DYNAMICAL SYSTEMS APPLIED TO ASYMPTOTIC GEOMETRY

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ABSTRACT. In the paper we discuss two questions about smooth expanding dynamical systems on the circle. (i) We characterize the sequences of asymptotic length ratios which occur for systems with Hölder continuous derivative. The sequence of asymptotic length ratios are precisely those given by a positive Hölder continuous function s (solenoid function) on the Cantor set C of 2-adic integers satisfying a functional equation called the matching condition. The functional equation for the 2-adic integer Cantor set is

$$s(2x+1) = \frac{s(x)}{s(2x)}\left(1 + \frac{1}{s(2x-1)}\right) - 1.$$

We also present a one-to-one correspondence between solenoid functions and affine classes of 2-adic quasiperiodic tilings of the real line that are fixed points of the 2-amalgamation operator. (ii) We calculate the precise maximum possible level of smoothness for a representative of the system, up to diffeomorphic conjugacy, in terms of the functions s and $cr(x) = (1 + s(x))/(1 + (s(x+1))^{-1})$. For example, in the Lipschitz structure on C determined by s, the maximum smoothness is $C^{1+\alpha}$ for $0 < \alpha \le 1$ if, and only if, s is α-Hölder continuous. The maximum smoothness is $C^{2+\alpha}$ for $0 < \alpha \le 1$ if, and only if, cr is $(1 + \alpha)$-Hölder. A curious connection with Mostow type rigidity is provided by the fact that s must be constant if it is α-Hölder for $\alpha > 1$.

CONTENTS

1. INTRODUCTION

One could say that this paper is about the space $A(2)$ of sequences $\{a_1, a_2, \ldots\}$ of positive real numbers satisfying

    (i) a_n/a_m is exponentially near 1 if $n - m$ is divisible by a high power of two, and

    (ii) $a_3, a_5, a_7, \ldots$ is constructed from a_1 and $a_2, a_4, a_6, \ldots$ by the recursion

$$(1) \qquad\qquad a_{2n+1} = \frac{a_n}{a_{2n}} \left(1 + \frac{1}{a_{2n-1}}\right) - 1.$$

The only explicit element in $A(2)$ that we know is $\{1, 1, 1, \ldots\}$. However, the following theorem shows that $A(2)$ is a dense subset of a separable infinite dimensional complex Banach manifold of [20].

Theorem 1. *The set $A(2)$ is canonically isomorphic to*

    A) the set of all possible affine structures on the leaves of the dyadic solenoid $\tilde{S}(2)$ that are transversely Hölder continuous and invariant by the natural dynamics $\tilde{E}(2) : \tilde{S}(2) \to \tilde{S}(2)$.

    B) the set of all C^r structures on the circle S invariant by the "doubling the angle" expanding dynamics $E(2) : S \to S$, $r > 1$.

    C) the set of all positive Hölder continuous functions s on the Cantor set C of 2-adic integers satisfying

$$s(2x + 1) = \frac{s(x)}{s(2x)} \left(1 + \frac{1}{s(2x - 1)}\right) - 1.$$

    D) the set of all affine classes of 2-adic quasiperiodic tilings of the real line that are fixed points of the 2-amalgamation operator.

    E) the set of all affine classes of 2-adic quasiperiodic fixed grids of the real line.

See proof of Theorem 1 in Section 2.7 (in [20] are studied the uniformly asymptotically affine (uaa) and the analytic structures on the circle invariant by the dynamics of $E(2)$ leaving the C^r case for this paper). The connection between the sequences of $A(2)$ and $C)$ appears from restricting s in $C)$ to the dense subset of natural numbers in the Cantor set of 2-adic integers. The connection between the sequences of $A)$ with $D)$ and $E)$ follows from the existence of a dense leaf in the solenoid, with a natural binary grid, which is expanded by the dynamics in a manner combinatorially like $x \to 2x$ acting on $\{n/2^k\} \subset \mathbb{R}$. Then the connection between $A(2)$ and $A)$ follows from using the sequences $\{a_0, a_1, a_2, \ldots\}$ to define ratios of consecutive lengths between integral points of the grid. The functional equation makes the doubling map look affine between the integral grid and its double. The 2-adic continuity allows the complete affine structure induced by pullback to impress itself on the other leaves of the solenoid $\tilde{S}(2)$. The passage from A) to B) uses the fact that the solenoid $\tilde{S}(2)$ with its dynamics is the inverse limit system associated to the diagram

$$
\begin{array}{ccccccc}
\cdots \xrightarrow{E(2)} & S & \xrightarrow{E(2)} & S & \xrightarrow{E(2)} & S \\
& \uparrow E(2) & & \uparrow E(2) & & \uparrow E(2) \\
\cdots \xrightarrow{E(2)} & S & \xrightarrow{E(2)} & S & \xrightarrow{E(2)} & S
\end{array}
$$

Thus, $\tilde{S}(2)$ projects to S, and the affine structures on the leaves of $\tilde{S}(2)$ determine a *canonical family of solenoidal charts on S* invariant by the dynamics $E(2)$ on S. This canonical family of charts on the circle is compact modulo affine normalization. The connection between $B)$ and $A)$ associates to each C^r structure U of the circle S invariant by $E(2)$ a unique canonical family of solenoid charts $\mathcal{F}_U$ with the property that the solenoidal charts are contained in the structure U. The conection between $B)$ and $C)$ is given by an explicit construction of a solenoid function s_U using the expanding property of $E(2)$ with respect to the C^r structure U (see Lemmas 3 and 5).

In order to state the next theorem, we introduce the following definitions. The *ultra-metric* $|\mathbf{u}|_s : C \times C \to \mathbb{R}_0^+$ is defined as follows. Let $x = \sum_{m=0}^{\infty} x_m 2^m \in C$ and $y = \sum_{m=0}^{\infty} y_m 2^m \in C$ be such that $x_n \ldots x_0 = y_n \ldots y_0$ and $x_{n+1} \neq y_{n+1}$. For $0 \leq i \leq n$, let $A_i = \sum_{m=0}^{i} x_m 2^m$ and $E_i = \sum_{m=0}^{i} 2^m$. We define

$$|\mathbf{u}|_s(x,y) = \inf_{0 \leq i \leq n} \left\{ 1 + \sum_{j=A_i}^{E_i} \prod_{l=A_i}^{j} s(l) + \sum_{j=0}^{A_i-1} \prod_{l=j}^{A_i-1} s(l) \right\} .$$

We present a geometric interpretation of the ultra-metric in Section 2.8. For $\beta > 0$, we say that a function $f : C \to \mathbb{R}$ is β-*Hölder*, with respect to the metric $|\mathbf{u}| = |\mathbf{u}|_s$, if there is a constant $d \geq 0$ such that $|f(y) - f(x)| \leq d \left(|\mathbf{u}|(x,y) \right)^{\beta}$ for all $x, y \in C$. We say that f is β-*hölder*, with respect to the metric $|\mathbf{u}|$, if there is a continuous function $\epsilon : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, with $\epsilon(0) = 0$, such that $|f(y) - f(x)| \leq \epsilon \left(|\mathbf{u}|(x,y) \right) \left(|\mathbf{u}|(x,y) \right)^{\beta}$ for all $x, y \in C$. By f being *Lipschitz* we mean that f is 1-Hölder, and by f being *lipschitz* we mean that f is 1-hölder. Of course on the real line, with respect to the Euclidean metric, β-Hölder for $\beta > 1$ or lipschitz implies constancy. We define the *solenoid cross ratio function* $cr(x) : C \to \mathbb{R}^+$ by $cr(x) = (1 + s(x))(1 + (s(x+1))^{-1})$.

Theorem 2. *For every C^r structure U of the circle S invariant by $E(2)$, the overlap maps and the expanding map $E(2) : S \to S$ attain its maximum of smoothness with respect to the canonical family of solenoid charts $\mathcal{F}_U$ contained in U. Table 1 presents explicit conditions in terms of the corresponding solenoid function $s = s_U$ which determine the degree of smoothness of the overlap homeomorphisms and of $E(2)$ in $\mathcal{F}_U$, and vice-versa.*

The regularity of the solenoidal chart overlap maps and $E(2) : S \to S$.	Condition on the functions s and cr, using the $|\mathbf{u}|_s$ ultra-metric on C.
have α-*Hölder* 1st *derivative* $0 < \alpha \leq 1$	s is α-*Hölder*
have α-*Hölder* 1st *derivative* $0 < \alpha \leq 1$	cr is α-*Hölder*
have *Lipschitz* 1st *derivative*	s is *Lipschitz*
have α-*Hölder* 2nd *derivative* $0 < \alpha \leq 1$	cr is $(1+\alpha)$-*Hölder*
have *Lipschitz* 2nd *derivative*	cr is 2-*Hölder*
Affine	s is *lipschitz*

Table 1.

See proof of Theorem 2 in Section 2.8. The scaling and solenoid functions give a deeper understanding of the smooth structures of one dimensional dynamical systems (cf. [1], [2], [3], [9], [18] and [21]) and also of two dimensional dynamical systems (cf. [13] and [15]).

1.1. **Smoothness of diffeomorphisms and ratio distortions of grids.** To prove Theorem 2, we show some of the relations proposed in [19] between distinct degrees of smoothness of a homeomorphism of a real line with distinct bounds of the ratio and cross ratio distortions of intervals of a fixed grid that we pass to describe.

Given $B \geq 1$, $M > 1$ and $\Omega : \mathbb{N} \to \mathbb{N}$, a (B, M) *grid*

$$\mathcal{G}_\Omega = \{ I_\beta^n \subset I : n \geq 1 \text{ and } \beta = 1, \ldots, \Omega(n) \}$$

of a closed interval I is a collection of *grid intervals* I_β^n *at level* n with the following properties: (i) The grid intervals are closed intervals; (ii) For every $n \geq 1$, the union $\cup_{\beta=1}^{\Omega(n)} I_\beta^n$ of all grid intervals I_β^n, at level n, is equal to the interval I; (iii) For every $n \geq 1$, any two distinct grid intervals at level n have disjoint interiors; (iv) For every $1 \leq \beta < \Omega(n)$, the intersection of the grid intervals I_β^n and $I_{\beta+1}^n$ is only an endpoint

common to both intervals; (v) For every $n \geq 1$, the set of all endpoints of the intervals I_β^n at level n is contained in the set of all end points of the intervals I_β^{n+1} at level $n + 1$; (vi) For every $n \geq 1$ and for every $1 \leq \beta < \Omega(n)$, we have $B^{-1} \leq |I_{\beta+1}^n|/|I_\beta^n| \leq B$; (vii) For every $n \geq 1$ and for every $1 \leq \alpha \leq \Omega(n)$, the grid interval I_α^n contains at least two grid intervals at level $n + 1$, and contains at most M grid intervals also at level $n + 1$.

Let $h : I \to J$ be a homeomorphism between two compact intervals I and J on the real line, and let $\mathcal{G}_\Omega$ be a grid of I. Let I_β and $I_{\beta'}$ be two intervals contained in the real line. The *logarithmic ratio distortion* $lrd(I_\beta, I_{\beta'})$ is given by

$$lrd(I_\beta, I_{\beta'}) = \log\left(\frac{|I_\beta|}{|I_{\beta'}|}\frac{|h(I_{\beta'})|}{|h(I_\beta)|}\right) .$$

We say that two closed intervals I_β and $I_{\beta'}$ are *adjacent* if their intersection $I_\beta \cap I_{\beta'}$ is only an endpoint common to both intervals. Let I_β, $I_{\beta'}$ and $I_{\beta''}$ be contained in the real line, such that I_β is adjacent to $I_{\beta'}$, and $I_{\beta'}$ is adjacent to $I_{\beta''}$. The *cross ratio* $cr(I_\beta, I_{\beta'}, I_{\beta''})$ is determined by

$$cr(I_\beta, I_{\beta'}, I_{\beta''}) \;\; = \;\; \log\left(1 + \frac{|I_{\beta'}|}{|I_\beta|}\frac{|I_\beta| + |I_{\beta'}| + |I_{\beta''}|}{|I_{\beta''}|}\right) .$$

The *cross ratio distortion* $crd(I_\beta, I_{\beta'}, I_{\beta''})$ is given by

$$crd(I_\beta, I_{\beta'}, I_{\beta''}) = cr(h(I_\beta), h(I_{\beta'}), h(I_{\beta''})) - cr(I_\beta, I_{\beta'}, I_{\beta''}) .$$

Theorem 3. *Let $h : I \to J$ be a homeomorphism between two compact intervals I and J on the real line, and let $\mathcal{G}_\Omega$ be a grid of I.*

> *(i) If h has the degree of smoothness presented in a line of Table 2, and $dh(x) \neq 0$ for all $x \in I$ (not applicable for quasisymmetric and (uaa) homeomorphisms), then the logarithmic ratio distortion satisfy the bounds presented in the same line with respect to all grid intervals. Conversely, if the logarithmic ratio distortion satisfies the bounds presented in a line of Table 2 with respect to all grid intervals, then $h : I \to J$ has the degree of smoothness presented in the same line, and $dh(x) \neq 0$ for all $x \in I$ (not applicable for quasisymmetric and (uaa) homeomorphisms).*

The smoothness of h	The order of $lrd\left(I_\beta^n, I_{\beta+1}^n\right)$
Quasisymmetric	$O(1)$
(uaa)	$o\left(\left|I_\beta^n\right|\right)\left|I_\beta^n\right|^{-1}$
$C^{1+\alpha}$	$O\left(\left|I_\beta^n\right|^\alpha\right)$
$C^{1+Lipschitz}$	$O\left(\left|I_\beta^n\right|\right)$
Affine	$o\left(\left|I_\beta^n\right|\right)$

Table 2.

> *(ii) If h has the degree of smoothness presented in a line of Table 3, and $dh(x) \neq 0$ for all $x \in I$ (not applicable for quasisymmetric and (uaa) homeomorphisms), then the cross ratio distortion satisfy the bounds presented in the same line with respect to all grid intervals. Conversely, if the cross ratio distortion satisfies the bounds presented in a line of Table 3 with respect to all grid intervals, then, for every closed interval K contained in the interior of I, the homeomorphism $h|K$ restricted to K has the degree of smoothness presented in the same line, and $dh(x) \neq 0$ for all $x \in I$ (not applicable for quasisymmetric and (uaa) homeomorphisms).*

4

The smoothness of h	The order of $\operatorname{crd}\left(I_\beta^n, I_{\beta+1}^n, I_{\beta+2}^n\right)$
Quasisymmetric	$\mathrm{O}(1)$
(uaa)	$\mathrm{o}\left(\left\|I_\beta^n\right\|\right)\left\|I_\beta^n\right\|^{-1}$
$C^{1+\alpha}$	$\mathrm{O}\left(\left\|I_\beta^n\right\|^\alpha\right)$
$C^{2+\alpha}$	$\mathrm{O}\left(\left\|I_\beta^n\right\|^{1+\alpha}\right)$
$C^{2+Lipschitz}$	$\mathrm{O}\left(\left\|I_\beta^n\right\|^2\right)$

Table 3.

In Section 3, we present the definitions of the degrees of smoothness presented in Tables 2 and 3, and we prove Theorem 3 in Section 3.6. We point out that some of the difficulties and usefulness of these results come from the fact that (i) we just compute the bounds of the ratio and cross ratio distortions with respect to a countable set of intervals fixed by a grid, and (ii) we do not restrict the grid intervals, at the same level, to have necessarily the same lengths. In hyperbolic dynamics, these grids are naturally determined by Markov partitions.

In [6], [8] and [10] other relations are also presented between distinct degrees of smoothness of a homeomorphim of the real line with distinct bounds of ratio and cross ratio distortions of intervals.

1.2. Interval arithmetics. Throughout the paper, we use the notation $\phi \leq \mathrm{O}(\psi(x))$ to indicate that for all x, $|\phi(x)| < c|\psi(x)|$ where $c \geq 1$ is a constant depending only upon quantities that are explicitly mentioned. Thus, $\phi(n) < \mathrm{O}(\mu^n)$ means that $|\phi(n)| < c\mu^n$ for some constant c as above. We also use the notation of interval arithmetic for some inequalities where:

(i) if I and J are intervals then $I + J$, $I.J$ and I/J have the obvious meaning as intervals,

(ii) if $I = \{x\}$ then we often denote I by x, and

(iii) $I \pm \epsilon$ denotes the interval consisting of those x such that $|x - y| < \epsilon$ for some $y \in I$.

Thus $\phi(n) \in 1 \pm \mathcal{O}(\nu^n)$ means that there exists a constant $c > 0$ depending only upon explicitly mentioned quantities such that for all $n \geq 0$, $1 - c\nu^n < \phi(n) < 1 + c\nu^n$. Similarly, the notation $\phi \leq \mathrm{o}(\psi(x))$ indicates that for all x, $|\phi(x)| < \epsilon(|\psi(x)|)|\psi(x)|$ where $\epsilon : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function, with $\epsilon(0) = 0$, depending only upon quantities that are explicitly mentioned.

2. EXPANDING DYNAMICS OF THE CIRCLE

In this section we prove a more general version of Theorems 1 and 2 applicable to expanding circle maps of degree d, with $d \geq 2$.

2.1. $C^{1+H\ddot{o}lder}$ structures U for the expanding circle map E. In this section, we present the definition of a $C^{1+H\ddot{o}lder}$ expanding circle map E with respect to a structure U and give its characterization in terms of the ratio distortion of E at small scales with respect to the charts in U.

The *expanding circle map* $E = E(d) : S \to S$ with degree $d \geq 2$ is given by $E(z) = z^d$ in complex notation. Let $p \in S$ be one of the fixed points of the expanding circle map E. The *Markov intervals of the expanding circle map E* are the adjacent closed intervals $I_0, \ldots, I_{d-1}$ with non empty interior such that only their boundaries are contained in the set $\{E^{-1}(p)\}$ of pre-images of the fixed point $p \in S$. Choose the interval I_0 such that $I_0 \cap I_{d-1} = \{p\}$. Let the *branch expanding circle map $E_i : I_i \to S$* be the restriction of the expanding circle map E to the Markov interval I_i, for all $0 \leq i < d$. Let the interval

5

$I_{\alpha_1...\alpha_n}$ be $E_{\alpha_n}^{-1} \circ ... \circ E_{\alpha_1}^{-1}(S)$. The nth-*level of the interval partition of the expanding circle map E* is the set of all closed intervals $I_{\alpha_1...\alpha_n} \in S$.

A $C^{1+H\ddot{o}lder}$ diffeomorphism $h : I \to J$ is a $C^{1+\epsilon}$ diffeomorphism for some $\epsilon > 0$ (the notion of a quasisymmetric homeomorphism and of a $C^{1+\epsilon}$ diffeomorphism $h : I \to J$ are presented, respectively, in sections 3.1 and 3.4.)

Definition 1. *The expanding circle map $E : S \to S$ is $C^{1+H\ddot{o}lder}$ with respect to a structure U on the circle S if for every finite cover U' of U,*

> *(i) there is an $\epsilon > 0$ with the property that for all charts $u : I \to \mathbb{R}$ and $v : J \to \mathbb{R}$ contained in U' and for all intervals $K \subset I$ such that $E(K) \subset J$, the maps $v \circ E \circ u^{-1}|u(K)$ are $C^{1+\epsilon}$ and their $C^{1+\epsilon}$ norms are bounded away from zero and infinity;*
>
> *(ii) for every chart $u : I \to \mathbb{R}$ contained in U' and for every map $u_{iso} : I \to \mathbb{R}$, which is an isometry with respect to the lengths on the circle $S \subset \mathbb{R}^2$ determined by the Euclidean norm on $\mathbb{R}^2$, the composition $u_{iso} \circ u^{-1}$ is a quasisymmetric homeomorphism.*

We note that the above condition (ii) is equivalent to demand that there are constants $c > 0$ and $\nu > 1$ such that, for every $n > 0$ and every $x \in S$, $|(v \circ E^n \circ u)'(x)| > c\nu^n$, where $u : I \to \mathbb{R}$ and $v : J \to \mathbb{R}$ are any two charts in U' such that $x \in u(I)$ and $E^n \circ u(x) \in K$.

Lemma 1. *The expanding circle map $E : S \to S$ is $C^{1+H\ddot{o}lder}$ with respect to a structure U if, and only if, for every finite cover U' of U, there are constants $0 < \mu < 1$ and $b > 1$ with the following property: for all charts $u : J \to \mathbb{R}$ and $v : K \to \mathbb{R}$ contained in U' and for all adjacent intervals $I_{\alpha_1...\alpha_n}$ and $I_{\beta_1...\beta_n}$ at level n of the interval partition such that $I_{\alpha_1...\alpha_n}, I_{\beta_1...\beta_n} \subset J$ and $E(I_{\alpha_1...\alpha_n}), E(I_{\beta_1...\beta_n}) \subset K$, we have that*

$$(2) \qquad b^{-1} < \frac{|u(I_{\alpha_1...\alpha_n})|}{|u(I_{\beta_1...\beta_n})|} < b \text{ and } \left| \log \frac{|u(I_{\alpha_1...\alpha_n})| \ |v(E(I_{\beta_1...\beta_n}))|}{|u(I_{\beta_1...\beta_n})| \ |v(E(I_{\alpha_1...\alpha_n}))|} \right| \le O(\mu^n).$$

Lemma 1 follows from Theorem 3 in Section 3.

