HAUSDORFF DIMENSION AND CONFORMAL MEASURES
OF FEIGENBAUM JULIA SETS

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ABSTRACT. We show that contrary to anticipation suggested by the dictionary between rational maps and Kleinian groups and by the "hairiness phenomenon", there exist many Feigenbaum Julia sets $J(f)$ whose Hausdorff dimension is strictly smaller than two. We also prove that for any Feigenbaum Julia set, the Poincaré critical exponent δ_{cr} is equal to the hyperbolic dimension $\mathrm{HD}_{\mathrm{hyp}}(J(f))$. Moreover, if area $J(f) = 0$ then $\mathrm{HD}_{\mathrm{hyp}}(J(f)) = \mathrm{HD}(J(f))$. In the stationary case, the last statement can be reversed: if area $J(f) > 0$ then $\mathrm{HD}_{\mathrm{hyp}}(J(f)) < 2$. We also give a new construction of conformal measures on $J(f)$ that implies that they exist for any $\delta \in [\delta_{\mathrm{cr}}, \infty)$, and analyze their scaling and dissipativity/conservativity properties.

CONTENTS

1. INTRODUCTION

1.1. **Statement of the results.** One of the first questions usually asked about a fractal subset of $\mathbb{R}^n$ is whether it has the maximal possible Hausdorff dimension, n. It certainly happens if the set has positive Lebesgue measure. On the other hand, it is easy to construct fractal sets of zero measure but of dimension n. Moreover, this phenomenon is often observable for fractal sets produced by conformal dynamical systems, iterated rational functions or Kleinian groups. In particular, the analogy with Kleinian groups suggested that the Julia sets of Feigenbaum maps should have Hausdorff dimension two. In this paper we will show that this is not always the case.

A quadratic polynomial (or more generally, a quadratic-like map) is called *Feigenbaum* if it is infinitely renormalizable of bounded combinatorial type with *a priori* bounds (see §2 for a precise definition). Feigenbaum polynomials are remarkable dynamical systems whose geometry has been

a focus of research for the past 25 years. By analogy with Kleinian groups, it was anticipated that their Julia sets have dimension 2, and this problem has been around for a while. However, in this paper we will show that this is not always the case:

Theorem A (HD < 2). *There exist Feigenbaum quadratic polynomials whose Julia sets $J(f)$ have Hausdorff dimension strictly less than 2.*

Maps to which this theorem applies are infinitely renormalizable maps with "high" combinatorics. To the best of our knowledge, previous examples of Julia sets of Hausdorff dimension smaller than 2 were related to one of two mechanisms:

(1) Definite expansion along the critical orbit,
(2) Porosity (presence of definite holes in small scales) of the Julia set.

Note that both mechanisms fail for Feigenbaum maps.

Remark. We show in [AL] that in fact $HD(J(f))$ can be arbitrary close to 1 for some Feigenbaum Julia sets. Namely, when the combinatorics of the renormalization gets close to the Chebyshev one (of the map $z \mapsto z^2 - 2$), the dimension gets close to 1.

It is still unknown whether there exist Feigenbaum Julia sets of positive area, or the ones with zero area but Hausdorff dimension two. Curiously, the affirmative answer to the former question would imply the affirmative answer to the latter one (subject of certain *a priori* bounds assumptions):

Theorem B. *Assume that a renormalization horseshoe $\mathcal{A}$ (see §2.5) contains maps g_+ and g_0 such that $\mathrm{area}(J(g_+)) > 0$ while $\mathrm{area}(J(g_0)) = 0$. Then it contains a map f such that $\mathrm{area}(J(f)) = 0$ but $\mathrm{HD}(J(f)) = 2$.*

In the course of this paper, we give a criterion for area $J(f) > 0$ which can provide an efficient numerical test on this property.

Along with the Hausdorff dimension, there are other natural geometric objects and quantities associated with Julia sets:

• The *Poincaré series*

$$(1.1) \qquad \Xi(\delta) \equiv \Xi_\delta(f, z) = \sum_{n=0}^{\infty} \sum_{f^n \zeta = z} \frac{1}{|Df^n(\zeta)|^\delta}, \quad z \in \mathbb{C} \setminus \mathcal{O},$$

where $\mathcal{O}$ is the postcritical set of f.

• The *critical exponent* $\delta_{\mathrm{cr}} = \delta_{\mathrm{cr}}(f)$ which separates convergent exponents of the Poincaré series from the divergent ones.

• The *hyperbolic dimension* $\mathrm{HD}_{\mathrm{hyp}}(J)$ of the Julia set $J = J(f)$, that is, the supremum of the dimensions of all invariant hyperbolic subsets of J.

• *Conformal measures* on the Julia sets, that is the measures that transform according to the rule:

$$(1.2) \qquad \mu(fX) = \int_X |Df|^\delta d\mu$$

for any measurable set X such that $f|X$ is injective. By Sullivan's Theorem [S3], the Julia set J supports a conformal measure with exponent δ_{cr}.

• The *minimum exponent* $\delta_* = \delta_*(f)$ is the infimum of the exponents of all conformal measures on the Julia set.

The following general relation between the above quantities has been known:

$$(1.3) \qquad \delta_* = \mathrm{HD}_{\mathrm{hyp}}(J) \leq \mathrm{HD}(J),$$

where the equality is due to Denker and Urbanski [DU] while the inequality is trivial. If moreover the Julia set has zero area then the inequality

$$\text{HD}(J) \leq \delta_{\text{cr}} \tag{1.4}$$

holds for a large class of maps (including Feigenbaum maps), a result due to Bishop[1] [B] (dynamically interpreted in [GS]). However, it is not known whether these quantities in general coincide. In the case of Feigenbaum maps, (and in fact, for quite a general class of rational maps (see §10)), we can resolve this issue:

Theorem C (Critical Exponents and Dimensions). *Let f be a Feigenbaum map. Then*

$$\delta_{\text{cr}}(f) = \delta_*(f) = \text{HD}_{\text{hyp}}(J).$$

Moreover, if area $J(f) = 0$*, then*

$$\text{HD}_{\text{hyp}}(J) = \text{HD}(J).$$

It is quite surprising that the last assertion can be reversed in the case of periodic combinatorics:

Theorem D (Positive Area and Hyperbolic Dimension). *Let f be a Feigenbaum map with periodic combinatorics. If* area$(J) > 0$ *then*

$$\text{HD}_{\text{hyp}}(J) < \text{HD}(J) = 2.$$

At this moment we cannot tell whether this result is positive or negative (that is, whether it gives an interesting property of certain Julia sets or indicates that those Julia sets should not exist).

Putting Theorems C and D together we obtain the following amusing statement: *A Feigenbaum map with periodic combinatorics has a Julia set of zero Lebesgue measure if and only if its hyperbolic and Hausdorff dimensions coincide.*

It is known (Prado [P]) that any conformal measure on a Feigenbaum Julia set is ergodic (and hence there exists at most one conformal measure for any given exponent δ). Another traditional question asked about quasi-invariant measures is whether they are dissipative or conservative[2]. Recall that an ergodic measure is called *conservative* if it satisfies the conclusion of the Poincaré Recurrence Theorem and is called *dissipative* otherwise.

Theorem E (Conformal Measures). *Let f be a Feigenbaum map. Then for any $\delta \geq \delta_*(f)$, the Julia set $J(f)$ supports a unique conformal measure μ_δ with exponent δ. If μ_δ is conservative then $\delta = \delta_* < 2$. Moreover, both dissipative and conservative alternatives with $\delta = \delta_* < 2$ are realizable for some Feigenbaum quadratic-like maps.*

This gives first examples of dissipative δ_*-conformal measures.

1.2. **Methods.** In this paper we develop an efficient method to estimate the Poincaré series by comparing its values on different renormalization levels. For periodic points of renormalization this method leads to a Recursive Quadratic Estimate for the Poincaré series. Let A^n be the fundamental annuli on consecutive renormalization levels. Then we show that the average Poincaré series,

$$\omega \equiv \omega_m(\delta) = \frac{1}{\text{area}\, A^m} \int_{A^m} \Xi_\delta(x) dx$$

satisfies an estimate $\omega \leq P_\delta(\omega)$, where P_δ is a quadratic polynomial (with positive coefficients) whose coefficients continuously depend on δ near 2. It gives a bound of the average Poincaré series

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[1]Actually Bishop has proved the *a priori* stronger result that (under the assumption that the Julia set has zero Lebesgue measure) $\delta_{\text{cr}} \geq \overline{\text{MD}}$, the *upper Minkowski dimension*. Together with this result of Bishop, our Theorem C below shows that the Minkowski dimension of a Feigenbaum Julia set is equal to its Hausdorff dimension.

[2]For Kleinian groups this question is related to existence of the Green function on the associated 3-manifold (see [Ah]).

ω by a real fixed point of P_δ if it exists. Moreover, coefficients of P_2 involve areas of points that escape annuli A^m, and one can see that if substantial masses of points escape the A^m then $P_2(x)$ has an attracting fixed point. Persistence of this point under a perturbation of δ yields convergence of the Poincaré series for δ slightly smaller than 2, implying that $\delta_{\mathrm{cr}} < 2$.

More generally, this leads us to the following *Trichotomy*. For any Feigenbaum map with periodic combinatorics one of the following three possibilities occurs:

1) *Lean* case: substantial masses of points escape the annuli A^n to the *outer* regions. In this case

$$\mathrm{HD}(J) = \mathrm{HD}_{\mathrm{hyp}}(J) < 2.$$

2) *Balanced* case: there is a balance between masses of points escaping *in* and *out* the fundamental annuli. In this case

$$\mathrm{area}\, J = 0 \quad \text{but} \quad \mathrm{HD}_{\mathrm{hyp}}(J) = 2.$$

3) *Black hole* case: the critical point acts like a black hole that attracts huge masses of matter, so that substantial masses escape the annuli A^n to the *inner* regions. In this case

$$\mathrm{area}\, J > 0 \quad \text{but} \quad \mathrm{HD}_{\mathrm{hyp}}(J) < 2.$$

In the case of Feigenbaum maps with high combinatorics, the critical point is lean which yields Theorem A. Theorem D is incorporated into the black hole case. *Heuristically*, phase transition from the lean case to the black hole case can occur only through the balanced case indicating that Theorem B should hold.

Theorem C is based upon a new construction of a δ-conformal measure by taking a properly truncated Poincaré series $\Xi_\delta^{tr}(f, z)$, putting the corresponding measure on the preimages of z, and letting $z \to 0$. This allows us to demonstrate that conformal measures exist for any $\delta \geq \delta_*$ which is the key step of the proof (see also Theorem E).

Dissipativity of the δ-conformal measure is equivalent to convergence of the Poincaré series Ξ_δ. A simple argument shows that $\Xi_{\delta_*}(Rf, z)$ is convergent, provided the dimension does not drop under the renormalization:

$$\mathrm{HD}_{\mathrm{hyp}}(J(Rf)) = \mathrm{HD}_{\mathrm{hyp}}(J(f)) \equiv \delta_*,$$

which holds, e.g., for renormalization fixed points $f = Rf$. On the other hand, if the dimension drops,

$$(1.5) \qquad\qquad \mathrm{HD}_{\mathrm{hyp}}(J(Rf)) < \mathrm{HD}_{\mathrm{hyp}}(J(f)),$$

then we find a linear recursive estimate for the Poincaré series (this time averaging over the conformal measure μ_{δ_*} on the annuli A^n), which would yield convergence of Ξ_δ for some $\delta < \delta_*$ if the measure μ_{δ_*} were dissipative (contradicting that δ_* is the minimum exponent).

Finally, we give two proofs that the drop condition (1.5) is satisfied for some Feigenbaum maps, which completes the proof of Theorem E. One of them, based on methods of quasiconformal deformations, shows that

$$\sup_{f \in \mathcal{H}} \mathrm{HD}_{\mathrm{hyp}}(f) = 2,$$

where $\mathcal{H}$ is any hybrid class of Feigenbaum quadratic-like maps. Another one is based on a dynamical interpolation in an appropriate renormalization horseshoe.

Our methods also work in the generalized renormalization contexts, with Fibonacci maps as main examples. They can be also applied to the real dynamics, where it is known that "wild attractors" exist [BKNS] (real analogue of the phenomenon of "positive area Julia sets"), which yields a real life version of Theorems B and D.

Also, in the search for Feigenbaum Julia sets of positive area, it might be a good idea to consider maps of higher degree (that is, ≥ 2) as well. Let us remark with this respect that all the results

and proofs of this paper are still valid for (unicritical) Feigenbaum maps of arbitrary degree (with only terminological and notational adjustments).

1.3. **Some history.** Geometric measures and Hausdorff dimension of limit and Julia sets are popular themes in holomorphic dynamics.

First examples of a conformal dynamical system with the limit set of measure zero but Hausdorff dimension 2 appeared in the works of Thurston [Th] and Sullivan [S2] on Kleinian groups. On the other hand, the limit set of a geometrically finite Kleinian group[3] has Hausdorff dimension less than 2 [Bo], [S4], [T]. In this context, the Hausdorff dimension problem was eventually resolved in full by Bishop and Jones [BJ]: the limit set $\Lambda(\Gamma)$ of a (finitely generated) Kleinian group has Hausdorff dimension 2 if and only if the group is geometrically infinite. Also, it has been recently announced that the limit set of *any* finitely generated Kleinian group has zero area, which settles a long-standing Ahlfors Conjecture, see [Ag, CG].

Another basic class of conformal dynamical systems is given by iterates of a single endomorphism, rational or transcendental. The exponential map $z \mapsto \lambda e^z$, $\lambda < e^{-1}$, was the first example of such an endomorphism whose Julia set has zero area [EL1] but Hausdorff dimension 2 [Mc1]. Examples of entire functions with Julia sets of positive area were given in [EL2, Mc1]. For other interesting properties of the measure and dimension of transcendental Julia sets, see [L2, R, K, SZ, UZ].

The first examples of rational Julia sets with Hausdorff dimension two were constructed by Shishikura [Sh1] (see also [Mc5]). Indeed, Shishikura demonstrated that this phenomenon is generic for quadratic polynomials $z \mapsto z^2 + c$ with c on the boundary of the Mandelbrot set. All these Julia sets have zero area [L3, Sh2].

On the other hand, the works of McMullen [Mc4] (see also [Pe]) and Przytycki [Prz] (see also [PR, GS]) showed that the situation in the iteration theory is more complicated than in the theory of Kleinian groups: There exist geometrically infinite Julia sets with Hausdorff dimension strictly less than two. Here a map is called geometrically finite if all the critical points on the Julia set are non-recurrent (see [LM]). The Hausdorff dimension of any geometrically finite Julia set is strictly less than 2, provided it is not the whole sphere (see [CJY, U, Mc5]).

As we have already mentioned, it is still unknown whether there exists a polynomial map whose Julia set has positive area. However, it was known that many Feigenbaum Julia sets have zero area [Y]. (It is essentially the same class of maps for which *a priori* bounds had been established in [L5] and to which our Theorem A applies.)

McMullen [Mc3] has demonstrated that in the dictionary between rational maps and Kleinian groups, Feigenbaum maps correspond to hyperbolic 3-manifolds that fiber over the circle. Using this analogy, he proved that the critical point lies "deep" (in some precise geometric sense) inside any Feigenbaum Julia set. This strongly suggested that these sets should have the maximal possible Hausdorff dimension. However, as we have already mentioned, these points can be lean in the measure-theoretic sense which leads to the opposite conclusion.

For Kleinian groups, conformal measures were introduced by Patterson [Pa] and Sullivan [S1]. See [N] for a discussion of their ergodic properties. Ergodic and conservativity/dissipativity properties of conformal measures on Julia sets were studied, among other papers, in [L1, Mc2, P, Ba, BM, GS] (part of this work was motivated by [BL] concerning real dynamics).

For the relations between the critical exponent and Hausdorff dimension for Kleinian groups see [BJ] (compare Theorem C).

1.4. **Organization of the paper.** We begin the paper (§2) with a basic material concerning the Hausdorff dimension, Poincaré series, conformal measures, and renormalization of quadratic-like maps. In §3 we study scaling and ergodic properties of conformal measures, and prove Theorem C and part of Theorem E.

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[3]We follow the classical convention that the limit set of a Kleinian group is not the whole sphere.

Section 4 is the central in the paper: here we derive the Recursive Quadratic Estimates. In §5 we deduce from them the Trichotomy. It immediately yields Theorem D and provides us with the "leanness" condition for Theorem A.

In §6 and §7 we complete a proof of Theorem E. In §7.2 we study how the Hausdorff dimension $h(g) \equiv \mathrm{HD}(J(g))$ varies within hybrid classes of Feigenbaum maps. One of the consequences of our analysis is the following: If f has stationary combinatorics then the function $h(g)$ is either identically equal to 2, or else $\mathrm{Im}\, h = [h(f_*), 2)$ where f_* is the associated renormalization fixed point.

In §8, we prove Theorem B.

In §9 we describe a class of well controlled Feigenbaum maps [L5] whose Julia sets are lean at the critical point [Y], which completes the proof of the first assertion of Theorem A. The description is given in terms of the principal nest of the Yoccoz puzzle.

§10 contains various remarks and some open problems.

The paper is concluded with two appendices. Appendix A is of technical nature: here we construct fundamental domains for Feigenbaum maps satisfying nice Markov properties. In Appendix B we analyze conformal measures on towers which yields sharper scaling properties of conformal measures on the Julia sets.

2. Background

2.1. Notations and terminology.
$\mathbb{C}$ is the complex plane;

$\mathbb{N} = \{0, 1, \dots\}$ is the set of natural numbers;

$\mathbb{Z}$ is the set of integers;

$\mathbb{D}_r(z) = \{\zeta : |z - \zeta| < r\}$, $\mathbb{D}_r \equiv \mathbb{D}_r(0)$, $\mathbb{D} \equiv \mathbb{D}_1$;

$U \Subset V$ means that U is *compactly contained* in V, i.e., $\mathrm{cl}\, U$ is compact and is contained in V.

A domain is a connected open set. By a (topological) disk we will mean a simply connected domain $U \subset \mathbb{C}$. A Jordan disk is a topological disk whose boundary is a Jordan curve. *Filling* of a domain D is the union of all Jordan disks with boundary contained in D: it is the smallest simply connected set containing D. Given a family of topological disks, we say that they have a *bounded shape* if they are quasi-disks with uniformly bounded dilatation. A (topological) annulus $A \subset \mathbb{C}$ is a doubly connected domain. Any topological annulus can be conformally uniformized by a round annulus $\{z : r < |z| < R\}$ (where possibly $r = 0$ or $R = \infty$). The modulus of the topological annulus, $\mathrm{mod}(A)$, is defined as $\log(R/r)$.

The Lebesgue measure ($\equiv$ area) of a measurable set $X \subset \mathbb{C}$ will be denoted by $\mathrm{area}\, X$ or $|X|$. If $X, Y \subset \mathbb{C}$ are measurable and $|Y| > 0$, we let $p(X|Y) = |X \cap Y|/|Y|$.

Distortion of a (holomorphic) univalent map $f : U \to \mathbb{C}$ is defined as

$$\sup_{z, \zeta \in U} \frac{|Df(z)|}{|Df(\zeta)|}.$$

"Quasi-conformal" will be abbreviated as "qc".

For a natural $s \geq 2$, let Σ_s denote the space of two-sided sequences $(\epsilon_k)_{k=-\infty}^{\infty}$ in s symbols: $\epsilon_k \in \{1, \dots, s\}$. The shift transformation σ on this space is called a *Bernoulli shift*.

The forward orbit of a point z under f is denoted as $\mathrm{orb}(z) \equiv \mathrm{orb}_f z = \{f^n z\}_{n=0}^{\infty}$;

Its limit set is denoted as $\omega(z)$.

$a \asymp b$ means that $C^{-1} < a/b < C$ with a constant $C > 0$ independent of particular a and b under consideration.

Assertion that "some property is satisfied for $\delta \approx a$" naturally means that it is satisfied for all δ sufficiently close to a.

2.2. Hyperbolic sets.

Let $X \subset \mathbb{C}$ be a compact set invariant under a certain holomorphic map f defined in a neighborhood of X. It is called *hyperbolic* if there exist $C > 0$ and $\lambda > 1$ such that for all $z \in X$

$$|Df^n(z)| \geq C\lambda^n, \ n = 0, 1, \ldots$$

2.3. Quadratic-like maps.

Let $\mathbb{U} \Subset \mathbb{V}$ be two topological disks. A (holomorphic) branched covering of degree d $f : \mathbb{U} \to \mathbb{V}$ is called a *polynomial-like map* of degree d. If $d = 2$, the map is called *quadratic-like*.

The *filled Julia* set $K(f)$ of a polynomial-like map is defined as the set of non-escaping points (where "escaping" means landing at the fundamental annulus $\mathbb{V} \smallsetminus \mathbb{U}$ under some iterate of f), and the *Julia set* $J(f)$ is defined as the boundary of $K(f)$. In the quadratic-like case, these sets are either Cantor or connected depending on whether the critical point itself is escaping or not.

Any quadratic-like map has two fixed points counted with multiplicity. In the case of connected Julia set these two points can be dynamically distinguished. One of them, usually denoted by α, is either non-repelling or *dividing*, i.e., removing of it makes the Julia set disconnected. Another one, denoted by β, is always *non-dividing*.

If a quadratic-like map is considered up to choice of domains $\mathbb{U}$ and $\mathbb{V}$, then it should be more carefully called a quadratic-like *germ*. More precisely, one says that two quadratic-like maps with connected Julia sets represent the same germ if they have a common Julia set and coincide in a neighborhood of it. For such a germ, let $\mathrm{mod}(f) = \sup \mathrm{mod}(\mathbb{V} \smallsetminus \mathbb{U})$ where the supremum is taken over all possible choices of domains $\mathbb{U}$ and $\mathbb{V}$. We will not make notational difference between maps and germs.

We will assume below that 0 is the critical point of f.

Two quadratic-like maps/germs f and g are called *hybrid equivalent* if there is a qc conjugacy h between them with $\bar{\partial}h = 0$ a.e. on the filled Julia set $K(f)$. By the Straightening Theorem [DH], any quadratic-like map f with connected Julia set is hybrid equivalent to a unique quadratic polynomial $z \mapsto z^2 + \chi(f)$, with $c = \chi(f)$ on the Mandelbrot set M. Let $\mathcal{H}_f$ stand for the hybrid class of f.

2.4. Feigenbaum maps.

A quadratic-like germ $f : \mathbb{U} \to \mathbb{V}$ is called *renormalizable* if there exist $p = p(f) > 1$ and topological disks $\mathbb{U}' \Subset \mathbb{V}'$ containing the critical point such that:

- $g \equiv f^p : \mathbb{U}' \to \mathbb{V}'$ is a quadratic-like map with connected Julia set $J(g)$ called a *pre-renormalization* of f;
- The "little Julia sets" $f^k J(g)$, $k = 0, 1, \ldots, p-1$, are pairwise disjoint except perhaps touching at their β-fixed points.

If the little Julia sets indeed touch then one says that f is *immediately renormalizable* (or that the renormalization is of *satellite type*). Otherwise the renormalization is called *primitive*.

A pre-renormalization considered up to affine conjugacy is called a *renormalization* of f. The renormalization of f with minimal possible period is denoted Rf.

The Mandelbrot set contains a plenty of canonical homeomorphic copies of itself. A quadratic-like map f is renormalizable if and only if its straightening $\chi(f)$ belongs to one of these copies. The choice of this copy specifies the *combinatorics* of the renormalization.

Now one can inductively define an infinitely renormalizable map f; its successive renormalizations are denoted by $R^m f$, $m = 0, 1, 2, \ldots$. If all these renormalizations have the same combinatorial type,

one says that f has a *stationary type*. If the sequence of the combinatorics is periodic, then f has a *periodic type*. If the periods of all renormalizations are bounded then f has a *bounded type*.

One says that an infinitely renormalizable map has *a priori* bounds if $\mod(R^n f) \geq \epsilon > 0$, $n = 0, 1, 2, \ldots$. Conjecturally *a priori* bounds are valid for all infinitely renormalizable maps of bounded type. So far, this conjecture has been confirmed for real maps (see [S5, MS]) and for a class of complex maps of high type (see [L5] and §9).

An infinitely renormalizable quadratic-like map of bounded type with *a priori* bounds will be also called a *Feigenbaum map*.

2.5. **Renormalization horseshoe.** A renormalizable quadratic-like map is called the *renormalization fixed* (respectively, *periodic*) *point* if $Rf = f$ (respectively $R^p f = f$ for some p). More generally, let us pick a finite family $\mathcal{M}$ of little Mandelbrot copies M_k, $k = 1, \ldots, s$. Assume that to any two-sided sequence $\bar{M} = (M_{k(n)})_{k=-\infty}^{\infty} \in \Sigma_s$ we can associate a quadratic-like germ $g_{\bar{M}}$ with uniform *a priori* bounds which is renormalizable with combinatorics $M_{k(0)}$ and such that $Rf_{\bar{M}} = f_{\sigma(\bar{M})}$ where $\sigma : \Sigma_s \to \Sigma_s$ is the Bernoulli shift. Then the family $\mathcal{A}_{\mathcal{M}}$ of these quadratic-like germs is called the *renormalization horseshoe* associated to the given family of copies. Notice that by definition, $\mathcal{A}$ is invariant under the renormalization, and $R : \mathcal{A}_{\mathcal{M}} \to \mathcal{A}_{\mathcal{M}}$ is topologically conjugate to the Bernoulli shift σ.

We say that a finite family $\mathcal{M}$ of little Mandelbrot copies is *fine* if for any one-sided sequence $\bar{M}_+ = (M_{k(n)})_{k=0}^{\infty}$ of these copies there is a quadratic-like germ which is renormalizable with combinatorics $\bar{M}_+$ with *a priori* bounds. For instance, any family of real Mandelbrot copies is nice.

Theorem 2.1 ([S5, Mc3])**.** *Any fine family $\mathcal{M}$ of Mandelbrot copies has an associated renormalization horseshoe $\mathcal{A}_{\mathcal{M}}$.*

Notice that periodic points of renormalization are dense in the horseshoe $\mathcal{A}_{\mathcal{M}}$.

Theorem 2.2 ([L8])**.** *The horseshoe $\mathcal{A}_{\mathcal{M}}$ is hyperbolic with codimension-one stable foliation. Moreover, local stable manifolds of points $f \in \mathcal{A}_{\mathcal{M}}$ coincide with their hybrid classes.*

2.6. **Nest of little Julia sets.** Let f be a Feigenbaum quadratic-like map. Let us consider a sequence $f_m : \mathbb{U}^m \to \mathbb{V}^m$ of m-fold pre-renormalizations of f. Let $J \equiv J^0 \supset J^1 \supset J^2 \supset \ldots$ stand for the corresponding nest of little Julia sets, $J^m = J(f_m)$. Then

$$\mathcal{O} \equiv \mathcal{O}(f) = \omega(0) = \bigcap_{m \geq 0} \bigcup_{i \geq 0} f^i(J^m)$$

is the *postcritical set* of f.

