RENORMALIZATION IN THE HÉNON FAMILY, I: UNIVERSALITY BUT NON-RIGIDITY

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Dedicated to Mitchell Feigenbaum on the occasion of his 60th birthday

Abstract. In this paper geometric properties of infinitely renormalizable real Hénon-like maps F in $\mathbb{R}^2$ are studied. It is shown that the appropriately defined renormalizations R^nF converge exponentially to the one-dimensional renormalization fixed point. The convergence to one-dimensional systems is at a super-exponential rate controlled by the average Jacobian and a universal function $a(x)$. It is also shown that the attracting Cantor set of such a map has Hausdorff dimension less than 1, but contrary to the one-dimensional intuition, it is not rigid, does not lie on a smooth curve, and generically has unbounded geometry.

Contents

1. Introduction

Since the universality discoveries, made in the mid-1970's by Feigenbaum [F1, F2] and, independently, by Coullet and Tresser [CT, TC], these fundamental phenomena have attracted a great deal of attention from mathematicians, pure and applied, and physicists (see [Cv] for a representative sample of theoretical and experimental articles in early 1980's on the subject).

Date: August 24, 2005.

Stony Brook IMS Preprint #2005/07
August 2005

However, a rigorous study of these phenomena has been surprisingly difficult and technically sophisticated and so far has only been thoroughly carried out in the case of one-dimensional maps, on the interval or the circle, with one critical point (see [FMP, L, Ma, McM, S, VSK, Y] and references therein).

Rigorous exploration of universality for dissipative two-dimensional systems was begun in the article by Collet, Eckmann and Koch [CEK]. It is shown in this article that the one-dimensional renormalization fixed point f_* is also a hyperbolic fixed point for nearby dissipative two-dimensional maps: this explained (at least, at the physical level) parameter universality observed in families of such systems. A subsequent paper by Gambaudo, van Strien and Tresser [GST] demonstrates that, similarly to the one-dimensional situation, infinitely renormalizable two-dimensional maps which are close to f_* have an attracting Cantor set $\mathcal{O}$ on which the map acts as the adding machine. However, the geometry of these Cantor sets and global topology of the maps in question have not yet received an adequate deal of attention.

In this paper we begin a more systematic study of the geometry of infinitely renormalizable dissipative two-dimensional dynamical systems.[1] What we have discovered is that for these maps universality features (some of which have specific two-dimensional nature) can coexist with unbounded geometry and lack of rigidity (which make them quite different from the familiar one-dimensional counterparts).

We consider a class $\mathcal{H}$ of Hénon-like maps of the form

$$F\colon (x,y) \mapsto (f(x) - \varepsilon(x,y), x),$$

where $f(x)$ is a unimodal map subject of certain regularity assumptions, and ε is small. If f is renormalizable then the renormalization of F is defined as $RF = H^{-1} \circ (F^2|_U) \circ H$, where U is a certain neighborhood of the "critical value" $v = (f(0), 0)$ and H is an explicit *non-linear* change of variables (§ 3.5). [2]

It is shown that the degenerate map $F_*(x,y) := (f_*(x), x)$, where f_* is the fixed point of the one-dimensional renormalization operator, is a hyperbolic fixed point for R with a one-dimensional unstable manifold (consisting of one-dimensional maps) and that the renormalizations $R^n F$ of infinitely renormalizable maps converge at a super-exponential rate toward the space of unimodal maps (Theorem 4.1 and 4.3). For any infinitely renormalizable map F of class $\mathcal{H}$ there exists a hierarchical family of pieces $\{B^n_\sigma\}$, 2^n on each level, organized by inclusion in the dyadic tree, such that

$$\mathcal{O} = \mathcal{O}_F = \bigcap_n \bigcup_\sigma B^n_\sigma$$

is an attracting Cantor set on which F acts as the adding machine (Corollary 5.5). This recasts the results of [CEK, GST] in our setting.

[1] Here only period-doubling renormalization will be considered and we will refer to it simply as "renormalization."

[2] The set-up in this article is different from that of [CEK]: a different normalization of Hénon-like maps is used, and renormalization is done near the "critical value" rather than the "critical point" using, at least initially, a non-linear change of variable. We found the theory quite sensitive to specific choices such as this.

Furthermore, the diameters of the pieces B^n_σ shrink at least exponentially with rate $O(\lambda^{-n})$, where $\lambda = 2.6\dots$ is the universal scaling factor of one-dimensional renormalization (Lemma 5.1). This implies that

$$\mathrm{HD}(\mathcal{O}) < \log 2/\log\lambda < 1,$$

which makes it possible to control distortion of the renormalizations (Lemma 6.1). Ultimately, this leads to the following asymptotic formula for the renormalizations (Theorem 7.9):

$$R^n F(x,y) = (f_n(x) - b^{2^n}\, a(x)\, y\, (1 + O(\rho^n)),\ x\,),$$

where $f_n \to f_*$ exponentially fast,

$$b = b_F = \exp\int_{\mathcal{O}} \log \operatorname{Jac} F\, d\mu,$$

is the *average Jacobian* of F (here μ is the unique invariant measure on $\mathcal{O}$ and the Jacobian is the absolute value of the determinant), $\rho \in (0,1)$, and $a(x)$ is a *universal* function. This is a new universality feature of two-dimensional dynamics: as f_* controls the zeroth order shape of the renormalizations, $a(x)$ gives the first order control.

On the other hand, we will show in the second half of the paper that there are some striking differences between the one- and two-dimensional situations (§ 8 – § 11). For example, *the Cantor set $\mathcal{O}$ is not rigid* (Theorem 10.1). Indeed, if the average Jacobians of F and G are different, say $b_F < b_G$, then a conjugacy $h\colon \mathcal{O}_F \to \mathcal{O}_G$ does not admit a smooth extension to $\mathbb{R}^2$: there is a definite upper bound

$$\alpha \le \frac{1}{2}\left(1 + \frac{\log b_G}{\log b_F}\right) < 1$$

on the Hölder exponent of h. Thus, in dimension two, universality and rigidity phenomena do not necessarily coexist. The above estimate on the Hölder exponent of the conjugation also applies to degenerate maps (i.e., one-dimensional) F giving the upper bound $1/2$ on the Hölder exponent of h.

Remark 1.1. One can compare this non-rigidity phenomenon with non-rigidity of circle maps. In 1961 Arnold constructed an analytic diffeomorphism of the circle with irrational rotation number whose conjugation with the corresponding rigid rotation is not absolutely continuous, see [Ar], [H]. However, this phenomenon is quite different from the one discussed here as it is related to the unbounded combinatorics (Liouville rotation number) of the circle diffeomorphism in question.

It was even more surprising to us that generically the Cantor set $\mathcal{O}$ does not have bounded geometry and so is not quasiconformally equivalent to the standard Cantor set (Theorem 11.1).[3] Even worse, the Cantor sets of generic infinitely renormalizable Hénon-like maps have unbounded geometry in some places, but in some other places they have a universal bounded geometry which is similar to their one-dimensional counterparts. (For instance,

[3] In fact, it seems to be quite a challenge to construct a *single example* of a Hénon map of the class we consider whose Cantor set would have bounded geometry (see Problem 5 in §13). It seems to go against the common intuition as one can find quite a few results in the literature obtained under the assumption of bounded geometry, compare [CGM, Mo]

around the tip we always recover the universal scaling factor.) Moreover, the Cantor set $\mathcal{O}$ cannot be embedded into a smooth planar curve (Theorem 9.7).

These properties, so different from their one-dimensional counterparts, come from a *tilting* and *bending phenomenon*: near the "tip" of Hénon-like maps renormalization boxes are not rectangles but rather slightly tilted and bent parallelograms. This tilt significantly affects the b-scale geometry of $\mathcal{O}$. Since the Jacobian b is replaced with b^{2^n} under the n-fold renormalization, the geometry gets affected at arbitrarily small scales. These phenomena are explored in §10, §11 and §12.

The bent of the boxes forces us to use *non-affine* change of variables to make renormalizations converge to a universal limit. However, we show in Theorem 8.2 that appropriate quadratic changes of coordinates would be sufficient. The renormalization limit obtained by this means would not correspond to the fixed point of the usual renormalization around the critical point, but rather to the one around the critical value.

In §9 we show that a non-degenerate Hénon-like map in question does not have continuous invariant line fields on the Cantor set $\mathcal{O}$ (Corollary 9.4). It implies that contrary to the "rigidity intuition", the Cantor set $\mathcal{O}$ does not lie on a smooth curve. It also implies that the $\mathrm{SL}(2,\mathbb{R})$-cocycle $z \mapsto DF(z)/\sqrt{\operatorname{Jac} F(z)}$ is non-uniformly hyperbolic over the adding machine $F : \mathcal{O} \to \mathcal{O}$ (Theorem 9.6). It seems to be previously unknown whether such cocycles exist.

On the positive side, as we show in the Theorem 12.1 , the Cantor set $\mathcal{O}$ has *Hölder geometry* in an appropriate meaning of this term.

In the forthcoming Part II, the global topological structure of infinitely renormalizable Hénon maps will be discussed.

To conclude, it should be mentioned that intensive investigation of stochastic attractors in the Hénon family has been carried out during the past two decades by Benedicks, Carleson, Viana, Young, and others (see [BC, BDV, WY]). This study has been concerned with stochastic maps with positive entropy, which are very different from the zero entropy maps studied here. We hope that, similarly to what has happened in the one-dimensional theory, the renormalization point of view will shed new light on stochastic phenomena as well.

Acknowledgment. We thank Jun Hu for sharing with us his viewpoint on Hénon renormalization, which is reflected in § 3.4 of this paper. We thank Charles Tresser for infinite renormalization discussions, and the referee for many useful comments. We also thank all the foundations that have supported us in the course of this work: the Guggenheim Memorial Foundation, Clay Mathematics Institute, FAPESP, NSERC and NSF.

2. General notation and terminology

Let $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $I = [-1, 1] \subset \mathbb{R}$, and
$\mathbb{D}_r = \{z \in \mathbb{C} : \ |z| < r\}$.
A *rectangle* in $\mathbb{R}^2$ or $\mathbb{C}^2$ will mean a rectangle with vertical and horizontal sides.

The letters x and y will be used not only for real variables but also for their complexifications. The partial derivatives will be denoted by ∂_x, ∂_y, ∂^2_{xx}, etc.

For a smooth self-map F of $\mathbb{R}^2$ or $\mathbb{C}^2$, $\operatorname{Jac} F$ stands for $|\det DF|$.

The coordinate projections in $\mathbb{R}^2$ or $\mathbb{C}^2$ will be denoted by π_1 and π_2. We let $\mathcal{F}^h$ and $\mathcal{F}^v$ be respectively the foliations by horizontal and vertical real or complex lines in $\mathbb{R}^2$ or $\mathbb{C}^2$. A self-map of $\mathbb{R}^2$ or $\mathbb{C}^2$ is *horizontal* if it preserves the horizontal foliation $\mathcal{F}^h$.

A smooth map f of an interval is called *unimodal* if it has a single critical point. In what follows, we will assume that *all the unimodal maps under consideration have a non-degenerate critical point and have negative Schwarzian derivative*, see [dMvS].

A self-map H of $\mathbb{R}^2$ (from some family under consideration) is said to have *bounded non-linearity* if it may be represented as $A \circ \Phi$, where A is affine and $\|\Phi - \mathrm{id}\|_{C^2} \le K$, where K is independent of the particular map is question.

The notation "dist" will be used for different metrics in different spaces, as long as there is no danger of confusion. The sup-norm in the space $\mathcal{A}^c_\Omega$ of bounded holomorphic functions on $\Omega \subset \mathbb{C}^n$ is denoted by $\|\cdot\|_\Omega$, or, if there is no danger of ambiguity, simply by $\|\cdot\|$. If Ω is symmetric with respect to the real subspace $\mathbb{R}^n$, $\mathcal{A}_\Omega$ stands for the real slice of $\mathcal{A}^c_\Omega$ consisting of functions that are real on the real subspace.

A set X is called *invariant* under a map f if $f(X) \subset X$. $A \Subset B$ means that A *is compactly contained in* B, i.e., the closure $\bar{A}$ is a compact subset of B. Notation $Q_1 \asymp Q_2$ means, as usual, that $C^{-1} \le Q_1/Q_2 \le C$ for some constant $C > 0$.

For reader's convenience, more special notations are collected in §15.

3. Hénon renormalization

In this section, after briefly recalling the main definitions of one-dimensional renormalization, the class of Hénon-like maps is introduced and renormalization for such maps is defined. First a *renormalizable map* is defined and this definition parallels the one-dimensional definition: a certain topological disk is invariant under the second iterate of the map. To define the *renormalization* of the map, we consider the second iterate restricted to the invariant disk and apply an appropriate non-linear change of coordinates in order to obtain a Hénon-like map of the same class.

3.1. **Renormalization of unimodal maps.** A unimodal map $f\colon I \to I$ with critical point $c \in I$ is called *renormalizable* if there exists a closed interval $J \subset \operatorname{int} I$ containing the critical point such that $J \cap f(J) = \emptyset$ and $f^2(J) \subset J$. Then $f^2\colon J \to J$ is a unimodal map.

We choose $J_c = [f^4(c), f^2(c)]$ to be the smallest interval as above, and call $f^2\colon J_c \to J_c$ appropriately rescaled (to bring J_c back to the unit size) the *renormalization* $R_c f$ of f. This is the classical *period-doubling* renormalization, and this is the only renormalization type discussed in this paper. However, we will also use the operator R_v in the discussion of period doubling renormalization. It is defined as follows. Let $J_v = [f^3(c), f(c)]$ to be the smallest closed interval invariant under f^2 which contains the critical value $f(c)$, and call $f^2\colon J_v \to J_v$ appropriately rescaled (to bring J_v back to the unit size) the *renormalization* $R_v f$ of f. The operator R_v renormalizes around the "critical value " and R_c around the "critical point".

Let $r \in \mathbb{Z}_+ \cup \{\omega\}$ and let $\mathcal{U}^r$ denote the space of C^r-smooth unimodal maps $f\colon I \to I$ such that:

(a) the critical point is mapped to 1 and 1 is mapped to -1 and

(b) there is a unique expanding fixed point $\alpha \in (-1, 1)$ with negative multiplier.

The subspace of renormalizable maps is denoted by $\mathcal{U}^r_0$, and the renormalization operators $R_c, R_v\colon \mathcal{U}^r_0 \to \mathcal{U}^r$ assign to each map their renormalizations.

For $r \geq 3$, the renormalization operator R_c has a unique fixed point $f_* \in \mathcal{U}_0^\omega$. It satisfies the functional equation $f_* = \lambda f_*^2(\lambda^{-1}x)$, where $\lambda = 2.6\ldots$ is the *universal scaling factor*. We let $\sigma = \lambda^{-1}$.

The fixed point f_* is hyperbolic under the renormalization operator, with a codimension-one stable manifold $\mathcal{W}^s(f_*)$ consisting of infinitely renormalizable maps. For details, see [L] and references therein. The operator R_v has also a unique fixed point f^* (see Lemma 2.4 of [BMT]).

3.2. **Hénon-like maps.** Consider two intervals, I^h and I^v, and let $B = I^h \times I^v$. A smooth map $F\colon B \to \mathbb{R}^2$ is called *Hénon-like* if it maps vertical sections of B to horizontal arcs, while the horizontal sections are mapped to parabola-like arcs (i.e., graphs of unimodal functions over the y-axis). Examples of Hénon-like maps are given by small perturbations of unimodal maps of the form

$$F(x,y) = (f(x) - \varepsilon(x,y),\, x), \tag{3.1}$$

where $f\colon I^h \to I^h$ is unimodal and ε is small. Note that, in this case,

$$\operatorname{Jac} F = \left|\frac{\partial \varepsilon}{\partial y}\right|.$$

If $\partial\varepsilon/\partial y \neq 0$ then the vertical sections are mapped diffeomorphically onto horizontal arcs, so that F is a diffeomorphism onto a "thickening" of the graph $\Gamma_f = \{(f(x), x)\}_{x \in I^h}$ (Figure 3.1). In this case F is a diffeomorphism onto its image which will be briefly called a *Hénon-*

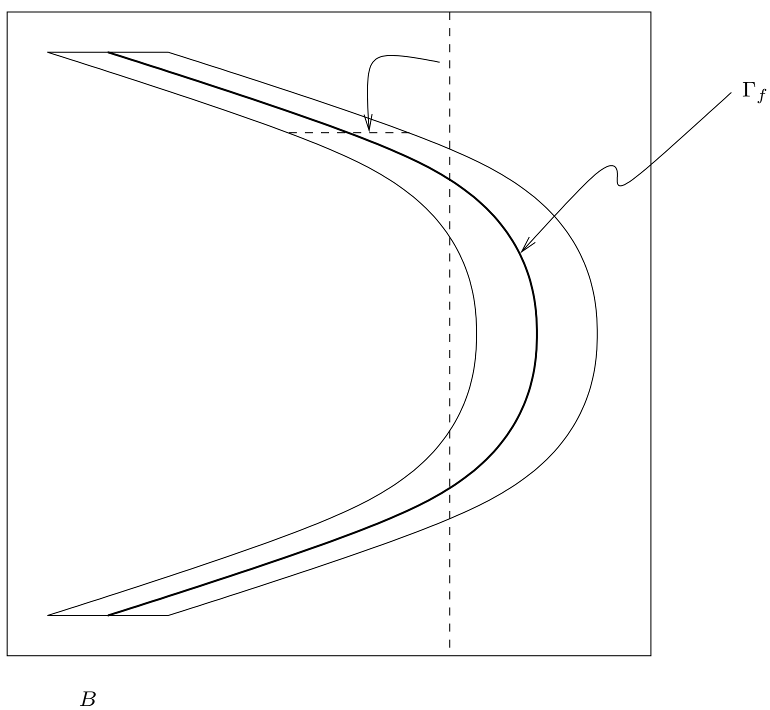

FIGURE 3.1. A Hénon-like map.

like diffeomorphism.

The classical Hénon family is obtained, up to affine normalization, letting $f(x)$ be a quadratic polynomial and $\varepsilon(x, y) = by$.