By using the Mean Value Theorem we obtain the following result for a $C^{1+H\ddot{o}lder}$ expanding circle map $E : S \to S$ with respect to a structure U. For every finite cover U' of U, there is an $\epsilon > 0$, with the property that for all charts $u : J \to \mathbb{R}$ and $v : K \to \mathbb{R}$ contained in U' and for all adjacent intervals I and I', such that $I, I' \subset J$ and $E^n(I), E^n(I') \subset K$, for some $n \ge 1$, we have

$$(3) \qquad \left| \log \frac{|u(I)||v(E^n(I'))|}{|u(I')||v(E^n(I))|} \right| \le O(|v(E^n(I)) \cup v(E^n(I'))|^{\epsilon}).$$

2.2. Solenoids $(\tilde{E}, \tilde{S})$.

In this section, we introduce the notion of a (thca) solenoid $(\tilde{E}, \tilde{S})$ and we prove that a $C^{1+H\ddot{o}lder}$ expanding circle map E with respect to a structure U determines a unique (thca) solenoid.

The sequence $\mathbf{x} = (..., x_3, x_2, x_1, x_0)$ is *an inverse path of the expanding circle map E* if $E(x_n) = x_{n-1}$, for all $n \ge 1$. The *topological solenoid $\tilde{S}$* consists of all inverse paths $\mathbf{x} = (..., x_3, x_2, x_1, x_0)$ of the expanding circle map E with the product topology. The topological solenoid is a compact set and is the twist product of the circle S with the Cantor set $\{0, ..., d-1\}^{\mathbb{Z}_{\ge 0}}$. The *solenoid map $\tilde{E}$* is the bijective map defined by

$$\tilde{E}(\mathbf{x}) = (... x_0, E(x_0)).$$

The *projection map $\pi = \pi_S : \tilde{S} \to S$* is defined by $\pi(\mathbf{x}) = x_0$. A *fiber over $x_0 \in S$* is the set of all points $\mathbf{x} \in \tilde{S}$ such that $\pi(\mathbf{x}) = x_0$. A fiber is topologically a Cantor set $\{0, ..., d-1\}^{\mathbb{Z}_{\ge 0}}$. A *leaf $\mathcal{L} = L_{\mathbf{z}}$* is the set of all points $\mathbf{w} \in \tilde{S}$ path connected to the point $\mathbf{z} \in \tilde{S}$. A *local leaf $\mathcal{L}'$* is a path connected subset of a leaf. The *monodromy map $\tilde{M} : \tilde{S} \to \tilde{S}$* is defined such that the local leaf starting on $\mathbf{x}$ and ending on $\tilde{M}(\mathbf{x})$ after being projected by π is an anti-clockwise arc starting on x_0, going around the circle
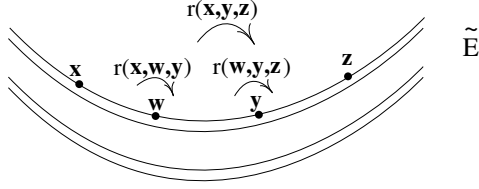
FIGURE 1. The matching condition for the leaf ratio function r.

once, and ending on the point x_0. Since the orbit of any point $\mathbf{x} \in \tilde{S}$ under $\tilde{M}$ is dense on its fiber (see Lemma 4 in Section 2.3), we get that all leaves $\mathcal{L}$ of the solenoid $\tilde{S}$ are dense.

Definition 2. *The solenoid $(\tilde{E}, \tilde{S})$ is transversely Hölder continuous affine (thca) if (i) every leaf $\mathcal{L}$ has an affine structure; (ii) the solenoid map $\tilde{E}$ preserves the affine structure on the leaves; and (iii) the ratio between adjacent leaves determined by their affine structure changes Hölder continuously along transversals.*

We say that $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *is a triple*, if the points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are distinct and are contained in the same leaf $\mathcal{L}$ of $\tilde{S}$. Let T be the set of all triples $(\mathbf{x}, \mathbf{y}, \mathbf{z})$. A function $r : T \to \mathbb{R}^+$ is *invariant by the action of the solenoid map $\tilde{E}$* if, and only if, for all triples $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in T$, we have $r(\mathbf{x}, \mathbf{y}, \mathbf{z}) = r(\tilde{E}(\mathbf{x}), \tilde{E}(\mathbf{y}), \tilde{E}(\mathbf{z}))$. A function $r : T \to \mathbb{R}^+$ *varies Hölder continuously along fibers* if and only if there are constants $c > 0$ and $0 < \mu < 1$ with the property that for all triples $(\mathbf{x}, \mathbf{y}, \mathbf{z}), (\mathbf{x}', \mathbf{y}', \mathbf{z}') \in T$, such that $x_n = x_n'$, $y_n = y_n'$ and $z_n = z_n'$, we have
$$|\log(r(\mathbf{x}, \mathbf{y}, \mathbf{z})) - \log(r(\mathbf{x}', \mathbf{y}', \mathbf{z}'))| \leq O(\mu^n).$$

Definition 3. *A Hölder leaf ratio function $r : T \to \mathbb{R}^+$ is a continuous function invariant by the action of the solenoid map $\tilde{E}$ that is Hölder continuously along fibers, and satisfies the following matching condition (see Figure 1): for all triples $(\mathbf{x}, \mathbf{w}, \mathbf{y}), (\mathbf{w}, \mathbf{y}, \mathbf{z}) \in T$,*
$$r(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{r(\mathbf{x}, \mathbf{w}, \mathbf{y}) r(\mathbf{w}, \mathbf{y}, \mathbf{z})}{1 + r(\mathbf{x}, \mathbf{w}, \mathbf{y})}.$$

Lemma 2. *There is a one-to-one correspondence between (thca) solenoids $(\tilde{E}, \tilde{S})$ and Hölder leaf ratio functions $r : T \to \mathbb{R}^+$.*

Proof: The affine structures on the leaves of the (thca) solenoid $\tilde{S}$ determine a function $r : T \to \mathbb{R}^+$ that varies continuously along leaves, and satisfies the matching condition. The converse is also true. Moreover, (i) the solenoid map $\tilde{S}$ preserves the affine structure on the leaves if and only if the function $r : T \to \mathbb{R}^+$ is invariant by the action of the solenoid map $\tilde{E}$ and (ii) the ratio between adjacent leaves determined by their affine structure changes Hölder continuously along transversals if and only if the function $r : T \to \mathbb{R}^+$ varies Hölder continuously along fibers. $\qquad\square$

Lemma 3. *A $C^{1+Hölder}$ expanding circle map $E : S \to S$ with respect to a structure U generates a Hölder leaf ratio function $r_U : T \to \mathbb{R}^+$.*

Proof: Let U' be a finite cover of U. For every triple $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in T$ and every n large enough, let $u_n : J_n \to \mathbb{R}$ be a chart contained in U' such that $x_n, y_n, z_n \in J_n$. Using (3), $r_U(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is well-defined by
$$r_U(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \lim_{n \to \infty} \frac{|u_n(y_n) - u_n(z_n)|}{|u_n(x_n) - u_n(y_n)|}.$$

By construction, r_U is invariant by the dynamics of the solenoid map and satisfies the matching condition. Again, using (3), we obtain that r_U is a continuous function varying Hölder continuously along transversals. Hence, r_U is a leaf ratio function. $\qquad\square$
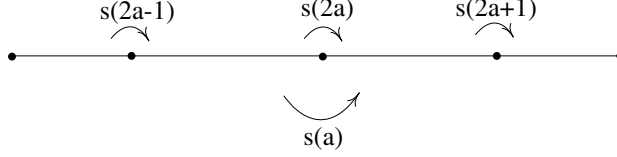
FIGURE 2. The matching condition for the solenoid function $(d = 2)$.

2.3. Solenoid functions $s : C \to \mathbb{R}^+$. In this section, we will introduce the notion of a solenoid function whose domain is a fiber of the solenoid. We will show that a Hölder leaf ratio function determines a Hölder solenoid function and that a Hölder solenoid function determines an element in the set of sequences $A(d)$.

Let $\sum_{i=-\infty}^{\infty} a_i d^i$ be a *d-adic number*. The d-adic numbers

$$\sum_{i=-\infty}^{n-1} (d-1)d^i + \sum_{i=n}^{\infty} a_i d^i \text{ and } (a_n + 1)d^n + \sum_{i=n+1}^{\infty} a_i d^i$$

such that $a_n + 1 < d$ are *d-adic equivalent*. The *d-adic set* $\tilde{\Omega}$ is the topological Cantor set $\{0, \ldots, d-1\}^{\mathbb{Z}}$ of all d-adic numbers modulo the above d-adic equivalence. The *product map* $d\times : \tilde{\Omega} \to \tilde{\Omega}$ is the multiplication by d of the d-adic numbers. The *add 1 map* $1+ : \tilde{\Omega} \to \tilde{\Omega}$ is the sum of 1 to the d-adic numbers.

Let the map $\tilde{\omega} : \tilde{\Omega} \to \tilde{S}$ be the homeomorphism between the d-adic set $\tilde{\Omega}$ and the solenoid $\tilde{S}$ defined as follows: $\tilde{\omega}(\sum_{i=-\infty}^{\infty} a_i d^i) = \mathbf{x} = (\ldots, x_1, x_0) \in \tilde{S}$, where $x_n = \cap_{i=1}^{\infty} E_{a_{n-1}}^{-1} \circ \ldots \circ E_{a_{n-i}}^{-1}(I_{a_{n-(i+1)}})$ for all $n \geq 0$ (recall that $I_{a_{n-(i+1)}}$ is a Markov interval of the expanding circle map E). Hence $x_n \in I_{a_n}$ for all $n \geq 0$. By construction, the map $\tilde{\omega} : \tilde{\Omega} \to \tilde{S}$ conjugates the product map $d\times : \tilde{\Omega} \to \tilde{\Omega}$ with the solenoid map $\tilde{E} : \tilde{S} \to \tilde{S}$, and conjugates the add 1 map $1+ : \tilde{\Omega} \to \tilde{\Omega}$ with the monodromy map $\tilde{M} : \tilde{S} \to \tilde{S}$.

Lemma 4. *Every orbit of the monodromy map is dense on its fiber.*

Proof: Since the add 1 map $1+ : \tilde{\Omega} \to \tilde{\Omega}$ is dense on the image $\tilde{\omega}^{-1}(F)$ of every fiber F of the solenoid $\tilde{S}$, the lemma follows. $\square$

Let Ω be the topological Cantor set $\{0, \ldots, d-1\}^{\mathbb{Z}^{\leq 0}}$ corresponding to all d-adic numbers of the form $\sum_{i=-\infty}^{-1} a_i d^i$ modulo the d-adic equivalence. *The projection map* $\pi_{\Omega} : \tilde{\Omega} \to \Omega$ is defined by $\pi_{\Omega}\left(\sum_{i=-\infty}^{\infty} a_i d^i\right) = \sum_{i=-\infty}^{-1} a_i d^i$. The map $\omega : \Omega \to S$ is defined by $\omega(\sum_{i=-\infty}^{-1} a_i d^i) = \cap_{i=1}^{\infty} E_{a_{-1}}^{-1} \circ \ldots \circ E_{a_{-i}}^{-1}(I_{a_{-(i+1)}})$. By construction,

$$\omega \circ \pi_{\Omega}\left(\sum_{i=-\infty}^{\infty} a_i d^i\right) = \pi_S \circ \tilde{\omega}\left(\sum_{i=-\infty}^{\infty} a_i d^i\right),$$

for all $\sum_{i=-\infty}^{\infty} a_i d^i \in \tilde{\Omega}$.

The set C is the topological Cantor set $\{0, \ldots, d-1\}^{\mathbb{Z}^{\geq 0}}$ corresponding to all d-adic integers of the form $\sum_{i=0}^{\infty} a_i d^i$.

Definition 4. *The solenoid function* $s : C \to \mathbb{R}^+$ *is a continuous function satisfying the following matching condition (see Figure 2), for all $a \in C$:*

$$
(4) \qquad s(a) = \frac{\prod_{i=1}^{d-1} s(da - i) \left(\sum_{j=0}^{d-1} \prod_{l=0}^{j} s(da + l)\right)}{1 + \sum_{j=1}^{d-1} \prod_{l=j}^{d-1} s(da - l)} .
$$

Lemma 5. *The Hölder leaf ratio function* $r : T \to \mathbb{R}^+$ *determines a Hölder solenoid function* $s_r : C \to \mathbb{R}^+$.

Proof: For all $\sum_{i=0}^{\infty} a_i d^i \in C$, we define

$$s_r \left(\sum_{i=0}^{\infty} a_i d^i \right) = r \left(\tilde{\omega} \left(\sum_{i=0}^{\infty} a_i d^i - 1 \right), \tilde{\omega} \left(\sum_{i=0}^{\infty} a_i d^i \right), \tilde{\omega} \left(\sum_{i=0}^{\infty} a_i d^i + 1 \right) \right).$$

The matching condition and the Hölder continuity of the leaf ratio function $r : T \to \mathbb{R}^+$ imply the matching condition and the Hölder continuity of the solenoid function $s_r : C \to \mathbb{R}^+$, respectively. $\qquad \square$

Lemma 6. *There is a one-to-one correspondence between Hölder solenoid functions $s : C \to \mathbb{R}^+$ and sequences $\{r_1, r_2, r_3, \ldots\} \in A(d)$ of positive real numbers with the following properties:*

(i) $r_n/r_m \le O(\mu^i)$ if $n - m$ is divisible by d^i, where $0 < \mu < 1$;
(ii) $r_1, r_2, \ldots$ satisfies

$$(5) \qquad\qquad r_a = \frac{\prod_{i=1}^{d-1} r_{da-i} \left(\sum_{j=0}^{d-1} \prod_{l=0}^{j} r_{da+l} \right)}{1 + \sum_{j=1}^{d-1} \prod_{l=j}^{d-1} r_{da-l}}.$$

A geometric interpretation of the sequences contained in the set $A(d)$ is given by the d-adic quasiperiodic tilings and grids of the real line defined in Section 2.4, below.

Proof: Given a Hölder solenoid function $s : C \to \mathbb{R}^+$, for all $i = \sum_{j=0}^{k} a_j d^j \ge 0$, we define r_i by

$$r_i = s \left(\sum_{j=0}^{k} a_j d^j \right).$$

The matching condition of the solenoid function $s : C \to \mathbb{R}^+$ implies that the ratios $r_1, r_2, \ldots$ satisfy (5). The Hölder continuity of the solenoid function $s : C \to \mathbb{R}^+$ implies condition (i). Conversely, for every d-adic integer $a = \sum_{i=0}^{\infty} a_i d^i \in C$, let $\underline{a}_n \in \mathbb{Z}_{\ge 0}$ be equal to $\sum_{i=0}^{n} a_i d^i$. Define the value $s(a)$ by

$$s(a) = \lim_{n \to \infty} r_{\underline{a}_n}.$$

Using condition (i) the above limit is well defined and the function $s : C \to \mathbb{R}^+$ is Hölder continuous. Using condition (ii) and the continuity of s we obtain that the function s satisfies the matching condition. $\qquad \square$

2.4. d-Adic quasiperiodic tilings and grids and amalgamation operators. In this section, we introduce d-adic quasiperiodic tilings of the real line that are fixed points of the d-amalgamation operator and d-adic quasiperiodic fixed grids of the real line, and we show that their affine classes are in one-to-one correspondence with (thca) solenoids.

A *tiling* $\mathcal{T} = \{I_\beta \subset \mathbb{R} : \beta \in \mathbb{Z}\}$ *of the real line* is a collection of *tiling intervals* I_β with the following properties: (i) The tiling intervals are closed intervals; (ii) The union $\cup_{\beta \in \mathbb{Z}} I_\beta$ of all tiling intervals I_β is equal to the real line; (iii) any two distinct tiling intervals have disjoint interiors; (iv) For every $\beta \in \mathbb{Z}$, the intersection of the tiling intervals I_β and $I_{\beta+1}$ is only an endpoint common to both intervals; (v) There is $B \ge 1$, such that for every $\beta \in \mathbb{Z}$, we have $B^{-1} \le |I_{\beta+1}|/|I_\beta| \le B$. We say that the tilings $\mathcal{T}_1 = \{I_\beta \subset \mathbb{R} : \beta \in \mathbb{Z}\}$ and $\mathcal{T}_2 = \{J_\beta \subset \mathbb{R} : \beta \in \mathbb{Z}\}$ of the real line are in the same affine class if there is an affine map $h : \mathbb{R} \to \mathbb{R}$ such that $h(I_\beta) = J_\beta$ for every $\beta \in \mathbb{Z}$. The *tiling sequence* $\underline{r} = (r_m)_{m \in \mathbb{Z}}$ is given by $r_m = |I_{m+1}|/|I_m|$. We note that a tiling sequence $\underline{r}$ determines an affine class of tilings $\mathcal{T}$ and vice-versa. We say that a tiling sequence is *d-adic quasiperiodic* if there is $0 < \mu < 1$ such that $|r_j - r_k| \le O(\mu^i)$, when $(j - k)/d^i$ is
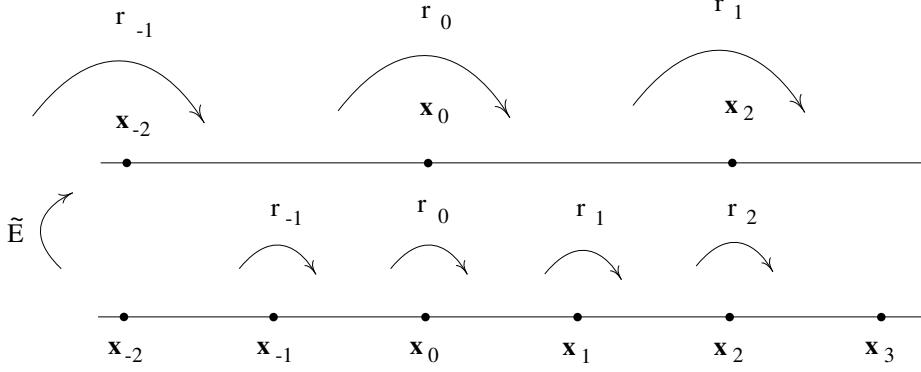
9

FIGURE 3. The leaf $\mathcal{L}$ fixed by the solenoid map $\tilde{E}$.

an integer. Let $\mathbf{T}$ denote the set of all tiling sequences. The *d-amalgamation operator* $A_d : \mathbf{T} \to \mathbf{T}$ is defined by $A_d(\underline{r}) = \underline{s}$, where

$$s_i = r_{d(i-1)+1,di} \frac{1 + \sum_{m=di+1}^{d(i+1)-1} r_{di+1,m}}{1 + \sum_{m=d(i-1)+1}^{di-1} r_{d(i-1)+1,m}},$$

for all $i \in \mathbb{Z}$.

Definition 5. *A d-adic quasiperiodic tiling $\mathcal{T}$ of the real line is a tiling such that the corresponding tiling sequence $\underline{r}$ is d-adic quasiperiodic. A tiling $\mathcal{T}$ of the real line is a fixed point of the d-amalgamation operator if the corresponding tiling sequence is a fixed point of the d-amalgamation operator, i.e. $A_d(\underline{r}) = \underline{r}$.*

Remark 1. *The tiling sequence $\underline{r} = (r_m)_{m \in \mathbb{Z}}$ of a d-adic quasiperiodic tiling of the real line that is a fixed point of the d-amalgamation operator determines a sequence $r_1, r_2, \ldots$ in $A(d)$.*

A *d-grid* $\mathcal{G}$ of the real line is a collection of intervals I_β^n satisfying properties (i) to (vii) of a (B,d)-grid $\mathcal{G}_\Omega$, for some $B \geq 1$, such that every interval I_β^n is the union of d grid intervals at level $n+1$, and $\Omega(n) = \infty$. We note that every level n of a grid forms a tiling of the real line. We say that the grids $\mathcal{G}_1 = \{I_\beta^n\}$ and $\mathcal{G}_2 = \{J_\beta^n\}$ of the real line are in the same affine class if there is an affine map $h : \mathbb{R} \to \mathbb{R}$ such that $h(I_\beta^n) = J_\beta^n$ for every $\beta \in \mathbb{Z}$ and every $n \in \mathbb{N}$. The *d-grid sequence* $\ldots \underline{r}^2 \underline{r}^1$ is given by $\underline{r}^n = (r_m^n)_{m \in \mathbb{Z}}$ where $r_m^n = |I_{m+1}^n|/|I_m^n|$. The following remark gives a geometric interpretation of the d-amalgamation operator.

Remark 2.   (i) *If $\ldots \underline{r}^2 \underline{r}^1$ is a d-grid sequence then $A_d(\underline{r}^{n+1}) = \underline{r}^n$ for every $n \geq 1$.*
    (ii) *If $\ldots \underline{r}^2 \underline{r}^1$ is a sequence such that $A_d(\underline{r}^{n+1}) = \underline{r}^n$ then the sequence determines an affine class of d-grids.*

Definition 6. *A d-adic quasiperiodic fixed grid $\mathcal{G}$ of the real line is a d-grid of the real line such that the corresponding grid sequence $\ldots \underline{r}^2 \underline{r}^1$ is constant, i.e. $\underline{r}^1 = \underline{r}^n$ for every $n \geq 1$, and $\underline{r}^1$ is d-adic quasiperiodic.*

Hence, all the levels of a d-adic quasiperiodic fixed grid $\mathcal{G}$ of the real line determine the same d-adic quasiperiodic tiling of the real line, up to affine equivalence, that is a fixed point of the d-amalgamation operator.