Lemma 2.3 (see e.g., [Mc3])**.** *There exist quadratic-like pre-renormalizations $f_m : \mathbb{U}^m \to \mathbb{V}^m$ with the following properties:*
  (P1) $\mathrm{cl}(\mathbb{V}^m \smallsetminus \mathbb{U}^m) \cap \mathcal{O} = \emptyset$;
  (P2) *Unbranched a priori bounds:* $\mod(\mathbb{V}^m \smallsetminus \overline{\mathbb{U}}^m) \geq \mu > 0$ *and* $\mod(\mathbb{U}^m \smallsetminus J^m) \geq \mu > 0$;
  (P3) *The disks $\mathbb{U}^m$ and $\mathbb{V}^m$ are quasidisks with bounded dilatation;*
  (P4) $\mathrm{diam}\, J^m \asymp \mathrm{diam}\, \mathbb{V}^m$.
All the constants depend only on the combinatorial and a priori bounds for f.

If f is a renormalization fixed point then there exists a dilation $\lambda > 1$ such that $f_{m+1} = \lambda f_m(\lambda^{-1} z)$, and the above picture becomes scaling invariant: $\mathbb{V}^m = \lambda^m \mathbb{V}^0$, $\mathbb{U}^m = \lambda^m \mathbb{U}^0$, and $J^m = \lambda^m J^0$. In the case of renormalization periodic point, the scaling invariance holds on the subsequences of levels congruent modulo the period.

In what follows, the nests of domains $\mathbb{V}^m$ and $\mathbb{U}^m$ will always be assumed to satisfy properties (P1)-(P4). In the case of periodic combinatorics, they will be additionally assumed to be scaling invariant on the appropriates subsequences of levels.

2.7. **Nice fundamental domains.** Let us consider a Feigenbaum map $f : \mathbb{U} \to \mathbb{V}$ with pre-renormalizations $f_n : \mathbb{U}^n \to \mathbb{V}^n$. Along with the domains $\mathbb{V}^n$ and $\mathbb{U}^n$, we will also need simply connected domains V^n and U^n satisfying the following "nice" topological and geometric properties:

(C1) $\mathbb{V}^n \cap \mathcal{O}(f) \subset V^n \subset \mathbb{U}^n$,

(C2) $V^{n+1} \subset U^n \equiv f_n^{-1}(V^n)$,

(C3) $f^k(\partial V^n) \cap V^n = \emptyset$, $k \geq 0$ (compare with "nice intervals" of Martens [M1]).

(G1) $A^n \equiv V^n \smallsetminus U^n$ is far from the postcritical set $\mathcal{O}(f)$: it has bounded hyperbolic diameter in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$.

(G2) $\mathrm{area}(A^n) \asymp \mathrm{area}(U^n) \asymp (\mathrm{diam}(U^n))^2 \asymp (\mathrm{diam}(V^n))^2$.

The bounds (G1)-(G2) together with the unbranched *a priori* bounds will be called *geometric bounds*. By *combinatorial bounds* we will mean a bound on the renormalization period of all renormalizations of f.

In Appendix A we will construct such a nest of nice fundamental domains for any Feigenbaum map.

Remark 2.1. Condition (C3) implies that the first landing map to V^n is particularly simple: its restriction to each component of its domain is a univalent map onto V^n. Due to (P1) each univalent pullback of V^n extends to a univalent pullback of $\mathbb{V}^n$, so by the Koebe Distortion Lemma the first landing map to V^n has bounded distortion.

Another consequence of (C3) is the structure of the first return map to V^n. We have that $U^n \subset V^n$ and that $f_n : U^n \to V^n$ is a double covering. Thus U^n is one of the components of the domain of the first return map to V^n. All other components are univalent pullbacks of V^n.

2.8. **Hausdorff dimension.** Given a $\delta \geq 0$, the Hausdorff δ-measure h_δ is defined as follows

$$h_\delta(X) = \lim_{\epsilon \to 0} \inf \sum (\mathrm{diam}\, U_i)^\delta,$$

where the infimum is taken over all coverings of X by sets U_i of diameter at most $\epsilon > 0$. For any X there is a unique critical exponent separating infinite and vanishing values of the measure h_δ. This exponent is called the Hausdorff dimension of X and is denoted by $\mathrm{HD}(X)$.

Given a Borel measure μ, the Hausdorff dimension $\mathrm{HD}(\mu)$ is defined as the infimum of the $\mathrm{HD}(X)$ as X runs over all measurable sets of full measure. *Local dimension* $\mathrm{HD}_\mu(z)$ of a measure μ at a point z is defined as

$$(2.1) \qquad \lim_{r \to 0} \frac{\log \mu(\mathbb{D}_r(z))}{\log r},$$

provided this limit exists. If the limit does not exist, one can still consider *upper* and *lower* local dimensions, $\overline{\mathrm{HD}}_\mu(z)$ and $\underline{\mathrm{HD}}_\mu(z)$, by taking upper and lower limits in (2.1).

The *hyperbolic dimension* $\mathrm{HD}_{\mathrm{hyp}}(X)$ of an invariant set X is defined as the supremum of the dimensions of all invariant hyperbolic subsets of X.

Theorem 2.4 ([Z]). *For any polynomial-like map $f : U \to V$ with connected Julia set, there is an alternative:*

- *Either $J(f)$ is a real analytic curve;*
- *or $\mathrm{HD}(J(f)) > 1$.*

2.9. **Conformal measures and critical exponents.** The *Poincaré series* $\Xi_\delta(z) \equiv \Xi_\delta(f, z)$, $z \in \mathbb{C} \smallsetminus \mathcal{O}$, was defined in the Introduction (1.1). Note that by the Koebe Distortion Lemma it has a bounded oscillation on any compact set $K \subset \mathbb{V} \smallsetminus \mathcal{O}$:

$$(2.2) \qquad \Xi_\delta(f, z) \leq C\Xi_\delta(f, \zeta), \quad z, \zeta \in K,$$

where the constant C depends only on the hyperbolic diameter of K in $\mathbb{V} \smallsetminus \mathcal{O}$,

By definition, the *critical exponent* $\delta_{cr} = \delta_{cr}(f)$ separates convergent values of δ in Ξ_δ from the divergent ones: the Poincaré series is convergent for $\delta > \delta_{cr}$ and divergent for $\delta < \delta_{cr}$. Notice that by (2.2), δ_{cr} is independent of the particular choice of $z \in \mathbb{V} \smallsetminus \mathcal{O}$. For $\delta = \delta_{cr}$ the Poincaré series can behave in both ways, and f is called of *convergent* or *divergent* type depending on it.

Taking a wandering disk $D \Subset \mathbb{V} \smallsetminus J$ (i.e., $f^{-n}D \cap f^{-m}D = \emptyset$ for $m \neq n$), we see that

$$\Xi_2(z) \asymp \bigcup_{n=0}^{\infty} \text{area}(f^{-n}(D)) < \infty,$$

so that $\delta_{cr} \leq 2$. One can also show that $\delta_{cr} > 0$. In fact, we have:

Lemma 2.5. *If the Julia set $J(f)$ is connected then $\Xi_1 = \infty$, so that $\delta_{cr} \geq 1$.*

Proof. Let $\Gamma_n = \partial(f^{-n}\mathbb{U})$. Then $\Xi_1 \asymp \sum \text{length}(\Gamma_n) = \infty$. $\qquad\qquad\square$

A quasi-invariant measure μ is called δ-*conformal* if its Radon-Nicodim Jacobian $d(f^*\mu)/d\mu$ is equal to $|Df|^\delta$, which is equivalent to the transformation rule (1.2). This transformation rule and density of preimages on the Julia set imply that $\text{supp}(\mu) = J(f)$ for any conformal measure μ on the Julia set.

Theorem 2.6 (Sullivan [S3]). *Any polynomial-like map f has at least one δ_{cr}-conformal measure μ on its Julia set.*

We have also defined in the Introduction the *minimal exponent* $\delta_* = \delta_*(f)$, the infimum of the exponents δ for which f admits a δ-conformal measure on the Julia set.

Theorem 2.7 (Denker-Urbansky [DU]). *For any polynomial-like map,*

$$\delta_*(f) = \text{HD}_{\text{hyp}}(J(f)).$$

Moreover, f admits a δ_*-conformal measure on its Julia set.

Let us also mention the following simple property:

Lemma 2.8. *The exponent $\delta_*(f) = \text{HD}_{\text{hyp}}(f)$ depends lower semicontinuously on the quadratic-like map f.*

Recall that an f-quasi-invariant measure μ on J is called *ergodic* if there is no decomposition $J = X_1 \cup X_2$ into two invariant measurable subsets of positive measure.

Theorem 2.9 (Prado [P]). *Let f be a Feigenbaum map. Then any conformal measure μ on $J(f)$ is ergodic. Hence for any given exponent δ, there exists at most one δ-conformal measure.*

2.10. **Poincaré series in the hyperbolic case.** Let us collect here some basic estimates on Poincaré series. They show that in the hyperbolic case the Poincaré series cannot blow up: it depends continuously on both δ and f.

Let Δ be the set of all continuous maps f defined over a compact set $X \subset \mathbb{C}$ that admit a holomorphic extension to a neighborhood of X. We say that $f_n : X_n \to \mathbb{C}$ converge to $f : X \to \mathbb{C}$ is $X_n \to X$ in the Hausdorff topology and there exists a neighborhood U of X such that for every n sufficiently big, f_n admits a holomorphic extension to U and the $f_n : U \to \mathbb{C}$ converge to $f : U \to \mathbb{C}$ uniformly. We will assume that X has no isolated points (so that derivatives of a function $f : X \to \mathbb{C}$ of class Δ are well defined).

Lemma 2.10. *Let $f : X \to \mathbb{C}$ be a map of class Δ without critical points. If $\delta > 0$, $K > 0$ are such that*

$$(2.3) \qquad\qquad\qquad\qquad \Xi_\delta(f, x) < K,$$

for every $x \in X$. Then for every $\tilde{K} > K$, there exists $\tilde{\delta} < \delta$ such that if $\tilde{f} : \tilde{X} \to \mathbb{C}$ is close to $f : X \to \mathbb{C}$ then for every $x \in \tilde{X}$, we have $\Xi_{\tilde{\delta}}(\tilde{f}, x) < \tilde{K}$.

Proof. For every $x \in X$, let $N_x \geq 0$ be minimal such that

$$(2.4) \qquad \sum_{f^{N_x}(y)=x} |Df^{N_x}(y)|^{-\delta} < \frac{1}{2K}.$$

Notice that N_x is upper semicontinuous, so $N = \sup N_x < \infty$. It follows that

$$(2.5) \qquad \sum_{f^N(y)=x} |Df^N(y)|^{-\delta} = \sum_{f^{N_x}z=x} |Df^{N_x}(z)|^{-\delta} \sum_{f^{N-N_x}(y)=z} |Df^{N-N_x}(y)|^{-\delta} < \frac{1}{2}.$$

Choose an integer $M > 0$ such that $2^{M-1} > K/(\tilde{K} - K)$. If $\tilde{f} : \tilde{X} \to \mathbb{C}$ is close to $f : X \to \mathbb{C}$ and $\tilde{\delta}$ is close to δ, we still have, for every $x \in \tilde{X}$,

$$(2.6) \qquad \sum_{\tilde{f}^N(y)=x} |D\tilde{f}^N(y)|^{-\tilde{\delta}} < \frac{1}{2},$$

$$(2.7) \qquad \sum_{m=0}^{MN-1} \sum_{\tilde{f}^m(y)=x} |D\tilde{f}^m(y)|^{-\tilde{\delta}} < K.$$

Hence
(2.8)

$$\Xi_{\tilde{\delta}}(\tilde{f}, x) = \sum_{m=0}^{MN-1} \sum_{\tilde{f}^m(y)=x} |D\tilde{f}^m(y)|^{-\tilde{\delta}} + \sum_{m=0}^{N-1} \sum_{\tilde{f}^m(z)=x} \left(|D\tilde{f}^m(z)|^{-\tilde{\delta}} \sum_{k=M}^{\infty} \sum_{\tilde{f}^{kN}(y)=z} |D\tilde{f}^{kN}(y)|^{-\tilde{\delta}} \right)$$

$$< K + \sum_{k=M}^{\infty} K2^{-k} < \tilde{K}.$$

$\square$

Lemma 2.11. *Let $f : X \to \mathbb{C}$ be a map of class Δ without critical points. If the maximal invariant set $Q = \cap_{n \geq 0} f^{-n}(X)$ is hyperbolic then there exists $K > 0$ such that for every $x \in X$, we have $\Xi_2(f, x) < K$.*

Proof. Let $\epsilon > 0$ be such that f admits a holomorphic extension to an ϵ-neighborhood U of X and that $f|U$ has no critical points.

Let $Q_m = \cap_{0 \leq j \leq m} f^{-j}(X)$ so that $Q = \cap Q_m$. Hyperbolicity of $\cap_{m \geq 0} f^{-m}(X)$ implies that there exists $\rho > 0$ such that if $y \in Q_m$ then there exists a Jordan domain $D_y^m \subset U$ such that $f^m : D_y^m \to \mathbb{D}_\rho(f^m(y))$ is univalent and the diameter of $f^k(D_y^m)$, $0 \leq k \leq m$ is exponentially small in $m - k$.

It is easy to see that Q has empty interior (otherwise $\{f^m|\operatorname{int} Q\}$ would form a normal family, contradicting the hyperbolicity of $f|Q$). For $x \in X$, let us fix a compact set $Z_x \subset \mathbb{D}_\rho(x) \setminus Q$ of positive Lebesgue measure. We may assume that $\{Z_x\}_{x \in X}$ form a finite family. If $y \in Q_m$, let $Z_y^m = (f^m|D_y^m)^{-1}(Z_{f^m(x)})$. Since the diameters of the $\{y\} \cup Z_y^m$ go to 0 as m grows, there exists $M > 0$ such that $Z_y^m \cap Z_x = \emptyset$ whenever $y \in Q_m$, $x \in X$, and $m \geq M$. It follows that $Z_y^m \cap Z_{y'}^{m'} = \emptyset$ if $|m' - m| \geq M$. We also have $Z_y^m \cap Z_{y'}^m = \emptyset$ whenever $f^m(y) = f^m(y')$ and $y \neq y'$.

Notice that, by the Koebe Distortion Lemma, $|Df^m(y)|^{-2} \leq C|Z_y^m|/|Z_{f^m(y)}|$ for some $C > 0$. Thus, for every $x \in X$,

$$\sum_{\substack{m \geq 0}} \sum_{\substack{f^m(y)=x, \\ y \in Q_m}} |Df^m(y)|^{-2} \leq C \sum_{\substack{m \geq 0}} \sum_{\substack{f^m(y)=x, \\ y \in Q_m}} \frac{|Z_y^m|}{|Z_x|} \leq CM \frac{|U|}{|Z_x|}.$$

$\square$

Corollary 2.12. *Let $f : X \to \mathbb{C}$ be a map of class Δ without critical points. If the maximal invariant set $Q = \cap_{n \geq 0} f^{-n}(X)$ is hyperbolic then there exists $\delta < 2$, $K > 0$ such that for every $x \in X$ we have $\Xi_\delta(f, x) < K$. Moreover, δ and K can be chosen uniform over a compact family of maps as above.*

3. CRITICAL EXPONENT, HYPERBOLIC DIMENSION, AND DISSIPATIVITY

In what follows, f is assumed to be a Feigenbaum map, unless otherwise is explicitly stated.

3.1. Dissipativity. Recall that a measure μ is called *dissipative* if there exists a wandering set X of positive measure, i.e., $\mu(X) > 0$ and $f^{-n}X \cap f^{-m}X = \emptyset$ for all $m > n \geq 0$.

Lemma 3.1. *Let μ be a δ-conformal measure on J. Then the following properties are equivalent:*
  (i) *μ is dissipative;*
  (ii) *The Poincaré series $\Xi_\delta(z)$, $z \in \mathbb{V} \setminus \mathcal{O}$, is convergent;*
  (iii) *$f^n z \to \mathcal{O}$ for almost all z;*
  (iv) *For any little Julia set J^m, almost any $\mathrm{orb}(z)$ is eventually absorbed by the cycle of little Julia sets, $\cup_{k=0}^{p-1} J^m$ (here p is the period of J^m);*
  (v) *$\mu(J^m) > 0$ for all $m \in \mathbb{N}$;*
  (vi) *$\mu(J^1) > 0$.*
  If μ is conservative then almost all orbits are dense in $J(f)$.

Proof. (i) $\implies$ (ii). Let X be a wandering set of positive measure. Taking an appropriate local branch of f^{-n} on X, we can easily construct a wandering set $X' \subset \mathbb{D}(z, r)$ such that $\mathbb{D}(z, 2r) \cap \mathcal{O} = \emptyset$. Then by the Koebe Distortion Lemma,

$$\Xi_\delta(z) \asymp \sum \mu(f^{-n}(X')) < \infty.$$

(ii) $\implies$ (iii). Take any disk $D = \mathbb{D}(z, r)$ such that $\mathbb{D}(z, 2r) \cap \mathcal{O} = \emptyset$. Then

$$\sum \mu(f^{-n}(D)) \asymp \Xi_\delta(z) < \infty.$$

By the Borel-Cantelli Lemma, for a.e. $\zeta \in J$, the $\mathrm{orb}(\zeta)$ visits D only finitely many times, which implies (iii).

(iii) $\implies$ (iv). For a pre-renormalization $f^p : \mathbb{U}^m \to \mathbb{V}^m$ with period p, let $\mathbb{U}_k^m = f^k(\mathbb{U}^m)$, $k = 0, 1, \ldots, p$ (so that $\mathbb{U}_0^m = \mathbb{U}^m$ and $\mathbb{U}_p^m = \mathbb{V}^m$).

Since the little Julia sets $f^k J^m$ may touch each other only at their β-fixed points, which are not contained in the postcritical set $\mathcal{O}$, there is a choice of pre-renormalization such that the sets $\mathbb{U}_1^m \cap \mathcal{O}, \ldots \mathbb{U}_p^m \cap \mathcal{O}$ are pairwise disjoint. Then there exist neighborhoods $\Omega_k = f^k(\Omega_0) \subset \mathbb{U}_k^m$ of $\mathcal{O} \cap \mathbb{U}_k^m$ such that each Ω_k is disjoint from all Ω_i with $i \not\equiv k \mod p$.

If $f^n z \to \mathcal{O}$, then

$$f^n z \in \bigcup_{k=0}^{p-1} \Omega_k, \quad n \geq N.$$

Let us take some moment $n \geq N$ for which $f^n z \in \Omega_0$. Then $f^{n+p} z \in \Omega_p$ which is disjoint from all Ω_k, $k = 1, \ldots, p-1$. Hence $f^{n+p} \in \Omega_0$. It follows that $f^{n+pl} \in \Omega_0 \subset \mathbb{U}^m$ for all $l = 0, 1, \ldots$. Thus $f^n z \in J^m$.

(iv) $\implies$ (v). Take a little Julia set J^m. By (iv), $\bigcup_{n=0}^\infty f^{-n}(J^m)$ has full measure. This implies (v) by the δ-covariance of μ.

Of course, (v) $\implies$ (vi).

(vi) $\implies$ (i). Full preimage $f^{-1}(J^1)$ consists of two symmetric components, $f^{p-1}(J^1)$ and X. If $\mu(J^1) > 0$ then X is a wandering set of positive measure.

Let us prove the last assertion. If it is not valid then there is a disk D (intersecting J) and a forward invariant set $X \subset J$ of positive measure such that $D \cap X = \emptyset$. Since $f^N D = J$ for some N, D contains a set Y of positive measure such that $f^N Y \subset X$. Hence $f^n Y \cap Y = \emptyset$ for $n \geq N$. This easily implies that

$$\mu\{y \in Y : \ f^n y \notin Y, \ n = 1, 2, \dots \} > 0,$$

which gives us a wandering set of positive measure. $\square$

Since $\Xi_2(z) < \infty$, we conclude:

Corollary 3.2 (compare[L1]). *The 2-conformal measure δ_2 is always dissipative.*

3.2. **Minimal conformal measure.** Let $\mathrm{Spec}(f)$ denote the *conformal spectrum* of f, that is, the set of exponents δ for which there exists a δ-conformal measure on $J(f)$. Since weak* limits of conformal measures are conformal, we have:

Lemma 3.3. *For any quadratic-like map f, $\mathrm{Spec}(f)$ is a closed non-empty subset of $\mathbb{R}_+$.*

In particular, the minimal exponent

$$\delta_* = \delta_*(f) = \inf \mathrm{Spec}(f)$$

belongs to the conformal spectrum. The corresponding δ_* conformal measure μ_* will be called *minimal.*

All existing conformal measures with the possible exception of the minimal one (in case $\delta_* < 2$), are bound to be dissipative:

Lemma 3.4. *Any conformal measure μ_δ on J is dissipative for $\delta > \delta_*$.*

Proof. Assume μ_δ is conservative. Then by the last assertion of Lemma 3.1, for μ_δ-a.e. z, there exists a sequence of moments $n_k \to \infty$ such that $\mathrm{dist}(f^{n_k}z, \mathcal{O}) > 2r$, where $r > 0$ is independent of z. Let $D_k(z)$ be the component of $f^{-n_k}(\mathbb{D}_r(f^{n_k}z))$ containing z. Then by the Koebe Distortion Lemma, $D_k(z)$ is an oval of bounded shape such and the map $f^{n_k} : D_k \to \mathbb{D}_r(f^{n_k}z)$ has a bounded distortion (with an absolute bound). By covariance of conformal measures,

$$(3.1) \qquad \mu_\delta(D_k) \asymp (\mathrm{diam}\, D_k)^\delta \leq \epsilon_k (\mathrm{diam}\, D_k)^{\delta_*} \asymp \epsilon_k \, \mu_{\delta_*}(D_k),$$

where $\epsilon_k \to 0$ as $k \to \infty$.

By the Besikovich Covering Lemma (see [Mat, §2.7]), there is a covering of a set X of full μ_δ-measure with a family of balls D_k with intersection multiplicity bounded by some absolute constant N. Hence

$$\mu_\delta(J) = \mu_\delta(X) \leq \sum \mu_\delta(D_k) \leq \epsilon_k \sum \mu_{\delta^*}(D_k) \leq N\epsilon_k \, \mu_{\delta_*}(J).$$

Letting $k \to \infty$ we conclude that $\mu_\delta(J) = 0$ – contradiction. $\square$

As the minimal measure μ_* is concerned, both dissipative and conservative options are realizable, at least for quadratic-like maps. In the following statement we give a condition for dissipativity; later on (§6) we will deal with the conservative case.

Proposition 3.5. *Assume $\delta_*(f) = \delta_*(Rf) \equiv \delta_*$. Then the δ_*-conformal measure of Rf is dissipative (and thus all conformal measures of Rf are dissipative).*

Proof. Let $g : \mathbb{U}' \to \mathbb{V}'$ be a pre-renormalization of f. We let $\Xi_* \equiv \Xi_{\delta_*}$, and let μ_* and ν_* be δ_*-conformal measures for f and g respectively. We will use the measure μ_* to prove dissipativity of ν_*.

Take some disk $D = \mathbb{D}_r(z) \subset \mathbb{V}' \smallsetminus \mathbb{U}'$ that intersects $J(f)$. Since D is wandering under the dynamics of g,

$$\Xi_*(g, z) \asymp \sum \mu_*(g^{-n}(D)) < \infty.$$

By Lemma 3.1, ν_* is dissipative. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.6. *Let f be a Feigenbaum map which is recurrent under the dynamics of the renormalization operator (for instance, a renormalization fixed point). Then the δ_*-conformal measure of f is dissipative.*

Proof. Note that Theorem 2.7 implies that δ_* decays under renormalization: $\delta_*(Rf) \leq \delta_*(f)$ (since it is obviously true for the hyperbolic dimension).

Let $R^{n_k}f \to f$, and let g be a limit of $R^{n_k-1}f$. Then $Rg = f$, so that $\delta_*(g) \geq \delta_*(f)$. On the other hand, by the lower semicontinuity of δ_* (Lemma 2.8) we have $\delta_*(g) \leq \lim \delta_*(R^i(f)) \leq \delta_*(f)$. Thus $\delta_*(f) = \delta_*(g)$, and Proposition 3.5 completes the proof. $\qquad\qquad\square$

3.3. Conformal spectrum.

Proposition 3.7. *For any $\delta \geq \mathrm{HD}_{\mathrm{hyp}}(J)$, there exists a unique δ-conformal measure μ_δ. In particular, there exists always a unique 2-conformal measure μ_2.*

Proof. The uniqueness part follows from Theorem 2.9, so let us deal with the existence.

Let $S_n^r = \{\zeta : \ f^n\zeta = 0 \text{ and } |f^k\zeta| \geq r, \ k = 0, \dots n - 1\}$ and let $S^r = \cup S_n^r$. Let us consider the cut-off Poincaré series

$$\Xi_\delta^r(0) = \sum_{n=0}^{\infty} \sum_{\zeta \in S_n^r} \frac{1}{|Df^n(\zeta)|^\delta}.$$

Since the set $J^r = \{z : |f^n z| \geq r, \ n = 0, 1, \dots\}$ is hyperbolic,

$$\delta_{\mathrm{cr}}(f|J^r) = \mathrm{HD}(J^r) \leq \mathrm{HD}_{\mathrm{hyp}}(J),$$

It follows that the cut-off Poincaré series is convergent for $\delta > \mathrm{HD}_{\mathrm{hyp}}(J)$. Hence we can weigh the involved preimages of 0 by the terms of $\Xi_\delta^r(0)$ and consider the corresponding probability measure:

$$\mu_\delta^r = \frac{1}{\Xi_\delta^r(0)} \sum_{n=0}^{\infty} \sum_{\zeta \in S_n^r} \frac{\delta_\zeta}{|Df^n(\zeta)|^\delta}.$$

Let now $r \to 0$ ($\delta > \delta_*$ being fixed). Then $\Xi_\delta^r(0) \to \infty$. To see this, consider the pre-renormalizations $f_m : \mathbb{U}^m \to \mathbb{V}^m$ and pick some f_m-preimage ζ_m of 0. Since the derivatives $|Df_m(\zeta_m)|$ are bounded,

$$\lim_{r \to 0} \Xi_\delta^r(0) \geq \sum_{m=0}^{\infty} \frac{1}{|Df_m(\zeta_m)|^\delta} = \infty,$$

as was asserted.

Let us now consider a weak* limit μ_δ of measures μ_δ^r as $r \to 0$. We claim that this measure is δ-conformal. Let us first verify this outside the critical point. Let U be a small disk whose closure does not contain 0 and such that $f|U$ is injective. Then for sufficiently small $r > 0$,

$$f(S_n^r \cap U) = S_{n-1}^r \cap f(U),$$

and hence

$$\mu_\delta^r|f(U) = |Df|^\delta(\mu_\delta^r|U).$$

Passing to the limit, we conclude that

$$\mu_\delta|f(U) = |Df|^\delta(\mu_\delta|U),$$

provided $\mu_\delta(\partial U) = 0$. Since there is a base of disks with the latter property, the measure μ is δ-conformal everywhere outside 0.

But the critical value $f(0)$ cannot carry any mass, for otherwise each of the $f_m(0)$ would carry a definite mass, and the total mass of μ_δ would be infinite. (We also note that the critical point 0 cannot carry any mass either, for otherwise every preimage ζ_m would carry a definite mass $|Df_m|^{-\delta}\mu(0)$, and the total mass of μ_δ would be infinite.) This completes the proof.   □

We will make use of the easy part of Denker-Urbanski's Theorem:

Lemma 3.8. *For any quadratic-like map f, $\mathrm{HD}_{\mathrm{hyp}}(J) \leq \delta_*$.*

Proof. Let us consider a δ-conformal measure μ and a hyperbolic set $X \subset J$. We need to show that $\mathrm{HD}(X) \leq \delta$.