We will use the abbreviation $F = (f - \varepsilon, x)$ for equation (3.1). Thus, $F_f = (f, x)$ denotes the degenerate Hénon-like map collapsing B onto Γ_f.
[4]

3.3. **Spaces of maps.** Let $r \in \mathbb{Z}_+ \cup \{\omega\}$. The space of C^r-smooth Hénon-like maps $F\colon B \to \mathbb{R}^2$ of the form (3.1) is denoted by $\mathcal{H}^r$. Let $\mathcal{U}^r$ be the space of unimodal maps as defined above. In the real analytic case ($r = \omega$), if $U \subset \mathbb{C}$ is a neighborhood of I and $\kappa > 0$, then $\mathcal{U}_{U,\kappa} \equiv \mathcal{U}^{\omega}_{U,\kappa}$ denotes the subspace of maps $f \in \mathcal{U}_{U,\kappa}$ with critical point $c \in [-1, 1-\kappa]$ which admit a holomorphic extension to U and and can be factored as $Q \circ \phi$, where $Q(x) = 1 - x^2$ and ϕ is an $\mathbb{R}$-symmetric univalent map on U. Since $\phi(c) = 0$ and $\phi(1) = \sqrt{2}$, this space of univalent maps is normal, so that $\mathcal{U}_{U,\kappa}$ is compact.[5]

Let $\Omega^h, \Omega^v \subset \mathbb{D}_2 \subset \mathbb{C}$ be neighborhoods of I^h, I^v, respectively, and let $\Omega = \Omega^h \times \Omega^v \subset \mathbb{C}^2$. Let $\mathcal{H}_\Omega \equiv \mathcal{H}^{\omega}_{\Omega}$ stand for the class of Hénon-like maps $F \in \mathcal{H}^{\omega}$ of form (3.1) such that $f \in \mathcal{U}_{\Omega^h}$ and ε admits a holomorphic extension to Ω. The subspace of maps $F \in \mathcal{H}_\Omega$ with $\|\varepsilon\|_\Omega \le \bar{\varepsilon}$ will be denoted by $\mathcal{H}_\Omega(\bar{\varepsilon})$. If f in (3.1) is fixed, we will also use the notation $\mathcal{H}_\Omega(f, \bar{\varepsilon})$.

Realizing a unimodal map f as a degenerate Hénon-like map F_f yields an embedding of the space of unimodal maps $\mathcal{U}_{\Omega^h}$ into the space of Hénon-like maps $\mathcal{H}_\Omega$ making it possible to think of $\mathcal{U}_{\Omega^h}$ as a subspace of $\mathcal{H}_\Omega$.

3.4. **Renormalizable Hénon-like maps.** An orientation preserving Hénon-like map is *renormalizable* if it has two saddle fixed points — a *regular* saddle β_0, with positive eigenvalues, and a *flip* saddle β_1, with negative eigenvalues — such that the unstable manifold $W^u(\beta_0)$ intersects the stable manifold $W^s(\beta_1)$ at a single orbit (Figure 3.2).

For example, if f is a renormalizable unimodal map with both fixed points repelling, then a small Hénon-like perturbation of type (3.1) is a renormalizable Hénon-like map.

Given a renormalizable map F, consider an intersection point $p_0 \in W^u(\beta_0) \cap W^s(\beta_1)$, and let $p_n = F^n(p_0)$. Let D be the topological disk bounded by the arcs of $W^s(\beta_1)$ and $W^u(\beta_0)$ with endpoints at p_0 and p_1.

Lemma 3.1. *The disk D is invariant under F^2.*

Proof. The boundary of D consists of two arcs, $\ell^s \subset W^s(\beta_1)$ and $\ell^u \subset W^u(\beta_0)$ both having p_0 and p_1 for endpoints. Because β_1 is a flip saddle, $F^2(\ell^s) \Subset \ell^s$ and there is a neighborhood $U \supset \ell^s$ with $F^2(U \cap D) \subset D$. If $F^2(D)$ were not contained in D then $F^2(\ell^u)$ would have to intersect the boundary $\ell^u \cup \ell^s$ of D. The only possibility for this to happen would be that $F^2(\ell^u)$ intersects $\ell^s \setminus F^2(\ell^s)$. By hypothesis, this intersection consists of points in the orbit of p_0.

[4]Usually, in particular in §7, it is more convenient to consider unimodal maps only on their dynamical interval $[f^2(c), f(c)]$. However, without loss of generality we will assume that the unimodal maps have an extension defined on a *symmetric* interval bounded by an orientation preserving fixed point and a preimage. We also assume, again without loss of generality, that all Hénon-like maps have an extension containing the *regular* saddle point and its local stable manifold (compare §3.4).

[5]We fix once and for all a small $\kappa > 0$ such that $c \in [-1, 1-\kappa]$ for all maps of interest (like the renormalization fixed point and the infinitely renormalizable quadratic map), and we will suppress it from the notation.

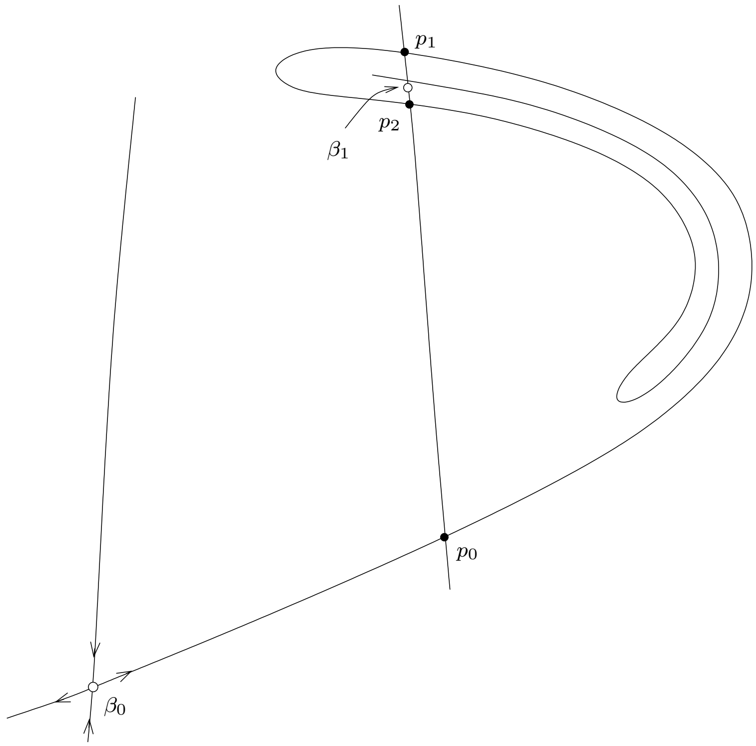

FIGURE 3.2. A renormalizable Hénon-like map.

But this would yield a contradiction, since $\ell^s \setminus F^2(\ell^s)$ contains only two points of the orbit of p_0, namely p_0 and p_1, which are not in $F^2(\ell^u)$. $\square$

Definition 3.1 (Pre-renormalization). The map $F^2|D$ is called a *pre-renormalization* of F.

Assume now that F is a small perturbation (3.1) of a twice renormalizable unimodal map. In this case, there is a preferred intersection point $p_0 \in W^s(\beta_1) \cap W^u(\beta_0)$. To define it, consider the *local stable manifold* $W^s_{\mathrm{loc}}(\beta_1)$, the component of the stable manifold $W^s(\beta_1) \cap B$ containing β_1. If ε is sufficiently small, then $W^s_{\mathrm{loc}}(\beta_1)$ is a nearly vertical smooth arc. Let now p_0 be the *lowest* intersection point of the unstable manifold $W^u(\beta_0)$ with $W^s_{\mathrm{loc}}(\beta_1)$, so that the arc of $W^u(\beta_0)$ between β_0 and p_0 does not intersect $W^s_{\mathrm{loc}}(\beta_1)$. This determines the preferred pre-renormalization $F^2|D$ of F.

3.5. **The Hénon renormalization operator.** We will now apply a carefully chosen nonlinear horizontal change of variables that will turn the pre-renormalization into a Hénon-like map of form (3.1).

The pre-renormalization is not Hénon-like, since it does not map the vertical foliation to the horizontal one. However, it is not far from it:

Lemma 3.2. *Let $f \in \mathcal{U}_{\Omega^h}$ with critical point c and let $U \Subset \Omega^h \smallsetminus \{c\}$ be an open set. There exist constants C and $\bar{\varepsilon} > 0$, depending only on Ω and U, such that for any $F \in \mathcal{H}_\Omega(f, \bar{\varepsilon})$, the leaves of the foliation $\mathcal{G} = F^{-2}(\mathcal{F}^h)$ in $U \times \Omega^v$ are graphs over sub-domains of Ω^v with vertical slope bounded by $C\|\operatorname{Jac} F\|_\Omega$.*

Proof. Since $\mathcal{U}_{\Omega^h}$ is a compact family of functions with a single critical point $c \notin \bar{U}$, we have $\kappa := \min_{x\in U} |Df(x)| > 0$, where κ depends only on Ω^h. Letting $r = \operatorname{dist}(\partial U, \partial\Omega^h)$, if $\|\varepsilon\|_\Omega < \bar{\varepsilon} := \kappa r/2$, then

$$\|\partial\varepsilon/\partial x\|_{U\times\Omega^v} < \kappa/2. \tag{3.2}$$

Since the foliation $F^{-2}(\mathcal{F}^h)$ is given by the level sets

$$f(x) - \varepsilon(x, y) = \text{const}$$

it follows from the Implicit Function Theorem and (3.2) that these level sets are holomorphic graphs over sub-domains of Ω^v with slopes satisfying

$$\left|\frac{\partial x}{\partial y}\right| = \left|\frac{\partial\varepsilon}{\partial y}\left(f'(x) - \frac{\partial\varepsilon}{\partial x}\right)^{-1}\right| \le \frac{2}{\kappa}\left|\frac{\partial\varepsilon}{\partial y}\right| = \frac{2}{\kappa}\operatorname{Jac} F(x, y).$$

□

For $U' \Subset U$, let $\Omega' \subset \Omega$ be the *saturation* of U' by the leaves of the foliation $\mathcal{G} \equiv F^{-2}(\mathcal{F}^h)$, that is, Ω' is the union of all leaves of $\mathcal{G}$ that intersect U'.

Corollary 3.3. *If $U' \Subset U$ is an open set such that*

$$\operatorname{dist}(\partial U', \partial U) > C\|\operatorname{Jac} F\| \operatorname{diam}\Omega.$$

then the leaves of $\mathcal{G}$ that intersect U' are holomorphic graphs over Ω^v.

Select neighborhoods $U' \Subset U \Subset \Omega^h$ as above so that they contain the interval $[\alpha, 1]$ and $f|_U$ is an expanding diffeomorphism with bounded non-linearity, with the bounds depending only on Ω and U. This is possible by compactness of $\mathcal{U}_{\Omega^h}$ and because unimodal maps with negative Schwarzian derivative are expanding on the interval $[\alpha, 1]$.

Lemma 3.4. *Given $U, U', \Omega, \Omega', \mathcal{G}$ as above, there exist $\bar{\varepsilon} > 0$, $C > 0$, and a domain $V \ni c$ with the following properties. Consider a Hénon-like map $F = (f - \varepsilon, x) \in \mathcal{H}_\Omega(f, \bar{\varepsilon})$ and define the horizontal diffeomorphism*

$$H(x, y) = (f(x) - \varepsilon(x, y), y). \tag{3.3}$$

Then there exists a unimodal map $g \in \mathcal{U}_V$ such that $\|g - f^2\|_V < C\bar{\varepsilon}$ and $G := H \circ F^2 \circ H^{-1}$ is a Hénon-like map $(x, y) \mapsto (g(x) - \delta(x, y), x)$ of class $\mathcal{H}_{V\times\Omega^v}$ with $\|\delta\|_{V\times\Omega^v} \le C\bar{\varepsilon}^2$.

Proof. Notice first that if ε is sufficiently small, then all maps $x \mapsto f(x) - \varepsilon(x, y)$ are diffeomorphisms on U for any $y \in \Omega^v$. Hence H is a diffeomorphism as well.

Let now

$$\phi_y(x) = \phi(x, y) = f(x) - \varepsilon(x, y), \tag{3.4}$$

and

$$v(x) = -\varepsilon(x, f^{-1}(x)).$$

A straightforward calculation gives us the following *Variational Formula*:

$$H \circ F^2 \circ H^{-1}(x, y) = \phi(\phi(x, \phi_y^{-1}(x)), x) =$$
$$(f^2(x) + v(f(x)) + f'(f(x))v(x) + O(\|\varepsilon\|^2),\ x), \tag{3.5}$$

which implies the assertion. □

Remark 3.1. Note that v is the restriction of the vector field $-\varepsilon\,\partial/\partial x$ to the graph Γ_f, and $v \circ f + (f' \circ f)v$ is the first variation of $f \mapsto f^2$ in the direction of v. Roughly speaking, the two-dimensional variation of $f \mapsto f^2$ in the direction of $-\varepsilon$ coincides, to the first order, with its one-dimensional variation in the direction of $v = -\varepsilon|\Gamma_f$. In symbols: $\delta_{-\varepsilon}(H \circ F_f^2 \circ H^{-1}) = F_{\delta_v f^2}$.

Remark 3.2. The residual term in (3.5) involves second derivatives of ε, but in the holomorphic setting they are estimated by $\|\varepsilon\|$.

Definition 3.2 (Renormalization)**.** Let J be the minimal interval such that $J \times I$ is invariant under $G = H \circ F^2 \circ H^{-1}$, let $s\colon J \to I$ be the orientation-reversing affine rescaling, and let $\Lambda(x, y) = (sx, sy)$. Then the *renormalization* RF is defined as $\Lambda \circ G \circ \Lambda^{-1}$ on the bidisk $\Lambda(V \times \Omega^v)$.

In the case of a degenerate map $F_f = (f, x)$ where f is a renormalizable unimodal map with critical point c, $J = [f^4(c), f^2(c)]$ is the same *dynamical interval* that we have used to define the period doubling renormalization for unimodal maps.

Let us summarize the above analysis:

Theorem 3.5. *Given a domain $\Omega \supset I$, there exist $\bar\varepsilon > 0$, $C > 0$, and a neighborhood sV of I with the following properties. Let $F = (f - \varepsilon, x)$ be a renormalizable Hénon-like map of class $\mathcal{H}_\Omega(\bar\varepsilon)$. Then the renormalization RF is a Hénon-like map of class $\mathcal{H}_W(g, C\bar\varepsilon^2)$, where $W = \Lambda(V \times \Omega^v)$ and g is a unimodal map such that* $\operatorname{dist}(R_c f, g) \le C\bar\varepsilon$. *The change of variable $\Lambda \circ H$ conjugating F^2 (appropriately restricted) to RF is an expanding map with bounded non-linearity, with all bounds depending only on Ω and $\bar\varepsilon$.*

Remark 3.3. Notice that if F is close to the renormalization fixed point $F_*(x) = (f_*(x), x)$ (see §3.1 and the next section), then the conjugacy $\Lambda \circ H$ expands the infinitesimal l_∞-norm at least by factor 2.6, as $\lambda = 2.6\dots$ is the dynamical scaling factor for the map f_*.

4. Hyperbolicity of the Hénon renormalization operator

In this section we show that the Hénon renormalization operator defined above has a hyperbolic fixed point

$$F_*(x, y) = (f_*(x), x), \tag{4.1}$$

where f_* is the fixed point of the one-dimensional renormalization operator. We also show that, starting with an infinitely renormalizable Hénon-like map $F = (f - \varepsilon, x)$ with ε sufficiently small, the renormalizations $R^n(F)$ converge super-exponentially fast to the subspace of degenerate (one-dimensional) maps and converge exponentially fast to the fixed point F_*. It follows that the local unstable manifold $\mathcal{W}^u(F_*)$ may be identified with the local unstable manifold $\mathcal{W}^u(f_*)$, of the one-dimensional renormalization operator, contained in the space of unimodal maps, and that the local stable manifold $\mathcal{W}^s(F_*)$ coincides with the set of infinitely renormalizable Hénon-like maps close to F_*.

Let $\mathcal{I}_\Omega(\bar\varepsilon)$ and $\mathcal{I}_\Omega(f, \bar\varepsilon)$ denote the subspaces of infinitely renormalizable Hénon-like maps (including degenerate ones) of classes $\mathcal{H}_\Omega(\bar\varepsilon)$ and $\mathcal{H}_\Omega(f, \bar\varepsilon)$ respectively.

Theorem 4.1. *Given a domain Ω, there is an $\bar{\varepsilon} > 0$ with the following property: for $F \in \mathcal{I}_\Omega(f, \bar{\varepsilon})$, there exists a domain $V \subset \Omega^h$ containing I and a sequence of unimodal maps $g_n \in \mathcal{U}_V$ such that, for all $n \geq 0$,*

$$\|g_n - f_*\|_V \leq C\rho^n \|f - f_*\|_V$$

and

$$\|R^n F - F_{g_n}\|_W = O(\bar{\varepsilon}^{2^n}),$$

where $W = V \times \Omega^v$ and $F_{g_n} = (g_n, x)$ is the degenerate Hénon-like map associated to g_n. All constants depend only on Ω and $\bar{\varepsilon}$. The constant $\rho < 1$ is universal.

Proof. By the renormalization theory of unimodal maps, it is possible to find a domain $V \Subset \Omega$ containing I and a number $N \in \mathbb{N}$ such that for any N times renormalizable unimodal map $f \in \mathcal{U}_V$ the following holds:
(i) $R_c^N f \in \mathcal{U}_V$ and $\operatorname{dist}(R_c^N f, f_*) < (1/4) \operatorname{dist}(f, f_*)$, where the distance is associated with the norm $\|\cdot\|_V$.

It follows easily from the definition of the renormalization operator and compactness of the space $\mathcal{U}_V$ that
(ii) There exists an $\bar{\varepsilon} > 0$ such that if $F \in \mathcal{I}_W(f, \bar{\varepsilon})$ for some unimodal map $f \in \mathcal{U}_V$, then f is N times renormalizable.