Lemma 7. *There is a one-to-one correspondence between (i) (thca) solenoids ; (ii) affine classes of d-adic quasiperiodic tilings of the real line that are fixed points of the d-amalgamation operator; (iii) affine classes of d-adic quasiperiodic fixed grids of the real line.*
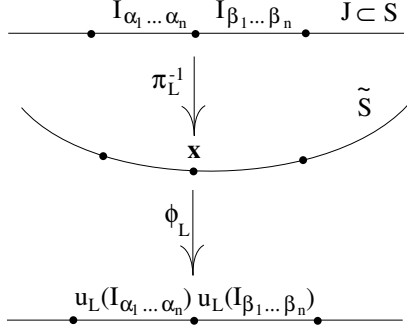
10

FIGURE 4. The solenoidal chart.

Proof: By construction, there is a one-to-one correspondence between (ii) affine classes of d-adic quasiperiodic tilings of the real line that are fixed points of the d-amalgamation operator and (iii) affine classes of d-adic quasiperiodic fixed grids of the real line. Let us prove that a (thca) solenoid determines canonically an affine class of d-adic quasiperiodic tilings of the real line that are fixed points of the d-amalgamation operator. Let $\mathcal{L}$ be a leaf of the (thca) solenoid $(\tilde{E}, \tilde{S})$ containing a fixed point $\mathbf{x}_0$ of the solenoid map $\tilde{E}$. The leaf $\mathcal{L}$ is marked by the points $\ldots, \mathbf{x}_{-1}, \mathbf{x}_0, \mathbf{x}_1, \ldots$ that project on the same point of the circle as the fixed point $\mathbf{x}_0$, and such that there is a local leaf $\mathcal{L}_m$ with extreme points $\mathbf{x}_m$ and $\mathbf{x}_{m+1}$ with the property that $\mathcal{L}_m$ does not contain any other point $\mathbf{x}_j$ for $m \neq j \neq m+1$. The affine structure on the leaf $\mathcal{L}$ determines the ratios $r_m = r(\mathbf{x}_{m-1}, \mathbf{x}_m, \mathbf{x}_{m+1})$ of the leaf ratio function $r : T \to \mathbb{R}^+$, for all $m \in \mathbb{Z}$. Since the solenoid map $\tilde{E}$ is affine and $\tilde{E}(\mathcal{L}) = \mathcal{L}$, the sequence of ratios $\underline{r} = (r_m)_{m \in \mathbb{Z}}$ is fixed by the amalgamation operator A_d (see Figure 3). The Hölder transversality of the solenoid $(\tilde{E}, \tilde{S})$ implies that the sequence $\underline{r}$ is d-adic quasiperiodic. Therefore, the sequence $\underline{r}$ determines an affine class of d-adic quasiperiodic tilings of the real line that are fixed point of the d-amalgamation operator, and so an affine class of d-adic quasiperiodic fixed grids of the real line. Conversely, an affine class of d-adic quasiperiodic fixed grids of the real line determines uniquely the affine structure of a leaf $\mathcal{L}$ that is fixed by the solenoid map $\tilde{E}$. Since the *grid sequence* $\ldots \underline{r}^2 \underline{r}^1$ is a fixed point of the amalgamation operator, i.e. $A_d(\underline{r}^n) = \underline{r}^{n-1}$, the solenoid map $\tilde{E}$ is affine on the leaf $\mathcal{L}$. By density of the leaf $\mathcal{L}$ on the solenoid $\tilde{S}$ and since the grid g_d is d-adic quasiperiodic, the affine structure of the leaf $\mathcal{L}$ extends to an affine structure transversely Hölder continuous on the solenoid $\tilde{S}$ such that the solenoid map $\tilde{E}$ leaves the affine structure invariant. $\square$

2.5. Solenoidal charts for the $C^{1+H\ddot{o}lder}$ expanding circle map E. In this section, we introduce the solenoidal charts which will determine a canonical structure for the expanding circle map.

Definition 7. *Let $\mathcal{L}$ be a local leaf with an affine structure and $\pi_{\mathcal{L}} = \pi_S|\mathcal{L}$ the homeomorphic projection of $\mathcal{L}$ onto an interval $J_{\mathcal{L}}$ of the circle S. Let $\phi_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}$ be a map preserving the affine structure of the leaf $\mathcal{L}$. A solenoidal chart $u_{\mathcal{L}} : J_{\mathcal{L}} \to \mathbb{R}$ on the circle S is defined by $u_{\mathcal{L}} = \phi_{\mathcal{L}} \circ \pi_{\mathcal{L}}^{-1}$ (see Figure 4).*

Lemma 8. *The solenoidal charts determined by a (thca) solenoid $(\tilde{E}, \tilde{S})$ produce a canonical structure U such that the expanding circle map E is $C^{1+H\ddot{o}lder}$.*

Proof: Let U' be a finite cover consisting of solenoidal charts. Let $I_{\alpha_1 \ldots \alpha_n}$ and $I_{\beta_1 \ldots \beta_n}$ be adjacent intervals at level n of the interval partition and $u_{\mathcal{L}} : J \to \mathbb{R}$ and $v_{\mathcal{L}'} : K \to \mathbb{R}$ solenoidal charts such that $I_{\alpha_1 \ldots \alpha_n}, I_{\beta_1 \ldots \beta_n} \subset J$ and $I_{\alpha_2 \ldots \alpha_n}, I_{\beta_2 \ldots \beta_n} \subset K$. Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be the points contained in $\mathcal{L}$ such that $\pi(\mathbf{x})$ and $\pi(\mathbf{y})$ are the endpoints of $I_{\alpha_1 \ldots \alpha_n}$, and $\pi(\mathbf{y})$ and $\pi(\mathbf{z})$ are the endpoints of $I_{\beta_1 \ldots \beta_n}$. Let $\mathbf{x}'$, $\mathbf{y}'$ and $\mathbf{z}'$ be the points contained in $\mathcal{L}'$ such that $\pi(\mathbf{x}')$ and $\pi(\mathbf{y}')$ are the endpoints of $I_{\alpha_2 \ldots \alpha_n}$, and $\pi(\mathbf{y}')$ and $\pi(\mathbf{z}')$ are the

11

endpoints of $I_{\beta_2\ldots\beta_n}$ (see Figure 4). By Lemma 2, the (thca) solenoid determines a leaf ratio function $r : T \to \mathbb{R}^+$ such that

$$(6) \qquad \frac{|u_{\mathcal{L}}(I_{\beta_1\ldots\beta_n})|}{|u_{\mathcal{L}}(I_{\alpha_1\ldots\alpha_n})|} \frac{|v_{\mathcal{L}'}(I_{\alpha_2\ldots\alpha_n})|}{|v_{\mathcal{L}'}(I_{\beta_2\ldots\beta_n})|} = \frac{r(\mathbf{x}, \mathbf{y}, \mathbf{z})}{r(\mathbf{x}', \mathbf{y}', \mathbf{z}')} \ .$$

By Lemma 5, using that $\tilde{E}$ is affine on leaves, the leaf ratio function $r : T \to \mathbb{R}^+$ determines a solenoid function $s_r : C \to \mathbb{R}^+$ such that

$$(7) \qquad \frac{r(\mathbf{x}, \mathbf{y}, \mathbf{z})}{r(\mathbf{x}', \mathbf{y}', \mathbf{z}')} = \frac{s\left(\tilde{\omega}^{-1}(\tilde{E}^n(\mathbf{y}))\right)}{s\left(\tilde{\omega}^{-1}(\tilde{E}^{n-1}(\mathbf{y}')) \right)} \ .$$

By Hölder continuity of the solenoid function,

$$(8) \qquad \left| \log \frac{s\left(\tilde{\omega}^{-1}(\tilde{E}^n(\mathbf{x}))\right)}{s\left(\tilde{\omega}^{-1}(\tilde{E}^{n-1}(\mathbf{y})) \right)} \right| \leq \mathrm{O}(\mu^n),$$

for some $0 < \mu < 1$. Putting (6), (7) and (8) together, and using that C is compact, we obtain that

$$(9) \qquad b^{-1} < \frac{|u_{\mathcal{L}}(I_{\alpha_1\ldots\alpha_n})|}{|u_{\mathcal{L}}(I_{\beta_1\ldots\beta_n})|} < b \text{ and } \left| \log \frac{|u_{\mathcal{L}}(I_{\alpha_1\ldots\alpha_n})|}{|u_{\mathcal{L}}(I_{\beta_1\ldots\beta_n})|} \frac{|v_{\mathcal{L}'}(I_{\beta_2\ldots\beta_n})|}{|v_{\mathcal{L}'}(I_{\alpha_2\ldots\alpha_n})|} \right| \leq \mathrm{O}(\mu^n)$$

for some $b \geq 1$. Hence, by Lemma 1, the expanding circle map E is $C^{1+H\ddot{o}lder}$ with respect to the structure U produced by the solenoidal charts. $\qquad\square$

Lemma 9. *The Hölder solenoid function $s : C \to \mathbb{R}^+$ determines a set of solenoidal charts which produce a structure U such that the expanding circle map E is $C^{1+H\ddot{o}lder}$.*

Proof: For every triple $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ such that there are $n \in \mathbb{Z}$ and $a \in C$ with the property that

$$(\tilde{E}^n(\mathbf{x}), \tilde{E}^n(\mathbf{y}), \tilde{E}^n(\mathbf{z})) = (\tilde{\omega}(a-1), \tilde{\omega}(a), \tilde{\omega}(a+1))$$

we define $r(\mathbf{x}, \mathbf{y}, \mathbf{z})$ equal to $s(a)$. Hence, the ratios r are invariant under the solenoid map $\tilde{E}$. Since the solenoid function satisfies the matching condition, the above ratios r determine an affine structure on the leaves of the solenoid. By construction, the solenoidal charts determined by this affine structure on the leaves satisfy (9), and so by Lemma 1, the expanding circle map E is $C^{1+H\ddot{o}lder}$ with respect to the structure U produced by the solenoidal charts. $\qquad\square$

2.6. Smooth properties of solenoidal charts. We will prove that the solenoidal charts maximize the smoothness of the expanding circle map with respect to all charts in the same $C^{1+H\ddot{o}lder}$ structure.

Let U be a $C^{1+H\ddot{o}lder}$ structure for the expanding circle map E. By Lemmas 2 and 3, the structure U determines a (thca) solenoid $(\tilde{E}, \tilde{S})_U$.

Lemma 10. *Let U be a $C^{1+H\ddot{o}lder}$ structure for the expanding circle map E, and let V be the set of all solenoidal charts determined by the (thca) solenoid $(\tilde{E}, \tilde{S})_U$. Then, the set V is contained in U and the degree of smoothness of the expanding circle map E when measured in terms of a cover U' of U attains its maximum when $U' \subset V$.*

Proof: Let the expanding circle map $E : S \to S$ be C^r, for some $r > 1$, with respect to a finite cover U' of the structure U. We shall prove that the solenoidal charts $v_{\mathcal{L}} : I \to \mathbb{R}$ are C^r compatible with the charts contained in U', proving the theorem. Let $\mathcal{L}$ be a local leaf that projects by $\pi_{\mathcal{L}} = \pi_S|\mathcal{L}$ homeomorphically on an interval I contained in the domain J of a chart $u : J \to \mathbb{R}$ of U'. For n large enough, let $u_n : J_n \to \mathbb{R}$ be a chart in U' such that $I_n = \pi_S(\tilde{E}^{-n}(\mathcal{L})) \subset J_n$. Let $\lambda_n : u_n(I_n) \to (0,1)$ be the restriction to the interval $u_n(I_n)$ of an affine map sending the interval $u_n(I_n)$ onto the interval $(0,1)$ (see Figure 5). Let $e_n : (0,1) \to \mathbb{R}$ be the C^r map defined by $e_n = u \circ E^n \circ u_n^{-1} \circ \lambda_n^{-1}$. The map e_n is the composition of a contraction λ_n^{-1} followed by an expansion $u \circ E^n \circ u_n^{-1}$.

FIGURE 5. The construction of the solenoidal charts from the $C^{1+H\ddot{o}lder}$ structure U.

Therefore, by the usual blow-down blow-up technique (see [9]), the map $e : (0,1) \to \mathbb{R}$ given by $e = \lim_{n\to\infty} e_n$ is a C^r homeomorphism. Hence, the map $v_{\mathcal{L}} : I \to \mathbb{R}$ defined by $e^{-1} \circ u$ is a solenoidal chart and is C^r compatible with the charts contained in U'. $\square$

2.7. **Proof of Theorem 1.** Theorem 1 follows as a corollary of Lemma 11, below, by taking $d = 2$.

Lemma 11. *The following sets are canonically isomorphic:*

(i) *The set of all $C^{1+H\ddot{o}lder}$ structures U for the expanding circle map*
$E : S \to S$ *of degree $d \geq 2$;*
(ii) *The set of all (thca) solenoids $(\tilde{E}, \tilde{S})$;*
(iii) *The set of all Hölder leaf ratio functions $r : T \to \mathbb{R}^+$;*
(iv) *The set of all Hölder solenoid functions $s : C \to \mathbb{R}^+$;*
(v) *The set of all sequences $\{r_0, r_1, \ldots\} \in A(d)$;*
(vi) *The set of all affine classes of d-adic quasiperiodic tilings of the real*
line that are fixed points of the d-amalgamation operator;
(vii) *The set of all affine classes of d-adic quasiperiodic fixed grids of the*
real line.

Proof: The proof of this lemma follows from the following diagram, where the implications are determined by the lemmas indicated by their numbers:

$$
\begin{array}{ccccc}
(i) & \overset{8}{\Longleftarrow} & (ii) & \overset{7}{\Longleftrightarrow} & (vi), (vii) \\
9\Uparrow & \overset{3}{\searrow} & \Updownarrow 2 & & \\
(v) \quad \overset{6}{\Longleftrightarrow} \quad (iv) & & \overset{5}{\Longleftarrow} \quad (iii) & &
\end{array}
$$

$\square$

2.8. **Proof of Theorem 2.** In this section, we use Theorem 3 to prove Theorem 2 (we will show Theorem 3 in Section 3.6). In fact, we prove a more general version of Theorem 2, which applies to expanding circle maps of degree $d \geq 2$, using the following generalization of the ultra-metric to the set C of all d-adic integers. Let $a = \sum_{m=0}^{\infty} a_m d^m \in C$ and $b = \sum_{m=0}^{\infty} b_m d^m \in C$ be such that $a_n \ldots a_0 = b_n \ldots b_0$ and $a_{n+1} \neq b_{n+1}$. For $0 \leq i \leq n$, let $A_i = \sum_{m=0}^{i} a_m d^m$ and $E_i = \sum_{m=0}^{i} (d-1) d^m$. We define the *ultra-metric* by

$$
|\mathbf{u}|_s (a, b) = \inf_{0 \leq i \leq n} \left\{ 1 + \sum_{j=A_i}^{E_i} \prod_{l=A_i}^{j} s(l) + \sum_{j=0}^{A_i-1} \prod_{l=j}^{A_i-1} s(l) \right\} .
$$

Let $\mathbf{p}$ be the fixed point of the solenoid map $\tilde{E}$ such that $\pi(\mathbf{p})$ is the fixed point of the expanding circle map chosen in Section 2.1 to generate the Markov partition of E. Let $\mathcal{L}_{\mathbf{p}}$ be the local leaf starting on $\mathbf{p}$ and ending on its image $\tilde{M}(\mathbf{p})$ by the monodromy map

$\tilde{M}$. Let $z : J = \pi_S(\mathcal{L}_\mathbf{p}) \to (0,1)$ be the corresponding solenoidal chart. The geometric interpretation of the ultra-metric $|\mathbf{u}|_s$ is given by the following equality

$$|\mathbf{u}|_s(a,b) = \inf_{0 \le i \le n} \{|z(I_{a_i \ldots a_0})|\} .$$

Proof of Theorem 2. Let U be a $C^{1+H\ddot{o}lder}$ structure for the expanding circle map E, and let V be the set of all solenoidal charts determined by the (thca) solenoid $(\tilde{E}, \tilde{S})_U$. By Lemma 10, the set V is contained in U and the degree of smoothness of the expanding circle map E when measured in terms of a cover U' of U attains its maximum when $U' \subset V$. Let $\mathcal{L}$ and $\mathcal{L}'$ be two local leaves and $u : J = \pi_S(\mathcal{L}) \to \mathbb{R}$ and $v : J' = \pi_S(\mathcal{L}') \to \mathbb{R}$ the corresponding solenoidal charts. If $J \cap J' \ne \emptyset$, let $I_{\beta_1 \ldots \beta_n} \subset J \cap J'$ be any interval at any level n of the interval partition. Let the points $\mathbf{x} \in \mathcal{L}$ and $\mathbf{y} \in \mathcal{L}'$ be such that $\pi_S(\mathbf{x}) = \pi_S(\mathbf{y}) \in S$ is the right endpoint of the interval $I_{\beta_1 \ldots \beta_n}$. Let a be the point $\tilde{\omega}(\tilde{E}^n(\mathbf{x})) \in C$ and b the point $\tilde{\omega}(\tilde{E}^n(\mathbf{y})) \in C$. By definition of the metric $|\mathbf{u}|_s$, there is a constant $D_0 = D_0(\mathcal{L}, \mathcal{L}') \ge 1$, such that $D_0^{-1}|z(I_{\beta_1 \ldots \beta_n})| \le |\mathbf{u}|_s(a,b) \le D_0|z(I_{\beta_1 \ldots \beta_n})|$ with respect to the solenoidal chart $z : J = \pi_S(\mathcal{L}_\mathbf{p}) \to (0,1)$ defined above. By Lemma 10, the overlap maps $z \circ u^{-1}$ and $z \circ v^{-1}$ are $C^{1+H\ddot{o}lder}$ smooth. Therefore, by Lemma 1, there is a constant $D_1 = D_1(\mathcal{L}, \mathcal{L}') \ge 1$ such that

(10) $$D_1^{-1} \le \frac{|\mathbf{u}|_s(a,b)}{|u(I_{\beta_1 \ldots \beta_n})|} \le D_1 \text{ and } D_1^{-1} \le \frac{|\mathbf{u}|_s(a,b)}{|v(I_{\beta_1 \ldots \beta_n})|} \le D_1 .$$

Let $I_{\beta_1' \ldots \beta_n'}$ and $I_{\beta_1'' \ldots \beta_n''}$ be adjacent intervals at level n of the interval partition, such that $I_{\beta_1' \ldots \beta_n'}$ is also adjacent to $I_{\beta_1 \ldots \beta_n}$. By proof of Lemma 9,

(11) $$s(a) = \frac{|u(I_{\beta_1' \ldots \beta_n'})|}{|u(I_{\beta_1 \ldots \beta_n})|}, \ s(a+1) = \frac{|u(I_{\beta_1'' \ldots \beta_n''})|}{|u(I_{\beta_1' \ldots \beta_n'})|},$$

and

(12) $$s(b) = \frac{|v(I_{\beta_1' \ldots \beta_n'})|}{|v(I_{\beta_1 \ldots \beta_n})|}, \ s(b+1) = \frac{|v(I_{\beta_1'' \ldots \beta_n''})|}{|v(I_{\beta_1' \ldots \beta_n'})|}.$$

The interval partition of the expanding circle map E generates a grid g_u in the set $u(J \cap J')$. Therefore, using (10), (11), (12) and Theorem 3, the equivalences presented in Tables 2 and 3 imply that the overlap maps $h = v \circ u^{-1} : u(J \cap J') \to v(J \cap J')$ satisfy the equivalences presented in Table 1. $\qquad\square$

3. Smoothness of diffeomorphisms and cross ratio distortion of grids

In the following subsections, we introduce the definitions of the degrees of smoothness of a homeomorphism $h : I \to J$ presented in Tables 2 and 3, and we prove the corresponding equivalences between the degrees of smoothness of h with the ratio and cross ratio distortions of intervals contained in a grid of I as presented in Tables 2 and 3. In Section 3.6, we prove Theorem 3.

Let I_β and $I_{\beta'}$ be two intervals contained in the real line. We define the *ratio* $r(I_\beta, I_{\beta'})$ between the intervals I_β and $I_{\beta'}$ by

$$r(I_\beta, I_{\beta'}) = \frac{|I_{\beta'}|}{|I_\beta|} .$$

Let I_β, $I_{\beta'}$ and $I_{\beta''}$ be contained in the real line, such that I_β is adjacent to $I_{\beta'}$, and $I_{\beta'}$ is adjacent to $I_{\beta''}$. Recall that the *cross ratio* $cr(I_\beta, I_{\beta'}, I_{\beta''})$ is given by

$$cr(I_\beta, I_{\beta'}, I_{\beta''}) = \log\left(1 + \frac{|I_{\beta'}|}{|I_\beta|}\frac{|I_\beta| + |I_{\beta'}| + |I_{\beta''}|}{|I_{\beta''}|}\right) .$$

We note that

$$cr(I_\beta, I_{\beta'}, I_{\beta''}) = \log\left((1 + r(I_\beta, I_{\beta'}))(1 + r(I_{\beta''}, I_{\beta'}))\right) .$$

Let $h : I \subset \mathbb{R} \to J \subset \mathbb{R}$ be a homeomorphism, and let $\mathcal{G}_\Omega$ be a grid of the compact interval I. We will use the following definitions and notations throughout all this section:

(i) We will denote by J_β^n the interval $h(I_\beta^n)$ where I_β^n is a grid interval. We will denote by $r(n, \beta)$ the ratio $r(I_\beta^n, I_{\beta+1}^n)$ between the grid intervals I_β^n and $I_{\beta+1}^n$, and we will denote by $r_h(n, \beta)$ the ratio $r(J_\beta^n, J_{\beta+1}^n)$.

