Complete polynomial vector fields on $\mathbb{C}^2$, Part I

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Abstract

In this work, under a mild assumption, we give the classification of the complete polynomial vector fields in two variables up to algebraic automorphisms of $\mathbb{C}^2$. The general problem is also reduced to the study of the combinatorics of certain resolutions of singularities. Whereas we deal with $\mathbb{C}$-complete vector fields, our results also apply to $\mathbb{R}$-complete ones thanks to a theorem of Forstneric [Fo].

Key-words: line at infinity - vector fields - singular foliations

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1 Introduction

Recall that a *holomorphic flow* on $\mathbb{C}^2$ is a holomorphic mapping $\Phi : \mathbb{C} \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ satisfying the two conditions below:

- $\Phi(0, p) = p$ for every $p \in \mathbb{C}^2$;
- $\Phi(T_1 + T_2, p) = \Phi(T_1, \Phi(T_2, p))$.

A holomorphic flow Φ on $\mathbb{C}^2$ induces a *holomorphic vector field* X on $\mathbb{C}^2$ by the equation

$$X(p) = \left. \frac{d\Phi(T, p)}{dT} \right|_{T=0} .$$

Conversely a holomorphic vector field X on $\mathbb{C}^2$ is said to be *complete* if it is associated to a holomorphic flow Φ. Since every polynomial vector field of degree 1 is complete, we assume that X has degree 2 or greater. A polynomial vector field X can be considered as a meromorphic vector field on $\mathbb{CP}(2)$ therefore inducing a singular holomorphic foliation $\mathcal{F}_X$ on $\mathbb{CP}(2)$. The singularities of $\mathcal{F}_X$ lying in the "line at infinity" Δ will be denoted by $p_1, \ldots, p_k$. A singularity p_i as above is called *dicritical* if there are infinitely many analytic curves invariant by $\mathcal{F}_X$ and passing through p_i. The first result of this paper is the following:

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Theorem A *Let X be a complete polynomial vector field on $\mathbb{C}^2$ with degree 2 or greater and let $\mathcal{F}_X$ be the singular foliation induced by X on $\mathbb{CP}(2)$. Assume that $\mathcal{F}_X$ has a dicritical singularity in Δ. Then X is conjugate by a polynomial automorphism to one of the following vector fields:*

1. *$P(y)x^{\epsilon}\partial/\partial x$, where $P(y)$ is a polynomial in y and $\epsilon = 0, 1$.*
2. *$x^n y^m (mx\partial/\partial x - ny\partial/\partial y)$, where $\text{g.c.d}(m, n) = 1$ and $m, n \in \mathbb{N}$;*

In view of Theorem A, we just need to consider vector fields all of whose singularities belonging to Δ are not dicritical. Again let X be such a vector field and let $\mathcal{F}_X$ be its associated foliation. Consider a singularity p_i of $\mathcal{F}_X$ in the line at infinity Δ and a vector field $\widetilde{X}$ obtained through a finite sequence of blowing-ups of X beginning at p_i. Denote by $\mathcal{E}$ the corresponding exceptional divisor and by D_i, $i = 1, \ldots, l$, its irreducible components which are all rational curves. We say that X has adapted poles at p_i if, for every sequence of blow-ups as above and every irreducible component D_i of the corresponsding exceptional divisor $\mathcal{E}$, either X vanishes identically on D_i or D_i consists of pole of $\widetilde{X}$ (in other words $\widetilde{X}$ is not regular on D_i). Clearly the great majority of polynomial vector fields have adapted poles at its singularities at infinity. We then have:

Theorem B *Let X be a complete polynomial vector field on $\mathbb{C}^2$ and denote by p_i, $i = 1, \ldots, k$ the singularities of the associated foliation $\mathcal{F}_X$ belonging to the line at infinity Δ. Suppose that X has adapted poles at each p_i. Then $\mathcal{F}_X$ possesses a dicritical singularity in the line at infinity Δ.*

There is a good amount of literature devoted to complete vector fields on $\mathbb{C}^2$, in particular our results are complementary to the recent results obtained by Cerveau and Scardua in [Ce-Sc]. Note however that the points of view adopted in both papers are almost disjoint. For more information on complete polynomial vector fields the reader can consult the references at the end as well as references in [Ce-Sc].

After Theorems A and B, in order to classify all complete polinomial vector fields on $\mathbb{C}^2$ we just have to consider vector fields which do not have adapted poles at one of the singularities $p_1, \ldots, p_k$. In particular we can always assume that none of these singularities is dicritical.

Let us close this Introduction by giving a brief description of the structure of the paper. First we observe that the method developed here may be pushed forward to deal with vector fields which do not have adapted poles. Indeed the assumption that X has adapted poles in Δ is used only in Section 6. More precisely, from Section 2 to Section 5, the classification of complete polynomial vector fields is reduced to a problem of understanding the possible configurations of rational curves arising from blow-ups of the singularities of $\mathcal{F}_X$ in the line at infinity Δ. The role of our main assumption is to make the "combinatorics" of these configurations simpler so as to allow for the complete description given in Section 6. It is reasonable to expect that a more detailed study of these configurations will lead to the general classification of complete polynomial vector fields.

Another feature of our method is its local nature. Indeed most of our results are local and therefore have potential to be applied in other situations (especially to other Stein surfaces). We mention in particular the results of Sections 3 and 5 (cf. Theorem (3.6), Proposition (5.1)). Also the local vector fields $Z_{1,11}$, $Z_{0,12}$, $Z_{1,00}$ introduced in Proposition (5.1) might have additional interest.

This paper is also a sequence of [Re4] where it was observed, in particular, that the problem of understanding complete polynomial vector fields on $\mathbb{C}^2$ can be unified with classical problems in Complex Geometry through the notion of *meromorphic semi-complete vector fields*. The method employed in the proof of our main result relies heavily on this connection. Indeed an important part of the proof the preceding theorems is a discussion of semi-complete singularities of meromorphic vector fields. The study of these singularities was initiated in [Re4] but the present discussion is based on a different setting.