Take some $\delta > \operatorname{dist}(f, f_*)$. Let $\bar{\varepsilon}$ be so small that property (ii) holds and $C\bar{\varepsilon} < \min(1/2, \delta/4)$, where C is the constant from Theorem 3.5 applied to R^N. Let g be a unimodal map approximating $R^N F$ as given by Theorem 3.5. Then

$$\operatorname{dist}(g, f_*) < \operatorname{dist}(g, R_c^N f) + \operatorname{dist}(R_c^N f, f_*) <$$
$$< C\bar{\varepsilon} + (1/4) \operatorname{dist}(f, f_*) < \delta/2.$$

Moreover, $R^N F \in \mathcal{H}_W(g, C\bar{\varepsilon}^2) = \mathcal{H}_W(g, \bar{\varepsilon}_1)$ with

$$C\bar{\varepsilon}_1 = (C\bar{\varepsilon})^2 < (1/4)(\delta/2).$$

Hence it is possible to repeat the argument above with $R^N F$ in place of F, g in place of f, $\delta/2$ in place of δ, and ε_1 in place of ε. In this way we construct inductively a sequence of N-times renormalizable unimodal maps $g_k \in \mathcal{U}_V$ such that $\operatorname{dist}(g_k, f_*) < \delta/2^k$ and $\operatorname{dist}(R^{Nk} F, g_k) = O(\bar{\varepsilon}^{2^k})$. The conclusion follows. $\square$

By a standard trick (see, e.g., [PS, Prop. 3.3]), one can adapt the metric $\|\cdot\|$ to the dynamics in such a way that R becomes strongly contracting:

Lemma 4.2. *There is a metric on $\mathcal{I}_\Omega(\bar{\varepsilon})$, equivalent to $\|\cdot\|_\Omega$, and $\rho \in (0,1)$ such that*

$$\operatorname{dist}(RF, F_*) \leq \rho \operatorname{dist}(F, F_*)$$

for all $F \in \mathcal{I}_\Omega(\bar{\varepsilon})$. $\square$

The space $\mathcal{H}_\Omega(\bar{\varepsilon})$ is naturally a real analytic Banach manifold modeled on the space $\mathcal{A}_\Omega$, with functions ε serving as local charts on $\mathcal{H}_\Omega(f, \bar{\varepsilon})$. It is obvious from the definition that the renormalization operator $R\colon \mathcal{H}_\Omega(\bar{\varepsilon}) \to \mathcal{H}_\Omega(\bar{\varepsilon})$ is real analytic.

By the unimodal renormalization theory, the fixed point f_* is a quadratic-like map on some domain $\Omega_* \subset \mathbb{C}$ (see e.g, [B] and references therein). Moreover, f_* is a hyperbolic fixed point of R_c in any space $\mathcal{U}_V$ with $V \Subset \Omega_*$,

Theorem 4.3. *Assume $\Omega^h \Subset \Omega_*$. Then the map F_* is the hyperbolic fixed point for the Hénon renormalization operator R acting on $\mathcal{H}_\Omega$, with one-dimensional unstable manifold $\mathcal{W}^u(F_*) = \mathcal{W}^u(f_*)$ contained in the space of unimodal maps. Moreover, the differential $DR(F_*)$ has vanishing spectrum on the quotient $T\mathcal{H}_\Omega / T\mathcal{U}_{\Omega^h}$.*

Proof. Let $E = T\mathcal{H}_\Omega / T\mathcal{U}_{\Omega^h}$, and let $A : E \to E$ be the operator induced by $DR(F_*)$. Then Theorem 3.5 implies that $\|A^n\| = O(\bar{\varepsilon}^{2^n})$, and hence $\operatorname{spec}(A) = \{0\}$. □

Corollary 4.4. *The set $\mathcal{I}_\Omega(\bar{\varepsilon})$ of infinitely renormalizable Hénon-like maps coincides with the stable manifold*

$$\mathcal{W}^s(F_*) = \{F \in \mathcal{H}_\Omega(\bar{\varepsilon}) : \; R^n F \to F_* \text{ as } n \to \infty\},$$

which is a codimension-one real analytic submanifold in $\mathcal{H}_\Omega(\bar{\varepsilon})$. □

Corollary 4.5. *For all Ω and $\bar{\varepsilon}$ as above, the intersection of $\mathcal{I}_\Omega(\varepsilon)$ with the Hénon family*

$$F_{a,b} \colon (x, y) \mapsto (f_a(x) - by, x)$$

is a real analytic curve intersecting transversally the one-dimensional slice $b = 0$ at a_*, the parameter value for which f_{a_*} is infinitely renormalizable.

Proof. By the unimodal renormalization theory, the stable manifold $\mathcal{W}(f_*) = \mathcal{W}^s(F_*) \cap \mathcal{U}_\Omega$ intersects transversally the quadratic family $\mathcal{Q} = \{f_a\}$ at a single point, a_*. By the hyperbolicity of the unimodal renormalization operator, $R^n(\mathcal{Q})$ is close to $\mathcal{W}^u(f_*)$ for big n's. Since $\mathcal{W}^u(f_*) = \mathcal{W}^u(F_*)$, the $R^n(\mathcal{Q})$ are transverse to $\mathcal{W}^s(F_*)$ for big n's as well. It follows that $\mathcal{Q}$, and hence the whole Hénon family, is transverse to $\mathcal{W}^s(F_*)$. □

Let us finish this section with a complexification of the previous results. Let $\mathcal{H}^c_\Omega(f_*, \bar{\varepsilon})$ stand for the space of maps of form $F = (f_* - \varepsilon, x)$, where $f_* \in \mathcal{U}_{\Omega^h}$ is the unimodal renormalization fixed point and $\varepsilon \in \mathcal{A}^c_\Omega$ is a holomorphic function on Ω (not necessarily real on the real line) with $\|\varepsilon\|_\Omega < \bar{\varepsilon}$. This neighborhood of F_* has a natural complex structure inherited from $\mathcal{A}^c_\Omega$, and the renormalization operator R extends to a holomorphic map on this space.

Theorem 4.6. *The degenerate map F_* is a hyperbolic fixed point of the renormalization operator R acting on $\mathcal{H}^c_\Omega(\bar{\varepsilon})$ with a codimension-one holomorphic stable manifold $\mathcal{I}^c_\Omega(\bar{\varepsilon}) \equiv \mathcal{W}^s_c(F_*)$, the complexification of $\mathcal{I}_\Omega(\bar{\varepsilon}) = \mathcal{W}^s(F_*)$.* □

The maps $F \in \mathcal{I}^c_\Omega$ will still be called infinitely renormalizable (complex) Hénon-like. Note that the renormalization of the complex maps can be described geometrically in the same way as for real maps, that is, as restriction of F^2 to an appropriate bidisk, conjugating it by a horizontal map H (given by the same formula) and rescaling.

5. The critical Cantor set

Here we begin the study of the attracting set for infinitely renormalizable Hénon-like maps. As in dimension one, it is a Cantor set on which the map acts like the dyadic adding machine. We show that its Hausdorff dimension is bounded from above by 0.73 and that it depends holomorphically on the map. We will see in Sections 10 and 11 that there are some fundamental differences between these Cantor sets and their one-dimensional counterparts.

Consider an infinitely renormalizable complex Hénon-like map $F \in \mathcal{I}^c_\Omega(\bar{\varepsilon})$, where Ω and $\bar{\varepsilon}$ are selected so that the previous results apply.

5.1. **Branches.** Let $\Psi_v^1 \equiv \psi_v^1 := H^{-1} \circ \Lambda^{-1}$ be the change of variable conjugating the renormalization RF to F^2 appropriately restricted, and let $\Psi_c^1 \equiv \psi_c^1 = F \circ \psi_v$. The subscripts v and c indicate that these maps are associated to the critical *value* and the *critical* point, respectively.

Remark 5.1. Note that while the maps Ψ_v^1 preserve the horizontal foliation $\mathcal{F}^h$, the maps Ψ_c^1 preserve the vertical one, $\mathcal{F}^v$. Indeed, by definition (3.3), H maps $F^{-1}(\mathcal{F}^v)$ to $\mathcal{F}^v$. Hence

$$(\Psi_c^1)^{-1}(\mathcal{F}^v) = \Lambda \circ H(F^{-1}(\mathcal{F}^v)) = \mathcal{F}^v.$$

Similarly, let ψ_v^2 and ψ_c^2 be the corresponding changes of variable for RF, let

$$\Psi_{vv}^2 = \psi_v^1 \circ \psi_v^2, \quad \Psi_{cv}^2 = \psi_c^1 \circ \psi_v^2, \quad \Psi_{vc}^2 = \psi_v^1 \circ \psi_c^2, \quad \ldots.$$

and, proceeding this way, construct, for any $n = 1, 2, \ldots$, 2^n maps

$$\Psi_w^n = \psi_{w_1}^1 \circ \cdots \circ \psi_{w_n}^n, \quad w = (w_1, \ldots, w_n) \in \{v, c\}^n.$$

The notation $\Psi_w^n(F)$ will also be used to emphasize dependence on the map F under consideration, and we will let $W = \{v, c\}$ and $W^n = \{v, c\}^n$ be the n-fold Cartesian product.

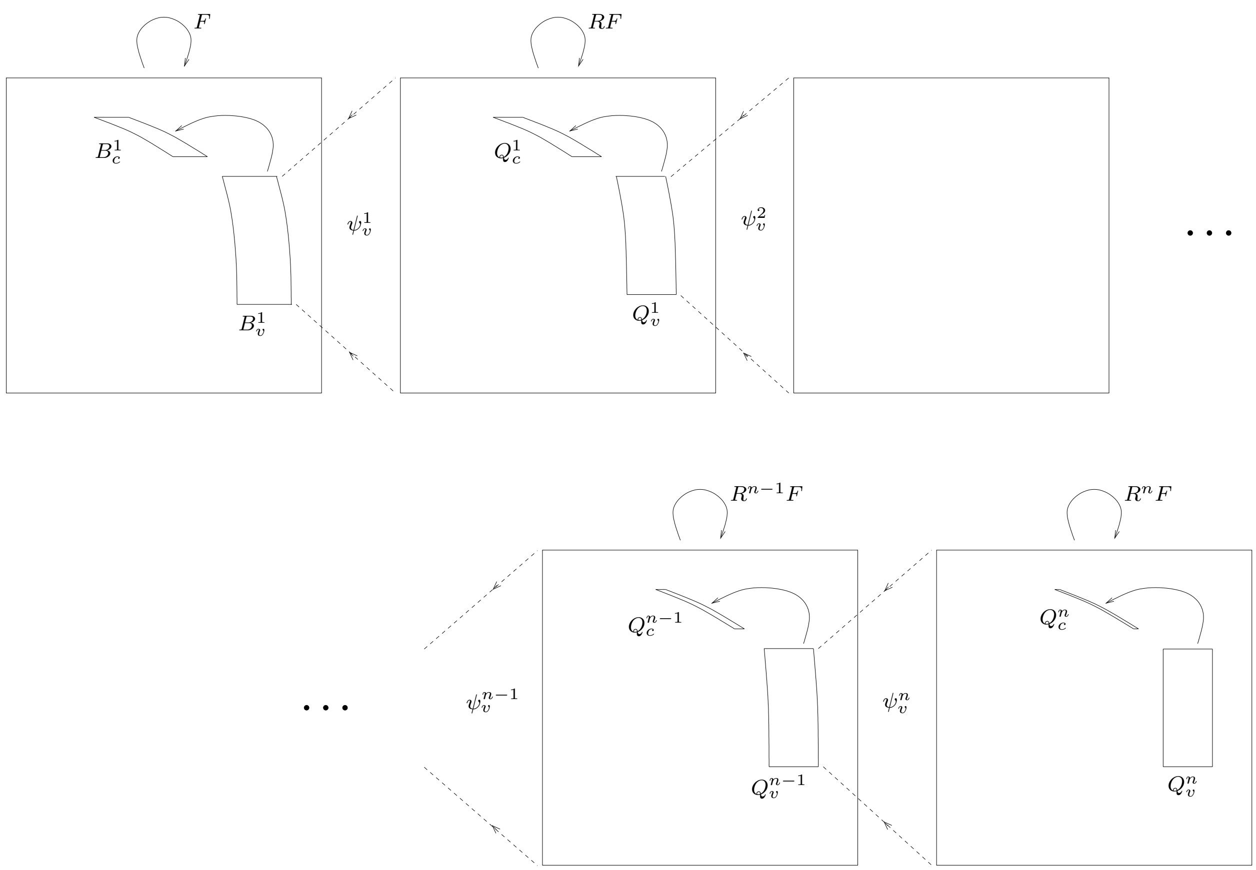

FIGURE 5.1. The renormalization microscope

Recall that $\sigma = \lambda^{-1}$ where λ is the universal scaling factor.

Lemma 5.1. *Let $F \in \mathcal{I}_\Omega^c(\bar{\varepsilon})$, $n \geq 1$, and $w \in W^n$. There exist $C > 0$ and a domain in $\mathbb{C}^2$, depending only on Ω and $\bar{\varepsilon}$, on which the holomorphic map Ψ_w^n is defined and $\|D\Psi_w^n\| \leq C\sigma^n$.*

Proof. In the notation from equation (3.4) we have:

$$H^{-1}(x,y) = (\phi_y^{-1}(x), y) \quad \text{and} \quad F \circ H^{-1}(x,y) = (x, \phi_y^{-1}(x)).$$

The map ϕ_y^{-1} is uniformly contracting on a neighborhood of the interval J, so that $|\partial\phi_y^{-1}/\partial x|$ is bounded away from 1. On the other hand, $\partial\phi_y^{-1}/\partial y$ is comparable with $\partial\varepsilon/\partial y$, which is small. It follows that the maps $\psi_v = H^{-1} \circ \Lambda^{-1}$ and $\psi_c = F \circ H^{-1} \circ \Lambda^{-1}$ uniformly contracts the infinitesimal l_∞-metric at least as strongly as Λ^{-1}, that is, by a factor $\sigma(1+O(\mathrm{dist}(F, F_*))$.

Since $R^n F \to F_*$ exponentially fast, the maps $\psi^k_{w_k}$, $w_k \in W$, contract the infinitesimal l_∞ normal by a factor $\sigma(1+O(\rho^k))$, where $\rho \in (0,1)$. Hence the compositions Ψ^n_w of these maps are contracting by a factor $O(\sigma^n)$. □

5.2. **Pieces.** Let us define $B^1_v \equiv B^1_v(F) = \psi^1_v(B)$ and $B^1_c \equiv B^1_c(F) = F(B^1_v)$. Then $F(B^1_c) \subset B^1_v$. We will let $Q^n_w = B^1_w(R^n F)$, $n \in \mathbb{Z}_+$, $w \in W$. Let Q^∞_w stand for the corresponding pieces for the degenerate limit map (4.1). Note that the pieces Q^n_w depend on F while the pieces Q^∞_w do not, and that the piece Q^∞_c is in fact an arc on the parabola-like curve $x = f_*(y)$.

Lemma 5.2. *Let $F \in I^c_\Omega(\bar\varepsilon)$. The pieces Q^n_v and Q^n_c have disjoint projections to both of the coordinate axes. Moreover, they converge exponentially, in the Hausdorff topology, to the pieces Q^∞_v and Q^∞_c, respectively.*

Proof. The first statement follows easily from the definition of renormalization. The second one follows from the exponential convergence $R^n F \to F_*$. □

The sets $B^n_w \equiv B^n_w(F) = \Psi^n_w(B)$, where $w \in W^n$, will be called *pieces*. They are closed topological disks. For each $n \in \mathbb{N}$, there are 2^n such pieces and forming the n^{th}-*generation* or n^{th}-*level* pieces. W^n can be viewed as the additive group of residues mod 2^n by letting

$$w \mapsto \sum_{k=0}^{n-1} w_{k+1} 2^k,$$

where the symbols v and c are interpreted as 0 and 1 respectively. Let $p\colon W^n \to W^n$ be the operation of adding 1 in this group.

Lemma 5.3. (1) *The above families of pieces are nested:*

$$B^n_{w\nu} \subset B^{n-1}_w, \quad w \in W^{n-1},\ \nu \in W.$$

(2) *The pieces B^n_w, $w \in W^n$, are pairwise disjoint.*
(3) *Under F, the pieces are permuted as follows. $F(B^n_w) = B^n_{p(w)}$ unless $p(w) = v^n$. If $p(w) = v^n$, then $F(B^n_w) \subset B^n_{v^n}$.*

Proof. The first assertion holds by construction:

$$B^n_{w\nu} = \Psi^n_{w\nu}(B) = \Psi^{n-1}_w \circ \psi^n_\nu(B) \subset B^{n-1}_w.$$

The second follows by induction. For all maps under consideration we have by Lemma 5.2 that $B^1_v(F)$ and $B^1_c(F)$ are disjoint. Assume that the pieces of the n^{th} generation are disjoint for all maps under consideration. This implies that the pieces $B^{n+1}_{wv} \subset B^1_v$, $w \in W^n$, are pairwise disjoint, as they are images of the disjoint pieces $B^n_w(RF)$ by the map ψ^1_v. Applying F, we see that the pieces $B^{n+1}_{wc} \subset B^1_c$, $w \in W^n$, are pairwise disjoint as well. The assertion follows because B^1_c and B^1_v are also disjoint.

Let us inductively check the third assertion. For $n = 1$, we have:

$$B_c^1 = F(B_v^1) \text{ and } F(B_c^1) = F^2(B_v^1) \subset B_v^1.$$

Consider now the pieces $B_w^n(RF)$, $w \in W^n$, of level n for RF. Assume inductively that they are permuted by RF as required. Then the pieces $B_{vw}^{n+1} = \psi_v^1(B_w^n(RF))$, $w \in W^n$, are permuted in the same fashion under F^2. Moreover, $B_{cw}^{n+1} = \psi_c^1(B_w^n(RF)) = F(B_{vw}^{n+1})$, and the conclusion follows. □

Furthermore, Lemma 5.1 implies:

Lemma 5.4. *There exists $C > 0$, depending only on Ω and $\bar{\varepsilon}$, such that for all $w \in W^n$,* $\operatorname{diam} B_w^n \le C\sigma^n$.