(ii) Let I_β be an interval contained in I (not necessarily a grid interval). The *average derivative* $dh(I_\beta)$ is given by

$$dh(I_\beta) = \frac{|h(I_\beta)|}{|I_\beta|} \ .$$

We will denote by $dh(n, \beta)$ the average derivative $dh(I_\beta^n)$ of the grid interval I_β^n.

(iii) The *logarithmic average derivative* $ldh(I_\beta)$ is given by

$$ldh(I_\beta) = \log(dh(I_\beta)) \ .$$

We will denote by $ldh(n, \beta)$ the logarithmic average derivative $ldh(I_\beta^n)$ of the grid interval I_β^n.

(iv) Let I_β and $I_{\beta'}$ be intervals contained in I (not necessarily grid intervals). We recall that the logarithmic ratio distortion $lrd(I_\beta, I_{\beta'})$ is given by

$$lrd(I_\beta, I_{\beta'}) = \log\left(\frac{|I_\beta|}{|I_{\beta'}|} \frac{|h(I_{\beta'})|}{|h(I_\beta)|}\right) \ .$$

Hence, we have

$$lrd(I_\beta, I_{\beta'}) = \log\frac{r(J_\beta, J_{\beta'})}{r(I_\beta, I_{\beta'})} = \log\frac{dh(I_{\beta'})}{dh(I_\beta)} \ .$$

We will denote by $lrd(n, \beta)$ the logarithmic ratio distortion $lrd(I_\beta^n, I_{\beta+1}^n)$ of the grid intervals I_β^n and $I_{\beta+1}^n$.

(v) Let the intervals I_β, $I_{\beta'}$ and $I_{\beta''}$ in I (not necessarily grid intervals) be such that I_β is adjacent to $I_{\beta'}$ and $I_{\beta'}$ is adjacent to $I_{\beta''}$. We recall that the cross ratio distortion $crd(I_\beta, I_{\beta'}, I_{\beta''})$ is given by

$$crd(I_\beta, I_{\beta'}, I_{\beta''}) = cr(h(I_\beta), h(I_{\beta'}), h(I_{\beta''})) - cr(I_\beta, I_{\beta'}, I_{\beta''}) \ .$$

We note that

$$(13) \qquad crd(I_\beta, I_{\beta'}, I_{\beta''}) = \log\left(\frac{1 + r(h(I_\beta), h(I_{\beta'}))}{1 + r(I_\beta, I_{\beta'})} \frac{1 + r(h(I_{\beta''}), h(I_{\beta'}))}{1 + r(I_{\beta''}, I_{\beta'})}\right) \ .$$

For all grid intervals I_β^n, $I_{\beta+1}^n$ and $I_{\beta+2}^n$, we will denote by $cr(n, \beta)$ and $cr_h(n, \beta)$ the cross ratios $cr(I_\beta^n, I_{\beta+1}^n, I_{\beta+2}^n)$ and $cr(J_\beta^n, J_{\beta+1}^n, J_{\beta+2}^n)$ respectively. We will denote by $crd(n, \beta)$ the cross ratio distortion given by $cr_h(n, \beta) - cr(n, \beta)$.

Remark 3. *(a) We will call properties (vi) and (vii) of a (B,M) grid $\mathcal{G}_\Omega$ of an interval I, the bounded geometry property of the grid.*
(b) By the bounded geometry property of a (B,M) grid $\mathcal{G}_\Omega$, there are constants $0 < B_1 < B_2 < 1$, just depending upon B and M, such that

$$B_1 < \frac{|I_\beta^{n+1}|}{|I_\alpha^n|} < B_2 \ ,$$

for all $n \geq 1$ and for all grid intervals I_α^n and I_β^{n+1} such that $I_\beta^{n+1} \subset I_\alpha^n$.
(c) We call a $(1,2)$ grid $\mathcal{G}_\Omega$ of I a symmetric grid of I, i.e. (i) all the intervals at the same level n have the same length, and (ii) each grid interval at level n is equal to the union of two grid intervals at level $n+1$.

15

3.1. Quasisymmetric homeomorphisms. The definition of a quasisymmetric homeomorphism that we introduce in this paper is more adapted to our problem and, apparently, is stronger than the usual one, where the constant d of the quasisymmetric condition in Definiton 8, below, is taken to be equal to 1. However, in Lemma 13, we will prove that they are equivalent.

Definition 8. *Let $d \geq 1$ and $k \geq 1$. The homeomorphism $h : I \to J$ satisfies the (d, k) quasisymmetric condition if*

$$
(14) \qquad \left| \log \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)} \frac{\delta_1}{\delta_2} \right| \leq \log(k) ,
$$

for all $x - \delta_1, x, x + \delta_2 \in I$ with $\delta_1 > 0$, $\delta_2 > 0$ and $d^{-1} \leq \delta_2 / \delta_1 \leq d$. The homeomorphism h is quasisymmetric if for every $d \geq 1$ there exists $k_d \geq 1$ such that h satisfies the (d, k_d) quasisymmetric condition.

Lemma 12. *Let $h : I \to J$ be a homeomorphism and let $\mathcal{G}_\Omega$ be a grid of a compact interval I. The following statements are equivalent:*

    (i) The homeomorphism $h : I \to J$ is quasisymmetric.
    (ii) There is $k(\mathcal{G}_\Omega) > 1$ such that

$$
(15) \qquad\qquad |r_h(n, \beta)| \leq k(\mathcal{G}_\Omega) ,
$$

    for every $n \geq 1$ and every $1 \leq \beta \leq \Omega(n)$.

Let $\mathcal{G}_\Omega$ be a grid of I. From Lemma 12, we obtain that a homeomorphism $h : I \to J$ is quasisymmetric if, and only if, the set of all intervals J^n_β form a (B, M) grid for some $B \geq 1$ and $M > 1$.

Lemma 13. *If, for some $d_0 \geq 1$ and $k_0 \geq 1$, a homeomorphism $h : I \to J$ satisfies the (d_0, k_0) quasisymmetric condition, then h is quasisymmetric.*

Lemma 14. *Let $h : I \to J$ be a homeomorphism and $\mathcal{G}_\Omega$ a grid of the compact interval I.*

    (i) If $h : I \to J$ is quasisymmetric then there is $C_0 \geq 0$ such that

$$
cr_h(n, \beta) \leq C_0 ,
$$

    for every $n \geq 1$ and every $1 \leq \beta < \Omega(n) - 1$.
    (ii) If there is $C_0 > 1$ such that, for every $n \geq 1$ and every $1 \leq \beta < \Omega(n) - 1$,

$$
cr_h(n, \beta) \leq C_0 ,
$$

    then, for every closed interval K contained in the interior of I, the homeomorphism h restricted to K is quasisymmetric.

Before proving Lemmas 12, 13 and 14, we will state and prove Lemma 15 which we will use in the proof of Lemma 12 and, later, in the proof of Lemma 18.

Lemma 15. *Let $\alpha > 1$ and $d \geq 1$. Let $\mathcal{G}_\Omega$ be a (B, M) grid of a compact interval I. Let $x - \delta_1, x, x + \delta_2$ contained in I be such that $\delta_1 > 0$, $\delta_2 > 0$ and $d^{-1} \leq \delta_2/\delta_1 \leq d$. Then, there are intervals L_1, L_2, R_1 and R_2 with the following properties:*

    (i)

$$
(16) \qquad\qquad L_1 \subset [x - \delta_1, x] \subset L_2 \text{ and } R_1 \subset [x, x + \delta_2] \subset R_2 .
$$

    (ii)

$$
(17) \qquad\qquad \alpha^{-1} < \frac{|L_1|}{\delta_1} < \frac{|L_2|}{\delta_1} < \alpha \text{ and } \alpha^{-1} < \frac{|R_1|}{\delta_2} < \frac{|R_2|}{\delta_2} < \alpha .
$$

(iii) Let $n_0 = n_0(x - \delta_1, x, x + \delta_2, \mathcal{G}_\Omega) \geq 1$ be the biggest integer such that

$$[x - \delta_1, x + \delta_2] \subset I_\beta^{n_0} \cup I_{\beta+1}^{n_0}$$

for some $1 \leq \beta < \Omega(n_0)$. Then, there are integers $n_1 = n_1(\alpha, B, M, d)$ and $n_2 = n_2(\alpha, B, M, d)$ such that

$$\begin{aligned} L_1 &= \cup_{i=l+1}^{m-1} I_i^{n_0+n_1} \;, \; L_2 = \cup_{i=l}^{m} I_i^{n_0+n_1} \;, \\ R_1 &= \cup_{i=m+1}^{r-1} I_i^{n_0+n_1} \;, \; R_2 = \cup_{i=m}^{r} I_i^{n_0+n_1} \;, \end{aligned}$$

for some l, m, r with the property that $l < m < r$ and $r - l \leq n_2$.

Proof of Lemma 15. Let $0 < B_1 = B_1(B, M) < B_2 = B_2(B, M) < 1$ be as in Remark 3. By construction of n_0, there is $I_\epsilon^{n_0+1}$ with the property that $I_\epsilon^{n_0+1} \subset [x - \delta_1, x, x + \delta_2]$. In particular, we have that either $I_\epsilon^{n_0+1} \subset I_\beta^{n_0}$ or $I_\epsilon^{n_0+1} \subset I_{\beta+1}^{n_0}$. Thus, using the bounded geometry property of a grid and Remark 3, we obtain that

$$(18) \qquad\qquad B^{-1} B_1 |I_\beta^{n_0}| \leq |I_\epsilon^{n_0+1}| \leq \delta_2 + \delta_1 \;.$$

Since $d^{-1} \leq \delta_2/\delta_1 \leq d$, by inequality (18), we get

$$\begin{aligned} \delta_1 &\geq (1 + D)^{-1}(\delta_2 + \delta_1) \\ (19) \qquad &\geq (1 + D)^{-1} B^{-1} B_1 |I_\beta^{n_0}| \;. \end{aligned}$$

Since $[x - \delta_1, x + \delta_2] \subset I_\beta^{n_0} \cup I_{\beta+1}^{n_0}$, by the bounded geometry property of a grid, we obtain that

$$\begin{aligned} \delta_1 &\leq |I_\beta^{n_0}| + |I_{\beta+1}^{n_0}| \\ (20) \qquad &\leq (1 + B)|I_\beta^{n_0}| \;. \end{aligned}$$

By inequalities (19) and (20), there is $A = A(B_0, B_1, d) > 1$ such that

$$(21) \qquad\qquad A^{-1}|I_\beta^{n_0}| \leq \delta_1 \leq A|I_\beta^{n_0}| \;.$$

Similarly, we have

$$(22) \qquad\qquad A^{-1}|I_\beta^{n_0}| \leq \delta_2 \leq A|I_\beta^{n_0}| \;.$$

Take $0 < \theta(\alpha) < 1$ such that $\alpha^{-1} \leq 1 - \theta < 1 + \theta \leq \alpha$. Let $n_1 = n_1(B, B_2, A, \theta)$ be the smallest integer such that

$$(23) \qquad\qquad B_2^{n_1} \leq B^{-1} \theta A^{-1}/2 \;.$$

Let $l < m < r$ be such that $x - \delta_1 \in I_l^{n_0+n_1}$, $x \in I_m^{n_0+n_1}$ and $x + \delta_2 \in I_r^{n_0+n_1}$. Then, by the bounded geometry property of a grid, there is $n_2 = 2Mn_1 \geq 1$ such that $r - l \leq n_2$. Hence, the intervals

$$\begin{aligned} L_1 &= \cup_{i=l+1}^{m-1} I_i^{n_0+n_1} \;, \; L_2 = \cup_{i=l}^{m} I_i^{n_0+n_1} \;, \\ R_1 &= \cup_{i=m+1}^{r-1} I_i^{n_0+n_1} \;, \; R_2 = \cup_{i=m}^{r} I_i^{n_0+n_1} \end{aligned}$$

satisfy property (i) and property (iii) of Lemma 15. Let us prove that the intervals L_1, L_2, R_1 and R_2 satisfy property (ii) of Lemma 15. By the bounded geometry property of a grid and inequality 23, we get

$$(24) \qquad\qquad |I_i^{n_0+n_1}| \leq B B_2^{n_1} |I_\beta^{n_0}| \leq \theta A^{-1} |I_\beta^{n_0}|/2 \;,$$

for all $l \leq i \leq r$. Thus, by inequalities (21) and (24), we get

$$\begin{aligned} |L_1|/\delta_1 &\geq (\delta_1 - |I_l^{n_0+n_1}| - |I_m^{n_0+n_1}|)/\delta_1 \\ &\geq (\delta_1 - \theta A^{-1} |I_\beta^{n_0}|)/\delta_1 \\ (25) \qquad &\geq 1 - \theta \;. \end{aligned}$$

17

Again, by inequalities (21) and (24), we get

$$
\begin{aligned}
|L_2|/\delta_1 &\leq (\delta_1 + |I_l^{n_0+n_1}| + |I_m^{n_0+n_1}|)/\delta_1 \\
&\leq (\delta_1 + \theta A^{-1}|I_\beta^{n_0}|)/\delta_1 \\
&\leq 1 + \theta .
\end{aligned}
$$

(26)

Similarly, using inequalities (22) and (24), we obtain that

(27) $$|R_1|/\delta_2 \geq 1 - \theta \text{ and } |R_2|/\delta_2 \leq 1 + \theta .$$

Noting that $\alpha^{-1} \leq 1 - \theta < 1 + \theta \leq \alpha$ and putting together inequalities (25), (26) and (27), we obtain that the intervals L_1, L_2, R_1 and R_2 satisfy property (ii) of Lemma 15. $\qquad\square$

Proof of Lemma 12. *Let us prove that statement (i) implies statement (ii).* For every level $n \geq 1$ and every $1 \leq \beta < \Omega(n)$, let $x - \delta_1, x, x + \delta_2 \in I$ be such that $I_\beta^n = [x - \delta_1, x]$ and $I_{\beta+1}^n = [x, x + \delta_2]$. Hence,

$$
\frac{r_h(n, \beta)}{r(n, \beta)} = \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)} .
$$

Since $h : I \to J$ is (k, B) quasisymmetric, for some $k = k(B)$, we have

$$
k^{-1} < \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)} \frac{\delta_1}{\delta_2} < k ,
$$

and so, we get

(28) $$k^{-1} < r_h(n, \beta)/r(n, \beta) < k .$$

Since, by the bounded geometry property of a grid $\mathcal{G}_\Omega$, there is $B \geq 1$ such that $B^{-1} \leq r(n, \beta) \leq B$, we get $k^{-1}B^{-1} \leq r_h(n, \beta) \leq kB$.

Let us prove that statement (ii) implies statement (i). Let $B \geq 1$ and $M > 1$ be as in the bounded geometry property of a grid. Let $d \geq 1$. Let $x - \delta_1, x, x + \delta_2 \in I$, be such that $\delta_1 > 0$, $\delta_2 > 0$ and $d^{-1} \leq \delta_2/\delta_1 \leq d$. Let L_1, L_2, R_1 and R_2 be the intervals as constructed in Lemma 15 with the constant $\alpha = 2$ in Lemma 15. Hence, we have that

$$
\begin{aligned}
|L_1| &= |I_l^{n_0+n_1}| \left(1 + \sum_{i=l+1}^{m-2} \prod_{j=l}^{i} r(n_0 + n_1, j) \right) , \\
|L_2| &= |I_l^{n_0+n_1}| \left(1 + \sum_{i=l}^{m-1} \prod_{j=l}^{i} r(n_0 + n_1, j) \right) , \\
|R_1| &= |I_l^{n_0+n_1}| \left(\sum_{i=m}^{r-2} \prod_{j=l}^{i} r(n_0 + n_1, j) \right) , \\
|R_2| &= |I_l^{n_0+n_1}| \left(\sum_{i=m-1}^{r-1} \prod_{j=l}^{i} r(n_0 + n_1, j) \right) .
\end{aligned}
$$

Hence, by monotonicity of the homeomorphism h, we obtain that

(29) $$\frac{|h(R_1)|}{|h(L_2)|} \frac{|L_1|}{|R_2|} \leq \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)} \frac{\delta_1}{\delta_2} \leq \frac{|h(R_2)|}{|h(L_1)|} \frac{|L_2|}{|R_1|} .$$

Since, by the bounded geometry property of a grid, $B^{-1} < r(n_0 + n_1, j) < B$ and, by Lemma 15, $l < m < r$ and $r - l \leq n_2(B, M, d)$, we get that there is $C_1 = C_1(B, n_2) > 1$

18

such that

$$C_1^{-1} \leq \frac{|L_1|}{|R_2|} = \frac{1 + \sum_{i=l+1}^{m-2} \prod_{j=l+1}^{i} r(n_0 + n_1, j)}{\sum_{i=m-1}^{r-1} \prod_{j=l+1}^{i} r(n_0 + n_1, j)} \leq C_1 \ ,$$

$$(30) \qquad C_1^{-1} \leq \frac{|L_2|}{|R_1|} = \frac{1 + \sum_{i=l}^{m-1} \prod_{j=l}^{i} r(n_0 + n_1, j)}{\sum_{i=m}^{r-2} \prod_{j=l}^{i} r(n_0 + n_1, j)} \leq C_1 \ .$$

By inequality (15) of statement (ii), there is $k = k(\mathcal{G}_\Omega) > 1$ such that $k^{-1} < r_h(n_0 + n_1, j) < k$ for every $1 \leq j < \Omega(n_0 + n_1)$. Hence, there is $C_2 = C_2(k, n_2) > 1$ such that

$$C_2^{-1} \leq \frac{|h(R_1)|}{|h(L_2)|} = \frac{\sum_{i=m}^{r-2} \prod_{j=l}^{i} r_h(n_0 + n_1, j)}{1 + \sum_{i=l}^{m-1} \prod_{j=l}^{i} r_h(n_0 + n_1, j)} \leq C_2 \ ,$$

$$(31) \qquad C_2^{-1} \leq \frac{|h(R_2)|}{|h(L_1)|} = \frac{\sum_{i=m}^{r-2} \prod_{j=l}^{i} r_h(n_0 + n_1, j)}{1 + \sum_{i=l}^{m-1} \prod_{j=l}^{i} r_h(n_0 + n_1, j)} \leq C_2 \ .$$

Putting together equations (29), (30) and (31), we obtain that

$$C_1^{-1} C_2^{-1} \leq \frac{|h(R_1)|}{|h(L_2)|} \frac{|L_1|}{|R_2|} \leq \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)} \frac{\delta_1}{\delta_2} \leq \frac{|h(R_2)|}{|h(L_1)|} \frac{|L_2|}{|R_1|} \leq C_1 C_2 \ .$$

$\square$

Proof of Lemma 13: If a homeomorphism $h : I \to J$ satisfies the (d_0, k_0) quasisymmetric condition for some $d_0 \geq 1$ and $k_0 \geq 1$ then h satisfies statement (ii) of Lemma 12 with respect to a symmetric grid (see definition of a symmetric grid in Remark 3). Hence, by statement (i) of Lemma 12, the homeomorphism h is quasisymmetric. $\square$

Proof of Lemma 14. *Let us prove statement (i).* By Lemma 12, there is $C_1 \geq 1$ such that $C_1^{-1} \leq r_h(n, \beta) \leq C_1$ for every level n and every $1 \leq \beta < \Omega(n)$. Therefore, there is $C_2 > 0$ such that, for every level n and every $1 \leq \beta < \Omega(n) - 1$,

$$(32) \qquad |cr_h(n, \beta)| = \left| \log\left((1 + r_h(n, \beta))(1 + r_h(n, \beta + 1))^{-1} \right) \right| \leq C_2 \ .$$

Let us prove statement (ii). By the bounded geometry property of a grid, there is $n_0 \geq 1$ large enough such that the grid intervals $I_1^{n_0}$ and $I_{\Omega(n)-1}^{n_0}$ do not intersect the interval L. The grid $\mathcal{G}_\Omega$ of I induces, by restriction, a grid of the interval $L' = \cup_{\beta=2}^{\Omega(n)-2} I_\beta^{n_0}$ which contains L. Hence, by Lemma 12, it is enough to prove that there is $C_1 \geq 1$ such that $C_1^{-1} \leq r_h(n, \beta) \leq C_1$ for every grid interval $I_\beta^n \subset L'$. Now, we will consider separately the following two possible cases: either (i) $r_h(n, \beta) \leq 1$ or (ii) $r_h(n, \beta) > 1$.

Case (i). Let $r_h(n, \beta) = |J_{\beta+1}^n|/|J_\beta^n| \leq 1$. By hypotheses of statement (ii), there is $C_2 > 1$ such that

$$cr_h(n, \beta - 1) = \log\left(1 + \frac{|J_\beta^n|}{|J_{\beta-1}^n|} \frac{|J_{\beta-1}^n| + |J_\beta^n| + |J_{\beta+1}^n|}{|J_{\beta+1}^n|} \right) \leq C_2 \ .$$

Hence, there is $C_3 > 1$ such that

$$1 \leq \frac{|J_\beta^n|}{|J_{\beta+1}^n|} \leq \frac{|J_\beta^n|}{|J_{\beta+1}^n|} \frac{|J_{\beta-1}^n| + |J_\beta^n| + |J_{\beta+1}^n|}{|J_{\beta-1}^n|} \leq C_3 \ ,$$

and so $C_3^{-1} \leq r_h(n, \beta) \leq 1$.

