BIFURCATIONS IN THE SPACE OF EXPONENTIAL MAPS

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ABSTRACT. This article investigates the parameter space of the exponential family $z \mapsto \exp(z)+\kappa$. We prove that the boundary (in $\mathbb{C}$) of every hyperbolic component is a Jordan arc, as conjectured by Eremenko and Lyubich as well as Baker and Rippon, and that ∞ is not accessible through any nonhyperbolic ("queer") stable component. The main part of the argument consists of demonstrating a general "Squeezing Lemma", which controls the structure of parameter space near infinity. We also prove a second conjecture of Eremenko and Lyubich concerning bifurcation trees of hyperbolic components.

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1. INTRODUCTION

This article is one step in a program to describe the dynamics of the family of exponential maps $E_\kappa : z \mapsto \exp(z) + \kappa$ and the structure of its parameter space. This simplest family among transcendental entire maps has received special attention over the years, much like the quadratic family has among polynomials. We believe that an understanding of exponential dynamics will be useful in the study of more general classes of entire functions.

Such an understanding is important not only in its own right, but also because the iteration of transcendental functions has links to many other areas in dynamical systems

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and function theory. We content ourselves with giving some examples. Features of exponential dynamics appear in the study of parabolic implosion [Sh], which is one of the most prominent current topics in polynomial dynamics. The family λte^{-t}, a close relative of the exponential, is the second simplest model in population dynamics (the first being the logistic family). Similarly, the standard Arnol'd family of circle maps, which can be complexified to a family of self-maps of $\mathbb{C}^*$, plays a prominent role in the renormalization of critical circle maps. Finally, anyone interested in finding the roots of an entire function should consider studying its Newton method, i.e. iteration of a transcendental meromorphic function.

The reason that the exponential family is a natural candidate to begin with is the same that has made the quadratic family a favorite object of study: in both cases the maps possess only one *singular value*. The singular values (i.e., the critical and asymptotic values) of a function play an important role in the study of its dynamics: a restriction on the number of singular values generally limits the amount of different dynamical features that can appear for the same map. Therefore, the simplest non-trivial parameter space of holomorphic functions is given by the quadratic family, in which each function has only a single simple critical point in $\mathbb{C}$. Similarly, exponential maps are the only transcendental maps with only one singular value in $\mathbb{C}$ (see e.g. [M2, Appendix D]). Furthermore, the exponential family is the limit of the families of unicritical polynomials, parametrized as $z \mapsto (1+\frac{z}{d})^d + c$. This makes it an excellent candidate for applying methods that have proved useful in the study of these polynomials, as first developed for the Mandelbrot set in Douady and Hubbard's famous Orsay Notes [DH].

Parameter space for exponential maps was first studied in the articles [BR] and [EL1, EL2] (the latter treated general parameter spaces of transcendental entire maps which have finitely many singular values). The preprint [DGH] (recently published as [BDG1, BDG2]) was the first to view the exponential family as a limit of the unicritical polynomial family. As in the polynomial cases, a special interest lies in studying hyperbolic components. (Since our maps have just one singular value, a *hyperbolic component* in parameter space can be described simply as a component of the open subset consisting of maps which have an attracting periodic orbit.) The articles mentioned above already include several important facts about these components. Every hyperbolic component W is simply connected and the multiplier map $\mu : W \to \mathbb{D}^*$, which maps each parameter to the multiplier of its unique attracting cycle, is a universal covering. Therefore, there is a conformal isomorphism $\Phi_W\colon W \to \mathbb{H}^-$ with $\mu = \exp\circ\Phi_W$, where $\mathbb{H}^-$ is the left half plane. Φ_W is uniquely defined by this condition up to addition of $2\pi i\mathbb{Z}$; a preferred choice for Φ_W was described in [Sch1, Sch2]; compare Proposition 5.6.

Both in [BR] and in [EL1], it was conjectured that the boundary of a hyperbolic component is always a Jordan curve through ∞ (see Figure 1). In this article, we shall prove this fact. (Here and throughout the article, all closures are understood to be taken on the Riemann sphere $\hat{\mathbb{C}}$.)

1.1. Theorem (Boundaries of Hyperbolic Components)**.**
Let W be a hyperbolic component, and let $\Phi_W : W \to \mathbb{H}^-$ be a conformal isomorphism with

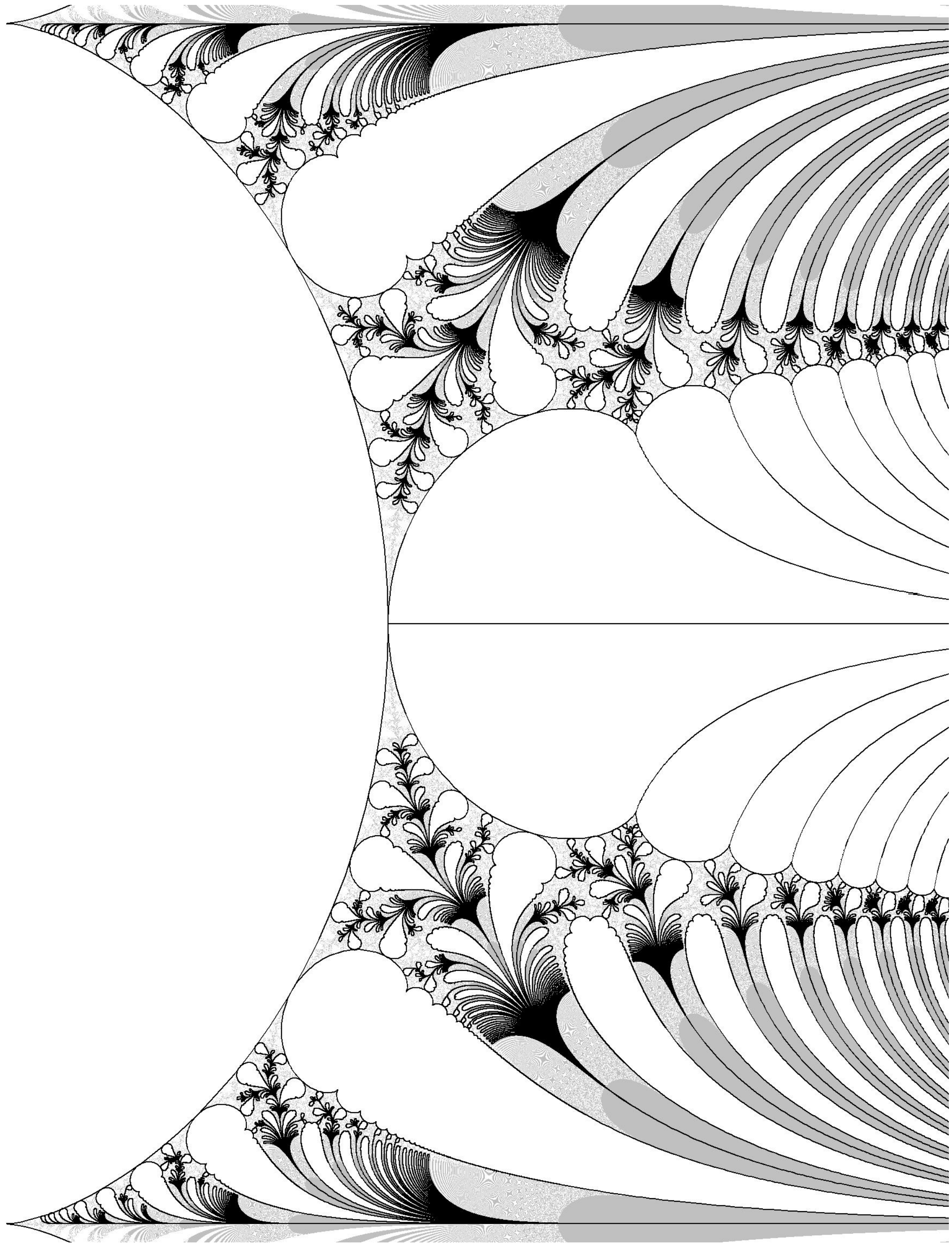

FIGURE 1. Several hyperbolic components in the strip $\{\operatorname{Im}\kappa \in [0, 2\pi]\}$. Within the period two component in the center of the picture, the curves corresponding to real positive multipliers, i.e. the components of the set $\mu^{-1}\big((0,1)\big)$, have been drawn in.

$\mu = \exp \circ \Phi_W$. *Then W extends to a homeomorphism $\Phi_W : \overline{W} \to \overline{\mathbb{H}^-}$ with $\Phi_W(\infty) = \infty$. In particular, $\partial W \cap \mathbb{C}$ is a Jordan arc tending to ∞ in both directions.*

It is not difficult to show (see Section 2) that the map $\Psi_W := \Phi_W^{-1}$ extends to a continuous map $\Psi_W : \overline{\mathbb{H}^-} \to \overline{W}$ with $\Psi_W(\infty) = \infty$. Thus the main difficulty lies in showing that any curve $\gamma : [0, \infty) \to W$ with $|\gamma(t)| \to \infty$ satisfies $\Phi_W(\gamma(t)) \to \infty$, or equivalently lies in the same homotopy class of W as $\Psi_W\big((-\infty, 0)\big)$. The basic idea is to use the structure of parameter space near ∞ to exclude any other direction in which γ could tend to ∞. This is the same technique used in the original (unpublished) proof of Theorem 1.1 [Sch1], as outlined in [Sch3]. (This proof, however, used landing properties of periodic parameter rays which were also established in [Sch1]; our proof does not require any results of this sort.) In fact, we extend Theorem 1.1 to the following remarkable result, which shows that there are only two possibilities for a curve to infinity in exponential parameter space which does not intersect boundaries of hyperbolic components.

1.2. Theorem (Squeezing Lemma)**.**
Let $\gamma : [0, \infty) \to \mathbb{C}$ be a curve in parameter space with $\lim_{t\to\infty} |\gamma(t)| = \infty$. Suppose that, for all t, $\gamma(t)$ does not have an indifferent periodic orbit. Then either

- *for all sufficiently large t, the singular value of $E_{\gamma(t)}$ escapes to ∞ under iteration, or*
- *γ lies in a hyperbolic component W and $\lim_{t\to\infty} \Phi_W(\gamma(t)) \to \infty$.*

The Squeezing Lemma is useful beyond the application in this article. A more precise version of it (Theorem 3.5) can be used to show that parameter rays cannot land at ∞. This is important in order to complete the classification of escaping parameters begun in [F]; see [R1, Section 5.12] or [FRS].

As in any other complex analytic parameter space, the most prominent open problem concerning exponential maps is to show that hyperbolicity is dense, which has been conjectured by Eremenko and Lyubich [EL1].

1.3. Conjecture (Hyperbolicity Conjecture)**.**
The set of hyperbolic parameters is dense.

Of course this conjecture is far from resolved even for quadratic polynomials. However, one can ask simpler questions about the possible structure of hypothetical non-hyperbolic components. For example, the Squeezing Lemma immediately shows that these cannot contain any curves to infinity.

1.4. Corollary (Nonaccesibility of ∞ in Queer Components)**.**
Suppose that W is a non-hyperbolic stable component. If W is unbounded, then ∞ is not accessible from W. ■

We believe that a stronger version of the Squeezing Lemma is true, which could be used to prove that any non-hyperbolic component must be bounded. This would imply connectedness of the bifurcation locus and is the subject of ongoing research.

The structure of this article is as follows. We start by using the Squeezing Lemma to prove Theorem 1.1 in Section 2. In Section 3, we review results from [SZ1] and [Sch2], and formulate a more precise version of the Squeezing Lemma (Theorem 3.5). Its statement breaks down into three parts, which are proved in the rest of the article. The first of these can be proved by an elementary argument (Section 4). The other two parts (proved in Section 6 and 7) require information on the structure of *wakes* in exponential parameter space. These results from [RS] are reviewed in Section 5.

Finally, in Section 8, we prove a second conjecture of Eremenko and Lyubich from [EL1]: there are infinitely many *bifurcation trees* of hyperbolic components. We also show that the boundary of a hyperbolic component W is an analytic curve, with the possible exception of cusps in the points of $\Phi_W^{-1}(\mathbb{Z})$. Furthermore, we indicate further developments which are related to our results, such as the landing of periodic parameter (and dynamic) rays.

Some remarks on notation. We have chosen to parametrize our exponential maps as $z \mapsto E_\kappa(z) = \exp(z) + \kappa$. Traditionally, they have often been parametrized as $\lambda \exp$, which is conjugate to E_κ if $\lambda = \exp(\kappa)$. We prefer our parametrization mainly because the behavior of exponential maps at ∞, and in particular the asymptotics of dynamic rays, do not depend on the parameter in this parametrization. Note that this is also the case in the usual parametrization of quadratic polynomials as $z \mapsto z^2 + c$. Another conceptual advantage is that the parameter κ equals the singular value of E_κ, and thus the picture in the parameter plane reflects the situation in the dynamical plane. Finally, our parametrization causes our statements on hyperbolic components to have fewer exceptions for the components of periods 1 and 2. Note that E_κ and $E_{\kappa'}$ are conformally conjugate if and only if $\kappa - \kappa' \in 2\pi i\mathbb{Z}$.