Let

$$\mathcal{O} \equiv \mathcal{O}_F = \bigcap_{n=1}^{\infty} \bigcup_{w \in W^n} B_w^n.$$

Let us also consider the *diadic group* $W^\infty = \varprojlim W^n$. The elements of W^∞ are infinite sequences $(w_1 w_2 \dots)$ of symbols $v \equiv 0$ and $c \equiv 1$ that can be also represented as formal power series

$$w \mapsto \sum_{k=0}^{\infty} w_{k+1} 2^k.$$

The integers $\mathbb{Z}$ are embedded into W^∞ as finite series. The *adding machine* $p : W^\infty \to W^\infty$ is the operation of adding 1 in this group. The discussion above yields that the map F acts on the invariant Cantor set $\mathcal{O}$ as the dyadic adding machine (as in the one-dimensional case, compare [Mi]):

Corollary 5.5. *The map $F|\mathcal{O}$ is topologicaly conjugate to $p : W^\infty \to W^\infty$. The conjugacy is given by the following homeomorphism $h : W^\infty \to \mathcal{O}$:*

$$h : w = (w_1 w_2 \dots) \mapsto \bigcap_{n=1}^{\infty} B_{w_1 \dots w_n}^n.$$

Furthermore,

$$HD(\mathcal{O}) \le \frac{\log 2}{\log \lambda} \le 0.73.$$

We call $\mathcal{O}$ the *critical Cantor set*[6] of F. Let us finish this section with a remark on the dependence of this Cantor set on F:

Lemma 5.6. *The critical Cantor set $\mathcal{O}_F \subset \Omega$ moves holomorphically as F ranges over $\mathcal{I}_\Omega^c(\bar{\varepsilon})$.*

Proof. Each contraction $\Psi_w^n = \Psi_w^n(F)$, $w \in W^n$, has a unique attracting fixed point $\alpha_w^n(F)$. By the Implicit Function Theorem, this point depends holomorphically on F.

By Lemma 5.5, any point of $\mathcal{O}_F$ can be encoded as $\alpha_w^\infty(F)$, where $w = (w_1, w_2 \dots) \in W^\infty$. Lemma 5.4 implies that $\alpha_{w_1 \dots w_n}^n(F) \to \alpha_w^\infty(F)$ as $n \to \infty$, at an exponential rate uniform

[6]This Cantor set consists of the "critical points" of F. More precisely, we will show in the forthcoming notes that generically $\mathcal{O}$ is the set of singularities of the unstable lamination of F.

in F. Since uniform limits of holomorphic functions are holomorphic, $\alpha_w^\infty(F)$ depends holomorphically on F.

Moreover, since the coding $h : W^\infty \to \mathcal{O}_F$ is injective, $\alpha_w^\infty(F) \neq \alpha_v^\infty(F)$ if $v \neq w$, and the conclusion follows. □

6. The average Jacobian

In this section we consider the average Jacobian b of an infinitely renormalizable Hénon-like map with respect to the unique invariant measure supported on its critical Cantor set. It is shown that the characteristic exponents of this measure are 0 and $\log b$ and that b is a natural parameter for infinitely renormalizable maps.

We continue to consider infinitely renormalizable Hénon-like maps and assume, moreover, that they are diffeomorphisms. They are, however, allowed to be complex. Lemma 5.1 and the standard distortion estimate imply:

Lemma 6.1 (Distortion Lemma)**.** *There exist constants C and $\rho < 1$ such that for any piece B^n_w and for any $y, z \in B^n_w$, $w \in W^n$ the following holds:*

$$\log \left| \frac{\operatorname{Jac} F^k(y)}{\operatorname{Jac} F^k(z)} \right| \le C\rho^n, \quad k = 1, 2, \dots, 2^n.$$

Since $F|_{\mathcal{O}}$ is the adding machine, it has a unique invariant measure μ. Let us consider the average Jacobian with respect to this measure:

$$b = \exp \int \log \operatorname{Jac} F \, d\mu.$$

Corollary 6.2. *For any piece B^n_w and any point $z \in B^n_w$,*

$$\operatorname{Jac} F^{2^n}(z) = b^{2^n}(1 + O(\rho^n)),$$

where ρ is as in Lemma 6.1.

Proof. Since

$$\int_{B^n_w} \log \operatorname{Jac} F^{2^n} \, d\mu = \int_{\mathcal{O}} \log \operatorname{Jac} F \, d\mu = \log b,$$

there exists a point $\zeta \in B^n_w$ such that

$$\log \operatorname{Jac} F^{2^n}(\zeta) = \log b / \mu(B^n_w) = 2^n \log b,$$

and the assertion follows from Lemma 6.1. □

The two characteristic exponents, $\chi_- \le \chi_0$, of the measure μ are given by

Theorem 6.3. *The characteristic exponents of μ are $\chi_- = \log b$ and $\chi_0 = 0$.*

Proof. Let G_n be the n-th renormalization of F. This map is smoothly conjugate to the restriction of F^{2^n} to the piece $B^n_{v^n}$. Let μ_n be the normalized restriction of μ to $B^n_{v^n}$, and let ν_n be the invariant measure on the critical Cantor set of G_n. Note that these two measures are preserved by the conjugacy. Then

$$2^n \chi_0 = \chi_0(F^{2^n} | B^n_{v^n}, \mu_n) = \chi_0(G_n, \nu_n) \le \int \log \|DG_n\| \, d\nu_n \le C,$$

since the maps G_n have uniformly bounded C^1-norms.

Hence $\chi_0 \le 0$. If $\chi_0 < 0$, both characteristic exponents of F would be negative and it would then follow from the Pesin theory that μ is supported on a periodic cycle[7] which is clearly not the case. Hence $\chi_0 = 0$. The formula for the other exponent now follows from the relation $\chi_0 + \chi_- = \log b$. □

Let us now take a look at the dependence of the average Jacobian on parameters. Consider a holomorphic one-parameter family of complex Hénon-like maps $F_t \in \mathcal{I}^c_\Omega(\bar\varepsilon)$,

$$F_t\colon (x,y) \mapsto (f(x) - t\,\varepsilon_t(x,y),\, x), \quad |t| < r,\ (x,y) \in \Omega,$$

such that

(i) F_t are real for real t;
(ii) $\varepsilon_t(x,y) = \gamma(x,y)\psi_t(x,y)$, where $\psi_t(x,y) = 1 + O(t)$;
(iii) $\partial\gamma/\partial y > 0$ on B and $\partial\gamma/\partial y \ne 0$ on Ω.

Let us consider the *complex Jacobian*,

$$\operatorname{Jac}^c F_t = \det DF_t = t\frac{\partial\varepsilon_t}{\partial y} = t\frac{\partial\gamma}{\partial y} + O(t^2).$$

By property (iii), it does not vanish for sufficiently small r, and hence F_t are complex diffeomorphisms. Moreover, for real t, they preserve orientation of B.

Lemma 6.4. *For sufficiently small $r > 0$, the average Jacobian $b_t \equiv b(F_t)$, $t \in (0,r)$, admits a holomorphic extension to the complex disk $\mathbb{D}_r$. Moreover,*

$$b'(0) = \exp \int_{O(f)} \log \frac{\partial\gamma}{\partial y}\, d\mu \ne 0.$$

Proof. We can define the average complex Jacobian by the following explicit formula:

$$b^c(F_t) = \exp \int_{\mathcal{O}_t} \log \operatorname{Jac}^c F_t\, d\mu_t =$$

$$= t \exp \int_{\mathcal{O}_t} \log \frac{\partial\gamma}{\partial y}\, d\mu_t \cdot \exp \int_{\mathcal{O}_t} \log \psi_t(x,y)\, d\mu_t,$$

where μ_t is the F_t-invariant measure on the critical Cantor set $\mathcal{O}_t = \mathcal{O}_{F_t}$. Since $\psi_t = 1 + O(t)$, there is a well defined holomorphic branch of $\log\psi_t(x,y)$ on the domain $\mathbb{D}_r \times \Omega$ which is positive on $(-r,r) \times B$. Since by Lemma 5.6 the Cantor set $\mathcal{O}_t$ moves holomorphically with t, the two integrals on the right-hand side of the formula above depend holomorphically on t. Since the second factor in that product goes to 1 as $t \to 0$, the result follows. □

Thus, in the Hénon-like families as above, the average Jacobian b can be used (consistently with the common intuition) as a holomorphic parameter that measures the distance to the reference unimodal map.

[7] Indeed, in this case the Pesin local stable manifold $W = W^s_{\text{loc}}(x)$ (see e.g., [PS]) of a typical point $x \in \mathcal{O}$ would be a neighborhood of x. Then for some big n, f^n would be a contracting map of W into itself, and the $\operatorname{orb} x$ would converge to an attracting cycle.

7. Universality around the tip

This section is central in our paper. We prove here that the renormalizations of Hénon-like maps near the tip have the following shape:

$$R^n F = (f_n - b^{2^n} a(x)\, y\, (1 + O(\rho^n)),\, x),$$

where $a(x)$ is a *universal* function associated with the unimodal fixed point f_*. To establish this Universality Law, we study closely the Renormalization Microscope constructed in Section 5. Lemma 7.6, Lemma 7.7, and Corollary 7.10 are the main technical results of this section; they quantify the *tilting* phenomenon mentioned earlier. These lemmas will also be crucial in the next sections when the non-rigidity and the existence of critical Cantor sets with unbounded geometry is established.

7.1. **Some universal one-dimensional functions.** Recall that $f_*\colon I \to I$ stands for the one-dimensional renormalization fixed point normalized so that $f_*(c_*) = 1$ and $f_*^2(c_*) = -1$, where $c_* \in I$ is the critical point of f_*. We let $J_c^* = [-1, f_*^4(c_*)]$ be the smallest renormalization interval of f_*, and we let $s\colon J_c^* \to I$ be the orientation reversing affine rescaling. The smallest renormalization interval around the critical value is denoted by $J_v^* = f_*(J_c^*) = [f_*^3(c_*), 1]$. Then $s \circ f_* : J_v^* \to [-1, 1]$ is an expanding diffeomorphism. Let us consider the inverse contraction

$$g_*\colon I \to J_v^*, \quad g_* = f_*^{-1} \circ s^{-1},$$

where f_*^{-1} stands for the branch of the inverse map that maps J_c^* onto J_v^*. The function g_* is the non-affine branch of the so called "presentation function" (see [BMT] and references therein). Note that 1 is the unique fixed point of g_*.

Let $J_c^*(n) \subset I$ be the smallest periodic interval of period 2^n that contains c_* and $J_v^*(n) \subset I$ be the smallest periodic interval of period 2^n that contains 1.

Let $G_*^n\colon I \to I$ be the diffeomorphism obtained by rescaling affinely the image of g_*^n. The fact that g_* is a contraction implies that the following limit exists

$$u_* = \lim_{n\to\infty} G_*^n\colon I \to I,$$

where the convergence is exponential in the C^3-topology. In fact, this function linearizes g_* near the attracting fixed point 1 (see, e.g., [M, Theorem 8.2]).

Lemma 7.1. *For every $n \geq 1$*

(1) $J_v^*(n) = g_*^n(I)$,
(2) $R_v^n f_* = G_*^n \circ f_* \circ (G_*^n)^{-1}$.

Moreover,

(3) $u_* \circ f^* = f_* \circ u_*$.

Proof. The proof of the first two items is by induction. Notice that the definition of g_* implies directly

$$f_*^2 | J_v^* = g_* \circ f_* \circ (g_*)^{-1}.$$

Let $h_n : I \to J_v^*(n)$ be the conjugation between the two infinitely renormalizable maps $f_*^{2^n} | J_v^*(n)$ and f_*,

$$f_*^{2^n} | J_v^*(n) = h_n \circ f_* \circ (h_n)^{-1}.$$

Note, $h_1 = g_*$. A calculation shows,

$$h_{n+1} = h_n \circ g_*.$$

To do this calculation, first notice that

$$J_v^*(n+1) = h_n(J_v^*).$$

Hence,

$$\begin{aligned} f_*^{2^{n+1}}|J_v^*(n+1) &= f_*^{2^n}|J_v^*(n) \circ f_*^{2^n}|J_v^*(n+1) \\ &= h_n \circ f_*^2|J_v^* \circ (h_n)^{-1} \\ &= (h_n \circ g_*) \circ f_*^2 \circ (h_n \circ g_*)^{-1}. \end{aligned}$$

Now, $R_v^n f_*$ is obtained by rescaling $f_*^{2^n}|J_v^*(n)$. In particular,

$$R_v^n f_* = G_*^n \circ f_* \circ (G_*^n)^{-1}.$$

This finishes the proof of item (1) and (2). The convergence of the sequence G_*^n to u_* implies that $R_v^n f_*$ converges. The limit has to be the unique fixed point f^* of R_v. This finishes the proof of (3). $\square$

Notice that $|J_c^*(n)| = \sigma^n$ and $f_*(J_c^*(n)) = J_v^*(n) = g_*^n(I)$. Hence,

Corollary 7.2. $\dfrac{dg_*}{dx}(1) = \sigma^2$.

Along with u_*, we consider its rescaling

$$v_* : I \to \mathbb{R}, \quad v_*(x) = \frac{1}{u_*'(1)}(u_*(x) - 1) + 1,$$

normalized so that $v_*(1) = 1$ and $\dfrac{dv_*}{dx}(1) = 1$.

Lemma 7.3. *Let $\rho \in (0,1)$, $C > 0$. Let us consider a sequence of smooth functions $g_k : I \to I$, $k = 1, \dots, n$, such that $\|g_k - g_*\|_{C^3} \le C\rho^k$. Let $g_k^n = g_k \circ \cdots \circ g_n$, and let $G_k^n = a_k^n \circ g_k^n : I \to I$, where a_k^n is the affine rescaling of $\operatorname{Im} g_k^n$ to I. Then $\|G_k^n - G_*^k\|_{C^2} \le C_1 \rho^{n-k}$, where C_1 depends only on ρ and C.*

Proof. Let $I_0 = I$ and $I_j = [x_j, y_j] \subset I$ such that $g_j(I_j) = I_{j-1}$. Rescale affinely the domain and image of $g_j \colon I_j \to I_{j-1}$ and denote the normalized diffeomorphism by $h_j \colon [-1,1] \to [-1,1]$. Let

$$I_j^* = [x_j^*, 1] = g_*^{n-j}([-1,1])$$

and rescale the domain and image of $g_* \colon I_j^* \to I_{j-1}^*$ and denote the normalized diffeomorphism by $h_j^* \colon [-1,1] \to [-1,1]$. Note that

$$h_k^* \circ h_{k+1}^* \circ \cdots \circ h_n^* \to u_*,$$

where the convergence in the C^2 topology is exponential in $n-k$. In the following estimates we will use a uniform constant $\rho < 1$ for exponential estimates. Let $\Delta x_j = x_j - x_j^*$ and $\Delta y_j = 1 - y_j$. Then

$$x_{j-1} = g_*(x_j^*) + g_*'(z) \cdot \Delta x_j + O(\rho^j).$$

Hence, using a similar argument for Δy_j,

$$|\Delta x_j|, |\Delta y_j| = O(\rho^j).$$

Because, g_j and g_* are contractions we have

$$|I_j|, |I_j^*| = O(\rho^{n-j}).$$

We will represent a diffeomorphism $\phi : I \to J$ by its nonlinearity

$$\eta_\phi = \frac{D^2\phi}{D\phi}.$$

Let η_j and η^* be the nonlinearities of g_j and g_*. Notice that

$$\|\eta_j - \eta^*\|_{C^1} = O(\rho^j).$$

Furthermore, let $\mathbb{I}_j : [-1,1] \to I_j$ and $\mathbb{I}_j^* : [-1,1] \to I_j^*$ be the affine orientation preserving rescalings. Using this notation

$$\eta_j(\mathbb{I}_j(x)) = \eta^*(\mathbb{I}_j^*(x)) + D\eta^*(z) \cdot \big(\mathbb{I}_j(x) - \mathbb{I}_j^*(x)\big) + O(\rho^j),$$

for some $z \in [\mathbb{I}_j(x), \mathbb{I}_j^*(x)]$. Hence,

$$\eta_j(\mathbb{I}_j(x)) = \eta^*(\mathbb{I}_j^*(x)) + O(\rho^j).$$

The nonlinearities of h_j and h_j^* are given by

$$\eta_{h_j} = |I_j| \cdot \eta_j(\mathbb{I}_j),$$

and similarly

$$\eta_{h_j^*} = |I_j^*| \cdot \eta^*(\mathbb{I}_j^*).$$

Now

$$|\eta_{h_j}(x) - \eta_{h_j^*}(x)| = O((|I_j| - |I_j^*|) + \rho^j \cdot |I_j^*|).$$

Hence

$$|\eta_{h_j}(x) - \eta_{h_j^*}(x)| = \begin{cases} O(\rho^{n-j}) & : \quad j \le (n+k)/2 \\ O(\rho^j) & : \quad j > (n+k)/2. \end{cases}$$

It follows that

$$\sum_{j=k}^{n} \|\eta_{h_j} - \eta_{h_j^*}\|_{C^0} = O(\rho^{n-k}).$$

Note that we can estimate $\|\eta_{h_j}\|_{C^1}$ by using

$$D\eta_{h_j} = |I_j|^2 D\eta_{h_j}(\mathbb{I}_j).$$

The resulting estimate allows to use a reshuffling argument, see Appendix , Lemma 14.1, which finishes the proof of the Lemma. □

7.2. **Asymptotics of the Ψ-functions.** Fix an infinitely renormalizable Hénon-like map $F \in \mathcal{I}_\Omega(\bar\varepsilon)$ to which we can apply Theorem 4.1. For such an F, we have a well defined *tip*:

$$\tau \equiv \tau(F) = \bigcap_{n \ge 0} B^n_{v^n},$$

where the pieces B^n_w are introduced in §5.2. Let us consider the tips of the renormalizations, $\tau_k = \tau(R^k F)$. To simplify the notations, we will translate these tips to the origin by letting

$$\Psi_k \equiv \Psi_k^{k+1} = \Psi_v^1(R^k F)\,(z + \tau_{k+1}) - \tau_k.$$

Denote the derivative of the maps Ψ_k at 0 by $D_k \equiv D_k^{k+1}$ and decompose it into the unipotent and diagonal factors:

$$D_k = \begin{pmatrix} 1 & t_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{pmatrix}. \tag{7.1}$$

Let us factor this derivative out from Ψ_k:

$$\Psi_k = D_k \circ (\mathrm{id} + \mathbf{s}_k),$$

where $\mathbf{s}_k(z) = (s_k(z), 0) = O(|z|^2)$ near 0. The convergence Theorem 4.1 and the explicit expression for the Ψ-functions (see (3.3) and §5.1) imply:

Lemma 7.4. *There exists $\rho < 1$ such that for $k \in \mathbb{Z}_+$ the following estimates hold:*
(1) $\alpha_k = \sigma^2 \cdot (1 + O(\rho^k))$, $\quad \beta_k = -\sigma \cdot (1 + O(\rho^k))$, $\quad t_k = O(\bar{\varepsilon}^{2^k})$;
(2) $|\partial_x s_k| = O(1)$, $\quad |\partial_y s_k| = O(\bar{\varepsilon}^{2^k})$;
(3) $|\partial^2_{xx} s_k| = O(1)$, $\quad |\partial^2_{xy} s_k| = O(\bar{\varepsilon}^{2^k})$, $\quad |\partial^2_{yy} s_k| = O(\bar{\varepsilon}^{2^k})$.