Case (ii). Let $r_h(n, \beta) = |J_{\beta+1}^n|/|J_\beta^n| > 1$. By hypotheses, there is $C_2 > 1$ such that

$$cr_h(n, \beta) = \log\left(1 + \frac{|J_{\beta+1}^n|}{|J_\beta^n|} \frac{|J_\beta^n| + |J_{\beta+1}^n| + |J_{\beta+2}^n|}{|J_{\beta+2}^n|} \right) \leq C_2 \ .$$

Hence, there is $C_3 > 1$ such that

$$1 \leq \frac{|J_{\beta+1}^n|}{|J_\beta^n|} \leq \frac{|J_{\beta+1}^n|}{|J_\beta^n|} \frac{|J_\beta^n| + |I_{\beta+1}^n| + |I_{\beta+2}^n|}{|J_{\beta+2}^n|} \leq C_3 \;,$$

and so $1 < r_h(n, \beta) \leq C_3$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3.2. **Horizontal and vertical translations of ratio distortions.** Lemmas 16 and 17 are the key to understand the relations between ratio and cross ratio distortions. We will use them in the following subsections.

Lemma 16. *Let $h : I \subset \mathbb{R} \to J \subset \mathbb{R}$ be a quasisymmetric homeomorphism and $\mathcal{G}_\Omega$ a grid of the closed interval I. Then, the logarithmic ratio distortion and cross ratio distortion satisfy the following estimates:*

$$(33) \qquad\qquad \frac{r_h(n, \beta)}{r(n, \beta)} \;\in\; 1 + lrd(n, \beta) \pm \mathrm{O}(lrd(n, \beta)^2)$$

$$(34) \qquad\qquad \frac{r(n, \beta)}{r_h(n, \beta)} \;\in\; 1 - lrd(n, \beta) \pm \mathrm{O}(lrd(n, \beta)^2)$$

$$
\begin{aligned}
crd(n, \beta) \;\in\;\; & \frac{lrd(n, \beta)}{1 + r(n, \beta)^{-1}} - \frac{lrd(n, \beta + 1)}{1 + r(n, \beta + 1)} \pm \mathrm{O}(lrd(n, \beta)^2, lrd(n, \beta + 1)^2) \\
(35) \qquad =\;\; & \frac{|I_{\beta+1}^n| lrd(n, \beta)}{|I_\beta^n| + |I_{\beta+1}^n|} - \frac{|I_{\beta+1}^n| lrd(n, \beta + 1)}{|I_{\beta+1}^n| + |I_{\beta+2}^n|} \pm \mathrm{O}(lrd(n, \beta)^2, lrd(n, \beta + 1)^2) \;.
\end{aligned}
$$

In what follows, we will use the following notations:

$$
\begin{aligned}
L_1(n, \beta, p) \;&=\; \max_{0 \leq i \leq p} \{ lrd(n, \beta + i)^2 \} \\
L_2(n, \beta, p) \;&=\; \max_{0 \leq i_1 \leq i_2 < p} \{ |lrd(n, \beta + i_1) lrd(n, \beta + i_2)| \} \\
C(n, \beta, p) \;&=\; \max_{0 \leq i < p} \{ |crd(n, \beta + i)| \} \\
M_1(n, \beta, p) \;&=\; \max\{ L_1(n, \beta, p), C(n, \beta, p) \} \\
M_2(n, \beta, p) \;&=\; \max\{ L_2(n, \beta, p), C(n, \beta, p) \} \;.
\end{aligned}
$$

Lemma 17. *Let $h : I \subset \mathbb{R} \to J \subset \mathbb{R}$ be a quasisymmetric homeomorphism and let $\mathcal{G}_\Omega$ be a grid of the closed interval I. Then, the logarithmic ratio distortion and the cross ratio distortion satisfy the following estimates:*

> (i) *(lrd-horizontal translations) There is a constant $C(i) > 0$, not depending upon the level n and not depending upon $1 \leq \beta \leq \Omega(n)$, such that*

$$
\begin{aligned}
lrd(n, \beta + i) \;\in\;\; & \left(\prod_{k=0}^{i-1} r(n, \beta + k) \right) \frac{1 + r(n, \beta + i)}{1 + r(n, \beta)} lrd(n, \beta) \pm C(i) M_1(n, \beta, i) \\
(36) \qquad =\;\; & \frac{|I_{\beta+i}^n| + |I_{\beta+i+1}^n|}{|I_\beta^n| + |I_{\beta+1}^n|} lrd(n, \beta) \pm C(i) M_1(n, \beta, i) \;.
\end{aligned}
$$

> (ii) *(lrd-vertical translations) Let I_α^{n-1} and $I_{\alpha+1}^{n-1}$ be two adjacent grid intervals. Take $\beta = \beta(n, \alpha)$ and $p = p(n, \alpha)$ such that $I_\beta^n, \ldots, I_{\beta+p}^n$ are all the grid intervals contained in the union $I_\alpha^{n-1} \cup I_{\alpha+1}^{n-1}$. Then, for every $0 \leq i < p$, we have*

$$(37) \qquad lrd(n - 1, \alpha) \in \frac{|I_\alpha^{n-1}| + |I_{\alpha+1}^{n-1}|}{|I_{\beta+i}^n| + |I_{\beta+i+1}^n|} lrd(n, \beta + i) \pm \mathrm{O}(M_2(n, \beta, p)) \;.$$

Proof of Lemma 16: *Let us prove inequality (33).* By Taylor series expansion, we have that $\log(x) = x - 1 \pm O((\log x)^2)$ for every x in a small neighbourhood of 1. Hence, using that h is quasisymmetric, we get

$$
\begin{aligned}
lrd(n, \beta) &= \log\left(\frac{r_h(n, \beta)}{r(n, \beta)}\right) \\
&\in \frac{r_h(n, \beta)}{r(n, \beta)} - 1 \pm O(lrd(n, \beta)^2) \ .
\end{aligned}
$$

Let us prove inequality (34). By Taylor series expansion, we have that $1/(1+x) \in 1 - x \pm O(x^2)$ for every x in a small neighbourhood of 0. Thus, using that h is quasisymmetric, we obtain that

$$
\begin{aligned}
\frac{r(n, \beta)}{r_h(n, \beta)} &= \frac{1}{1 + r_h(n, \beta)r(n, \beta)^{-1} - 1} \\
&\in 1 - \left(\frac{r_h(n, \beta)}{r(n, \beta)} - 1\right) \pm O\left(\left(\frac{r_h(n, \beta)}{r(n, \beta)} - 1\right)^2\right) \ .
\end{aligned}
$$

Hence, using inequality (33), we get

$$
\frac{r(n, \beta)}{r_h(n, \beta)} \in 1 - lrd(n, \beta) \pm O(lrd(n, \beta)^2) \ .
$$

Let us prove inequality (35). By definition of cross ratio distortion, we have

$$
crd(n, \beta) = \log\frac{1 + r_h(n, \beta)}{1 + r(n, \beta)} + \log\frac{1 + r_h(n, \beta + 1)^{-1}}{1 + r(n, \beta + 1)^{-1}} \ .
$$

By Taylor series expansion, we have that $\log(x + 1) = x \pm O(x^2)$ for every x in a small neighbourhood of 0. By the bounded geometry property of a grid, there is $C > 1$ such that $C^{-1} \leq 1 + r(n, \beta)^{-1} \leq C$ for every level $n \geq 1$ and $\beta = 1, \ldots, \Omega(n)$. Hence, using inequality (33), we get

$$
\begin{aligned}
\log\frac{1 + r_h(n, \beta)}{1 + r(n, \beta)} &= \log\left(1 + \frac{r_h(n, \beta)r(n, \beta)^{-1} - 1}{1 + r(n, \beta)^{-1}}\right) \\
\text{(38)} \qquad &\in \frac{lrd(n, \beta))}{1 + r(n, \beta)^{-1}} \pm O(lrd(n, \beta)^2) \ .
\end{aligned}
$$

Similarly, using inequality (34), we obtain

$$
\begin{aligned}
\log\frac{1 + r_h(n, \beta + 1)^{-1}}{1 + r(n, \beta + 1)^{-1}} &= \log\left(1 + \frac{r(n, \beta + 1)r_h(n, \beta + 1)^{-1} - 1}{1 + r(n, \beta + 1)}\right) \\
\text{(39)} \qquad &\in \frac{-lrd(n, \beta + 1)}{1 + r(n, \beta + 1)} \pm O(lrd(n, \beta + 1)^2) \ .
\end{aligned}
$$

Putting together equations (38) and (39), we get

$$
\begin{aligned}
crd(n, \beta) &= \log\frac{1 + r_h(n, \beta)}{1 + r(n, \beta)} + \log\frac{1 + r_h(n, \beta + 1)^{-1}}{1 + r(n, \beta + 1)^{-1}} \\
&\in \frac{lrd(n, \beta))}{1 + r(n, \beta)^{-1}} - \frac{lrd(n, \beta + 1)}{1 + r(n, \beta + 1)} \pm O(lrd(n, \beta)^2, lrd(n, \beta + 1)^2) \ .
\end{aligned}
$$

$\square$

Proof of Lemma 17. *Let us prove inequality (36).* Using inequality (35), we get

$$
\begin{aligned}
lrd(n, \beta + i + 1) &\in lrd(n, \beta + i)\frac{1 + r(n, \beta + i + 1)}{1 + r(n, \beta + i)^{-1}} \pm O(M_1(n, \beta + i, 1)) \\
&\subset lrd(n, \beta + i)r(n, \beta + i)\frac{1 + r(n, \beta + i + 1)}{1 + r(n, \beta + i)} \pm O(M_1(n, \beta + i, 1)) \ .
\end{aligned}
$$

Hence, we obtain

$$lrd(n, \beta+i) \quad \in \quad lrd(n,\beta) \prod_{k=0}^{i-1} \left(r(n, \beta+k) \frac{1 + r(n, \beta+k+1)}{1 + r(n, \beta+k)} \right) \pm C(i) M_1(n, \beta, i)$$

$$\subset \quad lrd(n,\beta) \frac{1 + r(n, \beta+i)}{1 + r(n, \beta)} \prod_{k=0}^{i-1} r(n, \beta+k) \pm C(i) M_1(n, \beta, i) \;,$$

where the constant $C(i) > 0$ does not depend upon n and upon $1 \le \beta \le \Omega(n)$. Since

$$\frac{1 + r(n, \beta+i)}{1 + r(n, \beta)} \prod_{k=0}^{i-1} r(n, \beta+k) = \frac{|I_{\beta+i}^n| + |I_{\beta+i+1}^n|}{|I_\beta^n| + |I_{\beta+1}^n|} \;,$$

we get

$$lrd(n, \beta+i) \quad \in \quad lrd(n,\beta) \frac{1 + r(n, \beta+i)}{1 + r(n, \beta)} \prod_{k=0}^{i-1} r(n, \beta+k) \pm C(i) M_1(n, \beta, i)$$

$$= \quad lrd(n,\beta) \frac{|I_{\beta+i}^n| + |I_{\beta+i+1}^n|}{|I_\beta^n| + |I_{\beta+1}^n|} \pm C(i) M_1(n, \beta, i) \;.$$

Let us prove inequality (37). Let $0 < m = m(n, \alpha) < p$ be such that $I_\beta^n, \ldots, I_{\beta+m}^n$ are all the grid intervals contained in I_α^{n-1} and $I_{\beta+m+1}^n, \ldots, I_{\beta+p}^n$ are all the grid intervals contained in I_α^{n-1}. For simplicity of exposition, we introduce the following definitions:

(i) We define $a_0 = 0$, $a_{h,0} = 0$ and, for every $0 < j < p$, we define

$$a_j = \frac{|I_{\beta+j}^n|}{|I_\beta^n|} = \prod_{i=0}^{j-1} r(n, \beta+i) \text{ and } a_{h,j} = \frac{|J_{\beta+j}^n|}{|J_\beta^n|} = \prod_{i=0}^{j-1} r_h(n, \beta+i) \;.$$

(ii) We define

$$R = \frac{|I_\alpha^{n-1}|}{|I_\beta^n|} \;, \quad R' = \frac{|I_{\alpha+1}^{n-1}|}{|I_\beta^n|} \;, \quad R_h = \frac{|J_\alpha^{n-1}|}{|J_\beta^n|} \;, \quad R_h' = \frac{|J_{\alpha+1}^{n-1}|}{|J_\beta^n|} \;.$$

Thus,

$$R = \sum_{j=0}^{m-1} a_j \;, \quad R' = \sum_{j=m}^{p-1} a_j \;, \quad R_h = \sum_{j=0}^{m-1} a_{h,j} \;, \quad R_h' = \sum_{j=m}^{p-1} a_{h,j} \;.$$

(iii) We define

$$E = \sum_{j=1}^{m-1} a_j \left(\sum_{i=0}^{j-1} lrd(n, \beta+i) \right) \text{ and } E' = \sum_{j=m}^{p-1} a_j \left(\sum_{i=0}^{j-1} lrd(n, \beta+i) \right) \;.$$

We will separate the proof of inequality (37) in three parts. In the first part, we will prove that

(40) $$lrd(n-1, \alpha) \in \frac{E'}{R'} - \frac{E}{R} \pm \mathrm{O}(L_2(n, \beta, p)) \;.$$

In the second part, we will prove that

(41) $$\frac{E'}{R'} - \frac{E}{R} \in lrd(n,\beta) \frac{|I_\alpha^n| + |I_{\alpha+1}^n|}{|I_\beta^n| + |I_{\beta+1}^n|} \pm \mathrm{O}(M_1(n, \beta, p)) \;.$$

In the third part, we will use the previous parts to prove inequality (37) in the case where $i = 0$. Then, we will use inequality (36) to extend, for every $0 \le i < p$, the proof of inequality (37).

First part. By inequality (33), we have that

$$r_h(n, \beta+i) \in r(n, \beta+i)(1 + lrd(n, \beta+i)) \pm \mathrm{O}(lrd((n, \beta+i)^2) \;.$$

22

Hence, for every $1 \le j < p$, we get

$$
\begin{aligned}
a_{h,j} &= \prod_{i=0}^{j-1} r_h(n, \beta + i) \\
&\in \prod_{i=0}^{j-1} \left(r(n, \beta + i)(1 + lrd(n, \beta + i)) \pm \mathrm{O}(lrd((n, \beta + i)^2)) \right) \\
&\subset \prod_{i=0}^{j-1} r(n, \beta + i) \left(1 + \sum_{i=0}^{j-1} lrd(n, \beta + i) \pm \mathrm{O}(L_2(n, \beta + 1, j)) \right) \\
&\subset a_j + a_j \sum_{i=0}^{j-1} lrd(n, \beta + i) \pm \mathrm{O}\left(a_j L_2(n, \beta + 1, j)\right) .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
R_h &= \sum_{j=0}^{m-1} a_{h,j} \\
&\in \sum_{j=0}^{m-1} a_j + \sum_{j=1}^{m-1} a_j \sum_{i=0}^{j-1} lrd(n, \beta + i) \pm \mathrm{O}\left(\sum_{j=0}^{m-1} a_j L_2(n, \beta, j) \right) \\
(42) \qquad &\subset R + E \pm \mathrm{O}(RL_2(n, \beta, m))) .
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
R'_h &= \sum_{j=m}^{p-1} a_{h,j} \\
&\in \sum_{j=m}^{p-1} a_j + \sum_{j=m}^{p-1} a_j \sum_{i=0}^{j-1} lrd(n, \beta + i) \pm \mathrm{O}\left(\sum_{j=m}^{n-1} a_j L_2(n, \beta, j) \right) \\
(43) \qquad &\subset R' + E' \pm \mathrm{O}(R' L_2(n, \beta, p)) .
\end{aligned}
$$

By inequalities (42) and (43), we obtain that

$$
\begin{aligned}
lrd(n - 1, \alpha) &= \log \frac{R'_h}{R'} \frac{R}{R_h} \\
&\in \log \frac{R' + E' \pm \mathrm{O}(R' L_2(n, \beta, p))}{R'} - \log \frac{R + E \pm \mathrm{O}(RL_2(n, \beta, m))}{R} \\
&\subset \frac{E'}{R'} - \frac{E}{R} \pm \mathrm{O}(L_2(n, \beta, p)) .
\end{aligned}
$$

Second part. By inequality (36), for every $1 \le j < p$, we obtain

$$
\begin{aligned}
\sum_{i=0}^{j-1} lrd(n, \beta + i) &\in \sum_{i=0}^{j-1} \left(\frac{a_i(1 + r(n, \beta + i))}{1 + r(n, \beta)} lrd(n, \beta) \pm \mathrm{O}(M_1(n, \beta, i)) \right) \\
&\subset \frac{lrd(n, \beta)}{1 + r(n, \beta)} \sum_{i=0}^{j-1} (a_i + a_{i+1}) \pm \mathrm{O}(M_1(n, \beta, j)) .
\end{aligned}
$$

Hence, we obtain that

$$
\begin{aligned}
E \;\; &= \;\; \sum_{j=1}^{m-1} a_j \sum_{i=0}^{j-1} lrd(n, \beta + i) \\[2mm]
&\in \;\; \sum_{j=1}^{m-1} a_j \left(\frac{lrd(n,\beta)}{1 + r(n,\beta)} \sum_{i=0}^{j-1} (a_i + a_{i+1}) \pm \mathrm{O}(M_1(n,\beta,j)) \right) \\[2mm]
&\subset \;\; \frac{lrd(n,\beta)}{1 + r(n,\beta)} \sum_{j=1}^{m-1} a_j \sum_{i=0}^{j-1} (a_i + a_{i+1}) \pm \mathrm{O}\left(\sum_{j=1}^{m-1} a_j M_1(n,\beta,j) \right) \\[2mm]
&\subset \;\; \frac{lrd(n,\beta)}{1 + r(n,\beta)} R(a_1 + \ldots + a_{m-1}) \pm \mathrm{O}(R M_1(n,\beta,m)) \; .
\end{aligned}
$$
(44)

Similarly, we have

$$
\begin{aligned}
E' \;\; &= \;\; \sum_{j=m}^{p-1} a_j \sum_{i=0}^{j-1} lrd(n, \beta_i) \\[2mm]
&\in \;\; \frac{lrd(n,\beta)}{1 + r(n,\beta)} \sum_{j=m}^{p-1} a_j \sum_{i=0}^{j-1} (a_i + a_{i+1}) \pm \mathrm{O}\left(\sum_{j=m}^{p-1} a_j M_1(n,\beta,j) \right) \\[2mm]
&\subset \;\; \frac{lrd(n,\beta)}{1 + r(n,\beta)} R'(1 + 2a_1 + \ldots + 2a_{m-1} + a_m + \ldots + a_{p-1}) \\[2mm]
&\quad \pm \mathrm{O}(R' M_1(n,\beta,p)) \; .
\end{aligned}
$$
(45)

Putting together inequalities (44) and (45), we obtain that

$$
\begin{aligned}
\frac{E'}{R'} - \frac{E}{R} \;\; &\in \;\; \frac{lrd(n,\beta)}{1 + r(n,\beta)}(1 + a_1 + \ldots + a_{p-1}) \pm \mathrm{O}(M_1(n,\beta,p)) \\[2mm]
&\subset \;\; lrd(n,\beta) \frac{|I_\alpha^n| + |I_{\alpha+1}^n|}{|I_\beta^n| + |I_{\beta+1}^n|} \pm \mathrm{O}(M_1(n,\beta,p)) \; .
\end{aligned}
$$

Third part. In the case where $i = 0$, inequality (37) follows, from putting together inequalities (40) and (41), since

$$
\begin{aligned}
lrd(n-1,\alpha) \;\; &\in \;\; \frac{E'}{R'} - \frac{E}{R} \pm \mathrm{O}(L_2(n,\beta,p)) \\[2mm]
&\subset \;\; lrd(n,\beta) \frac{|I_\alpha^n| + |I_{\alpha+1}^n|}{|I_\beta^n| + |I_{\beta+1}^n|} \pm \mathrm{O}(M_2(n,\beta,p)) \; .
\end{aligned}
$$

By inequality (36), for every $0 < i < p$, we have

$$
\frac{|I_\alpha^{n-1}| + |I_{\alpha+1}^{n-1}|}{|I_\beta^n| + |I_{\beta+1}^n|} lrd(n,\beta) \in \frac{|I_\alpha^{n-1}| + |I_{\alpha+1}^{n-1}|}{|I_{\beta+i}^n| + |I_{\beta+i+1}^n|} lrd(n, \beta + i) \pm \mathrm{O}(M_1(n,\beta,p)) \; .
$$

Thus,

$$
\begin{aligned}
lrd(n-1,\alpha) \;\; &\in \;\; lrd(n,\beta) \frac{|I_\alpha^n| + |I_{\alpha+1}^n|}{|I_\beta^n| + |I_{\beta+1}^n|} \pm \mathrm{O}(M_2(n,\beta,p)) \\[2mm]
&\subset \;\; lrd(n, \beta+i) \frac{|I_\alpha^n| + |I_{\alpha+1}^n|}{|I_{\beta+i}^n| + |I_{\beta+i+1}^n|} \pm \mathrm{O}(M_2(n,\beta,p)) \; .
\end{aligned}
$$

$\square$

3.3. Uniformly asymptotically affine (uaa) homeomorphisms.

The definition of uniformly asymptotically affine homeomorphism that we introduce in this paper is more adapted to our problem and, apparently, is stronger than the usual one for symmetric maps, where the constant d of the (uua) condition in Definition 9, below, is taken to be equal to 1. However, in Lemma 19, we will prove that they are equivalent.