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2 Basic notions and results

The local orbits of a polynomial vector field X induce a singular holomorphic foliation $\mathcal{F}_X$ on $\mathbb{C}^2$. Besides, considering $\mathbb{CP}(2)$ as the natural compactification of $\mathbb{C}^2$ obtained by adding the line at infinity Δ, the foliation $\mathcal{F}_X$ extend to a holomorphic foliation, still denoted by $\mathcal{F}_X$, on the whole of $\mathbb{CP}(2)$. This extension may or may not leave the line at infinity Δ invariant. On the other hand, the vector field X possesses a *meromorphic* extension to $\mathbb{CP}(2)$, also denoted by X, whose *pole divisor* coincides with Δ. Note that the meromorphic extension of X to $\mathbb{CP}(2)$ happens to be holomorphic if and only if the degree of X is 1 or if X has degree 2 and the line at infinity Δ is not invariant by $\mathcal{F}_X$ (for further details cf. below). Let us make these notions more precise.

Recall that a *meromorphic* vector field Y on a neighborhood U of the origin $(0, \ldots, 0) \in \mathbb{C}^n$ is by definition a vector field of the form

$$Y = F_1 \frac{\partial}{\partial z_1} + \cdots + F_n \frac{\partial}{\partial z_n} \, ,$$

where the F_i's are meromorphic functions on U (i.e. $F_i = f_i/g_i$ with f_i, g_i holomorphic on U). Note that Y may not be defined on the whole U even though we consider ∞ as a value since F_i may have indeterminacy points. We denote by D_Y the union of the sets $\{g_i = 0\}$. Of course D_Y is a divisor consisting of poles and indeterminacy points of Y which is called the *pole divisor* of Y

Definition 2.1 *The meromorphic vector field Y is said to be semi-complete on U if and only if there exists a meromorphic map $\Phi_{sg} : \Omega \subseteq \mathbb{C} \times U \to U$, where Ω is an open set of $\mathbb{C} \times U$, satisfying the conditions below.*

1.

$$\left. \frac{d\Phi_{sg}(T, x)}{dT} \right|_{T=0} = Y(x) \text{ for all } x \in U \setminus D_x;$$

2. $\Phi_{sg}(T_1+T_2,x) = \Phi_{sg}(T_1,\Phi_{sg}(T_2,x))$ *provided that both sides are defined;*

3. If (T_i,x) *is a sequence of points in* Ω *converging to a point* $(\hat{T},x)$ *in the boundary of* Ω*, then* $\Phi_{sg}(T_i,x)$ *converges to the boundary of* $U\setminus D_Y$ *in the sense that the sequence leaves every compact subset of* $U\setminus D_Y$*.*

The map Φ_{sg} is called the meromorphic semi-global flow associated to Y (or induced by Y).

Assume we are given a meromorphic vector field Y defined on a neighborhood of $(0,0)\in\mathbb{C}^2$. It is easy to see that Y has the form $Y = fZ/g$ where f,g are holomorphic functions and Z is a holomorphic vector field having at most an isolated singularity at the origin. Naturally we suppose that f,g do not have a (non-trivial) common factor, so that f,g and Z are unique (up to a trivial, i.e. inversible, factor). Next, let $\mathcal{F}$ denote the local singular foliation defined by the orbits of Z. We call $\mathcal{F}$ the foliation associated to Y and note that either $\mathcal{F}$ is regular at the origin or the origin is an isolated singularity of $\mathcal{F}$. An analytic curve $\mathcal{S}$ passing through the origin and invariant by $\mathcal{F}$ is said to be a *separatrix* of $\mathcal{F}$ (or of Y,Z).

The rest of this section is devoted to establishing some preliminary results concerning both theorems in the Introduction. Particular attention will be paid to meromorphic semi-complete vector fields which appear when we restrict X to a neighborhood of the line at infinity Δ. As it will be seen, a large amount of information on X arises from a detailed study of this restriction. To begin with, let us recall the notion of *time-form* dT of a meromorphic vector field and the basic lemma about integrals of dT over curves. For other general facts about meromorphic semi-complete vector fields the reader is referred to Section 2 of [Re4].

Let Y be a meromorphic vector field defined on an open set U and let $\mathcal{F}$ denote its associated foliation. The regular leaves of $\mathcal{F}$ (after excluding possible punctures corresponding to *zeros* or poles of Y) are naturally equipped with a *foliated* holomorphic 1-form dT defined by imposing $dT.Y=1$. As a piece of terminology, whenever the 1-form dT is involved, the expression "regular leaf of $\mathcal{F}$" should be understood as a regular leaf L (in the sense of the foliation $\mathcal{F}$) from which the intersection with the set of *zeros* or poles of Y was deleted. Hence the restriction of dT to a regular leaf L is, by this convention, always holomorphic. Note also that dT is "foliated" in the sense that it is defined only on the tangent spaces of the leaves. We call dT the *time-form* associated to (or induced by) Y. Lemma (2.2) below is the most fundamental result about semi-complete vector fields (cf. [Re1], [Re4]).

Lemma 2.2 *Let* Y*,* U*,* $\mathcal{F}$ *and* dT *be as above. Consider a regular leaf* L *of* $\mathcal{F}$ *and an embedded (open) curve* $c:[0,1]\to L$*. If* Y *is semi-complete on* U *then the integral of* dT *over* c *does not vanish.* □

Let us now go back to a complete polynomial vector field X on $\mathbb{C}^2$ whose degree is $d\in\mathbb{N}$. Set

$$X = X_0 + X_1 + \cdots + X_d \tag{1}$$

where X_i, $i=1,\ldots,d$, stands for the homogeneous component of degree i of X. With this notation the vector fields whose associated foliations $\mathcal{F}_X$ do not leave the line at infinity Δ

invariant admit an elementary characterization namely: $\mathcal{F}_X$ does not leave Δ invariant if and only if X_d has the form $F(x,y)(x\partial/\partial x + y\partial/\partial y)$, where F is a homogeneous polynomial of degree $d-1$. Furthermore, viewing X as a meromorphic vector field on $\mathbb{CP}(2)$, a direct inspection shows that the order of the pole divisor Δ is $d-1$ provided that Δ is invariant under $\mathcal{F}_X$. If Δ is not invariant under $\mathcal{F}_X$ then this order is $d-2$. On the other hand, given a point $p \in \Delta$ and a neighborhood $U \subset \mathbb{CP}(2)$ of p, it is clear that X defines a meromorphic semi-complete vector field on U.

Our first lemma shows that we can suppose that the line at infinity is invariant by the associated foliation $\mathcal{F}_X$. Whereas the proof is elementary, we give a detailed account since some basic ideas will often be used later on.