Note that since all the maps under consideration are holomorphic, the bounds on their derivatives follow from the bounds on the maps themselves.

Let now

$$\Psi_k^n = \Psi_k \circ \cdots \circ \Psi_{n-1}, \quad B_k^n = \operatorname{Im} \Psi_k^n.$$

Since by Lemma 5.1

$$\operatorname{diam}(B_k^n) = O(\sigma^{n-k}) \quad \text{for} \qquad k < n,$$

we conclude:

Corollary 7.5. *Let $k < n$. For $z \in B_{k+1}^n$ we have:*

$$|\partial_x s_k(z)| = O(\sigma^{n-k}), \quad |\partial_y s_k(z)| = O(\bar{\varepsilon}^{2^k} \cdot \sigma^{n-k}).$$

Let us now consider the derivatives of the maps Ψ_k^n at the origin:

$$D_k^n = D_k \circ D_{k+1} \circ \cdots D_{n-1}.$$

Since the unipotent matrices form a normal subgroup in the group of upper-triangular matrices, we can reshuffle this composition and obtain:

$$D_k^n = \begin{pmatrix} 1 & t_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\sigma^2)^{n-k} & 0 \\ 0 & (-\sigma)^{n-k} \end{pmatrix} (1 + O(\rho^k)). \tag{7.2}$$

Factoring the derivatives D_k^n out from Ψ_k^n, we obtain:

$$\Psi_k^n = D_k^n \circ (\mathrm{id} + \mathbf{S}_k^n), \tag{7.3}$$

where $\mathbf{S}_k^n(z) = (S_k^n(z), 0) = O(|z|^2)$ near 0.

Lemma 7.6. *For $k < n$, we have:*
(1) $|\partial_x S_k^n| = O(1)$, $\quad |\partial_y S_k^n| = O(\bar{\varepsilon}^{2^k})$;
(2) $|\partial^2_{xx} S_k^n| = O(1)$, $\quad |\partial^2_{yy} S_k^n| = O(\bar{\varepsilon}^{2^k})$, $\quad |\partial^2_{xy} S_k^n| = O(\bar{\varepsilon}^{2^k}\, \sigma^{n-k})$.

Proof. Let

$$z_{k+1}^n = \begin{pmatrix} x_{k+1}^n \\ y_{k+1}^n \end{pmatrix} = \Psi_{k+1}^n(z)$$

By (7.2) and (7.3),

$$\begin{aligned} x_{k+1}^n &= K_1\,(\sigma^2)^{n-k-1}\,(x + S_{k+1}^n(x,y)) + K_2\,t_k\,(-\sigma)^{n-k-1}\,y \\ y_n^{k+1} &= K_3\,(-\sigma)^{n-k-1}\,y, \end{aligned}$$

where $K_i = K_i(k,n) = O(1)$ (and the constants K_i below have the same meaning).

Moreover, since

$$D_k^n \circ (\mathrm{id} + \mathbf{S}_k^n) = \Psi_k^n = \Psi_k \circ \Psi_{k+1}^n = D_k \circ (\mathrm{id} + \mathbf{s}_k) \circ \Psi_{k+1}^n =$$
$$D_k^n \circ (\mathrm{id} + \mathbf{S}_{k+1}^n) + D_k \circ \mathbf{s}_k \circ \Psi_{k+1}^n,$$

we obtain:

$$S_k^n(z) = S_{k+1}^n(z) + K_4\,(\lambda^2)^{n-k-1}\,s_k(z_{k+1}^n)$$

(recall that $\lambda = \sigma^{-1}$).

The proof proceeds by relating the partial derivatives of S_k^n to the derivatives of S_{k+1}^n and s_k. For instance, by differentiating the last equation taking into account the above expressions for x_{k+1}^n and y_{k+1}^n, we obtain:

$$\frac{\partial S_k^n}{\partial y} = (1 + K_5\,\frac{\partial s_k}{\partial x})\,\frac{\partial S_n^{k+1}}{\partial y} + K_6\,t_k\,(-\lambda)^{n-k-1}\,\frac{\partial s_k}{\partial x} + K_7\,(-\lambda)^{n-k-1}\,\frac{\partial s_k}{\partial y},$$

where the partial derivatives of s_k are computed at z_{k+1}^n. Now Corollary 7.5 implies

$$\left|\frac{\partial S_k^n}{\partial y}\right| \le (1 + O(\rho^{n-k}))\,\left|\frac{\partial S_{k+1}^n}{\partial y}\right| + C\,\bar\varepsilon^{2^k},$$

and hence for all $k < n$,

$$\left|\frac{\partial S_k^n}{\partial y}\right| \le C\,\bar\varepsilon^{2^k},$$

as was asserted. The bound for $\partial S_k^n/\partial y$ is obtained in a similar way.

Since the functions S_k^n are holomorphic and defined on a fixed domain, the first two bounds on the second derivatives follow. However, the bound on the mixed derivative does not follow from this general reasoning. Differentiating $\partial S_k^n/\partial y$ (taking into account the expressions for x_{k+1}^n and y_{k+1}^n), we obtain:

$$\begin{aligned} \frac{\partial^2 S_k^n}{\partial xy} = &\left(1 + K_5 \frac{\partial s_k}{\partial x}\right)\,\frac{\partial^2 S_{k+1}^n}{\partial xy} + \\ &\left(1 + \frac{\partial S_{k+1}^n}{\partial x}\right)\,(\sigma^2)^{n-k-1}\,\frac{\partial^2 s_k}{\partial x^2}\left(K_8\,\frac{\partial S_{k+1}^n}{\partial y} + K_9\,t_k \lambda^{n-k-1}\right) + \\ &K_{10}\left(1 + \frac{\partial S_{k+1}^n}{\partial x}\right)\,(-\sigma)^{n-k-1}\,\frac{\partial^2 s_k}{\partial xy}, \end{aligned}$$

where the partial derivatives of s_k are calculated at x_{k+1}^n. Using Corollary 7.5 and the previous estimates on the first partial derivatives of S_k^n, we obtain

$$\left|\frac{\partial^2 S_k^n}{\partial xy}\right| \le (1 + O(\rho^{n-k}))\cdot\left|\frac{\partial^2 S_{k+1}^n}{\partial xy}\right| + C\cdot\bar\varepsilon^{2^k}\cdot\sigma^{n-k}.$$

Hence,

$$\left|\frac{\partial^2 S_k^n}{\partial xy}\right| \leq C \cdot \bar{\varepsilon}^{2^k} \cdot \sigma^{n-k}.$$

$\square$

We are now ready to describe the asymptotical behavior of the Ψ-functions using the universal one-dimensional functions from §7.1. Let us normalize the function v_* so that it fixes 0 rather than 1:

$$\mathbf{v}_*(x) = v_*(x+1) - 1.$$

Lemma 7.7. *There exists $\rho < 1$ such that for all $k < n$ and $y \in I$,*

$$|\mathrm{id} + S_k^n(\cdot, y) - \mathbf{v}_*(\cdot)| = O(\bar{\epsilon}^{2^k} \cdot y + \rho^{n-k})$$

and

$$\left|1 + \frac{\partial S_k^n}{\partial x}(\cdot, y) - \frac{\partial \mathbf{v}_*}{\partial x}(\cdot)\right| = O(\rho^{n-k}).$$

Proof. By Lemma 7.6,

$$\left|\frac{\partial^2 S_k^n}{\partial yx}\right| = O(\bar{\varepsilon}^{2^k} \sigma^{n-k}) = O(\sigma^{n-k})$$

and

$$\left|\frac{\partial S_k^n}{\partial y}\right| = O(\bar{\epsilon}^{2^k}).$$

Hence it is enough to verify the desired convergence on the horizontal section passing through the tip:

$$\mathrm{dist}_{C^1}(\mathrm{id} + S_k^n(\cdot, 0),\, \mathbf{v}_*(\cdot)) = O(\rho^{n-k}).$$

Let us normalize g_* so that 0 becomes its fixed point with 1 as multiplier:

$$\mathbf{g}_*(x) = \frac{g_*(x+1) - 1}{g_*'(1)}.$$

Now, $\mathrm{id} + S_k^n(\cdot, 0)$ is the rescaling of $\Psi_k^n(\cdot, 0)$ normalized so that the fixed point 0 has multiplier 1. By Theorem 4.1,

$$\mathrm{dist}_{C^3}(\mathrm{id} + s_k(\cdot, 0), \mathbf{g}_*(\cdot)) = O(\rho^k).$$

Hence, by Lemma 7.3,

$$\mathrm{dist}_{C^1}(\mathrm{id} + S_k^n(\cdot, 0), \mathbf{g}_*^{n-k}(\cdot)) = O(\rho^{n-k}).$$

Since $\mathbf{g}^n \to \mathbf{v}_*$ exponentially fast, the conclusion follows. $\square$

Proposition 7.8. *There exists a coefficient $a_F \in \mathbb{R}$ and an absolute constant $\rho \in (0, 1)$ such that*

$$\left|(x + S_0^n(x, y)) - (\mathbf{v}_*(x) + a_F y^2)\right| = O(\rho^n).$$

Proof. The image of the vertical interval $y \mapsto (0, y)$ under the map $\mathrm{id} + \mathbf{S}_0^n$ is the graph of a function $w_n : I \to \mathbb{R}$ defined by

$$w_n(y) = S_0^n(0, y).$$

By the second part of Lemma 7.7 we have:

$$|(x + S_0^n(x, y)) - (\mathbf{v}_*(x) + w_n(y))| = O(\rho^n).$$

Let us show that the functions w_n converge to a parabola. The identity

$$D_0^{n+1} \circ (\mathrm{id} + \mathbf{S}_0^{n+1}) = \Psi_0^{n+1} = \Psi_0^n \circ \Psi_n = D_0^n \circ (\mathrm{id} + \mathbf{S}_0^n) \circ D_n \circ (\mathrm{id} + \mathbf{s}_n),$$

implies

$$\mathbf{S}_0^{n+1} = \mathbf{s}_n + D_n^{-1} \circ \mathbf{S}_0^n \circ D_n \circ (\mathrm{id} + \mathbf{S}_n),$$

so that

$$w_{n+1}(y) = s_n(0, y) + \frac{1}{\alpha_n} S_0^n(\alpha_n s_n(0, y) + \beta_n t_n y,\ \beta_n y), \tag{7.4}$$

where α_n, β_n and t_k are the entries of D_n, see equation (7.1). The estimate of $\partial_y s_n$ from Lemma 7.4 implies:

$$s_n(0, y) = e_n y^2 + O(\bar{\varepsilon}^{2^n} y^3), \tag{7.5}$$

where $e_n = O(\bar{\epsilon}^{2^n})$. The estimate of $\partial^2_{xy} S_0^n$ from Lemma 7.6 implies:

$$\frac{\partial S_0^n}{\partial x}(0, y) = O(\bar{\varepsilon}^{2^n} y).$$

Hence

$$\begin{aligned}
&S_0^n(\alpha_n s_n(0, y) + \beta_n t_n y,\ \beta_n y) \\
&= S_0^n(0, \beta_n y) + \frac{\partial S_0^n}{\partial x}(0, \beta_n y)(\alpha_n s_n(0, y) + \beta_n t_n y) + O(\bar{\varepsilon}^{2^n} y^3) \\
&= S_0^n(0, \beta_n y) + q_n y^2 + O(\bar{\varepsilon}^{2^n} y^3) = w_n(\beta_n y) + q_n y^2 + O(\bar{\varepsilon}^{2^n} y^3),
\end{aligned}$$

where $q_n = O(\bar{\varepsilon}^{2^n})$. Incorporating this and (7.5) into (7.4), we obtain:

$$w_{n+1}(y) = \frac{1}{\alpha_n} w_n(\beta_n y) + c_n y^2 + O(\bar{\varepsilon}^{2^n} y^3),$$

where $c_n = O(\bar{\epsilon}^{2^n})$. Writing w_n in the form

$$w_n(y) = a_n y^2 + A_n(y) y^3,$$

we obtain:

$$a_{n+1} = \frac{\beta_n^2}{\alpha_n} a_n + c_n$$

and

$$\|A_{n+1}\| \le \frac{|\beta_n|^3}{\alpha_n} \|A_n\| + O(\bar{\varepsilon}^{2^n}).$$

Now the first item of Lemma 7.4 implies that $a_n \to a_F$ and $\|A_n\| \to 0$ exponentially fast. $\square$

7.3. **Universality.** We are ready to prove the main positive result of this paper:

Theorem 7.9 (Universality)**.** *For any $F \in \mathcal{I}_\Omega(\bar{\varepsilon})$ with sufficiently small $\bar{\varepsilon}$, we have:*

$$R^n F = (f_n(x) - b^{2^n} a(x)\, y\, (1 + O(\rho^n)),\ x\ ),$$

where $f_n \to f_*$ exponentially fast, b is the average Jacobian, $\rho \in (0, 1)$, and $a(x)$ is a universal function. Moreover, a is analytic and positive.

Proof. Let $F_n \equiv R^n F = (f_n - \varepsilon_n, x)$. The function $\Psi^n \equiv \Psi^n_v$ conjugates the renormalization F_n to the iterate F^{2^n} on the piece $B^n \equiv B^n_v$. (Here Ψ^n is the original Ψ-function rather than the normalized one, Ψ^n_0.) According to the chain rule,

$$\begin{aligned} \partial_y \varepsilon_n(z) &= \operatorname{Jac} F_n(z) = \operatorname{Jac} F^{2^n}(\Psi^n(z)) \frac{\operatorname{Jac} \Psi^n(z)}{\operatorname{Jac} \Psi^n(F_n z)} \\ &= b^{2^n} \frac{\operatorname{Jac} \Psi^n(z)}{\operatorname{Jac} \Psi^n(F_n z)} (1 + O(\rho^n)), \end{aligned} \tag{7.6}$$

where the last equality follows from Lemma 6.2.

Let $D^n \equiv D^n_0$, $\mathbf{S}^n \equiv \mathbf{S}^n_0$, $S^n \equiv S^n_0$. Let us consider affine maps $T^n : z \mapsto z - \tau_n$ and $L^n : z \mapsto (D^n)^{-1}(z - \tau)$ as local charts on B and B^n respectively. Various maps presented in these local charts will be written in the boldface, so that

$$\mathbf{F}_n = T^n \circ F_n \circ (T^n)^{-1}, \quad \mathbf{\Psi}^n \equiv \mathrm{id} + \mathbf{S}^n = L^n \circ \Psi^n \circ (T^n)^{-1}.$$

Since affine maps do not distort the Jacobian, we have:

$$\frac{\operatorname{Jac} \Psi^n(z)}{\operatorname{Jac} \Psi^n(F_n z)} = \frac{\operatorname{Jac} \mathbf{\Psi}^n(\mathbf{z})}{\operatorname{Jac} \mathbf{\Psi}^n(\mathbf{F}_n \mathbf{z})} = \frac{1 + \partial_x S^n(\mathbf{z})}{1 + \partial_x S^n(\mathbf{F}_n \mathbf{z})}, \tag{7.7}$$

where $\mathbf{z} = Tz$.

By Lemma 7.7,

$$1 + \partial_x S^n \to \mathbf{v}'_* \tag{7.8}$$

exponentially fast. By Theorem 4.1, $\tau_n \to \tau_\infty \equiv (c_*, 1)$ exponentially fast, so that T_n converges exponentially to the translation $T^\infty : z \mapsto z - \tau_\infty$. Applying Theorem 4.1 once again, we conclude that $\mathbf{F}_n \to (\mathbf{f}_*, x)$ exponentially fast, where $\mathbf{f}_*(x) = f_*(x+1) - 1$. Putting this together with (7.7) and (7.8), we conclude:

$$\frac{\operatorname{Jac} \Psi^n(z)}{\operatorname{Jac} \Psi^n(F_n z)} \to \frac{\mathbf{v}'_*(\mathbf{x})}{\mathbf{v}'_*(\mathbf{f}_*(\mathbf{x}))} = \frac{v'_*(x)}{v'_*(f_*(x))} \equiv a(x),$$

where $z = (x, y)$, $\mathbf{x} = x - 1$, and convergence is exponential. Since v_* is an analytic diffeomorphism, the function $a(x)$ is analytic and non-vanishing.