Definition 9. *Let $d \geq 1$ and $\epsilon : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous function with $\epsilon(0) = 0$. The homeomorphism $h : I \to J$ satisfies the (d, ϵ) uniformly asymptotically affine condition if*

$$(46) \qquad \left| \log \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)} \frac{\delta_1}{\delta_2} \right| \leq \epsilon(\delta_1 + \delta_2) \ ,$$

for all $x - \delta_1, x, x + \delta_2 \in I$, such that $\delta_1 > 0$, $\delta_2 > 0$ and $d^{-1} < \delta_2 / \delta_1 < d$. The map h is uniformly asymptotically affine (uaa) if for every $d \geq 1$ there exists ϵ_d such that h satisfies the (d, ϵ_d) uniformly asymptotically affine condition.

Lemma 18. *Let $h : I \to J$ be a homeomorphism and I a compact interval. The following statements are equivalent:*

    (i) The homeomorphism $h : I \to J$ is (uaa).
    (ii) There is a sequence γ_n converging to zero, when n tends to infinity, such that

$$(47) \qquad |lrd(n, \beta)| \leq \gamma_n \ ,$$

    for every $n \geq 1$ and every $1 \leq \beta < \Omega(n)$.

Lemma 19. *If $h : I \to J$ satisfies the (d_0, ϵ_{d_0}) uniformly asymptotically affine condition then the homeomorphism h is (uaa).*

Lemma 20. *Let $h : I \to J$ be a homeomorphism and $\mathcal{G}_\Omega$ a grid of the compact interval I.*

    (i) If $h : I \to J$ is (uaa) then there is a sequence α_n converging to zero, when n tends to infinity, such that

$$|crd(n, \beta)| \leq \alpha_n \ ,$$

    for every $n \geq 1$ and every $1 \leq \beta < \Omega(n) - 1$.
    (ii) If there is a sequence α_n converging to zero, when n tends to infinity, such that for every $n \geq 1$ and every $1 \leq \beta < \Omega(n) - 1$

$$|crd(n, \beta)| \leq \alpha_n \ ,$$

    then, for every closed interval K contained in the interior of I, the homeomorphism h is (uaa) in K.

Proof of Lemma 18. *Let us prove that statement (i) implies statement (ii).* Let $\mathcal{G}_\Omega$ be a (B, M) grid of I. We have that

$$(48) \qquad B^{-1} \leq r(n, \beta) \leq B \ ,$$

for every level $n \geq 1$ and every $1 \leq \beta < \Omega(n)$. For every level $n \geq 1$ and every $1 \leq \beta < \Omega(n)$, let $x - \delta_1, x, x + \delta_2 \in I$ be such that $I_\beta^n = [x - \delta_1, x]$ and $I_{\beta+1}^n = [x, x + \delta_2]$. Hence,

$$\frac{r_h(n, \beta)}{r(n, \beta)} = \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)} \frac{\delta_1}{\delta_2} \ .$$

Since $h : I \to J$ is (B, ϵ_B) uniformly asymptotically affine, we get

$$(49) \qquad lrd(n, \beta) < \epsilon_B(|I_\beta^n| + |I_{\beta+1}^n|) \ .$$

By Remark 3, there is $B_2 = B_2(B, M) < 1$ such that $|I_\beta^n| \leq B_2^n |I|$ and $|I_{\beta+1}^n| \leq B_2^n |I|$. Let $\alpha_n = \epsilon_B(2B_2^n|I|)$. Hence, by inequality (49), we have

$$
\begin{aligned}
lrd(n, \beta) \quad &< \quad \epsilon_B(|I_\beta^n| + |I_{\beta+1}^n|) \\
&< \quad \epsilon_B(2B_2^n|I|) \\
&< \quad \alpha_n \; ,
\end{aligned}
$$

for every n and every $1 \leq \beta < \Omega(n)$. Since $\epsilon_B(0) = 0$ and ϵ_B is continuous at 0, we get that $\alpha_n = \epsilon_B(2B_2^n|I|)$ converges to zero, when n tends to infinity.

Let us prove that statement (ii) implies statement (i). Let $\mathcal{G}_\Omega$ be a (B, M) grid of I. Let $d \geq 1$. Let $x - \delta_1, x, x + \delta_2 \in I$, be such that $\delta_1 > 0$, $\delta_2 > 0$ and $d^{-1} \leq \delta_2/\delta_1 \leq d$. For every $\alpha > 1$, let L_1, L_2, R_1 and R_2 be the intervals as constructed in Lemma 15. By inequality (16) and by monotonicity of the homeomorphism h, we obtain that

$$
(50) \qquad \frac{|h(R_1)|}{|h(L_2)|}\frac{|L_1|}{|R_2|} \leq \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)}\frac{\delta_1}{\delta_2} \leq \frac{|h(R_2)|}{|h(L_1)|}\frac{|L_2|}{|R_1|} \; .
$$

By inequality (17),

$$
(51) \qquad 1 \leq \frac{|L_2|}{|L_1|}\frac{|R_2|}{|R_1|} \leq \alpha^4
$$

By inequalities (50) and (51), we get

$$
(52) \qquad \alpha^{-4}\frac{|h(R_1)|}{|h(L_2)|}\frac{|L_2|}{|R_1|} \leq \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)}\frac{\delta_1}{\delta_2} \leq \alpha^4\frac{|h(R_2)|}{|h(L_1)|}\frac{|L_1|}{|R_2|} \; .
$$

Recalling equality (29) in the proof of Lemma 12, we have

$$
\frac{|h(R_2)|}{|h(L_1)|}\frac{|L_1|}{|R_2|} = \frac{\sum_{i=m-1}^{r-1}\prod_{j=l}^{i} r_h(n_0 + n_1, j)}{\sum_{i=m-1}^{r-1}\prod_{j=l}^{i} r(n_0 + n_1, j)}\frac{1 + \sum_{i=l+1}^{m-2}\prod_{j=l}^{i} r(n_0 + n_1, j)}{1 + \sum_{i=l+1}^{m-2}\prod_{j=l}^{i} r_h(n_0 + n_1, j)} \; ,
$$

$$
(53)
$$

$$
\frac{|h(R_1)|}{|h(L_2)|}\frac{|L_2|}{|R_1|} = \frac{\sum_{i=m}^{r-2}\prod_{j=l}^{i} r_h(n_0 + n_1, j)}{\sum_{i=m}^{r-2}\prod_{j=l}^{i} r(n_0 + n_1, j)}\frac{1 + \sum_{i=l}^{m-1}\prod_{j=l}^{i} r(n_0 + n_1, j)}{1 + \sum_{i=l}^{m-1}\prod_{j=l}^{i} r_h(n_0 + n_1, j)} \; .
$$

By inequality (47), there is $C_0 \geq 1$ and there is a sequence γ_n converging to zero, when n tends to infinity, such that

$$
(54) \qquad \frac{r_h(n_0 + n_1, j)}{r(n_0 + n_1, j)} \in 1 \pm C_0\gamma_{n_0+n_1} \; ,
$$

for every $n_0 + n_1$ and for every $1 \leq j < \Omega(n_0 + n_1)$. Without loss of generality, we will consider that γ_n is a decreasing sequence. Hence, by inequalities (53) and (54), there is $C_1 = C_1(C_0, n_2) > 1$ such that

$$
\left|\log \frac{|h(R_1)|}{|h(L_2)|}\frac{|L_2|}{|R_1|}\right| \leq C_1\gamma_{n_0+n_1} \quad \text{and} \quad \left|\log \frac{|h(R_2)|}{|h(L_1)|}\frac{|L_1|}{|R_2|}\right| \leq C_1\gamma_{n_0+n_1} \; .
$$

Therefore, by inequality (52), we obtain that

$$
\left|\log \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)}\frac{\delta_1}{\delta_2}\right| \leq C_1\gamma_{n_0+n_1} + 4\log(\alpha) \; .
$$

For every $m = 1, 2, \ldots$, let $\alpha_m = \exp(1/8m)$. Hence, we get

$$
(55) \qquad \left|\log \frac{h(x + \delta_2) - h(x)}{h(x) - h(x - \delta_1)}\frac{\delta_1}{\delta_2}\right| \leq C_1\gamma_{n_0+n_1} + 1/(2m) \; .
$$

By Lemma 15, $n_0 = n_0(x - \delta_1, x, x + \delta_2) \geq 1$ is the biggest integer such that $[x - \delta_1, x + \delta_2] \subset I_\beta^{n_0} \cup I_{\beta+1}^{n_0}$. Hence, there is $I_\alpha^{n_0+1} \subset [x - \delta_1, x + \delta_2]$, Thus, $|I_\alpha^{n_0+1}| \leq \delta$, where $\delta = \delta_1 + \delta_2$. By Remark 3, there is $0 < B_1(B, M) < 1$ such that $|I_\alpha^{n_0+1}| \geq B_1^{n_0+1}|I|$. Hence, we get that

$B_1^{n_0+1}|I| \le |I_\alpha^{n_0+1}| \le \delta$, and so

$$n_0 \ge \frac{\log\left(\delta B_1^{-1}|I|^{-1}\right)}{\log(B_1)} \ .$$

Therefore, there is a monotone sequence $\delta_m > 0$ converging to zero, when m tends to infinity, with the following property: if $\delta_1 + \delta_2 \le \delta_m$ then $n_0 = n_0(x - \delta_1, x, x + \delta_2)$ is sufficiently large such that $C_1\gamma_{n_0+n_1} \le 1/(2m)$. Hence, by inequality (55), for every $m \ge 1$ and every $\delta_0 + \delta_1 \le \delta_m$, we have

$$(56) \qquad \left| \log \frac{h(x+\delta_2) - h(x)}{h(x) - h(x-\delta_1)} \frac{\delta_1}{\delta_2} \right| \le C_1\gamma_{n_0+n_1} + 1/(2m) \le 1/m \ .$$

Therefore, we define the continuous function $\epsilon_D : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

  (i) $\epsilon_d(\delta_m) = 1/(m-1)$ for every $m = 2, 3, \ldots$;
  (ii) ϵ_d is affine in every interval $[\delta_m, \delta_m - 1]$;
  (iii) Since I is a compact interval and h is a homoeomorphism, there is an
        extension of ϵ_d to $[\delta_2, \infty)$ such that inequality (46) is satisfied.

By inequality (56), we get that ϵ_d satisfies inequality (46). $\qquad \square$

Proof of Lemma 19: Similarly to the proof that statement (i) implies statement (ii) of Lemma 18, we obtain that if $h : I \to J$ satisfies the (d_0, ϵ_{d_0}) uniformly asymptotically affine condition then satisfies statement (ii) of Lemma 18 with respect to a symmetric grid (see definition of a symmetric grid in Remark 3). Since statement (ii) implies statement (i) of Lemma 12, we get that the homeomorphism h is (uaa). $\qquad \square$

Before proving Lemma 20, we will state and prove Lemma 21 which we will use in the proof of Lemma 20.

Lemma 21. *Let $h : I \subset \mathbb{R} \to J \subset \mathbb{R}$ be a homeomorphism and $\mathcal{G}_\Omega$ a grid of the closed interval I. For every level n and every $0 \le i < \Omega(n) - 1$, let $a(n,i)$ and $b(n,i)$ be given by*

$$a(n,i) = \frac{1 + r_h(n,i)}{1 + r(n,i)} \text{ and } b(n,i) = \exp(-crd(n,i)) \ .$$

(i) Then, for every $1 \le i < \Omega(n) - 1$, we have

$$(57) \qquad a(n,i)a(n,i-1)b(n,i-1) = \frac{r_h(n,i)}{r(n,i)} \ .$$

(ii) Let $n \ge 1$ and $\beta, p \in \{2, \ldots, \Omega(n) - 1\}$ have the following properties:
   (a) There is $\epsilon > 1$ such that $a(n,\beta) \ge \epsilon$.
   (b) There is $\gamma < 1$ such that $\gamma \le b(n, \beta + i) \le \gamma^{-1}$, for every
       $0 \le i < p$.
   Then, for every $1 \le i \le p$, we have

$$a(n, \beta + i) \ \ge \ 1 + \frac{(\epsilon - 1)\gamma^i}{2} \prod_{k=1}^{i} r(n, \beta + k)$$

$$(58) \qquad \qquad + \frac{(\epsilon - 1)\gamma^i B^{-i}}{2} + B(\gamma - 1)\frac{1 - (B\gamma^{-1})^i}{1 - (B\gamma^{-1})} \ ,$$

where $B \ge 1$ is given by the bounded geometry property of the grid.

Proof: *Let us prove inequality (57).* By hypotheses, we have

$$
\begin{aligned}
b(n, i-1) &= \exp(-c r d(n, i-1)) \\
&= \frac{1 + r(n, i-1)}{1 + r_h(n, i-1)} \frac{1 + r(n, i)^{-1}}{1 + r_h(n, i)^{-1}} \\
&= a(n, i-1)^{-1} \frac{1 + r(n, i)}{1 + r_h(n, i)} \frac{r_h(n, i)}{r(n, i)} \\
&= a(n, i-1)^{-1} a(n, i)^{-1} \frac{r_h(n, i)}{r(n, i)} \; .
\end{aligned}
$$

Thus,

$$
b(n, i-1) a(n, i-1) a(n, i) = \frac{r_h(n, i)}{r(n, i)} \; .
$$

Let us prove inequality (58). By definition of $a(n, i)$ and by equality (57), we have

$$
\begin{aligned}
a(n, i) &= \frac{1 + r_h(n, i)}{1 + r(n, i)} \\
b(n, i-1) a(n, i-1) a(n, i) &= \frac{r_h(n, i)}{r(n, i)}
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
a(n, i)(1 + r(n, i)) &= 1 + r_h(n, i) \\
r_h(n, i) &= b(n, i-1) a(n, i-1) a(n, i) r(n, i) \; .
\end{aligned}
$$

Thus,

$$
a(n, i)(1 + r(n, i)) = 1 + b(n, i-1) a(n, i-1) a(n, i) r(n, i) \; ,
$$

and so

$$
a(n, i) = (1 - r(n, i)(b(n, i-1)(a(n, i-1) - 1) + b(n, i-1) - 1)^{-1} \; .
$$

Therefore, for every $n \geq 1$, $\beta, p \in \{2, \ldots, \Omega(n) - 1\}$ and $1 \leq i \leq p$, we get

$$
a(n, \beta + i) - 1 \geq r(n, \beta + i)(b(n, \beta + i - 1)(a(n, \beta + i - 1) - 1) + b(n, \beta + i - 1) - 1) \; .
$$

Hence, by induction in $1 \leq i \leq p$, we get

$$
\begin{aligned}
a(n, \beta + i) - 1 &\geq (a(n, \beta) - 1) \prod_{k=1}^{i} r(n, \beta + k) b(n, \beta + k - 1) \\
(59) \qquad &+ r(n, \beta + i) \sum_{k=1}^{i} (b(n, \beta + k - 1) - 1) \prod_{l=k}^{i-1} r(n, \beta + l) b(n, \beta + l) \; .
\end{aligned}
$$

Using that $B^{-1} < r(n, \beta + k) < B$ by the bounded geometry property of the grid, we get

$$
\begin{aligned}
(a(n, \beta) - 1) \prod_{k=1}^{i} r(n, \beta + k) b(n, \beta + k - 1) &\geq (\epsilon - 1) \gamma^{i} \prod_{k=1}^{i} r(n, \beta + k) \\
(60) \qquad &\geq \frac{(\epsilon - 1)\gamma^{i}}{2} \prod_{k=1}^{i} r(n, \beta + k) + \frac{(\epsilon - 1)\gamma^{i} B^{-i}}{2} \; .
\end{aligned}
$$

Furthermore, noting that $\gamma - 1 < 0$, we have

$$
\begin{aligned}
(61) \qquad r(n, \beta + i) \sum_{k=1}^{i} (b(n, \beta + k - 1) - 1) \prod_{l=k}^{i-1} r(n, \beta + l) b(n, \beta + l) \\
\geq B(\gamma - 1) \sum_{k=1}^{i} (B \gamma^{-1})^{i-k} \\
(62) \qquad \geq B(\gamma - 1) \frac{1 - (B \gamma^{-1})^{i}}{1 - (B \gamma^{-1})} \; .
\end{aligned}
$$

28

Putting inequalities (59), (60) and (61) together, we obtain that

$$a(n, \beta + i) - 1 \geq \frac{(\epsilon - 1)\gamma^i}{2} \prod_{k=1}^{i} r(n, \beta + k) + \frac{(\epsilon - 1)\gamma^i B^{-i}}{2} + B(\gamma - 1)\frac{1 - (B\gamma^{-1})^i}{1 - (B\gamma^{-1})} \ .$$

$\square$

Proof of Lemma 20. *Let us prove statement (i)*. By Lemma 18, there is a sequence α_n converging to zero, when n tends to infinity, such that

(63) $$|lrd(n, \beta)| \leq \gamma_n \ ,$$

for every $n \geq 1$ and every $1 \leq \beta < \Omega(n)$. By inequality (35) in Lemma 16, we have that

(64) $$crd(n, \beta) \in \frac{lrd(n, \beta)}{1 + r(n, \beta)^{-1}} - \frac{lrd(n, \beta + 1)}{1 + r(n, \beta + 1)} \pm \mathrm{O}(lrd(n, \beta)^2, lrd(n, \beta + 1)^2) \ .$$

By the bounded geometry property of a grid, there is $B \geq 1$ such that $B^{-1} \leq r(n, \beta) \leq B$. Thus, there is $C_0 > 1$ such that

(65) $$C_0^{-1} \leq \frac{1}{1 + r(n, \beta)^{-1}} \leq C_0 \text{ and } C_0^{-1} \leq \frac{1}{1 + r(n, \beta + 1)} \leq C_0 \ .$$

Therefore, putting together inequalities (63), (64) and (65), we obtain that there is $C_1 > 1$ such that $|crd(n, \beta)| \leq C_1 \gamma_n$, for every level n and every $1 \leq \beta < \Omega(n) - 1$.