The function $F : (0,\infty) \to (0,\infty), t \mapsto e^t - 1$ will be fixed throughout the article as a model function for exponential growth. If $\gamma : [0,\infty) \to \mathbb{C}$ is a curve, we shall say that $\lim_{t\to\infty} \gamma(t) = +\infty$ if $\operatorname{Re}\gamma(t) \to +\infty$ and $\operatorname{Im}\gamma$ is bounded; analogously for $-\infty$. The n-th iterate of any function f will be denoted by f^n. As already mentioned, the closure and boundary of a subset of the plane will always be taken on the Riemann sphere $\hat{\mathbb{C}}$. We conclude any proof and any result which immediately follows from previously proved theorems by the symbol $\blacksquare$. A result which is cited without proof is concluded by $\square$.

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2. The Boundary of a Hyperbolic Component

2.1. Definition (Bifurcation Locus and Stable Components).
The set $\mathcal{B} := \{\kappa_0 \in \mathbb{C} :$ *the family* $\kappa \mapsto E_\kappa^n(\kappa)$ *is not normal in* $\kappa_0\}$ *is called the* bifurcation locus *of exponential maps. A component* W *of* $\mathbb{C} \setminus \mathcal{B}$ *is called* hyperbolic *if all parameters in* W *have an attracting periodic orbit. Otherwise,* W *is called* nonhyperbolic *or* queer*.*

It is known that *Misiurewicz parameters*, i.e. parameters for which the singular value is preperiodic, and *indifferent parameters*, that is, parameters with an indifferent periodic orbit, are dense in $\mathcal{B}$ [EL2, Y].

Every attracting parameter lies in some hyperbolic component, and all parameters within the same hyperbolic component W have the same period. As mentioned in the introduction, any hyperbolic component W is simply connected and the multiplier map $\mu : W \to \mathbb{D}^*$ is a universal covering map [EL2]. Thus we can find a biholomorphic map Φ_W from W to the left halfplane $\mathbb{H}^-$ which satisfies $\exp \circ \Phi_W = \mu$. We will often also consider the inverse $\Psi_W := \Phi_W^{-1}$ of this map. Note that Φ_W and Ψ_W are uniquely defined only up to post- resp. precomposition by a translation of $2\pi i\mathbb{Z}$; this ambiguity will be removed in Proposition 5.6 by specifying a preferred choice of parametrization.

Hyperbolic components in exponential parameter space do not have "centers": E_κ has no critical points, so the attracting multiplier is never 0. In some sense, their center is at ∞:

2.2. Lemma (Small Multipliers).
Let W *be a hyperbolic component of period* n*. If* (κ_i) *is a sequence of parameters in a hyperbolic component* W *such that* $\mu(\kappa_j) \to 0$*, then* $\kappa_j \to \infty$*. In particular, every hyperbolic component is unbounded.*

Proof. Let $\kappa \in W$ and let $a_0 \mapsto a_1 \mapsto \dots \mapsto a_n = a_0$ be the attracting orbit of E_κ. A simple calculation shows that

$$\mu(\kappa) = \prod_{j=1}^{n} \exp(a_j). \tag{1}$$

Since $|\mu(\kappa)| \leq 1$, there is some j such that $\operatorname{Re} a_j \leq 0$; without loss of generality let us suppose that $j = 0$. This implies that $|a_1| \leq |\kappa| + 1$. Denote $f(t) = \exp(t) + |\kappa|$; then $|a_k| \leq f^{k-1}(|\kappa| + 1) \leq f^{n-1}(|\kappa| + 1)$ for $1 \leq k \leq n$. Thus the product (1) is bounded from below in terms of $|\kappa|$, which concludes the proof. ■

The hyperbolic components of periods one and two were described in [BR]; see Figure 1. We will content ourselves here by noting that there is a unique component W of period one, which contains a left halfplane. Indeed, the map Ψ_W is given by $\Psi_W(z) = z - \exp(z)$: for $\kappa = z - \exp(z)$, the point z is a fixed point with multiplier $\mu = \exp(z)$.

We shall now turn our attention to points on the boundary of a hyperbolic component.

2.3. Lemma (Indifferent Parameters).
Let W *be a hyperbolic component of period* n *and* $\kappa_0 \in \partial W \cap \mathbb{C}$*. Then* κ_0 *has an indifferent*

cycle of period dividing n*. Furthermore, as* $\kappa \to \kappa_0$ *in* $\overline{W}$*, the nonrepelling cycles of* E_κ *converge to the indifferent cycle of* E_{κ_0}*.*

Proof. Let $\kappa_j \to \kappa_0$ in $\overline{W}$. By (1), for every j there exists some point z_j on the nonrepelling orbit of E_{κ_j} with $\operatorname{Re} z_j \leq 0$. Thus the sequence $E_{\kappa_j}(z_j) = \exp(z_j) + \kappa_i$ is bounded and has some limit point $z \in \mathbb{C}$, which is a nonrepelling fixed point of $E_{\kappa_0}^n$. Since E_{κ_0} has at most one nonrepelling orbit [EL2, Theorem 5], and since z cannot be attracting for $\kappa_0 \in \partial W$, the claim follows. ∎

Similarly to the definition of external (and internal) rays for polynomials, the foliation of the punctured disk by radial rays gives rise to a foliation of the hyperbolic component by *internal rays*. These rays are of a natural interest when studying the boundaries of hyperbolic components.

2.4. Definition (Internal Rays)**.**
Let W *be a hyperbolic component and let* $h \in \mathbb{R}$*. The curve*

$$\Gamma_{W,h} : (-\infty, 0) \to \mathbb{C}, t \mapsto \Psi_W(t + 2\pi i h)$$

is called an internal ray at height h *(or at angle* α*, where* α *is the fractional part of* h*). We say that an internal ray* $\Gamma_{W,h}$ lands *at a point* $\kappa \in \hat{\mathbb{C}}$ *if* $\kappa = \lim_{t\to 0} \Gamma_{W,h}(t)$*.*

Remark. By Lemma 2.2, $\lim_{t\to-\infty} \Gamma_{W,h}(t) = \infty$. Note that the height of an internal ray is uniquely defined only if we fix a particular choice of Ψ_W; compare Proposition 5.6 and the comment thereafter.

2.5. Lemma (Continuous Extension)**.**
Let W *be a hyperbolic component. Then* Ψ_W *extends to a continuous surjection* $\Psi_W : \overline{\mathbb{H}^-} \to \overline{W}$ *with* $\Psi_W(\infty) = \infty$*. In particular, every internal ray of* W *lands in* $\hat{\mathbb{C}}$*.*

Proof. It is sufficient to show that Ψ_W has a continuous extension to $\overline{H}$; surjectivity then follows from the compactness of $\overline{H}$.

We must thus show that $\lim_{z\to z_0} \Psi_W(z)$ exists for every $z_0 \in i\mathbb{R} \cup \{\infty\}$. Let $h \in \mathbb{R}$ and let L denote the limit set of $\Psi_W(z)$ as $z \to ih$. By Lemma 2.3, every parameter $\kappa_0 \in L \cap \mathbb{C}$ has an indifferent periodic point a with multiplier $\mu = \exp(ih)$. We claim that the set of such parameters κ is discrete.

Indeed, suppose there is a sequence $\kappa_j \in L$ which converges nontrivially to some parameter $\kappa_0 \in \mathbb{C}$. Let a_0 be an indifferent periodic point of E_{κ_0}; by Lemma 2.3, we can pick a sequence $a_j \to a_0$ such that each a_j is on the indifferent periodic orbit of E_{κ_j}. The points (κ_j, a_j) lie in the zero set of the function of two complex variables given by $f(\kappa, a) := \big(E_\kappa^n(a) - a, (E_\kappa^n)'(a) - \mu\big)$. Since f is analytic, this implies that, for all κ in a neighborhood of κ_0, there is a solution of $f(\kappa, z) = 0$. In other words, all parameters in a neighborhood of κ_0 are indifferent, which is absurd.

Thus L is contained in a totally disconnected set. Since L is connected, this implies $|L| = 1$ as required.

Finally, let us show that $\Psi_W(z)$ has no accumulation points in $\mathbb{C}$ as $z \to \infty$. Suppose then that (z_j) is a sequence in $\mathbb{H}^-$ such that $\Psi_W(z_j)$ converges to some point $\kappa_0 \in \partial W \cap \mathbb{C}$. Then by Lemma 2.3, κ_0 has an indifferent orbit of multiplier, say, $e^{2\pi i h}$. It is easy to see that we can continue the multiplier of this orbit to an analytic function on a finite sheeted cover of a neighborhood U of κ_0. In particular, there are finitely many connected subsets of U in which the orbit becomes attracting, which means that (if U was chosen small enough) the set $\Psi_W^{-1}(U)$ consists of finitely many bounded components. Since $z_j \in \Psi_W^{-1}(U)$ for large j, it follows that $z_j \not\to \infty$, as required. ■

2.6. Corollary (Landing Points of Internal Rays)**.**
Let W be a hyperbolic component. Then every point of $\partial W \cap \mathbb{C}$ is the landing point of a unique internal ray; in particular every component of $\partial W \cap \mathbb{C}$ is a Jordan arc extending to ∞ in both directions.

Proof. The fact that every boundary point is the landing point of an internal ray follows immediately from Lemma 2.5. We need to show that no two internal rays can land at the same boundary point $\kappa_0 \in \mathbb{C}$. Suppose that $\Psi_W(ih) = \Psi_W(ih') = \kappa_0$ for some $h < h'$.

Connect ih and ih' by a curve in $\mathbb{H}^-$; the image of this curve under Ψ_W is then a simple closed curve γ in $\overline{W}$ which intersects ∂W only in κ_0. By the F. and M. Riesz theorem [M1, Theorem A.3], there exists some $h_1 \in (h, h')$ with $\kappa_1 := \Psi_W(h_1) \neq \kappa_0$. The indifferent parameter κ_1 lies in the bifurcation locus $\mathcal{B}$ and is separated from ∞ by the curve γ. Since Misiurewicz parameters are dense in $\mathcal{B}$, there exists some Misiurewicz parameter κ_2 which is also enclosed by γ. Indifferent parameters are also dense in $\mathcal{B}$, so we can find some indifferent parameter κ with $|\kappa_2 - \kappa| < \operatorname{dist}(\kappa_2, \partial W)$. The parameter κ lies on the boundary of some hyperbolic component, which is thus separated from ∞ by γ. This contradicts Lemma 2.2.

We have shown that Ψ_W is injective on $\Psi_W^{-1}(\partial W \cap \mathbb{C})$. It follows easily that every component C of $\partial W \cap \mathbb{C}$ can be written as $C = \Psi_W(I)$, where I is a component of $\Psi_W^{-1}(\partial W \cap \mathbb{C})$. In other words, $I = i \cdot (h, h')$ with $\Psi_W(ih) = \Psi_W(ih') = \infty$, and thus C is a Jordan arc tending to ∞ in both directions. ■

We are now prepared to use the Squeezing Lemma to prove our main theorem.

Proof of Theorem 1.1, using Theorem 1.2. We need to show that $\Psi_W : \overline{\mathbb{H}^-} \to \overline{W}$ is injective. By Corollary 2.6, it only remains to show that $\Psi_W(2\pi i h) \in \mathbb{C}$ for every $h \in \mathbb{R}$. So suppose by contradiction that $h \in \mathbb{R}$ with $\Psi_W(2\pi i h) = \infty$, and define a curve $\gamma\colon [0, \infty) \to W$ by $\gamma(t) := \Gamma_{W,h}(-1/t)$.

Then $|\gamma(t)| \to \infty$ as $t \to \infty$, while $\Phi_W(\gamma(t)) = -1/t + 2\pi i h \to 2\pi i h$. This contradicts the Squeezing Lemma. ■

3. Combinatorics of Exponential Maps

Dynamic rays. Let E_κ be any exponential map. The *set of escaping points* of E_κ is defined to be

$$I := I(E_\kappa) := \{z \in \mathbb{C} : |E_\kappa^n(z)| \to \infty\}.$$

It is known that the Julia set $J(E_\kappa)$ is the closure of $I(E_\kappa)$ [E, EL2]. Note that, because $|E_\kappa^n(z)| = |\exp(\operatorname{Re} E_\kappa^{n-1}(z)) + \kappa|$, $z \in I(E_\kappa)$ if and only if $\operatorname{Re} E_\kappa^n(z) \to +\infty$.