Plugging the last formula into (7.6), we obtain:

$$\partial_y \varepsilon_n(z) = b^{2^n} a(x) (1 + O(\rho^n)).$$

Integration of this formula yields:

$$\varepsilon_n(x, y) = c_n(x) + b^{2^n} a(x)\, y\, (1 + O(\rho^n)),$$

and since $\|c_n(x)\|$ is super-exponentially small, it can be incorporated into the unimodal term $f_n(x)$. □

Corollary 7.10. *The numbers t_k defined by equation (7.2) satisfy*

$$t_k \asymp -b^{2^k}.$$

Proof. Consider $\Psi_k = (\Lambda_k \circ H_k)^{-1}$, where Λ_k and H_k are used to define $R^{k+1}F$. Recall

$$\Lambda_k(x, y) = \begin{pmatrix} s_k(x) \\ s_k(y) \end{pmatrix}$$

and

$$H_k(x,y) = \begin{pmatrix} f_k(x) - \epsilon_k(x,y) \\ y \end{pmatrix}$$

where s_k is an orientation reversing affine map with $s \asymp -1$ as derivative. Then

$$D_k^{-1} = D\Lambda_k \circ DH_k = \begin{pmatrix} \cdot & -s\partial_y \varepsilon_k(\tau_k) \\ 0 & \cdot \end{pmatrix}.$$

The representation of D_k from (7.1) gives

$$\begin{pmatrix} 1 & -t_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{pmatrix} \begin{pmatrix} \cdot & -s\partial_y \varepsilon_k(\tau_k) \\ 0 & \cdot \end{pmatrix}.$$

This implies

$$t_k = \alpha_k \cdot s \cdot \partial_y \varepsilon_k(\tau_k),$$

where $s \asymp -1$. Now equation (7.6) and Lemma 7.4(1) imply

$$t_k \asymp -\partial_y \varepsilon_k(\tau_k) = -\operatorname{Jac} F_n(\tau_k) \asymp -b^{2^k}.$$

$\square$

8. Affine rescaling and quadratic change of variable

The renormalization procedure described in the previous sections differs in two ways from the standard unimodal period-doubling renormalization. First, we are renormalizing around the tip of the Hénon map which becomes the critical value in the degenerate case. Secondly, we use non-linear changes of coordinates Ψ_0^n to define $R^n F$. This was necessary for the renormalizations to be Hénon-like maps again. In this section we will show that in fact, a quadratic change of coordinates can be used to produce renormalizations converging to a degenerate universal map. (However, affine rescalings would not be sufficient!) This universal map is not the usual fixed point of renormalization around the critical point, but rather the fixed point of renormalization around the critical value.

Let us now introduce the promised quadratic change of coordinates. Take an infinitely renormalizable $F \in \mathcal{I}_\Omega(\bar\varepsilon)$ with sufficiently small $\bar\varepsilon$, so that the results from § 7 apply to the maps Ψ_0^n. As in that section, let us consider translations $T_n : z \mapsto z - \tau_n$ (where τ_n is the tip of $F_n \equiv R^n F$), and the affine local charts

$$L_0^n = (D_0^n)^{-1} \circ T_0 : B_v^n(F) \to \mathbb{R}^2.$$

Let us represent the maps F_n and Ψ_0^n in these charts:

$$\mathbf{F}_n = T^n \circ F_n \circ (T^n)^{-1}, \quad \mathbf{\Psi}_0^n = \mathrm{id} + \mathbf{S}_0^n = L_0^n \circ \Psi_0^n \circ T_n^{-1}.$$

Let us define the $n^{th}$$-$*affine* renormalization of F as follows:

$$R^n_{\text{aff}} F = L_0^n \circ [F|\, B_v^n(F)]^{2^n} \circ (L_0^n)^{-1} = \mathbf{\Psi}_0^n \circ \mathbf{F}_n \circ (\mathbf{\Psi}_0^n)^{-1}.$$

Note that the domain of the n^{th}-affine renormalizations is the $\operatorname{Im} \mathbf{\Psi}_0^n$.[8]

We also let $T_\infty : z \mapsto z - 1$ and

$$\mathbf{F}_* = T_\infty \circ F_* \circ T_\infty^{-1}$$

[8]Note that R^n_{aff} is *not* the n-fold iterate of some R_{aff}.

By Proposition 7.8, the maps $\mathbf{\Psi}_0^n$ converge to

$$\mathbf{V}_{*,a_F} : (x, y) \mapsto (\mathbf{v}_*(x) + a_F y^2, y),$$

exponentially fast. Furthermore, by Theorem 4.1, $\mathbf{F}_n \to \mathbf{F}_*$ exponentially fast. Hence

Theorem 8.1. *Let $F \in \mathcal{I}_\Omega(\bar{\varepsilon})$ be infinitely renormalizable with sufficiently small $\bar{\varepsilon}$. Then*

$$R^n_{\text{aff}} F \to \mathbf{V}_{*,a_F} \circ \mathbf{F}_* \circ \mathbf{V}^{-1}_{*,a_F}$$

exponentially fast.

Consider the quadratic change of coordinates $Q_F : \mathbb{R}^2 \to \mathbb{R}^2$,

$$Q_F : (x, y) \mapsto (x - a_F y^2, y),$$

and define $H_n : B^n_v(F) \to \mathbb{R}^2$ as the composition:

$$H_n = Q_F \circ L^n_0.$$

Conjugating F^{2^n} by these quadratic changes of variable, we obtain the desired renormalizations:

$$R^n_{\text{qd}} F = H_n \circ F^{2^n} \circ H_n^{-1}.$$

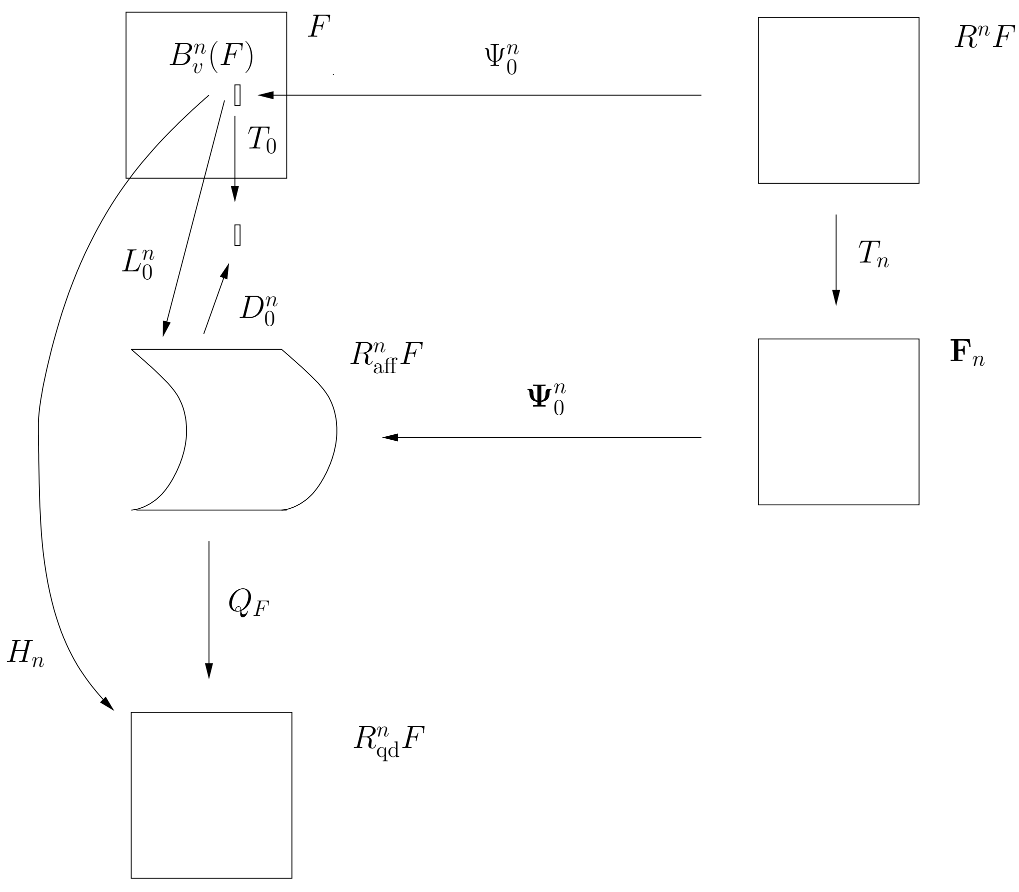

FIGURE 8.1. Changes of coordinates

Theorem 8.2. *Let $F \in \mathcal{I}_\Omega(\bar\varepsilon)$ be infinitely renormalizable with sufficiently small $\bar\varepsilon$. Then*

$$R^n_{\mathrm{qd}} F(x,y) \to (l \circ f^* \circ l^{-1}(x), \mathbf{v}_*^{-1}(x))$$

exponentially fast, where $l(x) = (x-1)/u'_*(1)$.

Proof. Let $\mathbf{V}_* = \mathbf{V}_{*,0}$. Proposition 7.8 tells us that

$$Q_F \circ \mathbf{\Psi}_0^n \to \mathbf{V}_*,$$

exponentially fast. This implies that

$$R^n_{\mathrm{q}d} = (Q_F \circ \mathbf{\Psi}_0^n) \circ \mathbf{F}_n \circ (Q_F \circ \mathbf{\Psi}_0^n)^{-1} \to \mathbf{V}_* \circ \mathbf{F}_* \circ \mathbf{V}_*^{-1}$$

exponentially fast. Applying Lemma 7.1(3) and the relation between u_* and $\mathbf{v}_*$, we obtain:

$$\mathbf{V}_* \circ \mathbf{F}_* \circ \mathbf{V}_*^{-1} : (x,y) \mapsto (l \circ f^* \circ l^{-1}(x), \mathbf{v}_*^{-1}(x)),$$

and the theorem follows. □

Remark 8.1. In the forthcoming Part II we will construct the stable manifold $W^s(\tau_F)$ at the tip τ_F and will show that the number a_F is equal to its curvature at τ_F.

Remark 8.2. The horizontal width of the box $B^n_v(F)$ is proportional to the square of its vertical size. This box, a narrow strip containing the tip, is aligned along $W^s(\tau_F)$. Any affine change of coordinates which brings this box roughly to the unit size is boundedly related to the affine map L^n_0. In the case when $a_F \neq 0$, these scalings are not capable to "unbend" the boxes $B^n_v(F)$. (As a model, notice that the rescaling of the parabola $x = ay^2$ by a linear map $(x,y) \mapsto (\sigma^2 x, \sigma y)$ does not change the curvature a.) Thus, the renormalizations obtained by affine changes of variable will always remember the curvature a_F. Hence they cannot have a universal limit.

9. Non-existence of continuous invariant line fields

In this section, $F \in \mathcal{H}_\Omega(\bar\varepsilon)$ stands for an infinitely renormalizable *non-degenerate* Hénon-like map to which the results of §4 apply. Then by the results of §5, it possesses the Cantor attractor $\mathcal{O} = \mathcal{O}_F$ on which it acts as the adding machine. We will show that F does not have continuous invariant line fields on $\mathcal{O}$. This has several interesting consequences:

- Contrary to a common intuition, the attractor $\mathcal{O}$ does not lie on a smooth curve.
- The $\mathrm{SL}(2,\mathbb{R})$-cocycle

$$z \mapsto DF(z)/\sqrt{\operatorname{Jac} F(z)} \tag{9.1}$$

is not uniformly hyperbolic over $\mathcal{O}$. By Theorem 6.3, it has non-vanishing characteristic exponents $\pm\dfrac{1}{2}\log b$, so it is non-uniformly hyperbolic. It seems to be the first example of a non-uniformly hyperbolic $\mathrm{SL}(2,\mathbb{R})$-cocycle over the adding machine.

Lemma 9.1. *If F has a continuous invariant line field on $\mathcal{O}_F$ then there exists $n_0 \geq 1$ such that for any $n \geq n_0$, the renormalization $R^n F$ has a continuous invariant direction field on $\mathcal{O}_{R^n F}$.*

Proof. Note first that any continuous F-invariant line field on $\mathcal{O} = \mathcal{O}_F$ can be pulled back to a continuous invariant line field on $\mathcal{O}_{R^nF}$ for any renormalizations R^nF.

Furthermore, since the set $\mathcal{O}$ is totally discontinuous, any continuous invariant line field on it can be continuously orientated. Then there exist a partition of $\mathcal{O}$ into two clopen sets $\mathcal{O}^+$ and $\mathcal{O}^-$ such that $F|\mathcal{O}^+$ preserves the orientation of the field, while $F|\mathcal{O}^-$ reverses it. Since the pieces B^n_w uniformly shrink as $n \to \infty$, for n large enough each $B^n_w \cap \mathcal{O}_F$ is contained either in $\mathcal{O}^+$ or in $\mathcal{O}^-$. Hence $F^{2^n}|B^n_v$ either preserves or reverses the orientation of the line field. It follows that the renormaliztion R^nF either preserves or reverses the induced orientation of the line field on $\mathcal{O}_{R^nF}$. In either case we conclude that the next renormalization, $R^{n+1}F$, preserves the induced orientation. $\square$

For any matrix

$$A = \begin{pmatrix} a & -\delta \\ 1 & 0 \end{pmatrix}, \quad \delta > 0, \tag{9.2}$$

let us consider its induced action on the circle S^1 of directions in $\mathbb{R}^2$. parametrized by the angle θ. (We will keep the same notation, A, for the induced action.) Let L and R stand for the left- and right-hand semi-circles of S^1, while U and D stand for the upper and lower semi-circles. Then $A(R) = U, \quad A(L) = D$, and in the projective coordinate $t = x/y = \operatorname{ctg}\theta$ both maps, $A : R \to U$ and $A : L \to D$, assume the form

$$t \mapsto a - \frac{\delta}{t}. \tag{9.3}$$

For $\alpha \in (0, \pi/2)$, let us consider two symmetric direction cones:

$$C^+_\alpha = (\alpha, \pi - \alpha) \equiv \{\theta \in S^1 : \alpha \le \theta \le \pi - \alpha\}; \quad C^-_\alpha = -C^+_\alpha.$$

Lemma 9.2. *There exists an angle $\alpha \in (0, \pi/2)$ with the following property. Let $X = \{F^n(z_0)\}^\infty_{n=-\infty}$ be any two-sided orbit of F in $\mathcal{O}$, and let $z \mapsto \theta(z)$ be an invariant direction field over X. Then there exist points $z^\pm \in X$ such that $z^\pm \in C^\pm_\alpha$.*

Proof. Let us write the differential of F in form (9.2):

$$A_z \equiv DF(z) = \begin{pmatrix} a(z) & -\delta(z) \\ 1 & 0 \end{pmatrix}.$$

Let $\bar{a} = \max_{z\in\mathcal{O}} |a(z)|$ Without loss of generality we can assume that $\delta(z) < \bar{a}$ everywhere (replacing F by its renormalization if needed). Let

$$\kappa = \max\{2|\bar{a}|, 1\}; \quad \alpha = \operatorname{arcctg}\kappa \in (0, \pi/4].$$

We let Q_i, $i = 1, \dots, 4$, be the four quadrants in S^1:

$$Q_1 = [0, \pi/2], \dots, Q_4 = [3\pi/2, 2\pi].$$

Assume that $\theta(z) \notin C^+_\alpha$ for any $z \in X$.

Note that $A_z[0, \alpha] \subset C^+_\alpha$ for any $z \in \mathcal{O}$. Indeed, in the projective coordinate $t = \operatorname{ctg}\theta$, the cone C^+_α is given by equation $|t| \le \kappa$. By (9.3), we have: $|\operatorname{ctg} A_z(0)| = |a(z)| < \kappa$ so that $A_z(0) \in C^+_\alpha$. If $A_z(\alpha) < \pi/2$, then obviously $A_z(\alpha) \in C^+_\alpha$ as well. Otherwise by (9.3) we have:

$$|\operatorname{ctg} A_z(\alpha)| \le |\operatorname{ctg} A_z(\pi/4)| = |a(z) - \delta(z)| \le 2|\bar{a}| \le \kappa,$$

and thus $A_z(\alpha) \in C^+_\alpha$ again.

By invariance of the direction field, we conclude that $\theta(z) \notin [0, \alpha]$ for $z \in X$. Hence $\theta(z) \notin Q_1$ for $z \in X$.

Since $A_z(Q_4) = [0, A_z(0)] \subset C_\alpha^+ \cup Q_1$, we conclude that $\theta(z) \notin Q_4$.

At this point we already know that $\theta(z) \in [\pi - \alpha, \pi] \cup Q_3 \equiv P$ for $z \in X$. But then

$$\theta(z) = A_{F^{-1}z}(\theta(F^{-1}z)) \subset D, \quad z \in X,$$

and hence $\theta(z) \in P \cap D = Q_3$.

By replacing F with its renormalizaion, we can bring it arbitrary closely to the degenerate fixed point F_*. Thus, we can assume that the Cantor attractor $\mathcal{O}_F$ is close to $\mathcal{O}_{F_*}$ in the first place, which implies (together with minimality of $\mathcal{O}_F$) that $a(z) = f'(z) - \partial_x \varepsilon(z) < 0$ for some $z \in X$. But then $A_z(Q_3) \subset Q_4$ for this point z, and we arrive at a contradiction.

We have proved the assertion for the positive cone C_α^+. The one for the negative cone follows by central symmetry of the cocycle. ☐

Proposition 9.3. *There are no continuous invariant direction fields on $\mathcal{O}_F$.*

Proof. Suppose there exists a continuous invariant direction field on $\mathcal{O}_F$. Then there exists such a field for every renormalization. By Lemma 9.2, for each n we can find a pair of points $z_n, \zeta_n \in \mathcal{O}_{R^n F}$ such that $\theta(z_n) \in C_\alpha^+$ while $\theta(\zeta_n) \in C_\alpha^-$.

Now project these points to the box B_v^n by the maps Ψ_v^n making use of equation (7.2) and Lemma 7.6. We obtain two sequences of points, $\hat{z}_n$ and $\hat{\zeta}_n$, converging to the tip τ_F. The direction field at $\hat{z}_n$ points upward at angle $\theta(z_n) = \pi/2 + O(b_F)$ while the direction field at $\hat{\zeta}_n$ points downward at angle $\theta(\zeta_n) = -\pi/2 + O(b_F)$. Thus, the direction field is not continuous at the tip of F. ☐

Lemma 9.1 and Proposition 9.3 imply the desired:

Corollary 9.4. *The map F does not have a continuous invariant line field on the critical Cantor set $\mathcal{O}_F$.*

It immediately yields:

Theorem 9.5. *The map F is not partially hyperbolic on $\mathcal{O}_F$ in the sense that the contracting and neutral line fields corresponding to the characteristic exponents* $\log b$ *and* 0 *(see Theorem 6.3) are discontinuous.*

Theorem 9.6. *The* $\mathrm{SL}(2, \mathbb{R})$*-cocycle (9.1) is non-uniformly hyperbolic over $\mathcal{O}$.*

Theorem 9.7. *There are no smooth curves containing $\mathcal{O}_F$.*

Proof. If $\mathcal{C}$ is a smooth curve containing $\mathcal{O}_F$, then its tangent lines $l(z)$ give us a continuous line field on $\mathcal{O}_F$. Since $\mathcal{O}_F$ does not have isolated points,

$$l(z) = \lim_{\zeta \to z} l(z, \zeta),$$

where $l(z, \zeta)$ is the line passing through z and $\zeta \in \mathcal{O}_F$, $\zeta \neq z$. It follows that the line field $l(z)$ is invariant over $\mathcal{O}_F$, contradicting Corollary 9.4. ☐

10. Non-rigidity of the critical Cantor set

We will show that the invariant Cantor set $\mathcal{O}$ of an infinitely renormalizable Hénon-like map is not rigid. In fact, there is a definite upper bound smaller than 1 on the Hölder exponent of the conjugacy between two such Cantor sets of any two Hénon-like maps with different average Jacobians.