Let us prove statement (ii). Let us suppose, by contradiction, that there is $\epsilon_0 > 0$ such that $|lrd(n(j), \beta(j))| > \epsilon_0$, where $I_{\beta(j)}^{n(j)} \subset K$ and $n(j)$ tends to infinity, when j tends to infinity. Hence, there is a subsequence m_j such that either $lrd(n(m_j), \beta(m_j)) < -\epsilon_0$ for every $j \geq 1$, or $lrd(n(m_j), \beta(m_j)) > \epsilon_0$ for every $j \geq 1$. For simplicity of notation, we will denote $n(m_j)$ by n_j, and $\beta(m_j)$ by β_j. It is enough to consider the case where $lrd(n_j, \beta_j) > \epsilon_0$ (if necessary, after re-ordering all the indices). Thus, there is $\epsilon = \epsilon(\epsilon_0) > 1$ such that, for every $j \geq 1$,

(66) $$\frac{1 + r_h(n_j, \beta_j)}{1 + r(n_j, \beta_j)} > \epsilon \ .$$

Let $a(n, i)$ and $b(n, i)$ be defined as in Lemma 21:

(67) $$a(n, i) = \frac{1 + r_h(n, i)}{1 + r(n, i)}$$

$$b(n, i) = \exp(-crd(n, \beta)) = \frac{1 + r(n, i)}{1 + r_h(n, i)} \frac{1 + r(n, i + 1)^{-1}}{1 + r_h(n, i + 1)^{-1}} \ .$$

Hence, we have that $a(n_j, \beta_j) \geq \epsilon$ for every $j \geq 1$. By hypotheses, the cross ratio distortion $crd(n, \beta)$ converges uniformly to zero when n tends to infinity. Thus, there is an increasing sequence γ_n converging to one, when n tends to infinity, such that

(68) $$\gamma_n \leq b(n, i) \leq \gamma_n^{-1} \ ,$$

for every $1 \leq i < \Omega(n) - 1$. Let $\eta = \min\{(\epsilon - 1)/4, 1/2\}$. For every j large enough, let p_j be the maximal integer with the following properties: (i) $\gamma_{n_j}^{p_j} \geq \eta$; (ii) $\gamma_{n_j}^{p_j}(\epsilon - 1)/2 \geq \eta$; and (iii), letting $B \geq 1$ be as given by the bounded geometry property of the grid,

$$\frac{(\epsilon - 1)\gamma^i B^{-p_j}}{2} \geq B(1 - \gamma)\frac{1 - (B\gamma^{-1})^{p_j}}{1 - (B\gamma^{-1})} \ .$$

Since γ_{n_j} converges to one, when j tends to infinity, we obtain that p_j also tends to infinity, when j tends to infinity. By properties (ii) and (iii) of η and by inequality (58), for every j large enough, and for every $1 \leq i \leq p_j$, we have

(69) $$a(n_j, \beta_j + i) \geq 1 + \eta \prod_{k=1}^{i} r(n_j, \beta_j + k) > 1 \ .$$

For every $j \geq 1$, let N_j be the smallest integer such that there are four grid intervals $I_{\alpha_j-1}^{N_j}$, $I_{\alpha_j}^{N_j}$, $I_{\alpha_j+1}^{N_j}$ and $I_{\alpha_j+2}^{N_j}$ such that

$$I_{\beta_j}^{n_j} \subset I_{\alpha_j-1}^{N_j} \text{ and } I_{\alpha_j}^{N_j} \cup I_{\alpha_j+1}^{N_j} \cup I_{\alpha_j+2}^{N_j} \subset \cup_{i=1}^{p_j-1} I_{\beta_j+i}^{n_j} .$$

Since the grid intervals $I_{\beta_j}^{n_j}, \ldots, I_{\beta_j+p(j)-1}^{n_j}$ are contained in at most four grid intervals at level $N_j - 1$, we obtain that

$$4M^{n_j-(N_j-1)} \geq p_j .$$

where $M > 1$ is given by the bounded geometry property of the grid. Thus, $n_j - N_j$ tends to infinity, when j tends to infinity. Let us denote by $RD(j)$ the following ratio:

$$
\begin{aligned}
RD(j) &= \frac{|I_{\alpha_j}^{N_j}|}{|J_{\alpha_j}^{N_j}|} \frac{|J_{\alpha_j+1}^{N_j}| + |J_{\alpha_j+2}^{N_j}|}{|I_{\alpha_j+1}^{N_j}| + |I_{\alpha_j+2}^{N_j}|} \\
&= \frac{r_h(N_j, \alpha_j)(1 + r_h(N_j, \alpha_j+1))}{r(N_j, \alpha_j)(1 + r(N_j, \alpha_j+1))} .
\end{aligned}
$$

By the bounded geometry property of a grid, we have $B^{-1} < r(N_j, \alpha_j + i) < B$ for every $-1 \leq i \leq 3$ and $j \geq 0$. By Lemma 12 and statement (ii) of Lemma 14, there is $k_0 > 1$ such that $k_0^{-1} < r_h(N_j, \alpha_j + i) < k_0$ for every $-1 \leq i \leq 3$ and $j \geq 0$. Hence, there is $k = k(B, k_0) > 1$ such that for every $j \geq 0$, we have

$$(70) \qquad\qquad\qquad k^{-1} \leq RD(j) \leq k .$$

Now, we are going to prove that $RD(j)$ tends to infinity, when j tends to infinity, and so we will get a contradiction. Let $e_1 < e_2 < e_3 < e_4$ be such that

$$I_{\alpha_j}^{N_j} = \cup_{i=e_1}^{e_2} I_{\beta_j+i}^{n_j} , \quad I_{\alpha_j+1}^{N_j} = \cup_{i=e_2+1}^{e_3} I_{\beta_j+i}^{n_j} , \quad I_{\alpha_j+2}^{N_j} = \cup_{i=e_3+1}^{e_4} I_{\beta_j+i}^{n_j} .$$

Hence, we get

$$(71) \qquad RD(j) = \frac{|I_{\alpha_j}^{N_j}|}{|J_{\alpha_j}^{N_j}|} \frac{|J_{\alpha_j+1}^{N_j}| + |J_{\alpha_j+2}^{N_j}|}{|I_{\alpha_j+1}^{N_j}| + |I_{\alpha_j+2}^{N_j}|} = \frac{R_1(j)}{R_{h,1}(j)} \frac{R_{h,2}(j) + R_{h,3}(j)}{R_2(j) + R_3(j)} ,$$

where

$$
\begin{aligned}
R_1(j) &= \frac{|I_{\alpha_j}^{N_j}|}{|I_{\beta_j+e_2}^{n_j}|} = 1 + \sum_{q=e_1}^{e_2-2} \prod_{i=q+1}^{e_2-1} r(n_j, \beta_j + i)^{-1} \\
R_{h,1}(j) &= \frac{|J_{\alpha_j}^{N_j}|}{|J_{\beta_j+e_2}^{n_j}|} = 1 + \sum_{q=e_1}^{e_2-2} \prod_{i=q+1}^{e_2-1} r_h(n_j, \beta_j + i)^{-1} \\
R_2(j) &= \frac{|I_{\alpha_j+1}^{N_j}|}{|I_{\beta_j+e_2}^{n_j}|} = \sum_{q=e_2}^{e_3-1} \prod_{i=e_2}^{q} r(n_j, \beta_j + i) \\
R_{h,2}(j) &= \frac{|J_{\alpha_j+1}^{N_j}|}{|J_{\beta_j+e_2}^{n_j}|} = \sum_{q=e_2}^{e_3-1} \prod_{i=e_2}^{q} r_h(n_j, \beta_j + i) \\
R_3(j) &= \frac{|I_{\alpha_j+2}^{N_j}|}{|I_{\beta_j+e_2}^{n_j}|} = \sum_{q=e_3}^{e_4-1} \prod_{i=e_2}^{q} r(n_j, \beta_j + i) \\
R_{h,3}(j) &= \frac{|J_{\alpha_j+2}^{N_j}|}{|J_{\beta_j+e_2}^{n_j}|} = \sum_{q=e_3}^{e_4-1} \prod_{i=e_2}^{q} r_h(n_j, \beta_j + i) .
\end{aligned}
$$

Hence, by inequalities (67) and (69), for every $1 \leq i \leq p_j$, we get

$$(72) \qquad\qquad\qquad \frac{r_h(n_j, \beta_j + i)}{r(n_j, \beta_j + i)} > 1 .$$

Thus, we deduce that

$$
\begin{aligned}
R_{h,1}(j) &= 1 + \sum_{q=e_1}^{e_2-2} \prod_{i=q+1}^{e_2-1} r(n_j, \beta_j + i)^{-1} \frac{r(n_j, \beta_j + i)}{r_h(n_j, \beta_j + i)} \\
&\leq 1 + \sum_{q=e_1}^{e_2-2} \prod_{i=q+1}^{e_2-1} r(n_j, \beta_j + i)^{-1} \\
&= R_1(j) \ .
\end{aligned}
$$

(73)

By inequality (72), we obtain

$$
\begin{aligned}
R_{h,2}(j) &= \sum_{q=e_2}^{e_3-1} \prod_{i=e_2}^{q} r(n_j, \beta_j + i) \frac{r_h(n_j, \beta_j + i)}{r(n_j, \beta_j + i)} \\
&\geq R_2(j) \ .
\end{aligned}
$$

(74)

Now, let us bound $R_{h,3}(j)$ in terms of $R_3(j)$. Putting together inequalities (57) and (69), we obtain

$$
\begin{aligned}
\frac{r_h(n,i)}{r(n,i)} &= b(n, i-1)a(n,i)a(n, i-1) \\
&\geq b(n, i-1)a(n,i) \ .
\end{aligned}
$$

(75)

Noting that $e_3 - e_2 < p_j$, and by inequality (68) and property (i) of η, we get

$$
\prod_{i=e_2}^{e_3-1} b(n_j, \beta_j + i - 1) \geq \gamma^{p_j} \geq \eta \ .
$$

Hence, by inequalities (69) and (75), we get

$$
\begin{aligned}
\prod_{i=e_2}^{e_3-1} \frac{r_h(n_j, \beta_j + i)}{r(n_j, \beta_j + i)} &\geq \prod_{i=e_2}^{e_3-1} b(n_j, \beta_j + i - 1)a(n_j, \beta_j + i) \\
&\geq \eta \prod_{i=e_2}^{e_3-1} \left(1 + \eta \prod_{k=1}^{i} r(n_{m_i}, \beta_j + k) \right) \\
&\geq \eta \left(1 + \eta \sum_{i=e_2}^{e_3-1} \prod_{k=1}^{i} r(n_{m_i}, \beta_j + k) \right) \\
&\geq \eta^2 \frac{|I_{\alpha_j+1}^{N_j}|}{|I_{\beta_j+1}^{n_j}|} \ .
\end{aligned}
$$

(76)

Noting that $I_{\beta_j+1}^{n_j} \subset I_{\alpha_j-1}^{N_j} \cup I_{\alpha_j}^{N_j}$ and by the bounded geometry property of the grid, we get

$$
\frac{|I_{\alpha_j+1}^{N_j}|}{|I_{\beta_j+1}^{n_j}|} \geq B^{-2} B_2^{N_j - n_j} \ ,
$$

(77)

where $B_2 < 1$ is given in Remark 3. Putting together inequalities (76) and (77), we obtain that

$$
\prod_{i=e_2}^{e_3-1} \frac{r_h(n_j, \beta_j + i)}{r(n_j, \beta_j + i)} \geq \eta^2 B^{-2} B_2^{N_j - n_j} \ .
$$

Hence,

$$R_{h,3}(j) = \prod_{i=e_2}^{e_3-1} \frac{r_h(n_j, \beta_j + i)}{r(n_j, \beta_j + i)} r(n_j, \beta_j + i) \sum_{q=e_3}^{e_4-1} \prod_{i=e_3}^{q} \frac{r_h(n_j, \beta_j + i)}{r(n_j, \beta_j + i)} r(n_j, \beta_j + i)$$

$$\geq \eta^2 B^{-2} B_2^{N_j - n_j} \prod_{i=e_2}^{e_3-1} r(n_j, \beta_j + i) \sum_{q=e_3}^{e_4-1} \prod_{i=e_3}^{q} r(n_j, \beta_j + i)$$

$$(78) \qquad = \eta^2 B^{-2} B_2^{N_j - n_j} R_3(j) \ .$$

Noting that $R_2(j) R_3(j)^{-1} = |I_{\alpha_j+1}^{N_j}||I_{\alpha_j+2}^{N_j}|^{-1}$ and by the bounded geometry property of the grid, we obtain

$$B^{-1} \leq R_2(j) R_3(j)^{-1} \leq B \ .$$

Therefore, putting together inequalities (73), (74) and (78), we obtain that

$$RD(j) = \frac{R_1(j)}{R_{h,1}(j)} \frac{R_{h,2}(j) + R_{h,3}(j)}{R_2(j) + R_3(j)}$$

$$\geq \frac{R_2(j) + \eta^2 B^{-2} B_2^{N_j - n_j} R_3(j)}{R_2(j) + R_3(j)}$$

$$\geq \frac{1 + \eta^2 B^{-3} B_2^{N_j - n_j}}{1 + B} \ .$$

Since $B_2^{N_j - n_j}$ tends to infinity, when j tends to infinity, we get that $RD(j)$ also tends to infinity, when j tends to infinity. However, by inequality (70), this is absurd. $\qquad \square$

3.4. C^{1+r} **diffeomorphisms.** Let $0 < r \leq 1$. We say that a homeomorphism $h : I \to J$ is C^{1+r} if its differentiable and its first derivative $dh : I \to \mathbb{R}$ is r-Hölder continuous, i.e. there is $C \geq 0$ such that, for every $x, y \in I$,

$$|dh(y) - dh(x)| \leq C|y - x|^r \ .$$

In particular, if $r = 1$ then dh is Lipschitz.

Lemma 22. *Let $h : I \to J$ be a homeomorphism, and let I be a compact interval with a grid $\mathcal{G}_\Omega$.*

(i) *For $0 < r \leq 1$, the map h is a C^{1+r} diffeomorphism if, and only if, for every $n \geq 1$ and for every $1 \leq \beta < \Omega(n)$, we have that*

$$(79) \qquad\qquad |lrd(n, \beta)| \leq \mathrm{O}(|I_\beta^n|^r) \ .$$

(ii) *The map h is affine if, and only if, for every $n \geq 1$ and every $1 \leq \beta < \Omega(n)$, we have that*

$$(80) \qquad\qquad |lrd(n, \beta)| \leq \mathrm{o}(|I_\beta^n|) \ .$$

Lemma 23. *Let $0 < r \leq 1$. Let $h : I \to J$ be a homeomorphism and $\mathcal{G}_\Omega$ a grid of the compact interval I.*

(i) *If $h : I \to J$ is a C^{1+r} diffeomorphism then, for every $n \geq 1$ and every $1 \leq \beta < \Omega(n) - 1$, we have that*

$$(81) \qquad\qquad |crd(n, \beta)| \leq \mathrm{O}(|I_\beta^n|^r) \ .$$

(ii) *If, for every $n \geq 1$ and every $1 \leq \beta < \Omega(n) - 1$, we have that*

$$(82) \qquad\qquad |crd(n, \beta)| \leq \mathrm{O}(|I_\beta^n|^r) \ ,$$

then, for every closed interval K contained in the interior of I, the homeomorphism $h|K$ restricted to K is a C^{1+r} diffeomorphism.

Proof of Lemma 22: By the Mean Value Theorem, if h is a C^{1+r} diffeomorphism then for every $n \geq 1$ and for every grid interval I_β^n we get that $lrd(n, \beta) \in \pm O(|I_\beta^n|^r)$, and so inequality (79) is satisfied. If h is affine then, for every $n \geq 1$ and for every grid interval I_β^n, we get that $lrd(n, \beta) = 0$, and so inequality (80) is satisfied.

Let us prove that inequality (79) implies that h is C^{1+r}. For every point $P \in I$, let $I_{\alpha_1}^1, I_{\alpha_2}^2, \ldots$ be a sequence of grid intervals $I_{\alpha_n}^n$ such that $P \in I_{\alpha_n}^n$ and $I_{\alpha_n}^n \subset I_{\alpha_{n-1}}^{n-1}$ for every $n > 1$. Let us suppose that $I_{\alpha_{n-1}}^{n-1} = \cup_{i=0}^j I_{\alpha_n+i}^n$ for some $j = j(\alpha_n) \geq 1$. By inequality (79) and using the bounded geometry of the grid, we obtain that

$$
\begin{aligned}
\frac{dh(n-1, \alpha_{n-1})}{dh(n, \alpha_n)} &= \frac{1 + \sum_{i=1}^j \prod_{k=1}^i r_h(n, \alpha_n + i)}{1 + \sum_{i=1}^j \prod_{k=1}^i r(n, \alpha_n + i)} \\
&\in \frac{1 + \sum_{i=1}^j \prod_{k=1}^i r(n, \alpha_n + i)(1 \pm O(|I_{\alpha_n+i}^n|))}{1 + \sum_{i=1}^j \prod_{k=1}^i r(n\alpha_n + i)} \\
&\subset O(|I_{\alpha_n}^n|^r) \ .
\end{aligned}
$$

A similar argument to the one above implies that for all $I_{\alpha_n}^n \subset I_{\alpha_{n-1}}^{n-1}$, we have

$$
dh(n, \alpha_n) \in dh(n-1, \alpha_{n-1}) \pm O(|I_{\alpha_{n-1}}^{n-1}|^r) \ .
$$

Hence, using the bounded geometry property of a grid, for every $m \geq 1$ and for every $n \geq m$, we get

$$
(83) \qquad dh(n, \alpha_n) \in dh(m, \alpha_m) \pm O(|I_{\alpha_m}^m|^r) \ .
$$

Thus, the average derivative $dh(n, \alpha_n)$ converges to a value d_P, when n tends to infinity. Let us prove that h is differentiable at P and that $dh(P) = d_P$. Let L be any interval such that the point $P \in L$. Take the largest $m \geq 1$ such that there is a grid interval I_γ^m with the property that $L \subset \cup_{j=-1,0,1} I_{\gamma+j}^m$. By the bounded geometry property of a grid, there is $C \geq 1$, not depending upon P, L and I_γ^m, such that

$$
(84) \qquad C^{-1} \leq \frac{|I_\gamma^m|}{|L|} \leq C \ .
$$

Then, using inequality (79) and the bounded geometry of the grid, for every $j = \{-1, 0, 1\}$, we obtain that

$$
|ldh(m, \gamma + j) - ldh(m, \gamma)| \leq O(|L|^r) \ ,
$$

and so

$$
(85) \qquad dh(m, \gamma + j) \in dh(m, \gamma) \pm O(|L|^r) \ .
$$

For every $n \geq m$, take the smallest sequence of adjacent grid intervals $I_{\beta_n}^n, \ldots, I_{\beta_n+i_n}^n$, at level n, such that $L \subset \cup_{i=0}^{i_n} I_{\beta_n+i}^n \subset \cup_{j=-1,0,1} I_{\gamma+j}^m$. By inequalities (83) and (85), for every $I_{\beta_n+i}^n \subset I_{\gamma+j(i)}^m$ we get that

$$
\begin{aligned}
dh(m, \beta_n + i) &\in dh(m, \gamma + j(i)) \pm O(|I_{\gamma+j(i)}^m|^r) \\
&\subset d_P \pm O(|L|^r) \ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{h(L)}{L} &= \lim_{n\to\infty} \sum_{i=0}^{i_n} \frac{|I_{\beta_n+i}^n|}{|L|} dh(n, \beta_n + i) \\
&\in \lim_{n\to\infty} \sum_{i=0}^{i_n} \frac{|I_{\beta_n+i}^n|}{|L|} (d_P \pm O(|L|^r)) \\
(86) \qquad &\subset d_P \pm O(|L|^r) \ .
\end{aligned}
$$

33

Therefore, for every $P \in I$, the homeomorphism h is differentiable at P and $dh(P) = d_P$. Let us check that dh is r-Hölder continuous. For every $P, P' \in I$, let L be the closed interval $[P, P']$. Using inequality (86), we obtain that

$$
\begin{aligned}
dh(P') - dh(P) \quad &\in \quad \frac{h(L)}{L} - \frac{h(L)}{L} \pm \mathrm{O}(|L|^r) \\
&\subset \quad \pm \mathrm{O}(|L|^r) \ ,
\end{aligned}
$$

and so dh is r-Hölder continuous.

Let us prove that inequality (80) implies that h is affine. A similar argument to the one above gives us that h is differentiable and that

$$
(87) \qquad\qquad |dh(P') - dh(P)| \leq \mathrm{o}(|P' - P|) \ ,
$$

for every $P, P' \in I$. Hence, we get that

$$
\begin{aligned}
|dh(P') - dh(P)| \quad &\leq \quad \lim_{n \to \infty} \sum_{i=0}^{n-1} \left| dh\left(P + \frac{(i+1)(P'-P)}{n} \right) - dh\left(P + \frac{i(P'-P)}{n} \right) \right| \\
&\leq \quad \lim_{n \to \infty} n \, \mathrm{o}\left(\frac{P'-P}{n} \right) = 0 \ ,
\end{aligned}
$$

and so h is an affine map. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Proof of Lemma 23: *Proof of statement (i)*: By Lemma 22, for every $n \geq 1$ and for every $1 \leq \beta < \Omega(n)$, we have that $|lrd(n, \beta)| \leq \mathrm{O}(|I_\beta^n|^r)$. Hence, by the bounded geometry property of a grid and by inequality (35), we get $|crd(n, \beta)| \leq \mathrm{O}(|I_\beta^n|^r)$.

Proof of statement (ii): Let K be a closed interval contained in the interior of I. By Lemmas 18 and 20, there is a decreasing sequence of positive reals ϵ_n which converges to 0, when n tends to ∞, such that

$$
(88) \qquad\qquad |lrd(n, \beta)| < |\epsilon_n| \ ,
$$

for all $n \geq 1$ and for all grid interval I_β^n intersecting K. For every grid interval I_α^{n-1} intersecting K, let $k_1 = k_1(n, \alpha)$ and $k_2 = k_2(n, \alpha)$ be such that $\cup_{\beta=k_1}^{k_2} I_\beta^n = I_\alpha^{n-1} \cup I_{\alpha+1}^{n-1}$. Let the integers β and i be such that $k_1 \leq \beta \leq k_2$ and $k_1 \leq \beta + i \leq k_2$. By the bounded geometry property of a grid, and by inequalities (36) and (81), we get

$$
lrd(n, \beta + i) \in \pm \mathrm{O}\left(|lrd(n, \beta)| + (|I_\beta^n| + |I_{\beta+1}^n|)^r \right) \ .
$$

Therefore,

$$
(89) \qquad\qquad L_2(n, \beta, p) \in \pm \mathrm{O}\left(lrd(n, \beta)^2 + (|I_\beta^n| + |I_{\beta+1}^n|)^{2r} \right) \ .
$$

By inequalities (37) and (89), we get

$$
(90) \ \ lrd(n-1, \alpha) \in \frac{|I_\alpha^{n-1}| + |I_{\alpha+1}^{n-1}|}{|I_{\beta+i}^n| + |I_{\beta+i+1}^n|} lrd(n, \beta+i) \pm \mathrm{O}\left(lrd(n, \beta)^2 + (|I_\beta^n| + |I_{\beta+1}^n|)^r \right) \ .
$$

Let us suppose, by contradiction, that there is a sequence of grid intervals $I_{\beta_j}^{n_j}$ and a sequence of positive reals $|e_j|$ which tends to infinity, when j tends to infinity, such that

$$
(91) \qquad\qquad lrd(n_j, \beta_j) = e_j(|I_{\beta_j}^{n_j}| + |I_{\beta_j+1}^{n_j}|)^r \ .
$$

Using that the number of grid intervals at every level n is finite, we obtain that there exists a subsequence m_j of j such that $I_{\beta_{m_{j+1}}}^{n_{m_{j+1}}} \subset I_{\beta_{m_j}}^{n_{m_j}}$. Therefore, there exists a sequence of grid intervals $I_{\alpha_1}^1, I_{\alpha_2}^2, \ldots$ with the following properties:

    (i) for every $i \geq 1$, $I_{\alpha_{i+1}}^{i+1} \subset I_{\alpha_i}^i$;

(ii) for every $i \geq 1$, let a_i be determined such that

$$(92) \qquad lrd(i, \alpha_i) = a_i(|I_{\alpha_i}^i| + |I_{\alpha_i+1}^i|)^r \ .$$

Then, there is a subsequence m_j of j such that $|a_i| \leq |a_{m_j}|$ for every $1 \leq i \leq m_j$, and $|a_{m_j}|$ tends to infinity, when j tends to infinity.

