track and  $\rho$  be a  $\iota$ -measure ratio function. We say that  $J: \mathbf{B}_{o}^{\iota'} \to \mathbb{R}^{+}$  is a  $(\rho, \delta, P)$   $\iota$ -measure-length ratio cocycle if  $J = \kappa/(\kappa \circ f_{\iota'})$  where  $\kappa$  is a positive Hölder continuous function on  $\mathbf{B}_{o}^{\iota'}$  and is bounded away from 0, and

(7.1) 
$$\sum_{f_{\iota'}\eta=\xi} J(\eta)\rho(f_{\iota'}\eta:m(f_{\iota'}\eta))^{1/\delta}e^{P/\delta} < 1 ,$$

for every  $\eta \in \mathbf{B}_o^{\iota'}$ .

We note that in (7.1), the mother of  $\eta$  is not defined because  $\eta$  is a leaf primary cylinder segment, and so we used instead the mother of the leaf 2-cylinder  $f_{\iota'}\eta$ .

Let us consider a  $C^{1+}$  self-renormalisable structure S on  $\mathbf{B}^{\iota}$ , and fix a bounded atlas  $\mathcal{B}$  for S. Let  $\delta > 0$ . By Theorem 6.1, the  $C^{1+}$  self-renormalisable structure  $S C^{1+}$ -realises a Gibbs measure  $\nu = \nu_{S,\delta}$  as a natural invariant measure  $\mu = \mu_{S,\delta} = i_*\nu$  with pressure  $P = P(S,\delta)$ . Let  $\rho = \rho_{S,\delta}$  be the corresponding  $\iota$ -measure ratio function (see Lemma 6.8). Since  $\mu$  is a natural geometric measure, for every  $\xi \in \mathbf{B}_o^{\iota'}$ , the ratios  $|\xi_n|_i e^{-nP/\delta}/\mu(\xi_n)^{1/\delta}$  are uniformly bounded away from 0 and  $\infty$ , where the length  $|\xi_n|_i$  is measured in any chart  $i \in \mathcal{B}$  containing  $\xi_n$  in its domain. Therefore,

$$\kappa_i(\xi_n) = \frac{|\xi_n|_i e^{-nP/\delta}}{\mu(\xi_n)^{1/\delta}}$$

is well-defined. By Lemma 6.7, we get

$$\frac{\mu_{\iota}(\xi_n)}{\mu_{\iota}(m\xi_n)} \in (1 \pm \mathcal{O}(\varepsilon^n))\rho(f_{\iota'}\xi : m(f_{\iota'}\xi))$$

for some  $0 < \varepsilon < 1$ . Hence, the ratios  $\mu_{\iota}(\xi_n)/\mu_{\iota}(m\xi_n)$  converge exponentially fast along backward orbits  $\xi$  of cylinders. By (4.1), we get that  $|\xi_n|/|m\xi_n|$  also converge exponentially fast along backward orbits  $\xi$  of cylinders. Therefore, it follows that there is a Hölder function  $J_{\mathcal{S},\delta}: \mathbf{B}'_o \to \mathbb{R}$ such that

(7.2) 
$$\frac{\kappa_i(\xi_n)}{\kappa_i(m\xi_n)} \in (1 \pm \mathcal{O}(\varepsilon^n)) J_{\mathcal{S},\delta}(\xi)$$

for some  $0 < \varepsilon < 1$ .

**Lemma 7.1.** Let  $\mathbf{B}^{\iota}$  be a gap train-track. Let S be a  $C^{1+}$  self-renormalisable structure, and  $\delta > 0$ . Let  $\mu_{S,\delta}$  be the natural geometric measure with pressure  $P = P(S, \delta)$ , and  $\rho = \rho_{S,\delta}$  the corresponding  $\iota$ -measure ratio function. The function  $J_{S,\delta} : \mathbf{B}_o^{\iota'} \to \mathbb{R}^+$  given by (7.2) is a  $(\rho, \delta, P)$   $\iota$ -measure-length ratio cocycle.

*Proof.* If I is an n-cylinder in  $\mathbf{B}^{\iota}$ , then  $\sum_{mI'=I} |I'| < |I|$ , where the lengths are measured in the same chart. Thus, since  $|I'| = \kappa_i(I')\mu_\iota(I')^{1/\delta}e^{(n+1)P/\delta}$  we deduce that

$$\sum_{mI'=I} \frac{\kappa_i(I')}{\kappa_i(I)} \left(\frac{\mu_\iota(I')}{\mu_\iota(I)}\right)^{1/\delta} e^{P/\delta} < 1.$$

For every  $\xi \in \mathbf{B}'_o$ , we have that  $\tau_{\iota'}\eta = \xi$  if, and only if,  $\eta_{n+1} \subset \xi_n$  for every  $n \ge 1$ . Hence, the Hölder continuous function  $J = J_{\mathcal{S},\delta}$  satisfies (7.1).

Now, suppose that  $\xi \in \mathbf{B}_o^{\iota'}$  is such that there exists  $p \ge 1$  with the property that  $\xi_{n_p} \subset \xi_n$  for every  $n \ge 1$ . By (7.2), we get

$$\frac{\kappa_{i_0}(\xi_{jp})}{\kappa_{i_0}(\xi_{(j-1)p})} = \prod_{l=0}^{p-1} \frac{\kappa_{i_{l+1}}(\xi_{(j-1)p+l+1})}{\kappa_{i_l}(\xi_{(j-1)p+l})}$$
  

$$\in (1 \pm \mathcal{O}(\nu^{(j-1)p})) \prod_{l=0}^{p-1} J(f_{\iota'}^l(\xi))$$

where  $i_0, \ldots, i_{p-1}$  are charts contained in a bounded atlas of S. Thus, for all 1 < m < M, we have

$$\frac{\kappa_{i_0}(\xi_{Mp})}{\kappa_{i_0}(\xi_{mp})} = \prod_{n=0}^{M-m-1} \frac{\kappa_{i_0}(\xi_{(n+m+1)p})}{\kappa_{i_0}(\xi_{(n+m)p})}$$
$$\in (1 \pm \mathcal{O}(\nu^{mp})) \left[\prod_{l=0}^{p-1} J(f_{\iota'}^l(\xi))\right]^{M-m}$$

Since the term on the left of this equation is uniformly bounded away from 0 and  $\infty$ , it follows that  $\prod_{l=0}^{p-1} J(f_{\iota'}^l(\xi)) = 1$ . From Livšic's theorem (e.g. see [12]) we get that  $J_{\mathcal{S},\delta} = \kappa/(\kappa \circ f_{\iota'})$  where  $\kappa$  is a positive Hölder continuous function on  $\mathbf{B}_o^{\iota'}$  and is bounded away from 0.

7.2. Gap ratio function. Let  $\mathbf{B}^{\iota}$  be a gap train-track. Let  $\mathcal{G}^{\iota'}$  be the set of all pairs  $(\xi_1 : \xi_2) \in \mathbf{B}_o^{\iota'} \times \mathbf{B}_o^{\iota'}$  such that  $mf_{\iota'}\xi_1 = mf_{\iota'}\xi_2$ . The metric  $d_{\Lambda}$  induces a natural metric  $d_{\mathcal{G}^{\iota'}}$  on  $\mathcal{G}^{\iota'}$  given by

$$d_{\mathcal{G}^{L'}}((\xi_1:\xi_2),(\eta_1:\eta_2)) = \max\{d_{\Lambda}(\xi_1,\eta_1),d_{\Lambda}(\xi_2,\eta_2)\}$$

**Definition 7.2.** A function  $\gamma : \mathcal{G}^{\iota'} \to \mathbb{R}^+$  is an  $\iota$ -gap ratio function if it satisfies the following conditions:

(i)  $\gamma(\xi_1:\xi_2)$  is uniformly bounded away from 0 and  $\infty$ ; (ii)  $\gamma(\xi_1:\xi_2) = \gamma(\xi_1:\xi_3)\gamma(\xi_3:\xi_2)$ ; (iii) there are  $0 < \theta < 1$  and C > 1 such that

(7.3) 
$$|\gamma(\xi_1:\xi_2) - \gamma(\eta_1:\eta_2)| \le C \left( d_{\mathcal{G}^{\iota'}}((\xi_1:\xi_2),\gamma(\eta_1:\eta_2)) \right)^{\theta}$$

We note that part (ii) of this definition implies that  $\gamma(\xi_1:\xi_2) = \gamma(\xi_2:\xi_1)^{-1}$ .

Let  $\mathcal{S}$  be a  $C^{1+}$  self-renormalisable structure on  $\mathbf{B}^{\iota}$  and  $\mathcal{B}$  a bounded atlas for  $\mathcal{S}$ . Then the gap ratio function  $\gamma_{\mathcal{S}}$  is well-defined by

(7.4) 
$$\gamma_{\mathcal{S}}(\xi:\eta) = \lim_{n \to \infty} \frac{|\pi_{\mathbf{B}^{\iota}} f_{\iota}^{n} \xi|_{i_{n}}}{|\pi_{\mathbf{B}^{\iota}} f_{\iota}^{n} \eta|_{i_{n}}}$$

where  $i_n \in \mathcal{B}$  contains in its domain the *n*-cylinder  $mf_{\iota}^n \xi$  (we note that  $mf_{\iota}^n \xi = mf_{\iota}^n \eta$ ).

7.3. Ratio functions. We are going to construct the ratio function of a  $C^{1+}$  self-renormalisable structure from the gap ratio function and measure-length ratio cocycle.

**Lemma 7.2.** Let  $\mathbf{B}^{\iota}$  be a gap train-track. Let S be a  $C^{1+}$  self-renormalisable structure. Let  $r_S$  be the corresponding  $\iota$ -ratio function. Let  $\delta > 0$  and let  $\mu = \mu_{S,\delta}$  be the natural geometric measure with pressure  $P = P(S, \delta)$  and  $\rho_{S,\delta}$  the corresponding  $\iota$ -measure ratio function. Let  $J_{S,\delta}$  and  $\gamma_S$ be the corresponding  $\iota$ -gap ratio function and  $\iota$ -measure-length ratio cocycle. Then the following equalities are satisfied: (i) Let I be an  $\iota$ -leaf n-cylinder contained in the  $\iota$ -leaf (n-1)-cylinder L. Then

(7.5) 
$$r_{\mathcal{S}}(I:L) = J_{\mathcal{S},\delta}(\xi_I) \,\rho_{\mathcal{S},\delta}(I:L)^{1/\delta} \,e^{P/\delta}$$

where  $\xi_I = f_\iota^{n-1} I \in \mathbf{B}_o^\iota$ . (ii) Let I be an n-cylinder

(ii) Let I be an n-cylinder and K an n-gap and both contained in a (n-1)-cylinder L. Then

(7.6) 
$$r_{\mathcal{S}}(I:K) = r_{\mathcal{S}}(I:L) \frac{\sum_{G \subset L} \gamma_{\mathcal{S}}(G:K)}{1 - \sum_{D \subset L} r_{\mathcal{S}}(D:L)}$$

where the sum in the numerator is over all n-gaps  $G \subset L$  and the sum in the denominator is over all n-cylinders  $D \subset L$ .

Proof. For every n-cylinder  $I \subset \mathbf{B}^{\iota}$ , define  $\kappa_i(I) = |I|_i e^{-nP/\delta}/\mu_\iota(I)^{1/\delta}$  and let  $J_{\mathcal{S},\delta}$  be the associate measure-length cocycle. Let I be an  $\iota$ -leaf n-cylinder, L the  $\iota$ -leaf (n-1)-cylinder containing I. Choose  $p \in I$  and let  $U_m$  be the  $\iota'$ -leaf m-cylinders containing p. Let  $A_m$  be the rectangle  $[I, U_m]$  and  $B_m$  be the rectangle  $[L, U_m]$ . Then  $f_{\iota'}^{m-1}A_m$  and  $f_{\iota'}^{n-1}B_m$  are  $\iota'$ -spanning rectangles of some Markov rectangle. Let  $a_m$  and  $b_m$  be the projections of these into  $\mathbf{B}^{\iota}$ . Then by the invariance of  $\mu, \mu(A_m)/\mu(B_m) = \mu_{\iota}(a_m)/\mu_{\iota}(b_m)$  and therefore

$$\rho_{\mathcal{S},\delta}(I:L)^{1/\delta} = \lim_{m \to \infty} \frac{\mu(A_m)^{1/\delta}}{\mu(B_m)^{1/\delta}} \\ = \lim_{m \to \infty} \frac{\mu_{\iota}(a_m)^{1/\delta}}{\mu_{\iota}(b_m)^{1/\delta}} \\ = \lim_{m \to \infty} \frac{\kappa(a_m)^{-1} |a_m|_{i_m} e^{-(n+m)P/\delta}}{\kappa(b_m)^{-1} |b_m|_{i_m} e^{-(n+m-1)P/\delta}} \\ = J_{\mathcal{S},\delta}(\xi_I)^{-1} r_{\mathcal{S}}(I:L) e^{-P/\delta}$$

where  $|a_m|_{i_m}$  and  $|b_m|_{i_m}$  are measured in a chart  $i_m$  of the bounded atlas on  $\mathbf{B}^{\iota}$ , and  $\xi_I$  is the leaf primary cylinder segment  $f_{\iota}^{n-1}(I)$ . Thus, equation (7.5) is satisfied.

We note that the ratio of the size of K to the size  $\ell$  of the totality of gaps G in L is given by  $(\sum_{G \subset L} \gamma_{\mathcal{S}}(G:K))^{-1}$  where  $\gamma_{\mathcal{S}}$  is the gap ratio function and the sum is over all *n*-gaps in L. But since the complement of the gaps in L is the union of *n*-cylinders we have that the ratio of  $\ell$  to the size of L is  $1 - \sum_{D \subset L} r_{\mathcal{S}}(D:L)$  where the sum is over all *n*-cylinders D in L. Thus we deduce that for  $r_{\mathcal{S}}(I:K)$  we should take

(7.7) 
$$r_{\mathcal{S}}(I:K) = r_{\mathcal{S}}(I:L) \frac{\sum_{G \subset L} \gamma_{\mathcal{S}}(G:K)}{1 - \sum_{D \subset L} r_{\mathcal{S}}(D:L)}$$

which proves (7.6).

7.4. Cocycle-gap pairs. In this section, we are going to construct a gap-cocyle map b which reflects the cylinder-gap condition of an  $\iota$ -solenoid function (see Section 3.9), i.e the ratios are well-defined along the  $\iota$ -boundaries of the Markov rectangles. Hence, r is an  $\iota$ -ratio function.