A complete classification of the set of escaping points of an exponential map was given in [SZ1]. To describe this result, let us introduce some combinatorial notation. An infinite sequence $\underline{s} = s_1 s_2 \dots$ of integers is called an *external address*; we say that a point $z \in \mathbb{C}$ has external address $\underline{s}$ if

$$\operatorname{Im} E_\kappa^n(z) \in \big((2s_{n+1}-1)\pi, (2s_{n+1}+1)\pi\big)$$

for all $n \geq 0$. An external address $\underline{s}$ is called *exponentially bounded* if there exists some $x > 0$ such that

$$2\pi|s_n| < F^{n-1}(x)$$

for all $n \geq 0$. (Recall that $F(t) = \exp(t) - 1$.)

3.1. Theorem and Definition (Classification of Escaping Points [SZ1])**.**
Let $\kappa \in \mathbb{C}$, and suppose that $\kappa \notin I(E_\kappa)$. For every exponentially bounded address $\underline{s}$ there exists $t_{\underline{s}} \geq 0$ and a curve $g_{\underline{s}} := g_{\underline{s}}^\kappa : (t_{\underline{s}}, \infty) \to I(E_\kappa)$ or $g_{\underline{s}} := g_{\underline{s}}^\kappa : [t_{\underline{s}}, \infty) \to I(E_\kappa)$ (called the dynamic ray *at address $\underline{s}$) with the following properties.*

(a) *The trace of $g_{\underline{s}}$ is a path connected component of $I(E_\kappa)$;*
(b) *for large t, $g_{\underline{s}}(t)$ has external address $\underline{s}$;*
(c) *$E_\kappa(g_{\underline{s}}(t)) = g_{\sigma(\underline{s})}(F(t))$;*
(d) *$g_{\underline{s}}(F^{n-1}(t)) = F^{n-1}(t) + 2\pi i s_n + O(e^{-F^{n-1}(t)})$ as t or n tend to ∞.*

Conversely, every path connected component of $I(E_\kappa)$ is such a dynamic ray.

Now suppose the singular value does escape. Then there still exist dynamic rays $g_{\underline{s}}$ with properties (a) to (d) for all exponentially bounded addresses $\underline{s}$. However, there are countably many $\underline{s}$ for which $g_{\underline{s}}$ is not defined for all $t > t_{\underline{s}}$ (resp. $t \geq t_{\underline{s}}$). More precisely, there exist $\underline{s}^0$ and $t^0 \geq t_{\underline{s}^0}$ such that $\kappa = g_{\underline{s}^0}(t^0)$. For every $\underline{s} \in \sigma^{-n}(\underline{s}^0)$, the ray $g_{\underline{s}}$ is not defined for $t \leq F^{-n}(t_0)$. Every path connected component of the escaping set is either a dynamic ray or is mapped into $g_{\underline{s}^0}$ by some forward iterate of E_κ. □

As usual, we say that a ray $g_{\underline{s}}$ *lands* at a point $z \in \hat{\mathbb{C}}$ if $\lim_{t \to t_{\underline{s}}} g_{\underline{s}}(t) = z$.

We need to be able to recognize a point which is on the dynamic ray $g_{\underline{s}}$. Such a criterion is given by the following result, which is an extension of [SZ1, Theorem 4.4].

3.2. Lemma (Fast Points are on Rays)**.**
Let $\kappa \in \mathbb{C}$, and let $x \geq \max(\operatorname{Re}\kappa - 1, 2\pi + 6)$. Suppose that $z_0 \in I(E_\kappa)$ such that $\operatorname{Re} E_\kappa^n(z_0) \geq F^n(x)$ for all $n \geq 0$. Then $z_0 = g_{\underline{s}}(t)$, where $t \geq x$ and $\underline{s}$ is the external address of z_0.

Remark. It is sufficient to require $x > Q(|\kappa|)$ with $Q(|\kappa|) = \log^+|\kappa| + O(1)$ (see [R3]).

Sketch of proof. By [R3, Theorem 4.3], $z_0 = g_{\underline{s}}(t)$ for some $t \geq t_{\underline{s}}$. The fact $t \geq x$ follows by applying [SZ1, Theorem 4.4] to some forward iterate of z_0. □

Parameter rays. Given the importance of escaping points in providing structure in the dynamical plane, it is natural to ask about their analog in parameter space: *escaping parameters*; i.e. those for which the singular value escapes.

3.3. Definition (Parameter Rays)**.**
Let $\underline{s}$ be an exponentially bounded external address, let $t \geq t_{\underline{s}}$. Then we define

$$G_{\underline{s}}(t) := \{\kappa : g^{\kappa}_{\underline{s}}(t) = \kappa\}.$$

The set $G_{\underline{s}} := \bigcup_t G_{\underline{s}}(t)$ is called the parameter ray *at address $\underline{s}$.*

In [F] it was shown that, for $t > t_{\underline{s}}$, the set $G_{\underline{s}}(t)$ consists of a single point, which depends continuously (and even differentiably) on t. Thus $G_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{C}$ is a (differentiable) curve, justifying the term "parameter ray". Using the Squeezing Lemma, this result is extended to escaping endpoints, and thus to a classification of all escaping parameters, in [FRS].

For the purposes of this article, we will not require these results about parameter rays. However, we will sometimes use them heuristically in order to explain the strategy of a proof.

Vertical order and intermediate external addresses. The existence of dynamic rays provides a structure to the Julia set and the dynamical plane. This structure can be used to describe the behavior of a curve to ∞ under iteration, provided that the curve does not contain escaping parameters. Indeed, suppose that $\mathcal{C}$ is any disjoint family of curves $\gamma : [0, \infty) \to \mathbb{C}$ with $\operatorname{Re}\gamma(t) \to +\infty$. Then $\mathcal{C}$ is equipped with a natural vertical order: of any two curves in $\mathcal{C}$, one is *above* the other. More precisely, define $\mathbb{H}^+_R := \{z \in \mathbb{C} : \operatorname{Re} z > R\}$ for $R > 0$. If $\gamma \in \mathcal{C}$ and R is large enough, then the set $\mathbb{H}^+_R \setminus \gamma$ has exactly two unbounded components, one above and one below γ. Any other curve of $\mathcal{C}$ must (eventually) tend to ∞ within one of these.

For the family of dynamic rays, it is easy to see that this vertical order coincides with the lexicographic order on external addresses. We want to use this structure to assign combinatorics to curves to ∞ in Fatou components. To do this, we shall add *intermediate external addresses* to our repertoire. An intermediate external address is a finite sequence of the form

$$\underline{s} = s_1 s_2 \ldots s_{n-1}\infty,$$

where $n \geq 2$, $s_1, \ldots, s_{n-2} \in \mathbb{Z}$ and $s_{n-1} \in \mathbb{Z} + \frac{1}{2}$. When we wish to make the distinction, we will refer to an external address in the original sense as an "infinite" external address. For concise notation, we will always take the terms "exponentially bounded", "unbounded" or "bounded" — which are not useful for intermediate addresses — to mean that the address is infinite and has the corresponding property.

We denote the space of all infinite and intermediate external addresses by $\mathcal{S}$. The point of this definition is that the space $\mathcal{S}$ is order-complete with respect to the lexicographic order; i.e., every bounded subset has a supremum. We will also often consider ∞ as an intermediate external address of length 1, and define $\overline{\mathcal{S}} := \mathcal{S} \cup \{\infty\}$. For more details, we refer the reader to [RS, Section 3].

Now let $\gamma : [0,\infty) \to \mathbb{C} \setminus I(E_\kappa)$ be a curve to ∞. If $\lim_{t\to\infty} \gamma(t) = +\infty$, we define

$$\begin{aligned}\operatorname{addr}(\gamma) :=& \inf\{\underline{s} \in \mathcal{S} : \underline{s} \text{ is exponentially bounded and } g_{\underline{s}} \text{ is above } \gamma\}\\ =& \sup\{\underline{s} \in \mathcal{S} : \underline{s} \text{ is exponentially bounded and } g_{\underline{s}} \text{ is below } \gamma\}.\end{aligned}$$

(Supremum and infimum exist by the order-completeness of $\mathcal{S}$; they are equal since exponentially bounded addresses are dense in $\mathcal{S}$.) If γ does not tend to $+\infty$, then — since γ does not intersect dynamic rays — $\operatorname{Re}\gamma(t)$ is bounded from above. In this case, we set $\operatorname{addr}(\gamma) = \infty$. Note that, whenever $\operatorname{addr}(\gamma) \neq \infty$, $\operatorname{addr}(E_\kappa \circ \gamma) = \sigma(\operatorname{addr}(\gamma))$.

Attracting dynamics. Now suppose that E_κ has an attracting or parabolic periodic cycle. Then the singular value κ is contained in some periodic Fatou component which we call the *characteristic Fatou component*. Let $U_0 \mapsto U_1 \overset{\approx}{\mapsto} \dots \overset{\approx}{\mapsto} U_n = U_0$ be the cycle of Fatou components, labeled such that U_1 is the characteristic component. Since U_1 contains a neighborhood of the singular value, U_0 contains an entire left half plane. In particular, U_0 contains a horizontal curve along which $\operatorname{Re}(z) \to -\infty$, which is unique up to homotopy. Its pullback γ to U_1 under E_κ^{n-1} is a curve to $+\infty$, and we define the *intermediate external address of* κ to be $\operatorname{addr}(\kappa) := \operatorname{addr}(\gamma)$. Note that $\underline{s} := \operatorname{addr}(\kappa)$ is an intermediate external address of length n because $\operatorname{addr}(E_\kappa^{n-1}(\gamma)) = \infty$.

It is easy to see that $\operatorname{addr}(\kappa)$ depends only on the hyperbolic component W which contains κ; this address will therefore also be denoted by $\operatorname{addr}(W)$. (The same is true of other combinatorial objects which we shall later associate to κ.) Its significance lies in the following theorem, which is the main result of [Sch2]. (The existence part of this theorem has also appeared in [DFJ] and is a special case of Lemma 4.4 below.)

3.4. Theorem and Definition (Classification of Hyperbolic Components [Sch2])**.** *For every intermediate external address $\underline{s}$, there exists exactly one hyperbolic component W with* $\operatorname{addr}(W) = \underline{s}$*. We denote this component by* $\operatorname{Hyp}(\underline{s})$*. The vertical order of hyperbolic components coincides with the lexicographic order of their external addresses.* □

To explain the last statement, let $\Phi_W : W \to \mathbb{H}^-$ with $\exp \circ \Phi_W = \mu$, as before. The homotopy class of all curves $\gamma : [0,\infty) \to W$ with $\Phi_W(\gamma(t)) \to \infty$ is called the *preferred homotopy class of* W. As above, these preferred homotopy classes have a vertical order, which is the order referred to in the above theorem.

A Precise Version of the Squeezing Lemma. Let $\gamma : [0,\infty) \to \mathbb{C}$ be a curve in parameter space. Suppose that $\lim_{t\to\infty} |\gamma(t)| = \infty$ and that γ does not contain any indifferent parameters. Then, as above, we can associate to γ an address

$$\begin{aligned}\operatorname{addr}(\gamma) := \inf\{\underline{s} \in \mathcal{S} :\ & \underline{s} \text{ is an intermediate external address and}\\ & \text{every curve in the preferred homotopy class of } \operatorname{Hyp}(\underline{s}) \text{ is above } \gamma\}.\end{aligned}$$

Note that, for every $k \in \mathbb{Z} + \frac{1}{2}$, the line $\{\operatorname{Im}(\kappa) = 2\pi k\}$ is contained in the union of the unique period 1 component $\operatorname{Hyp}(\infty)$ (for $\operatorname{Re}(\kappa) < 1$), the period 2 component $\operatorname{Hyp}(k\infty)$ (for $\operatorname{Re}\kappa > 1$) and their common parabolic boundary point $1 + 2\pi i k$. Therefore, if $\operatorname{Re}(\gamma(t)) \to +\infty$, then $\operatorname{Im}(\gamma(t))$ is necessarily bounded, and $\operatorname{addr}(\gamma)$ starts with a finite entry. In all other cases, $\mathrm{Re}(\gamma(t))$ is bounded above and γ is contained in $\mathrm{Hyp}(\infty)$; in this case, $\mathrm{addr}(\gamma) = \infty$.

Using this definition, we can now state a more precise version of the Squeezing Lemma.