Lemma 2.3 *Consider a complete polynomial vector field X on $\mathbb{C}^2$ and denote by $\mathcal{F}_X$ the foliation induced by X on $\mathbb{CP}(2)$. Assume that the line at infinity Δ is not invariant under $\mathcal{F}_X$. Then the degree of X is at most* 1.

Proof: First we set $X = FZ$ where F is a polynomial of degree $0 \leq n \leq d$ and Z is a polynomial vector field of degree $d-n$ and isolated *zeros*. In other words, we have $Z = P\partial/\partial x + Q\partial/\partial y$ where P, Q for polynomials P, Q without non-trivial common factors.

First we suppose for a contradiction that d is strictly greater than 2. In view of the preceding discussion, the line at infinity Δ is the polar divisor of X and has order $d-2 \geq 1$. Let $\mathcal{C} \subset \mathbb{CP}(2)$ be the algebraic curve induced in affine coordinates by $F=0$. Finally consider a "generic" point $p \in \Delta$ and a neighborhood U of p such that $U \cap \mathcal{C} = \emptyset$.

Let $\mathcal{F}_X$ be the singular foliation induced by X on $\mathbb{CP}(2)$ and notice that p is a regular point of $\mathcal{F}_X$. Besides the leaf L containing p is transverse at p to Δ. Thus we can introduce coordinates (u,v) on U, identifying p with $(0,0) \in \mathbb{C}^2$, in which X becomes

$$X(u,v) = u^{2-d}.f.\frac{\partial}{\partial u}$$

where f is a holomorphic function such that $f(0,0) \neq 0$ and $\{u=0\} \subset \Delta$ (here we use the fact that $U \cap \mathcal{C} = \emptyset$). The axis $\{v=0\}$ is obviously invariant under $\mathcal{F}$ and the time-form $dT_{\{v=0\}}$ induced on $\{v=0\}$ by X is given by $dT_{\{v=0\}} = u^{d-2}du/f(u,0)$. Since $d \geq 3$ and $f(0,0) \neq 0$, it easily follows the existence of an embedded curve $c : [0,1] \rightarrow \{v=0\} \setminus (0,0)$ on which the integral of $dT_{\{v=0\}}$ (cf. Remark (3.1)). The resulting contradiction shows that $d \leq 2$.

It only remains to check the case $d=2$. Modulo a linear change of coordinates, we have $X = X_0 + X_1 + x(x\partial/\partial x + y\partial/\partial y)$. The above calculation shows that the natural extension of X to $\mathbb{CP}(2)$ is, in fact, holomorphic. Therefore X is complete on $\mathbb{CP}(2)$ since $\mathbb{CP}(2)$ is compact. Besides a generic point p of Δ is regular for X and has its (local) orbit transverse to Δ. It follows that points in the orbit of p reaches Δ in finite time. Thus the restriction of X to the affine $\mathbb{C}^2$ cannot be complete as its flow goes off to infinity in finite time. □

In view of Lemma (2.3) all complete polynomial vector fields considered from now on will be such that the associated foliation $\mathcal{F}_X$ leaves the line at infinity Δ invariant. In particular, in the sequel, the extension of X to $\mathbb{CP}(2)$ is strictly meromorphic. Furthermore the pole divisor is constituted by the line at infinity Δ and has order $d-1$ (where d is the degree of X).

Lemma 2.4 *Let X be as above and let X_d be its top-degree homogeneous component (as in (1)). Then X_d is semi-complete on the entire $\mathbb{C}^2$.*

Proof: For each integer $k \in \mathbb{N}$, we consider the homothety $\Lambda_k(x,y) = (kx, ky)$ of $\mathbb{C}^2$. The vector fields defined by $\Lambda_k^* X$ are obviously complete, and therefore semi-complete, on $\mathbb{C}^2$. Next we set $X^k = k^{1-d}\Lambda_k^* X$. Since X^k and $\Lambda_k^* X$ differ by a multiplicative constant, it is clear that X^k is complete on $\mathbb{C}^2$. Finally, when k goes to infinity, X^k converges uniformly on $\mathbb{C}^2$ towards X_d. Since the space of semi-complete vector fields is closed under uniform convergence (cf. [G-R]), it results that X_d is semi-complete on $\mathbb{C}^2$. The lemma is proved. $\square$

The lemma above has an immediate application. In fact, a homogeneous polynomial vector field has a 1-parameter group of symmetries consisting of homotheties. Hence these vector fields can essentially be integrated. In other words, it is possible to describe all homogeneous polynomial vector fields which are semi-complete on $\mathbb{C}^2$. This classification was carried out in [G-R] and, combined with Lemma (2.4), yields:

Corollary 2.5 *Let X and X_d be as in the statement of Lemma (2.4). Then, up to a linear change of coordinates of $\mathbb{C}^2$, X_d has one of the following normal forms:*

1. $X_d = y^a f(x,y)\partial/\partial x$ *where f has degree strictly less than* 3 *and* $a \in \mathbb{N}$.
2. $X_d = x(x\partial/\partial x + ny\partial/\partial y)$, $n \in N$, $n \neq 1$.
3. $X_d = x^i y^j (mx\partial/\partial x - ny\partial/\partial y)$ *where* $m, n \in \mathbb{N}^*$ *and* $ni - mj = -1, 0\, 1$. *We also may have* $X_d = (xy)^a (x\partial/\partial x - y\partial/\partial y)$, $a \in \mathbb{N}$.
4. $X_d = x^2\partial/\partial x - y(nx - (n+1)y)\partial/\partial y$, $n \in \mathbb{N}$.
5. $X_d = [xy(x-y)]^a [x(x-2y)\partial/\partial x + y(y-2x)\partial/\partial y]$, $a \in \mathbb{N}$.
6. $X_d = [xy(x-y)^2]^a [x(x-3y)\partial/\partial x + y(y-3x)\partial/\partial y]$, $a \in \mathbb{N}$.
7. $X_d = [xy^2(x-y)^3]^a [x(2x-5y)\partial/\partial x + y(y-4x)\partial/\partial y]$, $a \in \mathbb{N}$. $\square$

As an application of Corollary (2.5), we shall prove Lemma (2.6) below. This lemma estimates the number of singularities that the foliation $\mathcal{F}_X$ induced by X on $\mathbb{CP}(2)$ may have in the line at infinity. This estimate will be useful in Section 7. Also recall that the line at infinity is supposed to be invariant by $\mathcal{F}_X$ (cf. Lemma (2.3)).