By a standard distortion argument using δ-covariance of μ, we obtain: $\mu_\delta(\mathbb{D}_r(z)) \asymp r^\delta$ for all disks $\mathbb{D}_r(z)$ centered at $z \in J$ of radius $r \leq 1$.

By the Besikovich Covering Lemma, for any $\epsilon > 0$, there exist a covering of X with disks $\mathbb{D}_{r_i}(z_i)$ of radii $r_i < \epsilon$ with intersection multiplicity bounded by N. Then

$$\sum r_i^\delta \asymp \sum \mu_\delta(\mathbb{D}_{r_i}(z_i))^\delta \leq N\mu_\delta(X).$$

Hence $h_\delta(X) < \infty$, so that $\delta \geq \mathrm{HD}(X)$.   □

Theorem 3.9. *For any Feigenbaum map, $\mathrm{HD}_{\mathrm{hyp}}(J) = \delta_* = \delta_{\mathrm{cr}}$.*

Proof. We will show that $\mathrm{HD}_{\mathrm{hyp}}(J) \leq \delta_* \leq \delta_{\mathrm{cr}} \leq \mathrm{HD}_{\mathrm{hyp}}(J)$.

The first inequality is the content of Lemma 3.8.

The middle inequality, $\delta_* \leq \delta_{\mathrm{cr}}$, holds since by Theorem 2.6 there exists a δ_{cr}-conformal measure, and by definition, δ_* is the minimum exponent of conformal measures on J.

To prove the last inequality, $\delta_{\mathrm{cr}} \leq \mathrm{HD}_{\mathrm{hyp}}(J)$, take any $\delta > \mathrm{HD}_{\mathrm{hyp}}(J)$. Then by Lemma 3.7, there exists a δ-conformal measure on J. By Lemma 3.4, it is dissipative. By Lemma 3.1, the Poincaré series $\Xi_\delta(z)$, $z \in \mathbb{C} \setminus \mathcal{O}$, is convergent. By definition, $\delta_{\mathrm{cr}} \leq \delta$, and the conclusion follows.   □

Remark 3.1. Our exposition is organized in such a way that it does not use explicitly the constructive part of Theorem 2.7 of Denker and Urbanski (which provides a minimal conformal measure μ_*), so that along the lines we obtain a new proof of it (in the Feigenbaum setting).

3.4. **Scaling at the critical point.** Let μ be a δ-conformal measure on the Julia set $J(f)$, and let

$$\sigma = \limsup_{r \to 0} \frac{\log \mu(\mathbb{D}_r)}{\log r}.$$

Lemma 3.10. *If $\mathrm{area}(J(f)) = 0$ then $\sigma \leq \delta$.*

Proof. Let us consider a fundamental annulus $A^1 = V^1 \setminus U^1$ of the pre-renormalization g of f, where V^1 and U^1 are domains constructed in §2.7. Then $\mu(A^1) > 0$ since $A^1 \cap J(f) \neq \emptyset$ and $\mathrm{supp}\,\mu = J(f)$. We also have $\mu(\partial A^1) = 0$.[4] Given a point $z \in U^1 \setminus J(g)$, there is a unique moment $n(z)$ such that $g^{n(z)} \in A^1$. Since $\mathrm{area}(J(g)) = 0$, $n(z)$ is well defined for a.e. $z \in U^1$.

Let us tile $\overline{A}^1$ by finitely many compact sets $\Delta_i \subset \overline{A}^1$ of positive μ-measure that have boundary of zero μ-measure, such that each Δ_i is contained in a Jordan domain $\tilde{\Delta}_i \subset \mathbb{V}^1 \setminus \mathcal{O}(f)$.

Pulling the Δ_i back by the appropriate branches of $g^{-n(z)}|\tilde{\Delta}_i$, we obtain compact sets Π_j covering almost all of U^1. Since U^1 is a nice domain, the Π_j have pairwise disjoint interiors. Moreover, there exists $\tilde{\Pi}_j \supseteq \Pi_j$ such that if $g^{n_j}(\Pi_j) = \Delta_i$ then $g^{n_j}|\tilde{\Pi}_j$ is a univalent map onto $\tilde{\Delta}_i$

---

[4]Indeed, we have $f^n(\partial A^1) \cap U^1 = \emptyset$, $n \geq 0$, which implies that $\mu(\partial A^1) = 0$ in both dissipative and conservative cases.

Since the $\tilde{\Pi}_j$ provide a definite Koebe space around Π_j, we can apply the Koebe Distortion Lemma to conclude that each Π_j is mapped with bounded distortion by some g^{n_j} onto the appropriate Δ_i. Together with the δ-covariance of μ, this yields:

$$(3.2) \qquad \mu(\Pi_j) \asymp \frac{\mu(\Delta_i)}{|Dg^{n_j}(z)|^{\delta}} \asymp (\operatorname{diam}\Pi_j)^{\delta}.$$

Take some $\epsilon > 0$. Then there exist arbitrary small disks $\mathbb{D}_r$ such that $\mu(\mathbb{D}_r) \le r^{\sigma-\epsilon}$. Since the hyperbolic diameter of Π_j in $\tilde{\Pi}_j$ is bounded and $0 \notin \tilde{\Pi}_j$, we have: $\operatorname{diam}\Pi_j \le C\operatorname{dist}(\Pi_j, 0)$. Hence there is a scaling factor $k > 1$ independent of r such that if $\Pi_j \cap \mathbb{D}_{r/k} \neq \emptyset$ then $\Pi_j \subset \mathbb{D}_r$. Since the Π_j have disjoint interiors and have boundaries of zero μ-measure, we conclude:

$$(3.3) \qquad \sum \mu(\Pi_j) \le \mu(\mathbb{D}_r) \le r^{\sigma-\epsilon}$$

where the summation is taken over those Π_j that intersect $\mathbb{D}_{r/k}$. Since these Π_j cover almost all of $\mathbb{D}_{r/k}$,

$$r^2 \asymp \operatorname{area}(\mathbb{D}_{r/k}) \le \sum \operatorname{area}\Pi_j \asymp \sum (\operatorname{diam}\Pi_j)^2.$$

Hence $r^{\delta} \le C\sum(\operatorname{diam}\Pi_j)^{\delta} \asymp \sum \mu(\Pi_j)$. Comparing it with (3.3), we conclude that $\sigma - \epsilon \le \delta$. Since it holds for any $\epsilon > 0$, we are done. $\qquad \square$

3.5. Hyperbolic dimension.

Proposition 3.11. *Let μ be a δ-conformal measure on J. If $\operatorname{area}(J) = 0$ then for any $z \in J$, $\underline{\mathrm{HD}}_\mu(z) \le \delta$.*

Proof. By Lemma 3.10, for any $\epsilon > 0$ there exists a $c > 0$ such that

$$(3.4) \qquad \mu(\mathbb{D}_r) \ge cr^{\delta+\epsilon}, \quad r \le \operatorname{diam} J.$$

Take a point $z \in J$. If the $\operatorname{orb}(z)$ does not accumulate on 0, then obviously $\mathrm{HD}_\mu(z) = \delta$. If it does, then consider the first entries of the $\operatorname{orb}(z)$ to the domains V^m from §2.7 and pull them back to z. We obtain a shrinking nest of neighborhoods D^m of z with bounded shape.

Let l_m be such that $f^{l_m}(D^m) = V^m$. Then it follows from the properties (C2-3) of the domains V^m that either $f^{l_m}(D^{m+1}) \subset U^m$, or $f^{l_m}(D^{m+1}) \subset (V^m \smallsetminus U^m)$. In either case, by property (G2) of the domains V^m, $\operatorname{area}(f^{l_m}(D_{m+1}))/\operatorname{area}(f^{l_m}(D_m))$ is bounded away from 1. Since $f^{l_m} : D_m \to V^m$ has bounded distortion (it admits a univalent extension onto $\mathbb{V}^m$), we conclude that $\operatorname{area}(D_{m+1})/\operatorname{area}(D_m)$ is bounded away from 1. So $\operatorname{diam}(D_m)$ decays exponentially fast.

Since $\operatorname{diam}(V^m) > C\lambda^k$ for some $C > 0$, $\lambda > 0$, it follows that $\operatorname{diam}(D^m) \le c\operatorname{diam}(V^m)^{\kappa}$ for some $c > 0$, $\kappa > 0$. It easily follows from (3.4), bounded distortion, and δ-covariance of μ that $\mu(D^m) \ge c(\operatorname{diam} D^m)^{\delta+\epsilon}$, and the conclusion follows. $\qquad \square$

Corollary 3.12. *If $\operatorname{area}(J) = 0$ then $\mathrm{HD}(J) = \mathrm{HD}_{\mathrm{hyp}}(J)$.*

Proof. By Theorem 2.7, it is enough to prove that for any δ-conformal measure μ on J, $\mathrm{HD}(J) \le \delta$. Take some $\epsilon > 0$. By Proposition 3.11, any point $z \in J$ is a center of arbitrary small disks $\mathbb{D}_r(z)$ such that $\mu(\mathbb{D}_r(z)) \ge r^{\delta+\epsilon}$. By the Besicovich Covering Lemma, J admits an arbitrary fine covering by such disks $D_k = \mathbb{D}(z_k, r_k)$ with intersection multiplicity at most N. Hence $\sum r_k^{\delta+\epsilon} \le N\mu(J)$, $h_\delta(J) < \infty$, and we are done. $\qquad \square$

Remark 3.2. This result also follows from Corollary 3.9 and the work of Bishop [B].

3.6. All dimensions and exponents are equal. Combining Theorem 3.9 with Corollary 3.12, we obtain:

Theorem 3.13. *Assume that a Feigenbaum Julia set J has zero area. Then*

$$\delta_{\mathrm{cr}} = \delta_* = \mathrm{HD}_{\mathrm{hyp}}(J) = \mathrm{HD}(J).$$

3.7. **Continuity properties of the critical exponent.** Let us formulate a couple of useful technical lemmas. Theorem 3.9 and Lemma 2.8 yield:

Corollary 3.14. *The critical exponent $\delta_{\mathrm{cr}}(f)$ depends lower semicontinuously on the Feigenbaum map f.*

The situation within one hybrid class is even better:

Lemma 3.15. *The critical exponent $\delta_{\mathrm{cr}}(g)$ is continuous on any hybrid class $\mathcal{H}_f$. Moreover, it is uniformly continuous over compact subsets K of quadratic-like germs: For any $\epsilon > 0$ there exists $\sigma > 0$ such that if $f, g \in K$, $g \in \mathcal{H}_f$ and $\mathrm{dist}(f, g) < \sigma$ then $|\delta_{\mathrm{cr}}(f) - \delta_{\mathrm{cr}}(g)| < \epsilon$.*

Proof. If $f, g \in K$, $g \in \mathcal{H}_f$ and $\mathrm{dist}(f, g) < \sigma$ then f and g are conjugate by a K-qc homeomorphism h, where dilatation K is close to 1. This map is κ-Hölder continuous, with exponent $\kappa = 1/K$.

Let $f : \mathbb{U} \to \mathbb{V}$ and $\mathcal{O} = \mathcal{O}(f)$. Consider a round disk D compactly contained in $\mathbb{V} \smallsetminus \mathcal{O}$ centered around some point x. If $f^m(y) = x$, let D_y be the component of $f^{-m}(x)$ containing y. Then by the Koebe Distortion Lemma, the maps $f^m : D_y \to D$ and $g^m : h(D_y) \to h(D)$ have bounded distortion. Thus $|Df^m(y)| \asymp (\mathrm{diam}\, D_y)^{-1}$ and $|Dg^m(h(y))| \asymp (\mathrm{diam}\, h(D_y))^{-1}$. This implies that

$$M^{-1}|Df^m(y)|^\kappa \leq |Dg_p^m(y)| \leq M|Df_p^m(y)|^K.$$

Hence $K^{-1} \leq \delta_{\mathrm{cr}}(f_p)/\delta_{\mathrm{cr}}(g_p) \leq K$, and we are done. $\qquad\square$

4. Recursive Quadratic Estimates

Let $X_{m,n}$, $m < n$ be the set of points in U^m that land in V^n under iterates of f_m, and let $X_n = X_{0,n}$. Let $Y_{m,n}$, $m < n$ be the set of points in A^n that never return to V^n under iterates of f_m, and let $Y_n = Y_{0,n}$.

We define the following quantities:

$$(4.1) \qquad \eta_{m,n} \equiv \eta_{m,n}(f) = \frac{|X_{m,n}|}{|U^m|}, \quad \eta_n = \eta_{0,n},$$

$$(4.2) \qquad \xi_{m,n} \equiv \xi_{m,n}(f) = \frac{|Y_{m,n}|}{|A^n|}, \quad \xi_n = \xi_{0,n},$$

$$(4.3) \qquad \rho_{m,n} \equiv \rho_{m,n}(f) = \frac{|U^n|}{|U^m|}.$$

Let us start with a couple of relations between these quantities:

Lemma 4.1. *Let f be a Feigenbaum map. There exists $C > 0$ only depending on the geometric bounds and $C_0 > 0$, depending also on the combinatorial bounds, such that*

$$(4.4) \qquad \max\{C_0^{-1}, 1 - C_0\xi_{m,n}\} \leq \frac{\eta_{m,n+1}}{\eta_{m,n}} \leq 1 - C^{-1}\xi_{m,n}.$$

Proof. We may assume that $m = 0$. The first landing map $\phi : X_n \to V^n$ sends each component of X_n univalently onto V^n with bounded distortion. Clearly, $\phi^{-1}(Y_n) \cap X_{n+1} = \emptyset$, so that

$$(4.5) \qquad \frac{\eta_{n+1}}{\eta_n} = p(X_{n+1}|X_n) \leq 1 - C^{-1}\xi_n.$$

Notice that $|V^{n+1}|/|V^n| > q$, where q depends also on the combinatorial bounds, so we get

$$(4.6) \qquad \frac{\eta_{n+1}}{\eta_n} > C_0^{-1}.$$

To conclude the remaining inequality, it is enough to show that $p(X_{n+1}|V^n) > 1 - C_0\xi_n$. Let S^n be the set of points in V^n which land in A^n and later return to V^n, and let W^n be the set of

connected components of S^n which are not contained in V^{n+1}. Then $|W^n|/|V^n| < 1 - q$, where q depends also on the combinatorial bounds. We set $\psi : W^n \to V^n$ to be the composition of the first landing to A^n followed by the first return to V^n. The restriction of ψ to each connected component of W^n is a univalent map onto V^n, moreover any iterate of ψ has bounded distortion by the Koebe Distortion Lemma (the restriction of ψ to each component of W^n extends to a univalent map onto $\mathbb{V}^n$). Since $|W^n|/|V^n| < 1 - q$, we have $|\psi^{-k}(W^n)| \le e^{-qk}|V^n|$.

Since almost every $x \in V^n$ either escapes through Y_n or eventually passes through V^{n+1}, we have

$$(4.7) \qquad \frac{|V^n \smallsetminus X_{n+1}|}{|V^n|} \le \sum_{k=0}^{\infty} \frac{|\psi^{-k}(Y_n)|}{|V^n|} \le C \sum_{k=0}^{\infty} e^{-qk}\xi_n \le C_0 \xi_n.$$

$\square$

Let $\Xi^{[j]}$ be the Poincaré series Ξ truncated at level j, so that summation in (1.1) is taken over $n \le j$ instead of all n. Let

$$(4.8) \qquad \omega_{m,n}(f, \delta) \equiv \omega_{m,n}(\delta) = \frac{1}{|A^n|} \int_{A^n} \Xi_\delta(f_m, x)dx, \quad \omega_n = \omega_{0,n}.$$

We will use $\omega^{[j]}$ to denote the truncation of ω at level j, that is, taking the integral of $\Xi^{[j]}$ instead of Ξ.

Our key estimates are presented in Lemmas 4.2 and 4.4.

Lemma 4.2. *Let f be a Feigenbaum map. Then for any $0 \le l < m < n$, there exists $C > 0$, only depending on the geometric bounds, and $\epsilon > 0$, depending on the geometric and combinatorial bounds, and on $n - l$, $m - l$, such that if $2 - \epsilon \le \delta \le 2$ then*

$$(4.9) \qquad \omega_{l,m}^{[j+1]}(\delta) \le C\frac{\eta_{l,m}}{\rho_{l,m}} + (1 - C^{-1}(\eta_{m,n} + \xi_{l,m}))\omega_{l,m}^{[j]}(\delta) + C\rho_{m,n}\xi_{m,n}\omega_{l,n}^{[j]}(\delta),$$

$$(4.10) \qquad \omega_{l,n}^{[j+1]}(\delta) \le C \left(\frac{\eta_{l,m}}{\rho_{l,m}} + \omega_{l,m}^{[j]}(\delta) \right) \omega_{m,n}^{[j]}(\delta).$$

Let $\tau_{m,n}$, $m < n$ be the infimum of $|Df_m^k(x)|$ over all $x \in U^m$, $k \ge 0$ such that $f_m^k(x) \in V^n$ and $f_m^j(x) \notin V^n$, $0 \le j \le k$, and let $\tau_n = \tau_{0,n}$. Let $\upsilon_{m,n}$, $m < n$ be the infimum of $|Df_m^k(x)|$ over all $x \in A^n$, $k \ge 1$ such that $f_m^k(x) \in A^m$ and $f_m^j(x) \notin A^m \cup A^n$, $1 \le j \le k$, and let $\upsilon_n = \upsilon_{0,n}$. Let us show that the product $\tau\upsilon$ is exponentially large in terms of $n - m$.

Lemma 4.3. *Let f be a Feigenbaum map. Then for every $0 \le m < n$, there exist $C > 0$, $\theta < 1$, only depending on the geometric bounds, such that for every $m < n$ we have*

$$(4.11) \qquad \tau_{m,n}\upsilon_{m,n} \ge C^{-1}\theta^{m-n}.$$

Proof. We may assume that $m = 0$. Let $f^k(x)$ be the first landing of x in V^n. Then there exists a domain $D \subset U$ containing x such that $f^k : D \to V^n$ is univalent and has bounded distortion. Let now $z \in A^n$ and let $y = (f^k|D)^{-1}(z)$. We claim that if $f^s(z) \in A$ then $|Df^{k+s}(y)| \ge C^{-1}\theta^{-n}$, which implies the result (notice that $|Df^k(x)||Df^s(z)| \ge C^{-1}|Df^{k+s}(y)|$ by bounded distortion of $f^k|D$).

To see this, let W be the complement of the postcritical set in some definite neighborhood of V, and consider the Poincaré metric $\| \cdot \|_W$ on W. Since A is away from the postcritical set, we have

$$|Df^{k+s}(y)| \ge C^{-1}\|Df^{k+s}(y)\|_W.$$

It follows from the Schwarz Lemma that $f : U \to V$ does not contract $\| \cdot \|_W$, so

$$\|Df(f^j(y))\|_W \ge 1, \quad 0 \le j \le k + s - 1.$$

Closer inspection, see [Mc3, Prop. 6.9] shows that whenever $f^j(y) \in A^i$ with $1 \leq i \leq n$ we have $\|Df(f^j(y))\|_W \geq \theta^{-1}$. Thus

$$\|Df^{k+s}(y)\|_W \geq \theta^{-n}.$$

$\square$

Lemma 4.4. *Let f be a Feigenbaum map. Then for every $0 \leq l < m < n$, there exist $C > 0$, only depending on the geometric bounds, such that if $\delta \leq 2$ then*

$$(4.12) \quad \omega_{l,m}(\delta) \geq C^{-1}\tau_{l,m}^{2-\delta}\frac{\eta_{l,m}}{\rho_{l,m}} + \max\{1 - C(\eta_{m,n} + \xi_{l,m}), 0\}\omega_{l,m}(\delta) + C^{-1}\upsilon_{m,n}^{2-\delta}\xi_{m,n}\rho_{m,n}\omega_{l,n}(\delta),$$

$$(4.13) \quad \omega_{l,n}(\delta) \geq C^{-1}\left(\tau_{l,m}^{2-\delta}\frac{\eta_{l,m}}{\rho_{l,m}} + \omega_{l,m}(\delta)\right)\omega_{m,n}(\delta).$$

4.1. **Proof of Lemmas 4.2 and 4.4.** We may assume that $l = 0$. To simplify notation, truncation of the Poincaré series (labeled by j or $j + 1$) will be implicit in what follows.

We define $Z_{m,n}$ as the set of points in A^m that do not return to A^m before passing through A^n. We let

$$(4.14) \qquad \kappa_{m,n} = \frac{|Z_{m,n}|}{|A^m|}.$$

Notice that

$$(4.15) \qquad \kappa_{m,n} \asymp \xi_m + \eta_{m,n}$$

In this section, an orbit (under iteration of f_r) will denote a sequence $(x_0, ..., x_k)$, $k \geq 0$, where $x_k \in V^r$ and $f_r(x_i) = x_{i+1}$, $i < k$. Unless explicitly stated otherwise, the orbits considered will be orbits under iteration of f.

An orbit will be said to be trivial if its length is 0.

Given a family $\mathcal{F}$ of orbits $(x_0, ..., x_k)$ under iteration of some f_r, we define a function $\mathbb{C} \to [0, \infty]$

$$(4.16) \qquad \Xi_\delta(\mathcal{F})(z) = \sum_{(x_0,...,x_k=z)\in\mathcal{F}} |Df_r^k(x_0)|^{-\delta}$$

(notice that we do not make f_r explicit in this notation).

We shall need some special notation for certain families of orbits. By $D \leftarrow E$, we will understand the family of orbits $(x_0, ..., x_k)$ with $x_0 \in E$ and $x_k \in D$. The family of orbits $(x_0, ..., x_k)$ with $x_0 \in E$, $x_k \in D$ and $x_1, ..., x_{k-1} \in S$ will be denoted $D \underset{S}{\leftarrow} E$. A "plus sign" over the arrow will indicate that only non-trivial orbits are considered. The justaposition of arrows will denote composition in the natural way. Thus

$$(4.17) \qquad D \underset{S}{\overset{+}{\leftarrow}} D \underset{S}{\leftarrow} E,$$

denotes the family of orbits $(x_0, ..., x_k)$, with $x_0 \in E$, $x_k \in D$, and such that there exists $0 \leq i < k$ such that $x_i \in D$ and $x_1, ..., x_{i-1}, x_{i+1}, ..., x_{k-1} \in S$.

When considering orbits under some f_r other than f, this will be indicated above the arrow, for instance, $D \underset{S}{\overset{f_r}{\leftarrow}} E$ or $D \overset{f_r,+}{\leftarrow} E$.

Given a function $\phi : \mathbb{C} \to [0, \infty]$ and a measurable set $N \subset \mathbb{C}$, we let

$$(4.18) \qquad I(\phi|N) = \int_N \phi(x)dx$$

and if $|N| > 0$, we let

$$(4.19) \qquad E(\phi|N) = \frac{1}{|N|}I(\phi|N).$$

More generally, if μ is a measure on $\mathbb{C}$ with $\mu(\mathbb{V}) < \infty$, we let

$$(4.20) \qquad E_\mu(\phi|N) = \frac{1}{\mu(N)} \int_N \phi(x) d\mu(x)$$

whenever $\mu(N) > 0$.

In this notation, we have for instance

$$(4.21) \qquad \omega_m(\delta) = E(\Xi_\delta(A^m \leftarrow U)|A^m).$$

Let us say that a family $\mathcal{F}$ of orbits is hyperbolic if $(x_0, ..., x_k) \in \mathcal{F}$ implies that $|Df^k(x_0)| \geq C\lambda^k$, $C > 0$, $\lambda > 1$. Notice that $\mathcal{F}$ is hyperbolic if and only if there exists a neighborhood W of 0 such that if $(x_0, ..., x_k) \in \mathcal{F}$ then $x_0, ..., x_{k-1} \notin W$.

Let $\delta_\mathcal{F}$ be the infimum over all δ such that $\Xi_\delta(\mathcal{F})$ is uniformly bounded. By Corollary 2.12, $\delta_\mathcal{F} < 2$ if $\mathcal{F}$ is hyperbolic.

Lemma 4.5. *With $\delta_\mathcal{F}$ defined as above, we have:*
   (1) *The map $\delta \mapsto \sup \Xi_\delta(\mathcal{F})$ is a continuous convex function of $\delta > \delta_\mathcal{F}$,*
   (2) *Let μ be a finite measure on $\mathbb{V}$, and let*

$$(4.22) \qquad \nu_\delta = \int \sum_{(x_0,...,x_k=x)\in\mathcal{F}} \frac{\delta_{x_0}}{|Df^k(x_0)|^\delta} d\mu(x)$$

   which is a finite measure for $\delta > \delta_\mathcal{F}$. Then for any Borelian set N, $\delta \mapsto \nu_\delta(N)$ is a continuous convex function of $\delta > \delta_\mathcal{F}$. In particular $\delta \mapsto \nu_\delta$, $\delta > \delta_\mathcal{F}$ is continuous in the weak topology.*

Proof. Follows immediately from the convexity of $\delta \mapsto x^{-\delta}$. $\qquad\qquad\qquad\square$

In the most common application of the previous lemma, we will take μ as Lebesgue and $\mathcal{F}$ as a set of hyperbolic first returns or landings, say $A^m \xleftarrow[U \smallsetminus V^n]{+} A^m$. In this case, the measure ν_2 is the restriction of Lebesgue measure to the set of points that land or return.

4.1.1. *Proof of (4.9).* Let

$$(4.23) \qquad B \equiv B^{m,n} = U \smallsetminus (A^m \cup A^n).$$

We clearly have

$$(4.24) \qquad \omega_m(\delta) = E(\Xi_\delta(A^m \leftarrow U)|A^m) = 1 + E(\Xi_\delta(A^m \xleftarrow{+} U)|A^m),$$

where the 1 accounts for the trivial orbits.

Since U is the disjoint union of A^n, A^m and B, we can split $A^m \xleftarrow{+} U$, the set of non-trivial orbits that terminate in A^m (under iteration by f) in three groups:
   (1) $A^m \xleftarrow[B]{} B$ which consists of orbits $(x_0, ..., x_k)$ terminating in A^m such that $x_0, ..., x_{k-1}$ do not intersect $A^m \cup A^n$,
   (2) $A^m \xleftarrow[B]{} A^n \leftarrow U$ which consists of orbits $(x_0, ..., x_k)$ such that $x_0, ..., x_{k-1}$ intersects $A^n \cup A^m$ and the last such intersection happens in A^n,
   (3) $A^m \xleftarrow[B]{+} A^m \leftarrow U$ which consists of non-trivial orbits $(x_0, ..., x_k)$ such that $x_0, ..., x_{k-1}$ intersects $A^m \cup A^n$ and the last such intersection happens in A^m.

Together with (4.24), this decomposition gives

$$(4.25)$$
$$\omega_m(\delta) = 1 + E(\Xi_\delta(A^m \xleftarrow[B]{} B)|A^m) + E(\Xi_\delta(A^m \xleftarrow[B]{} A^n \leftarrow U)|A^m) + E(\Xi_\delta(A^m \xleftarrow[B]{+} A^m \leftarrow U)|A^m).$$

Notice that

$$(4.26) \qquad I(\Xi_2(A^m \underset{B}{\leftarrow} B)|A^m),$$

is the area of the set of points in B that land in A^m before passing through V^n, which is certainly smaller than $\eta_m|U|$. Thus we have

$$(4.27) \qquad E(\Xi_2(A^m \underset{B}{\leftarrow} B)|A^m) \leq \eta_m \frac{|U|}{|A^m|} = \frac{\eta_m}{\rho_m} \frac{|U^m|}{|A^m|}.$$

By hyperbolicity of the set of orbits that avoid V^n, we have

$$(4.28) \qquad E(\Xi_\delta(A^m \underset{B}{\leftarrow} B)|A^m) \leq C \frac{\eta_m}{\rho_m}, \quad \delta \approx 2,$$

by continuity of the left hand side in $\delta \approx 2$.