3.5. Theorem (Squeezing Lemma)**.**
Let $\gamma : [0, \infty) \to \mathbb{C}$ be a curve in parameter space with $|\gamma(t)| \to \infty$. Suppose that γ contains no indifferent parameters. Then

(a) *$\underline{s} := \mathrm{addr}(\gamma)$ is either intermediate or exponentially bounded.*
(b) *If $\underline{s}$ is exponentially bounded, then $\gamma(t)$ is escaping for all sufficiently large t. More precisely, $\gamma(t) \in G_{\underline{s}}(\tau)$ for some $\tau \geq \mathrm{Re}\,\gamma(t) - 1$.*
(c) *If $\underline{s}$ is intermediate, then $\gamma \subset \mathrm{Hyp}(\underline{s})$ and $\lim_{t\to\infty} \Phi_{\mathrm{Hyp}(\underline{s})}(\gamma(t)) \to \infty$. (In other words, γ lies in the preferred homotopy class of $\mathrm{Hyp}(\underline{s})$.)*

Note that the proof of Theorem 1.1 in [Sch1] (outlined in [Sch3]) consists of proving the Squeezing Lemma in the case where γ lies in some hyperbolic component. This is done by distinguishing three cases, based on whether $\underline{s} := \mathrm{addr}(\gamma)$ is bounded, unbounded or intermediate. Our proof for the exponentially bounded case (b) is a variant of the bounded case in [Sch1]: there exists a parameter ray at address $\underline{s}$, and it is squeezed between nearby parameter rays (in [Sch1]) resp. by hyperbolic components (here). In the intermediate case, both proofs use the bifurcation structure of hyperbolic components, showing that there are curves which separate γ from the preferred homotopy class of $\mathrm{Hyp}(\underline{s})$. However, in our case — where we do not know that γ starts out in some hyperbolic component — we need an additional tool (Lemma 5.10 below) to ensure that the real parts on these curves tend to $+\infty$.

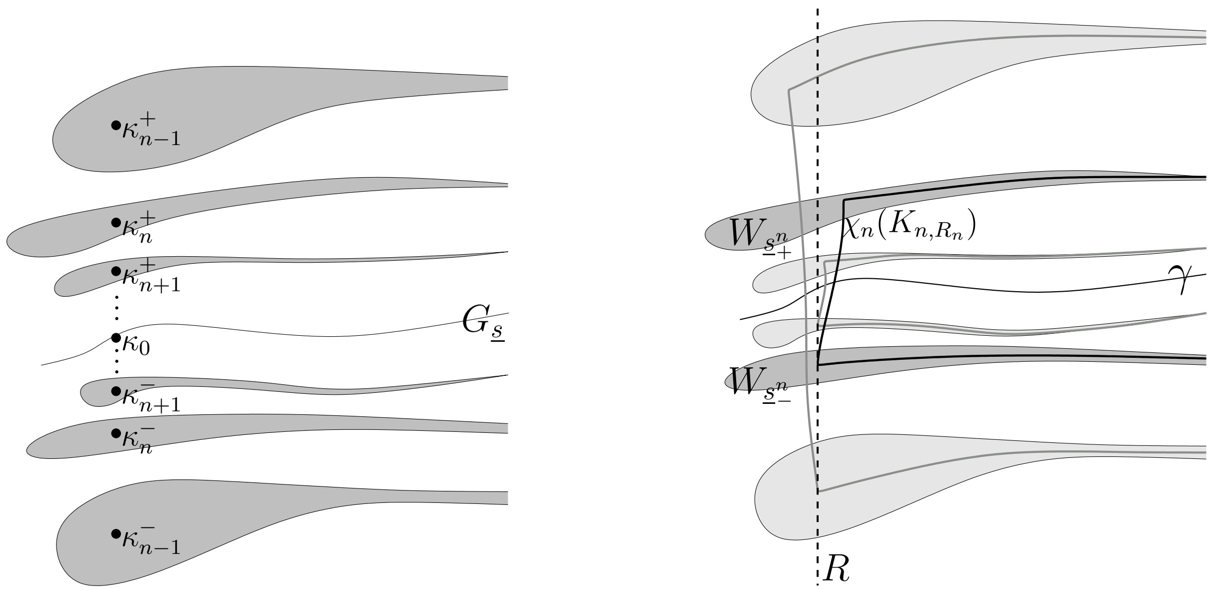
It remains to exclude the unbounded resp. exponentially unbounded cases. Both proofs use the concept of *internal addresses* to show that $\underline{s}$ is contained in the nested intersection of wakes of infinitely many hyperbolic components such that this intersection contains only $\underline{s}$, and hence no hyperbolic components. This suffices to exclude the case of unbounded addresses for curves in hyperbolic components. For our proof, we again use Lemma 5.10 to show that, in the exponentially unbounded case, these wakes have real parts tending to $+\infty$, and thus their intersection is empty.

4. Squeezing around Parameter Rays

In this section, we shall prove part (b) of the Squeezing Lemma, which we restate here for convenience.

4.1. Theorem (Curves at Exponentially Bounded Addresses)**.**
Let $\gamma : [0, \infty) \to \mathbb{C}$ be a curve in parameter space which does not contain any indifferent parameters. Suppose that $\gamma(t) \to \infty$ and that $\underline{s} := \mathrm{addr}(\gamma)$ is exponentially bounded. Then, if t is large enough, $\gamma(t)$ is escaping; in fact $\gamma(t) \in G_{\underline{s}}(\tau)$ with $\tau \to \infty$ as $t \to \infty$.

This fact was originally proved — for bounded external addresses — by approximating the parameter ray at a given address above and below by other parameter rays. This is the origin of the name "Squeezing Lemma". Our proof will instead squeeze from above



(a) Idea of the proof of Theorem 4.1

(b) Illustration of Lemma 4.4

FIGURE 2. Squeezing along parameter rays

and below using hyperbolic components. It is a generalization of the proof of existence of hyperbolic components [Sch2, Theorem 3.5]; the essence of the argument goes back as far as [BR, Section 7] (which, however, did not use a combinatorial description to distinguish these components).

Before we come to the technical details of the proof, let us outline the general idea. Let us suppose for a moment that we know that the "parameter ray" $G_{\underline{s}}$ is indeed a ray with $\operatorname{addr}(G_{\underline{s}}) = \underline{s}$. When we pick such a parameter κ_0 sufficiently far to the right, the singular orbit will escape to infinity exponentially fast and with external address $\underline{s}$. Now, for every sufficiently large n, we should be able to perturb κ_0 slightly to parameters $\kappa_n^\pm$ so that the n-th image of the singular value moves to the lines $\{\operatorname{Im} z = 2\pi(s_{n+1} \pm \frac{1}{2})\}$, and thus is mapped (far) to the left in the next step. The parameters $\kappa_n^\pm$ should then have attracting orbits and intermediate external addresses $s_1 \dots s_n(s_{n+1} \pm \frac{1}{2})\infty$, and the size of the perturbation of κ_0 will tend to zero as $n \to \infty$. The perturbed parameters can be connected to ∞ within their hyperbolic components, and thus the curve γ has to pass in between every pair (κ_n^-, κ_n^+), and therefore contains κ_0. (Compare Figure 2(a).)

In order to carry out this plan, we need to be able to identify parameters in a given hyperbolic component. We shall call a parameter $\kappa \in \mathbb{C}$ *exponentially κ-bounded for n steps* if $\operatorname{Re}\kappa \geq 4$, $|\operatorname{Im}(E_\kappa^{k-1}(\kappa))| < F^{k-1}(\operatorname{Re}\kappa - 2)$ for $1 \leq k \leq n$ and $\operatorname{Re}(E_\kappa^{k-1}(\kappa)) > 0$ for $1 \leq k \leq n-1$. We shall also say that κ has *initial external address $s_1 \dots s_n$* if $\operatorname{Im}(E_\kappa^{k-1}(\kappa)) \in \big((2s_k - 1)\pi, (2s_k + 1)\pi\big)$ for $1 \leq k \leq n$.

4.2. Lemma (Exponential Growth)**.**
Let $\kappa \in \mathbb{C}$ such that $\operatorname{Re}\kappa \geq 2$. If $z \in \mathbb{C}$ with $\operatorname{Re} z \leq \operatorname{Re}\kappa + 1$, then

$$|E_\kappa^{k-1}(z)| \leq F^{k-1}(\operatorname{Re}(\kappa) + 2) \tag{2}$$

for all $k \geq 2$. Furthermore, suppose that κ is exponentially κ-bounded for $n \geq 1$ steps. Then, for $1 \leq k \leq n$,

$$|\operatorname{Re}(E_\kappa^{k-1}(\kappa))| \geq F^{k-1}(\operatorname{Re}(\kappa) - 1). \tag{3}$$

Proof. First note that, for $t, x \geq 2$,

$$F(t) \geq 2t + 2 \tag{4}$$

$$F(x+1) \geq 2F(x) + 1 \geq F(x) + 2F(x-1) + 1. \tag{5}$$

The first claim follows inductively from the observation that, for any $z \in \mathbb{C}$,

$$\begin{aligned} |E_\kappa(z)| + 1 \leq \exp(\operatorname{Re} z) + 1 + |\kappa| &\leq F(\max(\operatorname{Re}\kappa, \operatorname{Re} z)) + 2 + \operatorname{Re}\kappa + |\operatorname{Im}\kappa| \\ &\leq F(\max(\operatorname{Re}\kappa, \operatorname{Re} z)) + 2\operatorname{Re}\kappa \\ &\underset{(4)}{\leq} 2F(\max(\operatorname{Re}\kappa, \operatorname{Re} z)) \\ &\underset{(5)}{\leq} F(\max(\operatorname{Re}\kappa, \operatorname{Re} z) + 1). \end{aligned}$$

Now suppose κ is exponentially κ-bounded for n steps. We set $x := \operatorname{Re}\kappa - 1$, $t_0 := x - 1$ and $z_k := E_\kappa^{k-1}(\kappa)$. In order to prove (3), we prove inductively that $|\operatorname{Re}(z_k)| \geq F^{k-1}(x) + 1$. Indeed, suppose that $k \geq 1$ such that this claim is true for k. Then $|\operatorname{Re}(z_{k+1})| \geq |z_{k+1}| - F^k(t_0)$ by hypothesis, and we can estimate

$$\begin{aligned} |\operatorname{Re}(z_{k+1}) - 1| &\geq |z_{k+1}| - F^k(t_0) \\ &\geq \exp(\operatorname{Re}(z_k)) - |\kappa| - F^k(t_0) \\ &\geq F(\operatorname{Re}(z_k)) - \operatorname{Re}(\kappa) - |\operatorname{Im}(\kappa)| - F^k(t_0) \\ &\geq F(F^{k-1}(x) + 1) - (2t_0 + 2) - F^k(t_0) \\ &\underset{(4)}{\geq} F(F^{k-1}(x) + 1) - 2F^k(t_0) \\ &\geq F(F^{k-1}(x) + 1) - 2F(F^{k-1}(x) - 1) \underset{(5)}{\geq} F^k(x) + 1. \end{aligned}$$

(In the fourth and fifth step, we used the facts that $|\operatorname{Im}\kappa| \leq \operatorname{Re}\kappa - 2$ and $t_0 \geq \operatorname{Re}\kappa \geq 2$, respectively.) ■

The following lemma generalizes [Sch2, Lemma 3.4].

4.3. Lemma (Identifying Hyperbolic Components)**.**
Suppose that $\kappa_0 \in \mathbb{C}$ is exponentially κ_0-bounded for $n \geq 2$ steps, and consider the map $f_n : \kappa \mapsto E_\kappa^{n-1}(\kappa)$. Then

$$|f_n'(\kappa_0)| \geq F^{n-1}(\operatorname{Re}(\kappa_0) - 1). \tag{6}$$

If $\operatorname{Re}(E_{\kappa_0}^{n-1}(\kappa_0)) < 0$*, then* E_{κ_0} *has an attracting periodic orbit of exact period* n*. The multiplier of this orbit tends to* 0 *as either* $\operatorname{Re}(\kappa_0)$ *or* n *tend to* ∞*. If* $s_1 \dots s_{n-2}$ *is the initial external address of* κ_0 *and* $s_{n-1} \in \mathbb{Z} + \frac{1}{2}$ *with* $\operatorname{Im} E_{\kappa_0}^{n-1}(\kappa_0) \in \big((2s_{n-1}-1)\pi, (2s_{n-1}+1)\pi\big)$*, then* $\operatorname{addr}\kappa_0 = s_1 \dots s_{n-1}\infty$*.*

Proof. Let us again abbreviate $z_k := E_{\kappa_0}^{k-1}(\kappa_0)$. We shall show (6) by induction in n. So suppose that either $n = 2$, or that $n > 2$ and (6) is true for $n-1$. By the chain rule,

$$f_n'(\kappa_0) = \exp(z_{n-1}) f_{n-1}'(\kappa_0) + 1.$$

Furthermore, $|f_{n-1}'(\kappa_0)| \geq 1$ either trivially (if $n = 2$) or by the induction hypothesis (if $n > 2$). Thus (3) implies that

$$|f_n'(\kappa_0)| \geq |\exp(z_{n-1})| - 1 = F(\operatorname{Re}(z_{n-1})) \geq F^{n-1}(\operatorname{Re}(\kappa_0) - 1).$$

Now suppose that $\operatorname{Re} z_n < 0$. By Lemma 4.2,

$$\operatorname{Re} z_n < -F^{n-1}(\operatorname{Re}\kappa_0 - 1).$$

The fact that E_{κ_0} is attracting (with exact period n) was proved in [Sch2, Lemma 3.4] under the stronger assumption that $\operatorname{Im} z_k$ is bounded. The proof remains essentially the same, so we shall only sketch it without working out the precise estimates. The image of the left half plane $\mathbb{H}^+ := \{z \in \mathbb{C} : \operatorname{Re} z < \operatorname{Re} z_n + 1\}$ under E_{κ_0} is a punctured disk D around κ_0 with radius at most $\exp(1 - F^{n-1}(\operatorname{Re}\kappa_0 - 1))$. Using (2), it is not difficult to show that, for any point $z \in D$,

$$\left|\big(E_{\kappa_0}^{n-1}(z)\big)'\right| = \prod_{k=1}^{n-1} \exp(\operatorname{Re}(E_{\kappa_0}^{k-1}(z))) \leq \exp(F^{n-1}(\operatorname{Re}\kappa_0 - 2)).$$

Let $U := E_{\kappa_0}^{n-1}(D) = E_{\kappa_0}^{n}(\mathbb{H}^+)$; it follows that

$$\operatorname{diam} U \leq \exp(1 + F^{n-1}(\operatorname{Re}\kappa_0 - 2) - F^{n-1}(\operatorname{Re}(\kappa_0) - 1)) \leq 1$$

Thus $U \Subset \mathbb{H}^+$, and U contains an attracting periodic point a of exact period $n+1$. As n or $\operatorname{Re}\kappa_0$ gets large, the diameter of U tends to 0. Since U contains the image of $\mathbb{D}_1(z_n)$, the multiplier $\mu = \big(E_{\kappa_0}^{n-1}\big)'(a)$ must also tend to 0 by Koebe's theorem.