Let  $\mathbf{B}^{\iota}$  be a gap train-track and  $\mathbf{B}^{\iota'}$  a no-gap train-track (as in the case of codimension one attractors or repellors). Let  $\mathcal{Q}$  be the set of all periodic orbits O which are contained in the  $\iota$ -boundaries of the Markov rectangles. For every periodic orbit  $O \in \mathcal{Q}$ , let us choose a point x = x(O) belonging to the orbit O. Let us denote by p(x) the smallest period of x. Let us denote by M(1, x) and M(2, x) the Markov rectangles containing the point x. Let us denote by  $l_i(x)$  the  $\iota$ -leaf *i*-cylinder segment of Markov rectangle M(1, x) containing the point x. Let  $A(f_{\iota}^i(x))$  be the smallest  $\iota$ -leaf segment containing all the  $\iota$ -boundary leaf segments of Markov rectangles intersecting the global leaf segment passing through the point  $f_{\iota}^{i}(x)$ . Let q(x) be the smallest integer which is a multiple of p(x), such that

$$A(f_{\iota}^{i}(x)) \subset f_{\iota}^{q(x)+i}(l_{i}(x))$$

for every  $0 \leq i < p(x)$ . Let us denote the  $\iota$ -leaf segments  $f_{\iota}^{q(x)+i}(l_i(x))$  by  $L_i(x)$ . We note that when using the notation  $L_i(x)$ , we will always consider i to be  $i \mod p(x)$ . For every  $j \in \{1, 2\}$ , let J(j, x)be the primary  $\iota'$ -leaf segment contained in M(j, x) with x as an endpoint such that R(j, i, x) = $[f_{\iota}^{q(x)+i}(l_i(x)), f_{\iota}^{q(x)+i}(J(j, x))]$  is a rectangle for every  $0 \leq i < p(x)$ . Let  $\operatorname{Co}(j, i, x) \subset \mathbf{B}_o^{\iota'}$  be the set of all  $\iota$ -primary leaves  $\xi$  of Markov rectangles M such that  $f_{\iota'}\xi \subset L_i(x)$  and  $f_{\iota'}M \cap R(j, i, x)$  has nonempty interior. Let  $\operatorname{Gap}(j, i, x) \subset \mathcal{G}^{\iota'}$  be the set of all sister pairs  $(\xi_1, \xi_2)$  such that  $mf_{\iota'}\xi_1(=mf_{\iota'}\xi_2)$ is an  $\iota$ -primary leaf of a Markov rectangle M with the property that  $M \cap R(j, i, x)$  has non-empty interior. Let  $\operatorname{Co}_j = \bigcup_{O \in \mathcal{Q}} \bigcup_{i=0}^{p(x(O))-1} \operatorname{Co}(j, i, x(O))$  and  $\operatorname{Gap}_j = \bigcup_{O \in \mathcal{Q}} \bigcup_{i=0}^{p(x(O))-1} \operatorname{Gap}(j, i, x(O))$ . Let  $\rho$  be an  $\iota$ -measure ratio function with corresponding Gibbs measure  $\nu$ . Let  $\mathcal{D}_j(\rho, \delta, P)$  be the set of all pairs  $(\gamma_j, J_j)$  with the following properties:

- (i)  $\gamma_j : \operatorname{Gap}_j \to \mathbb{R}^+$  is a map;
- (ii)  $J_j : \operatorname{Co}_j \to \mathbb{R}^+$  is a map satisfying property (7.1), with respect to  $(\rho, \delta, P)$ , for every  $\xi \in \operatorname{Co}_j$  such that  $\xi \subset \bigcup_{O \in \mathcal{Q}} \bigcup_{i=0}^{p(x(O))-1} L_i(x(O))$ .
- (iii) For every  $x(O) \in \mathcal{Q}$ , letting x = x(O),  $\prod_{l=0}^{p(x)-1} J_j(f_{\iota'}^l I^j(i,x) = 1$ , where  $I^j(i,x) \subset Co_j$  is a  $\iota'$ -primary leaf segment containing the periodic point  $f_{\iota'}^i(x)$ .

For every  $x(O) \in Q$ , let x = x(O) and let A(1, x) and A(2, x) be the p(x)-cylinders of M(1, x) and M(2, x), respectively, containing the point x. The points

$$\pi_{\mathbf{B}_{o}^{t'}}A(j,x), \pi_{\mathbf{B}_{o}^{t'}}f_{\iota}A(j,x), \dots, \pi_{\mathbf{B}_{o}^{t'}}f_{\iota}^{p(x)-1}A(j,x)$$

in  $\mathbf{B}_{o}^{\iota'}$  form a periodic orbit, under  $f_{\iota}$ , with period p(x), where  $\pi_{\mathbf{B}_{o}^{\iota'}} : \Lambda \to \mathbf{B}_{o}^{\iota'}$  is the natural projection. The primary cylinders contained in the sets  $\operatorname{Co}(j, i, x)$  are pre-orbits of the points  $\pi_{\mathbf{B}_{o}^{\iota'}}f_{\iota}^{i}A(j, x)$  in  $\mathbf{B}_{o}^{\iota'}$ , under  $f_{\iota}$ . Hence, we note that, if  $\prod_{i=0}^{p(x)-1} J(\pi_{\mathbf{B}_{o}^{\iota'}}f_{\iota}^{i}A(j, x)) = 1$ , then, by Livšic's theorem (e.g. see [12]), there is a map k such that, for every  $\xi \in \operatorname{Co}(j, i, x)$ ,  $J(\xi) = k(\xi)/(k \circ f_{\iota'})(\xi)$ .

We say that C is an *out-gap segment of a rectangle* R if C is a gap segment of R and is not a leaf n-gap segment of any Markov rectangle M such that  $M \cap R$  is a rectangle with non-empty interior.

We say that C is a leaf n-cylinder segment of a rectangle R, if C is a leaf n-cylinder segment of a Markov rectangle M such that  $M \cap R$  is a rectangle with non-empty interior. We say that C is a leaf n-gap segment of a rectangle R, if C is a leaf n-gap segment of a Markov rectangle M such that  $M \cap R$  is a rectangle with non-empty interior. We say that C is an n-leaf segment of a rectangle R, if C is a leaf n-cylinder segment of R or if C is a leaf n-gap segment of R.

**Lemma 7.3.** Let  $(\gamma_1, J_1) \in \mathcal{D}_1(\rho, \delta, P)$ . Let x = x(O), where  $O \in \mathcal{Q}$ . For every  $i \in \{0, 1, \dots, p(x)-1\}$  and for all 2-leaf segments  $C \subset L_i(x)$  of R(1, i, x), the ratios r(C : mC) are uniquely determined such that they are invariant under f, satisfy the matching condition, and satisfy equalities (7.5) and (7.6).

*Proof.* If  $C \subset L_i(x)$  is a leaf 2-cylinder segment of R(1, i, x), then we define the ratio r(C : mC), using (7.5), by

(7.8) 
$$r(C:mC) = J(\xi_C) \,\rho(C:mC)^{1/\delta} \, e^{P/\delta}$$

where  $\xi_C = f_i C \in \text{Co}_1$ . For every sister pair  $(\xi_1 : \xi_2) \in \text{Gap}_1$  we define the ratio  $r(f_{i'}\xi_1 : f_{i'}\xi_2)$ equal to  $\gamma(\xi_1 : \xi_2)$ . If  $C \subset L_i(x)$  is a leaf 2-gap segment of R(1, i, x), then we define the ratio r(C : mC) by

(7.9) 
$$r(C:mC) = \frac{1 - \sum_{D \subset mC} r(D:mC)}{\sum_{\substack{G \subset mC}} r(G:C)},$$

where the sum, in the numerator, is over all 2-cylinders  $D \subset mC$  of R(1, i, x), and the sum, in the denominator, is over all 2-gaps  $G \subset mC$  of R(1, i, x). Hence,

$$\sum_{C \subset mC} r(C:mC) = 1$$

where the sum is over all 2-leaf segments  $C \subset mC$  of R(1, i, x).

**Lemma 7.4.** Let  $(\gamma_1, J_1) \in \mathcal{D}_1(\rho, \delta, P)$ . Let x = x(O), where  $O \in \mathcal{Q}$ , and let  $i \in \{0, 1, \ldots, p(x)-1\}$ . For all  $n \ge 0$ , and for all out-gaps and all 2-leaf segments  $C \subset f_{\iota}^{n+i}\ell_i(x)$  of  $f_{\iota}^{n+i}M(1,x)$ , the ratios  $r(C: f_{\iota}^{n+i}\ell_i(x))$  are uniquely determined such that they are invariant under f, satisfy the matching condition, and satisfy equalities (7.5) and (7.6).

Proof. Let us denote  $f_{\iota}^{n}M(1,x)$  by  $M_{n}$  and  $f_{\iota}^{n}\ell_{i}(x)$  by  $L_{i}^{n}$ . The proof follows by induction on  $n \geq 0$ . For the case n = 0, the ratios  $r(C : L_{i}^{n+i})$  are uniquely determined by Lemma 7.3. Let us prove that the ratios  $r(C : L_{i}^{n+1+i})$  are uniquely determined using the induction hypotheses with respect to n. For every out-gap and every primary cylinder segment  $C \subset L_{i}^{n+1+i}$  of  $f_{\iota}^{n+i}M(1,x)$ ,  $f_{\iota'}C$  is a out-gap or a 2-leaf segment. Hence, by the induction hypotheses, the ratio  $r(f_{\iota'}C : L_{i}^{n+i})$  is well-defined. Therefore, using the invariance of f, we define

(7.10) 
$$r(C:L_i^{n+1+i}) = r(f_{\iota'}C:L_i^{n+i})$$

For every 2-leaf segment  $C \subset L_i^{n+1+i}$  of  $f_i^{n+i}M(1,x)$ , the ratio r(C:mC) is well-defined by Lemma 7.3. Hence, by (7.10), we define

$$r(C: L_i^{n+1+i}) = r(C: mC)r(mC: L_i^{n+1+i})$$

which ends the proof of the induction.

**Lemma 7.5.** Let  $(\gamma_1, J_1) \in \mathcal{D}_1(\rho, \delta, P)$ . Let x = x(O), where  $O \in \mathcal{Q}$ , and let  $i \in \{0, 1, \dots, p(x)-1\}$ . Let  $n \geq 0$  and  $j \in \{0, \dots, n\}$ . For all out-gaps and all j + 2-leaf segments  $C \subset f_{\iota'}^n L_i(x)$  of  $f_{\iota'}^n R(1, i, x)$ , the ratios  $r(C : f_{\iota'}^n L_i(x))$  are uniquely determined such that they are invariant under f, satisfy the matching condition, and satisfy equalities (7.5) and (7.6).

Proof. The proof follows by induction in  $n \ge 0$ . For the case n = 0, noting that  $L_i(x) = f_i^{q(x)+i} \ell_i(x)$ , the ratios  $r(C:L_i(x))$  are well-defined by Lemma 7.4. Hence, using the matching condition, the ratio  $r(f_{\iota'}^{n+1}L_{i+1}(x):f_{\iota'}^nL_i(x))$  is well-defined. Let us prove that for all out-gaps and j + 2-leaf segments  $C \subset f_{\iota'}^{n+1}L_i(x)$  of  $f_{\iota'}^{n+1}R(1,i,x)$ , with  $1 \le j \le n+1$ , the ratios  $r(C:f_{\iota'}^{n+1}L_i(x))$  are uniquely determined using the induction hypotheses with respect to n. By the induction hypotheses and by the matching condition, the ratio  $r(f_{\iota}C:f_{\iota'}^nL_i(x))$  is well-defined. By invariance of f, we define  $r(C:f_{\iota'}^{n+1}L_i(x)) = r(f_{\iota}C:f_{\iota'}^nL_i(x))$ . which ends the proof of the induction.

Let us attribute the ratios for the cylinders and gaps of R(2, i, x) such that they agree with the ratios previously defined in R(1, i, x).

**Lemma 7.6.** Let  $(\gamma_1, J_1) \in \mathcal{D}_1(\rho, \delta, P)$ . Let x = x(O), where  $O \in \mathcal{Q}$ , and let  $i \in \{0, 1, \dots, p(x)-1\}$ . Let  $n \ge 0$  and  $j \in \{1, \dots, n\}$ . For all out-gaps and all j+2-leaf segments  $C \subset f_{\iota'}^n L_i(x) \setminus f_{\iota'}^{n+1} L_{i+1}(x)$  of  $f_{\iota'}^n R(2, i, x)$ , the ratios  $r(C : f_{\iota'}^n L_i(x))$  are uniquely determined such that they are invariant under f, satisfy the matching condition, satisfy equalities (7.5) and (7.6), and are well-defined along the  $\iota$ -boundaries of the Markov rectangles. Hence, r is an  $\iota$ -ratio function.

*Proof.* The proof follows by induction in  $n \ge 0$ . Let us prove the case n = 0. By construction,  $L_i(x) \supset A(f_\iota^i(x))$ , i.e  $L_i(x)$  contains all the  $\iota$ -boundary leaf segments of Markov rectangles intersecting the global leaf segment passing through the point  $f_\iota^i(x)$ . Hence, if  $G_2 \subset L_i(x) \setminus f_{\iota'}L_{i+1}(x)$  is an out-gap of R(2, i, x), then there is an out-gap or a leaf 2-gap segment  $G_1$  of R(1, i, x) such

that  $G_1 = G_2$ . Therefore, we define  $r(G_2 : L_i(x)) = r(G_1 : L_i(x))$ . Since  $L_i(x) \supset A(f_i^i(x))$ , if  $G_2 \subset L_i(x) \setminus f_{i'}L_{i+1}(x)$  is a leaf 2-gap segment of R(2, i, x) then there is an out-gap or a leaf 2-gap segment  $G_1$  of R(1, i, x) such that  $G_1 = G_2$ . Hence, we define  $r(G_2 : L_i(x)) = r(G_1 : L_i(x))$ . If  $C_2 \subset L_i(x) \setminus f_{i'}L_{i+1}(x)$  is a leaf 2-cylinder segment of R(2, i, x), then there is a primary leaf segment or a leaf 2-cylinder segment  $C_1$  of R(1, i, x) such that  $C_2 = C_1$ . Therefore, we define  $r(C_2 : L_i(x)) = r(C_1 : L_i(x))$ . Let us prove that for all out-gaps and j + 2-leaf segments  $C \subset f_{i'}^{n+1}L_i(x) \setminus f_{i'}^{n+2}L_{i+1}(x)$  of  $f_{i'}^{n+1}R(2, i, x)$ , with  $1 \leq j \leq n+1$ , the ratios  $r(C : f_{i'}^{n+1}L_i(x))$  are uniquely determined using the induction hypotheses with respect to n. By the induction hypotheses and by the matching condition, the ratio  $r(f_i C : f_{i'}^n L_i(x))$  is well-defined. By invariance of f, we define  $r(C : f_{i'}^{n+1}L_i(x)) = r(f_i C : f_{i'}^n L_i(x))$ . which ends the proof of the induction.

**Lemma 7.7.** Let  $(\gamma_1, J_1) \in \mathcal{D}_1(\rho, \delta, P)$ . Let x = x(O), where  $O \in \mathcal{Q}$ , and let  $i \in \{0, 1, \ldots, p(x)-1\}$ . For all out-gaps and all 2-leaf segments  $C \subset L_i(x)$  of R(2, i, x), the ratios  $r(C : L_i(x))$  are uniquely determined such that they are invariant under f, satisfy the matching condition, satisfy equalities (7.5) and (7.6), and are well-defined along the  $\iota$ -boundaries of the Markov rectangles. Hence, r is an  $\iota$ -ratio function.

*Proof.* By construction of  $L_i(x) \setminus f_{\iota'}L_{i+1}(x)$ , there is k = k(n, i, x) such that  $L_i(x) \setminus f_{\iota'}L_{i+1}(x) = \bigcup_{l=1}^k D_l$ , where  $D_l$  are out-gaps, primary leaf segments and 2-leaf segments of R(1, i, x). Therefore,  $f_{\iota'}^n L_i(x) \setminus f_{\iota'}^{n+1}L_{i+1}(x) = \bigcup_{l=1}^k f_{\iota'}^n D_l$  where  $f_{\iota'}^n D_l$  are out-gaps and j+2-leaf segments of  $R(1, i, f_{\iota'}^n(x))$  with  $0 \le j \le n$ . Hence, by Lemma 7.6 and using the matching condition, the ratio  $r(f_{\iota'}^n L_i(x) \setminus f_{\iota'}^{n+1}L_{i+1}(x) : f_{\iota'}^n L_i(x))$  is well-defined. Hence, using the matching condition, we define

$$r(f_{\iota'}^{n+1}L_{i+1}(x):f_{\iota'}^{n}L_{i}(x)) = 1 - r(f_{\iota'}^{n}L_{i}(x) \setminus f_{\iota'}^{n+1}L_{i+1}(x):f_{\iota'}^{n}L_{i}(x))$$

Therefore, using again the matching condition, we define

(7.11) 
$$r(f_{\iota'}^{n+1}L_{i+n+1}(x):L_i(x)) = \prod_{j=0}^n r(f_{\iota'}^{j+1}L_{i+j+1}(x):f_{\iota'}^jL_{i+j}(x)) .$$

Let M(i, x) be the 2-cylinder of R(2, i, x) containing the point x. Take N > 0, large enough, such that  $f_{\iota'}^{N+1}L_{i+N+1}(x) \subset M(i, x)$ . Hence, there is m = m(N, i, x) such that  $M(i, x) = (\bigcup_{l=0}^{m} D_l) \cup f_{\iota'}^{N+1}L_{i+N+1}(x)$  where  $D_l$  are out-gaps or j + 2-leaf segments of R(1, i, x) for some  $0 \leq j \leq N$ . Hence, by Lemma 7.6, (7.11) and using the matching condition, the ratio is well-defined by

$$r(M(i,x):L_i(x)) = \sum_{l=0}^m r(D_l:L_i(x)) + r(f_{\iota'}^{N+1}L_{i+N+1}(x):L_i(x))$$

If  $C \subset L_i(x) \setminus M(i, x)$  is a out-gap or a 2-leaf segment of R(2, i, x), then, by Lemma 7.6, the ratio is well-defined by  $r(C : L_i(x))$ . By construction of the ratios, in Lemmas 7.3-7.7, they are compatible with the cylinder-gap condition.