Lemma 2.6 *Let X be a complete polynomial vector field on $\mathbb{C}^2$ and denote by $\mathcal{F}_X$ the foliation induced by X on $\mathbb{CP}(2)$. Then the line at infinity contains at most* 3 *singularities of* $\mathcal{F}_X$.

Proof: We consider the change of coordinates $u = 1/x$ and $v = y/x$ and note that in the coordinates (u, v) the line at infinity Δ is represented by $\{u = 0\}$.

First we set $X_d = F.Y_d$ where F is a polynomial of degree k and Y_d is a polynomial vector field of degree $d - k$. Next let us consider the algebraic curve $\mathcal{C} \subset \mathbb{CP}(2)$ induced on $\mathbb{CP}(2)$ by the affine equation $F = 0$. We also consider the foliation $\mathcal{F}_d$ induced on $\mathbb{CP}(2)$ by Y_d. Finally denote by $\Delta \cap \mathrm{Sing}(\mathcal{F}_X)$ (resp. $\Delta \cap \mathrm{Sing}(\mathcal{F}_d)$) the set of singularities of $\mathcal{F}_X$ (resp. $\mathcal{F}_d$) belonging to Δ. An elementary calculation with the coordinates (u, v) shows that

$$\Delta \cap \mathrm{Sing}(\mathcal{F}_X) \subseteq (\Delta \cap \mathrm{Sing}(\mathcal{F}_d)) \cup (\mathcal{C} \cap \Delta)\,.$$

Now a direct inspection in the list of Corollary (2.5) implies that the set $(\Delta \cap \mathrm{Sing}(\mathcal{F}_d)) \cup (\mathcal{C} \cap \Delta)$ consists of at most 3 points. The proof of the lemma is over. $\square$

3 Simple semi-complete singularities

In this section we shall begin the study of a certain class of semi-complete singularities. The results obtained here will largely be used in the remaining sections. In the sequel Y stands for a meromorphic vector field defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$. We always set $Y = fZ/g$ where f, g are holomorphic functions without common factors and Z is a holomorphic vector field for which the origin is either a regular point or an isolated singularity. Also $\mathcal{F}$ will stand for the foliation associated to Y (or to Z). We point out that the decomposition $Y = fZ/g$ is unique up to an inversible factor.

If Z is singular at $(0, 0)$, we can consider its eigenvalues at this singularity. Three cases can occur:

a- Both eigenvalues, λ_1, λ_2 of Z vanish at $(0, 0)$.

b- Exactly one eigenvalue, λ_2, vanishes at $(0, 0)$.

c- None of the eigenvalues λ_1, λ_2 vanishes at $(0, 0)$

Whereas Z is defined up to an inversible factor, all the cases **a**, **b** and **c** are well-defined. In the case **c**, the precise values of λ_1, λ_2 are not well-defined but so is their ratio λ_1/λ_2. Following a usual abuse of notation, in the case **c**, we shall say that the foliation $\mathcal{F}$ associated to Z has eigenvalues λ_1, λ_2 different from *zero*. In other words, given a singular holomorphic foliation $\mathcal{F}$, we say that $\mathcal{F}$ has eigenvalues λ_1, λ_2 if there exists Z as before having λ_1, λ_2 as its eigenvalues at $(0, 0)$. The reader will easily check that all the relevant notions discussed depend only on the ratio λ_1/λ_2. A singularity is said to be *simple* if it has at least one eigenvalue different from *zero*. A simple singularity possessing exactly one eigenvalue different from *zero* is called a *saddle-node*.

More generally the *order* of $\mathcal{F}$ at $(0, 0)$ is defined as the order of Z at $(0, 0)$, namely it is the degree of the first non-vanishing homogeneous component of the Taylor series of Z based at $(0, 0)$. It is obvious that the order of $\mathcal{F}$ does not depend on the vector field with isolated singularities Z chosen.

Remark 3.1 A useful fact is the non-existence, in dimension 1, of *strictly meromorphic* semi-complete vector fields. In other words, if $Y = f(x)\partial/\partial x$ with f meromorphic, then Y is not semi-complete on a neighborhood of $0 \in \mathbb{C}$. In fact, fixed a neighborhood U of $0 \in \mathbb{C}$, we have $f(x) = x^{-n}.h(x)$ where $n \geq 1$, h is holomorphic and $h(0) \neq 0$. Thus the corresponding time-form is $dT = x^n dx/h$. It easily follows the existence of an embedded curve $c : [0,1] \to U \setminus \{(0,0)\}$ on which the integral the integral of dT vanishes. Similarly we can also prove that Y is not semi-complete provided that $0 \in \mathbb{C}$ is an essential singularity of f.

Summarizing the preceding discussion, the fact that Y is semi-complete implies that f is holomorphic at $0 \in \mathbb{C}$. Consider now that f is holomorphic but $f(0) = f'(0) = f''(0) = 0$. An elementary estimate (cf. [Re1]) shows that in this case Y is not semi-complete. Finally when Y is semi-complete but $f(0) = f'(0) = 0$ it is easy to see that Y is conjugate to $x^2\partial/\partial x$ (cf. [G-R]). These elementary results give a complete description of semi-complete singularities in dimension 1.

Let us say that $P = P_\alpha/P_\beta$ is a *homogeneous rational function* if P_α and P_β are homogeneous polynomials (possibly with different degrees). The next lemma is borrowed from [Re4]

Lemma 3.2 *Consider the linear vector field $Z = x\partial/\partial x + \lambda y\partial/\partial y$. Suppose that $P = P_\alpha/P_\beta$ is a (non-constant) homogeneous rational function and that $\lambda \notin \mathbb{R}_+$. Suppose also that PZ is semi-complete. Then one has*

1. λ is rational, i.e. $\lambda = -n/m$ for appropriate coprime positive integers m, n.