We have the obvious estimate

$$(4.29) \qquad E(\Xi_\delta(A^m \underset{B}{\leftarrow} A^n \leftarrow U)|A^m) \leq E(\Xi_\delta(A^m \underset{B}{\leftarrow} A^n)|A^m) \sup_{A^n} \Xi_\delta(A^n \leftarrow U),$$

By bounded oscillation of the Poincaré series in A^n, we have

$$(4.30) \qquad \inf_{A^n} \Xi_\delta(A^n \leftarrow U) \asymp \sup_{A^n} \Xi_\delta(A^n \leftarrow U) \asymp E(\Xi_\delta(A^n \leftarrow U)|A^n) = \omega_n(\delta).$$

Moreover,

$$(4.31) \qquad I(\Xi_2(A^m \underset{B}{\leftarrow} A^n)|A^m)$$

is the area of the set of points in A^n that land in A^m before passing through V^n. This is precisely $|A^n|\xi_{m,n}$, so

$$(4.32) \qquad E(\Xi_2(A^m \underset{B}{\leftarrow} A^n)|A^m) = \xi_{m,n} \frac{|A^n|}{|A^m|} = \xi_{m,n}\rho_{m,n} \frac{|U^m|}{|A^m|} \frac{|A^n|}{|U^n|} \leq C\xi_{m,n}\rho_{m,n}.$$

By hyperbolicity of the set of orbits avoiding U^n, we get

$$(4.33) \qquad E(\Xi_\delta(A^m \underset{B}{\leftarrow} A^n)|A^m) \leq C\xi_{m,n}\rho_{m,n}, \quad \delta \approx 2.$$

Estimates (4.29), (4.30) and (4.33) imply

$$(4.34) \qquad E(\Xi_\delta(A^m \underset{B}{\leftarrow} A^n \leftarrow U)|A^m) \leq C\xi_{m,n}\rho_{m,n}\omega_n(\delta), \quad \delta \approx 2.$$

Let us estimate the last term of (4.25). We have the identity

$$(4.35) \qquad \frac{E(\Xi_2(A^m \underset{B}{\overset{+}{\leftarrow}} A^m \leftarrow U)|A^m)}{E(\Xi_2(A^m \leftarrow U)|A^m)} = \frac{E(\Xi_2(A^m \smallsetminus Z_{m,n} \leftarrow U)|A^m)}{E(\Xi_2(A^m \leftarrow U)|A^m)} = 1 - \frac{E(\Xi_2(Z_{m,n} \leftarrow U)|A^m)}{E(\Xi_2(A^m \leftarrow U)|A^m)}.$$

By bounded oscillation of Poincaré series, we have

$$(4.36) \qquad \frac{E(\Xi_\delta(Z_{m,n} \leftarrow U)|A^m)}{E(\Xi_\delta(A^m \leftarrow U)|A^m)} \asymp \kappa_{m,n}.$$

By hyperbolicity of the set of orbits that do not pass through V^n, we have

$$(4.37)$$
$$E(\Xi_\delta(A^m \underset{B}{\overset{+}{\leftarrow}} A^m \leftarrow U)|A^m) \leq (1 - C^{-1}\kappa_{m,n})E(\Xi_\delta(A^m \leftarrow U)|A^m) = (1 - C^{-1}\kappa_{m,n})\omega_m(\delta), \quad \delta \approx 2.$$

Plugging (4.28), (4.34) and (4.37) in (4.25), we get

$$(4.38) \qquad \omega_m(\delta) = 1 + C\frac{\eta_m}{\rho_m} + (1 - C^{-1}\kappa_{m,n})\omega_m(\delta) + C\xi_{m,n}\rho_{m,n}\omega_n(\delta), \quad \delta \approx 2$$

which implies (4.9) (notice that $\eta_m \geq \rho_m$, so the 1 can be put inside $C\eta_m/\rho_m$).

4.1.2. *Proof of (4.10).* Let us split $A^n \leftarrow U$ into two groups of orbits:

(1) $A^n \xleftarrow{\quad U \smallsetminus A^m \quad} U \smallsetminus A^m$,

(2) $A^n \xleftarrow{\quad U \smallsetminus A^m \quad} A^m \leftarrow U$.

We further split $A^n \xleftarrow{\quad U \smallsetminus A^m \quad} U \smallsetminus A^m$ into two groups:

(1a) $A^n \xleftarrow{\quad U \smallsetminus A^m \quad} U^m$,

(1b) $A^n \xleftarrow{\quad U \smallsetminus A^m \quad} U^m \xleftarrow{\quad U \smallsetminus V^m \quad} U \smallsetminus V^m$.

Notice that we can write

$$(4.39) \qquad \left(A^n \xleftarrow{\quad U \smallsetminus A^m \quad} A^m \right) = \left(A^n \xleftarrow{\quad U \smallsetminus A^m \quad} U^m \xleftarrow{\quad U \smallsetminus V^m \quad} A^m \right).$$

Thus we have

$$(4.40) \qquad \omega_n(\delta) = E(\Xi_\delta(A^n \leftarrow U)|A^n) = E(\Xi_\delta(A^n \xleftarrow{\quad U \smallsetminus A^m \quad} U^m)|A^n)$$
$$+ E(\Xi_\delta(A^n \xleftarrow{\quad U \smallsetminus A^m \quad} U^m \xleftarrow{\quad U \smallsetminus V^m \quad} U \smallsetminus V^m)|A^n)$$
$$+ E(\Xi_\delta(A^n \xleftarrow{\quad U \smallsetminus A^m \quad} U^m \xleftarrow{\quad U \smallsetminus V^m \quad} A^m \leftarrow U)|A^n).$$

This gives the estimate

$$(4.41)$$
$$\frac{E(\Xi_\delta(A^n \leftarrow U)|A^n)}{E(\Xi_\delta(A^n \xleftarrow{\; U \smallsetminus A^m \;} U^m)|A^n)} \leq 1 + \sup_{U^m} \Xi_\delta(U^m \xleftarrow{\; U \smallsetminus V^m \;} U \smallsetminus V^m) + \sup_{U^m} \Xi_\delta(U^m \xleftarrow{\; U \smallsetminus V^m \;} A^m \leftarrow U).$$

Since f_m is the first return map from U^m to V^m, there is a natural correspondence between the orbits of $A^n \xleftarrow{\quad U \smallsetminus A^m \quad} U^m$ and $A^n \xleftarrow{\; f_m \;} U^m$. It follows that

$$(4.42) \qquad \Xi_\delta(A^n \xleftarrow{\quad U \smallsetminus A^m \quad} U^m) = \Xi_\delta(A^n \xleftarrow{\; f_m \;} U^m),$$

thus

$$(4.43) \qquad E(\Xi_\delta(A^n \xleftarrow{\; U \smallsetminus A^m \;} U^m)|A^n) = E(\Xi_\delta(A^n \xleftarrow{\; f_m \;} U^m)|A^n) = \omega_{m,n}(\delta).$$

By the Markov property[5] of the first landing map to V^m, we have

$$(4.44)$$
$$\sup_{U^m} \Xi_2(U^m \xleftarrow{\; U \smallsetminus V^m \;} U \smallsetminus V^m) \asymp \inf_{U^m} \Xi_2(U^m \xleftarrow{\; U \smallsetminus V^m \;} U \smallsetminus V^m) \asymp E(\Xi_2(U^m \xleftarrow{\; U \smallsetminus V^m \;} U \smallsetminus V^m)|U^m) \asymp \frac{\eta_m}{\rho_m}.$$

By hyperbolicity of the set of orbits avoiding U^m,

$$(4.45) \qquad \sup_{U^m} \Xi_\delta(U^m \xleftarrow{\quad U \smallsetminus V^m \quad} U \smallsetminus V^m) \leq C \frac{\eta_m}{\rho_m}, \quad \delta \approx 2.$$

Similarly,

$$(4.46) \qquad \sup_{U^m} \Xi_2(U^m \xleftarrow{\quad U \smallsetminus V^m \quad} A^m) \asymp \inf_{U^m} \Xi_2(U^m \xleftarrow{\quad U \smallsetminus V^m \quad} A^m) \asymp E(\Xi_2(U^m \xleftarrow{\quad U \smallsetminus V^m \quad} A^m)|U^m) \asymp 1,$$

and by hyperbolicity of the set of orbits avoiding U^m, we get

$$(4.47) \qquad E(\Xi_\delta(U^m \xleftarrow{\quad U \smallsetminus U^m \quad} A^m)|U^m) \leq C, \quad \delta \approx 2.$$

---

[5]Let us remark that one does not need to use the Markov property to conclude the upper bound in estimates (4.44) and (4.46).

Estimate (4.47) implies

$$(4.48) \qquad \sup_{U^m} \Xi_\delta(U^m \xleftarrow{\;\;\;\;}_{U \smallsetminus V^m} A^m \leftarrow U) \leq C \sup_{A^m} \Xi_\delta(A^m \leftarrow U), \quad \delta \approx 2,$$

and by bounded oscillation of the Poincaré series in A^m, we get

$$(4.49) \qquad \sup_{U^m} \Xi_\delta(U^m \xleftarrow{\;\;\;\;}_{U \smallsetminus V^m} A^m \leftarrow U) \leq C\omega_m(\delta), \quad \delta \approx 2.$$

Plugging (4.45) and (4.49) in (4.41) we get

$$(4.50) \qquad \frac{E(\Xi_\delta(A^n \leftarrow U)|A^n)}{E(\Xi_\delta(A^n \xleftarrow{\;\;\;\;}_{U \smallsetminus A^m} U^m)|A^n)} \leq 1 + C\frac{\eta_m}{\rho_m} + C\omega_m(\delta), \quad \delta \approx 2.$$

This and (4.43) imply (4.10).

4.1.3. *Proof of (4.12).* It is easy to see that if $x \in A^n$ and $f^k(x) \in A^m$ then $|Df^k(x)| \geq C^{-1}\theta^{m-n}$ for some $\theta < 1$. This follows from the Schwarz Lemma and the Koebe Distortion Lemma, see Lemma 4.1 for an analogous estimate.

For simplicity, we will assume that if $x \in A^m$ and $f^k(x) \in A^m$ then $|Df^k(x)| \geq 1$ (a slightly different argument allows to deal with the general case[6]). This assumption can be made without loss of generality, see Remark A.1.

As in the proof of (4.9), we shall use the identity (4.25).

Notice that the argument for the upper bound on $E(\Xi_2(A^m \xleftarrow{}_{B} B)|A^m)$, (4.27) also implies a lower bound of the same order,

$$(4.51) \qquad E(\Xi_2(A^m \xleftarrow{}_{B} B)|A^m) \geq C^{-1}\frac{\eta_m}{\rho_m}.$$

Similarly, the argument of (4.32) gives the bound

$$(4.52) \qquad E(\Xi_2(A^m \xleftarrow{}_{B} A^n)|A^m) \geq C^{-1}\xi_{m,n}\rho_{m,n}.$$

Thus

$$(4.53) \qquad E(\Xi_\delta(A^m \xleftarrow{}_{B} B)|A^m) \geq C^{-1}\frac{\eta_m}{\rho_m}\tau_m^{2-\delta},$$

$$(4.54) \qquad E(\Xi_\delta(A^m \xleftarrow{}_{B} A^n)|A^m) \geq C^{-1}\xi_{m,n}\rho_{m,n}\upsilon_{m,n}^{2-\delta}.$$

Recall equality (4.35). The expansion of the Euclidean metric along orbits of $A^m \xleftarrow{+}_{B} A^m$ implies that

$$(4.55) \qquad \frac{E(\Xi_\delta(A^m \xleftarrow{+}_{B} A^m \leftarrow U)|A^m)}{E(\Xi_\delta(A^m \leftarrow U)|A^m)} \geq \frac{E(\Xi_\delta(A^m \smallsetminus Z_{m,n} \leftarrow U)|A^m)}{E(\Xi_\delta(A^m \leftarrow U)|A^m)} = 1 - \frac{E(\Xi_\delta(Z_{m,n} \leftarrow U)|A^m)}{E(\Xi_\delta(A^m \leftarrow U)|A^m)},$$

for $\delta \leq 2$. By (4.36), we get

$$(4.56) \quad E(\Xi_\delta(A^m \xleftarrow{+}_{B} A^m \leftarrow U)|A^m) \geq (1 - C\kappa_{m,n})E(\Xi_\delta(A^m \leftarrow U)|A^m) = (1 - C\kappa_{m,n})\omega_m(\delta).$$

Plugging (4.53), (4.54) and (4.56) in (4.25), we obtain (4.12).

---

[6]Let W be the complement of the postcritical set of f in some definite neighborhood of V and let $\|\cdot\|_W$ denote the Poincaré metric on W. By the Schwarz Lemma, $f : U \to V$ expands $\|\cdot\|_W$. Let $\|\cdot\|_*$ denote the metric on $\mathbb{C}$ which is equal to the usual Euclidean metric outside $\cup A^r$ and in each A^r is given by $\|\cdot\|$, suitably rescaled so that the $\|\cdot\|_*$ area of A^r coincides with the usual Euclidean area of A^r. Since the A^r are away from the postcritical set, the metric $\|\cdot\|_*$ is comparable to the usual Euclidean metric everywhere. Moreover, we get improved inequality that if $x \in A^m$ and $f^k(x) \in A^m$ then $\|Df^k(x)\|_* \geq 1$ (instead of $|Df^k(x)| > C^{-1}$). If one defines Poincaré series, $\omega_{m,n}$, $\eta_{m,n}$, $\xi_{m,n}$,... with respect to the metric $\|\cdot\|_*$, the resulting objects are comparable to the Euclidean ones. Thus, it is enough to prove (4.12) for those modified objects, which can be done with the argument given below.

4.1.4. *Proof of (4.13).* We will use the identity (4.40), which leads to the lower bound (analogous to (4.41))

(4.57)
$$\frac{E(\Xi_\delta(A^n \leftarrow U)|A^n)}{E(\Xi_\delta(A^n \underset{U \smallsetminus A^m}{\longleftarrow} U^m)|A^n)} \geq 1 + \inf_{U^m} \Xi_\delta(U^m \underset{U \smallsetminus V^m}{\longleftarrow} U \smallsetminus V^m) + \inf_{U^m} \Xi_\delta(U^m \underset{U \smallsetminus V^m}{\longleftarrow} A^m \leftarrow U).$$

By (4.44) and (4.46),

(4.58)
$$\inf_{U^m} \Xi_\delta(U^m \underset{U \smallsetminus V^m}{\longleftarrow} U \smallsetminus V^m) \geq C^{-1} \tau_m^{2-\delta} \frac{\eta_m}{\rho_m},$$

(4.59)
$$\inf_{U^m} \Xi_\delta(U^m \underset{U \smallsetminus V^m}{\longleftarrow} A^m) \geq C^{-1}.$$

By bounded oscillation of Poincaré series in A^m and (4.59), we have

(4.60)
$$\inf_{U^m} \Xi_\delta(U^m \underset{U \smallsetminus V^m}{\longleftarrow} A^m \leftarrow U) \geq C^{-1} \omega_m(\delta).$$

Plugging (4.58) and (4.60) in (4.57), and using (4.43) one gets (4.13).

5. Trichotomy

Let f be a periodic point of renormalization of period p. In this case, we may assume that the fundamental annulus A^n form a periodic sequence up to rescaling by $\rho_n^{-1/2}$ (see Remark A.2). Then $\eta_{m,n}, \xi_{m,n},\dots$ only depend on $n - m$ and the congruence class of m modulo p.

The results in this section will be obtained from some particular cases of the estimates in Lemmas 4.2 and 4.4. Setting $n = 2m$ an integer multiple of p, the estimates can be rewritten as

(5.1)
$$\omega_m^{[j+1]}(\delta) \leq C\frac{\eta_m}{\rho_m} + (1 - C^{-1}(\eta_m + \xi_m)\omega_m^{[j]}(\delta) + C\xi_m\rho_m\omega_{2m}^{[j]}(\delta), \quad \delta \approx 2,$$

(5.2)
$$\omega_{2m}^{[j+1]}(\delta) \leq C\left(\frac{\eta_m}{\rho_m} + \omega_m^{[j]}(\delta)\right)\omega_m^{[j]}(\delta), \quad \delta \approx 2,$$

(5.3)
$$\omega_m(\delta) \geq C^{-1}\tau_m^{2-\delta}\frac{\eta_m}{\rho_m} + \max\{1 - C(\eta_m + \xi_m), 0\}\omega_m(\delta) + C^{-1}v_m^{2-\delta}\xi_m\rho_m\omega_{2m}(\delta),$$

(5.4)
$$\omega_{2m}(\delta) \geq C^{-1}\left(\tau_m^{2-\delta}\frac{\eta_m}{\rho_m} + \omega_m(\delta)\right)\omega_m(\delta).$$

Putting the respective estimates together, we get the following quadratic estimates:

(5.5)
$$\omega_m^{[j+1]}(\delta) \leq C\frac{\eta_m}{\rho_m} + (1 - C^{-1}(\eta_m + \xi_m) + C\eta_m\xi_m)\omega_m^{[j]}(\delta) + C\xi_m\rho_m\omega_m^{[j]}(\delta)^2, \quad \delta \approx 2,$$

(5.6)
$$\omega_m(\delta) \geq C^{-1}\tau_m^{2-\delta}\frac{\eta_m}{\rho_m} + (\max\{1 - C(\eta_m + \xi_m), 0\} + C^{-1}(\tau_m v_m)^{2-\delta}\eta_m\xi_m)\omega_m(\delta) + Cv_m^{2-\delta}\xi_m\rho_m\omega_m(\delta)^2.$$

Let us now explore those relations.

Lemma 5.1. *Let f be a periodic point of renormalization of period p. If $m = 0 \bmod p$ then there exists $C > 0$ (only depending on the geometric bounds) such that*

(1) *If the polynomial*

(5.7)
$$P : x \mapsto C\frac{\eta_m}{\rho_m} + (1 - C^{-1}(\eta_m + \xi_m) + C\eta_m\xi_m)x + C\xi_m\rho_m x^2$$

has a real positive fixed point then $\delta_{\mathrm{cr}} < 2$ and for δ close to 2, the smallest such fixed point gives an upper bound on $\omega_m(\delta)$.

(2) *If $\delta \geq \delta_{\mathrm{cr}}$, the polynomial*

$$(5.8) \quad Q_\delta : x \mapsto C^{-1}\tau_m^{2-\delta}\frac{\eta_m}{\rho_m} + (\max\{1 - C(\eta_m + \xi_m), 0\} + C^{-1}(\tau_m v_m)^{2-\delta}\eta_m\xi_m)x + Cv_m^{2-\delta}\xi_m\rho_m x^2$$

has a fixed point and the smallest (respectively, largest) such fixed point gives a lower bound (respectively, upper bound) to $\omega_m(\delta)$.

Proof. By repeated application of (5.5), for $\delta \approx 2$ and for every $j \geq 0$, we have $\omega_m^{[j]}(\delta) \leq P^{j+1}(0)$. If there exists a positive fixed point for P then the smallest such fixed point attracts the orbit of 0. This gives the first statement.

By (5.6), for $\delta > \delta_{\mathrm{cr}}$, $Q_\delta(\omega_m(\delta)) \leq \omega_m(\delta)$. Thus $\omega_m(\delta)$ is contained in the interval bounded by the fixed points of Q_δ. This still holds for $\delta = \delta_{\mathrm{cr}}$ by continuity of the coefficients of Q_δ. This implies the second statement. $\square$

Theorem 5.2 (Balanced case). *Let f be a periodic point of renormalization of period p. Then there exists $C > 0$, only depending on the geometric bounds, such that if $\delta_{\mathrm{cr}} = 2$ then*

$$(5.9) \qquad\qquad C^{-1} \leq \frac{\eta_m}{\xi_m} \leq C, \quad m \equiv 0 \mod p.$$

Moreover,

$$(5.10) \qquad\qquad \xi_m \asymp \eta_m \asymp \frac{1}{m}$$

for all $m \in \mathbb{N}$ with the constants depending also on p.

Proof. If (5.9) does not hold then

$$(5.11) \qquad\qquad \xi_m\eta_m << (\xi_m + \eta_m)^2,$$

and then the quadratic polynomial (5.7) has two positive fixed points. By the first part of Lemma 5.1, $\delta_{\mathrm{cr}} < 2$ contradicting the assumption. The same argument applied to the k-fold renormalization of f, $k = 0, 1, \ldots, p - 1$, gives

$$\eta_m \asymp \xi_m, \quad m \equiv k \mod p,$$

with the constant depending on k. Thus, $\eta_m \asymp \xi_m$ for all m, with the constant depending on p. By Lemma 4.1, this implies that

$$\eta_m - C\eta_m^2 \leq \eta_{m+1} \leq \eta_m - C^{-1}\eta_m^2.$$

Hence

$$C^{-1} \leq \frac{1}{\eta_{m+1}} - \frac{1}{\eta_m} \leq C,$$

and thus $1/\eta_m \asymp m$. $\square$

The following result gives Theorem D stated in the Introduction.

Corollary 5.3. *Let f be a periodic point of renormalization. If $|J(f)| > 0$ then $\delta_{\mathrm{cr}} < 2$.*

Proof. Since f is a renormalization periodic point, $f_m : U^m \to V^m$ is a rescaling of $f : U \to V$ for $m \equiv 0 \mod p$. Hence

$$\eta_m = p(X_m|U^m) \geq p(J(f_m)|U^m) = p(J(f)|U) > 0, \quad m \equiv 0 \mod p.$$

Then by Lemma 4.1, $\xi_m \to 0$, so that (5.9) is violated - this contradicts $\delta_{\mathrm{cr}} = 2$. $\square$

Lemma 5.4. *Let f be a periodic point of renormalization of period p and let $m \equiv 0 \mod p$. Then there exists $C > 0$ depending only on the geometric bounds such that*

$$(5.12) \qquad \omega_m(2) \geq C^{-1} \frac{\eta_m}{(\xi_m + \eta_m)\rho_m} \quad if \quad \eta_m \xi_m < C^{-1}, [7]$$

$$(5.13) \qquad \omega_m(2) \leq C \frac{\eta_m}{(\xi_m + \eta_m)\rho_m} \quad if \quad \left| \ln \frac{\eta_m}{\xi_m} \right| > C,$$

$$(5.14) \qquad \omega_{2m}(2) \asymp \left(\frac{\eta_m}{\rho_m} + \omega_m(2) \right) \omega_m(2) \asymp \omega_m(2)^2.$$

In particular, if $\eta_m \xi_m < C^{-1}$ and if either η_m/ξ_m or $\omega_m \rho_m$ is small, then both are small and indeed they are comparable.

Proof. If $\eta_m \xi_m$ is small then the linear coefficient of Q_2 is comparable with $\eta_m + \xi_m$, and a direct calculation of the smallest fixed point of Q_2 gives the lower bound (5.12) on $\omega_m(2)$.

If $|\ln(\eta_m/\xi_m)|$ is large, then (5.11) holds, and the polynomial P has two real fixed points. A direct calculation of the smallest one gives the upper bound (5.13) on $\omega_m(2)$.

By (5.2), (5.4),

$$(5.15) \qquad \omega_{2m}(2) \asymp \left(\frac{\eta_m}{\rho_m} + \omega_m(2) \right) \omega_m(2).$$

By (5.3), $\omega_m(2) > C^{-1}\eta_m/\rho_m$ which shows that $(\eta_m/\rho_m) + \omega_m(2) \asymp \omega_m(2)$ which together with (5.15) implies (5.14).

The last assertion is a straightforward consequence of (5.12) and (5.13). $\qquad \square$

Theorem 5.5 (Lean case)**.** *Let f be a periodic point of renormalization of period p. Then there exists $C > 0$, only depending on the geometric bounds, such that if for some $m \equiv 0 \mod p$ we have*

$$(5.16) \qquad \eta_m < C^{-1}\xi_m$$

then $\inf \xi_n > 0$.[8] Moreover, $\eta_n \to 0$ exponentially fast.

Proof. Since η_n monotonically decrease, assumption (5.16) implies that η_n (and hence $\eta_n \xi_n$) are small for $n \geq m$. It also implies by Lemma 5.4, that $\omega_m \rho_m$ is small.

By (5.14), $\omega_{2m} \leq K\omega_m^2$, so that $K\omega_{2m}\rho_{2m} \leq (K\omega_m\rho_m)^2$. Iterating this estimate k times, we conclude that

$$K\omega_{2^k m}\rho_{2^k m} \leq (K\omega_m\rho_m)^{2^k} \equiv q^{2^k}$$

where $q < 1$ provided $\omega_m \rho_m$ is small enough. By Lemma 5.4,

$$\frac{\eta_{2^k m}}{\xi_{2^k m}} = O(q^{2^k}).$$

Since η_n monotonically decrease, we conclude that they decay exponentially: $\eta_n = O(q^{n/2})$.

Furthermore, it follows from the renormalization periodicity that the ξ_{kp} monotonically decrease in k. It is also easy to check that $\xi_{n+1} \asymp \xi_n$. It follows that

$$\inf \xi_n = 0 \implies \xi_n \to 0.$$

Since η_n decay exponentially, $\limsup \xi_n > 0$ by Lemma 4.1. Hence ξ_n are bounded away from 0. $\qquad \square$

---

[7]Note that this assumption is always satisfied on high levels since $\eta_m \xi_m \to 0$ by Lemma 4.1

[8]An explicit lower bound for $\inf \xi_n$ in terms of the geometric and combinatorial bounds and the period of f can be worked out.

Lemma 5.6. *Let f be a periodic point of renormalization of period p and let $m \equiv 0 \mod p$. If area$(J(f)) = 0$ then there exists $C > 0$, only depending on the geometric bounds, such that*

$$(5.17) \qquad C^{-1} \frac{\eta_m}{\xi_m \rho_m} \leq \omega_m(2) \leq C \frac{\eta_m}{\xi_m \rho_m}.$$

Proof. Since the set Y_m is wandering,

$$\int_{Y_m} \Xi_2(f, x) dx = \text{area} \left(\bigcup_{n=0}^{\infty} f^{-n}(Y_m) \right) \equiv |Z_m|.$$

The set Z_m consists of points that land at A^m at least once but at most finitely many times. Hence

$$X_m \setminus J(f) \subset Z_m \subset X_m.$$

Since area of the Julia set is zero, $|Z_m| = |X_m|$ and hence

$$(5.18) \qquad \int_{Y_m} \Xi_2(f, x) dx = |X_m|.$$

By bounded oscillation of the Poincaré series,

$$(5.19) \qquad \omega_m(2) \asymp \frac{1}{|Y_m|} \int_{Y_m} \Xi_2(f, x) dx = \frac{|X_m|}{|Y_m|} \asymp \frac{\eta_m}{\xi_m \rho_m}.$$

$\square$

Lemma 5.7. *Let f be a periodic point of renormalization of period p. If $\delta_{\mathrm{cr}} < 2$ then there exists $\theta < 1$, only depending on δ_{cr} and the geometric bounds such that*

$$(5.20) \qquad \frac{\eta_m \xi_m}{(\eta_m + \xi_m)^2} \leq \theta^m, \quad m \equiv 0 \mod p.$$

Proof. By Lemma 5.1, the quadratic polynomial $Q_{\delta_{\mathrm{cr}}}$ has a real fixed point and hence has the positive discriminant. Together with Lemma 4.3 this implies (5.20). $\square$

Theorem 5.8 (Black hole case)**.** *Let f be a periodic point of renormalization of period p. Then there exists $C > 0$, only depending on the geometric bounds, such that if for some $m \equiv 0 \mod p$ we have*

$$(5.21) \qquad \eta_m > C \xi_m$$

then $\lim \eta_n > 0$.[9] Moreover, $\xi_n \to 0$ exponentially fast.