To see that κ_0 has the correct external address, connect z_n to $-\infty$ by a curve γ_n at constant imaginary parts, and consider the pullbacks γ_k of this curve along the orbit (z_k). We claim that, for $1 < k \leq n-1$, all points on γ_k have real parts larger than $\operatorname{Re}\kappa_0$. (This shows that these curves do not cross the partition boundaries, and therefore completes the proof.) Indeed, suppose by contradiction that $z \in \gamma_k$ with $\operatorname{Re} z \leq \operatorname{Re}\kappa_0$. Then, by (2), $|E_{\kappa_0}^{n-k}(z)| \leq F^{n-k}(\operatorname{Re}\kappa_0 + 2)$, which contradicts the fact that

$$|\operatorname{Re} E_{\kappa_0}^{n-k}(z)| \geq |\operatorname{Re} z_n| \geq F^{n-1}(\operatorname{Re}\kappa_0 - 1) \underset{(4)}{\geq} F^{n-2}(2\operatorname{Re}\kappa_0) > F^{n-2}(\operatorname{Re}\kappa_0 + 2). \qquad \blacksquare$$

For any infinite external address $\underline{s}$ and any $n \geq 2$, let us define

$$\underline{s}_{\pm}^{n} := s_1 s_2 \dots s_{n-2}(s_{n-1} \pm \frac{1}{2})\infty,$$

and consider, for any $R > 0$, the half strip

$$K_{n,R} := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq R, \mathrm{Im}(z) \in [(2s_{n-1} - 1)\pi, (2s_{n-1} + 1)\pi]\}.$$

The statement of the next lemma is somewhat technical, but the general idea is very simple. Our goal is to construct, for every n, an inverse branch χ_n of the map $f_n : \kappa \mapsto E_\kappa^{n-1}(\kappa)$, defined on a suitable K_{n,R_n}, such that the upper boundary of K_{n,R_n} is mapped into $\mathrm{Hyp}(\underline{s}_+^n)$ and the lower into $\mathrm{Hyp}(\underline{s}_-^n)$. A curve γ as in the statement of Theorem 4.1 will then have to (eventually) lie in the image of every χ_n, and as n grows, these will be thinner and thinner by the derivative estimate (6) of Lemma 4.3. (Compare Figure 2(b).)

4.4. Lemma (Parameters with Prescribed Singular Orbit)**.**
Let $\underline{s}$ be any exponentially bounded address, say $2\pi(|s_k| + 1) < F^{k-1}(x)$ with $x \geq 3$. Set $R := x + 2$ and let $n \geq 2$ be arbitrary. Then there is R_n with $|F^{-(n-2)}(R_n) - R| \leq 1$ and a conformal map χ_n from $K_n := K_{n,R_n}$ into parameter space such that

1. *For any $z \in K_n$, $\kappa := \chi_n(z)$ is exponentially κ-bounded for $n - 1$ steps and has initial external address $s_1 \dots s_{n-2}$.*
2. *$f_{n-1}(\chi_n(z)) = z$ for all z.*
3. $\mathrm{Re}\Big(\chi_n\big(R_n + (2s_{n-1} - 1)\pi i\big)\Big) = R$.

Remark. This proves the existence of hyperbolic components with an arbitrary intermediate external address; i.e. one half of Theorem 3.4.

Proof. For all κ, set $T_\kappa := \max\{x+1, \log(2(|\kappa|+2))\}$. A simple induction (quite similar to that of [SZ1, Lemma 3.3] or [R3, Lemma 4.1]) shows that there exists a branch φ_κ of $E_\kappa^{-(n-2)}$ on $K_{n,F^{n-2}(T_\kappa)}$ such that, for all $z \in K_{n,F^{n-2}(T_\kappa)}$,

- $\varphi_\kappa(z)$ has initial external address $s_1 \dots s_{n-2}$ and
- $|\mathrm{Re}(E_\kappa^{k-1}(\varphi_\kappa(z))) - F^{k-n+1}(\mathrm{Re}\, z)| < 1$ for $1 \leq k \leq n - 1$.

Observe that φ depends holomorphically on z and κ.

We claim that there exists κ_0 with $\mathrm{Re}\,\kappa_0 = R$ such that $\kappa_0 = \varphi_{\kappa_0}(r_0 + (2s_{n-1} - 1)\pi i)$ for some $r_0 \geq T_{\kappa_0}$. Indeed, note that, if $\mathrm{Re}\,\kappa = R$ and $\mathrm{Im}\,\kappa \in \big((2s_1 - 1)\pi, (2s_1 + 1)\pi\big)$, then $T_\kappa = x = R - 2$. Thus the complement of the set

$$A_\kappa := \{z \in \mathbb{C} : \mathrm{Re}\, z \leq R - 1\} \cup \{\varphi(r + (2s_{n-1} - 1)\pi i) : r \geq T_\kappa\}$$

has two unbounded components, one above and one below A_κ. For $\kappa = R + (2s_1 - 1)\pi i$, the singular value lies in the lower of these components, and for $\kappa = R + (2s_1 + 1)\pi i$ it lies in the upper. Thus, for an intermediate choice κ_0, the singular value must be contained in A_{κ_0}.

So let κ_0 and r_0 be as above, and set $R_n := r_0$, $K_n := K_{n,R_n}$ and $\chi_n(R_n + 2(s_{n-1} - 1)\pi i) := \kappa_0$. We claim that we can extend χ_n to an analytic function $\chi_n : K_n \to \mathbb{C}$ with $\varphi_{\chi_n(z)}(z) = \chi_n(z)$. First note that, whenever $\varphi_\kappa(z) = \kappa$, the parameter κ is exponentially κ-bounded for $n-1$ steps and has initial external address $s_1 \dots s_{n-2}$. Thus, by (6) and the implicit function theorem, we can locally extend any such solution to an analytic function of z.

Since K_n is simply connected, by the monodromy theorem it suffices to show that we can continue χ_n analytically along every curve $\gamma : [0,1] \to K_n$ with $\gamma(0) = r_0 + 2(s_{n-1} - 1)\pi i$. Let I be the maximum interval such that the solution can be continued along $\gamma|_I$. By the above remark, I is open as a subset of $[0,1]$. It thus remains to show that I is closed. So let t_0 be a limit point of I, and let κ be a limit point of $\chi_n(\gamma(t))$ as $t \to t_0$. Then, by continuity of φ, $\varphi_\kappa(\gamma(t_0)) = \kappa$. By (6), the set of such κ is discrete, and thus $\chi_n(\gamma(t)) \to \kappa$ as $t \to t_0$. ■

Proof of Theorem 4.1. Set $\underline{s} := \operatorname{addr}(\gamma)$, and choose $x \geq 3$ with $2\pi(|s_k|+1) \leq F^{k-1}(x)$. For every n, let χ_n, R_n and K_n be as in Lemma 4.4.

By Lemma 4.3, the curves $\gamma^n_\pm : t \mapsto \chi_n(t + 2(s_{n-1} \pm 1)\pi i)$ lie in the preferred homotopy classes of $\operatorname{Hyp}(\underline{s}^n_\pm)$. Note also that, by the estimate (6),

$$\operatorname{diam} \chi_n\Big(\big\{R_n + bi : b \in [(2s_{n-1} - 1)\pi, (2s_{n-1} + 1)\pi]\big\}\Big) \leq \frac{2\pi}{F^{n-2}(x+1)}.$$

Recall that γ tends to ∞ between the components $\operatorname{Hyp}(\underline{s}^n_-)$ and $\operatorname{Hyp}(\underline{s}^n_+)$, and thus between γ^n_- and γ^n_+. It follows that, for large t,

$$\gamma(t) \in \bigcap_n \chi_n(K_n).$$

By Lemmas 4.2 and 4.4, this means that $\gamma(t)$ is escaping, and the singular orbit of $E_{\gamma(t)}$ has external address $\underline{s}$. By (3) and Lemma 3.2, $\gamma(t) = g^{\gamma(t)}_{\underline{s}}(\tau)$ with $\tau \geq \operatorname{Re}\gamma(t) - 1$. ■

5. Bifurcations of Hyperbolic Components

In this section, we shall collect the combinatorial notions and results from [Sch2], [SZ2] and [RS] which we require to complete the proof of the Squeezing Lemma.

Itineraries and Kneading Sequences. An important aspect of attracting (as well as escaping and singularly preperiodic) exponential dynamics is the possibility to associate so-called *itineraries* to orbits of E_κ. The idea is that, whenever there is a curve (such as a dynamic ray or a curve in a Fatou component) starting at ∞ and ending at the singular value, the preimages of this curve will form a partition of the plane. Here we will only define the combinatorial analog of this notion and one of its important applications; compare [SZ2] for more details and motivation.

5.1. Definition ((Combinatorial) Itinerary)**.**
Let $\underline{s} \in \mathcal{S}$ *and* $\underline{r} \in \overline{\mathcal{S}}$. *Then the* itinerary of $\underline{r}$ with respect to $\underline{s}$ *is* $\operatorname{itin}_{\underline{s}}(\underline{r}) = \mathtt{u}_1\mathtt{u}_2\ldots$, *where*

$$\begin{cases} \mathtt{u}_k = \mathtt{j} & \text{if } \; \mathtt{j}\underline{s} < \sigma^{k-1}(\underline{r}) < (\mathtt{j}+1)\underline{s} \\ \mathtt{u}_k = {}^{\mathtt{j}}_{\mathtt{j}-1} & \text{if } \; \sigma^{k-1}(\underline{r}) = \mathtt{j}\underline{s} \\ \mathtt{u}_k = * & \text{if } \; \sigma^{k-1}(\underline{r}) = \infty. \end{cases}$$

Remark. Thus $\operatorname{itin}_{\underline{s}}(\underline{r})$ has the same length as $\underline{r}$; in particular, itineraries of intermediate external addresses are finite.

5.2. Proposition (Rays Landing Together [SZ2, Proposition 4.5])**.**
Let κ be an attracting or parabolic parameter and $\underline{s} := \operatorname{addr}(\kappa)$. Let $\underline{r}$ and $\widetilde{\underline{r}}$ be periodic external addresses. Then the dynamic rays $g_{\underline{r}}$ and $g_{\widetilde{\underline{r}}}$ land at a common point if and only if $\operatorname{itin}_{\underline{s}}(\underline{r}) = \operatorname{itin}_{\underline{s}}(\widetilde{\underline{r}})$. □

Remark. This result remains true without the assumption of periodicity; see [BD, R3].