**Definition 7.3.** Let  $(\gamma_1, J_1) \in \mathcal{D}_1(\rho, \delta, P)$ . Let x = x(O), where  $O \in \mathcal{Q}$ , and let  $i \in \{0, 1, \dots, p(x) - 1\}$ . Let the ratios  $r(C : L_i(x))$  for all out-gaps and all 2-leaf segments  $C \subset L_i(x)$  of R(2, i, x) be as given in Lemma 7.7. For all  $\xi \in \text{Co}(2, i, x)$ , letting  $I = f_{i'}\xi \subset L_i(x)$ , we define

$$J_2(\xi) = r(I:L_i(x))r(L_i(x):mI)\rho(I:K_i)^{-1/\delta}e^{-P/\delta}.$$

For all  $(C, D) \in \text{Gap}(2, i, x)$ , we define

$$\gamma(C:D) = r(f_{\iota'}C:L_i(x))r(L_i(x):f_{\iota'}D) .$$

**Lemma 7.8.** Let  $\mathcal{D}_1(\rho, \delta, P) \neq \emptyset$ . The gap cocycle map  $b = b_{\rho,\delta,P} : \mathcal{D}_1(\rho, \delta, P) \to \mathcal{D}_2(\rho, \delta, P)$  is well-defined by  $b(\gamma_1, J_1) = (\gamma_2, J_2)$  where  $\gamma_2$  and  $J_2$  are as given in Definition 7.3. Furthermore, the cocycle gap map b is a bijection.

Proof. Let us check that  $(\gamma_2, J_2)$  satisfies properties (i)-(iii) of  $\mathcal{D}_2(\rho, \delta, P)$ . By construction of the ratios r, in Lemmas 7.3-7.7,  $(\gamma_2, J_2)$  satisfies properties (i) and (ii) in the definition of  $\mathcal{D}_2(\rho, \delta, P)$ . Let us check property (iii). Let us denote by A and B the p(x)-cylinders of M(1, x) and M(2, x), respectively, containing the point x. By invariance of r, we have that  $r(A:B) = r(f_t^{p(x)}A:f_t^{p(x)}B)$ , and so  $r(A:f_t^{p(x)}A) = r(B:f_t^{p(x)}B)$ . By invariance of the  $\iota$ -measure ratio function  $\rho$ , we have that  $\rho(A:B) = \rho(f_t^{p(x)}A:f_t^{p(x)}B)$ , and so  $\rho(A:f_t^{p(x)}A) = \rho(B:f_t^{p(x)}B)$ . Since, by hypotheses  $\prod_{l=0}^{p(x)-1} J(m^i f_t^{p(x)-i}A) = 1$ , we get, from (7.5), that  $r(A:f_t^{p(x)}A) = \rho(A:f_t^{p(x)}A)e^{p(x)P/\delta}$ . Therefore,

$$r(B: f_{\iota}^{p(x)}B) = r(A: f_{\iota}^{p(x)}A)$$
$$= \rho(A: f_{\iota}^{p(x)}A)e^{p(x)P/\delta}$$
$$= \rho(B: f_{\iota}^{p(x)}B)e^{p(x)P/\delta}$$

and so, using (7.5), we obtain that  $\prod_{i=0}^{p(x)-1} J(\pi_{\mathbf{B}'_{o}} f^{i}_{\iota} B) = 1.$ 

**Definition 7.4.** Let  $\mathbf{B}^{\iota}$  be a gap train-track. Let  $\delta > 0$  and  $P \in \mathbb{R}$ . Let  $\rho$  be an  $\iota$ -measure ratio function and  $\nu = \nu_{\rho}$  the corresponding Gibbs measure on  $\Theta$ . We say that a pair  $(\gamma, J)$  is a  $(\nu, \delta, P)$   $\iota$  cocycle-gap pair if  $(\gamma, J)$  has the following properties:

- (i)  $\gamma$  is an  $\iota$ -gap ratio function.
- (ii) J is an  $\iota$  measure-length ratio cocyle.
- (iii) If  $\mathbf{B}^{\iota'}$  is a no-gap train-track then  $(\gamma, J)$  satisfies the following gap-cocyle prop-
- erty:  $b(\gamma | \text{Gap}_1, J | \text{Co}_1) = (\gamma | \text{Gap}_2, J | \text{Co}_2)$ , where  $b = b_{\nu,\delta,P}$  is the gap-cocyle map.

Let  $\mathcal{J}G^{\iota}(\nu, \delta, P)$  be the set of all  $(\nu, \delta, P)$   $\iota$  cocycle-gap pairs.

**Lemma 7.9.** Let  $\mathbf{B}^{\iota}$  be a gap train-track. Let  $\delta > 0$  and  $P \in \mathbb{R}$ . Let  $\rho$  be an  $\iota$ -measure ratio function with corresponding Gibbs measure  $\nu$ .

(i) If there is a  $(\rho, \delta, P)$  *i*-measure-length ratio cocycle, then the set  $\mathcal{J}G^{\iota}(\nu, \delta, P)$  is an infinite dimensional space.

(ii) If S is a  $C^{1+}$  self-renormalisable structure with natural geometric measure  $\mu_{S,\delta} = i_*\nu$  and pressure P, then  $(\gamma_S, J_{S,\delta}) \in \mathcal{J}G^{\iota}(\nu, \delta, P)$ .

(iii) If the set  $\mathcal{J}G^{\iota}(\nu, \delta, P) \neq \emptyset$ , then there is a well-defined injective map  $(\gamma, J) \rightarrow r(\gamma, J)$  which associates to each cocycle-gap pair  $(\gamma, J) \in \mathcal{J}G^{\iota}(\nu, \delta, P)$  an  $\iota$ -ratio function  $r(\gamma, J)$  satisfying (7.5) and (7.6).

**Remark 7.10.** Let  $0 < \delta < 1$  and P = 0. Let  $\rho$  be an  $\iota$ -measure ratio function with corresponding Gibbs measure  $\nu$ . Since J = 1 is a  $(\rho, \delta, P)$   $\iota$ -measure-length ratio cocycle, then, by Lemma 7.9, the set  $\mathcal{JG}^{\iota}(\nu, \delta, P)$  is an infinite dimensional space.

Proof of Lemma 7.9. Proof of (i). Choose a map  $\gamma_1 : \operatorname{Gap}_1 \to \mathbb{R}^+$ . Let  $J_0$  be a  $(\rho, \delta, P)$  *i*measure-length ratio cocycle, and let  $J_1 = J_0|\operatorname{Co}_1$ . Since  $(\gamma_1, J_1) \in \mathcal{D}_1(\rho, \delta, P)$ , by Lemma 7.8, the pair  $(\gamma_2, J_2) = b_{\rho,\delta,P}(\gamma_1, J_1) \in \mathcal{D}_2(\rho, \delta, P)$  is well-defined. Let  $k_0$  and  $k_2$  be maps such that  $J_0 = k_0/(k_0 \circ f_{\iota'})$  and  $J_2 = k_2/(k_2 \circ f_{\iota'})$ . For every  $x(O) \in \mathcal{Q}$ , let x = x(O), and let B be the p(x)cylinder of M(2, x) containing the point x. Recall that the points  $\pi_{\mathbf{B}'_o} B, \pi_{\mathbf{B}'_o} f_{\iota} B, \ldots, \pi_{\mathbf{B}'_o} f_{\iota}^{p(x)-1} B$ in  $\mathbf{B}'_o$  form a periodic orbit under  $f_{\iota}$ , with period p(x), and that the primary cylinders contained in the set  $\operatorname{Co}(2, i, x)$  are pre-orbits of the points  $\pi_{\mathbf{B}'_o} f_{\iota}^i B$  in  $\mathbf{B}'_o$ , under  $f_{\iota}$ . Therefore, there is a small neighbourhood V of  $\operatorname{Co}_2$  in  $\mathbf{B}'_o$ , there is  $\varepsilon > 0$ , small enough, and there is an Hölder continuous map  $k : \mathbf{B}'_o \to \mathbb{R}^+$  with the following properties:

(i)  $k | \operatorname{Co}_2 = k_2, k | (\mathbf{B}_o^{\iota'} \setminus V) = k_0 \text{ and } \operatorname{Co}_1 \subset \mathbf{B}_o^{\iota'} \setminus V.$ 

(ii) Let  $a = \min_{\xi \in Co_2} \{J_0(\xi), J_2(\xi)\}$  and  $b = \max_{\xi \in Co_2} \{J_0(\xi), J_2(\xi)\}$ , and let  $J = k/(k \circ f_{\iota'})$ . For every  $\xi \in V$ , we have that  $a - \varepsilon \leq J(\xi) \leq b + \varepsilon$ , and, so, J satisfies the cocycle-gap property.

Choosing an Hölder continuous map  $\gamma : \mathcal{G}^{\iota'} \to \mathbb{R}^+$  such that  $\gamma | \operatorname{Gap}_1 = \gamma_1$  and  $\gamma | \operatorname{Gap}_2 = \gamma_2$ and by property (i) above, the pair  $(\gamma, J)$  satisfies (7.1). Therefore, the pair  $(\gamma, J)$  is contained in  $\mathcal{J}G^{\iota}(\nu, \delta, P)$ . Using that (7.1) is an open condition, the above construction allows us to construct an infinite set of  $\iota$ -measure-length ratio cocycles and an infinite set of gap ratio functions such that the corresponding pairs are contained in  $\mathcal{J}G^{\iota}(\nu, \delta, P)$ .

Proof of (ii). Let S be a  $C^{1+}$  self-renormalisable structure with natural geometric measure  $\mu_{S,\delta} = i_*\nu$  and pressure P. By Lemma 7.1,  $J_{S,\delta}$  is a  $(\rho, \delta, P)$   $\iota$ -measure-length ratio cocycle and, by (7.4),  $\gamma_S$  is an  $\iota$ -gap ratio function. If  $\mathbf{B}^{\iota'}$  is a no-gap train-track, using (7.5) and (7.6), the pair  $(\gamma_S, J_{S,\delta})$  satisfies the cocycle-gap condition because the ratio function  $r_S$  associated to S is well-defined along the  $\iota$ -boundaries of the Markov rectangles.

Proof of (iii). The equations (7.5) and (7.6) give us an inductive construction, on the level n of the n-cylinders and n-gaps, of a ratio function r in terms of  $(\rho, J, \gamma, \delta, P)$  with the property that the ratio between a leaf n-cylinder segment C and a leaf n-cylinder or n-gap segment D with a common endpoint with C is bounded away from zero and infinity independent of n and of the cylinders and gaps considered. The construction gives that r is invariant under f. The Hölder continuity of  $\gamma$ , J and  $\rho$  implies that r satisfies (3.2). If  $\mathbf{B}^{t'}$  is a no-gap train-track, by the construction of the cocycle-gap condition, the ratio function r is well-defined along the  $\iota$ -boundaries of the Markov rectangles. Hence, r is an  $\iota$ -ratio function.  $\Box$ 

## 8. Realisations of Gibbs measures

In this section, we are going to give an explicit geometric construction of all  $C^{1+}$  hyperbolic diffeomorphisms which have a natural geometric measure, and we will prove the theorems stated in the introduction of the paper.

8.1. One-dimensional realisations. Let S be a  $C^{1+}$  self-renormalisable structure on a traintrack  $\mathbf{B}^{\iota}$ . In Lemma 6.8 we have shown that the map

$$(8.1) (\mathcal{S},\delta) \to \rho_{\mathcal{S},\delta}$$

is well-defined where  $\rho_{\mathcal{S},\delta}$  is the  $\iota$ -measure ratio function associated to a Gibbs measure  $\nu_{\mathcal{S},\delta} = \nu$ such that  $\mu_{\mathcal{S},\delta} = (i_{\iota})_* \nu_{\iota}$  is a natural geometric measure of  $\mathcal{S}$ .

**Lemma 8.1.** (Rigidity) Let  $\mathbf{B}^{\iota}$  be a no gap train-track (and  $\delta = 1$ ). The map  $S \to \rho_{S,\delta}$  is an one-to-one correspondence between  $C^{1+}$  self-renormalisable structures on  $\mathbf{B}^{\iota}$  and  $\iota$ -measure ratio functions. Furthermore,  $\rho_{S,\delta} = r_S$  where  $r_S$  is the ratio function determined by S.

However, if  $\mathbf{B}^{\iota}$  is a gap train-track then the set of pre-images of the map  $(\mathcal{S}, \delta) \to \rho_{\mathcal{S}, \delta}$  is an infinite dimensional space (see Lemma 8.3 below).

*Proof.* By Theorem 6.1, the  $C^{1+}$  self-renormalisable structure S realises a Gibbs measure  $\nu = \nu_{S,\delta}$ . By Lemma 6.8, we get that  $\rho_{S,\delta} = r_S$ . Since, by Lemma 4.1, the ratio function  $r_S$  determines uniquely the  $C^{1+}$  self-renormalisable structure S, the map  $S \to \rho_{S,\delta}$  is an one-to-one correspondence.

**Definition 8.1.** Let  $\mathbf{B}_{\iota}$  and  $\mathbf{B}_{\iota'}$  be (gap or no-gap) train-tracks. Let  $\rho$  be an  $\iota$ -measure ratio function and  $\nu = \nu_{\rho}$  on  $\Theta$  the corresponding Gibbs measure (see Remark 5.8). Let us denote by  $\mathcal{D}^{\iota}(\nu, \delta, P)$ the set of all  $C^{1+}$  self-renormalisable structures S with geometric natural measure  $\mu_{S,\delta} = (i_{\iota}) * \nu_{\iota}$ and pressure P. By Lemma 8.1, if  $\mathbf{B}^{\iota}$  is a no-gap train-track, and  $\delta = 1$  and P = 0, the set  $\mathcal{D}^{\iota}(\nu, \delta, P)$  is a singleton.

Let  $\mathbf{B}^{\iota}$  be a gap train-track and  $\mathcal{S} \ a \ C^{1+}$  self-renormalisable structure in  $\mathcal{D}^{\iota}(\nu, \delta, P)$ . In Theorem 7.2, we associate to the  $C^{1+}$  self-renormalisable structure  $\mathcal{S}$  a measure-length ratio cocycle  $J_{\mathcal{S}}$ , and, in Section 7.1, we associate to the  $C^{1+}$  self-renormalisable structure  $\mathcal{S}$  a gap ratio function  $\gamma_{\mathcal{S}}$ . By Lemma 7.9, if  $\mathbf{B}^{\iota'}$  is a no-gap train-track then the cylinder-gap condition of  $r_{\mathcal{S}}$  implies that the pair  $(\gamma_{\mathcal{S}}, J_{\mathcal{S}, \delta})$  satisfies the cocycle-gap condition. Therefore, the map

$$(8.2) \qquad \qquad \mathcal{S} \to (\gamma_{\mathcal{S}}, J_{\mathcal{S},\delta})$$

between  $C^{1+}$  self-renormalisable structures contained in  $\mathcal{D}^{\iota}(\nu, \delta, P)$  and pairs contained in  $\mathcal{J}G^{\iota}(\nu, \delta, P)$  is well-defined.