Proof. Assumption (5.21) allows us to apply (5.13). Since (5.21) and (5.13) are not compatible with (5.17), area of the Julia set must be positive and hence $\lim \eta_n > 0$.[10]

Moreover, by Corollary 5.3, $\delta_{\mathrm{cr}} < 2$ which implies by Lemma 5.7 that $\eta_n \xi_n \to 0$ exponentially fast, so $\xi_n \to 0$ exponentially fast as well. $\square$

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[9]A lower bound for $\lim \eta_n$ in terms of the geometric and combinatorial bounds and the period of f can be worked out.

[10]Notice that this conclusion gives a *criterion for positive measure* which might be checkable numerically. Let us remark that this conclusion is based on an *upper bound* on $\omega_m(2)$, which is derived from Lemma 4.2.

6. Conservativity

In order to complete the proof of Theorem E we must show that some Feigenbaum maps have a conservative δ_{cr} conformal measure. In this section we will provide a criterium for conservativity, and in the next section we will show that such criterium is satisfied by some Feigenbaum maps

Theorem 6.1. *Let f be a Feigenbaum map[11]. If $\delta_{\mathrm{cr}}(f) > \delta_{\mathrm{cr}}(R^n(f))$ then the $\delta_{\mathrm{cr}}(f)$ conformal measure on $J(f)$ is conservative.*

Proof. Let $\delta_{\mathrm{cr}} \equiv \delta_{\mathrm{cr}}(f)$ and let μ be the δ_{cr} conformal measure on $J(f)$. Let $Q \subset A^n$ be the set of points that return to A^n. By Lemma 3.1, μ is conservative if and only if $\mu(Q) = \mu(A^n)$. We will show that if $\mu(Q) < \mu(A^n)$ then

$$(6.1) \qquad E_\mu(\Xi_\delta^{[j+1]}(A^n \leftarrow U)|A^n) \leq C + \kappa E_\mu(\Xi_\delta^{[j]}(A^n \leftarrow U)|A^n), \quad \delta \approx \delta_{\mathrm{cr}},$$

for some $C > 0$, $\kappa < 1$. This implies that $E_\mu(\Xi_\delta(A^n \leftarrow U)|A^n) \leq C/(1-\kappa)$ for $\delta \approx \delta_{\mathrm{cr}}$ which contradicts the definition of δ_{cr} and concludes the proof.

Through the argument we will skip the notation for truncation. We start with the estimate

$$(6.2) \quad E_\mu(\Xi_\delta(A^n \leftarrow U)|A^n) = 1 + E_\mu(\Xi_\delta(A^n \underset{U \smallsetminus A^n}{\longleftarrow} U \smallsetminus A^n)|A^n) + E_\mu(\Xi_\delta(A^n \underset{U \smallsetminus A^n}{\overset{+}{\longleftarrow}} A^n \leftarrow U)|A^n).$$

Let

$$(6.3) \qquad \nu_\delta = \int_{A^n} \sum_{(x_0,\dots,x_k=x)\in\mathcal{F}} \frac{\delta_{x_0}}{|Df^k(x_0)|^\delta} d\mu(x), \quad \mathcal{F} = \left(A^n \underset{U \smallsetminus A^n}{\overset{+}{\longleftarrow}} A^n\right).$$

Then $\nu_{\delta_{\mathrm{cr}}} = \mu|Q$. Notice that

$$(6.4) \qquad \Xi_\delta(A^n \underset{U \smallsetminus A^n}{\overset{+}{\longleftarrow}} A^n) = \Xi_\delta(A^n \underset{U \smallsetminus A^n}{\longleftarrow} U^n \underset{U \smallsetminus V^n}{\longleftarrow} A^n) + \Xi_\delta(A^n \underset{U \smallsetminus V^n}{\overset{+}{\longleftarrow}} A^n).$$

The assumption of dissipativity of μ implies in particular that $\sup \Xi_{\delta_{\mathrm{cr}}}(V^n \underset{U \smallsetminus V^n}{\overset{+}{\longleftarrow}} A^n) < \infty$. By hyperbolicity of the set of orbits avoiding V^n, and by Lemma 2.10, we have that $\Xi_\delta(V^n \underset{U \smallsetminus V^n}{\overset{+}{\longleftarrow}} A^n)$ is uniformly bounded in a neighborhood of δ_{cr}. Notice that

$$(6.5) \qquad \Xi_\delta(A^n \underset{U \smallsetminus A^n}{\longleftarrow} U^n) = \Xi_\delta(A^n \overset{f_n}{\longleftarrow} U^n),$$

which by hypothesis is uniformly bounded in a neighborhood of δ_{cr}. Thus

$$(6.6) \qquad \Xi_\delta(A^n \underset{U \smallsetminus A^n}{\overset{+}{\longleftarrow}} A^n) < C, \quad \delta \approx \delta_{\mathrm{cr}}$$

and it follows immediately (see Lemma 4.5) that $\delta \mapsto \nu_\delta(N)$ is continuous in a neighborhood of δ_{cr} for any Borelian set N. If $\mu(Q) < \mu(A^n)$ it then follows that

$$(6.7) \qquad \frac{E_\mu(\Xi_\delta(A^n \underset{U \smallsetminus A^n}{\overset{+}{\longleftarrow}} A^n \leftarrow U)|A^n)}{E_\mu(\Xi_\delta(A^n \leftarrow U)|A^n)} = \frac{E_{\nu_\delta}(\Xi_\delta(Q \leftarrow U)|A^n)}{E_\mu(\Xi_\delta(A^n \leftarrow U)|A^n)} \leq \kappa < 1, \quad \delta \approx \delta_{\mathrm{cr}}.$$

Similarly to (6.6) we conclude that that

$$(6.8)$$
$$E_\mu(\Xi_\delta(A^n \underset{U \smallsetminus A^n}{\longleftarrow} U \smallsetminus A^n)|A^n) = E_\mu(\Xi_\delta(A^n \underset{U \smallsetminus A^n}{\longleftarrow} U^n)|A^n) + E_\mu(\Xi_\delta(A^n \underset{U \smallsetminus V^n}{\longleftarrow} U \smallsetminus V^n)|A^n)$$
$$+ E_\mu(\Xi_\delta(A^n \underset{U \smallsetminus A^n}{\longleftarrow} U^n \underset{U \smallsetminus V^n}{\longleftarrow} U \smallsetminus V^n)|A^n) < C, \quad \delta \approx \delta_{\mathrm{cr}}.$$

Plugging (6.8) and (6.7) into (6.2) we get (6.1). $\qquad\qquad\square$

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[11]It is enough to assume that f is n times renormalizable.

7. Varying the dimension

In this section we will give two proofs that for some Feigenbaum maps, $\delta_{\mathrm{cr}}(Rf) < \delta_{\mathrm{cr}}(f)$. Together with Theorem 6.1, it will justify the assertion of Theorem E that the δ_{cr}-conformal measure can be conservative (up to the proof of the existence of Lean Feigenbaum Julia sets, which will be completed in § 9).

7.1. Varying the dimension within the horseshoe.

Below we will make use of the continuity properties of the critical exponent (Lemmas 3.14 and 3.15) and an observation that it is non-increasing under the renormalization:

$$(7.1) \qquad \delta_{\mathrm{cr}}(Rf) \leq \delta_{\mathrm{cr}}(f)$$

Lemma 7.1. *Let f be a Feigenbaum map. If $\delta_{\mathrm{cr}}(f) = \delta_{\mathrm{cr}}(R(f))$ then $\delta_{\mathrm{cr}}(f) = \delta_{\mathrm{cr}}(R^n(f))$ for all $n \geq 0$.*

Proof. Assume that $\delta_{\mathrm{cr}}(f) = \delta_{\mathrm{cr}}(R(f)) > \delta_{\mathrm{cr}}(R^2(f))$. Then the $\delta_{\mathrm{cr}}(R(f))$ conformal measure on $J(R(f))$ is dissipative by the first relation (Proposition 3.5) and conservative by the second one (Theorem 6.1) - contradiction. $\square$

Let us consider the horseshoe $\mathcal{A} \equiv \mathcal{A}_{\mathcal{M}}$ associated to some fine collection $\mathcal{M}$ of little Mandelbrot copies (see §2.5). Notice that under the renormalization operator $R : \mathcal{A} \to \mathcal{A}$ the orbit of a generic point $f \in \mathcal{A}$ is dense in $\mathcal{A}$.

Lemma 7.2. *Assume that $\mathrm{orb}_R(f)$ is dense in $\mathcal{A}$. Then*

$$\delta_{\mathrm{cr}}(f) = \max_{g \in \mathcal{A}} \delta_{\mathrm{cr}}(g).$$

Proof. This follows from the lower semi-continuity of the critical exponent and (7.1) . $\square$

Lemma 7.3. *If the critical exponent is not constant over a renormalization horseshoe $\mathcal{A}$ (associated with some fine collection of Mandelbrot copies) then there exists $f \in \mathcal{A}$ such that $\delta_{\mathrm{cr}}(f) > \delta_{\mathrm{cr}}(Rf)$.*

Proof. Take two maps $f_+, f_- \in \mathcal{A}$ such that $\delta_{\mathrm{cr}}(f_+) > \delta_{\mathrm{cr}}(f_-)$. Since the critical exponent is lower semi-continuous, f_+ can be selected to be a periodic point of some period p of R. Let f belong to the intersection of the unstable manifold of f_+ with the stable manifold of f_-. Using semi-continuity again, we conclude that

$$\lim \delta_{\mathrm{cr}}(R^{-np}(f)) \geq \delta_{\mathrm{cr}}(f_+).$$

On the other hand, by monotonicity (7.1) and uniform continuity on hybrid classes (Lemma 3.15),

$$\lim \delta_{\mathrm{cr}}(R^n f) = \lim \delta_{\mathrm{cr}}(R^n f_-) \leq \delta_{\mathrm{cr}}(f_-).$$

$\square$

Example 7.1. Let f_p be the real fixed point of renormalization of period p with combinatorics closest to Chebyshev. By Lemma 2.5, $\Xi_1(f_p, z) = \infty$. On the other hand by Lemma 3.6 the δ_{cr}-conformal measure on of f_p is dissipative; hence by Lemma 3.2, $\Xi_{\delta_{\mathrm{cr}}}(f_p, z) < \infty$. Hence $\delta_{\mathrm{cr}}(f_p) > 1$.[12]

Moreover, it is shown in [AL] that $\lim \delta_{\mathrm{cr}}(f_p) = 1$. Thus we can choose p_1 and p_2 such that $\delta_{\mathrm{cr}}(f_{p_1}) > \delta_{\mathrm{cr}}(f_{p_2})$, so the horseshoe associated to the corresponding Mandelbrot copies satisfies the hypothesis of the previous lemma.

---

[12]This also follows, from Theorem 2.4 and Corollary 3.12, in the case of p large which we use here (since in this case the Julia set has zero Lebesgue measure).

7.2. **Varying the dimension within a hybrid class.** Recall that $\mathcal{H}_f$ stands for the hybrid class of a quadratic-like map f (see §2.3). We will study now how the dimension $g \mapsto \mathrm{HD}(J(g))$ vary over $\mathcal{H}_f$.

Lemma 7.4. *Let f be a Feigenbaum map. Then*

$$\sup_{g \in \mathcal{H}_f} \mathrm{HD}(J(g)) = 2.$$

Proof. Let us take a little square $\Delta \Subset \mathbb{V} \setminus \mathcal{O}$ intersecting the Julia set. It is easy to find four branches $g_i \equiv f_i^{-n_i} | \Delta$ of inverse iterates such that $D_i \equiv g_i(\Delta) \Subset \Delta$ and f^{n_i} is the first return map from D_i to Δ. Let $S = \cup D_i$ and let $G : S \to \Delta$ be the associated Bernoulli map, $G|D_i = f^{n_i}$. Let Q be the maximal G-invariant set contained in S. By the Schwarz Lemma, Q is hyperbolic for G, so that the forward saturation $\mathrm{orb}(Q)$ is a hyperbolic set for f.

Let us consider a diffeomorphism $H : \mathbb{C} \to \mathbb{C}$ such that $\tilde{\Delta} \equiv h(\Delta)$ is the standard square $[0,1] \times [0,1]$ of size 1 and $\tilde{D}_i \equiv H(D_i)$ are rescaled standard squares of size close to $1/4$ (and we can select H to be affine on $\mathbb{C} \setminus \Delta$). Let $\tilde{S} = \cup \tilde{D}_i$. Consider a piecewise affine $\tilde{G} : \tilde{S} \to \tilde{\Delta}$ that affinely maps each $\tilde{D}_i$ onto $\tilde{\Delta}$. Then the maximal invariant set $\tilde{Q}$ of $\tilde{G}$ has Hausdorff dimension $2 - \epsilon$.

By the standard pullback argument, the maps $G : S \to \Delta$ and $\tilde{G} : \tilde{S} \to \tilde{\Delta}$ are conjugate by some qc homeomorphism h which is affine on $\mathbb{C} \setminus \Delta$. Let us consider the corresponding G-invariant Beltrami differential $\mu = \bar{\partial}h/\partial h$.

Let $f : \mathbb{U} \to \mathbb{V}$ and let $W^m = f^{-m}\mathbb{U}$. For $m \geq 0$, define a Beltrami differential μ_m as follows. Let $\mu_m = \mu$ on $\mathbb{C} \setminus W^m$. Pull it back by the iterates of f to the domain $W^m \setminus J(f)$. Finally, let $\mu_m|J(f) = 0$. Since G is composed from iterates of f, this Beltrami differential is simultaneously invariant with respect to $f : W^m \to W^{m-1}$ and $G : S \to \Delta$. Since conformal pullbacks preserve the norm of Beltrami differentials, $\|\mu_m\|_\infty \leq \|\mu\|_\infty$.

We assume now that $\mathrm{area}(J(f)) = 0$, for otherwise there is nothing to prove. In this case $\mu_m \to \mu$ almost everywhere.

Let h_m be solution of the Beltrami equation $\bar{\partial}h_m/\partial h_m = \mu_m$ coinciding with h at three points. Then $h_m \to h$ uniformly on $\bar{\mathbb{C}}$.

Consider a quadratic-like map

$$g_m = h_m \circ f \circ h_m^{-1} : h_m(W^m) \to h_m(W^{m-1}).$$

and the corresponding expanding map

$$G_m = h_m \circ G \circ h_m^{-1} : h_m(S) \to h_m(\Delta).$$

Since the latter maps uniformly converge to $\tilde{G}$, $\mathrm{HD}(h_m(Q)) \to \mathrm{HD}(\tilde{Q}) = 2 - \epsilon$. The conclusion follows since the forward saturation of $h_m(Q)$ under g_m is a hyperbolic subset of $J(g_m)$. $\qquad\square$

Remark 7.1. In case of the Julia set of positive area, it would be interesting to obtain the same result for the hyperbolic dimension $\mathrm{HD}_{\mathrm{hyp}}(g)$ rather than $\mathrm{HD}(J(g))$.

Let us consider a fine family $\mathcal{M}$ of little Mandelbrot copies and a Feigenbaum map f with combinatorics $\bar{M}_+ = \{M_0, M_1, \dots\}$, $M_i \in \mathcal{M}$. We say that f has *recurrent combinatorics* if the sequence $\bar{M}_+$ is recurrent. Such a sequence can be extended to the "past" as a bi-infinite recurrent sequence $\bar{M} = \{M_k\}_{k \in Z}$. Take the map f_* in the renormalization horseshoe $\mathcal{A}_\mathcal{M}$ with combinatorics $\bar{M}$. It is recurrent under the renormalization dynamics and the R-orbits of f and f_* are forward asymptotic. We say that f_* is "associated" with f.

Theorem 7.5. *Let f be a Feigenbaum map with recurrent combinatorics, and let f_* be an associated map in the horseshoe $\mathcal{A}_\mathcal{M}$. Then the function $h : g \mapsto \mathrm{HD}(J(g))$, $g \in \mathcal{H}_f$, is either identically equal to 2, or*

$$\mathrm{Im}\, h = [\mathrm{HD}(J(f_*)), 2).$$

Proof. Since qc maps preserve the property $\mathrm{HD}(J) = 2$, the function h is either identically equal to 2, or always less than 2. Assume that the latter happens.

Since hybrid classes are connected and the Hausdorff dimension varies continuously under qc deformations, $\mathrm{Im}(h)$ is an interval. By Lemma 7.4, it must be either $[a, 2)$ or $(a, 2)$.

Since the function h is a decaying Lyapunov function for the renormalization (i.e., $h(Rg) \leq h(g)$), and by the recurrence assumption the orbit $R^n f$ accumulates on f_*, we conclude that $h(f) \geq h(f_*)$. Hence

$$h(f_*) = \min_{g \in \mathcal{H}_f} h(g),$$

and we are done. $\square$

Take, for instance, a renormalization fixed point f_* of lean type. Then $\mathrm{HD}(J(f_*)) < 2$ and there is a map $f \in \mathcal{H}_{f_*}$ with $\mathrm{HD}(J(f)) > \mathrm{HD}(J(f_*))$. Since $R^n f \to f_*$ and the critical exponent is continuous on hybrid classes, there is an n such that $\mathrm{HD}(R^{n+1}f) < \mathrm{HD}(R^n f)$, so that the δ_{cr}-conformal measure of $R^n f$ is conservative.

8. INTERPOLATION ARGUMENT

Let us now prove Theorem B.

Theorem 8.1. *Let $\mathcal{A}$ be the horseshoe associated to some fine collection of Mandelbrot copies. If there exist $g_+, g_0 \in \mathcal{A}$ such that $\mathrm{area}(J(g_+)) > 0$ and $\mathrm{area}(J(g_0)) = 0$, then for a generic $f \in \mathcal{A}$, $\delta_{\mathrm{cr}}(f) = 2$ and $\mathrm{area}(J(f)) = 0$.*

Proof. Any map in the stable manifold of g_0 has a Julia set of zero area. Since the stable manifold of g_0 is dense in $\mathcal{A}$ and area $J(g)$ depends upper semi-continuously on $g \in \mathcal{A}$, it follows that a generic map in $\mathcal{A}$ has a Julia set of zero area.

Fix $s > 0$ and let $f_{[s]} \in \mathcal{A}$ be such that $R^s(f_{[s]})$ belongs to the intersection of the stable manifold of g_0 and the local unstable manifold of $R^s(g_+)$, so that the combinatorics of $f_{[s]}$ coincides with that of g_+ on many top renormalization levels and coincides with that of g_0 on all sufficiently deep renormalization levels. Let us denote by $\phi_{n,s}$ the first "univalent" return map to the disk $V^{n,s} \equiv V^n(f_{[s]})$ (see §2.7), that is, $\phi_{n,s}(z) = f_{[s]}^k(z)$ if k is smallest possible such that there is a univalent branch of $f_{[s]}^{-k}|V^{n,s}$ onto a neighborhood of z. Since the maps $\phi_{n,s}$ are Markov and have bounded distortion (see Remark 2.1), the critical exponent $\delta_{\mathrm{cr}}(\phi_{n,s})$ makes sense (i.e., it is independent of the base point $z \in V^{n,s}$ in the Poincaré series $\Xi_\delta(\phi_{n,s}, z)$). Obviously $\delta_{\mathrm{cr}}(f_{[s]}) \geq \delta_{\mathrm{cr}}(\phi_{n,s})$. Let us show that we can choose n, s so that $\delta_{\mathrm{cr}}(\phi_{n,s})$ is close to 2, as this will imply the result by Lemma 7.2.

Let $D_k^{n,s}$ be the domain of $\phi_{n,s}^k$. We may assume that $|D\phi_{n,s}(x)| > 2$ for $x \in D_1^{n,s}$, see Remark A.1.

Let us show that we can choose n and s so that $p(D_1^{n,s}|V^{n,s})$ is close to 1. Since $J(g_+)$ has positive Lebesgue measure, its critical point is a density point of $J(g_+)$. Given $\xi > 0$, we can choose $\rho > 0$ so that $p(J(g_+)|\mathbb{D}_\rho) > 1 - \xi$. Take the minimal $n > 0$ such that $\mathrm{diam}(V^{n,s}) \leq \rho/2$ for all s sufficiently big. Then for s big, $\mathrm{diam}(V^{n,s}) \asymp \rho$ and so $V^{n,s} \supset \mathbb{D}_{q\rho}$ for some absolute $q > 0$.

Let $A^{n,s}$ be defined in the natural way (see § 2.7). Let $L^s \subset \mathbb{D}_\rho$ be the set of points whose positive orbit under f_s intersects $\mathbb{D}_{q\rho}$. Notice that $L^s \cap A^{n,s} \subset D_1^{n,s} \cap A^{n,s}$. Since the positive orbit of almost every $x \in J(g_+)$ (under iterates of g_+) accumulates on 0, $\liminf L^s \supset J(g_+) \cap \mathbb{D}_\rho$ up to a set of zero area. Thus we can choose s so that $p(L^s|\mathbb{D}_\rho) > 1 - \xi$. Hence $p(D_1^{n,s}|A^{n,s}) > 1 - C\xi$. By bounded distortion,

$$(8.1) \qquad p(D_1^{n,s}|B^{n,s}) > 1 - C\xi \quad \text{where} \quad B^{n,s} = \bigcup_{k \geq 0} f_{[s]}^{-k}(A^{n,s}).$$

Since $f_{[s]}$ is hybrid equivalent to g_0, $J(f_{[s]})$ has zero area. Thus $B^{n,s} = V^{n,s}$ modulo a set of zero area, so that (8.1) is equivalent to $p(D_1^{n,s}|V^{n,s}) > 1 - C\xi$ as desired.

Fix $\epsilon > 0$. Notice that if $p(D_1^{n,s}|V^{n,s})$ is close to 1 then $p(D_k^{n,s}|V^{n,s}) > (1-\epsilon)^k$. We have

$$
\int_{V^{n,s}} \Xi_{2-\gamma}(\phi_{n,s},x)dx = \int_{V^{n,s}} \sum_{k\geq 0}\sum_{\phi_{n,s}^k(y)=x} |D\phi_{n,s}^k(y)|^{\gamma-2}dx
$$

$$
\geq \int_{V^{n,s}} \sum_{k\geq 0}\sum_{\phi_{n,s}^k(y)=x} |D\phi_{n,s}^k(y)|^{-2}2^{k\gamma}dx = \sum_{k\geq 0}|D_k^{n,s}|2^{k\gamma}
$$

$$
\geq |V^{n,s}|\sum_{k\geq 0} 2^{k\gamma}(1-\epsilon)^k.
$$

Thus the critical exponent $2-\gamma$ of $\phi_{n,s}$ satisfies $2^\gamma(1-\epsilon) \leq 1$, so by choosing ϵ small we can get $\delta_{\mathrm{cr}}(\phi_{n,s})$ arbitrarily close to 2. $\qquad\square$

Remark 8.1. Another interpolation argument allows one to show that, under the hypothesis of the previous theorem, there exists $f \in \mathcal{A}$ with a dense orbit (or which is periodic) and with area $J(f) > 0$. Notice that if f has a dense orbit then it has automatically $\delta_{\mathrm{cr}}(f) = 2$ by the previous theorem and Lemma 7.2, and if f is periodic then $\delta_{\mathrm{cr}}(f) < 2$ (since we are in the Lean case).

9. LEAN CRITICAL POINTS

In this section we will show that there are many Feigenbaum Julia sets with zero area (Yarrington's thesis [Y]). We will restrict ourselves to a particular class of combinatorics (to be defined below), for which the estimates are simpler. As a consequence of our estimates, we will also conclude that the Lean case exists, completing the proof of the first assertion of Theorem A.

Let $F : \cup\Upsilon_j \to \Upsilon$ be a holomorphic map with the following properties:

(1) $\cup\Upsilon_j \subset \Upsilon$,
(2) For every $j \neq 0$, $F : \Upsilon_j \to \Upsilon$ is univalent,
(3) $f \equiv F : \Upsilon_0 \to \Upsilon$ is a quadratic-like map with connected Julia set.

We let $\Upsilon^k = f^{-k}(\Upsilon_0)$.

We say that F is κ-good if

(1) $\mathrm{mod}(\Upsilon \smallsetminus \overline{\Upsilon}_0) \geq \kappa$,
(2) The domain of the first return map to Υ_0 has density at most κ^{-1}.

Let P be the (monic) quadratic polynomial which is hybrid equivalent to $f : \Upsilon_0 \to \Upsilon$. Denote by D_r the domain bounded by the equipotential of height r for P. We construct the standard Yoccoz puzzle of P with the help of the equipotential at height 4, see for instance [L5]. Let $\hat{T}_1 \supset \hat{T}_2 \supset ...$ be the principal nest of P obtained in the standard way (see [L5]).

We will make the following combinatorial assumptions:

(1) P belongs to a selected finite family of Misiurewicz wakes, or, equivalently, $\hat{T}_1$ is mapped to the top level puzzle piece by an iterate F^n with a bounded n (see [L6, Sc]);
(2) P is renormalizable with no central cascades: there exists $\tau(P) > 0$ (the height of P) such that the first return of 0 to $\hat{T}_k$ belongs to $\hat{T}_{k+1}$ if and only if $k \geq \tau(P)$.

In what follows, the constants involved will depend on the choice of the Misiurewicz wake.

We will need the following simple plane topology argument.

Lemma 9.1. *Let $W_i \subset \mathbb{C}$, $i \geq 0$, be Jordan domains such that $\mathrm{diam}\,W_j \to 0$. Assume that $W_0 \cap W_i \neq \emptyset$, $i > 0$ while $\overline{W}_i \cap \overline{W}_j = \emptyset$ for $i > j > 0$. Consider a compact set $W \subset \mathbb{C}$ obtained by taking the closure of $\cup W_i$ and then filling in the holes. Then W is a Jordan disk.*

Proof. We may assume that W_0 is not contained in any W_i, for otherwise $W = W_i$. We can also discard the W_i, $i > 0$ that are contained in W_0. It follows that $\#(\partial W_i \cap \partial W_0) \geq 2$ for $i \geq 0$.

Note that $\partial W \subset \overline{\cup \partial W_i}$. Moreover, if a sequence of domains W_i accumulates on some $x \in \partial W$ then $x \in \partial W_0$. Hence $\partial W \subset \cup \partial W_i$.

It follows that each connected component of $\partial W \smallsetminus \partial W_0$ is an open arc γ_k contained in some ∂W_i, whose endpoints are contained in ∂W_0. (Warning: there can be infinitely many γ_k's in one W_i). Notice that the endpoints of γ_k cannot coincide (otherwise it would bound some W_i such that $\partial W_i \cap \partial W_0$ is a point), so $Q_k \equiv \overline{\gamma}_k \smallsetminus \gamma_k$ consists of two points. Since $\operatorname{diam} W_i \to 0$ we also have $\operatorname{diam} \gamma_k \to 0$.