2. P has one of the forms below:

2i *$P = x^c y^d$ where $mc - nd = 0$ or ± 1.*

2ii *If $\lambda = -1$, then P is $x^c y^d$ ($mc - nd = 0$ or ± 1) or $P = (x-y)(xy)^a$ for $a \in \mathbb{Z}$.* □

Remark 3.3 Consider a holomorphic vector field of the form

$$x^a y^b h(x,y)[x(1 + \text{h.o.t.})\partial/\partial x \;-\; y(1 + \text{h.o.t.})\partial/\partial y\,,$$

where $a, b \in \mathbb{Z}$ and h is holomorphic with $h(0,0) = 0$. Of course we suppose that h is not divisible by x, y. Next we assume that X is semi-complete on a neighborhood of $(0,0)$. Denote by h^k the homogeneous component of the first non-trivial jet of h at $(0,0)$. The same argument employed in the proof of Lemma (2.4), modulo replacing k by $1/k$, shows that the vector field $x^a y^b h^k(x\partial/\partial x - y\partial/\partial y)$ is semi-complete. From the preceding lemma it then follows that $h^k = x - y$ and $a = b$. However a much stronger conclusion holds: X admits the normal form

$$(xy)^a(x-y)g(x\partial/\partial x - y\partial/\partial y)\,,$$

where g is a holomorphic function satisfying $g(0,0) \neq 0$. In fact, in order to deduce the normal form above, we just need to check that $x(1 + \text{h.o.t.})\partial/\partial x \;-\; y(1 + \text{h.o.t.})\partial/\partial y$ is linearizable. After [M-M], this amounts to prove that the local holonomy of their separatrizes is the identity. However the integral of time-form on a curve c projecting onto a loop around

0 in $\{y=0\}$ is cearly equal to *zero*. Since X is semi-complete, such curve must be closed which means that the holonomy in question is trivial.

Of course the next step is to discuss the case $\lambda > 0$. However, at this point, we do not want to consider only linear vector fields. This discussion will naturally lead us to consider singularities having an infinite number of separatrizes. Recall that a singularity of a holomorphic foliation $\mathcal{F}$ is said to be *dicritical* if $\mathcal{F}$ possesses infinitely many separatrizes at p. Sometimes we also say that a vector field Y defined on a neighborhood of $(0,0) \in \mathbb{C}^2$ is dicritical to say that $(0,0)$ is a dicritical singularity of the foliation associated to Y. Let us begin with the following:

Lemma 3.4 *Consider a semi-complete meromorphic vector field Y defined on a neighborhood of $(0,0) \in \mathbb{C}^2$ and having the form*

$$Y = \frac{f}{g} Z$$

where f, g are holomorphic functions with $f(0,0)g(0,0) = 0$ and Z is a holomorphic vector field with an isolated singularity at $(0,0) \in \mathbb{C}^2$ whose eigenvalues are 1 and λ. Assume that $\lambda > 0$ but neither λ nor $1/\lambda$ belongs to $\mathbb{N}$. Then Z, Y are dicritical vector fields.

Proof: Note that Poincaré's linearization theorem [Ar] ensures that Z is linearizable. Therefore, in appropriate coordinates, we have $Y = f(x\partial/\partial x + \lambda y \partial/\partial y)/g$. If λ is rational equal to n/m, then Z admits the meromorphic first integral $x^n y^{-m}$ and therefore admits an infinite number of separatrizes.

In order to prove the lemma is now sufficient to check that λ is rational provided that Y is semi-complete. Let P_α (resp. P_β) be the first non-vanishing homogeneous component of the Taylor series of f (resp. g) at $(0,0) \in \mathbb{C}^2$. The same argument carried out in the proof of Lemma (2.4), modulo replacing k by $1/k$, shows that the vector field $Y^{\rm ho} = P_\alpha(x\partial/\partial x + \lambda y\partial/\partial y)/P_\beta$ is semi-complete on $\mathbb{C}^2$. We are going to see that this implies that λ is rational.

Suppose for a contradiction that λ is not rational. In this case the only separatrizes of $Y^{\rm ho}$ are the axes $\{x=0\}$, $\{y=0\}$. Since in dimension 1 there is no meromorphic semi-complete vector field, it follows that the *zero set* of P_β has to be invariant under the foliation $\mathcal{F}_{Y^{\rm ho}}$ associated to $Y^{\rm ho}$. Thus P_β must have the form $x^a y^b$ for some $a, b \in \mathbb{N}$. Therefore we can write P as $x^c y^d Q(x,y)$ where Q is a homogeneous polynomial.

Observe that the orbit L of $Y^{\rm ho}$ (or $\mathcal{F}_{Y^{\rm ho}}$) passing through the point (x_1, y_1) $(x_1 y_1 \neq 0)$ is parametrized by $\mathsf{A} : T \mapsto (x_1 e^T, y_1 e^{\lambda T})$. The restriction to L of the vector field PZ is given in the coordinate T by $P(x_1 e^T, y_1 e^{\lambda T})\partial/\partial T$. Because λ is not rational, the parametrization A is an one-to-one map from $\mathbb{C}$ to L. It results that the one-dimensional vector field $x_1^c y_1^d e^{(c+\lambda d)T} Q(x_1 e^T, y_1^{\lambda T})\partial/\partial T$ is semi-complete on the entire $\mathbb{C}$. On the other hand the function $T \mapsto e^{(c+\lambda d)T} Q(x_1 e^T, y_1^{\lambda T})$ is defined on the whole of $\mathbb{C}$. Since λ is not rational and Q is a polynomial, we conclude that this function has an essential singularity at infinity. This contradicts the fact that this function corresponds to a semi-complete vector field (cf. Remark (3.1)). The lemma is proved. $\square$

Let us now consider the case $\lambda \in \mathbb{N}$ since the case $1/\lambda \in \mathbb{N}$ is analogous. Thus we denote by 1 and $n \in \mathbb{N}^*$ the eigenvalues of Z at $(0,0)$. Such Z is either linearizable or conjugate to its Poincaré's-Dulac normal form [Ar]

$$(nx + y^n)\partial/\partial x \; + \; y\partial/\partial y \, . \tag{2}$$

When Z is linearizable, it has infinitely many separatrizes. Thus we are, in fact, interested in the case in which Z is conjugate to the Poincaré-Dulac normal form (2). In particular $\{y = 0\}$ is the unique separatrix of Z (or Y).

Lemma 3.5 *Let Y be $Y = fZ/g$ where Z is a vector field as in (2) and f, g are holomorphic functions satisfying $f(0,0)g(0,0) = 0$. Assume that Y is semi-complete and that the regular orbits of Y can contain at most one singular point whose order is necessarily 1. Then $n = 1$. Furthermore, up to an inversible factor, $g(x, y) = y$ and $\{f = 0\}$ defines a smooth analytic curve which is not tangent to $\{y = 0\}$.*

Proof: Since the divisor of poles of Y is contained in the union of the separatrizes, it follows that Y has the form

$$Y = y^k F(x,y)[(nx + y^n)\partial/\partial x \; + \; y\partial/\partial y] \, ,$$

where $k \in \mathbb{Z}$ and F is a holomorphic function which is not divisible by y. Clearly $k < 0$, otherwise the first homogeneous component of Y would not be semi-complete (cf. Corollary (2.5) and Remark (3.1)).