**Definition 8.2.** The  $(\delta_{\iota}, P_{\iota})$ -bounded solenoid equivalence class of a Gibbs measure  $\nu$  is the set of all solenoid functions  $\sigma_{\iota}$  with the following properties: There is  $C = C(\sigma_{\iota}) > 0$  such that for every pair  $(\xi, D) \in \operatorname{msc}_{\iota}$ 

$$\left|\delta_{\iota} \log s_{\iota}(D_{\Lambda} \cap \xi_{\Lambda} : \xi_{\Lambda}) - \log \rho_{\xi}(D) - nP_{\iota}\right| < C ,$$

where (i)  $\rho$  is the *i*-extended measure scaling function of  $\nu$ , (ii)  $s_i$  is the scaling function determined by  $\sigma_i$ , (iii)  $\xi_{\Lambda} = i(\pi_{\iota'}^{-1}\xi)$  is an  $\iota'$ -leaf primary cylinder segment and (iv)  $D_{\Lambda} = i(\pi_{\iota}^{-1}D)$  and so  $D_{\Lambda} \cap \xi_{\Lambda}$  is an *i*-leaf *n*-cylinder segment.

**Remark 8.2.** Let  $\sigma_{1,\iota}$  and  $\sigma_{2,\iota}$  be two solenoid functions in the same  $(\delta_{\iota}, P_{\iota})$ -bounded solenoid equivalence class of a Gibbs measure  $\nu$ . Using the fact that  $\sigma_{1,\iota}$  and  $\sigma_{2,\iota}$  are bounded away from zero, we obtain that the corresponding scaling functions also satisfy inequality (3.6) for all pairs  $(J, m^{i}J)$  where J is an  $\iota$ -leaf (i + 1)-cylinder. Hence, the solenoid functions  $\sigma_{1,\iota}$  and  $\sigma_{2,\iota}$  are in the same bounded equivalence class (see Definiton 3.2).

By Lemma 8.3, below, the set  $\mathcal{J}G^{\iota}(\nu, \delta, P)$  gives a parametrization of all  $C^{1+}$  self-renormalisable structures  $\mathcal{S}$  which are pre-images of the  $\iota$ -measure ratio function  $\rho_{\nu,\iota}$ , under the map  $\mathcal{S} \to \rho_{\mathcal{S},\delta}$ given in (8.1), with a natural geometric measure  $\mu_{\iota} = (i_{\iota})_*\nu_{\iota}$  and pressure  $P(\mathcal{S}, \delta) = P$ . Hence,  $\mathcal{J}G^{\iota}(\nu, \delta, P)$  forms a moduli space for the set of all  $C^{1+}$  self-renormalisable structures in  $\mathcal{D}^{\iota}(\nu, \delta, P)$ .

**Lemma 8.3.** (Flexibility) Let  $\mathbf{B}_{\iota}$  be a gap train-track. Let  $\rho$  be an  $\iota$ -measure ratio function and  $\nu = \nu_{\rho}$  the corresponding Gibbs measure on  $\Theta$ .

(i) Let  $\delta > 0$  and  $P \in \mathbb{R}$  be such that  $\mathcal{J}G^{\iota}(\nu, \delta, P) \neq \emptyset$ . The map  $S \to (\gamma_S, J_{S,\delta})$  determines a one-to-one correspondence between  $C^{1+}$  self-renormalisable structures in  $\mathcal{D}^{\iota}(\nu, \delta, P)$  and cocycle-gap pairs in  $\mathcal{J}G^{\iota}(\nu, \delta, P)$ .

(ii) A  $C^{1+}$  self-renormalisable structure S is contained in  $\mathcal{D}^{\iota}(\nu, \delta, P)$  if, and only if, the  $\iota$ -solenoid function  $\sigma_{S}$  is contained in the  $(\delta, P)$ -bounded solenoid equivalence class of  $\nu$  (see Definition 8.2).

Proof of (i). Let us prove that  $(J, \gamma) \in \mathcal{J}G^{\iota}(\nu, \delta, P)$  determines an unique  $C^{1+}$  self-renormalisable structure  $\mathcal{S}$  with a natural geometric measure  $\mu_{\mathcal{S},\delta} = (i_{\iota})_*\nu_{\iota}$ . By Lemma 7.9, the pair  $(J, \gamma)$ determines an unique  $\iota$ -ratio function  $r = r_{\iota}(J, \gamma)$ . By Lemma 4.1, the  $\iota$ -ratio function r determines an unique  $C^{1+}$  self-renormalisable structure  $\mathcal{S}$  with an atlas  $\mathcal{B}(r)$ . Let us prove that  $\mu_{\iota} = (i_{\iota})_*\nu_{\iota}$  is a natural geometric measure of  $\mathcal{S}$  with the given  $\delta$  and P. Let  $\rho$  be the  $\iota$ -measure ratio function associated to the Gibbs measure  $\nu$ . By Lemma 5.2, for every leaf n-cylinder or n-gap segment Iwe obtain that

(8.3) 
$$\mu_{\iota}(I) = \mathcal{O}(\rho(I \cap \xi : \xi))$$

for every  $\xi \in \pi_{\iota'}(I)$ . On the other hand, by construction of the ratio function  $r_{\iota}$  and using (7.5), we get

$$\rho(I \cap \xi : \xi) = e^{-nP} r(I \cap \xi : \xi)^{\delta} \prod_{j=0}^{n-1} \left( J\left(\tau_{\iota'}^{j}(\xi)\right) \right)^{-\delta} .$$

Since J is a Hölder cocycle, it follows that  $\prod_{j=0}^{n-1} J\left(\tau_{\iota'}^{j}(\xi)\right) = k(\xi)/k(\tau_{\iota'}^{n}(\xi))$  is uniformly bounded away from zero and infinity, where k is an Hölder continuous positive function. By (4.1), we get that

(8.4) 
$$r(I \cap \xi : \xi) = \mathcal{O}\left(|I|_j\right)$$

where  $j \in \mathcal{B}(r)$  and I is contained in the domain of j. Hence,

(8.5) 
$$\rho(I \cap \xi : \xi) = \mathcal{O}\left(|I|_j^{\delta} e^{-nP}\right)$$

Putting together equations (8.3) and (8.5), we deduce that  $\mu_{\iota}(I) = \mathcal{O}\left(|I|_{j}^{\delta}e^{-nP}\right)$ , and so  $\mu_{\iota} = (i_{\iota})^{*}\nu_{\iota}$  is a natural geometric measure of  $\mathcal{S}$  with the given  $\delta$  and P.

Proof of (ii). Let  $\mathcal{S}$  be a  $C^{1+}$  self-renormalisable structure in  $\mathcal{D}^{\iota}(\nu, \delta, P)$ . Then, putting together (8.4) and (8.5), there is  $\kappa > 0$  such that

$$|\delta \log r_{\iota}(I \cap \xi : \xi) - \log \rho(I \cap \xi : \xi) - np| < \kappa$$

for every leaf *n*-cylinder *I* and  $\xi \in \pi_{\iota'}(I)$ . Thus the solenoid function  $r|\mathbf{S}^{\iota}$  is in the  $(\delta, P)$ -bounded solenoid equivalence class of  $\nu$ .

Conversely, let S be a  $C^{1+}$  self-renormalisable structure in the  $(\delta, P)$ -bounded solenoid equivalence class of  $\nu$  and  $\mu_{\iota} = (i_{\iota})_* \nu_{\iota}$ , i.e. there is  $\kappa > 0$  such that

(8.6) 
$$|\delta \log r_{\iota}(I \cap \xi : \xi) - \log \rho(I \cap \xi : \xi) - np| < \kappa$$

for every leaf *n*-cylinder *I* and  $\xi \in \pi_{\iota'}(I)$ . Hence, using (8.3) and (8.4) in (8.6), we get  $\mu_{\iota}(I) = \mathcal{O}\left(|I|_j^{\delta} e^{-nP}\right)$ . Since  $\mu_{\iota} = (i_{\iota})_* \nu_{\iota}$  we get that  $\mathcal{S}$  is contained in  $\mathcal{D}^{\iota}(\nu, \delta, P)$ .  $\Box$ 

8.2. Two-dimensional realisations. We start by giving the definition of a natural geometric measure for a  $C^{1+}$  hyperbolic diffeomorphism.

**Definition 8.3.** For  $\iota \in \{s, u\}$ , if  $\mathbf{B}^{\iota}$  is a gap train-track assume  $0 < \delta_{\iota} < 1$ , and if  $\mathbf{B}^{\iota}$  is a no-gap train-track take  $\delta_{\iota} = 1$ .

(i) Let g be a  $C^{1+}$  hyperbolic diffeomorphism in  $\mathcal{T}(f,\Lambda)$ . We say that g has a natural geometric measure  $\mu = \mu_{g,\delta_s,\delta_u}$  with pressures  $P_s = P_s(g,\delta_s,\delta_u)$  and  $P_u = P_u(g,\delta_s,\delta_u)$  if, there is  $\kappa > 1$  such that for all leaf  $n_s$ -cylinder  $I_s$ , for all leaf  $n_u$ -cylinder  $I_u$ ,

(8.7) 
$$\kappa^{-1} < \frac{\mu(R)}{|I_u|^{\delta_u} |I_s|^{\delta_s} e^{-n_s P_s - n_u P_u}} < \kappa ,$$

where R is the  $(n_s, n_u)$ -rectangle  $[I_s, I_u]$  and where the lengths  $|\cdot|$  are measured in the stable and unstable  $C^{1+}$  foliated lamination atlasses  $\mathcal{A}_s(g, \rho)$  and  $\mathcal{A}_u(g, \rho)$  of g with respect to some Riemannian metric  $\rho$ .

(ii) We say that a  $C^{1+}$  hyperbolic diffeomorphism with a natural geometric measure  $\mu = \mu_{g,\delta_s,\delta_u}$  with pressures  $P_s = P_s(g,\delta_s,\delta_u)$  and  $P_u = P_u(g,\delta_s,\delta_u)$  is a  $C^{1+}$  realisation of a Gibbs measure  $\nu = \nu_{g,\delta_s,\delta_u}$  if  $\mu = i_*\nu$ . We denote by  $\mathcal{T}(\nu,\delta_s,P_s,\delta_u,P_u)$  the set of all these  $C^{1+}$  hyperbolic diffeomorphisms  $g \in \mathcal{T}(f,\Lambda)$ .

A  $C^{1+}$  hyperbolic diffeomorphism  $g \in \mathcal{T}(f,\Lambda)$  determines an unique pair  $(\mathcal{S}(g,s),\mathcal{S}(g,u))$ of  $C^{1+}$  stable and unstable self-renormalisable structures (see Lemma 4.2). By Theorem 6.1, for  $\delta_s > 0$  and  $\delta_u > 0$ , the pair  $(\mathcal{S}(g,s),\mathcal{S}(g,u))$  of self-renormalisable structures determines an unique pair of natural geometric measures  $(\mu_{\mathcal{S}(g,s),\delta_s},\mu_{\mathcal{S}(g,u),\delta_u})$  corresponding to a unique pair of Gibbs measures  $(\nu_{\mathcal{S}(g,s),\delta_s},\nu_{\mathcal{S}(g,u),\delta_u})$ . Furthermore, by Lemma 6.8, the self-renormalisable structures  $(\mathcal{S}(g,s),\mathcal{S}(g,u))$  determine a pair of measure ratio functions  $(\rho_{\mathcal{S}(g,s),\delta_s},\rho_{\mathcal{S}(g,u),\delta_u})$  of  $(\nu_{\mathcal{S}(g,s),\delta_s},\nu_{\mathcal{S}(g,u),\delta_u})$ .

**Lemma 8.4.** Let g be a  $C^{1+}$  hyperbolic diffeomorphism contained in  $\mathcal{T}(f, \Lambda)$ . The following statements are equivalent:

(i) g has a natural geometric measure μ<sub>g,δ<sub>s</sub>,δ<sub>u</sub>;
(ii) g is a C<sup>1+</sup> realisation of a Gibbs measure ν<sub>g,δ<sub>s</sub>,δ<sub>u</sub>;
(iii) ν<sub>S(g,s),δ<sub>s</sub></sub> = ν<sub>S(g,u),δ<sub>u</sub></sub>;
(iv) The s-measure ratio function ρ<sub>S(g,s),δ<sub>s</sub></sub> is dual to the u-measure ratio function ρ<sub>S(g,u),δ<sub>u</sub></sub>.
</sub></sub>

Furthermore, if g has a natural geometric measure  $\mu_{g,\delta_s,\delta_u}$ , then  $(\pi_s)_*\mu_{g,\delta_s,\delta_u} = \mu_{\mathcal{S}(g,s),\delta_s}$  and  $(\pi_u)_*\mu_{g,\delta_s,\delta_u} = \mu_{\mathcal{S}(g,u),\delta_u}$ .

*Proof.* By Lemma 6.9, (iii) is equivalent to (iv). By definition if g is a  $C^{1+}$  realisation of a Gibbs measure  $\nu_{g,\delta_s,\delta_u}$  then  $\mu_{g,\delta_s,\delta_u} = i_*\nu_{g,\delta_s,\delta_u}$  is a natural geometric measure of g, and so (ii) implies (i). Let us prove first that (i) implies (ii) and (iii), and secondly that (iii) implies (i). Then the last paragraph of this lemma follows from (8.10) below which ends the proof.

(i) implies (ii) and (iii). Let  $\mu_{g,\delta_s,\delta_u}$  be the natural geometric measure of g. Since the stable and unstable lamination atlasses  $\mathcal{A}_s(g,\rho)$  and  $\mathcal{A}_u(g,\rho)$  of g are  $C^{1+}$  foliated (see Section 2.13) and by construction of the  $C^{1+}$  train-track atlasses  $\mathcal{B}_s(g,\rho)$  and  $\mathcal{B}_u(g,\rho)$ , in Section 4.2, we obtain that there is  $\kappa_1 \geq 1$  with the property that, (for  $\iota = s$  and u) and for every  $\iota$ -leaf  $n_{\iota}$ -cylinder I,

(8.8) 
$$\kappa_1^{-1} |I|_{\rho} \le |I'|_j \le \kappa_1 |I|_{\rho}$$

where  $I' = \pi_{\mathbf{B}^{\iota}}(I)$ , where  $|I'|_j$  is measured in any chart  $j \in \mathcal{B}_{\iota}(g, \rho)$  and where  $|I|_{\rho}$  is the length in the Riemannian metric  $\rho$  of the minimal full  $\iota$ -leaf containing I. Let  $I'_{\Lambda}$  be the  $(1, n_{\iota})$ -rectangle in  $\Lambda$  such that  $\pi_{\mathbf{B}^{\iota}}(I'_{\Lambda}) = I'$ . Noting that  $(\pi_{\mathbf{B}^{\iota}})_* \mu_{g,\delta_s,\delta_u}(I') = \mu_{g,\delta_s,\delta_u}(I'_{\Lambda})$ , by (8.7) and (8.8), there is  $\kappa_2 \geq 1$  such that

(8.9) 
$$\kappa_2^{-1} \leq \frac{(\pi_{\mathbf{B}^\iota})_* \mu_{g,\delta_s,\delta_u}(I')}{|I'|_{\delta_\iota}^{\delta_\iota} e^{-n_\iota P_\iota}} \leq \kappa_2 ,$$

for every  $n_{\iota}$ -cylinder I' on the train-track. By Theorem 6.1, the natural geometric measure determined by the  $C^{1+}$  self-renormalisable structure  $\mathcal{S}(g,\iota)$  and by  $\delta_{\iota} > 0$  is uniquely determined by (8.9). Hence,

(8.10) 
$$(\pi_{\mathbf{B}^s})_* \mu_{g,\delta_s,\delta_u} = \mu_{\mathcal{S}(g,s),\delta_s} \text{ and } (\pi_{\mathbf{B}^u})_* \mu_{g,\delta_s,\delta_u} = \mu_{\mathcal{S}(g,u),\delta_u} .$$

Therefore, the Gibbs measures  $\nu_{\mathcal{S}(g,s),\delta_s}$  and  $\nu_{\mathcal{S}(g,u),\delta_u}$  on  $\Theta$  are equal which proves (iii), and  $\mu_{g,\delta_s,\delta_u} = i_*\nu_{\mathcal{S}(g,s),\delta_s} = i_*\nu_{\mathcal{S}(g,u),\delta_u}$  which proves (ii).