Let δ_k and δ_k' denote the two components of $\partial W_0 \smallsetminus Q_k$. One of them, say, δ_k, is such that $\gamma_k \cup Q_k \cup \delta_k$ bounds a domain contained in W (since W is a full compact set) which is disjoint from W_0. This shows that each δ_k is contained in $\operatorname{int} W \cap \partial W_0$. Hence $\partial \delta_k \cap \delta_i = \emptyset$ for $k \neq i$, so that all the arcs δ_k are disjoint. Since they lie on the Jordan curve ∂W_0, $\operatorname{diam} \delta_k \to 0$.

Let $h_k : \delta_k \to \gamma_k$ be a homeomorphism fixing the endpoints, and let $h : \partial W_0 \to \mathbb{C}$ be defined by $h|\delta_k = h_k$ and $h|\partial W_0 \smallsetminus \cup \delta_k = \operatorname{id}$. Since $\operatorname{diam} \gamma_k \to 0$ and $\operatorname{diam} \delta_k \to 0$, h is continuous. It is also injective by definition, so it is a homeomorphism onto its image. Notice that $h(\partial W_0) \supset \partial W$ by construction. On the other hand, ∂W can not be a proper subset of $h(\partial W_0)$ (otherwise it would not disconnect the plane). Thus $\partial W = h(\partial W_0)$ is a Jordan curve as was asserted. $\qquad\square$

Remark 9.1. Notice that the proof constructs an explicit homeomorphism between ∂W^0 and ∂W, which is close to the identity when $\sup \operatorname{diam}(W_i)$ is small, and which can be extended to a homeomorphism of $\overline{W}^0$ to $\overline{W}$ which is also close to the identity.

We will say that $F : \cup \Upsilon_j \to \Upsilon$ is normalized if $F(z) = F(0) + z^2 + O(z^3)$ near 0.

Lemma 9.2. *For every $\eta > 0$, there exists $\kappa = \kappa(\eta) > 0$ with the following property. Let F be κ-good and normalized. There exists a homeomorphism $h : D_4 \to \mathbb{C}$ which conjugates $P : D_2 \to D_4$ to $F : h(D_2) \to h(D_4)$ such that*

(1) *h is η close to the identity,*
(2) *If S is a non-central component of the domain of the first return map to Υ_0, S' is a component of a preimage of S by some iterate of f, and Q is a Yoccoz puzzle piece for P intersecting $h^{-1}(S')$, then $h^{-1}(S') \subset Q$.*

Proof. Let M^0 be the domain bounded by the equator of $\Upsilon \smallsetminus \Upsilon_0$, and $M^k = f^{-k}(M^0)$. If κ is big, then ∂M^0 is close to a circle centered at 0, and $f|M^1$ is essentially quadratic. It follows that there exists r and a quasiconformal map $\hat{h} : \overline{D}_r \smallsetminus D_{r^{1/2}} \to \overline{M}^0 \smallsetminus M^1$ with dilatation close to 1 which conjugates $f|\partial M^1$ and $P|\partial D_{r^{1/2}}$ (notice that $r \geq \kappa$ and indeed $\ln r$ is close to $2 \operatorname{mod}(M^0 \smallsetminus M^1)$). By the pullback argument, we obtain an extension $\hat{h} : \overline{D}_r \to \overline{M}^0$ with the same dilatation which is a hybrid conjugacy. By the normalization at the origin, $\hat{h}|D_{256}$ is uniformly close to the identity in the Euclidean metric, provided κ is big.

Let $\mathcal{S}$ be the collection of non-central components of the domain of the first return map to Υ^0, and let $\mathcal{S}'$ be the collection of components of preimages of some $S \in \mathcal{S}$ by some iterate of f. Notice that if $S \in \mathcal{S}$ then there exists $\tilde{S} \subset \Upsilon^0 \smallsetminus \Upsilon^1$ such that the first return map $\phi : S \to \Upsilon^0$ extends to a univalent map $\phi : \tilde{S} \to \Upsilon$. If $S' \in \mathcal{S}$ is such that $f^k(S') = S$ then $S' \subset \Upsilon^k \smallsetminus \Upsilon^{k+1}$ and the map $f^k : S' \to S$ extends to a bigger domain $\tilde{S}' \subset \Upsilon^k \smallsetminus \Upsilon^{k+1}$ so that $f^k : \tilde{S}' \to \tilde{S}$ is univalent. Notice that $\operatorname{mod}(\tilde{S}' \smallsetminus S') = \operatorname{mod}(\Upsilon \smallsetminus \Upsilon^0) \geq \kappa$, so the hyperbolic diameter of S' in $\tilde{S}'$ is very small. By the Schwarz Lemma, since $\tilde{S}' \subset \Upsilon^k \smallsetminus \Upsilon^{k+1} \subset M^k \smallsetminus M^{k+2}$, the hyperbolic diameter of S' in $M^k \smallsetminus \overline{M}^{k+2}$ is also very small.

Notice that if $S' \in \mathcal{S}'$ intersects $\partial \hat{h}(D_4)$ then $M^k \subset \mathbb{C} \smallsetminus \overline{\hat{h}(D_4)}$ and $M^{k+2} \subset \hat{h}(D_4)$, so $M^k \smallsetminus M^{k+2} \subset \hat{h}(D_{256}) \smallsetminus \hat{h}(D_{\sqrt{2}})$, and the hyperbolic diameter of S' in $\hat{h}(D_{256}) \smallsetminus \hat{h}(D_{\sqrt{2}})$ is small. Let

B be the union of $\hat{h}(D_4)$ with all $S' \subset \mathcal{S}'$ which intersect $\hat{h}(D_4)$, and let B^0 be obtained from $\overline{B}$ by filling in the holes and taking the interior. By the previous lemma, B^0 is a Jordan domain. Moreover, ∂B^0 is close to $\partial \hat{h}(D_4)$ in the hyperbolic metric of $\hat{h}(D_{256}) \smallsetminus \hat{h}(D_{\sqrt{2}})$. Let $B^k = f^{-k}(B^0)$. Then (using Remark 9.1) there exists a homeomorphism $\tilde{h} : \overline{D}_4 \smallsetminus D_2 \to \overline{B}^0 \smallsetminus B^1$ that conjugates $P|\partial D_2$ to $f|\partial B^1$, such that $\tilde{h} \circ \hat{h}^{-1} : \hat{h}(D_4) \smallsetminus \hat{h}(D_2) \to \overline{B}^0 \smallsetminus B^1$ is uniformly close to the identity in the hyperbolic metric of $\hat{h}(D_{256}) \smallsetminus \hat{h}(D_{\sqrt{2}})$.

The image by $\tilde{h}$ of the external rays γ landing at the α fixed point of P divide $\overline{B}^0 \smallsetminus B^1$ into finitely many (say, k) topological rectangles. Applying the previous lemma $k-1$ times, we obtain a new homeomorphism $h : \overline{D}_4 \smallsetminus D_2 \to \overline{B}^0 \smallsetminus B^1$, such that $h(\gamma \cap \overline{D}_4 \smallsetminus D_2)$ is disjoint from $\cup_{S' \in \mathcal{S}'} S'$ for all external rays γ. We may still require (see Remark 9.1) that $h \circ \hat{h}^{-1} : \hat{h}(D_4) \smallsetminus \hat{h}(D_2)$ is uniformly close to the identity in the hyperbolic metric of $\hat{h}(D_{256}) \smallsetminus \hat{h}(D_{\sqrt{2}})$.

Using the pullback argument, we can extend h to a homeomorphism $h : \overline{D}_4 \smallsetminus K(P) \to \overline{B}^0 \smallsetminus K(f)$. Since $P|D_2 \smallsetminus K(P)$, $f|B^1 \smallsetminus K(f)$ are unbranched coverings, we conclude that $h \circ \hat{h}^{-1} : \hat{h}(D_4) \smallsetminus K(f) \to \overline{B}^0 \smallsetminus K(f)$ is uniformly close to the identity in the hyperbolic metric of $\hat{h}(D_{256}) \smallsetminus K(f)$. In particular, $h \circ \hat{h}^{-1}$ extends through $K(f)$ as the identity, so h extends to a homeomorphism $h : \overline{D}_4 \to \overline{B}^0$ which is uniformly close to $\hat{h}$ in the hyperbolic metric of $\hat{h}(D_{256})$. The map h has the desired properties by construction. $\square$

We can use h to transfer the principal nest $\hat{T}_1 \supset \hat{T}_2 \supset ...$ for P to the principal nest $T_1 \supset T_2 \supset ...$ for f, $T_k = h(\hat{T}_k)$. Let us say that F has (N, τ_0) *bounded combinatorics* if the first return time of $T_{\tau_0 + 1}$ to T_{τ_0} is bounded by N.

Lemma 9.3. *For every $N > 0$, $\tau_0 > 0$, there exists $\kappa > 0$, $\xi > 0$, with the following property. If F is κ-good and has (N, τ_0) bounded combinatorics then*

(1) $|T_{\tau_0}| > \xi$ *if F is normalized,*
(2) *The density of the domain of the first return map to T_{τ_0} is at most $1 - \xi$.*

Proof. Let us split the returning set of T_{τ_0} under the dynamics of F into two parts: X consists of points that return under the dynamics of f, and Y is the complement.

Let us first show that $T_{\tau_0} \smallsetminus X$ contains a disk of definite radius. Parameters N, τ_0 specify a compact set of polynomials with the property that the domain of the first return map to $\hat{T}_{\tau_0}$ is not dense. In particular, there exists $\chi > 0$ such that for any such polynomial, the non-returning set of $\hat{T}_{\tau_0}$ contains a disk of radius χ. Taking $\eta < \chi/10$ and $\kappa = \kappa(\eta)$ given by Lemma 9.2, we conclude that $T_{\tau_0} \smallsetminus X$ contains a disk of radius $\chi/2$. In particular this implies the first statement, since $|T_{\tau_0}| \geq \chi^2/4$.

Since the area of T_{τ_0} is bounded by 100, the density of X in T_{τ_0} is at most $1 - \chi^2/400$. If $x \in Y$, there exists $k > 0$ minimal such that $f^k(x)$ belongs to some non-central component S of the domain of the first return map to Υ^0. The first return map $\phi : S \to \Upsilon^0$ extends to a univalent map $\phi : \tilde{S} \to \Upsilon$, where $\tilde{S} \subset \Upsilon^0 \smallsetminus \Upsilon^1$. Let S', $\tilde{S}'$ be the components of $f^{-k}(S)$, $f^{-k}(\tilde{S})$ containing x. Notice that $f^k : \tilde{S}' \to \tilde{S}$ is univalent, so $f^k : S' \to S$ has small distortion. It follows that the density of $Y \cap S'$ in S' is at most $1 - \sigma$ for some absolute σ. The domains $S'(x)$ obtained in this way form a disjoint cover of Y which does not intersect X. It follows that the density of $X \cup Y$ in T_{τ_0} is at most $1 - (\sigma\chi^2/400)$. $\square$

Let $\tau(F) = \tau(P)$ be the height of F.

Lemma 9.4 (see [L5])**.** *There exists $\kappa > 0$, $\rho > 0$ such that if F is κ-good then $\mathrm{mod}(T_k \smallsetminus \overline{T}_{k+1}) > \rho k$, $k \leq \tau(F) + 2$.*

For the proof of the next lemma, we will need some combinatorial preparation for dealing with first return maps. Let us denote the first return map to T_k by $R_k : \cup T_k^j \to T_k$, where j takes values in $\mathbb{Z}$ and we set $T_{k+1} = T_k^0$. Let Ω be the set of finite sequences of non-zero integers $\underline{d}$. The length of $\underline{d}$ will be denoted $|\underline{d}|$ (we allow $|\underline{d}| = 0$). If $\underline{d} = (j_1, ..., j_m)$, let $T_k^{\underline{d}} = \{x \in T_k, R_k^{i-1}(x) \in T_k^{j_i}, 1 \le i \le m\}$, and let $R_k^{\underline{d}} = R^m | T_k^{\underline{d}}$, so that $R_k^{\underline{d}} : T_k^{\underline{d}} \to T_k$ is univalent. Let $W_k^{\underline{d}} = (R_k^{\underline{d}})^{-1}(T_{k+1})$. The map $L_k : \cup W_k^{\underline{d}} \to T_{k+1}$, $L_k | W_k^{\underline{d}} = R_k^{\underline{d}}$ is the first landing map from T_k to T_{k+1}, and its distortion is small provided $\mathrm{mod}(T_k \smallsetminus T_{k+1})$ is big.

Lemma 9.5. *There exists $\kappa > 0$, $\tau > 0$, $\lambda_1 < 1$ such that for $\tau \le k \le \tau(F) + 2$ we have*

(1) *Let Y_k be the union of T_k^j such that the distortion of $R_k | T_k^j$ is at least $1 + \lambda_1^k$. Then $|Y_k| \le \lambda_1^k |T_k|$.*

(2) *Let Z_{k+1} be the union of connected components of $(R_k | T_{k+1})^{-1}(\cup T_k^j)$ where the distortion of R_k^2 is at least $1 + \lambda_1^k$. Then $|Z_{k+1}| \le \lambda_1^k |T_{k+1}|$.*

Proof. Let $R_k | T_k^j = F^m$. It is easy to see that $F^{m-1} | F(T_k^j)$ extends to a univalent map onto T_{k-1}. By the previous lemma, $F^{m-1} | F(T_k^j)$ has exponentially small distortion in k, $\tau \le k \le \tau(F) + 2$.

By the previous lemma, if $\tau \le k \le \tau(F)$, the hyperbolic diameter of any T_k^j in T_k is exponentially small. Thus, the distortion of $F | T_k^j$ is also exponentially small in k, provided T_k^j is not exponentially close to 0 in the hyperbolic metric of T_k. We conclude that Y_k is contained in an exponentially small hyperbolic neighborhood of 0 in T_k. The first assertion follows.

Let D be a component of $(R_k | T_{k+1})^{-1}(\cup T_k^j)$. By the previous lemma, the hyperbolic diameter of D in T_{k+1} is exponentially small. If the distortion of $R_k | D$ is not exponentially small, then the distortion of either $R_k | (R_k(D))$ or of $R_k | D$ is not exponentially small. In the first case, $D \subset (R_k | T_{k+1})^{-1}(Y_k)$. In the second case, we conclude as before that D is contained in an exponentially small hyperbolic neighborhood of 0 in T_{k+1}. Thus Z_{k+1} is contained in the union of 3 sets with exponentially small hyperbolic diameter in T_{k+1}. The second assertion follows. $\square$

Lemma 9.6. *There exists $\tau_0 > 0$ such that for every $N > 0$, there exists $\kappa > 0$, $\lambda < 1$ with the following property. If F is κ-good, has (N, τ_0) bounded combinatorics, and $\tau_0 \le k \le \tau(F) + 2$ then the density of the domain of the first return map to T_k is at most λ^k.*

Proof. In the proof, λ's will denote constants which are definitely smaller than one, depending only on ρ from Lemma 9.4.

Let Y_k, Z_k, and λ_1 be as in the previous lemma. If $D \not\subset Z_{k+1}$ is a pullback of some T_k^j under $R_k | T_{k+1}$ then the distortion of the map $R_k^2 : D \to T_k$ is at most $1 + \lambda_1^k$. Hence

$$1 - p(\cup T_{k+1}^j | D) \ge (1 + \lambda_1^k)^{-1}(1 - p(\cup W_k^{\underline{d}} | T_k)).$$

It follows that

(9.1) $$1 - p(\cup T_{k+1}^j | T_{k+1}) \ge (1 - p(Z_{k+1} | T_{k+1}))(1 + \lambda_1^k)^{-1}(1 - p(\cup W_k^{\underline{d}} | T_k)).$$

By Lemma 9.5, $1 - p(\cup T_{k+1}^j | T_{k+1}) \ge (1 + \lambda_1^k)^{-1}(1 - p(\cup T_k^j | T_k))$. By Lemma 9.3, there exists $\xi > 0$ such that $1 - p(\cup T_{\tau_0}^j | T_{\tau_0}) \ge \xi$, so there exists $\sigma > 0$ such that

(9.2) $$1 - p(\cup T_k^j | T_k) \ge \sigma, \quad \tau_0 \le k \le \tau(F) + 2.$$

Notice that for $j \ne 0$ the map $R_k : T_k^j \to T_k$ carries the set $T_k^j \cap \bigcup W_k^{\underline{d}}$ to $\bigcup W_k^{\underline{d}}$. Hence, if $T_k^j \subset T_k \smallsetminus Y_k$ then by the first part of Lemma 9.5, $p(\cup W_k^{\underline{d}} | T_k^j) \le (1 + \lambda_1^k) p(\cup W_k^{\underline{d}} | T_k)$. Hence

(9.3) $$p(\cup W_k^{\underline{d}} | T_k) \le p(Y_k | T_k) + (1 - \sigma)(1 + \lambda_1^k) p(\cup W_k^{\underline{d}} | T_k),$$

so if k is big so that $(1 - \sigma)(1 + \lambda_1^k) \leq 1 - (\sigma/2)$, (9.2) and (9.3) imply

$$(9.4) \qquad p(\cup W_k^d \,|\, T_k) \leq \min\{1 - \sigma, 2\sigma^{-1} p(Y_k \,|\, T_k)\}.$$

Hence $|\cup W_k^{\frac{d}{k}}| \leq \lambda_2^k |T_k|$. Putting this into (9.1), we get $|\cup T_k^j| \leq \lambda_3^k |T_k|$ as desired. $\qquad\square$

Lemma 9.7. *There exists $\tau_0 > 0$, such that for every $N > 0$, there exists $\kappa > 0$, $\tau > \tau_0$, with the following property. If F is κ-good, has (N, τ_0) bounded combinatorics and $\tau(F) \geq \tau$ then the first return map to $T_{\tau(F)}$ is κ-good.*

Proof. Let κ, λ be as in Lemma 9.6, let ρ as in Lemma 9.4, and let τ satisfy $\lambda^\tau < \kappa^{-1}$, $\tau\rho \geq \kappa$. $\qquad\square$

Theorem 9.8 (Lean case exists). *There exists $\tau_0 > 0$, such that for every $N > 0$, there exists $\tau > \tau_0$ with the following property. Let f is an infinitely renormalizable quadratic-like map such that all renormalizations of f have (N, τ_0) bounded combinatorics and height at least τ. Then the Julia set of f has Lebesgue measure zero. Moreover, if f is a Feigenbaum map then $\mathrm{HD}(J(f)) < 2$.*

Proof. Let τ_0, N, κ, and τ be as in the previous lemma. It is enough to prove the result for quadratic maps. In this case, we can consider a quadratic-like restriction $f = F : \Upsilon^0 \to \Upsilon$ which is κ-good. By the previous lemma, there exists a domain $\Upsilon_{(1)} = T_{\tau(F)}$ such that the first return map to $\Upsilon_{(1)}$ is κ-good. Repeating this procedure, we obtain a sequence of domains $\Upsilon_{(n)}$ around 0 such that the first return map F_n to $\Upsilon_{(n)}$ is κ-good. We also let $\Upsilon_{(0)} = \Upsilon$. Let $\Upsilon_{(n)}^0$ be the central domain of the first return map to $\Upsilon_{(n)}$. Let $f_n = F_n : \Upsilon_{(n)}^0 \to \Upsilon_{(n)}$.

Let $M_{(n)}^0$ be the domain bounded by the equator of $\Upsilon_{(n)} \smallsetminus \Upsilon_{(n)}^0$. Let $M_{(n)}^k = f_n^{-k}(M_{(n)}^0)$. Let $\mathbb{V}^n = M_{(n)}^{k_n}$ where k_n is minimal with $\mathrm{diam}(M_{(n)}^{k_n}) < 100\,\mathrm{diam}(J^n)$, and let $\mathbb{U}^n = f_n^{-1}(\mathbb{V}^n)$. Then $\mathbb{U}^n$ and $\mathbb{V}^n$ have the Properties (P1-4) of Lemma 2.3 (see the proof of Lemma 9.2). Let $V^n = f_n^{-k_n-2}(\Upsilon_{(n)}^0)$, $U^n = f_n^{-1}(V^n)$, and $A^n = V^n \smallsetminus U^n$. Then U^n and V^n satisfy the conditions (C1-3) and (G1-2) of § 2.7.

We claim that the density of the domain of the first return map to V^n is bounded by some $\lambda < 1$. The argument is similar to the proof of assertion (2) of Lemma 9.3. If $x \in A^n$ is such that $F_n^k(x) \in V^n$ for some k, then there exists $k \geq k_n + 4$, and there exists $k_n + 3 \leq r_x \leq k - 1$ such that $F_n^l(x) \in \Upsilon_{(n)} \smallsetminus \Upsilon_{(n)}^0$, $k_n + 3 \leq l \leq r_x$ and $F_n^{r_x+1}(x)$ belongs to the domain of the first return map (under F_n) to $\Upsilon_{(n)}^0$. Moreover, there exists $\tilde{S}_x \ni x$ such that $F_n^{r_x+1}|\tilde{S}_x$ is a univalent map onto $\Upsilon_{(n)}$. Let $S_x = (F_n^{r_x+1}|\tilde{S}_x)^{-1}(\Upsilon_{(n)}^0)$. Then $S_x \subset V^n \smallsetminus U^n$ and the S_x form a disjoint cover of the domain of the first return map to $V^n \smallsetminus U^n$. Since $F_n^{r_x+1} : S_x \to \Upsilon_{(n)}^0$ has bounded distortion, one sees that there exists a definite (at least $1 - C\kappa^{-1} > 0$) probability that a point in A^n never returns to V^n. Since $\mathrm{area}(A^n) \asymp \mathrm{area}(V^n)$, this concludes the claim.

Let Δ_n be the set of points in $\mathbb{V}^0$ that visit V^n. Each branch of the first landing map $\phi_n : \Delta_n \to V^n$ has a univalent extension to $\mathbb{V}^n$ and thus has bounded distortion. Notice that if $x \in \Delta_{n+1}$ then $\phi_n(x)$ belongs to the domain of the first return map to V^n. This implies that $p(\Delta_{n+1}|\Delta_n) < \tilde{\lambda} < 1$, so $\mathrm{area}(\Delta_n)$ decreases exponentially fast. By [L1], almost every $x \in J(f)$ is such that $0 \in \omega(x)$, so $\mathrm{area}(J) = \mathrm{area}(\cap \Delta_n) = 0$.

Assume now that f is Feigenbaum. In the notation of §4, we have shown that $\xi_n \geq 1 - C\kappa^{-1}$. The same argument applied to F_m shows that $\xi_{m,n} \geq 1 - C\kappa^{-1}$. By Lemma 4.1, $\eta_{m,n}$ is exponentially small in $n - m$. Fix some $\epsilon > 0$ small. For $0 < \rho < 1$ small, let

$$(9.5) \qquad \omega_\rho^{[j]}(\delta) = \sup \omega_{m,n}^{[j]}(\delta), \quad \xi_\rho = \sup \xi_{m,n}, \quad \eta_\rho = \sup \eta_{m,n},$$

where we take the supremums over pairs $m < n$ such that $\rho^{1+\epsilon} < \rho_{m,n} < \rho^{1-\epsilon}$ (for every m, there exists such an n provided ρ is sufficiently small). Notice that such pairs satisfy $-C^{-1} \ln \rho \leq n - m \leq -C \ln \rho$ (where C depends on f through the geometric and combinatorial bounds), so $\eta_\rho < \rho^\upsilon$ for

some $\upsilon > 0$. Moreover, $\xi_\rho \geq 1 - C\kappa^{-1}$. Putting together (4.9) and (4.10), we get that for every δ close to 2 (depending on ρ), we have $\omega_\rho^{[j+1]}(\delta) \leq P_\rho(\omega_\rho^{[j]}(\delta))$, where

$$(9.6) \qquad P_\rho(x) = C\frac{\eta_\rho}{\rho^{1+\epsilon}} + (1 - C^{-1}(\eta_\rho + \xi_\rho) + C\eta_\rho\xi_\rho\rho^{-2\epsilon})x + C\xi_\rho\rho^{1-\epsilon}x^2.$$

If $\epsilon > 0$ is small, then for every ρ sufficiently small P_ρ has a positive fixed point s that attracts 0, and we have $\lim_{j\to\infty} \omega_\rho^{[j]}(\delta) \leq s$, $\delta \approx 2$. This implies that $\delta_{\mathrm{cr}}(f) < 2$, so $\mathrm{HD}(J(f)) < 2$ by Theorem C. $\qquad\qquad\square$

10. Various remarks and open problems

In this section we will formulate some related results, possible further generalizations, and open problems. Some of these issues will be elaborated in forthcoming notes.

10.1. **Outstanding Problems.** The following problems still remain open:

Problem 1. Are there Feigenbaum Julia sets with positive area?

We are inclined to believe that the answer is affirmative.

Problem 2. Are there Feigenbaum Julia sets with zero area but Hausdorff dimension 2?

According to Theorem B, the affirmative answer to the first question (together with certain *a priori* bounds) implies the affirmative answer to the second.

Problem 3. What is the area and dimension of the Julia set of the classical Feigenbaum quadratic polynomial F_2 corresponding to the doubling renormalizations?

It is unlikely that Problem 3 can be resolved without computer assistance.

All results of this paper extend naturally (with the same proofs) to the setting of unicritical polynomial-like maps of some fixed degree $d \geq 2$ (specific constants may depend on the degree). Let $F_d : z \mapsto z^d + c_d$ (where $d \geq 2$ is an even integer) be the doubling Feigenbaum map of degree d, and let $J_d = J(F_d)$.

Problem 4. What is $\mathrm{area}(J_d)$ for big d?

In case $\mathrm{area}(J_d) = 0$, it would be interesting to find out whether $\mathrm{HD}(J_d) = 2$ for big d. (According to [LS], $\mathrm{HD}_{\mathrm{hyp}}(J_d) \to 2$ as $d \to \infty$.)

10.2. **Hausdorff dimension in the parameter space.** By our method one can produce a fair amount of Feigenbaum parameters with small Julia set:

Theorem 10.1. *The set of parameters $c \in M$ such that $P_c : z \mapsto z^2 + c$ is a Feigenbaum map with $\mathrm{HD}(J(f)) < 2$ has Hausdorff dimension at least 1.*

To see this, it is enough to use the method of [L7] to obtain large sets of Feigenbaum maps of high combinatorial type.

10.3. **Hyperbolic dimension and critical exponents.** Our methods of proving that $\mathrm{HD}_{\mathrm{hyp}}(J) = \mathrm{HD}(J)$ and $\delta_{\mathrm{cr}} = \delta_*$ can be applied to much broader class of rational maps: it will be elaborated in forthcoming notes.

10.4. **Absolutely continuous invariant measures.** Construction of invariant geometric measures is probably the most important traditional problem of ergodic theory. Here is an answer in our situation:

Theorem 10.2. *For any Feigenbaum map, there is a σ-finite measure λ equivalent to the δ_{cr}-conformal measure μ.*

To construct this measure, one can use the method of [KL].

Problem 5. Can the invariant geometric measure of a Feigenbaum map be finite? (Of course, it may happen only in the conservative case.)