Let us denote $|I_{\alpha_i}^i| + |I_{\alpha_i+1}^i|$ by B_i. Using inequality (90) inductively, we get

$$(93) \qquad lrd(m_j, \beta_{m_j}) \quad \in \quad \frac{B_{m_j}}{B_1} lrd(1, \alpha_1) \pm \mathrm{O}\left(\sum_{i=2}^{m_j} \frac{B_{m_j}}{B_i}(lrd(i, \alpha_i)^2 + B_i^r)\right) \ .$$

By the bounded geometry property of a grid, there is $0 < \theta < 1$ such that

$$(94) \qquad \frac{B_k}{B_i} \leq \theta^{k-i} \ ,$$

for every $1 \leq i \leq m_j$ and for every $1 \leq k \leq m_j$. Noting that $|a_1| \leq |a_{m_j}|$, by inequalities (92) and (94), we get

$$\frac{B_{m_j}}{B_1} lrd(1, \alpha_1) \quad = \quad \frac{a_1 B_1^r B_{m_j}}{B_1}$$

$$(95) \qquad\qquad\qquad\qquad \in \quad \pm \mathrm{O}\left(|a_{m_j}| B_{m_j}^r \theta^{(1-r)m_j}\right)$$

By inequality (88), $a_i B_i \leq \epsilon_i$, and $|a_i| \leq |a_{m_j}|$ for $i \leq m_j$. Hence, by inequalities (92) and (94), we obtain that

$$\frac{B_{m_j}}{B_i}(lrd(i, \alpha_i)^2 + B_i^r) \quad = \quad \frac{a_i(a_i B_i^r)(B_i^r B_{m_j}) + B_i^r B_{m_j}}{B_i}$$

$$(96) \qquad\qquad\qquad\qquad \in \quad \pm \mathrm{O}\left((|a_{m_j}|\epsilon_i + 1)B_{m_j}^r \theta^{(1-r)(m_j-i)}\right)$$

Using inequalities (95) and (96) in inequality (93), we get

$$\frac{|lrd(m_j, \beta_{m_j})|}{|a_{m_j}| B_{m_j}^r} \quad \leq \quad \mathrm{O}\left(\theta^{(1-r)m_j} + \sum_{i=2}^{m_j}\left((\epsilon_i + |a_{m_j}|^{-1})\theta^{(1-r)(m_j-i)}\right)\right)$$

$$(97) \qquad\qquad\qquad \leq \quad \mathrm{O}\left(\theta^{(1-r)m_j} + \frac{|a_{m_j}|^{-1}}{1 - \theta^{1-r}} + \sum_{i=2}^{m_j}\left(\epsilon_i \theta^{(1-r)(m_j-i)}\right)\right) \ .$$

Since ϵ_i converges to zero, when i tends to infinity, inequality (97) implies that there is $j_0 \geq 0$ such that, for every $j \geq j_0$, we get

$$|lrd(m_j, \beta_{m_j})| < |a_{m_j}| B_{m_j}^r \ ,$$

which contradicts (92). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3.5. C^{2+r} diffeomorphisms. Let $0 < r \leq 1$. We say that a homeomorphism $h : I \to J$ is C^{2+r} if its twice differentiable and its second derivative $d^2 h : I \to \mathbb{R}$ is r-Hölder continuous.

Lemma 24. *Let $0 < r \leq 1$. Let $h : I \to J$ be a homeomorphism and $\mathcal{G}_\Omega$ a grid of the compact interval I.*

(i) *If $h : I \to J$ is C^{2+r} then*

$$|crd(n, \beta)| \leq \mathrm{O}(|I_\beta^n|^{1+r}) \ ,$$

for every $n \geq 1$ and every $1 \leq \beta < \Omega(n) - 1$.

(ii) *If, for every $n \geq 1$ and every $1 \leq \beta < \Omega(n) - 1$, we have that*

$$(98) \qquad |crd(n, \beta)| \leq \mathrm{O}(|I_\beta^n|^{1+r}) \ ,$$

then, for every closed interval K contained in the interior of I, the homeomorphism $h|K$ restricted to K is C^{2+r}.

Before proving Lemma 24, we will state and prove Lemma 25 which we will use later in the proof of Lemma 24.

Lemma 25. *Let $\mathcal{G}_\Omega$ be a grid of the closed interval I. Let $h : I \subset \mathbb{R} \to J \subset \mathbb{R}$ be a homeomorphism such that for every $n \geq 1$ and every $1 \leq \beta < \Omega(n) - 1$,*

$$(99) \qquad\qquad |crd(n,\beta)| \leq \mathrm{O}(|I_\beta^n|^{1+r}) \,,$$

where $0 \leq r < 1$. Then, for every closed interval K contained in the interior of I, the logarithmic ratio distortion and the cross ratio distortion satisfy the following estimates:

> *(i) There is a constant $C(i) > 0$, not depending upon the level n and not depending upon $1 \leq \beta \leq \Omega(n)$, such that*

$$(100) \qquad lrd(n,\beta+i) \quad \in \quad \frac{|I_{\beta+i}^n| + |I_{\beta+i+1}^n|}{|I_\beta^n| + |I_{\beta+1}^n|} lrd(n,\beta) \pm C(i)|I_\beta^n|^{1+r} \,.$$

> *(ii) Let I_α^{n-1} and $I_{\alpha+1}^{n-1}$ be two adjacent grid intervals. Let I_β^n and $I_{\beta+1}^n$ be grid intervals contained in the union $I_\alpha^{n-1} \cup I_{\alpha+1}^{n-1}$. Then,*

$$(101) \qquad lrd(n-1,\alpha) \in \frac{|I_\alpha^{n-1}| + |I_{\alpha+1}^{n-1}|}{|I_\beta^n| + |I_{\beta+1}^n|} lrd(n,\beta) \pm \mathrm{O}(|I_\beta^n|^{1+r}) \,.$$

Proof of Lemma 25: By Lemma 23, for every $0 < s < 1$, the homeomorphism $h|K$ is C^{1+s}, and so the map $\psi : I \to \mathbb{R}$ is well-defined by $\psi(x) = \log dh(x)$. By bounded geometry property of a grid and by inequality (99), for every integer i, there is a positive constant $E_1(i)$ such that

$$(102) \qquad\qquad |crd(n,\beta+j_1)| \leq E_1(i)(|I_\beta^n|^{1+r}) \,,$$

for every grid interval I_β^n and $0 \leq j_1 \leq i$. Take $s < 1$ such that $2s = 1 + r$ and $0 \leq j_2 \leq i$. By inequality (99) and statement (ii) of Lemma 23, h is C^{1+s}. Hence, using the bounded geometry property of a grid and statement (i) of Lemma 22, we obtain that

$$|lrd(n,\beta+j_1)lrd(n,\beta+j_2)| \quad \leq \quad \mathrm{O}(|I_{\beta+j_1}^n|^s|I_{\beta+j_2}^n|^s)$$
$$(103) \qquad\qquad\qquad\qquad\qquad\qquad \leq \quad E_2(i)(|I_\beta^n|^{1+r})$$

where $E_2(i)$ is a positive constant depending upon i. Using inequalities (102) and (103) in (36), we get inequality (100). Furthermore, using inequalities (102) and (103) in (37), we get inequality (101). $\qquad\qquad\square$

Proof of Lemma 24: *Proof of statement (i)*: Let h be C^{2+r} and let $\psi : I \to \mathbb{R}$ be given by $\psi(x) = \log dh(x)$. For every $n \geq 1$, let $I_\gamma^n = [x,y]$, $I_{\gamma+1}^n = [y,z]$ and $I_{\gamma+1}^n = [z,w]$ be adjacent grid intervals, at level n. By Taylor series, we get

$$
\begin{aligned}
|h(I_\gamma^n)| &\in |I_\gamma^n|dh(y) + |I_\gamma^n|^2 d^2h(y) \pm \mathrm{O}(|I_\gamma^n|^{2+r}) \\
|h(I_{\gamma+1}^n)| &\in |I_{\gamma+1}^n|dh(y) - |I_{\gamma+1}^n|^2 d^2h(y) \pm \mathrm{O}(|I_{\gamma+1}^n|^{2+r}) \\
|h(I_{\gamma+1}^n)| &\in |I_{\gamma+1}^n|dh(z) + |I_{\gamma+1}^n|^2 d^2h(z) \pm \mathrm{O}(|I_{\gamma+1}^n|^{2+r}) \\
|h(I_{\gamma+2}^n)| &\in |I_{\gamma+2}^n|dh(z) - |I_{\gamma+2}^n|^2 d^2h(z) \pm \mathrm{O}(|I_{\gamma+2}^n|^{2+r}) \,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{|h(I_{\gamma+1}^n)|}{|I_{\gamma+1}^n|} \frac{|I_\gamma^n|}{|h(I_\gamma^n)|} &\in \frac{dh(y) - |I_{\gamma+1}^n|d^2h(y) \pm \mathrm{O}(|I_{\gamma+1}^n|^{1+r})}{dh(y) + |I_\gamma^n|d^2h(y) \pm \mathrm{O}(|I_\gamma^n|^{1+r})} \\
&\in 1 - (|I_\gamma^n| + |I_{\gamma+1}^n|)\frac{d\psi(y)}{2} \pm \mathrm{O}((|I_\gamma^n| + |I_{\gamma+1}^n|)^r) \,,
\end{aligned}
$$

and so

$$lrd(n,\gamma) \in -(|I_\gamma^n| + |I_{\gamma+1}^n|)\frac{d\psi(y)}{2} \pm \mathrm{O}((|I_\gamma^n| + |I_{\gamma+1}^n|)^r) \,.$$

Similarly, we get

$$lrd(n,\gamma+1) \in -(|I_{\gamma+1}^n| + |I_{\gamma+2}^n|)\frac{d\psi(z)}{2} \pm \mathrm{O}((|I_{\gamma+1}^n| + |I_{\gamma+2}^n|)^r).$$

Therefore, by inequality (35), the cross ratio distortion $c(n, \gamma) \in \pm O(|I_\gamma^n|^r)$.

Proof of statement (ii): We prove statement (ii), first in the case where $0 < r < 1$ and secondly in the case where $r = 1$.

Case $0 < r < 1$: By Lemma 23, for every $0 < s < 1$, the homeomorphism $h|K$ is C^{1+s}, and so the map $\psi : I \to \mathbb{R}$ is well-defined by $\psi(x) = \log dh(x)$. For every point $P \in I$, let $I_{\alpha_1}^1, I_{\alpha_2}^2, \ldots$ be a sequence of grid intervals $I_{\alpha_n}^n$ such that $P \in I_{\alpha_n}^n$ and $I_{\alpha_n}^n \subset I_{\alpha_{n-1}}^{n-1}$ for every $n > 1$. By the bounded geometry property of a grid and by inequality (98), for every grid interval $I_\beta^n \subset \cup_{i=-1,0,1} I_{\alpha_{n-1}+i}^{n-1}$, we have that

(104)
$$|crd(n, \beta)| \le O(|I_{\alpha_n}^n|^{1+r}) \ .$$

By inequality (101), we have

$$\frac{lrd(n-1, \alpha_{n-1})}{|I_{\alpha_{n-1}}^{n-1}| + |I_{\alpha_{n-1}+1}^{n-1}|} \in \frac{lrd(n, \alpha_n)}{|I_{\alpha_n}^n| + |I_{\alpha_n+1}^n|} \pm O(|I_{\alpha_n}^n|^r) \ .$$

Hence, by the bounded geometry property of a grid, for every $m \ge 1$ and for every $n \ge m$, we get that

(105)
$$\frac{lrd(n, \alpha_n)}{|I_{\alpha_n}^n| + |I_{\alpha_n+1}^n|} \in \frac{lrd(m, \alpha_m)}{|I_{\alpha_m}^m| + |I_{\alpha_m+1}^m|} \pm O(|I_{\alpha_m}^m|^r) \ .$$

Thus, $lrd(n, \alpha_n)/|I_{\alpha_n}^n| + |I_{\alpha_n+1}^n|$ converges to a value d_P, when n tends to infinity. Let us prove that ψ is differentiable at P and that $d\psi(P) = 2d_P$. Let $L = [x, y]$ be any interval such that the point $P \in L$. Take the largest $m \ge 1$ such that there is a grid interval I_γ^m with the property that $L \subset \cup_{j=-1,0,1} I_{\gamma+j}^m$. By the bounded geometry property of a grid, there is $C \ge 1$, not depending upon P, L and I_γ^m, such that

(106)
$$C^{-1} \le \frac{|I_\gamma^m|}{|L|} \le C \ .$$

For every $n \ge m$, take the smallest sequence of adjacent grid intervals $I_{\beta_n}^n, \ldots, I_{\beta_n+i_n}^n$, at level n, such that $L \subset \cup_{i=0}^{i_n} I_{\beta_n+i}^n \subset \cup_{j=-1,0,1} I_{\gamma+j}^m$. Hence, by definition of the logarithmic ratio distortion, we get

$$\psi(x) = \lim_{n \to \infty} ldh(I_{\beta_n}^n)$$

and

$$\psi(y) = \lim_{n \to \infty} ldh(I_{\beta_n+i_n}^n) \ .$$

Therefore,

$$\frac{\psi(y) - \psi(x)}{y - x} = \lim_{n \to \infty} \frac{ldh(I_{\beta_n+i_n}^n) - ldh(I_{\beta_n}^n)}{y - x}$$

(107)
$$= \lim_{n \to \infty} \frac{\sum_{i=0}^{i_n - 1} lrd(I_{\beta_n+i}^n)}{y - x} \ .$$

By inequalities (105) and (106), for every $I_{\beta_n+i}^n \subset I_{\gamma+j(i)}^m$, we get

$$lrd(n, \beta_n + i) \in \left(|I_{\beta_n+i}^n| + |I_{\beta_n+i+1}^n|\right) \left(\frac{lrd(m, \gamma + j(i))}{|I_{\gamma+j(i)}^m| + |I_{\gamma+j(i)+1}^m|} \pm O(|I_{\gamma+j(i)}^m|^r) \right)$$

(108)
$$\subset \left(|I_{\beta_n+i}^n| + |I_{\beta_n+i+1}^n|\right) (d_P \pm O(|L|^r)) \ .$$

Putting together (107) and (108), we obtain that

$$\frac{\psi(y) - \psi(x)}{y - x} \in \lim_{n \to \infty} (d_P \pm O(|L|^r)) \frac{\sum_{i=0}^{i_n - 1} |I_{\beta_n+i}^n| + |I_{\beta_n+i+1}^n|}{y - x}$$

$$\subset \lim_{n \to \infty} (d_P \pm O(|L|^r)) \frac{|I_{\beta_n}^n| + |I_{\beta_n+i_n}^n| + 2\sum_{i=1}^{i_n - 1} |I_{\beta_n+i}^n|}{y - x}$$

(109)
$$\subset 2d_P \pm O(|L|^r) \ .$$

37

Therefore, for every $P \in I$, the homeomorphism ψ is differentiable at P and $d\psi(P) = 2d_P$. Let us check that $d\psi$ is r-Hölder continuous. For every $P, P' \in I$, let L be the closed interval $[P, P']$. Using (109), we obtain that

$$
\begin{aligned}
d\psi(P') - d\psi(P) \quad &\in \quad \frac{\psi(P') - \psi(P)}{P' - P} - \frac{\psi(P') - \psi(P)}{P' - P} \pm \mathrm{O}(|L|^r) \\
&\subset \quad \pm \mathrm{O}(|L|^r) \,,
\end{aligned}
$$

and so $d\psi$ is r-Hölder continuous.

Case $r = 1$: By the above argument, h is C^{2+s} for every $0 < s < 1$ and so, in particular, h is $C^{1+Lipschitz}$. Thus, by Lemma 22, for every $n \geq 1$ and every $1 \leq \beta \leq \Omega(n) - 1$ we get that

$$
|lrd(n, \beta)| \leq \mathrm{O}(|I_\beta^n|) \,,
$$

which implies that inequality (101) is also satisfied for $r = 1$. Now, a similar argument to the one above gives that $d\psi$ is Lipschitz. $\qquad\square$

3.6. **Proof of Theorem 3.** In this section, we prove Theorem 3.

Proof of Theorem 3: The equivalences presented for quasisymmetric homeomorphisms follow from Lemma 12 with respect to ratio distortion and from Lemma 14 with respect to cross ratio distortion, noting that the ratios $r(n, \beta)$ and the cross ratios $cr(n, \beta)$ are uniformly bounded by the bounded geometry property of the grid. The equivalences presented for uniformly asymptotically affine (uaa) homeomorphisms follow from Lemma 18 with respect to ratio distortion and from Lemma 20 with respect to cross ratio distortion. The equivalences presented for $C^{1+\alpha}$, $C^{1+Lipschitz}$ and affine diffeomorphisms follow from Lemma 22 with respect to ratio distortion and from Lemma 23 with respect to cross ratio distortion. The equivalences presented for $C^{2+\alpha}$ and $C^{2+Lipschitz}$ diffeomorphisms follow from Lemma 24. $\qquad\square$

References

[1] T. Bedford and A.M. Fisher, Ratio geometry, rigidity and the scenery process for hyperbolic cantor sets. *SUNY StonyBrook* preprint.

[2] G. Cui, F. P. Gardiner and Y. Jiang, Scaling functions of degree 2 circle endomorphisms. Ahlfors-Bers Colloquium at Storrs, November, 2001.

[3] M. J. Feigenbaum, Quantitative universality for a class of nonlinear transformations, *J. Stat. Phys.* **19** (1978), 25-52.

[4] F. Ferreira, Diferenciabilidade, Renormalização e Universalidade em Sistemas Dinâmicos Hiperbólicos. Porto Ph.D. Thesis 2000.

[5] F. Gardiner and D. Sullivan, Lacunary series as quadratic differentials. Proceedings of the Symposium in honor of Wilhelm Magnus at Polytechnic Institute of Brooklyn.

[6] F. Gardiner and D. Sullivan, Symmetric structures on a closed curve. *Amer. J. of Math.* **114** (1992), 683-736.

[7] J. Hu and D. Sullivan, Topological conjugacy of circle diffeomorphisms. *Erg. Th. Dyn. Syst.* **17** (1997), 173-186.

[8] W. de Melo and S. van Strien, *One-dimensional dynamics.* Springer-Verlag 1991.

[9] A. A. Pinto and D.A.Rand, Classifying C^{1+} structures on dynamical fractals: 1 The moduli space of solenoid functions for Markov maps on train tracks. *Erg. Th. Dyn. Syst.* (1995), **15**, 685-696.

[10] A. A. Pinto and D.A.Rand, Classifying C^{1+} structures on dynamical fractals: 2 Embedded trees. *Erg. Th. Dyn. Syst.* (1995), **15**, 969-992.

[11] A.A. Pinto and D.A. Rand, Existence, uniqueness and ratio decomposition for Gibbs states via duality. *Ergod. Th. & Dynam. Sys.* (2001) **21** 533-543.

[12] A.A. Pinto and D.A. Rand, Smoothness of holonomies for codimension 1 hyperbolic dynamics. *Bull. London Math. Soc.* (2002) **34** 341-352.

[13] A.A. Pinto and D.A. Rand, Teichmüller spaces and HR structures for hyperbolic surface dynamics. *Ergod. Th. & Dynam. Sys.* (2002) **22** 1905-1931.

[14] A.A. Pinto and D.A. Rand, Rigidity of hyperbolic sets on surfaces. *Journal London Math Soc.* (2004) **2** 1-22.

[15] A.A. Pinto and D.A. Rand, Geometric measures for hyperbolic sets on surfaces. In preparation.

[16] M. Shub, Endomorphisms of compact differentiable manifolds. *Amer. J. of Math.* (1969).

[17] M. Shub and D. Sullivan, Expanding Endomorphism of the circle revisited. *Erg. Th. Dyn. Syst.* (1985), **5**, 285-289.

[18] D. Sullivan, Differentiable structures on fractal-like sets determined by intrinsic scaling functions on dual Cantor sets. Proceedings of Symposia in Pure Mathematics vol.**48** (1988), *American Mathematical Society*.

[19] D. Sullivan, Bounds, quadratic differentials, and renormalization conjectures. AMS Centennial Publications. Volume 2: Mathematics into the Twenty-first Century (1988 Centennial Symposium, August 8-12), *American Mathematical Society*, Providence, RI.,1991.

[20] D. Sullivan, Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers. *Topological Methods in Modern Mathematics*, (eds. L. Goldberg and A. Phillips) Publish or Perish, (1993), 543-563.

[21] E. B. Vul, Ya. G. Sinai, and K. M. Khanin: Feigenbaum universality and the thermodynamical formalism. *Uspekhi Mat. Nauk* **39**, 3-37 (1984).

[22] R. F. Williams: Expanding attractors. *Publ. I.H.E.S.* **43**, 169-203 (1974).

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