(iii) implies (i). Let us denote  $\nu_{\mathcal{S}(g,s),\delta_s} = \nu_{\mathcal{S}(g,u),\delta_u}$  by  $\nu$ . Let  $\mu = i_*\nu$ . For  $\iota \in \{s, u\}$ , by definition of a  $C^{1+}$  realisation of a Gibbs measure as a self-renormalisable structure  $\mathcal{S}(g, \delta_\iota)$ , for every  $\iota$ -leaf  $n_\iota$ -cylinder  $I_\iota$ , there is  $\kappa_3 \geq 1$  such that

$$\kappa_3^{-1} \le \frac{\mu_\iota(I'_\iota)}{|I'_\iota|_j^{\delta_\iota} e^{-n_\iota P_\iota}} \le \kappa_3 ,$$

where  $I'_{\iota} = \pi_{\mathbf{B}^{\iota}}(I)$  and  $|I'_{\iota}|_{j}$  is measured in any chart  $j \in \mathcal{B}_{\iota}(g, \rho)$ . Hence, by (8.8), for  $\iota = s$  and u, we obtain that

(8.11) 
$$\mu_{\iota}(I_{\iota}') = \mathcal{O}\left(|I_{\iota}|_{\rho}^{\delta_{\iota}} e^{-n_{\iota}P_{\iota}}\right)$$

Let R be the rectangle  $[I_s, I_u]$ . By Theorem 6.4,

$$\mu(R) = \int_{I'_{\iota'}} \rho_{\iota,\xi}(R:M) \mu_{\iota'}(d\xi) ,$$

where M is the Markov rectangle containing R. By Theorem 6.1 (i) and (ii), we get that  $\rho_{\iota,\xi}(R : M) = \mathcal{O}(\mu_{\iota}(I'_{\iota}))$  for every  $\xi \in \pi_{\mathbf{B}^{\iota'}}(R)$ . Hence

(8.12) 
$$\mu(R) = \mathcal{O}(\mu_s(I'_s)\mu_u(I'_u)) \;.$$

Putting together (8.11) and (8.12), we get

(8.13) 
$$\mu(R) = \mathcal{O}\left(|I_u|_{\rho}^{\delta_u}|I_s|_{\rho}^{\delta_s}e^{-n_sP_s-n_uP_u}\right)$$

and so  $\mu$  is a natural geometric measure.

**Lemma 8.5.** The map  $g \to (\mathcal{S}(g,s), \mathcal{S}(g,u))$  gives a one-to-one correspondence between  $C^{1+}$  conjugacy classes of hyperbolic diffeomorphisms contained in  $\mathcal{T}(\nu, \delta_s, P_s, \delta_u, P_u)$  and pairs of  $C^{1+}$  self-renormalisable structures contained in  $\mathcal{D}^s(\nu, \delta_s, P_s) \times \mathcal{D}^u(\nu, \delta_u, P_u)$ .

Proof. By Lemma 8.4, if  $g \in \mathcal{T}(\nu, \delta_s, P_s, \delta_u, P_u)$  then, for  $\iota \in \{s, u\}$ ,  $\mathcal{S}(g, \iota) \in \mathcal{D}^{\iota}(\nu, \delta_\iota, P_\iota)$ . Conversely, by Lemma 4.2, a pair  $(\mathcal{S}_s, \mathcal{S}_u) \in \mathcal{D}^s(\nu, \delta_s, P_s) \times \mathcal{D}^u(\nu, \delta_u, P_u)$  determines a  $C^{1+}$  hyperbolic diffeomorphism g such that  $\mathcal{S}(g, s) = \mathcal{S}_s$  and  $\mathcal{S}(g, u) = \mathcal{S}_u$  and  $\nu_{\mathcal{S}(g,s),\delta_s} = \nu_{\mathcal{S}(g,u),\delta_u} = \nu$ . Therefore, by Lemma 8.4, we obtain that g is a  $C^{1+}$  realisation of the Gibbs measure  $\nu$ .

**Lemma 8.6.** (Dual-rigidity) Let  $\mathbf{B}^{\iota'}$  be a no-gap train-track (and so  $\delta_{\iota'} = 1$  and  $P_{\iota'} = 0$ ). For every  $\delta_{\iota} > 0$  and every  $C^{1+} \iota$ -self-renormalisable structure  $\mathcal{S}_{\iota}$  there is an unique  $C^{1+} \iota'$ -self-renormalisable structure  $\mathcal{S}_{\iota}$  there is an unique  $C^{1+} \iota'$ -self-renormalisable structure  $\mathcal{S}_{\iota'}$  such that the  $C^{1+}$  hyperbolic diffeomorphism g corresponding to the pair ( $\mathcal{S}_s, \mathcal{S}_u$ ) =  $(\mathcal{S}(g,s), \mathcal{S}(g,u))$  has a natural geometric measure  $\mu_{g,\delta_s,\delta_u}$ . Furthermore,  $\mu_{\mathcal{S}_s,\delta_s} = (\pi_{\mathbf{B}^s})_* \mu_{g,\delta_s,\delta_u}$  and  $\mu_{\mathcal{S}_u,\delta_u} = (\pi_{\mathbf{B}^u})_* \mu_{g,\delta_s,\delta_u}$ .

Proof. By Theorem 6.1, a  $C^{1+}$  self-renormalisable structure  $S_{\iota}$  and  $\delta_{\iota} > 0$  determine an unique Gibbs measure  $\nu = \nu_{S_{\iota},\delta_{\iota}}$  and  $P_{\iota} \in \mathbb{R}$  such that  $S_{\iota} \in \mathcal{D}^{\iota}(\nu, \delta_{\iota}, P_{\iota})$  is a  $C^{1+}$  realisation of  $\nu$ . By Lemma 6.8, the  $C^{1+}$  self-renormalisable structure  $S_{\iota}$  determines an  $\iota$ -measure ratio function  $\rho_{S_{\iota},\delta_{\iota}}$  determines an unique  $\iota'$ -measure ratio function  $\rho_{\iota'}$  of  $\nu$  on  $\Theta$ . By Lemma 8.1, there is an unique  $C^{1+}$  self-renormalisable structure  $S_{\iota'}$ , with  $\iota'$ -measure ratio function  $\rho_{S_{\iota'},1} = \rho_{\iota'}$ , which is a  $C^{1+}$  realisation of the Gibbs measure  $\nu$ . By Lemma 4.2, the pair  $(S_s, S_u)$  determines a  $C^{1+}$  hyperbolic diffeomorphism g such that  $S(g,s) = S_s$  and  $S(g,u) = S_u$ . Hence,  $\nu_{S(g,s),\delta_s} = \nu$  and  $\nu_{S(g,u),\delta_u} = \nu$  which implies that  $\nu_{S(g,s),\delta_s} = \nu_{S(g,u),\delta_u}$ . Therefore, by Lemma 8.4, g is a  $C^{1+}$  realisation of the Gibbs measure  $\nu$  with natural geometric measure  $\mu_{g,\delta_s,\delta_u} = i_*\nu$ . Thus,  $\mu_{S_s,\delta_s} = (\pi_{\mathbf{B}^s})_*\mu_{g,\delta_s,\delta_u}$  and  $\mu_{S_u,\delta_u} = (\pi_{\mathbf{B}^u})_*\mu_{g,\delta_s,\delta_u}$ .

Recall the definition of the maps  $g \to (\mathcal{S}(g,s), \mathcal{S}(g,u))$  and  $\mathcal{S}(g,\iota) \to (\gamma_{\mathcal{S}(g,\iota)}, J_{\mathcal{S}(g,\iota),\delta_{\iota}})$  for  $\iota$  equal to s and u.

**Theorem 8.7.** (Flexibility) Let  $\mathbf{B}_{\iota}$  be a gap train-track. Let  $\nu$  be a Gibbs measure determining an  $\iota$ -measure ratio function. Let  $\delta_{\iota} > 0$  and  $P_{\iota} \in \mathbb{R}$  be such that  $\mathcal{J}G^{\iota}(\nu, \delta_{\iota}, P_{\iota}) \neq \emptyset$ .

(i) (Smale horseshoes) Let  $\delta_{\iota'} > 0$  and  $P_{\iota'} \in \mathbb{R}$  be such that  $\mathcal{J}G^{\iota'}(\nu, \delta_{\iota'}, P_{\iota'}) \neq \emptyset$ . The map

 $g \to (\gamma_{\mathcal{S}(g,s)}, J_{\mathcal{S}(g,s),\delta_s}, \gamma_{\mathcal{S}(g,u)}, J_{\mathcal{S}(g,u),\delta_u})$ 

gives an one-to-one correspondence between  $C^{1+}$  conjugacy classes of hyperbolic diffeomorphisms in  $\mathcal{T}(\nu, \delta_s, P_s, \delta_u, P_u)$  and pairs of stable and unstable cocycle-gap pairs in  $\mathcal{J}G^s(\nu, \delta_s, P_s) \times \mathcal{J}G^u(\nu, \delta_u, P_u)$ .

(ii) (Codimension one attractors and repellors) Let  $\delta_{\iota'} = 1$  and  $P_{\iota'} = 0$ . The map  $g \to (\gamma_{\mathcal{S}(g,\iota)}, J_{\mathcal{S}(g,\iota),\delta_{\iota}})$  gives an one-to-one correspondence between  $C^{1+}$  conjugacy classes of hyperbolic diffeomorphisms in  $\mathcal{T}(\nu, \delta_s, P_s, \delta_u, P_u)$  and pairs of stable and unstable cocycle-gap pairs in  $\mathcal{J}G^{\iota}(\nu, \delta_{\iota}, P_{\iota})$ .

*Proof.* Statement (i) follows from putting together the results of lemmas 8.3 and 8.5. Statement (ii) follows as statement (i) using the fact that, by Lemma 8.1, the  $C^{1+}$  self-renormalisable structure  $\mathcal{S}(g,\iota)$  uniquely determines  $\mathcal{S}(g,\iota')$  in this case.

#### 9. Eigenvalues

In this section, we show that the set of stable and unstable eigenvalues of all periodic points of the hyperbolic diffeomorphisms is a complete invariant of the Lipschitz conjugacy classes extending the results of De la Llave, Marco and Moriyon. Furthermore, we extend the eigenvalue formula of A. N. Livšic and Ja. G. Sinai from Anosov diffeomorphisms to  $C^{1+}$  hyperbolic diffeomorphisms.

## 9.1. Lipschitz conjugacy classes.

**Lemma 9.1.** Let  $\mathbf{B}^{\iota}$  be a (gap or a no-gap) train-track. Let  $\delta > 0$  and  $P \in \mathbb{R}$ . Let  $S_1 \in \mathcal{D}^{\iota}(\nu_1, \delta, P)$ and  $S_2 \in \mathcal{D}^{\iota}(\nu_2, \delta, P)$  be  $C^{1+}$  self-renormalisable structures The following statements are equivalent:

- (i) The  $C^{1+}$  self-renormalisable structures  $S_1$  and  $S_2$  are Lipschitz conjugate;
- (ii) The Gibbs measures  $\nu_1$  and  $\nu_2$  are equal;
- (iii) The solenoid functions  $s_{S_1}$  and  $s_{S_2}$  are in the same bounded equivalence class (Definition 3.2).

Proof that (i) is equivalent to (ii). Using (6.1), if  $\nu_1 = \nu_2$  then the  $C^{1+}$  self-renormalisable structure  $S_1$  is Lipschitz conjugate to  $S_2$ . Conversely, if  $S_1$  is Lipschitz conjugate to  $S_2$  then the  $C^{1+}$  self-renormalisable structure  $S_1$  (and  $S_2$ ) satisfies (6.1) with respect to the measures  $\mu_{\iota,1} = (i_{\iota})_*\nu_{\iota,1}$  and  $\mu_{\iota,2} = (i_{\iota})_*\nu_{\iota,2}$ . By Theorem 6.1, there is an unique  $\tau$ -invariant Gibbs measure satisfying (6.1) and so  $\nu_1 = \nu_2$ .

Proof that (ii) is equivalent to (iii). Using (3.6) and (4.1), we obtain that the  $C^{1+}$  self-renormalisable structures S and S' on  $\mathbf{B}^{\iota}$  are in the same Lipschitz equivalence class if, and only if, the corresponding solenoid functions  $r_{\mathcal{S},\iota}|\mathbf{S}^{\iota}$  and  $r_{\mathcal{S}',\iota}|\mathbf{S}^{\iota}$  are in the same bounded equivalence class. Hence, statement (ii) is equivalent to statement (iii).  $\Box$ 

**Lemma 9.2.** Let  $g_1$  and  $g_2$  be  $C^{1+}$  hyperbolic diffeomorphisms. The following statements are equivalent:

- (i) The diffeomorphism  $g_1$  is Lipschitz conjugate to  $g_2$ .
- (ii) For  $\iota$  equal to s and u,  $S(g_1, \iota)$  is Lipschitz conjugate to  $S(g_2, \iota)$ .
- (iii) For  $\iota$  equal to s and u, the solenoid functions  $s_{g_1,\iota}$  and  $s_{g_2,\iota}$  are in the same

bounded equivalence class (Definition 3.2).