10.5. **Hausdorff measure.** Conformal measures appear as a useful substitution of Hausdorff measures h_δ since in the non-hyperbolic dynamical situations the latter usually vanish. The following statement (which easily follows from Lemma B.8 from Appendix B) confirms this principle:

Proposition 10.3. *Let $\delta = \mathrm{HD}(J)$ for a Feigenbaum Julia set J with periodic combinatorics. If $\mathrm{area}(J) = 0$ then $h_\delta(J) = 0$.*

10.6. **Conservativity/dissipativity of towers.** The Trichotomy nicely translates to measure-theoretic properties of towers $\bar{f}$ (see Appendix B):

- *Lean case*: the Lebesgue measure is dissipative, supported on $\mathbb{C} \smallsetminus J(\bar{f})$, and almost all orbits escape to infinity under the tower dynamics;
- *Balanced case*: the Lebesgue measure is conservative under the tower dynamics, and supported on $\mathbb{C} \smallsetminus J(\bar{f})$;
- *Black hole case*: the Lebesgue measure is dissipative, supported on $J(\bar{f})$, and almost all orbits are attracted to the postcritical set $\mathcal{O}(\bar{f})$.

10.7. **Real Fibonacci maps and wild attractors.** In this section, we will consider a class $\mathcal{A}$ of "real generalized polynomial-like maps"

$$f : \bigcup_{j=0}^{s} I_j \to I, \quad s \geq 1,$$

where

(1) I_j are disjoint closed intervals contained in a 0-symmetric closed interval I, and moreover $I_j \subset \mathrm{int}\, I$;
(2) One of the I_j (conventionally I_0) is a 0-symmetric interval and $f : (I_0, \partial I_0) \to (I, \partial I)$ is a unimodal map which can be written as $\phi(|x|^l)$ where ϕ is a diffeomorphism with negative Schwarzian derivative onto a neighborhood of I, and $l > 1$ is a real number (the *criticality* of f);
(3) If $j \neq 0$ then $f : (I_j, \partial I_j) \to (I, \partial I)$ extends to a diffeomorphism with negative Schwarzian derivative onto a neighborhood of I;
(4) The critical point is recurrent and $\omega(0)$ is a minimal set;
(5) All intervals I_j intersect $\omega(0)$;
(6) f is not renormalizable in the classical sense: there exist no 0-symmetric intervals $T \subset I_0$ such that the first return map to T is a unimodal map.

We shall consider such maps up to rescaling. The *real Julia set* is defined as $J^r(f) = \cap f^{-n}(I)$. If it has positive length ($\equiv$ one-dimensional Lebesgue measure) then almost all orbits are attracted to $\omega(0)$, which is called a *wild attractor*.

The principal nest is an infinite sequence of intervals defined inductively by $I^0 = I$, $I^1 = I_0$, and I^{n+1} as the domain of the first return to I^n containing 0. Let I_j^n be the connected components of the first return map to I^n which intersect $\omega(0)$ (conventionally, we let $0 \in I_0^n$ so that $I_0^n = I^{n+1}$).

We let $f_n : \cup I^n_j \to I^n$ be the first return map, and we call f_{n+1} the *generalized renormalization* of f_n.

We say that f has a *bounded geometry* if all the intervals I^j_n and all the gaps in between are commensurable with I^n (with constants independent of j and n). In this case $0 < \mathrm{HD}(\omega(0)) < 1$, and one is interested in $\mathrm{HD}(J^r(f))$.

Maps of class $\mathcal{A}$ with bounded geometry are quite similar to Feigenbaum maps, and our methods apply to this setting as well. There is one (temporary?) advantage however: by [BKNS], we know that wild attractors do exist for some maps $f \in \mathcal{A}$, so that our results concerning Julia sets with positive measure become unconditional in this context.

Though we will concentrate on the real setting described above, maps of class $\mathcal{A}$ have natural complex counterparts: we refer the reader to [LMil] and [Bu], as well as to a more recent work by Smania [Sm]. Our methods apply equally well in this setting (where existence of Julia sets of positive Lebesgue measure is still an open problem).

Maps of class $\mathcal{A}$ with $1 < l \le 2$ cannot have bounded geometry (see [L4] for $l = 2$ and [Sh] for a recent generalization).

We say that $f \in \mathcal{A}$ is a *Fibonacci map* if for every n the domain of f_n has two components, I^n_0 and I^n_1, and $f_n|I^n_1 = f_{n-1}$ while $f_n|I^n_0 = f^2_{n-1}$. For $l = 2$, geometry of Fibonacci maps was analyzed in [LMil]. A Fibonacci map with criticality $l > 2$ has bounded geometry (see [KN]).

For $l > 2$ but close to 2, the intervals I^n_0 and I^n_1 are much smaller than I^n [KN], and in this case one can show that $J^r(f)$ is lean at the critical point. Our methods then imply:

Theorem 10.4. *There exists an $\epsilon > 0$ such that if f is a Fibonacci map of degree $l \in (2, 2 + \epsilon)$, then $\mathrm{HD}(J^r(f)) < 1$.*

On the other hand, it is proved in [BKNS] that if f is a Fibonacci map of large degree l, then the real Julia set $J^r(f)$ has positive length (so that f has a wild attractor). Then our methods imply:

Theorem 10.5. *If f is a Fibonacci map of sufficiently large degree l, then $\mathrm{HD}_{\mathrm{hyp}}(J^r(f)) < 1$.*

If l is an even integer, one can prove more by methods of complex dynamics:

Theorem 10.6 (see [Bu]). *If $l > 2$ is an even integer, then there exists a unique fixed point of the generalized renormalization with Fibonacci combinatorics. It attracts all other Fibonacci maps with the same criticality.*

It is generally believed that the same result is true with arbitrary criticalities:

Conjecture 10.7. *For every $l > 2$ there exists a unique fixed point of the generalized renormalization with the Fibonacci combinatorics.*

Martens' method [M2] can probably be applied to prove existence, but the uniqueness part appears to be wide open at this point.

Let us denote by f_l the fixed point of the Fibonacci generalized renormalization if it exists and is unique. By compactness (real *a priori* bounds), f_l depends continuously on l (over l's for which it is defined). Thus if Conjecture 10.7 holds, we can see a "phase transition" in the family $\{f_l\}_{l \in (2, \infty)}$.

Theorem 10.8. *Assume that Conjecture 10.7 holds. Then all three cases of the Trichotomy occur for f_l, $2 < l < \infty$:*

(1) *(Lean case) For an open set of l's containing $(2, 2 + \epsilon)$, $\mathrm{HD}_{\mathrm{hyp}}(J^r(f)) = \mathrm{HD}(J^r(f)) < 1$;*
(2) *(Balanced case) For a compact non-empty set of l's, $\mathrm{HD}_{\mathrm{hyp}}(J^r(f)) = \mathrm{HD}(J^r(f)) = 1$ but the length of $J^r(f)$ is zero;*
(3) *(Black hole case) For an open set of l's containing (K, ∞), $\mathrm{HD}_{\mathrm{hyp}}(J^r(f)) < 1$ but $J^r(f)$ has positive length.*

Proof. By Theorems 10.4 and 10.5, the Lean and Black hole cases do occur for some sets of l's containing intervals $(2, 2 + \epsilon)$ and (K, ∞) respectively. To see that they occur for open sets of parameters, notice that both are equivalent to a finite criteria (existence of n such that η_n/ξ_n is either small or large, with constants depending continuously on l, see Theorems 5.5 and 5.8). Since $(2, \infty)$ is connected, the Balanced case must also occur. □

Problem 6. Is there a unique value l_0 for which the Balanced case occurs?

It seems unlikely that the Balanced case can ever happen for an integer l.[13]

APPENDIX A. NICE FUNDAMENTAL DOMAINS

Let $f : \mathbb{U} \to \mathbb{V}$ be a Feigenbaum map, and let $f_n : \mathbb{U}^n \to \mathbb{V}^n$ be its pre-renormalizations satisfying properties (P1)-(P4) of Lemma 2.3. In this section we will construct a sequence of simply connected domains V^n satisfying properties (C1-3) and (G1-2) from §2.7. Our construction is reminiscent of Sinai's construction of Markov partitions [Si].

A.1. **From admissible pairs to nice domains.** Take some $n \in \mathbb{N}$, and let $\mathcal{F}^n$ be the family of inverse branches of f^k, $k \geq 0$, which are well defined on $\mathbb{V}^n$.

Let $X^n \subset \mathbb{V}^n$. For $g \in \mathcal{F}^n$ (an inverse branch of f^k), a set $g(X^n) \subset \mathbb{V}$ is called a *copy* of X^n of *depth* $k \geq 0$. In particular, let us consider the set $\mathcal{C}^n = \{g(0)\}_{g \in \mathcal{F}^n}$ of all "copies" of 0. There is a natural one-to-one correspondence between $\mathcal{C}^n$ and $\mathcal{F}^n$ so that we can use $\mathcal{C}^n$ to label all copies X_x^n of X^n, $x \in \mathcal{C}^n$.

Lemma A.1. *Let $X^n \subset \mathbb{V}^n$ be such that $f_n^{-1}(X^n) \subset X^n$. Then $\bigcup_k f^{-k}(X^n)$ is the union of all copies of X^n.*

Proof. Let $z \in \bigcup_k f^{-k}(X^n)$, and let k be the first landing moment of z to X^n. Let $D_k \equiv \mathbb{V}^n$, and let us define D_{k-j} for $1 \leq j \leq k$ inductively as the connected component of $f^{-1}(D_{k-j+1})$ containing $f^{k-j}(z)$. Let us show that $f^k : D_0 \to \mathbb{V}^n$ is univalent. If this is not the case, then $0 \in D_{k-j}$ for some $1 \leq j \leq k$. Take the minimal such j. Then it is easy to see that j is the renormalization period on level n and $D_{k-j} = \mathbb{U}^n$. Hence $f^{k-j}(z) \in f_n^{-1}(X^n) \subset X^n$, contradicting to the choice of k.

It follows that $(f^k|D_0)^{-1}(X^n)$ is a copy of X^n containing z. □

A pair (Y^n, X^n) of nested sets $X^n \subset Y^n \subset \mathbb{V}^n$ is called *admissible* if every copy of Y^n that intersects X^n is contained in Y^n.

Lemma A.2. *Let (Y^n, X^n) be an admissible pair.*
  (1) *Let $x, y \in \mathcal{C}^n$ be such that $\mathrm{depth}(x) \leq \mathrm{depth}(y)$. If $X_x^n \cap Y_y^n \neq \emptyset$ then $Y_y^n \subset Y_x^n$.*
  (2) *Assume X^n is a domain. Let W be a domain which is the union of some copies of X^n. If $X_x^n \subset W$ is a copy of X^n of smallest depth then $W \subset Y_x^n$.*

Proof. First assertion. Let $k = \mathrm{depth}(x)$. Then $f^k(X_x^n) = X^n$ intersects $f^k(Y_y^n) = Y_{f^k(y)}^n$, so $f^k(Y_y^n) \subset Y^n$. It follows that $Y_y^n \subset Y_x^n$.

Second assertion. Notice that there is a sequence $x = x_1, x_2, \ldots$ of points of $\mathcal{C}^n$ such that W is equal to the increasing union of domains $X_{x_1}^n \cup \cdots \cup X_{x_m}^n$. Thus, we can restrict ourselves to the case where W is itself a finite union of copies of X^n.

Let us carry induction in the number of copies, m. The assertion is obvious for $m = 1$. For the induction step, let us consider connected components U_l of $\bigcup_{i=2}^m X_{x_i}^n$. By the induction hypothesis,

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[13]For the same reason it is unlikely that there exists any example of a complex Feigenbaum/Fibonacci periodic point of renormalization which falls into the Balanced case, since there are only countably many parameters (combinatorics and degree) to play with.

$U_l \subset Y_{x_j}^n$ for some $j = j(l) \in [2, m]$. Since the union $U_l \cup X_x^n$ is connected, $U_l \cap X_x^n \neq \emptyset$; all the more, $Y_{x_j}^n \cap X_x^n \neq \emptyset$. By the first assertion, $Y_{x_j}^n \subset Y_x^n$. All the more, $U_l \subset Y_x^n$ and hence

$$W = X_x^n \cup \bigcup_l U_l \subset Y_x^n.$$

$\square$

Let us say that an open set $X \subset \mathbb{V}$ is *nice* if $f^k(\partial X) \cap X = \emptyset$ for all $k \geq 0$ (compare [M1]). Let us mention some nice properties of nice sets:

Lemma A.3. *Let f be a quadratic-like map. Then*
  (N1) *For any open set $X \subset \mathbb{V}$, $\cup f^{-k}(X)$ is nice;*
  (N2) *Components of a nice set are nice;*
  (N3) *If X is a nice domain then the filling of X is nice, provided f has no attracting periodic orbits;*
  (N4) *Intersection of two nice sets is nice;*
  (N5) *Preimages of nice sets are nice.*

Proof. We will show (N3), as all the other properties are obvious. Let Z be the filling of X. Then $\partial Z \subset \partial X$. If $z \in \partial Z$ is such that $f^k(z) \in Z$ then $f^k(z) \in D$ where D is a Jordan disk with $\partial D \subset X$. Since X is nice and ∂Z is connected, this implies that $f^k(\partial Z) \subset D$, and hence $f^k(Z) \subset D$ as well. Thus f has an attracting periodic point in D, contradiction. $\square$

Notice that property (C3) states precisely that V^n should be nice. The next lemma shows how to use certain admissible pairs to construct nice domains.

Lemma A.4. *Let (Y^n, X^n) be an admissible pair, where Y^n is a simply connected domain and X^n is a domain containing 0 satisfying $f_n^{-1}(X^n) \subset X^n$. Then the filled connected component S^n of $\cup f^{-k}(X^n)$ containing 0 is a nice simply connected domain contained in Y^n, and such that $f_n^{-1}(S^n) \subset S^n$.*

Proof. Let Z^n be the connected component of $\cup f^{-k}(X^n)$ containing 0. Z^n is nice by (N1) and (N2). By Lemma A.1, Z^n is the union of some copies $X_{x_i}^n$ of X^n, $x_i \in \mathcal{C}^n$. The smallest depth of these copies (that is, 0) is reached on $X^n \equiv X_0^n$ itself. By the second assertion of Lemma A.2, $Z^n \subset Y^n$. By (N3), S^n is nice as well. Since Y^n is simply connected, $S^n \subset Y^n$, and since $X^n \subset S^n$, $f_n^{-1}(S^n) \cap S^n \neq \emptyset$, the nice property implies that $f_n^{-1}(S^n) \subset S^n$. $\square$

A.2. **Separation between some Julia copies.** Let p_n be the renormalization period on level n, i.e., $f_n = f^{p_n}$. Let $J^n = J(f_n)$, and let $J_i^n = f^i(J^n)$, $i = 0, 1, \ldots, p_n - 1$. Notice that the J_i^n are copies of J^n. There are two cases to distinguish: whether the J_i^n are all disjoint (the *primitive* case) or not (the *satellite* case).

To distinguish those cases, recall that f_n has two fixed points, conventionally denoted α_n and β_n, such that $J^n \smallsetminus \{\alpha_n\}$ is disconnected while $J^n \smallsetminus \{\beta_n\}$ is connected. The satellite case happens if and only if $n > 0$ and f_{n-1} is *immediately renormalizable*, that is, $\alpha_{n-1} = \beta_n$. In this case, $J_i^n \cap J_j^n \neq \emptyset$, $0 \leq i < j \leq p_n - 1$ if and only if $j - i = 0 \bmod \frac{p_n}{p_{n-1}}$, and in this case $J_i^n \cap J_j^n = f^i(\beta_n)$.

A $(\nu \operatorname{diam} J)$-neighborhood of a set J will be called the ν-*enlargement* of J.

Lemma A.5. *There exists $\nu > 0$ such that if $J^n \cap J_i^n = \emptyset$ then the ν-enlargement of J^n is disjoint from J_i^n.*

Proof. Let $m \leq n$ be minimal such that $J_i^n \subset J_k^m$ with $J_k^m \cap J^m = \emptyset$. Then $J_k^m \subset J^{m-1}$ (in the primitive case) or $J_k^m \subset J^{m-2}$ (in the satellite case). By a simple compactness argument, J_k^m is disjoint from the ν-enlargement of J^m. All the more, the smaller set $J_i^n \subset J_k^m$ is disjoint from the ν-enlargement of $J^n \subset J^m$. $\square$

A.3. **Construction of admissible pairs.** Let ν be as in Lemma A.5, and fix $\rho \leq \nu$. Let $l \geq 0$ be minimal such that $\Gamma^n \equiv f_n^{-l}(\mathbb{U}^n)$ is contained in the intersection of the ρ-enlargement of J^n with $\mathbb{U}^{n-1}$. Notice that:

(1) $\mathrm{mod}(\Gamma^n \smallsetminus f_n^{-1}(\overline{\Gamma}^n))$ is definite (once ρ is fixed),
(2) If $J_i^n \cap \Gamma^n \neq \emptyset$ then either $J_i^n = J^n$ or f_{n-1} is immediately renormalizable and $J_i^n = f_{n-1}^j(J^n)$ for some j,
(3) If a copy Γ_x^n of Γ^n of depth $k > 0$ intersects J^n then f_{n-1} is immediately renormalizable and $f^k|\Gamma_x^n$ is an iterate of f_{n-1}.

    Indeed, $\Gamma_x^n \cap J^n \neq \emptyset$ implies that $f^k(\Gamma_x^n) \cap f^k(J^n) = \Gamma^n \cap f^k(J^n) \neq \emptyset$. By Lemma A.5, J^n touches $f^k(J^n)$, so f_{n-1} is immediately renormalizable, and $k = sp_{n-1}$ for some integer $s > 0$. To conclude, we must show that $f^{jp_{n-1}}(\Gamma_x^n) \subset \mathbb{U}^{n-1}$, $0 \leq j < s$. But if this is not the case then the connected set $f^{jp_{n-1}}(\Gamma_x^n)$ intersects both $J^{n-1} \supset f^{jp_{n-1}}(J^n)$ and $\partial\mathbb{U}^{n-1}$. This implies that further iterates $f^{rp_{n-1}}(\Gamma_x^n)$, $r > j$, have the same property, and in particular $\Gamma^n = f^k(\Gamma_x^n)$ intersects $\partial\mathbb{U}^{n-1}$, contradiction.

We will consider separately the primitive and satellite cases.

A.3.1. *Primitive case.* Assume that f_{n-1} is not immediately renormalizable. In this case, any copy of Γ^n of depth $k > 0$ is disjoint from J^n.

Let $Y^n = f_n^{-1}(\Gamma^n)$. Notice that ∂Y^n has bounded hyperbolic diameter in the hyperbolic metric of $\mathbb{V}^n \smallsetminus J^n$.

If Y_x^n is a copy of Y^n of positive depth, then the hyperbolic diameter of Y_x^n in $\mathbb{V} \smallsetminus J^n \supset \Gamma_x^n$ is bounded (since $\mathrm{mod}(\Gamma_x^n \smallsetminus \overline{Y}_x^n)$ is definite). Notice that the hyperbolic metrics of $\mathbb{V} \smallsetminus J^n$ and $\mathbb{V}^n \smallsetminus J^n$ are comparable on $\mathbb{U}^n \supset Y^n$. Hence if $Y_x^n \cap \partial Y^n \neq \emptyset$, then the hyperbolic diameter of $Y_x^n \cap Y^n$ in the hyperbolic metric of $\mathbb{V}^n \smallsetminus J^n$ is bounded by some K.

Let $s \geq 0$ be minimal such that the hyperbolic distance (in the hyperbolic metric of $\mathbb{V}^n \smallsetminus J^n$) from any $z \in X^n \equiv f_n^{-s}(Y^n)$ to ∂Y^n is greater than K. Then (Y^n, X^n) is admissible.

Notice also that the hyperbolic diameter of $\overline{Y}^n \smallsetminus X^n$ (in the hyperbolic metric of $\mathbb{V}^n \smallsetminus J^n$) is uniformly bounded.

A.3.2. *Satellite case.* Assume that f_{n-1} is immediately renormalizable.

Lemma A.6. *There exists a simply connected domain $Y^n \subset \Gamma^n$ such that*

(1) *Y^n is nice for f_{n-1} and $f_n^{-1}(Y^n) \subset Y^n$;*
(2) *There exists a finite set F of copies of β_n such that Y^n is a neighborhood of $J^n \smallsetminus F$;*
(3) *$\mathrm{mod}(\Gamma^n \smallsetminus \overline{Y}^n)$ is definite;*
(4) *The hyperbolic diameter of ∂Y^n in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$ is uniformly bounded;*
(5) *If $Y_x^n \cap \partial Y^n \neq \emptyset$ then $\Gamma_x^n \subset \Gamma^n \smallsetminus J^n$;*
(6) *If $Y_x^n \cap \partial Y^n \neq \emptyset$ then the diameter of Y_x^n in the hyperbolic metric of $\mathbb{V}^n \smallsetminus J^n$ is bounded.*

Proof. The properties described in Lemma 2.3 allow us to obtain a straightening of $f_{n-1} : \mathbb{U}^{n-1} \to \mathbb{V}^{n-1}$ with bounded dilatation, and thus to define equipotentials and external rays in $\mathbb{V}^{n-1}$. Let E^{n-1} be the domain bounded by the equipotential of level 2. Let us cut E^{n-1} by the external rays landing at α_{n-1}, and let Z^n be the connected component of 0. Let us show that $Y^n = f_n^{-l}(Z^n)$ has all the desired properties if l is large enough (independent of n).

Properties (1) and (2) are obvious. If l is large enough then $Y^n \subset \Gamma^n$. Moreover, then properties (3) and (4) are also valid for every fixed l.

Let us deal with property (5). Let us show first that if $Y_x^n \cap \partial Y^n \neq \emptyset$ then $\Gamma_x^n \cap J^n = \emptyset$. Indeed, if $\Gamma_x^n \cap J^n \neq \emptyset$ then $f^k : J_x^n \to J^n$, $k = \mathrm{depth}(x)$, is an iterate of f_{n-1}. But since Y^n is nice for f_{n-1}, $Y_x^n \cap \partial Y^n = \emptyset$ - contradiction.

Let us now show that if $Y_x^n \cap \partial Y^n \neq \emptyset$ then $\Gamma_x^n \cap \partial \Gamma^n = \emptyset$. Notice that $\operatorname{diam}(Y_x^n) \asymp \operatorname{diam}(\Gamma_x^n)$, $\operatorname{mod}(\Gamma_x^n \smallsetminus \overline{Y}_x^n) = \operatorname{mod}(\Gamma^n \smallsetminus \overline{Y}^n)$ is bounded from below (uniformly for all sufficiently big l), and ∂Y^n is contained in a ϵ-neighborhood of J^n (where $\epsilon \to 0$ when $l \to \infty$). Those geometric properties together show that if Γ_x^n intersects both boundaries, $\partial \Gamma^n$ and ∂Y^n (so that Y_x^n has size comparable with J^n), then $Y_x^n \cap \partial Y^n \neq \emptyset$ implies that $\Gamma_x^n \cap J^n \neq \emptyset$ for l big enough, contradiction. This proves (5).

The last property follows from the third, the fifth, and the fact that the hyperbolic metrics of $\mathbb{V} \smallsetminus J^n$ and of $\mathbb{V}^n \smallsetminus J^n$ are comparable on $\mathbb{U}^n$. $\qquad\square$

Let X^n be the connected component of 0 of the interior of

$$Y^n \smallsetminus \bigcup_{Y_x^n \cap \partial Y^n \neq \emptyset} Y_x^n.$$

Lemma A.7. *The pair (Y^n, X^n) is admissible. Moreover:*

(1) X^n *is simply connected,*
(2) X^n *is a neighborhood of $J^n \smallsetminus F$,*
(3) $f_n^{-1}(X^n) \subset X^n$,
(4) $\overline{Y}^n \smallsetminus X^n$ *has bounded hyperbolic diameter in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$.*

Proof. If $Y_x^n \cap X^n \neq \emptyset$ then $Y_x^n \cap \partial Y^n \neq \emptyset$ by definition of X^n, and hence $Y_x^n \subset Y^n$. Let us now check Properties (1-4).

(1). Notice that X^n is a bounded connected component of the complement of the closure of the connected set

$$\bigcup_{\partial Y_x^n \cap \partial Y^n \neq \emptyset} \partial Y_x^n,$$

so it is simply connected.

(2). By Properties (2) and (6) of Lemma A.6, X^n is a neighborhood of $J^n \smallsetminus F$.

(3). If $Y_x^n \cap \partial Y^n \neq \emptyset$ then by Property (5) of Lemma A.6, $\Gamma_x^n \subset \Gamma^n \smallsetminus J^n$ which implies that $\operatorname{depth}(x) \geq p_n$ and in particular $f_n(Y_x^n)$ is a copy of Y^n. Thus, if $Y_x^n \cap f_n^{-1}(X^n) \neq \emptyset$ and $Y_x^n \cap \partial Y^n \neq \emptyset$, then there is a copy of Y^n, $f_n(Y_x^n)$, intersecting X^n and $\partial(f_n(Y_x^n))$, contradicting the definition of X^n. Since $J^n \smallsetminus F \subset X^n$ (by (2)), $f_n^{-1}(X^n)$ is a connected set intersecting X^n, and by the definition of X^n it follows that $f_n^{-1}(X^n) \subset X^n$.

(4). By Property (6), $\overline{Y}^n \smallsetminus X^n$ has bounded hyperbolic diameter in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$. $\qquad\square$

A.4. **Construction of domains U^n and V^n.** Let (Y^n, X^n), $n \geq 0$ be given by the construction of the previous section. Let S^n be given by Lemma A.4. We can not set $V^n = S^n$ because this does not guarantee property (C2) (although it would possess the other properties).

This can be fixed as follows. Let Q^n be the connected component of 0 of $\cap_{m \leq n} S^m$. By (N2) and (N4), Q^n is nice. Let $V^n = f_n^{-1}(Q^n)$, and $U^n = f_n^{-1}(V^n)$. Then V^n is nice by (N5), so V^n has Property (C3). Notice that $J^n \smallsetminus F \subset V^n$, where F is a finite set of copies of β_n (empty if f_{n-1} is not immediately renormalizable). Since we also have $V^n \subset \mathbb{U}^n$, this implies (C1). To check (C2), assume there is $x \in V^n \cap \partial U^{n-1}$. Then $f_n(x) \in Q^n \subset Q^{n-1}$ and $f_{n-1}^2(x) \in \partial Q^{n-1}$. Notice that $f_n(x) = f_{n-1}^k(x)$ where $k = p_n/p_{n-1} \geq 2$. So $f_{n-1}^{k-2}(\partial Q^{n-1}) \cap Q^{n-1} \neq \emptyset$, contradicting that Q^{n-1} is nice.

We have so far checked the combinatorial properties (C1-3), and we will now verify the geometric properties (G1-2).

We first make a couple of remarks on the geometry of S^n. We have that ∂S^n has bounded hyperbolic diameter in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$. Since $\mathbb{V}^n \cap \mathcal{O}(f) \subset S^n$, this implies that the Euclidean distance from ∂S^n to $\mathcal{O}(f) \cup \partial \mathbb{V}^n$ is comparable with $\operatorname{diam}(J^n) \asymp \operatorname{diam}(\mathcal{O}(f) \cap \mathbb{V}^n) \asymp \operatorname{diam}(\mathbb{V}^n)$. In particular, S^n contains a neighborhood of 0 of radius comparable with $\operatorname{diam}(J^n)$. On the other

hand, $S^n \subset \mathbb{V}^n$ is contained in a disk of radius $\mathrm{diam}(\mathbb{V}^n) \asymp \mathrm{diam}(J^n)$. Since $\mathrm{diam}(J^n)$ decreases exponentially fast, we see that there exists $\upsilon \geq 0$ such that $S^n \subset S^m$ whenever $n \geq m + \upsilon$.