As in the case of external addresses, the itinerary which describes the behavior of the singular value is of particular importance:

5.3. Definition (Kneading Sequence)**.**
Let $\underline{s} \in \mathcal{S}$. Then the kneading sequence *of $\underline{s}$ is defined to be $\mathbb{K}(\underline{s}) := \operatorname{itin}_{\underline{s}}(\underline{s})$; we also define $\mathbb{K}(\infty) := *$. If $\underline{s}$ is an intermediate external address and $W = \operatorname{Hyp}(\underline{s})$ is the corresponding hyperbolic component, we also set $\mathbb{K}(W) := \mathbb{K}(\underline{s})$.*

Characteristic Rays and Wakes. If a quadratic polynomial has an attracting orbit, then the combinatorics (and hence the topology) of the Julia set can be described exactly in terms of the periodic rays which land together on the boundaries of periodic Fatou components. More precisely, if p is a hyperbolic quadratic polynomial, then there is a unique pair of periodic rays, called the *characteristic rays*, landing together on the boundary of the Fatou component containing the critical value and separating it from all other points on the attracting orbit. The corresponding parameter rays land together at the hyperbolic component containing p and bound the *wake* of this component.

The analogous dynamical statements were shown for exponential maps in [Sch2]; they will be used below to define a combinatorial notion of wakes. In fact, it is also true (see Section 8) in the exponential family that, on the boundary of every hyperbolic component, there are two parameter rays landing together which provide a natural definition of the *wake* of this component as a subset of exponential parameter space (Figure 3(a)). While we can and will not use this result in our proofs, it often provides a useful motivation for the combinatorial ideas.

5.4. Proposition (Characteristic Rays [Sch2, Theorem 6.2])**.**
Let $\underline{s}$ be an intermediate external address of length $n \geq 2$, and let $W = \operatorname{Hyp}(\underline{s})$ be the hyperbolic component of address $\underline{s}$. Then there exists a (unique) pair of periodic external addresses $\underline{s}^-$ and $\underline{s}^+$ of period n with the following properties.

- $\underline{s}^- < \underline{s} < \underline{s}^+$;
- $\operatorname{itin}_{\underline{s}}(\underline{s}^-) = \operatorname{itin}_{\underline{s}}(\underline{s}^+)$*, and the first $n-1$ entries of this common itinerary agree with $\mathbb{K}(\underline{s})$;*
- $\sigma^j(\underline{s}^-), \sigma^j(\underline{s}^+) \notin (\underline{s}^-, \underline{s}^+)$ *for all j (where $(\underline{s}^-, \underline{s}^+) = \{\underline{r} \in \mathcal{S} : \underline{s}^- < \underline{r} < \underline{s}^+\}$).*

These addresses are called the characteristic external addresses *of $\underline{s}$ (or W). Their (common) itinerary is called the* forbidden kneading sequence *of W and denoted by $\mathbb{K}^*(W)$. The interval $\mathcal{W}(W) := (\underline{s}^-, \underline{s}^+) \subset \mathcal{S}$ is called the* (combinatorial) wake *of W.* □

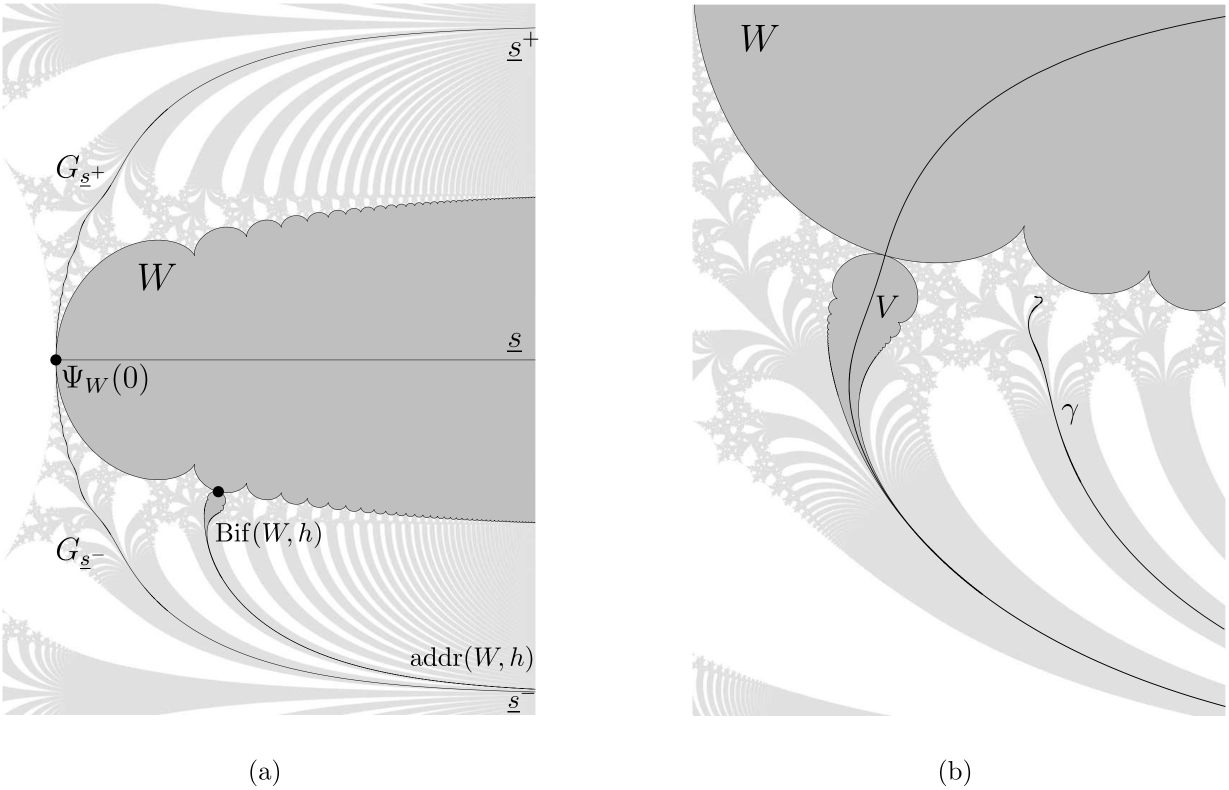


FIGURE 3. Illustration of Propositions 5.6 and 6.2 for the component $W = \mathrm{Hyp}(\frac{1}{2}\infty)$. In (a), the two parameter rays bounding the wake of W are shown; also drawn in is the child component $\mathrm{Bif}(W,h)$ with $h = 3/2$. (b) exemplifies Proposition 5.6; here the child component is $V := \mathrm{Bif}(W, \frac{1}{2}) = \mathrm{Hyp}(01\frac{1}{2}\infty)$. The curve running through these two components surrounds γ, which in this case is the parameter ray at address $\overline{02030}$.

5.5. Proposition (Characteristic Rays and Wakes [RS])**.**
Let W be any hyperbolic component, and let $\underline{s}^-$ and $\underline{s}^+$ be its characteristic external addresses. Then, for any $\underline{r} \in \mathcal{S}$, $\mathrm{itin}_{\underline{r}}(\underline{s}^+) = \mathrm{itin}_{\underline{r}}(\underline{s}^-)$ if and only if $\underline{r} \in \mathcal{W}(W)$. ☐

Bifurcation Structure. Finally, we will need a good description of the structure of hyperbolic components; particularly how these components are connected to each other. Again, suitable analogs of many of the following statements are well-known for Mandel- and Multibrot sets; see e.g. [M3, Sch4].

Recall that, for any hyperbolic component W, there exists a conformal isomorphism $\Psi_W : \mathbb{H}^- \to W$ with $\mu \circ \Psi_W = \exp$, which extends continuously to $\overline{\mathbb{H}^-}$ by Lemma 2.5. Recall also that this defines Ψ_W uniquely up to precomposition by an additive translation of $2\pi i\mathbb{Z}$. We will now describe the structure of the hyperbolic components which touch W in its parabolic boundary points, using a preferred choice for Ψ_W.

The statement we will utilize (Proposition 5.6 below) is somewhat abstract, so let us outline the most important ideas. Suppose that $h = \frac{p}{q} \in \mathbb{Q}\setminus\mathbb{Z}$ with $\kappa_0 := \Psi_W(2\pi i h) \in \mathbb{C}$. Then κ_0 is a parabolic parameter, and by arguments similar to those used in the Mandelbrot set [M3], there exists a unique component $\operatorname{Bif}(W,h)$ of period qn which touches W at κ_0 (here n is the period of W). Let us call the address of this component $\operatorname{addr}(W,h) := \operatorname{addr}(\operatorname{Bif}(W,h))$; this address lies in the wake of W, either above or below $\underline{s}$. As h increases, there will be some point h_0 at which $\operatorname{addr}(W,h)$ jumps from above $\underline{s}$ to below $\underline{s}$. Our preferred parametrization is exactly the one for which $h_0 = 0$; see Figure 3(a). (Also, every component which bifurcates from a period n component has period qn for some $q \geq 2$, and the bifurcation angle is $p/q \in \mathbb{Q}\setminus\mathbb{Z}$.)

The problem with this description is that we do not yet know that $\Psi_W(2\pi i h) \in \mathbb{C}$ for all h. However, the address $\operatorname{addr}(W,h)$ can be defined combinatorially for all $h \in \mathbb{Q}\setminus\mathbb{Z}$, regardless of whether $\Psi_W(2\pi i h) \in \mathbb{C}$ (and can be computed by a simple algorithm [RS]); this yields the following fundamental result.

5.6. Proposition (Bifurcation Structure)**.**
Let W be a hyperbolic component and $\underline{s} := \operatorname{addr}(W)$. If $n \geq 2$, then there exists a unique choice of Ψ_W — called the preferred parametrization *of W — and a unique map $\operatorname{addr}(W,\cdot) : \mathbb{Q}\setminus\mathbb{Z} \to \mathcal{S}$ with the following properties.*

(a) $\operatorname{addr}(W,\cdot)$ *is strictly increasing on $\{h > 0\}$ and (separately) on $\{h < 0\}$;*
(b) $\operatorname{addr}(W,\frac{p}{q})$ *is an intermediate external address of length qn;*
(c) *if $h \in \mathbb{Q}\setminus\mathbb{Z}$ is such that $\Psi_W(2\pi i h) \in \mathbb{C}$, then the parameter $\Psi_W(2\pi i h)$ lies on the boundary of $\operatorname{Bif}(W,h) := \operatorname{Hyp}(\operatorname{addr}(W,h))$;*
(d) $\overline{\mathcal{W}(W)} = \overline{\bigcup_{h\in\mathbb{Q}\setminus\mathbb{Z}} \mathcal{W}(\operatorname{Bif}(W,h))}$.
(e) $\lim_{h\to+\infty} \operatorname{addr}(W,h) = \lim_{h\to-\infty} \operatorname{addr}(W,h) = \underline{s}$*;*
(f) $\lim_{h\nearrow 0} \operatorname{addr}(W,h) = \underline{s}^+$ *and* $\lim_{h\searrow 0} \operatorname{addr}(W,h) = \underline{s}^-$.

If $n = 1$, then the preferred parametrization is, by definition, the map $\Psi_W(z) = z - \exp(z)$, which maps $\mathbb{R}^-$ to $(-\infty,-1)$. There exists a unique map $\operatorname{addr}(W,\cdot)$ which is strictly increasing on all of $\mathbb{Q}\setminus\mathbb{Z}$ and satisfies properties (b) to (d) above.

Remark. From now on, we shall always fix Ψ_W to be the preferred parametrization of W (note that this is the same preferred choice as described in [Sch2, Theorem 7.1]). In particular, with this definition the internal ray of height h is uniquely defined (recall the remark after Definition 2.4).

5.7. Definition (Types of Hyperbolic Components)**.**
The components $\operatorname{Bif}(W,h)$ *are called the* child components *of W. If W is a child component*

of some other hyperbolic component, it is also called a satellite component*; otherwise we say that W is a* primitive component*.*

Remark. The term *satellite* is more commonly used for a hyperbolic component which shares a parabolic boundary point with a hyperbolic component of lesser period. Theorem 6.3 shows that these definitions are equivalent; compare also Corollary 8.3.

5.8. Proposition (Child Components)**.**
Let W be a hyperbolic component, and let $V := \operatorname{Bif}(W, h)$ be a child component of W. Then $\Psi_W(2\pi i h) = \Psi_V(0)$. If $\kappa_0 := \Psi_W(2\pi i h) \in \mathbb{C}$, then W and V are the only hyperbolic components containing κ_0 on their boundary.

Furthermore, if two components have a common parabolic boundary point, then one is a child component of the other; no two hyperbolic components have a common child component. In particular, no two hyperbolic components of equal period have a common parabolic boundary point. □

Remark. We will show in Proposition 8.1 that no two components ever touch at irrational boundary points.

If W is a hyperbolic component, and V is a child component of W, then $\mathcal{W}(V)$ is called a *subwake* of W. By Proposition 5.6 (d), these subwakes exhaust most of the wake of W. The following statement makes this more precise.

5.9. Proposition (Subwakes Fill Wake)**.**
Let W be a hyperbolic component of period n, and let $\underline{s} \in \mathcal{W}(W) \setminus \{\operatorname{addr}(W)\}$. If $\underline{s}$ is not contained in any subwake of W, then $\underline{s}$ is a bounded infinite external address. □

We will also need a bound on parameters in a given wake, which is given by the following result.