Proof that (i) is equivalent to (ii). For all  $x \in \Lambda$ , let A be a small open set of M containing x, and let R be a rectangle (not necessarily a Markov rectangle) such that  $A \cap \Lambda \subset R$ . We construct an orthogonal chart  $j : R \to \mathbb{R}^2$  as follows. Let  $e_{g,s} : \ell^s(x, R) \to \mathbb{R}$  be a chart contained in  $\mathcal{A}^s(g, \rho)$ and  $e_{g,u} : \ell^u(x, R) \to \mathbb{R}$  be a chart contained in  $\mathcal{A}^u(g, \rho)$ . The orthogonal chart j on R is now given



FIGURE 16. An orthogonal chart.

by  $j(z) = (e_{g,s}[z, x]), e_{g,u}[x, z]) \in \mathbb{R}^2$  (see Figure 16). By [24], the orthogonal chart  $j: R \to \mathbb{R}^2$  has an extension  $\hat{j}: B \to \mathbb{R}^2$  to an open set B of the surface such that  $\hat{j}$  is  $C^{1+}$  compatible with the charts in the  $C^{1+}$  structure  $\mathcal{C}(g)$  of the surface M. Hence, using the orthogonal charts, any two  $C^{1+}$ hyperbolic diffeomorphisms  $g_1$  and  $g_2$  are Lipschitz conjugate if, and only if the charts in  $\mathcal{A}^{\iota}(g_1, \rho_1)$ are bi-Lipschitz compatible with the charts in  $\mathcal{A}^{\iota}(g_2, \rho_2)$  for  $\iota$  equal to s and u. By construction of the train-track atlasses  $\mathcal{B}^{\iota}(g_1, \rho_1)$  and  $\mathcal{B}^{\iota}(g_2, \rho_2)$  from the lamination atlasses  $\mathcal{A}^{\iota}(g_1, \rho_1)$  and  $\mathcal{A}^{\iota}(g_2, \rho_2)$ , the charts in  $\mathcal{A}^{\iota}(g_1, \rho_1)$  are bi-Lipschitz compatible with the charts in  $\mathcal{A}^{\iota}(g_2, \rho_2)$  if, and only if, the charts in  $\mathcal{B}^{\iota}(g_1, \rho_1)$  are bi-Lipschitz compatible with the charts in  $\mathcal{B}^{\iota}(g_2, \rho_2)$ . Hence, the  $C^{1+}$  hyperbolic diffeomorphisms  $g_1$  and  $g_2$  are Lipschitz conjugate if, and only if, for  $\iota$  equal to s and u, the corresponding  $C^{1+}$  self-renormalisable structures  $\mathcal{S}(g_1, \iota)$  and  $\mathcal{S}(g_2, \iota)$  are Lipschitz conjugate. Therefore, statement (i) is equivalent to statement (ii). *Proof that (ii) is equivalent to (iii)*. Follows from Lemma 9.1.  $\Box$ 

**Lemma 9.3.** Let  $\delta_s > 0$ ,  $\delta_u > 0$  and  $P_s, P_u \in \mathbb{R}$ .

(i) A C<sup>1+</sup> hyperbolic diffeomorphism g is contained in T(ν, δ<sub>s</sub>, P<sub>s</sub>, δ<sub>u</sub>, P<sub>u</sub>) if, and only if, for ι equal to s and u, the ι-solenoid function σ<sub>g,ι</sub> is contained in the (δ<sub>ι</sub>, P<sub>ι</sub>)-bounded solenoid equivalence class of ν (see Definition 8.2).
(ii) If g<sub>1</sub> ∈ T(ν<sub>1</sub>, δ<sub>s</sub>, P<sub>s</sub>, δ<sub>u</sub>, P<sub>u</sub>) and g<sub>2</sub> ∈ T(ν<sub>2</sub>, δ<sub>s</sub>, P<sub>s</sub>, δ<sub>u</sub>, P<sub>u</sub>) are C<sup>1+</sup> hyperbolic diffeomorphisms then g<sub>1</sub> is Lipschitz conjugate to g<sub>2</sub> if, and only if, ν<sub>1</sub> = ν<sub>2</sub>.

Proof of (i). By Lemma 4.2, the  $C^{1+}$  hyperbolic diffeomorphism g determines an unique pair  $(\mathcal{S}(g,s), \mathcal{S}(g,u))$  of  $C^{1+}$  self-renormalisable structures such that  $\sigma_{g,s} = \sigma_{\mathcal{S}(g,s),s}$  and  $\sigma_{g,u} = \sigma_{\mathcal{S}(g,u),u}$ . By Lemma 8.5,  $g \in \mathcal{T}(\nu, \delta_s, P_s, \delta_u, P_u)$  if, and only if,  $(\mathcal{S}(g,s), \mathcal{S}(g,u)) \in \mathcal{D}^s(\nu, \delta_s, P_s) \times \mathcal{D}^u(\nu, \delta_u, P_u)$ . By Lemma 8.3 (ii), for  $\iota$  equal to s and  $u, \mathcal{S}(g,\iota) \in \mathcal{D}^\iota(\nu, \delta_\iota, P_\iota)$  if, and only if,  $\mathcal{S}(g,\iota)$  is contained in the  $(\delta_\iota, P_\iota)$ -bounded solenoid equivalence class of  $\nu$  which ends the proof.

Proof of (ii). By Lemma 8.5,  $g_1 \in \mathcal{T}(\nu_1, \delta_s, P_s, \delta_u, P_u)$  and  $g_2 \in \mathcal{T}(\nu_2, \delta_s, P_s, \delta_u, P_u)$  if, and only if, for  $\iota$  equal to s and u,  $\mathcal{S}(g_1, \iota) \in \mathcal{D}^{\iota}(\nu_1, \delta_{\iota}, P_{\iota})$  and  $\mathcal{S}(g_2, \iota) \in \mathcal{D}^{\iota}(\nu_2, \delta_{\iota}, P_{\iota})$ . By Lemma 9.1,  $\mathcal{S}(g_1, \iota)$  and  $\mathcal{S}(g_2, \iota)$  are Lipschitz conjugate if, and only if,  $\nu_1 = \nu_2$ . Since, by Lemma 9.2,  $g_1$  and  $g_2$  are Lipschitz conjugate if, and only if, for  $\iota$  equal to s and u,  $\mathcal{S}(g_1, \iota)$  and  $\mathcal{S}(g_2, \iota)$  are Lipschitz conjugate, we get that  $g_1$  and  $g_2$  are Lipschitz conjugate if, and only if,  $\nu_1 = \nu_2$ .  $\Box$ 

9.2. Extending the result of De la Llave, Marco and Moriyon. Let  $\mathcal{P}$  be the set of all periodic points in  $\Lambda$  under f. Let p(x) be the (smallest) period of the periodic point  $x \in \mathcal{P}$ . For every  $x \in \mathcal{P}$  and  $\iota \in \{s, u\}$ , let  $j : J \to \mathbb{R}$  be a chart in  $\mathcal{A}(g, \rho_g)$  such that  $x \in J$ . The eigenvalue  $\lambda_{a,\iota}^{\iota}(x)$  of x is the derivative of the map  $j^{-1}f^pj$  at j(x).

For  $\iota \in \{s, u\}$ , by construction of the train-tracks,  $\mathcal{P}^{\iota} = \pi_{\mathbf{B}^{\iota}}(\mathcal{P})$  is the set of all periodic points in  $\mathbf{B}^{\iota}$  under the Markov map  $f_{\iota}$ . Furthermore,  $\pi_{\mathbf{B}^{\iota}}|\mathcal{P}$  is an injection and the periodic points  $x \in \Lambda$ and  $\pi_{\mathbf{B}^{\iota}}(x) \in \mathbf{B}^{\iota}$  have the same period  $p(x) = p(\pi_{\mathbf{B}^{\iota}}(x))$ . Let us denote  $\pi_{\mathbf{B}^{\iota}}(x)$  by  $x_{\iota}$ . Let  $\mathcal{S}_{\iota}$  be a  $C^{1+}$  self-renormalisable structure. Let  $j: J \to \mathbb{R}$  be a train-track chart of  $\mathcal{S}_{\iota}$  such that  $x_{\iota} \in J$ . The *eigenvalue*  $\lambda_{\mathcal{S}_{\iota}}(x_{\iota})$  of  $x_{\iota}$  is the derivative of the map  $j \circ \tau_{\iota}^{p(x_{\iota})} \circ j^{-1}$  at  $j(x_{\iota})$ , where  $\tau_{\iota}$  is the Markov map on the train-track  $\mathbf{B}^{\iota}$ .

For every  $x \in \mathcal{P}$ , every  $\iota \in \{s, u\}$  and every  $n \ge 0$ , let  $I_n^{\iota}(x)$  be an  $\iota$ -leaf (np(x) + 1)-cylinder segment such that  $x \in I_n^{\iota}(x)$  and  $f_{\iota}^{p(x)} I_{n+1}^{\iota}(x) = I_n^{\iota}(x)$ .

**Lemma 9.4.** For  $\iota \in \{s, u\}$ , let  $S_{\iota} \in \mathcal{D}^{\iota}(\nu, \delta_{\iota}, P_{\iota})$  be a  $C^{1+}$   $\iota$  self-renormalisable structure. For every  $x \in \mathcal{P}$ ,

(9.1) 
$$\lambda_{\mathcal{S}_{\iota}}(x_{\iota}) = r_{\mathcal{S}_{\iota}}(I_{0}^{\iota}(x):I_{1}^{\iota}(x))$$

(9.2) 
$$= \rho_{\nu,\iota} (I_0^{\iota} : I_1^{\iota})^{-1/\delta_{\iota}} e^{-p(x)P_{\iota}/\delta_{\iota}}$$

(9.3) 
$$= \rho_{\nu,\iota'} (I_0^{\iota'} : I_1^{\iota'})^{-1/\delta_\iota} e^{-p(x)P_\iota/\delta_\iota} ,$$

where  $r_{S_{\iota}}$  is the  $\iota$ -ratio function of  $S_{\iota}$ ,  $\rho_{\nu,\iota}$  is the  $\iota$ -measure ratio function of the Gibbs measure  $\nu$ , and  $\rho_{\nu,\iota'}$  is the  $\iota'$ -measure ratio function of the Gibbs measure  $\nu$ .

*Proof.* For every  $x \in \mathcal{P}$ , let us denote by p the period p(x) of x, and let us denote by  $I_n^\iota$  the interval  $I_n^\iota(x)$ . We note that the p-mother  $m^p I_{n+1}^\iota$  of  $I_{n+1}^\iota$  is  $I_n^\iota$ , and so  $f_\iota^p I_{n+1}^\iota = m^p I_{n+1}^\iota$ . By (4.1),

$$r_{\mathcal{S}_{\iota}}(I_0^{\iota}:I_1^{\iota}) = \lim_{n \to \infty} \frac{|I_n^{\iota}|}{|I_{n+1}^{\iota}|} .$$

Hence,

$$\begin{aligned} \lambda_{\mathcal{S}_{\iota}}(x_{\iota}) &= \lim_{n \to \infty} \frac{|f_{\iota}^{p} I_{n+1}^{\iota}|}{|I_{n+1}^{\iota}|} \\ &= \lim_{n \to \infty} \frac{|I_{\iota}^{\iota}|}{|I_{n+1}^{\iota}|} \\ &= r_{\mathcal{S}_{\iota}}(I_{0}^{\iota}:I_{1}^{\iota}) \end{aligned}$$

which proves (9.1). By Theorem 6.8, the  $\iota$ -measure ratio function  $\rho_{\mathcal{S}_{\iota},\delta_{\iota}}$  is the  $\iota$ -measure ratio function  $\rho_{\nu,\iota}$  of the Gibbs measure  $\nu$ . Hence, by (7.5), we get

(9.4)  
$$r_{\mathcal{S}_{\iota}}(I_{1}^{\iota}:I_{0}^{\iota}) = \prod_{l=0}^{p-1} r_{\mathcal{S}_{\iota}}(m^{l}I_{1}^{\iota}:m^{l+1}I_{1}^{\iota}) \\= \prod_{l=0}^{p-1} \left( J_{\mathcal{S},\delta_{\iota}}(\xi_{l})\rho_{\nu,\iota}(m^{l}I_{1}^{\iota}:m^{l+1}I_{1}^{\iota})^{1/\delta_{\iota}}e^{P_{\iota}/\delta_{\iota}} \right)$$

where  $\xi_l = f_l^{p-l} m^l I_1^{\iota} \in \mathbf{B}_o^{\iota}$ . We note that  $f_{\iota'}\xi_l = \xi_{l+1}$  and  $f_{\iota'}\xi_{p-1} = \xi_0$  in  $\mathbf{B}_o^{\iota'}$ . Since  $J_{\mathcal{S}_{\iota},\delta_{\iota}} = \kappa/(\kappa \circ f_{\iota'})$  for some function  $\kappa$ , we get

(9.5) 
$$\prod_{l=0}^{p-1} J_{\mathcal{S}_{\iota},\delta_{\iota}}(\xi_l) = \prod_{l=0}^{p-1} \frac{\kappa(\xi_l)}{\kappa(\xi_{l+1})} = 1$$

Furthermore,

(9.6) 
$$\prod_{l=0}^{p-1} \rho_{\nu,\iota}(m^l I_1^{\iota} : m^{l+1} I_1^{\iota}) = \rho_{\nu,\iota}(I_1^{\iota} : I_0^{\iota})$$

Using (9.5) and (9.6) in (9.4) we obtain that

$$r_{\mathcal{S}_{\iota}}(I_1^{\iota}:I_0^{\iota}) = \rho_{\nu,\iota}(I_1^{\iota}:I_0^{\iota})^{1/\delta_{\iota}} e^{pP_{\iota}/\delta_{\iota}}$$
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Therefore, by (9.1), we have

$$\begin{aligned} \lambda_{\mathcal{S}_{\iota}}(x_{\iota}) &= r_{\mathcal{S}_{\iota}}(I_{0}^{\iota}:I_{1}^{\iota}) \\ &= \rho_{\nu,\iota}(I_{0}^{\iota}:I_{1}^{\iota})^{-1/\delta_{\iota}}e^{-pP_{\iota}/\delta} \end{aligned}$$

which proves (9.2). By Lemma 6.7, there is  $0 < \varepsilon < 1$  such that for every  $n \ge 0$ 

(9.7) 
$$\rho_{\nu,s}(I_{n+1}^s:I_n^s) \in (1\pm\varepsilon^n) \frac{\mu([I_{n+1}^s,I_1^u])}{\mu([I_n^s,I_1^u])}$$

and

(9.8) 
$$\rho_{\nu,u}(I_{n+1}^u:I_n^u) \in (1\pm\varepsilon^n)\frac{\mu([I_1^s,I_{n+1}^u])}{\mu([I_1^s,I_n^u])}$$

Since  $f^{np}\left([I_1^s, I_{n+1}^u]\right) = \left([I_{n+1}^s, I_1^u]\right)$  and by invariance of  $\mu$ , we obtain that

(9.9) 
$$\frac{\mu([I_{n+1}^s, I_1^u])}{\mu([I_n^s, I_1^u])} = \frac{\mu([I_1^s, I_{n+1}^u])}{\mu([I_1^s, I_n^u])}$$

Putting together (9.7), (9.8) and (9.9), we obtain that

$$\rho_{\nu,s}(I_{n+1}^s:I_n^s) \in (1\pm\varepsilon^n)\rho_{\nu,u}(I_{n+1}^u:I_n^u).$$

Hence, by invariance of  $\rho_{\mathcal{S}_{s},s}$  and  $\rho_{\mathcal{S}_{s},u}$  under f, we obtain

$$\rho_{\nu,s}(I_1^s:I_0^s) = \lim_{n \to \infty} \rho_{\nu,s}(I_{n+1}^s:I_n^s) \\
= \lim_{n \to \infty} \rho_{\nu,u}(I_{n+1}^u:I_n^u) \\
= \rho_{\nu,u}(I_1^u:I_0^u)$$

which proves (9.3).

**Lemma 9.5.** Let  $\mathbf{B}^{\iota}$  be a (gap or a no-gap) train-track.

(i) The  $C^{1+}$  self-renormalisable structures  $S_1 \in \mathcal{D}^{\iota}(\nu_1, \delta, P)$  and  $S_2 \in \mathcal{D}^{\iota}(\nu_2, \delta, P)$ have the same eigenvalues for all periodic orbits if, and only if,  $\nu_1$  is equal to  $\nu_2$ . (ii) The set of eigenvalues of all periodic orbits of a  $C^{1+}$  self-renormalisable structure is a complete invariant of each Lipschitz conjugacy class.

Statement (ii) of the above lemma for Markov maps is also in [34].