Theorem 10.1. *Let F and $\tilde{F}$ be two infinitely renormalizable Hénon-like maps with average Jacobian b and $\tilde{b}$ resp. Assume $b > \tilde{b}$. Let ϕ be a homeomorphism which conjugates $F|_{\mathcal{O}_F}$ and $\tilde{F}|_{\mathcal{O}_{\tilde{F}}}$ with $\phi(\tau(\tilde{F})) = \tau(F)$. Then the Hölder exponent of ϕ is at most $\frac{1}{2}(1 + \ln b / \ln \tilde{b})$.*

Proof. We let $F_k = R^k F$ be the k-fold renormalization, $v_k = \tau(F_k)$ be its tip, $c_k = (F_k)^{-1}(v_k)$ be its "critical point", and $c_k^{k+n} = \Psi_k^{k+n}(c_{k+n})$. Furthermore, let $w_k = F_k(v_k)$ and $z_k^{k+n} = F_k(c_k^{k+n})$, see Figure 10.1. We will mark the corresponding objects of $\tilde{F}$ with the tilde.

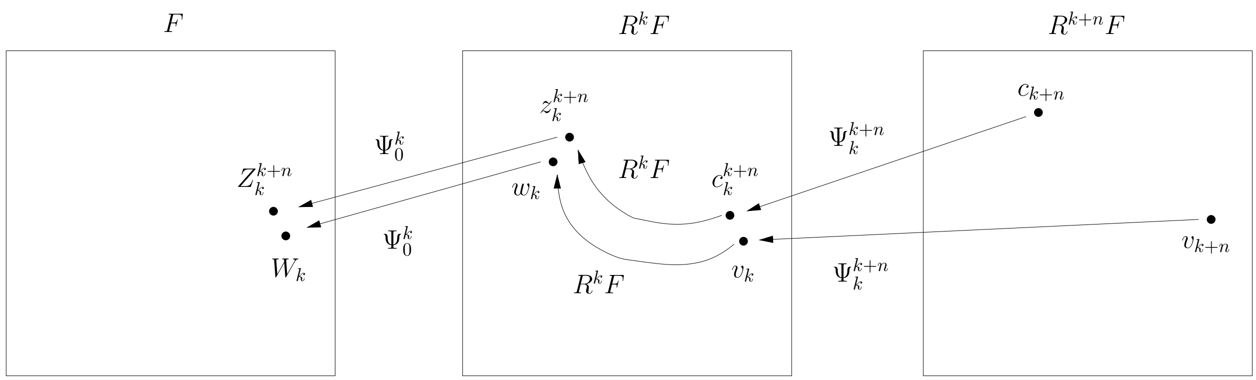

Figure 10.1

For large renormalization levels $k \geq 1$, we have: $b^{2^k} \gg \tilde{b}^{2^k}$. Choose now the scale $n = n(k) \geq 1$ satisfying

$$\sigma^{n+1} \leq \tilde{b}^{2^k} < \sigma^n.$$

Let $\Delta\tilde{x}$ and $\Delta\tilde{y}$ be the differences between the x- and y-coordinate of the points $\tilde{v}_k$ and $\tilde{c}_k^{k+n}$. Representation (7.2), Lemma 7.6 and Corollary 7.10 imply:

$$|\Delta\tilde{y}| \asymp \sigma^n$$

and

$$|\Delta\tilde{x}| = O(\sigma^{2n} + \tilde{b}^{2^k} \cdot |\Delta\tilde{y}|) = O(\sigma^{2n}).$$

Applying $\tilde{F}_k$ to these points using the Universality Theorem 7.9, we obtain :

$$\begin{aligned} \operatorname{dist}(\tilde{z}_k^{k+n}, \tilde{w}_k) &= O(|\Delta\tilde{x}| + |\Delta\tilde{y}| \cdot \frac{\partial\varepsilon_k}{\partial y}) \\ &= O(\sigma^{2n} + \sigma^n \tilde{b}^{2^k}) = O(\sigma^{2n}). \end{aligned}$$

(Notice that $\tilde{F}_k$ has compressed the vertical distance between $\tilde{v}_k$ and $\tilde{c}_k^{k+n}$ to make it comparable with the horizontal distance.)

Consider now points $\tilde{Z}_k^{k+n} = \Psi_0^k(\tilde{z}_k^{k+n})$ and $\tilde{W}_k = \Psi_0^k(\tilde{w}_k)$ in the domain of $\tilde{F}$. By Lemma 5.1, we have:

$$\operatorname{dist}(\tilde{W}_k, \tilde{Z}_k^{k+n}) = O(\sigma^{2n+k}).$$

Let us now estimate the distance between the corresponding points for F. For the same reason as above, we have: $|\Delta y| \asymp \sigma^n$. Furthermore, since the tilt of the box B_k^{n+k} is of order b^{2^k} (by Corollary 7.10), we obtain for some $\gamma > 0$:

$$|\Delta x| \geq 2\gamma \left(b^{2^k} |\Delta y| - \sigma^{2n} \right) \geq \gamma\, b^{2^k} \sigma^n,$$

where the last estimate uses that $b^{2^k} \gg \sigma^n$. Hence

$$|\pi_2(w_k) - \pi_2(z_k^{k+n})| = |\Delta x| \geq \gamma\, b^{2^k} \sigma^n,$$

where π_2 stands for the vertical projection. Using representation (7.2) and Lemma 7.6 once again, we obtain:

$$\operatorname{dist}(W_k, Z_k^{k+n}) \geq \gamma\, \sigma^{k+n} b^{2^k}$$

Any Hölder exponent $\alpha > 0$ for the conjugating homeomorphism has to satisfy

$$\operatorname{dist}(W_k, Z_k^{k+n}) \leq C\, (\operatorname{dist}(\tilde{W}_k, \tilde{Z}_k^n))^\alpha.$$

Hence

$$\sigma^k\, \tilde{b}^{2^k}\, b^{2^k} \leq C\, \left(\sigma^k\, \tilde{b}^{2^k}\, \tilde{b}^{2^k} \right)^\alpha$$

which implies the asserted bound:

$$\alpha \leq \frac{1}{2} \left(1 + \frac{\ln b}{\ln \tilde{b}} \right).$$

$\square$

Corollary 10.2. *Let F be an infinitely renormalizable Hénon-like map with the average Jacobian b and F_0 be a degenerate infinitely renormalizable Hénon-like map. Let ϕ be a homeomorphism which conjugates $F|_{\mathcal{O}_F}$ and $F_0|_{\mathcal{O}_{F_0}}$ with $\phi(\tau(F_0)) = \tau(F)$. Then the Hölder exponent of ϕ is at most $\frac{1}{2}$.* $\square$

11. Generic unbounded geometry

An infinitely renormalizable Hénon map has *bounded geometry* if

$$\operatorname{diam}(B_{w\nu}^n) \asymp \operatorname{dist}(B_{wv}^n, B_{wc}^n),$$

for $n \geq 1$ and $w \in W^{n-1}$ and $\nu \in W$. A slight modified version of this definition would require

$$\operatorname{diam}(B_{w\nu}^n \cap \mathcal{O}) \asymp \operatorname{dist}(B_{wv}^n \cap \mathcal{O}, B_{wc}^n \cap \mathcal{O}).$$

The following theorem holds for both definitions, with the same proof:

Theorem 11.1. *Let F_b, $b \in [0, 1]$, be a family of infinitely renormalizable Hénon-like maps parameterized by the average Jacobian, that is, $b_{F_b} = b$. Then for some $b_0 > 0$, the set of parameter values for which F_b does not have bounded geometry contains a dense G_δ subset in an interval $[0, b_0]$.*

Proof. Let us take $\bar{b} > 0$ so small that the estimates of §7 on Ψ_k^n hold for all F_b with $b \in [0, 2\bar{b}]$. For $n > k \geq 1$, let us consider the boxes $B_k^n = \Psi_k^n(B)$ in the domain of $F_k \equiv R^k F$, and let

$$P_k^n = \Psi_k^{n-1}(F_{n-1}(B_{n-1}^n)).^{9}$$

Note that $B_k^n \cup P_k^n \subset B_k^{n-1}$. As in §7.2, $\tau_k = \tau(F_k)$ stands for the tip of F_k. Let us also consider some point $c_k \in P_k$ moving continuously with the parameter (for instance, we can take the "critical point" $c_k = (F_k)^{-1}(v_k)$ of F_k), and let $c_k^n = \Psi_k^n(c_n) \in B_k^n$ (compare Figure 10.1).

Making use of representations (7.2) and (7.3), let us estimate the relative horizontal positions of the points τ_k and c_k^n. Let

$$z = (x, y) = (\mathrm{id} + \mathbf{S}_k^n)(\tau_n), \quad z_0 = (x_0, y_0) = (\mathrm{id} + \mathbf{S}_k^n)(c_n).$$

By Lemma 7.7, we have:

$$x - x_0 = \mathbf{v}_*(c_n) - \mathbf{v}_*(\tau_n) + O(\bar{b}^{2^k} + \rho^{n-k}),$$

which is a negative number of order 1, provided k and $n-k$ are sufficiently big ($\geq N$). Hence

$$\begin{aligned} \pi_1(c_k^n) - \pi_1(\tau_k) &= \pi_1(D_k^n(z - z_0)) \\ &= [\sigma^{2(n-k)}(x - x_0) + t_k(-\sigma)^{n-k}(y - y_0)]\,(1 + O(\rho^k)) \end{aligned}$$

Together with Corollary 7.10, the above estimates yield for even $n-k$:

$$\pi_1(c_k^n) - \pi_1(\tau_k) = \sigma^{2(n-k)}(x - x_0)[1 - b^{2^k}\sigma^{-(n-k)} r_{n,k}]\,(1 + O(\rho^k), \tag{11.1}$$

where $0 < r \leq r_{n,k} \leq \rho$ uniformly in b.

Let us now take any parameter $b_- \in (0, \bar{b})$ and any integer $k \geq N$. Let us find the biggest n such that $n-k$ is even and $\sigma^{n-k} > \rho(b_-)^{2^k}$. By (11.1), for the map F_{b_-}, the point c_k^n lies to the left of the tip τ_k. Let us increase b_- to a parameter b_+ such that $(b_+)^{2^k} = 2r^{-1}\sigma^{n-k}$. Then for F_{b_+}, the point c_k^n lies to the right of the tip τ_k. Hence there exists a parameter $b \in (b_-, b_+)$ for which c_n^k lies strictly below the tip τ_k.

Moreover,

$$b^{2^k} \asymp \sigma^{n-k}, \tag{11.2}$$

and the hyperbolic distance between b and b_- in the hyperbolic line $\mathbb{R}_+$ is small: $\ln(b/b_-) = O(2^{-k})$. Letting k run through all integers $N, N+1, \ldots$, we obtain a dense set of parameters $b \in (0, \bar{b})$ for which the point c_k^n lies strictly below the tip τ_k for some k, n. It follows that there is a open and dense subset $\Lambda_k \subset (0, \bar{b})$ of parameters for which some point $c_k^n \in P_k^n$[10] lies strictly below the tip τ_k for some $n > k$. Hence for any parameter b in the open G_δ-set $\Lambda = \cap \Lambda_k$, this happens for infinitely many levels k.

We are going to show that the geometry of the critical Cantor set degenerates for $b \in \Lambda$. It is convenient to shift the level by 1, so that we assume that $b \in \Lambda_{k+1}$. Let w_k and z_k^n be the images of the points τ_{k+1} and c_{k+1}^n under the the map $F_k \circ \Psi_k^{k+1}$ (which is equal to $\Psi_c^1(F_{k+1})$ in notation of §5.1). Since the maps Ψ_c^1 preserve the vertical foliation (see Remark 5.1), the points w_k and z_k^n also lie one strictly above the other.

[9] In notations of §5.2, $B_k^n = B_{v^{n-k}}^{n-k}(F_k)$, $P_k^n = B_{v^{n-k-1}c}^{n-k}(F_k)$.

[10] We keep the same notation for this point, though it is not necessarily the one chosen above

Since the point c^n_{k+1} lies strictly below τ_{k+1} on distance of order σ^{n-k}, the interval between the points $\Psi^{k+1}_k(c^n_{k+1})$ and $\Psi^{k+1}_k(\tau_{k+1})$ has length of order σ^{n-k} and slope of order $-b^{2^k}$ (see Lemma 7.4). Hence the distance between the horizontal projections of these two points is of order $\sigma^{n-k}b^{2^k}$. But it is equal to the distance between their F_k-images, z^n_k and w_k. Thus,

$$\operatorname{dist}(w_k, z^n_k) \asymp \sigma^{n-k}b^{2^k}.$$

Applying F_k once more, we obtain two point on the same horizontal line such that

$$\operatorname{dist}(F_k(w_k), F_k(z^n_k)) \asymp \sigma^{n-k}\, b^{2^{k+1}}. \tag{11.3}$$

Let us now estimate the sizes of the corresponding pieces. Let Q stand for either B^n_k or P^n_k. By (7.2), Proposition 7.8 and Corollary 7.10, it contains two points such that the interval joining them has length of order σ^{n-k} and tilt of order b^{2^k}. Hence

$$|\pi_1(Q)| \geq \gamma\sigma^{n-k}b^{2^k}$$

for some $\gamma > 0$. It follows that both projections of $F^2_k(Q)$ are at least that big (up to a constant). We are interested only in the vertical size:

$$|\pi_2(F^2_k(Q))| \geq \gamma\sigma^{n-k}b^{2^k}.$$

Comparing this with (11.3), we see that the distance between the points $F_k(w_k)$ and $F_k(z^n_k)$ is at least b^{2^k} times smaller than the vertical size of the pieces $F^2_k(B^n_k)$ and $F^2_k(P^k_n)$ that contain these points.

Finally, we should bring these two pieces to the domain of F by the map Ψ^k_0. Since this map contracts the horizontal distances stronger than the vertical ones, the gap between the images of the pieces will be even smaller compared to the size of the pieces (the gap will become at least $b^{2^k}\sigma^k$ times smaller than the size of the pieces).

The conclusion follows. □

12. Hölder geometry of the critical Cantor set

If $P = B^{n-1}_\sigma$, $n \geq 1$ and $\sigma \in \Sigma^{n-1}$, is a piece of an infinitely renormalizable Hénon-like map $F \in \mathcal{I}_\Omega(\bar\varepsilon)$ we call the distance $g = \operatorname{dist}(B^n_{\sigma v}, B^n_{\sigma c})$ the *gap* of the piece P. An infinitely renormalizable Hénon map has *Hölder bounded geometry* if there exist $\alpha > 0$ and $C > 0$ such that

$$g^\alpha \geq C \cdot \operatorname{diam}(P)$$

for every piece P of F.

Theorem 12.1. *Every infinitely renormalizable Hénon-like map $F \in \mathcal{I}_\Omega(\bar\varepsilon)$, with sufficiently small $\bar\varepsilon$, has Hölder bounded geometry.*

The proof of this Theorem will be by induction in the size of the pieces. The beginning of the induction is the following Proposition.

Proposition 12.2. *There exist constants $K, C > 0$ such that for every $F \in \mathcal{I}_\Omega(\bar\varepsilon)$ with sufficiently small $\bar\varepsilon$ and every piece P of F with gap g the following holds. If*

$$\operatorname{diam}(P) \geq K \cdot b_F$$

then

$$g \geq C \cdot \operatorname{diam}(P).$$

Remark 12.1. In the previous section we showed that the geometry of $\mathcal{O}_F$ might be unbounded. Proposition 12.2 states that this two-dimensional phenomenon becomes observable only at a scale of the order of b.

The proof of this Proposition relies on the following Lemma for which we need some notation. Given a piece P, let H and V stand for its horizontal and vertical projections. Let

$$q_P = \frac{|V|}{|H|}.$$

The piece P is obtained by repeatedly applying contractions, say $P = \Psi^n_{\omega_1\omega_2\dots\omega_n}(B)$. Let $P_k = \Psi^n_{\omega_k\omega_{k+1}\dots\omega_n}(B)$ be the corresponding piece of $F_k \equiv R^k F$, $k \le n$.

Lemma 12.3. *For every $K > 0$ there exists $C > 0$ such that if P is a piece of F with*

$$\operatorname{diam}(P) \ge K \cdot b_F$$

then

$$q_k = q_{P_k} \le C \cdot \frac{1}{b_F},$$

for $k \ge 1$.