5.10. Lemma (Bound on Parameter Wakes [R3, Corollary 5.8])**.**
Let $\kappa \in \mathbb{C}$, and suppose that two dynamic rays $g_{\underline{s}^1}$ and $g_{\underline{s}^2}$ have a common landing point. Suppose that there are $n \in \mathbb{N}$ and $M > F^{n-1}(6)$ such that, for every $k > 0$ and $j \in \{1, 2\}$, $\max_{k \le \ell < k+n} 2\pi |s^j_\ell| \ge M$. Then $|\kappa| > F^{-(n-1)}(M - \pi) - 2$.

Proof. We claim first that there is an $\ell \in \{1, 2\}$ such that the ray $g_{\underline{s}^\ell}$ contains a point which maps into the left half plane $\{z \in \mathbb{C} : \operatorname{Re} z \le \operatorname{Re} \kappa\}$ under iteration. Indeed, since $\underline{s}^1 \ne \underline{s}^2$, there exists some $j \ge 0$ such that the first entries of $\sigma^j(\underline{s}^1)$ and $\sigma^j(\underline{s}^2)$ differ. In order for the two rays $g_{\sigma^j}(\underline{s}^1)$ and $g_{\sigma^j}(\underline{s}^2)$ to land together, at least one of them must cross, or land on, one of the lines $\{\operatorname{Im} z = (2k-1)\pi\}$. Since each of these lines maps to $\{\kappa + t : t \in \mathbb{R}^-\}$, the claim follows.

Let $t_0 \ge t_{\underline{s}^\ell}$ be maximal such that there is an $m \ge 0$ with $\operatorname{Re} E_\kappa^m(g_{\underline{s}^\ell}(t_0))) \le \operatorname{Re} \kappa$. Then, for every $j \ge 0$, the forward images of the piece $g_{\sigma^j(\underline{s}^\ell)}\big([F^j(t_0), \infty)\big)$ do not intersect any strip boundaries; thus this piece is contained in the strip

$$\{z \in \mathbb{C} : \operatorname{Im} z \in \big((2s_{j+1} - 1)\pi, (2s_{j+1} + 1)\pi\big)\}.$$

Let $z := E_\kappa^m(g_{\underline{s}^\ell}(t_0))$. By the first part of Lemma 4.2, $|E_\kappa^j(z)| \leq F^j(|\kappa|+2)$ for all j. On the other hand, by assumption there exists some $j \leq n-1$ with $2\pi|s_{m+j+1}^\ell| \geq M$, and thus

$$M - \pi \leq |E_\kappa^j(z)| \leq F^j(|\kappa|+2) \leq F^{n-1}(|\kappa|+2). \qquad \blacksquare$$

6. Cutting Inside Wakes

We begin this section by showing that, for every hyperbolic component, every internal ray lands in $\mathbb{C}$, except possibly the ray at height $h = 0$. This suffices to prove our main result (Theorem 1.1) for all satellite components (see Definition 5.7). We conclude by proving part (c) of the Squeezing Lemma; i.e. the case of intermediate external addresses.

The main technique applied in this section is to use bifurcating components to cut a given curve off from ∞. Thus we need to make sure that sufficiently many such components exist. Luckily this is the case:

6.1. Proposition (Bifurcation Angles are Dense)**.**
Let W be a hyperbolic component, and let $\Psi_W : \mathbb{H}^- \to W$ be its preferred parametrization. Then the set $\{h \in \mathbb{Q} \setminus \mathbb{Z} : \Psi_W(2\pi i h) \in \mathbb{C}\}$ is dense in $\mathbb{R}$.

Proof. The set $\{h \in \mathbb{R} : \Psi_W(2\pi i h) \in \mathbb{C}\}$ is open by continuity of Ψ_W (Lemma 2.5), and dense by the F. and M. Riesz theorem. $\blacksquare$

We can now formulate the main tool we will use in this and the following section to transfer combinatorial information to the parameter plane. (Compare Figure 3(b).)

6.2. Proposition (Curves in the Wake of a Hyperbolic Component)**.**
Let $\gamma : [0,\infty) \to \mathbb{C}$ be a curve with $\gamma(t) \to \infty$ as $t \to \infty$, and suppose that γ contains no indifferent parameters. If W is a hyperbolic component with $\operatorname{addr}(\gamma) \in \mathcal{W}(W)$, then there exists a curve in $\mathbb{C}$ which separates γ from all hyperbolic components $\operatorname{Hyp}(\underline{t})$ with $\underline{t} \notin \mathcal{W}(W)$. This curve can be chosen to consist only of parameters in W, parameters in a single child component of W and one parabolic parameter on ∂W.

Proof. To fix ideas, let us assume that γ tends to ∞ either in or below the preferred homotopy class of W. By assumption, $\operatorname{addr}(\gamma) > \underline{s}^-$, so by Proposition 5.6 (f), there exists an $\varepsilon > 0$ such that $\operatorname{addr}(W,h) < \operatorname{addr}(\gamma)$ for all rational $h \in (0,\varepsilon)$. By Proposition 6.1, there exists some rational $h \in (0,\varepsilon)$ with $\kappa_0 := \Psi_W(2\pi i h) \in \mathbb{C}$. Since γ intersects at most one hyperbolic component, we can also assume that γ does not intersect the child component $V := \operatorname{Bif}(W,h)$. The desired curve is given by

$$\Gamma_{W,h} \cup \{\kappa_0\} \cup \Gamma_{V,0}. \qquad \blacksquare$$

6.3. Theorem (Landing of Non-Central Internal Rays)**.**
Let W be a hyperbolic component and let Ψ_W be its preferred parametrization. Then $\Psi_W(2\pi i h) \in \mathbb{C}$ for all $h \neq 0$.

Proof. Suppose that $h \neq 0$ and $\Psi_W(2\pi i h) = \infty$. Let us assume that $h > 0$; the case $h < 0$ is completely analogous. Consider the internal ray

$$\Gamma_{W,h} : (-\infty, 0) \to \mathbb{C}, t \mapsto \Psi_W(t + 2\pi i h).$$

Then the curve $\Gamma_{W,h} : (-\infty, -1] \to \mathbb{C}$ tends to ∞ in the preferred homotopy class of W as $t \to -\infty$ and thus has external address $\underline{t} := \operatorname{addr}(W)$. Since $\lim_{t\to 0} \Gamma_{W,h}(t) = +\infty$ by assumption, the curve $\Gamma_{W,h} : [-1, 0) \to \mathbb{C}$ also defines an external address

$$\underline{s} := \operatorname{addr}\Big(\Gamma_{W,h}\big([-1, 0)\big)\Big).$$

By Proposition 6.1, there exists a rational $h_0 > h$ for which $\kappa_0 := \Psi_W(t + 2\pi i h_0) \in \mathbb{C}$. Since $\Gamma_{W,h}$ surrounds κ_0, it also surrounds the child component $\operatorname{Bif}(W, h_0)$. This shows that $\underline{s} < \underline{t}$.

Similarly, there exists some $h_1 \in \mathbb{Q} \setminus \mathbb{Z}$ between 0 and h with $\kappa_1 := \Psi_W(2\pi i h_1) \in \mathbb{C}$. As in the proof of Proposition 6.2, the curve $\Gamma_{W,h_1} \cup \{\kappa_1\} \cup \Gamma_{\operatorname{Bif}(W,h_1),0}$ surrounds $\Gamma_{W,h}$. It follows that $\operatorname{addr}(\operatorname{Bif}(W, h_1)) \leq \underline{s} \leq \operatorname{addr}(W)$; in particular $\underline{s} \in \mathcal{W}(W)$.

By part (b) of the Squeezing Lemma (Theorem 4.1), $\underline{s}$ is either intermediate or infinite and not exponentially bounded; in particular $\mathbb{K}(\underline{s})$ is not periodic. By Corollary 5.9, there exists some child component V of W such that $\underline{s} \in \mathcal{W}(V)$. By Proposition 6.2, there is a curve which is disjoint from W and separates the piece $\Gamma_{W,h}([-1, 0))$ from W. This is a contradiction. ■

6.4. Corollary (Boundaries of Satellite Components)**.**
Suppose that V is a satellite component. Then $\Psi_V : \overline{\mathbb{H}^-} \to \overline{W}$ is a homeomorphism, and $\partial V \cap \mathbb{C}$ is a Jordan arc.

Proof. By Theorem 6.3, it only remains to show that $\Psi_V(0) \in \mathbb{C}$. Since V is a satellite component, there exists a hyperbolic component W and $h \in \mathbb{Q}\setminus\mathbb{Z}$ such that $V = \operatorname{Bif}(W, h)$. By Proposition 5.8 and Theorem 6.3, $\Psi_V(0) = \Psi_W(2\pi i h) \in \mathbb{C}$. ■

We will now prove part (c) of the Squeezing Lemma.

6.5. Theorem (Squeezing Lemma, Part (c))**.**
Suppose that $\gamma : [0, \infty) \to \mathbb{C}$ is a curve to ∞ which does not contain indifferent parameters. If $\underline{s} := \operatorname{addr}(\gamma)$ is intermediate, then γ lies in the preferred homotopy class of $\operatorname{Hyp}(\underline{s})$.

Proof. By the discussion preceeding Theorem 3.5, the case $\underline{s} = \infty$ is trivial. Let us then suppose, by contradiction, that $\underline{s} \neq \infty$ and that γ does not lie in the preferred homotopy class of $W := \operatorname{Hyp}(\underline{s})$. To fix our ideas, let us assume that γ tends to ∞ below this homotopy class.

If $\gamma \subset W$, then by Theorem 6.3, $\Phi_W(\gamma(t)) \to 0$ as $t \to \infty$, and for any $h \in \mathbb{Q} \setminus \mathbb{Z}$ with $h > 0$ it follows that $\operatorname{Bif}(W, h)$ tends to ∞ between the preferred homotopy class of W and γ, a contradiction.

So let us assume that γ does not intersect W. For every $j \in \mathbb{N}$, set $\kappa_j := \Psi_W(2\pi i(j+\frac{1}{2})) \in \mathbb{C}$. Denote by Γ_j^1 the internal ray of W landing at κ_j and by Γ_j^2 the central internal ray of the child component V_j bifurcating from κ_j. Then the curve

$$\Gamma_j := \Gamma_j^1 \cup \{\kappa_j\} \cup \Gamma_j^2$$

surrounds γ. We will derive a contradiction by showing that

$$\lim_{j\to\infty} \min\{|\kappa| : \kappa \in \Gamma_j\} = \infty.$$

Since Ψ_W is continuous in ∞, it is clear that

$$\lim_{j\to\infty} \inf\{|\kappa| : \kappa \in \Gamma_j^1\} = \infty.$$

It is thus sufficient to concentrate on the curves Γ_j^2. Denote the characteristic addresses of V_j by $\underline{r}_j$ and $\underline{\tilde{r}}_j$. These addresses are periodic of period $2n$ (where n is the period of W), and the corresponding dynamic rays land together for every parameter on Γ_j^2. Since all $\underline{r}_j$ and $\underline{\tilde{r}}_j$ are different, their largest entries must tend to ∞ as j becomes large. By Lemma 5.10, this implies that $\inf\{|\kappa| : \kappa \in \Gamma_j^2\}$ also tends to ∞. ■

7. Pushing Wakes to Infinity

In this section, we complete the proof of the Squeezing Lemma by showing that the address of any curve γ in exponential parameter space must be intermediate or exponentially bounded. In order to do this, we shall surround γ by curves in hyperbolic components, similarly to the previous section, such that these curves must lie farther and farther to the right, which yields a contradiction. However, we shall need a theorem relating a given external address to the wakes which contain it. The theory of *internal addresses*, which was introduced for polynomials in [LS], gives exactly such a description. We will use the following result (compare [LS, Proposition 5.4]).

7.1. Proposition (Finding Hyperbolic Components [RS])**.**
Let $\underline{s}$ be an unbounded infinite external address, and suppose that W is a hyperbolic component with $\underline{s} \in \mathcal{W}(W)$. Let k be maximal such that the first $k-1$ entries of $\underline{\mathtt{u}} := \mathbb{K}(\underline{s})$ and $\mathbb{K}^*(W)$ coincide (with the convention that $k = 1$ if $W = \mathrm{Hyp}(\infty)$ is the period 1 component). If $\mathtt{u}_k \in \mathbb{Z}$, then there exists a hyperbolic component V with $\underline{s} \in \mathcal{W}(V)$ and $\mathbb{K}^*(V) = \overline{\mathtt{u}_1 \dots \mathtt{u}_k}$. □

7.2. Theorem (Squeezing Lemma, Part (a))**.**
Suppose that $\gamma : [0,\infty) \to \mathbb{C}$ is a curve to ∞ which does not contain indifferent parameters. Then $\mathrm{addr}(\gamma)$ is either intermediate or exponentially bounded.