Proof of (i). By Lemma 9.1, the  $C^{1+}$  self-renormalisable structures  $S_1 \in \mathcal{D}^{\iota}(\nu_1, \delta, P)$  and  $S_2 \in \mathcal{D}^{\iota}(\nu_2, \delta, P)$  are Lipschitz conjugate if, and only if, the Gibbs measures  $\nu_1$  and  $\nu_2$  are equal. By Lemma 9.4, if the Gibbs measures  $\nu_1$  and  $\nu_2$  are equal, then  $S_1$  and  $S_2$  have the same eigenvalues for all periodic orbits. Hence, to finish the proof of statement (i), we are going to prove that if the  $C^{1+}$  self-renormalisable structures  $S_1$  and  $S_2$  have the same eigenvalues for all periodic orbits, then the  $C^{1+}$  self-renormalisable structures  $S_1$  and  $S_2$  have the same eigenvalues for all periodic orbits, then

Without loss of generality, let us assume that  $S_1$  and  $S_2$  are unstable  $C^{1+}$  self-renormalisable structures. For  $j \in \{1, 2\}$ , the (restricted) *u*-scaling function  $z_{u,j} : \Theta^u \to \mathbb{R}^+$  of S is well-defined by (see Section 4.6)

$$z_{u,j}(w_0w_1...) = \lim_{n \to \infty} \frac{|\pi_{\mathbf{B}^s} \circ f^{n+1} \circ \pi_{\mathbf{B}^u}^{-1} \circ i_u(w_0w_1...)|_{k_n}}{|\pi_{\mathbf{B}^s} \circ f^n \circ \pi_{\mathbf{B}^u}^{-1} \circ i_u(w_1w_2...)|_{k_n}},$$

where  $k_n$  is a train-track chart in a  $C^{1+}$  self-renormalisable atlas  $\mathcal{B}_j$  determined by  $\mathcal{S}_j$  such that the domain of the chart  $k_n$  contains  $\pi_{\mathbf{B}^s} \circ f^n \circ \pi_{\mathbf{B}^u}^{-1} \circ i_u(w_1w_2...)$ . For every stable-leaf (i+1)-cylinder J, let  $w(J) \in \Theta^u$  be such that  $i_u(w(J)) = \pi_{\mathbf{B}^u}(J)$ . Hence, for every  $l \in \{0, \ldots, i-1\}$  we have that

$$\pi_{\mathbf{B}^{u}}^{-1} \circ i_{u}(f_{u}^{l}w(J)) = f^{-i+l}(m^{l}J) ,$$

where  $f^{-i+l}(m^l J)$  are stable-leaf primary cylinders. By construction of the (restricted) *u*-scaling function  $z_{u,j}$  and of the *u*-scaling function  $s_{u,j}$  of  $S_j$ , we have that

(9.10) 
$$s_{u,j}(J:m^iJ) = \prod_{l=0}^{i-1} z_{u,j}(f_u^l(w(J))) .$$

Let  $\mathcal{P}_{\Theta^u}$  be the set of all periodic point under the shift. For every  $w = w_0 w_1 \dots \in \mathcal{P}_{\Theta^u}$  let p(w)be the smallest period of w. By construction of the train-tracks, for every w, there is a unique periodic point  $x(w) \in \Lambda$  with period p(w) with respect to the map f such that  $i_u(w) = \pi_{\mathbf{B}^u} x(w)$ . Furthermore, there is a unique periodic point  $\pi_s x(w) \in \mathbf{B}^s$  with period p(w) for the Markov map. By (9.10), for every  $w \in \mathcal{P}_{\Theta^u}$  we have that

(9.11) 
$$\prod_{i=0}^{p(w)-1} z_{u,j}(f_u^i(w)) = \lambda_{\mathcal{S}_j}(\pi_{\mathbf{B}^s} x(w)) \ .$$

Since the  $C^{1+}$  self-renormalisable structures  $S_1$  and  $S_2$  have the same eigenvalues for all periodic orbits, by (9.11), we have that

(9.12) 
$$\prod_{i=0}^{p(w)-1} \frac{z_{u,1}(f_u^i(w))}{z_{u,2}(f_u^i(w))} = 1 ,$$

for every  $w \in \mathcal{P}_{\Theta^u}$ . From Livšic's theorem (e.g. see [12]) we get that

(9.13) 
$$\frac{z_{u,1}(w)}{z_{u,2}(w)} = \frac{\kappa(w)}{\kappa \circ f_u(w)}$$

where  $\kappa : \Theta^u \to \mathbb{R}^+$  is a positive Hölder continuous function. By (9.10) and (9.13), for every stable-leaf (i+1)-cylinder J we obtain that

(9.14) 
$$\frac{s_{u,1}(J:m^{i}J)}{s_{u,2}(J:m^{i}J)} = \prod_{l=0}^{i-1} \frac{z_{u,1}(f_{u}^{l}(w(J)))}{z_{u,2}(f_{u}^{l}(w(J)))} = \frac{\kappa(w)}{\kappa \circ f_{u}^{i}(w)}.$$

Since  $\kappa$  is bounded away from zero and infinity, there is C > 1 such that for all  $w \in \Theta^u$  and  $i \ge 1$ we have that

(9.15) 
$$C^{-1} < \frac{\kappa(w)}{\kappa \circ \tau_u^i(w)} < C$$

Putting together (9.14) and (9.15), we obtain that

$$\frac{s_{u,1}(J:m^iJ)}{s_{u,2}(J:m^iJ)}$$

Therefore, the  $\iota$ -solenoid functions  $\sigma_{u,1} : \mathbf{S}^{\iota} \to \mathbb{R}^+$  and  $\sigma_{u,2} : \mathbf{S}^{\iota} \to \mathbb{R}^+$  corresponding to the  $C^{1+}$  selfrenormalisable structures  $S_1$  and  $S_2$  are in the same bounded equivalence class (see Definition 3.2). Hence, by Lemma 9.1, the  $C^{1+}$  self-renormalisable structures  $S_1$  and  $S_2$  are Lipschitz conjugate. *Proof of (ii)*. Statement (ii) follows from putting together Lemma 9.1 and Statement (i) of this lemma with P = 0.  $\Box$ 

# **Theorem 9.6.** (i) If $g \in \mathcal{T}(f, \Lambda)$ then $\lambda_{g,s}(x) = \lambda_{\mathcal{S}_s}(x_s)^{-1}$ and $\lambda_{g,u}(x) = \lambda_{\mathcal{S}_s}(x_u)$ where, for $\iota \in \{s, u\}, \lambda_{g,\iota}(x)$ is the eigenvalue of the $C^{1+}$ hyperbolic diffeomorphism g and $\lambda_{\mathcal{S}_{\iota}}$ is the eigenvalue of the $C^{1+}$ self-renormalisable structure $\mathcal{S}_{\iota} = \mathcal{S}(g, \iota)$ .

(ii) The set of stable and unstable eigenvalues of all periodic orbits of a  $C^{1+}$  hyperbolic diffeomorphism  $g \in \mathcal{T}(f, \Lambda)$  is a complete invariant of each Lipschitz conjugacy class.

Proof of (i). By construction of the train-track atlas  $\mathcal{B}^{\iota}(g,\rho_g)$  from the lamination atlas  $\mathcal{A}^{\iota}(g,\rho_g)$ in Section 4.2, if  $\lambda_{g,\iota}(x)$  is the eigenvalue of  $x \in \mathcal{P}$  then the eigenvalue of  $x_{\iota} \in \mathcal{P}_{\iota}$  is either  $\lambda_{g,\iota}(x)$  if  $\iota = u$ , or  $\lambda_{g,\iota}^{-1}$  if  $\iota = s$ .

Proof of (ii). By Lemma 9.5, the set of eigenvalues of all periodic orbits of a  $C^{1+}$  self-renormalisable structure is a complete invariant of each Lipschitz conjugacy class of  $C^{1+}$  self-renormalisable structures. Hence, using Lemma 9.2, we get that the set of stable and unstable eigenvalues of all periodic orbits of a  $C^{1+}$  hyperbolic diffeomorphism g is a complete invariant of each Lipschitz conjugacy class.  $\Box$ 

9.3. Extending the eigenvalue formula of A. N. Livšic and Ja. G. Sinai. In Lemma 9.4, we give an explicit formula for the stable and unstable eigenvalues of all periodic points of the  $C^{1+}$  hyperbolic diffeomorphisms in terms of measure ratio functions. We use this formula, in Theorem 9.7 below, to extend the eigenvalue formula of A. N. Livšic and Ja. G. Sinai for Anosov diffeomorphisms to  $C^{1+}$  hyperbolic diffeomorphisms.

**Theorem 9.7.** A  $C^{1+}$  hyperbolic diffeomorphism  $g \in \mathcal{T}(f, \Lambda)$  has a natural geometric measure  $\mu_{g,\delta_s,\delta_u}$  with pressures  $P_s = P_s(g,\delta_s,\delta_u)$  and  $P_u = P_u(g,\delta_s,\delta_u)$  if, and only if, for all  $x \in \Lambda$ 

(9.16) 
$$\lambda_{g,s}(x_s)^{-\delta_s} e^{p(x)P_s} = \lambda_{g,u}(x_u)^{\delta_u} e^{p(x)P_u}$$

*Proof.* By Theorem 6.1, the  $C^{1+}$  self-renormalisable structures S(g, s) and S(g, u) are  $C^{1+}$  realisations of Gibbs measures  $\nu_1 = \nu_{S(g,s),\delta_s}$  and  $\nu_2 = \nu_{S(g,u),\delta_s}$ . By Lemma 9.4 and the statement (i) of Theorem 9.6, for all  $x \in \mathcal{P}$  we have

(9.17) 
$$\lambda_{g,u}(x_u) = \lambda_{S(g,u)}(x_u) \\ = \rho_{\nu_2,u}(I_0^u : I_1^u)^{-1/\delta_u} e^{-p(x)P_u/\delta_u}$$

and

(9.18) 
$$\lambda_{g,s}(x_s) = \lambda_{S(g,s)}(x_s)^{-1} \\ = \rho_{\nu_1,u}(I_0^u : I_1^u)^{1/\delta_s} e^{p(x)P_s/\delta_s} .$$

Let us prove that if the  $C^{1+}$  hyperbolic diffeomorphism g has a natural geometric measure then (9.16) holds. Hence, by Lemma 8.4, the Gibbs measures  $\nu_1$  and  $\nu_2$  are equal. By (9.17), we have

(9.19) 
$$\rho_{\nu_1,u}(I_0^u:I_1^u) = \rho_{\nu_2,u}(I_0^u:I_1^u) \\ = \lambda_{g,u}(x_\iota)^{-\delta_u} e^{-p(x)P_u} .$$

By (9.18), we obtain that

(9.20) 
$$\rho_{\nu_1,u}(I_0^u:I_1^u) = \lambda_{g,s}(x_\iota)^{\delta_s} e^{-p(x)P_s}$$

Putting together (9.19) and (9.20), we obtain that

$$\lambda_{g,s}(x_s)^{-\delta_s} e^{p(x)P_s} = \lambda_{g,u}(x_u)^{\delta_u} e^{p(x)P_u} ,$$

and so (9.16) holds. Conversely, let us prove that if (9.16) holds then the  $C^{1+}$  hyperbolic diffeomorphism g has a natural geometric measure. Putting together (9.16) and (9.18), we obtain that

$$\lambda_{g,u}(x_u) = \rho_{\nu_1,u}(I_0^u : I_1^u)^{-1/\delta_u} e^{-nP_u/\delta_u}$$

Hence, the Gibbs measure  $\nu_1$  determines the same set of eivenvalues for all periodic orbits of self-renormalisable structures in  $\mathbf{B}^u$  as the Gibbs measure  $\nu_2$ . Therefore, by Lemma 9.5,  $\nu_1 = \nu_2$ 

and consequently, by Lemma 8.4, the  $C^{1+}$  hyperbolic diffeomorphism g has a natural geometric measure.

#### 10. Invariant Hausdorff measures

In this section, we present the proofs of all theorems stated in the Introduction. Let  $S_{\iota}$  be a  $C^{1+}$   $\iota$  self-renormalisable structure. By Lemma 6.3, a natural geometric measure  $\mu_{S_{\iota},\delta_{\iota}}$  with pressure  $P(S_{\iota},\delta_{\iota}) = 0$  is an invariant measure absolutely continuous with respect to the Hausdorff measure of  $\mathbf{B}^{\iota}$  and  $\delta_{\iota}$  is the Hausdorff dimension of  $\mathbf{B}^{\iota}$  with respect to the charts of  $S_{\iota}$ . Let us denote  $\mathcal{D}^{\iota}(\nu,\delta_{\iota},0)$  and  $\mathcal{J}G^{\iota}(\nu,\delta_{\iota},0)$  respectively by  $\mathcal{D}^{\iota}(\nu,\delta_{\iota})$  and  $\mathcal{J}G^{\iota}(\nu,\delta_{\iota})$ . By Theorem 6.1, for every  $C^{1+}$   $\iota$  self-renormalisable structure  $S_{\iota}$  there is an unique Gibbs measure  $\nu_{S_{\iota}}$  such that  $S_{\iota} \in \mathcal{D}^{\iota}(\nu,\delta_{\iota})$ . Using Lemma 8.5, we obtain that the sets  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s,\delta_u)$  defined in the introduction are equal to the sets  $\mathcal{T}(\nu, \delta_s, 0, \delta_u, 0)$  (see Definition 8.3).

Theorem 1.1 follows from Theorem 9.7, Theorem 1.8 follows from Theorem 8.7, Theorem 1.10 follows from Lemma 4.2, and Theorem 1.11 follows from Lemma 8.5.

Proof of Theorem 1.2. Proof of statement (i). By Lemma 9.3 (ii), the sets  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  are Lipschitz conjugacy classes in  $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ , and the map  $\nu \to \mathcal{T}(\nu, \delta_s, \delta_u)$  is injective. If  $g \in \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  then g has a natural geometric measure  $\mu_{g,\delta_s,\delta_u}$  with pressures  $P_s(g, \delta_s, \delta_u)$  and  $P_u(g, \delta_s, \delta_u)$  equal to zero. By Lemma 8.4, there is a Gibbs measure  $\nu = \nu_{g,\delta_s,\delta_u}$  on  $\Theta$  such that  $i_*\nu = \mu_{g,\delta_s,\delta_u}$  and so  $g \in [\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ . Hence, the map  $\nu \to \mathcal{T}(\nu, \delta_s, \delta_u)$  is surjective into the Lipschitz conjugacy classes in  $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ .

Proof of statement (ii). By Theorem 9.6 (ii), the set of stable and unstable eigenvalues of all periodic orbits of a  $C^{1+}$  hyperbolic diffeomorphisms  $g \in \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  is a complete invariant of each Lipschitz conjugacy class, and by statement (i) of this lemma the sets  $\mathcal{T}(\nu, \delta_s, \delta_u)$  are the Lipschitz conjugacy classes in  $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ .  $\Box$ 

Proof of Theorem 1.12. We will separate the proof in three parts. In part (i), we prove that if  $S_{\iota} \in \mathcal{D}^{\iota}(\nu, \delta_{\iota})$  then  $\sigma_{\nu,\iota}$  satisfies the properties indicated in Theorem 1.12. In part (ii), we prove the converse of part (i). In part (iii), we prove that  $\mathcal{D}^{\iota}(\nu, \delta_{\iota}) \neq \emptyset$  if, and ony if,  $\mathcal{D}^{\iota'}(\nu, \delta_{\iota'}) \neq \emptyset$ .

Part (i). Let  $S_{\iota} \in \mathcal{D}^{\iota}(\nu, \delta_{\iota})$ . By Lemma 6.8,  $S_{\iota}$  and  $\delta_{\iota}$  determine an unique  $\iota$ -measure ratio function  $\rho_{\nu,\iota}$  of the Gibbs measure  $\nu$ . Hence, the function  $\rho_{\nu,\iota}|\text{msol}^{\iota'}$  is the  $\iota$ -measure solenoid function  $\sigma_{\nu,\iota}$  of  $\nu$  and, by Lemma 5.7,  $\sigma_{\nu,\iota}$  satisfies the properties indicated in Theorem 1.12.