Let us first deal with the case $n = 1$. Note that we are going to strongly use Theorem (3.6) which is the next result to be proved. This theorem is concerned with the so-called *saddle-node* singularities which are those having exactly one eigenvalue different from *zero*. Blowing-up the vector field Y we obtain a new vector field $\widetilde{Y}$ defined and semi-complete on a neighborhood of the exceptional divisor $\pi^{-1}(0)$ (where π stands for the blow-up map). Denote by $\widetilde{\mathcal{F}}$ the foliation associated to $\widetilde{Y}$ and note that $\widetilde{\mathcal{F}}$ has a unique singularity $p \in \pi^{-1}(0)$. More precisely, $\widetilde{Y}$ on a neighborhood of p is given in standard coordinates (x, t) ($\pi(x, t) = (x, tx)$), by

$$H(x,t)[(x(1+t)\partial/\partial x \; - \; t^2\partial/\partial t] \, ,$$

where H is meromorphic function. Because of Theorem (3.6), we know that the restriction of $\widetilde{Y}$ to the exceptional divisor $\{x = 0\}$ has to be regular i.e. H is not divisible by x or x^{-1}. This implies that the order of F at $(0,0) \in \mathbb{C}^2$ is k and, in particular, $F(0,0) = 0$.

On the other hand, H has the form $t^k h(x,t)$ where h is holomorphic on a neighborhood of $x = t = 0$ and not divisible by t or t^{-1}. Again Theorem (3.6) shows that h has to be an inversible factor, i.e. $h(0,0) \neq 0$. In other words, the proper transform of the (non-trivial) analytic curve $F = 0$ intersects $\pi^{-1}(0)$ at points different from $x = t = 0$.

The restriction of $\widetilde{Y}$ to $\pi^{-1}(0)$ is a holomorphic vector field which has a singularity at $\{x = t = 0\}$ whose order is $2 - k$. In particular $k \in \{0, 1, 2\}$. The other singularities correspond to the intesection of $\pi^{-1}(0)$ with the proper transform of $\{F = 0\}$. The statement follows since the regular orbits of X can contain only one singular point whose order is 1. $\square$

In the rest of this section we briefly discuss the case of singularities as in **b**, that is, those singularities having exactly one eigenvalue different from *zero*. As mentioned they are called *saddle-nodes* and were classified in [M-R]. A consequence of this classification is the existence of a large moduli space. The subclass of saddle-nodes consisting of those associated to semi-complete holomorphic vector fields was characterized in [Re3]. In the sequel we summarize and adapt these results to meromorphic semi-complete vector fields.

To begin with, let ω be a singular holomorphic 1-form defining a saddle-node $\mathcal{F}$. According to Dulac [Dul], ω admits the normal form

$$\omega(x,y) = [x(1+\lambda y^p) + yR(x,y)]\, dy \;-\; y^{p+1}\, dx \;,$$

where $\lambda \in \mathbb{C}$, $p \in \mathbb{N}^*$ and $R(x,0) = o(|\ x\ |^p)$. In particular $\mathcal{F}$ has a (smooth) separatrix given in the above coordinates by $\{y = 0\}$. This separatrix is often referred to as the *strong invariant manifold* of $\mathcal{F}$. Furthermore there is also a *formal* change of coordinates $(x,y) \mapsto (\varphi(x,y), y)$ which brings ω to the form

$$\omega(x,y) = x(1+\lambda y^{p+1})\, dy \;- y^{p+1}\, dx \;. \tag{3}$$

The expression in (3) is said to be the *formal normal form* of $\mathcal{F}$. In these formal coordinates the axis $\{x = 0\}$ is invariant by $\mathcal{F}$ and called the *weak invariant manifold* of $\mathcal{F}$. Note however that the weak invariant manifold of $\mathcal{F}$ does not necessarily correspond to an actual separatrix of $\mathcal{F}$ since the change of coordinates $(x,y) \mapsto (\varphi(x,y), y)$ does not converge in general. Finally it is also known that a saddle-node $\mathcal{F}$ possesses at least one and at most two separatrizes (which are necessarily smooth) depending on whether or not the weak invariant manifold of $\mathcal{F}$ is convergent.

A general remark about saddle-nodes is the following one: denoting by π_2 the projection $\pi_2(x,y) = y$, Dulac's normal form implies that the fibers of π_2, namely the vertical lines, are transverse to the leaves of $\mathcal{F}$ away from $\{y = 0\}$. This allows us to define the monodromy of $\mathcal{F}$ as being the "first return map" to a fixed fiber.

As to semi-complete vector fields whose associated foliation $\mathcal{F}$ is a saddle-node, one has:

Theorem 3.6 *Suppose that Y is a meromorphic semi-complete vector field defined around $(0,0) \in \mathbb{C}^2$. Suppose that the foliation $\mathcal{F}$ associated to Y is a saddle-node. Then, up to an inversible factor, Y has one of the following normal forms:*

1. $Y = x(1+\lambda y)\partial/\partial x + y^2\partial/\partial y$, $\lambda \in \mathbb{Z}$.
2. $Y = y^{-p}[(x(1+\lambda y^p) + yR(x,y))\partial/\partial x + y^{p+1}\partial/\partial y]$.
3. $Y = y^{1-p}[(x(1+\lambda y^p) + yR(x,y))\partial/\partial x + y^{p+1}\partial/\partial y]$ *and the monodromy induced by $\mathcal{F}$ is trivial (in particular $\lambda \in \mathbb{Z}$ and the weak invariant manifold of $\mathcal{F}$ is convergent).*

Proof: The proof of this theorem relies heavily on the methods introduced in Section 4 of [Re2] and Section 4 of [Re3]. For convenience of the reader we summarize the argument below.

First we set $Y = fZ/g$ where Z is a holomorphic vector field with an isolated singularity at $(0,0)$. Note that when $f(0,0).g(0,0) \neq 0$ (i.e. when Y is holomorphic with an isolated singularity at $(0,0)$), then Y has the normal form 1. Indeed this is precisely the content of Theorem 4.1 in Section 4 of [Re3].