Proof. Suppose, by contradiction, that $\underline{s} := \mathrm{addr}(\gamma)$ is infinite but not exponentially bounded. The first step in our proof is to find a sequence W_k of hyperbolic components with $\underline{s} \in \mathcal{W}(W_k)$ such that the combinatorics of W_k becomes "large" in a suitable sense.

Set $\underline{\mathtt{u}} := \mathbb{K}(\underline{s})$; then the sequence $\underline{\mathtt{u}}$ is also not exponentially bounded. We can thus choose a subsequence $(\mathtt{u}_{n_k})$ of entries such that $|\mathtt{u}_j| < |\mathtt{u}_{n_k}|$ whenever $j < n_k$ and such that

$$F^{-n_k}(2\pi|\mathtt{u}_{n_k}|) \to \infty.$$

We claim that, for every k, there exists a hyperbolic component W_k with $\underline{s} \in \mathcal{W}(W_k)$ and

$$\mathbb{K}^*(W_k) = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n_k}}.$$

Indeed, let $m < n_k$ be maximal such that there exists a hyperbolic component U with $\underline{s} \in \mathcal{W}(U)$ with $\mathbb{K}^*(U) = \overline{\mathtt{u}_1 \dots \mathtt{u}_m}$. (The existence of such an m follows from Proposition 7.1.) By maximality of m and Proposition 7.1, $\mathbb{K}^*(U)$ and $\underline{\mathtt{u}}$ agree on the first $n_k - 1$ entries. On the other hand, all entries of $\mathbb{K}^*(U)$ are different from $\mathtt{u}_{n_k}$. The existence of a component W_k with the desired properties now follows by applying Proposition 7.1 to $\underline{s}$ and U.

By Proposition 6.2, there is a curve $\Gamma_k \subset \mathbb{C}$ which tends to $+\infty$ in both directions, which separates γ from a left half plane, and which consists only of parameters in W_k and a child component of W_k together with a common boundary point. We will show that the curves Γ_k tend to ∞ uniformly as $k \to \infty$.

Let $\underline{r}^k$ and $\underline{\tilde{r}}^k$ denote the characteristic addresses of W_k. Since $\mathbb{K}^*(W_k) = \overline{\mathtt{u}_1 \dots \mathtt{u}_{n_k}}$, for every j the jn_k-th entries of $\underline{r}^k$ and $\underline{\tilde{r}}^k$ are of size at least $|\mathtt{u}_{n_k}| - 1$. By Lemma 5.10, it follows that

$$\Gamma_k \subset \left\{\kappa : |\kappa| > \mathcal{F}^{-n_k+1}(2\pi|\mathtt{u}_{n_k}| - 3\pi) - 2\right\}.$$

On the other hand, $F^{-n_k+1}(2\pi|\mathtt{u}_{n_k}| - 3\pi) \to \infty$. This is a contradiction because every Γ_k surrounds γ. ■

This completes the proof of the Squeezing Lemma, and thus of Theorem 1.1. ■

8. Further Results

With Theorem 1.1 now proved, we discuss a number of important properties of the bifurcation structure in exponential parameter space: we prove that boundaries of hyperbolic components are analytic except at roots and co-roots, we solve another conjecture of Eremenko and Lyubich on "bifurcation trees", and we give an intrinsic definition of wakes in parameter space using landing properties of dynamic and parameter rays.

Analyticity of boundaries. Let W be a hyperbolic component of period n. We will call $\Psi_W(0)$ the *root* of W; the points of $\Psi_W(2\pi i(\mathbb{Z} \setminus \{0\})$ are called *co-roots*. We will show that the boundary of W is an analytic curve, except at its co-roots and possibly at its root point. Note that it is a priori clear that ∂W is a piecewise analytic curve. We need to rule out the existence of critical points of the multiplier map μ on ∂W.

8.1. Proposition (Closures Intersect at Parabolic Points)**.**
Let W be a hyperbolic component and let $\kappa \in \partial W$ be an irrationally indifferent parameter. Then no other hyperbolic component contains κ on its boundary.

Proof. Suppose that W_1 and W_2 are two hyperbolic components which have an irrational boundary parameter κ_0 in common. Let $h_0 \in \mathbb{R} \setminus \mathbb{Q}$ with $\kappa_0 = \Psi_{W_1}(ih_0)$, and suppose, to fix our ideas, that $h_0 > 0$. Then for every $h \in (0, h_0) \cap \mathbb{Q}$, the curve

$$\Gamma_{W_1,h} \cup \{\Psi_{W_1}(ih)\} \cup \Gamma_{\mathrm{Bif}(W_1,h),0}$$

separates κ_0, and thus also W_2, from every component which does not lie in the wake of W_1 (compare Proposition 6.2).

This proves that $\operatorname{addr}(W_2) \in \mathcal{W}(W_1)$. By symmetry, also $\operatorname{addr}(W_2) \in \mathcal{W}(W_1)$, which is a contradiction. ■

8.2. Corollary (Analytic Boundary)**.**
Let W be a hyperbolic component. Then the function $h \mapsto \Psi_W(2\pi i h)$ is analytic in $\mathbb{R} \setminus \mathbb{Z}$. Furthermore, ∂W has a cusp in every co-root of W. The boundary is analytic or has a cusp in the root of W depending on whether W is a satellite or primitive component, respectively.

Proof. Let $h \in \mathbb{R} \setminus \mathbb{Z}$; then the multiplier map μ extends analytically to a neighborhood of $\kappa := \Psi_W(2\pi i h)$. Since $\mu \circ \Psi_W = \exp$, analyticity of Ψ_W follows unless μ has a critical point in κ. In that case, let D be a small disk around $e^{2\pi i h}$. The preimage of $D \cap \mathbb{D}$ under μ has at least two components D_1, D_2 whose boundary contains κ. By Proposition 5.8 (if $\vartheta \in \mathbb{Q}$) resp. Proposition 8.1 (otherwise), W is the only component of period at most n containing κ in its boundary, and thus $D_1 \cup D_2 \subset W$. However, this is impossible by Corollary 2.6.

The statement about points of $\Psi_W(2\pi i \mathbb{Z})$ is proved in a similar way as for the Mandelbrot set [M3, Lemmas 6.1 and 6.2]. By an elementary local argument, for every root or co-root κ_0 there is a neighborhood U of κ_0 such that the multiplier map is defined at least on some double cover of $U \setminus \{\kappa_0\}$ (using the fact that every primitive parabolic parameter is at most a double parabolic, because there is only one singular orbit). In the satellite case, the lower-period multiplier is defined in a neighborhood of κ_0 in the κ-plane and has no critical point at κ_0, so W has analytic boundary. It follows that the higher-period multiplier has similar properties. Otherwise, κ_0 is on the boundary of a single hyperbolic component, so μ must be injective in a neighborhood of κ_0 on the double cover of U, and the claim follows as in [M3]. ■

Bifurcation trees. A consequence of Proposition 8.1 is the following characterization of satellite hyperbolic components.

8.3. Corollary (Satellite Components)**.**
Let W be a hyperbolic component. Then the following are equivalent

(a) *W is a satellite component.*
(b) *There is a hyperbolic component $V \neq W$ with $\Psi_W(0) \in \partial V$.*
(c) *There is a hyperbolic component V of period less than W such that $\partial V \cap \partial W \cap \mathbb{C} \neq \emptyset$.*

Proof. By Proposition 5.8, the root of every satellite component lies on a hyperbolic component of smaller period. If $\Psi_W(0)$ lies on the boundary of another hyperbolic component V, then it follows from Proposition 5.8 that W is a child component of V, and thus the period of V is smaller than that of W. Finally, suppose that V is a component of smaller period than W such that W and V have a common finite boundary point. Then this boundary point is parabolic by Proposition 8.1. Thus, by Proposition 5.8, W is a child component of V. ■

The *bifurcation forest* of hyperbolic components is the (infinite) graph with one vertex for each hyperbolic component and one edge for each pair of components whose closures have a finite intersection point. A *bifurcation tree* is any component of this graph. The periods of hyperbolic components make every bifurcation tree an oriented tree, which thus has a unique root point of lowest period. By Corollary 8.3, all vertices of the tree correspond to satellite components, with the exception of the root point of the tree, which always is a primitive component. Conversely, every primitive component is the root point of its own bifurcation tree. Moreover, different primitive components have disjoint bifurcation trees. It was conjectured in [EL1] that there are infinitely many bifurcation trees. We will now prove this fact.

8.4. Corollary (Infinitely Many Bifurcation Trees).
There are infinitely many bifurcation trees.

PROOF. It suffices to prove that there are infinitely many primitive components. We will show that, for every $k > 0$, the component $\operatorname{Hyp}(0(k+\frac{1}{2})\infty)$ is primitive. Indeed, note that $0(k+\frac{1}{2})\infty \in \mathcal{W}(\operatorname{Hyp}(\frac{1}{2}\infty)) = (\overline{01}, \overline{10})$. Thus $\operatorname{Hyp}(0+\frac{1}{2})\infty)$ does not bifurcate from the unique period 1 component $\operatorname{Hyp}(\infty)$, and is thus primitive. ■

Wakes and periodic parameter rays. In [Sch1, Corollaries IV.4.4 and IV.5.2], it was shown (without using the results of this article), that every parabolic parameter is the landing point of either one or two parameter rays at periodic addresses. (Recall the definition of the parameter rays $G_{\underline{s}}$ in Definition 3.3.) More precisely, suppose that κ is a parabolic parameter; say $\kappa = \Psi_W(2\pi i k)$ for some hyperbolic component W of period n and some $k \in \mathbb{Z}$. If $k = 0$, then κ is the landing point of exactly two periodic parameter rays, namely those at the characteristic addresses of W, which both have period n. If $k \in \mathbb{Z} \setminus \{0\}$, then κ is the landing point of a single periodic parameter ray, at the address $\operatorname{addr}(W,k) := \lim_{h\to k} \operatorname{addr}(W,h)$. These addresses are called *sector boundaries* of W.

Using Theorem 1.1 and the fact that every periodic external address is a characteristic address or a sector boundary [RS], we obtain the following result [Sch1, Theorem V.7.2].

8.5. Theorem (Periodic Parameter Rays Land).
Every periodic parameter ray lands at a parabolic parameter. Conversely, every parabolic parameter is the landing point of either one or two periodic parameter rays. □

We have so far only defined the *combinatorial* wake of a hyperbolic component $W = \operatorname{Hyp}(\underline{s})$, as the interval $(\underline{s}^-, \underline{s}^+)$. However, Theorem 8.5 (and the discussion preceding it) suggest the following definition of the wake as a subset of parameter space (which is analogous to the usual definition of wakes in the Mandelbrot set).

8.6. Definition (Wake)**.**
Let W be a hyperbolic component with characteristic addresses $\underline{s}^-$ and $\underline{s}^+$. Then the (parameter) wake *of W is defined to be the component of $\mathbb{C} \setminus \left(G_{\underline{s}^-} \cup G_{\underline{s}^+} \cup \Psi_W(0)\right)$ which contains W.*

For the Mandelbrot set, the parameter wake of a hyperbolic component W coincides with the set of all parameters for which the dynamic rays at the characteristic addresses of W land at a common repelling periodic point. For exponential maps, the equivalence of this definition to the previous one — and the landing of periodic dynamic rays for all expoential maps with nonescaping singular value — was recently shown in [R2], using Theorem 8.5 and holomorphic motions.

8.7. Theorem (Characterization of Wakes)**.**
Let W be a hyperbolic component with characteristic addresses $\underline{s}^-$ and $\underline{s}^+$. Then the parameter wake of W coincides with the set of all parameters κ for which $g_{\underline{s}^-}$ and $g_{\underline{s}^+}$ have a common repelling landing point. □

Nonhyperbolic components. The main open question about exponential parameter space, like for quadratic polynomials and the Mandelbrot set, is whether hyperbolic dynamics is dense. In other words, we need to show that there are no non-hyperbolic (or "queer") stable components. It seems possible to show at least that any non-hyperbolic component must be bounded: the first combinatorial step is to show that hyperbolic components, as well as parameter rays at periodic and preperiodic external addresses together with their landing points, disconnect the κ-plane into complementary components such that each component is separated from all external addresses with at most two exceptions, both of which must be exponentially bounded. In particular, if a non-hyperbolic component is unbounded, it must "squeeze" to ∞ very close to one or two parameter rays, and a stronger variant of the Squeezing Lemma might prevent this from happening. This would show that all non-hyperbolic components were bounded, and the bifurcation locus was connected. However, most features in exponential parameter spaces are unbounded (for example, all hyperbolic components and all puzzle pieces). If one could prove that every non-hyperbolic component had to be unbounded as well, this would settle density of hyperbolicity in an interesting way.

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