Part (ii). Conversely, if  $\nu$  has an  $\iota$ -solenoid function  $\sigma_{\nu,\iota}$  satisfying the properties indicated in Theorem 1.12, by lemmas 5.4 and 5.7,  $\sigma_{\nu,\iota}$  determines an unique  $\iota$ -measure ratio function  $\rho_{\nu,\iota}$ of  $\nu$ . If  $\mathbf{B}^{\iota}$  is a no-gap train-track, by Lemma 8.1, there is a  $C^{1+}$  self-renormalisable structure  $S_{\iota} \in \mathcal{D}^{\iota}(\nu, \delta_{\iota})$  with  $\delta_{\iota} = 1$ . If  $\mathbf{B}^{\iota}$  is a gap train-track then, by Remark 7.10, the set  $\mathcal{J}G^{\iota}(\nu, \delta_{\iota})$  is non-empty (in fact it is an infinite dimensional space). Hence, by Lemma 8.3, the set  $\mathcal{D}(\nu, \delta_{\iota})$  is also non-empty which ends the proof.

Part (iii). To prove that  $\mathcal{D}^{\iota}(\nu, \delta_{\iota}) \neq \emptyset$  if, and ony if,  $\mathcal{D}^{\iota'}(\nu, \delta_{\iota'}) \neq \emptyset$ , it is enough to prove one of the implications. Let us prove that if  $\mathcal{D}^{\iota}(\nu, \delta_{\iota}) \neq \emptyset$  then  $\mathcal{D}^{\iota'}(\nu, \delta_{\iota'}) \neq \emptyset$ , Let  $\mathcal{S}_{\iota} \in \mathcal{D}^{\iota}(\nu, \delta_{\iota})$ . By Lemma 6.8,  $\mathcal{S}_{\iota}$  and  $\delta_{\iota}$  determine an unique  $\iota$ -measure ratio function  $\rho_{\nu,\iota}$  of the Gibbs measure  $\nu$ . By Lemma 6.9, the  $\iota$ -measure ratio function  $\rho_{\nu,\iota}$  determines an unique dual  $\iota'$ -measure ratio function  $\rho_{\nu,\iota'}$  of  $\nu$ . Hence, the function  $\rho_{\nu,\iota'}|\text{msol}^{\iota'}$  is the  $\iota$ -measure solenoid function  $\sigma_{\nu,\iota'}$  of  $\nu$  and, by Lemma 5.7,  $\sigma_{\nu,\iota'}$  satisfies the properties indicated in Theorem 1.12. Now the proof follows as in part (ii), with  $\iota$  changed by  $\iota'$ , which shows that  $\sigma_{\nu,\iota'}$  determines a non-empty set  $\mathcal{D}^{\iota'}(\nu, \delta_{\iota'})$ .  $\Box$ 

Proof of Theorem 1.4 and Theorem 1.5. Proof that statement (i) implies statements (ii) and (iii). If  $g \in [\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ , by Lemma 8.5, the sets  $\mathcal{D}^s(\nu, \delta_s)$  and  $\mathcal{D}^u(\nu, \delta_u)$  are both non-empty. Hence, by Theorem 1.12, the stable measure solenoid function of the Gibbs measure  $\nu$  satisfies (ii) and the unstable measure solenoid function of the Gibbs measure  $\nu$  satisfies (iii).

Proof that statement (ii) implies statement (i), and that statement (iii) implies statement (i). By Theorem 1.12, the properties of the the  $\iota$ -solenoid function  $\sigma_{\nu,\iota}$  indicated in this theorem imply that  $\mathcal{D}^{\iota}(\nu, \delta_{\iota}) \neq \emptyset$ . Again, by Theorem 1.12 and  $\mathcal{D}^{\iota'}(\nu, \delta_{\iota'}) \neq \emptyset$ . Hence, by Lemma 8.5, the set  $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  is non-empty. Therefore, every  $g \in \mathcal{T}(\nu, \delta_s, \delta_u)$  is a  $C^{1+}$ -Hausdorff realisation of  $\nu$  which ends the proof.  $\Box$ 

Proof of Theorem 1.6. Let  $\nu$  be a Gibbs measure. By Theorem 1.12, the set  $\mathcal{D}^{s}(\nu, \delta_{s})$  and  $\mathcal{D}^{u}(\nu, \delta_{u})$ are both non-empty. Hence, by Lemma 8.5, the set  $\mathcal{T}(\nu, \delta_s, \delta_u)$  is also non-empty. Therefore, every  $g \in \mathcal{T}(\nu, \delta_s, \delta_u)$  is a  $C^{1+}$ -Hausdorff realisation of  $\nu$  which ends the proof.  $\Box$ 

10.1. Moduli space  $SOL^{\iota}$ . Recall the definiton of the set  $SOL^{\iota}$  given in Section 5.5. By Theorem 10.1, below, the set of all  $\iota$ -measure solenoid functions  $\sigma_{\nu}$  with the properties indicated in Theorem 1.12 determine an infinite dimensional metric space  $SOL^{\iota}$  which gives a nice parametrization of all Lipschitz conjugacy classes  $\mathcal{D}^{\iota}(\nu, \delta)$  of  $C^{1+}$  self-renormalisable structures  $\mathcal{S}_{\iota}$  with a given Hausdorff dimension  $\delta$ .

Theorem 1.7 follows from Theorem 10.1.

**Theorem 10.1.** If  $\mathbf{B}^{\iota}$  is a gap train-track assume  $0 < \delta_{\iota} < 1$  and if  $\mathbf{B}^{\iota}$  is a no-gap train-track assume  $\delta_{\iota} = 1$ .

- (i) The map  $S \to \rho_{S,\delta_{\iota}}$  induces an one-to-one correspondence between the sets  $\mathcal{D}^{\iota}(\nu, \delta_{\iota})$ and the elements of  $SOL^{\iota}$ .
- (ii) The map  $g \to \rho_{\mathcal{S}(g,\iota),\delta_{\iota}}$  induces an one-to-one correspondence between the sets  $[\nu]$  contained in  $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  and the elements of  $\mathcal{SOL}^{\iota}$ .

Proof of Theorem 10.1. Proof of (i). If  $\mathcal{S} \in \mathcal{D}^{\iota}(\nu, \delta_{\iota})$  then the Hausdorff dimension of  $\mathcal{S}$  is  $\delta_{\iota}$ , and  $\mathcal{S}$  determines an  $\iota$ -measure ratio function  $\rho_{\mathcal{S},\delta_{\iota}} = \rho_{\nu,\iota}$  which does not depend upon  $\mathcal{S} \in \mathcal{D}^{\iota}(\nu, \delta_{\iota})$ . By Lemma 5.7,  $\rho_{\nu,\iota}|\text{Msol}^{\iota}$  is an element of  $SOL^{\iota}$ . Hence, the map  $S \to \rho_{S,\delta_{\iota}}$  associates to each set  $\mathcal{D}^{\iota}(\nu, \delta)$  an unique element of  $\mathcal{SOL}^{\iota}$ . Conversely, let  $\hat{\sigma} \in \mathrm{Msol}^{\iota}$ . By Lemma 5.7,  $\hat{\sigma}$  determines an unique  $\iota$ -measure ratio function  $\rho_{\hat{\sigma}}$  such that  $\rho_{\hat{\sigma}} | \text{Msol}^{\iota} = \hat{\sigma}$ . By Lemma 5.1, the  $\iota$ -measure ratio function  $\rho_{\hat{\sigma}}$  determines a Gibbs measure  $\nu_{\hat{\sigma}}$ . If  $\mathbf{B}^{\iota}$  is a no-gap train-track then, by Lemma 8.1,  $\rho_{\hat{\sigma}}$ determines a non-empty set  $\mathcal{D}^{\iota}(\nu_{\hat{\sigma}}, \delta_{\iota})$ . If  $\mathbf{B}^{\iota}$  is a gap train-track then, by Remark 7.10, the set  $\mathcal{J}G^{\iota}(\nu, \delta_{\iota})$  is non-empty and so, by Lemma 8.3, the set  $\mathcal{D}^{\iota}(\nu_{\hat{\sigma}}, \delta_{\iota})$  is also non-empty. Therefore, each element  $\hat{\sigma} \in \text{Msol}^{\iota}$  determines an unique non-empty set  $\mathcal{D}^{\iota}(\nu_{\hat{\sigma}}, \delta_{\iota})$  of  $C^{1+}$  self-renormalisable structures  $\mathcal{S}$  with  $\rho_{\mathcal{S},\delta_{\iota}}|\mathrm{Msol}^{\iota} = \hat{\sigma}$ .

Proof of (ii). By Lemma 8.4, if  $g \in [\nu]$  then  $\mathcal{S}(g,\iota) \in \mathcal{D}^{\iota}(\nu,\delta_{\iota})$  and so, by statement (i) of this lemma,  $\rho_{\mathcal{S}(q,\iota),\delta_{\iota}}|Msol^{\iota}$  is an element of  $\mathcal{SOL}^{\iota}$  which does not depend upon  $g \in [\nu]$ . Conversely, let  $\hat{\sigma} \in Msol^{\iota}$ . By statement (i) of this lemma,  $\hat{\sigma}$  determines an  $\iota$ -measure ratio function  $\rho_{\hat{\sigma},\iota}$ , and a non-empty set  $\mathcal{D}^{\iota}(\nu_{\hat{\sigma}}, \delta_{\iota})$ . By Lemma 8.4,  $\rho_{\hat{\sigma},\iota}$  determines an unique dual  $\iota'$ -ratio function  $\rho_{\hat{\sigma},\iota'}$ associated to the Gibbs measure  $\nu_{\hat{\sigma}}$ . Again, by statement (i) of this lemma,  $\rho_{\hat{\sigma},\iota'}|\text{Msol}^{\iota'}$  determines a non-empty set  $\mathcal{D}^{\iota'}(\nu_{\hat{\sigma}}, \delta_{\iota'})$ . By Lemma 8.5, the set  $\mathcal{D}^{s}(\nu_{\hat{\sigma}}, \delta_{s}) \times \mathcal{D}^{u}(\nu_{\hat{\sigma}}, \delta_{u})$  determines an unique nonempty set  $[\nu_{\hat{\sigma}}] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$  of hyperbolic diffeomorphisms  $g \in [\nu_{\hat{\sigma}}]$  such that  $\rho_{\mathcal{S}(g,\iota),\delta_\iota} | \mathrm{Msol}^\iota = \hat{\sigma}$ . 

10.2. Moduli space of cocycle-gap pairs. By Lemma 9.1, each set  $\mathcal{D}^{\iota}(\nu, \delta)$  is a Lipschitz conjugacy class. Hence, by Theorem 1.13 proved below, if  $\mathbf{B}^{\iota}$  is a no-gap train-track then the Lipschitz conjugacy class consists of a single  $C^{1+}$  self-renormalisable structure. Furthermore, by Lemma 9.5, the set of eigenvalues of all periodic orbits of  $\mathcal{S}_{\iota}$  is a complete invariant of each set  $\mathcal{D}^{\iota}(\nu, \delta)$ .

Proof of Theorem 1.13. Statement (i) follows from Lemma 8.1. Now, let us prove statement (ii). By Remark 7.10, the set  $\mathcal{J}G^{\iota}(\nu, \delta)$  is an infinite dimensional space, and by Lemma 8.3, the set  $\mathcal{D}^{\iota}(\nu, \delta)$  is parametrized by the cocycle-gap pairs in  $\mathcal{J}G^{\iota}(\nu, \delta)$  which ends the proof.  $\Box$ 

Proof of Theorem 1.14. By Lemma 8.5, if  $g \in \mathcal{T}(\nu, \delta_s, \delta_u)$  then  $\mathcal{S}_{\iota'}(g) \in \mathcal{D}^{\iota'}(\nu, \delta_{\iota'})$ . Conversely, let  $\mathcal{S}_{\iota'}$  be a  $C^{1+}$  self-renormalisable structure contained in  $\mathcal{D}^{\iota'}(\nu, \delta_{\iota'})$ . By Lemma 8.5, a pair  $(\mathcal{S}_{\iota}, \mathcal{S}_{\iota'})$  determines a  $C^{1+}$  hyperbolic diffeomorphism  $g \in \mathcal{T}(\nu, \delta_s, \delta_u)$ . if, and only if,  $\mathcal{S}_{\iota'} \in \mathcal{D}^{\iota'}(\nu, \delta_{\iota'})$ . By Theorem 1.12, the set  $\mathcal{D}(\nu, \delta_{\iota})$  is non-empty. Noting that  $\delta_{\iota} = 1$ , it follows from Theorem 1.13 (ii) that the set  $\mathcal{D}^{\iota}(\nu, \delta_{\iota})$  contains only one  $C^{1+}$  self-renormalisable structure  $\mathcal{S}_{\iota}$  which finishes the proof.  $\Box$ 

10.3.  $\delta_{\iota}$ -bounded solenoid equivalence class of Gibbs measures. When we speak of a  $\delta_{\iota}$ bounded solenoid equivalence class of  $\nu$  we mean a  $(\delta_{\iota}, 0)$ -bounded solenoid equivalence class of a Gibbs measure  $\nu$  (see Definition 8.2). In Section 7, we use the cocycle-gap pairs to construct explicitly the solenoid functions in the  $\delta_{\iota}$ -bounded solenoid equivalence classes of the Gibbs measures  $\nu$ . By Theorem 1.15 (ii) proved below, given an  $\iota$ -solenoid function  $\sigma_{\iota}$  there is an unique Gibbs measure  $\nu$  such that  $\sigma_{\iota}$  belongs to the  $\delta_{\iota}$ -bounded solenoid equivalence class of  $\nu$ .

Proof of Theorem 1.15. Statement (i) follows from Lemma 9.3 (i). Statement (ii) follows from Lemma 8.1 if  $\mathbf{B}^{\iota}$  is a no-gap train-track, and from Lemma 8.3 (ii) if  $\mathbf{B}^{\iota}$  is a gap train-track. Statement (iii) follows from statement (ii) and Theorem 1.13.  $\Box$ 

Proof of Theorem 1.16. By Theorem 1.15 (ii), the  $\iota$ -solenoid function  $\sigma_{\iota}$  determines an unique  $C^{1+}$  self-renormalisable structure  $\mathcal{S}_{\iota} \in \mathcal{D}^{\iota}(\nu, \delta_{\iota})$ . By Theorem 1.12, the set  $\mathcal{D}^{\iota'}(\nu, \delta_{\iota'})$  is nonempty. Let  $\mathcal{S}_{\iota'} \in \mathcal{D}^{\iota'}(\nu, 1)$ . By Theorem 1.15 (ii), the  $C^{1+}$  self-renormalisable structure  $\mathcal{S}_{\iota'}$  determines an unique  $\iota'$ -solenoid function  $\sigma_{\iota'}$  such that, by Theorem 1.15 (i), the pair  $(\sigma_{\iota}, \sigma_{\iota'})$  determines an unique  $C^{1+}$  conjugacy class  $\mathcal{T}(\nu, \delta_s, \delta_u)$  of hyperbolic diffeomorphisms  $g \in \mathcal{T}(\nu, \delta_s, \delta_u)$  with an invariant measure  $\mu = i_*\nu$  absolutely continuous with respect to the Hausdorff measure.  $\Box$ 

# Acknowledgments

We are grateful to Dennis Sullivan, Mikhail Lyubich, Krerley Oliveira, Carlos Matheus, and Flávio Ferreira for a number of very fruitful and useful discussions on this work and also for their friendship and encouragement. We thank the Calouste Gulbenkian Foundation, PRODYN-ESF, POCTI by FCT and Ministério da Ciência e da Tecnologia, and Centro de Matemática da Universidade do Porto for their financial support of A. A. Pinto and the UK Engineering and Physical Sciences Research Council and the EU for their financial support of D. A. Rand. Part of this research was started during a visit by the authors to the IHES and CUNY and continued at the Isaac Newton Institute, IMPA and SUNY. We thank them for their hospitality.

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