Next we observe that the pole divisor $\{g = 0\}$ is contained in the strong invariant manifold of $\mathcal{F}$. To verify this assertion we first notice that $\{g = 0\}$ must be invariant under $\mathcal{F}$ as a consequence of the general fact that there is no one-dimensional meromorphic semi-complete vector field (cf. Remark (3.1)). Thus $\{g = 0\}$ is contained in the union of the separatrizes of $\mathcal{F}$. Next we suppose for a contradiction that the weak invariant manifold of $\mathcal{F}$ is convergent (i.e. defines a separatrix) and part of the pole divisor of Y. In this case the technique used in the proof of Proposition 4.2 of [Re2] applies word-by-word to show that the resulting vector field Y is not semi-complete. This contradiction implies that $\{g = 0\}$ must be contained in the strong invariant manifold of $\mathcal{F}$ as desired.

Combining the information above with Dulac's normal form, we conclude that Y possesses the form

$$Y = \frac{f}{y^k}\left[(x(1+\lambda y^p) + yR(x,y))\frac{\partial}{\partial x} + y^{p+1}\frac{\partial}{\partial y}\right] .$$

Now we are going to prove that $f(0,0) \neq 0$ i.e. f is an inversible factor. Hence we assume for a contradiction that $f(0,0) = 0$ but f is not divisible by y. Still according to the terminology of Section 4 of [Re2], we see that the "asymptotic order of the divided time-form" induced by Y on $\{y = 0\}$ is at least 2 since this form is $dx/(xf(x,0))$ (this is also a consequence of the fact that the index of the strong invariant manifold of a saddle-node is *zero*, cf. Section 5). However this order cannot be greater than 2 since Y is semi-complete. Furthermore when this order happens to be 2, the local holonomy of the separatrix in question must be the identity provided that Y is semi-complete. Nonetheless the local holonomy of the strong invariant manifold of a saddle-node is never the identity. In fact, using Dulac's normal form, an elementary calculation shows that this holonomy has the form $H(z) = z + z^p + \cdots$. We then conclude that $f(0,0) \neq 0$.

Therefore we have so far

$$Y = y^{-k}H(x,y)\left[(x(1+\lambda y^p) + yR(x,y))\frac{\partial}{\partial x} + y^{p+1}\frac{\partial}{\partial y}\right] ,$$

where H is holomorphic and satisfies $H(0,0) \neq 0$.

Recall that π_2 denotes the projection $\pi_2(x,y) = y$ whose fibers are transverse to the leaves of $\mathcal{F}$ away from $\{y = 0\}$. Let L be a regular leaf of $\mathcal{F}$ and consider an embedded curve $c : [0,1] \to L$. If dT_L stands for the time-form induced on L by Y, we clearly have

$$\int_c dT_l \; = \; \int_{\pi_2(c)} h(y)y^{p-k-1}dy \, ,$$

where $h(y) = H(0, y)$ so that $h(0) \neq 0$. Since the integral on the left hand side is never *zero*, it follows that $p - k - 1 = 0$ or 1. The case $k = p - 1$ does not require further comments. On the other hand, if $k = p - 2$, then the integral of dT_L over c is *zero* provided that $\pi_2(c)$ is a loop around the origin $0 \in \{x = 0\}$. This implies that c must be closed itself. In other words the monodromy of $\mathcal{F}$ with respect to the fibration induced by π_2 is trivial. Conversely it is easy to check that the normal forms 1, 2 and 3 in the statement are, in fact, semi-complete. This finishes the proof of the theorem. □

Before closing the section, it is interesting to translate the condition in the item 3 of Theorem (3.6) in terms of the classifying space of Martinet-Ramis [M-R]. Note however that this translation will not be needed for the rest of the paper.

Fix $p \in \mathbb{N}^*$ and consider the foliation $\mathcal{F}_{p,\lambda}$ whose leaves are "graphs" (over the y-axis) of the form

$$x = \text{const}\,.\, y^{\lambda} \exp(-1/px^p)\,.$$

Given $\lambda \in \mathbb{C}$, the moduli space of saddle-nodes $\mathcal{F}$ having p, λ fixed is obtained from the foliation above through the following data:

- p translations $z \mapsto z + c_i$, $z, c_i \in \mathbb{C}$ denoted by $g_1^+, \ldots, g_p^+$.
- p local diffeomorphisms $z \mapsto z + \cdots$, $z \in \mathbb{C}$, tangent to the identity denoted by $g_1^-, \ldots, g_p^-$.

These diffeomorphisms induce a permutation of (part of) the leaves of $\mathcal{F}_{p,\lambda}$. More precisely the total permutation (after one tour around $0 \in \mathbb{C}$) is given by the composition

$$g_p^- \circ g_p^+ \circ \ldots \circ g_1^- \circ g_1^+\,.$$

However recall that our saddle-node has trivial monodromy. One easily checks that this cannot happen in the presence of the ramification y^{λ}. Thus we conclude that λ belongs to $\mathbb{Z}$ and, in particular, the model $\mathcal{F}_{p,\lambda}$ introduced above has itself trivial monodromy. Hence the monodromy of $\mathcal{F}$ itself is nothing but $g_p^- \circ g_p^+ \circ \ldots \circ g_1^- \circ g_1^+$. In other words the condition is

$$g_p^- \circ g_p^+ \circ \ldots \circ g_1^- \circ g_1^+ = Id\,.$$

In particular note that, if $p = 1$, the above equation implies that $g_1^- = g_1^+ = Id$. This explains why in item 1 of Theorem (3.6) the saddle-node in question is analytically conjugate to its formal normal form.

4 Polynomial vector fields and first integrals

Here we want to specifically consider complete polynomial vector fields whose associated foliation $\mathcal{F}$ has a singularity in the line at infinity which admits infinitely many separatrizes. Recall that such a singularity is said to be dicritical. The main result of the section is Proposition (4.1) below.

Proposition 4.1 *Let X be a complete polynomial vector field on $\mathbb{C}^2$ and let $\mathcal{F}_X$ denote the foliation induced by X on $\mathbb{CP}(2)$. Assume that $\mathcal{F}_X$ possesses a dicritical singularity p, belonging to the line at infinity Δ. Then $\mathcal{F}_X$ has a meromorphic first integral on $\mathbb{CP}(2)$. Furthermore, modulo a normalization, the closure of the regular leaves of $\mathcal{F}_X$ is isomorphic to $\mathbb{CP}(1)$.*

We begin with a weakened version of this proposition which is the following lemma.