Proof. The piece P is of the n^{th} generation of F. Let $1 \le k \le n$ and $s \ge k$ be maximal such that

$$P_k = \Psi^{s-k}_{v^{s-k}}(P_s),$$

(where only "critical value" contractions were used). Then

$$P_s = \Psi^{n-s}_{c\omega_{s+1}\dots\omega_n}(B)$$

Let

$$P' = \Psi^{n-s}_{v\omega_{s+1}\dots\omega_n}(B) \subset B^1_v(F_s).$$

Note,

$$F_s(P') = P_s.$$

Let H_s, V_s and H', V' be the horizontal and vertical projections of P_s and P' respectively. From Theorem 7.9, for some uniform $A > 0$ and $K_1 > 0$

$$K \cdot b \le \operatorname{diam}(P_s) \le |V_s| + |H_s| \le |V_s| + A|H'| + K_1 b^{2^s}.$$

Because $|V_s| = |H'|$ we get

$$|H'| \ge K_3 \cdot b, \tag{12.1}$$

for some $K_3 > 0$. From Theorem 7.9 we get for some uniform $a > 0$ and $K_4 > 0$

$$|H_s| \ge a|H'| - K_4 b^{2^s}. \tag{12.2}$$

Now 12.1 and 12.2 imply

$$q_s = \frac{|V_s|}{|H_s|} \le \frac{|H'|}{a|H'| - K_4 b^{2^s}} = O(1).$$

From Proposition 7.8 and (7.2) we get

$$q_k = O(1/\sigma^{s-k}).$$

Using Lemma 5.1

$$\begin{aligned} b_F &\le \frac{1}{K}\cdot \operatorname{diam}(P) \le \frac{1}{K}\cdot \operatorname{diam}(P_k) \\ &\le \frac{1}{K}\cdot \operatorname{diam}(P_s)\cdot C\sigma^{s-k} \le \frac{C}{K}\cdot \sigma^{s-k}. \end{aligned}$$

And the Lemma follows. □

Proof of Proposition 12.2. Let P_k be a piece (of some F_k) of generation $n-k$. Let $G_h \subset H$ and $G_v \subset V$ be the minimal closed intervals such that $G_h \times V$ and $H \times G_v$ do intersect the two pieces of the next generation contained in P_k. Note, G_v (and G_h) is a degenerate interval if the pieces of the next generation have intersecting vertical (horizontal) projections. The following argument will show that this does not happen. Let

$$\Gamma^{\text{hor}}_{k,n} = \min_{P_k} \frac{|G_h|}{|H|}$$

$$\Gamma^{\text{ver}}_{k,n} = \min_{P_k} \frac{|G_v|}{|V|}$$

and

$$\Gamma_{k,n} = \min\{\Gamma^{\text{hor}}_{k,n}, \Gamma^{\text{ver}}_{k,n}\}.$$

Let $\mathcal{P}_{k,n}$ be the pieces of generation $n-k$ of F_k and

$$\mathcal{P}^c_{k,n} = \{P \in \mathcal{P}_{k,n} | P \in B^1_c(F_k)\}$$

and

$$\mathcal{P}^v_{k,n} = \{P \in \mathcal{P}_{k,n} | P \in B^1_v(F_k)\}.$$

Also define

$$\Gamma^{\text{hor},c}_{k,n} = \min_{P_k \in \mathcal{P}^c_{k,n}} \frac{|G_h|}{|H|},$$

$$\Gamma^{\text{hor},v}_{k,n} = \min_{P_k \in \mathcal{P}^v_{k,n}} \frac{|G_h|}{|H|}.$$

And similarly, define $\Gamma^{\text{ver},c}_{k,n}$ and $\Gamma^{\text{ver},v}_{k,n}$. Observe, using the specific normalization of Hénon-like maps (y'=x) and the fact that the functions $\psi^1_v(F_k)$ are affine in the vertical direction,

(1) $\Gamma^{\text{ver},v}_{k,n} = \Gamma^{\text{ver}}_{k+1,n}$,
(2) $\Gamma^{\text{ver},c}_{k,n} = \Gamma^{\text{hor},v}_{k,n}$,
(3) $\Gamma^{\text{hor},c}_{k,n} \ge \Gamma^{\text{ver},v}_{k,n}$.

The last property follows from Lemma 5.3 (3). These relations imply

$$\Gamma_{k,n} \ge \min\{\Gamma^{\text{ver}}_{k+1,n}, \Gamma^{\text{hor},v}_{k,n}\}. \tag{12.3}$$

Now we will express $\Gamma^{\text{hor},v}_{k,n}$ in terms of $\Gamma^{\text{hor}}_{k+1,n}$. Let $P \in \mathcal{P}_{k+1,n}$ and $G_h \subset H$ and V be the corresponding intervals. Let $\hat{P} = \psi^{k+1}_v(P)$ and $\hat{G}_h \subset \hat{H}$. Then, using Lemma 7.4, (7.1), and the tilt quantified in Corollary 7.10

$$|\hat{G}_h| \ge D_g|G_h| - K_1 \cdot |V| \cdot b^{2^{k+1}},$$

and

$$|\hat{H}| \le D_h|H| + K_1 \cdot |V| \cdot b^{2^{k+1}},$$

where

$$D_g = \frac{\partial \Psi_v^{k+1}}{\partial x}(x_g, y_0),$$

$$D_h = \frac{\partial \Psi_v^{k+1}}{\partial x}(x_h, y_0),$$

with $x_g \in G_h$, $x_h \in H$ appropriately chosen, $y_0 \in \partial V$, and $K_1 > 0$. Lemma 7.4(3) and Lemma 5.1 gives

$$\ln \frac{D_g}{D_h} = O(\sigma^{n-k}).$$

These estimates, together with Lemma 12.3 and the assumption that $\operatorname{diam}(P) \geq K \cdot b$, imply that for some constant $K_2, K_3 > 0$

$$\frac{|\hat{G}_h|}{|\hat{H}|} \geq \frac{|G_h|}{|H|} \cdot \exp(-K_2 \cdot \sigma^{n-k}) \cdot \frac{1 - K_3 \cdot b^{2^{k+1}-1} \cdot \frac{|H|}{|G_h|}}{1 + K_3 \cdot b^{2^{k+1}-1}}.$$

This implies

$$\Gamma_{k,n}^{\mathrm{hor},v} \geq \frac{e^{-K_2 \sigma^{n-k}}}{1 + K_3 \cdot b^{2^{k+1}-1}} \cdot \left[\Gamma_{k+1,n}^{\mathrm{hor}} - K_3 \cdot b^{2^{k+1}-1} \right]. \tag{12.4}$$

Equation (12.3) and (12.4) imply

$$\Gamma_{k,n} \geq \frac{e^{-K_2 \sigma^{n-k}}}{1 + K_3 \cdot b^{2^{k+1}-1}} \cdot \left[\Gamma_{k+1,n} - K_3 \cdot b^{2^{k+1}-1} \right]. \tag{12.5}$$

By iterating estimate (12.5) and using that $\Gamma_{n-1,n} \asymp 1$ we get $m > 0$ such that

$$\Gamma_{0,n} \geq m > 0,$$

for $n \geq 1$. This implies Proposition 12.2. □

The induction hypothesis (denoted by Ind_n, $n \geq 0$) we will use to prove Theorem 12.1 is the following. There exist $\alpha_n > 0$ and constants $C > 0$ and $K > 0$, independent of F and $n \geq 0$, such that the condition

$$\operatorname{diam}(P) \geq K \cdot b^{2^n},$$

on any piece P of F implies

$$g^{\alpha_n} \geq C \cdot \operatorname{diam}(P).$$

Proposition 12.2 states that Ind_0 holds with $\alpha_0 = 1$.

Assume that $\mathrm{Ind_j}$ holds for $j \leq n$. We are going to prove Ind_{n+1}. Consider a piece P_{n+1} of F with

$$\operatorname{diam}(P_{n+1}) \geq K \cdot b^{2^{n+1}}.$$

Because $\mathrm{Ind_j}$ holds for $j \leq n$ we may assume without loss of generality that $\operatorname{diam}(P_{n+1}) \leq K \cdot b^{2^n}$. This piece is obtained by applying a contraction $\Psi_c^1(RF)$ or $\Psi_v^1(RF)$ to a piece P_n of RF. Note that

$$\operatorname{diam}(P_n) \geq \operatorname{diam}(P_{n+1}) \geq K(b^2)^{2^n}.$$

Hence, if g_n is the gap of P_n, Ind_n implies

$$g_n^{\alpha_n} \geq C \cdot \operatorname{diam}(P_n).$$

Observe,

$$g_{n+1} \geq A \cdot b \cdot g_n,$$

for some constant $A > 0$. We need to find an estimate for $\alpha_{n+1} > 0$ such that

$$g_{n+1}^{\alpha_{n+1}} \geq C \cdot \operatorname{diam}(P_{n+1}). \tag{12.6}$$

We may assume $\alpha_{n+1} \leq \alpha_n$. The condition 12.6 holds if

$$(A \cdot b)^{\alpha_{n+1}} \cdot (C \cdot \operatorname{diam}(P_n))^{\frac{\alpha_{n+1}}{\alpha_n}} \geq C \cdot \operatorname{diam}(P_n). \tag{12.7}$$

Use the fact that for some $L > 0$

$$\operatorname{diam}(P_n) \leq L \cdot \frac{1}{b} \cdot \operatorname{diam}(P_{n+1}) \leq \frac{L}{K} \cdot b^{2^n - 1}$$

to reduce the condition 12.7 to the next sufficient condition for 12.6. Namely,

$$A^{\alpha_{n+1}} \geq (C \cdot L)^{1 - \frac{\alpha_{n+1}}{\alpha_n}} \cdot b^{(2^n - 1)\cdot(1 - \frac{\alpha_{n+1}}{\alpha_n}) - \alpha_{n+1}}. \tag{12.8}$$

Finally, this condition 12.8 reduces to the sufficient condition

$$-M \geq \ln b \cdot [(1 - \frac{\alpha_{n+1}}{\alpha_n}) \cdot (2^n - 1) - 1],$$

where $M > 0$ is some large constant. Now choose α_{n+1} such that

$$(1 - \frac{\alpha_{n+1}}{\alpha_n}) \cdot (2^n - 1) = m$$

is constant but sufficiently large and one obtains $\alpha_{n+1} > 0$ for which Ind_{n+1} holds. Moreover, the sequence $\alpha_n > 0$ decreases to some $\alpha > 0$. This finishes the proof of the Theorem 12.1.

13. Open Problems

Let us finish with some further questions that naturally arise from the previous discussion. The first two of them are probably very hard, while others should be more tractable.

(1) Prove that F_* is the only fixed point of the Hénon renormalization R, and $R^n F \to F_*$ exponentially for any infinitely renormalizable Hénon-like map F.
(2) Is it true that the trace of the unstable manifold $\mathcal{W}^u(F_*)$ by the two-parameter Hénon family $F_{c,b} : (x, y) \mapsto (x^2 + c - by, x)$ is a (real analytic) curve γ on which the Jacobian b assumes all values $0 < b < 1$. If so, does this curve converge to some particular point $(c, 1)$ as $b \to 1$?
(3) How good is the conjugacy $h \colon \mathcal{O}_F \to \mathcal{O}_G$ when $b_F = b_G$?
(4) Is the conjugacy $h \colon \mathcal{O}_F \to \mathcal{O}_G$ always Hölder? An equivalent question (due to Theorem 12.1) is whether the pieces B^n_σ decay no faster than exponentially in n? The answer is probably negative in general.
(5) Can $\mathcal{O}_F$ have bounded geometry when $b_F \neq 0$? If so, does this property depend only on the average Jacobian b_F?
(6) Does the Hausdorff dimension of $\mathcal{O}_F$ depend only on the average Jacobian b_F? (This question was suggested by A. Avila.)

14. Appendix: Shuffling

In this section we will briefly recall some analysis of long compositions of diffeomorphisms of the interval. It is convenient to represent a C^3 diffeomorphism $\phi : [-1,1] \to [-1,1]$ by its C^1 non-linearity

$$\eta_\phi = \frac{D^2\phi}{D\phi}.$$

The following Lemma was used in §7.

Lemma 14.1. *(Shuffling) For every $B > 0$ there exists $K > 0$ such that the following holds. Let $\phi_j, \phi_j^* : [-1,1] \to [-1,1]$, $j = 1, \dots, n$ be C^3 diffeomorphisms and let*

$$\Phi = \phi_n \circ \cdots \circ \phi_2 \circ \phi_1$$

and

$$\Phi^* = \phi_n^* \circ \cdots \circ \phi_2^* \circ \phi_1^*.$$

If

$$\sum_{j=1}^n \|\eta_j\|_{C^1} \le B$$

and

$$\sum_{j=1}^n \|\eta_j^*\|_{C^1} \le B$$

where $\eta_j^{(*)}$ is the non-linearity of $\phi_j^{(*)}$, then

$$\mathrm{dist}_{C^2}(\Phi, \Phi^*) \le K \sum_{j=1}^n \|\eta_j - \eta_j^*\|_{C^0}.$$

This Lemma is a consequence of the Sandwich-Lemma 10.5 from [Ma]. Here we will use a slightly different version of this Sandwich-Lemma, whose proof is exactly the same as the proof for the original formulation.

Lemma 14.2. *(Sandwich) For every $B > 0$ there exists $K > 0$ such that the following holds. Let $\phi_j, \phi : [-1,1] \to [-1,1]$, $j = 1, \dots, n$ be C^3 diffeomorphisms and let*

$$\Phi = \phi_n \circ \cdots \circ \phi_{k+1} \circ \phi_k \circ \cdots \circ \phi_2 \circ \phi_1$$

and

$$\Psi = \phi_n \circ \cdots \circ \phi_{k+1} \circ \phi \circ \phi_k \circ \cdots \circ \phi_2 \circ \phi_1.$$

If

$$\sum_{j=1}^n \|\eta_{\phi_j}\|_{C^1} + \|\eta_\phi\|_{C^1} \le B,$$

then

$$\|\eta_\Phi - \eta_\Psi\|_{C^0} \le K \|\eta_\phi\|_{C^0}.$$

The proof for the Shuffling-Lemma 14.1 consists of *sandwiching* the diffeomorphisms $\phi_k^* \circ \phi_k^{-1}$ between ϕ_{k+1} and ϕ_k, $k = 1, \dots, n$. In this way Φ is changed into Φ^*. To estimate the distance between these two diffeomorphism we need the following Lemma.

Lemma 14.3. *For every $B > 0$ there exists $K > 0$ such that the following holds. Let $\phi, \psi : [-1,1] \to [-1,1]$ be C^3 diffeomorphisms with*

$$\|\eta_\phi\|_{C^0} \le B$$

Then

$$\|\eta_{\psi\circ\phi^{-1}}\|_{C^0} \le K \cdot \|\eta_\psi - \eta_\phi\|_{C^0}$$

and

$$\|\eta_{\psi\circ\phi^{-1}}\|_{C^1} \le K \cdot \|\eta_\psi - \eta_\phi\|_{C^1}$$

Proof. The Chain-rule for non-linearities

$$\eta_{\psi\circ\phi}(x) = \eta_\psi(\phi(x)) \cdot D\phi(x) + \eta_\phi(x)$$

implies

$$\eta_{\phi^{-1}}(x) = -\eta_\phi(\phi^{-1}(x)) \cdot D\phi^{-1}(x).$$

Again the chain-rule gives

$$\eta_{\psi\circ\phi^{-1}} = D\phi^{-1} \cdot \left(\eta_\psi(\phi^{-1}) - \eta_\phi(\phi^{-1})\right).$$

Differentiation gives

$$\begin{aligned} D\eta_{\psi\circ\phi^{-1}} =& (D\phi^{-1})^2 \cdot \left(D\eta_\psi(\phi^{-1}) - D\eta_\phi(\phi^{-1})\right) \\ &+ D^2\phi^{-1} \cdot \left(\eta_\psi(\phi^{-1}) - \eta_\phi(\phi^{-1})\right). \end{aligned}$$

The bound $\|\eta_\phi\|_{C^0} \le B$ gives a bound on $\|\phi^{-1}\|_{C^2}$ and the Lemma follows. □

Now we are ready to prove the shuffling-Lemma 14.1. The Lemmas 14.2 and 14.3 imply the following estimate on the diffeomorphisms as defined in Lemma 14.1

$$\|\eta_\Phi - \eta_{\Phi^*}\|_{C^0} \le K \sum_{j=1}^{n} \|\eta_j - \eta_j^*\|_{C^0},$$

where $K = K(B)$. One can integrate non-linearities and obtain

$$\phi(x) = 2\frac{\int_{-1}^{x} e^{\int_{-1}^{s}\eta_\phi} ds}{\int_{-1}^{1} e^{\int_{-1}^{s}\eta_\phi} ds} - 1.$$

and

$$D\phi(x) = 2\frac{e^{\int_{-1}^{x}\eta_\phi} ds}{\int_{-1}^{1} e^{\int_{-1}^{s}\eta_\phi} ds}.$$

Notice that the Sandwich-Lemma 14.2 implies that

$$\|\eta_\Phi\|_{C^0}, \|\eta_{\Phi^*}\|_{C^0} \le K \cdot B.$$

This uniform bound and the two expressions above can be used to get the desired estimate on the C^2 distance between Φ and Φ^* in 14.1. We finished the proof of the Shuffling-Lemma.

15. List of special notations

β_0, β_1 saddle fixed points of a Hénon-like map F, §3.4
$b = b_F$ average Jacobian of F, §6
$B^n_w = B^n_w(F)$ renormalization pieces of level n, §5.2
D^n_k derivative at the tip, §7.2
$F(x,y) = (f(x) - \varepsilon(x,y), x)$ Hénon-like map, §3.2
f_* fixed point of the unimodal renormalization R_c, §3.1
f^* fixed point of the unimodal renormalization R_v, §3.1
F_* fixed point of the Hénon-like renormalization, §4
H non-linear part of coordinate change, §3.5
$\mathcal{H}_\Omega$ space of analytic Hénon-like maps, §3.3
$\mathcal{I}_\Omega(\overline{\epsilon})$ space of infintely renormalizable unimodal maps,
$\operatorname{Jac} F = |\partial\varepsilon/\partial y|$ Jacobian of F, §3.2
λ the universal scaling factor, §3.1
Λ scaling part of coordinate change, §3.5
$\mathcal{O} = \mathcal{O}_F$ the critical Cantor set, §5.2
$\psi^1_v = H^{-1} \circ \Lambda^{-1}$ coordinate change conjugating RF to F^2, §5.1
$\Psi^n_\omega = \Psi^n_\omega(F)$ coordinate change conjugating R^nF to F^{2^n}, §5.1
$\Psi_k = \Psi^{k+1}_k = \Psi^1_v(R^kF)$, §7.2
R_c renormalization operator near the "critical point", §3.1
R_v renormalization operator near the "critical value", §3.1
s_k non-linear part of the coordinate change Ψ_k, §7.2
S^n_k non-linear part of the coordinate change Ψ^n_k, §7.2
$\sigma = \lambda^{-1}$ the universal scaling factor, §3.1
t_k tilt, §7.2
$\tau = \tau_F$ tip, §7.2
$\mathcal{U}_U$ space of analytic unimodal maps, §3.3
v_* universal change of coordinates, §7.1

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