Lemma A.8. *The hyperbolic diameter of ∂Q^n in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$ is bounded.*

Proof. For $m < n$, the Euclidean distance of points in ∂S^m to $\mathcal{O}(f)$ is comparable with $\mathrm{diam}\, J^m \geq \mathrm{diam}(J^n)$. This implies that the hyperbolic distance in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$ of points in $\partial S^m \cap S^n$ to ∂S^n is bounded. Since $\partial Q^n \subset \partial S^n \cup \bigcup_{m<n}(S^n \cap \partial S^m)$, the result follows. $\qquad \square$

Lemma A.9. *The hyperbolic diameter of $\overline{V}^n \smallsetminus U^n$ in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$ is uniformly bounded.*

Proof. Recall that if $g : X \to Y$ is a covering map of degree n between hyperbolic Riemann surfaces and $Z \subset Y$ is such that $g^{-1}(Z)$ is connected then the hyperbolic diameter of $g^{-1}(Z)$ in X is at most n times the hyperbolic diameter d of Z in Y.

Let us show first that ∂V^n and ∂U^n have bounded hyperbolic diameter in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$. By the Schwarz Lemma, the hyperbolic diameter of ∂V^n in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$ is bounded by its hyperbolic diameter in $f_n^{-1}(\mathbb{V}^n \smallsetminus \mathcal{O}(f))$. Since $\partial V^n = f_n^{-1}(\partial Q^n)$ is connected, its hyperbolic diameter in $f_n^{-1}(\mathbb{V}^n \smallsetminus \mathcal{O}(f))$ is bounded by two times the hyperbolic diameter of ∂Q^n in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$, which is bounded by Lemma A.8. Applying this argument to $\partial U^n = f_n^{-1}(\partial V^n)$, we see that ∂U^n also has bounded hyperbolic diameter in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$.

This implies that the Euclidean distance of every $x \in \overline{V}^n \smallsetminus U^n$ to $\partial \mathbb{V}^n \cup \mathcal{O}(f)$ is comparable with $\mathrm{diam}\, J^n$. So the hyperbolic distance of every $x \in \overline{V}^n \smallsetminus U^n$ to $\partial V^n \cup \partial U^n$ in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$ is bounded. The result follows. $\qquad \square$

Property (G1) is given by Lemma A.9. Let us show that (C1-3) and (G1) imply (G2).

Since ∂U^n has bounded hyperbolic diameter in $\mathbb{V}^n \smallsetminus \mathcal{O}(f)$, $U^n \supset \mathbb{V}^n \cap \mathcal{O}(f)$, and $\mathrm{diam}(\mathbb{V}^n) \asymp \mathrm{diam}(\mathbb{V}^n \cap \mathcal{O}(f))$, it follows that U^n contains a disk D centered around 0 of radius comparable with $\mathrm{diam}(V^n)$. Thus $\mathrm{area}(U^n) \asymp (\mathrm{diam}(U^n))^2 \asymp (\mathrm{diam}(V^n))^2$. We must show that $\mathrm{area}(V^n \smallsetminus U^n) \asymp \mathrm{area}(V^n)$.

Lemma A.10. *There exists $k > 0$ (independent of n) and a round disk $W^n \subset V^n \smallsetminus f_n^{-k}(\mathbb{U}^n)$ with $\mathrm{diam}(W^n) \asymp \mathrm{diam}(J^n)$.*

Proof. This follows easily by a compactness argument, but we will show how it can be deduced from the construction, since this gives more explicit estimates.

In the primitive case, the construction implies that $V^n \supset X^n \supset f_n^{-l}(\mathbb{U}^n)$ for some l. The annulus $f_n^{-l-1}(\mathbb{V}^n \smallsetminus \overline{\mathbb{U}}^n)$ has a definite modulus, and encircles J^n, so it contains a round disk of diameter comparable with $\mathrm{diam}(J^n)$. We can take W^n as this disk and $k = l + 1$.

In the satellite case, notice that for every $\epsilon > 0$, there exists $r > 0$ such that $f_n^{-r}(\mathbb{U}^n) \smallsetminus Y^n$ can be tiled into a bounded number N (independent of ϵ) of sets of Euclidean diameter less than $\epsilon \, \mathrm{diam}(J^n)$ (contained in $\epsilon \, \mathrm{diam}(J^n)$-neighborhoods of points where ∂Y^n touches J^n). Notice that ∂V^n is contained in a bounded neighborhood of ∂Y^n in the hyperbolic metric of $\mathbb{V}^n \smallsetminus J^n$. This implies that we can choose l such that $f^{-l}(\mathbb{U}^n) \smallsetminus V^n$ is also the union of N sets with Euclidean diameter less than $\mathrm{diam}(J^n)/(100N)$. Since the annulus $f_n^{-l-1}(\mathbb{V}^n \smallsetminus \overline{\mathbb{U}}^n)$ has a definite modulus and encircles J^n, $f_n^{-l-1}(\mathbb{V}^n \smallsetminus \overline{\mathbb{U}}^n) \cap V^n$ contains a round disk of diameter comparable with $\mathrm{diam}(J^n)$. We can take W^n as this disk and $k = l + 1$. $\qquad \square$

Taking W and k as in Lemma A.10, we have $W \subset \cup_{j \leq k} f_n^{-j}(V^n \smallsetminus U^n)$, so $\mathrm{area}(V^n) \asymp \mathrm{area}(W) \leq C \, \mathrm{area}(V^n \smallsetminus U^n)$. This concludes the proof of (G2).

Remark A.1. Notice that this construction gives V^n with additional properties:

(A1) U^n contains a round disk around 0 of radius comparable with $\mathrm{diam}(V^n)$,

(A2) V^n is a neighborhood of $J^n \smallsetminus F$ where F is a finite (bounded) set of copies of β_n. Moreover, if f_{n-1} is not immediately renormalizable then $F = \emptyset$.

Moreover, in the construction we have freedom to choose ρ small so that Γ^n, and hence $V^n \subset Y^n \subset \Gamma^n$, is contained in the ρ-enlargement of J^n. This implies that if J^n_x is a copy of J^n of depth $k > 0$ which intersects $V^n \subset Y^n$ then $\text{diam}(J^n_x)$ is much smaller than $\text{diam}(J^n)$ (this is easy in the primitive case, and follows from bounded shape of Γ^n_x and assertion (5) in Lemma A.6 in the satellite case). Under these circumstances, we obtain:

(A3) If V^n_x is a univalent pullback of V^n of depth $k > 0$ contained in V^n then $\text{diam}(V^n_x)$ is much smaller than $\text{diam}(V^n)$ (since $\text{diam}(V^n_x) \asymp \text{diam}(J^n_x)$).

(A4) If V^n_x is a univalent pullback of V^n of depth $k > 0$ contained in V^n then $|Df^k(x)| > 2$ for every $x \in V^n_x$. This follows from the Koebe Distortion Lemma and (A3).

Remark A.2. If f is a periodic point of renormalization, $f_p(x) = \lambda f(\lambda^{-1}x)$, we may assume that $\mathbb{V}^{n+p} = \lambda \mathbb{V}^n$. In this case we can modify the above construction to obtain V^n satisfying $V^{n+p} = \lambda V^n$ as follows.

We can construct Y^n satisfying $Y^{n+p} = \lambda Y^n$. In the primitive case, we can easily get that $X^{n+p} = \lambda X^n$, but in the immediately renormalizable case we will only get $X^{n+p} \subset \lambda X^n$. To fix this, in the immediately renormalizable case we modify the construction by taking the connected component of 0 in the interior of $\cap_k \lambda^{-k} X^{n+kp}$ as the new definition of X^n. Then $X^{n+p} = \lambda X^n$, but this only gives $S^{n+p} \supset \lambda S^n$. We take $\cup_k \lambda^{-k} S^{n+kp}$ as the new definition of S^n. Then $S^{n+p} = \lambda S^n$, but we only have $Q^{n+p} = \lambda Q^n$ for large n. We take $Q^n = \lambda^{-r} Q^{n+rp}$ for r large. Then $V^n = f_n^{-1}(Q^n)$ satisfies $V^{n+p} = \lambda V^n$. All properties (C1-3), (G1-2) and (A1-4) can be easily verified as before.

Remark A.3. The only parameters determining the constants in this construction are the unbranched *a priori* bounds and the combinatorics. Actually, one can see that a lower bound on the unbranched *a priori* bounds implies an upper bound on the period of any immediate renormalizations. As a consequence, the combinatorics are only used to get a lower bound on ν (the spacing between non-touching J^n_i).

APPENDIX B. CONFORMAL MEASURES ON TOWERS

B.1. **Tower limits of conformal measures.** In this section, let $f : \mathbb{U} \to \mathbb{V}$ be a Feigenbaum map with stationary combinatorics, and let $f_n : \mathbb{U}^n \to \mathbb{V}^n$ be its pre-renormalizations with domains satisfying Lemma 2.3. Let $\mathbb{A}^n = \mathbb{V}^n \smallsetminus \mathbb{U}^n$. Consider rescalings

$$f_m^{(n)} = T_n \circ f_{n+m} \circ T_n^{-1} : \mathbb{U}_{(n)}^m \to \mathbb{V}_{(n)}^m, \quad n = 0, 1, 2, \ldots; \; m = -n, -n+1, \ldots,$$

that normalize the maps $f_0^{(n)} = T_n \circ f_n \circ T_n^{-1}$.

Let μ be a δ-conformal measure on $J(f)$. To study its local geometry, let us push it forward by dilations T_n, $n \in \mathbb{N}$, and normalize to be 1 on $\mathbb{V}_{(n)}^0$, $\mu_n = (T_n)_*(\mu)/\kappa_n$.

Lemma B.1. *The measures μ_n are uniformly bounded on compact subsets of $\mathbb{C}$.*

Proof. Take some $m, k \in \mathbb{Z}_+$, and consider domains $\mathbb{V}^m$ and $\mathbb{U}^{m+k}$. We will show that

(B.1) $$\mu(\mathbb{V}^m) \leq C_k \mu(\mathbb{U}^{m+k}),$$

which after rescalings implies the assertion. Moreover, taking $R^m f$, we see that it is enough to prove (B.1) for $m = 0$ with the constant C_k depending only on $\text{mod}(f)$.

Since the straightening of f is K-qc where K depends only on k and $\text{mod}(f)$, there exists an N depending only on $\text{mod}(f)$ such that $f^N \mathbb{U}^k = \mathbb{V}$. Moreover, $|Df^N(z)| \leq C$, where C depends on the same data. By δ-conformality of μ, we conclude that $\mu(\mathbb{V}) \leq C^\delta \mu(\mathbb{U}^k)$. $\square$

Hence the family of measures μ_m is precompact in the weak* topology on compact subsets. Let $\bar{\mu}$ be a limit of these measures. This measure is Radon (i.e., it assigns finite mass to compact sets) and δ-conformal for the whole tower $\bar{f} = \{f_m\}_{m \in \mathbb{Z}}$.

B.2. **Ergodicity.** A set $X \subset \mathbb{C}$ is called invariant with respect to the tower $\bar{f}$ if $f_m(X \cap \mathbb{U}^m) \subset X$ for all $m \in Z$. A measure $\bar{\mu}$ is called ergodic with respect to the tower if there is no decomposition $\mathbb{C} = X \cup Y$ into two disjoint measurable sets of positive measure invariant with respect to the tower.

Theorem B.2. *Let $\bar{\mu}$ be a conformal measure of the tower $\bar{f}$ which is fully supported on either the Julia set $J(\bar{f})$, or on its complement $\mathbb{C} \smallsetminus J(\bar{f})$. Then $\bar{\mu}$ is ergodic.*

Proof. If $\bar{\mu}$ is supported on the Julia set $J(\bar{f})$, then Theorem 2.9 implies that it is ergodic.

Assume that $\bar{\mu}$ is supported on $\mathbb{C} \smallsetminus J(\bar{f})$. Assume that there is a decomposition $\mathbb{C} \smallsetminus J(\bar{f}) = X \cup Y$ into two invariant measurable sets with positive measure. Without loss of generality we can assume that the inequality

(B.2) $$\bar{\mu}(Y \cap \mathbb{A}^m) \geq \bar{\mu}(X \cap \mathbb{A}^m)$$

holds for infinitely many levels $m \in -\mathbb{N}$.

Let us cut each $\mathbb{A}^m$ by a smooth arc γ of zero $\bar{\mu}$-measure to obtain a (topological) rectangle Δ^m. Then there exists a compact subset $S \subset X \cap \mathbb{U}^0$ of positive measure whose forward orbit does not intersect the cuts γ_m. It follows that for each $m \in -\mathbb{N}$, this set can be covered by disjoint preimages Π_i^m of Δ^m under iterates of f_m.

By [Mc3, Prop. 6.9], $\operatorname{diam} \Pi_i^m \to 0$ as $m \to -\infty$. Hence the unions $\Pi^m \equiv \cup_i \Pi_i^m$ shrink to S as $m \to -\infty$. It follows that $\bar{\mu}(\Pi^m \smallsetminus S) \to 0$. Hence there exists a rectangle $\Pi_{i(m)}^m$ such that $\operatorname{dens}(S|\Pi_{i(m)}^m) \to 1$ as $m \to -\infty$.

On the other hand, since the rectangles Π_i^m are mapped onto the Δ^m with bounded distortion, $\operatorname{dens}(S|\Pi_i^m) \leq 1 - \epsilon < 1$ for all m satisfying (B.2). This contradiction proves the result. $\square$

Corollary B.3. *Given an exponent δ, a tower can have at most one normalized δ-conformal measure supported on the Julia set $J(\bar{f})$, and at most one such a measure supported on the complement $\mathbb{C} \smallsetminus J(\bar{f})$.*

Thus, any tower conformal measure $\bar{\mu}$ may have at most two ergodic components, $\bar{\mu}|J(\bar{f})$ and $\bar{\mu}|\mathbb{C} \smallsetminus J(\bar{f})$.

B.3. **Scaling.** In what follows, $\rho \in (0,1)$ will stand for the scaling factor of the renormalization fixed point.

Lemma B.4 (Scaling covariance). *Let $\bar{\mu}$ be a conformal measure for the tower $\bar{f}$ fully supported either on $J(\bar{f})$, or on $\mathbb{C} \smallsetminus J(\bar{f})$. Then there exists a $\sigma = \sigma(\bar{\mu}) > 0$ such that $T_*\bar{\mu} = \rho^\sigma \bar{\mu}$, and for all $r > 0$, $\bar{\mu}(\mathbb{D}_r) \asymp r^\sigma$.*

Proof. Since the Feigenbaum dilation $Tz \mapsto \rho z$ conjugates f_m to f_{m-1}, the measure $T_*\bar{\mu}$ is also conformal with the same exponent. By Corollary B.3, $T_*\bar{\mu} = \kappa\bar{\mu}$ with some $\kappa > 0$. Hence $\bar{\mu}(\mathbb{D}_r) = \kappa\bar{\mu}(\mathbb{D}_{\rho r})$ for all $r > 0$. Thus

$$\bar{\mu}(\mathbb{D}_{\rho^m}) = \kappa^{-m}\bar{\mu}(\mathbb{D}), \quad m = \pm 1, \pm 2, \ldots,$$

and the statement follows with $\sigma = -(\log \kappa / \log \rho)$. $\square$

If $\bar{\mu}$ is a tower conformal measure with two ergodic components, then let

$$\sigma_- \equiv \sigma_-(\bar{\mu}) = \max\{\sigma(\bar{\mu}|J(\bar{f})), \sigma(\bar{\mu}|\mathbb{C} \smallsetminus J(\bar{f}))\};$$

$$\sigma_+ \equiv \sigma_+(\bar{\mu}) = \min\{\sigma(\bar{\mu}|J(\bar{f})), \sigma(\bar{\mu}|\mathbb{C} \smallsetminus J(\bar{f}))\}.$$

In the case of one ergodic component we set $\sigma_+ = \sigma_- = \sigma(\bar{\mu})$.

Corollary B.5. *For any tower δ-conformal measure $\bar{\mu}$, $\bar{\mu}(\mathbb{D}_r) \asymp r^{\sigma_+}$ for $r \leq 1$ and $\bar{\mu}(\mathbb{D}_r) \asymp r^{\sigma_-}$ for $r \geq 1$, where the constants are independent of $\bar{\mu}$.*

Let $\bar{\mu}_+$ be the tower conformal measure on the Julia set (if it exists), and $\bar{\mu}_-$ be such a measure on the complement (if exists). Both measures are normalized to be 1 on $\mathbb{V}$.

Corollary B.6. *Let μ be a conformal measure of f. Then*

$$\sigma_+ \leq \liminf_{r \to 0} \frac{\log \mu(\mathbb{D}_r)}{\log r} \leq \limsup_{r \to 0} \frac{\log \mu(\mathbb{D}_r)}{\log r} \leq \sigma_-.$$

Proof. The limit set of the measures μ_n consists of convex combinations $t\bar{\mu}_+ + (1-t)\bar{\mu}_-$. For any such combination $\bar{\mu}$, we have:

$$\rho^{\sigma_-}\bar{\mu}(\mathbb{D}_r) \leq \bar{\mu}(\mathbb{D}_{\rho r}) \leq \rho^{\sigma_+}\bar{\mu}(\mathbb{D}_r)$$

for any $r > 0$. Hence

$$\rho^{\sigma_- + \epsilon}\mu(\mathbb{D}_r) \leq \mu(\mathbb{D}_{\rho r}) \leq \rho^{\sigma_+ - \epsilon}\bar{\mu}(\mathbb{D}_r),$$

provided r is sufficiently small (depending on $\epsilon > 0$), which implies the assertion.   □

Lemma B.7. *If a tower conformal measure $\bar{\mu}$ is not ergodic, then*

$$\sigma_+ = \sigma(\bar{\mu}|J(\bar{f})) < \sigma(\bar{\mu}|\mathbb{C} \smallsetminus J(\bar{f})) = \sigma_-.$$

Proof. Let $X = \mathbb{C} \smallsetminus J(\bar{f})$, $Y = J(\bar{f})$. Assume that

$$\sigma_+ = \sigma(\bar{\mu}|X) \leq \sigma(\bar{\mu}|Y) = \sigma_-.$$

Then by Lemma B.4,

$$\bar{\mu}(X \cap \mathbb{A}^m) = \bar{\mu}(X \cap \mathbb{A}^0)\rho^{m\sigma_+} \quad \text{and} \quad \bar{\mu}(Y \cap \mathbb{A}^m) = \bar{\mu}(Y \cap \mathbb{A}^0)\rho^{m\sigma_-}.$$

Hence there exists a $c > 0$ such that $\bar{\mu}(Y \cap \mathbb{A}^m) \geq c\bar{\mu}(X \cap \mathbb{A}^m)$ for all $m \in -\mathbb{N}$. This leads to a contradiction in the same way as (B.2) led to a contradiction in the proof of Theorem B.2.   □

Lemma B.8. *Assume that* $\text{area}(J(f)) = 0$. *Let $\bar{\mu}$ be a tower δ-conformal measure.*
  (i) *If $\delta < 2$ then $\sigma_-(\bar{\mu}) < \delta$.*
  (ii) *If $\delta = 2$ then $\sigma_-(\bar{\mu}) \leq 2$. Moreover, $\sigma_+(\bar{\mu}) < 2$ unless $\bar{\mu}$ is the Lebesgue measure.*[14]

Proof. Since $\text{area}(J(f)) = 0$, there exists a compact set $S \subset \mathbb{D} \smallsetminus J(\bar{f})$ of positive Lebesgue measure.

Let $\Delta^m \subset \mathbb{A}^m$ be the topological rectangles obtained by cutting the annuli $\mathbb{A}^m$ by arcs of zero $\bar{\mu}$-measure (as in the proof of Theorem B.2). Take a point $z \in S$, and consider a sequence of moments $n_m = n_m(z) \to \infty$ such that $\zeta_m \equiv f^{n_m} z \in \Delta^m$, $m \leq 0$. Pulling this rectangle back by f^{n_m}, we obtain a rectangle $\Pi^m = \Pi^m(z)$ containing z.

(i) Let $\delta < 2$. Assume $\sigma_- \geq \delta$. Then

(B.3) $$\bar{\mu}(\Delta^m) \geq c(\text{diam}\,\Delta^m)^{\sigma_-} \geq c(\text{diam}\,\Delta^m)^{\delta}, \quad m \in -\mathbb{N}.$$

Let $g_m = f^{n_m} : \Pi^m \to \Delta^m$. By the dynamical δ-covariance and bounded distortion,

(B.4) $$\bar{\mu}(\Pi^m) \geq a\frac{\bar{\mu}(\Delta^m)}{|Dg_m(z)|^{\delta}} \geq ac\frac{(\text{diam}\,\Delta^m)^{\delta}}{|Dg_m(z)|^{\delta}} \asymp (\text{diam}\,\Pi^m)^{\delta}.$$

Hence

(B.5) $$\bar{\mu}(\Pi^m)/\text{area}(\Pi^m) \to \infty \text{ as } m \to -\infty.$$

But for a given m, different rectangles $\Pi^m(z)$, $z \in S$, are disjoint and cover the whole set S. It follows that $\bar{\mu}(S) = \infty$ - contradiction.

(ii) Let $\delta = 2$. Note that the Lebesgue measure is a 2-conformal measure supported on $\mathbb{C} \smallsetminus J(\bar{f})$. By ergodicity (Theorem B.2), $\bar{\mu}|\mathbb{C} \smallsetminus J(\bar{f})$ is proportional to it. Let $\bar{\nu} = \bar{\mu} + \text{area}$. By Lemma B.7, $\sigma_-(\bar{\nu}) = \sigma(\text{area}) = 2$ and $\sigma_+(\bar{\nu}) < 2$ unless $\bar{\nu}$ (and hence $\bar{\mu}$) is Lebesgue.   □

_____

[14]Thus, if $\bar{\mu}$ is supported on $J(\bar{f})$ then $\sigma(\bar{\mu}) < 2$.

Lemma B.9. *Assume* $\mathrm{area}(J) = 0$ *but* $\mathrm{HD}(J) = 2$. *Let* μ *be a 2-conformal measure on* J. *Then*

$$\frac{\log \mu(\mathbb{D}_r)}{\log r} \to 2 \quad as \quad r \to 0.$$

Proof. By Corollary B.6 and Lemma B.8,

$$\limsup_{r \to 0} \frac{\log \mu(\mathbb{D}_r)}{\log r} \leq 2.$$

Assume

$$\liminf_{r \to 0} \frac{\log \mu(\mathbb{D}_r)}{\log r} < 2,$$

so that $\mu(\mathbb{D}_{r_k}) \geq r_k^\sigma$ for some $\sigma < 2$ and a sequence $r_k \to 0$. Then

$$\mu(\mathbb{V}^{n_k}) \geq (\mathrm{diam}\, \mathbb{V}^{n_k})^\sigma \asymp \rho^{\sigma n_k}$$

for some sequence of $n_k \to \infty$.

Let $X_n = \cup_{k=0}^\infty f^{-k} \mathbb{V}^n$. Let us consider the landing map $\Lambda_n : X_n \to \mathbb{V}^n$ to $\mathbb{V}^n$ and the associated Poincaré series $\Xi_2(\Lambda_n, z)$, $z \in \mathbb{V}^n$. For $z \in \mathbb{V}^n$, we have:

$$\mu(\mathbb{V}^n)\, \Xi_2(\Lambda_n, z) \asymp \mu(X_n) = 1,$$

and hence

$$\Xi_2(\Lambda_{n_k}, z) \leq \rho^{-\sigma n_k}.$$

But then

$$\eta_{n_k} \equiv \mathrm{area}(X_{n_k}) \asymp \mathrm{area}(\mathbb{V}^{n_k})\, \Xi_2(\Lambda_{n_k}, z) \leq \rho^{2 n_k} \rho^{-\sigma n_k},$$

contradicting Theorem 5.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let μ be a 2-conformal measure on the Julia set J. We say that the Julia set is *measure-theoretic hairy* at the critical point if

$$\frac{\pi}{\mu(\mathbb{D}_r)} \mu \circ r \to \mathrm{area} \quad as \quad r \to 0.$$

Here convergence is understood in the weak*-topology on compact sets.

Proposition B.10. *If* $\mathrm{HD}(J) = 2$ *then* J *is measure-theoretic hairy.*

Proof. If $\mathrm{area}(J) > 0$ then the assertion is obvious since the critical point is the density point of J. So, assume $\mathrm{area}(J) = 0$. Then any limit measure $\bar{\mu}$ on the tower $\bar{f}$ is 2-covariant not only by the tower dynamics but by the ρ-scaling as well. Hence $\bar{\mu}(\mathbb{D}_r) \asymp r^2$. Spreading it around using distortion estimates, we see that $\bar{\mu}(\mathbb{D}_r(z)) \asymp r^2$ near any point $z \in \mathbb{C}$. Hence $\bar{\mu}$ is equivalent to the Lebesgue measure. But the only absolutely continuous 2-conformal measure is Lebesgue. $\qquad\square$

Let us consider a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(r) \to \infty$ as $r \to 0$. Let us say that it is *slowly varying* (near the origin) if for any $\kappa < 1$, $\phi(\kappa r)/\phi(r) \to 1$ as $r \to 0$. Clearly, such a function has sub-polynomial growth:

$$\frac{\log \phi(r)}{\log r} \to 0 \quad as \quad r \to 0.$$

Corollary B.11. *Assume* $\mathrm{area}(J) = 0$ *but* $\mathrm{HD}(J) = 2$. *Let* μ *be a 2-conformal measure on* J. *Then* $\mu(\mathbb{D}_r) = r^2 \phi(r)$, *where* ϕ *is a slowly varying (growing to* ∞*) function.*

Proof. Let us show that $\phi(r) \to \infty$. Otherwise there would be a sequence of levels $n_k \to \infty$ such that $\mu(\mathbb{V}^{n_k}) \leq C\, \mathrm{area}(\mathbb{V}^{n_k})$. Hence for $z \in \mathbb{V}^{n_k}$,

$$\mu(X_{n_k}) \asymp \mu(\mathbb{V}^{n_k})\, \Xi_2(z) \leq C\, \mathrm{area}(\mathbb{V}^{n_k})\, \Xi_2(z) \asymp \mathrm{area}(X_{n_k}).$$

(Here X_n is as in Lemma B.9.) But this is impossible since $\mu(X_{n_k}) = 1$ while $\mathrm{area}(X_{n_k}) \to 0$ as $k \to \infty$.

Furthermore, by Proposition B.10,

$$\frac{\mu(\mathbb{D}_{\kappa r})}{\mu(\mathbb{D}_r)} \to \kappa^2 \quad \text{as} \quad r \to 0,$$

which implies that ϕ is slowly varying. $\qquad \square$

In fact, we have:

Proposition B.12. *Under the above circumstances, $\phi(r) \asymp \log \frac{1}{r}$ for sufficiently small r.*

Proof. As in the proof of Lemma B.9, we have:

$$\Xi_2(\Lambda_n, z) \asymp \frac{1}{\mu(\mathbb{V}^n)} \asymp \frac{1}{\rho^{2n}\phi(\rho^n)})$$

and

$$\eta_n \equiv \operatorname{area}(X_n) \asymp \operatorname{area}(\mathbb{V}^n)\,\Xi_2(\Lambda_{n_k}, z) \asymp \frac{\rho^{2n}}{\rho^{2n}\phi(\rho^n)} = \frac{1}{\phi(\rho^n)}.$$

Since by Theorem 5.2, $\eta_n \asymp 1/n$, we obtain the right asymptotics for $r = \rho^n$. Since ϕ is slowly varying, we can interpolate it to the intermediate scales. $\qquad \square$

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