Lemma 4.2 *Let X, $\mathcal{F}_X$ be as in the statement of Proposition (4.1) and denote by $p \in \Delta$ a dicritical singularity of X. Then X possesses a meromorphic first integral on $\mathbb{C}^2$. Moreover, if this first integral is not algebraic, then the set of leaves of $\mathcal{F}_X$ that pass through p only once contains an open set.*

Proof: Suppose that $\mathcal{F}_X$ as above possesses a singularity $p \in \Delta$ with infinitely many separatrizes. We consider coordinates (u, v) around p such that $\{u = 0\} \subset \Delta$. Given a small neighborhood V of p, we consider the restriction $X_{|_V}$ of X to V. Clearly $X_{|_V}$ defines a meromorphic semi-complete vector field on V.

Obviously only a finite number of separatrizes of $\mathcal{F}_X$ going through p may be contained in the divisor of poles or zeros of X. Thus there are (infinitely many) separatrizes $\mathcal{S}$ of $\mathcal{F}_X$ at p which are (local) regular orbits of X. We fix one of these separatrizes $\mathcal{S}$. Recall that $\mathcal{S}$ has a Puiseux parametrization $\mathsf{A}(t) = (a(t), b(t))$, $\mathsf{A}(0) = (0, 0)$, defined on a neighborhood W of $0 \in \mathbb{C}$. Furthermore A is injective on W and a diffeomorphism from $W \setminus \{0\}$ onto $\mathcal{S} \setminus \{(0, 0)\}$. Since $\mathcal{S}$ is invariant under X, the restriction to $\mathcal{S} \setminus \{(0, 0)\}$ of X can be pulled-back by A to give a meromorphic vector field Z on W, i.e. $Z(t) = \mathsf{A}^* (X|_{\mathcal{S}})$ where $t \in W \setminus \{(0, 0)\}$ for a sufficiently small neighborhood W of $0 \in \mathbb{C}$. We also have that Z is semi-complete on W since $X_{|_V}$ is semi-complete on V and A is injective on W. It follows from Remark (3.1) that Z admits a holomorphic extension to $0 \in \mathbb{C}$ which is still denoted by Z. Moreover, letting $Z(t) = h(t)\partial/\partial t$, we cannot have $h(0) = h'(0) = h''(0) = 0$. However we must have at least $h(0) = 0$. Otherwise the semi-global flow of Z would reach the origin $0 \in \mathbb{C}$ in finite time. Since $\mathsf{A}(0)$ lies in the line at infinity Δ, it would follow that points in the orbit of X containing $\mathcal{S}$ reach Δ in finite time. This is impossible since X is complete on $\mathbb{C}^2$.

Therefore we have only two cases left, namely $h(0) = 0$ but $h'(0) \neq 0$ and $h(0) = h'(0) = 0$ but $h''(0) \neq 0$. Let us discuss them separately. First suppose that $h(0) = h'(0) = 0$ but $h''(0) \neq 0$. Modulo a normalization we can suppose that $\mathcal{S}$ is smooth at p. Again we denote by L the global orbit of X containing $\mathcal{S}$. By virtue of the preceding L is a Riemann surface endowed with a complete holomorphic vector field $X_{|_L}$ which has a singularity of order 2 (where $X_{|_L}$ stands for the restriction of X to L). It immediately results that L has to be compactified into $\mathbb{CP}(1)$. In other words the closure of L is a rational curve and, in particular, an algebraic invariant curve of $\mathcal{F}_X$. Since there are infinitely many such curves, Jouanolou's theorem [Jo] ensures that $\mathcal{F}_X$ has a meromorphic first integral (alternatively we can also apply Borel-Nishino's theorem, cf. [La]). Furthermore the level curves of this first integral are necessarily rational curves (up to normalization) as it follows from the discussion above.

Suppose now that $h(0) = 0$ but $h'(0) \neq 0$. In this case the vector field Z has a non-vanishing residue at $0 \in \mathbb{C}$. We then conclude that $\mathcal{S}$ possesses a *period* i.e. there exists a loop $c : [0,1] \to \mathcal{S}$ ($c(0) = c(1)$) on which the integral of the corresponding time-form is different from *zero*. If L is the global orbit of X containing $\mathcal{S}$ the preceding implies that L is isomorphic to $\mathbb{C}^*$.

On the other hand the characterization of singularities with infinitely many separatrizes obtained through Seidenberg's theorem [Se] (cf. Section 7 for further details) ensures that the set of orbits L as above has positive logarithmic capacity. In fact it contains an open set. Thus Suzuki's results in [Sz1], [Sz2] apply to provide the existence of a non-constant meromorphic first integral for X. If this first integral is not algebraic, then a "generic" orbit passes through p once but not twice. Otherwise the leaf would contain two singularities of X and therefore it would be a rational curve (up to normalization). In this case the mentioned Jouanolou's theorem would provide an algebraic first integral for $\mathcal{F}_X$. The proof of the proposition is over. □

The preceding lemma suggests a natural strategy to establish Proposition (4.1). Namely we assume for a contradiction that $\mathcal{F}_X$ does not have an algebraic first integral. Then we have to show that a generic leaf passing through a dicritical singularity p must return and cross Δ once again (maybe through another dicritical singularity). The resulting contradiction will then complete our proof.

Using Seidenberg's theorem, we reduce the singularities of $\mathcal{F}_X$ in Δ so that they all will have at least one eigenvalue different from *zero*. In particular we obtain a normal crossing divisor $\mathcal{E}$ whose irreducible components are rational curves, one of them being the proper transform of Δ (which will still be denoted by Δ). The other components were introduced by the punctual blow-ups performed and are denoted by D_i, $i = 1, \ldots, s$. The fact that p is a dicritical singularity implies that one of the following assertions necessarily holds:

1. There is a component D_{i_0} of $\mathcal{E}$ which is not invariant by $\widetilde{\mathcal{F}}_X$ (where $\widetilde{\mathcal{F}}_X$ stands for the proper transform of $\mathcal{F}_X$).

2. There is a singularity p_0 of $\widetilde{\mathcal{F}}$ in $\mathcal{E}$ which is dicritical and has two eigenvalues different from *zero*.

We fix a local cross section Σ through a point q of Δ which is regular for $\widetilde{\mathcal{F}}_X$. Note that a regular leaf L of $\widetilde{\mathcal{F}}_X$ necessarily meets Σ infinitely many times unless L is algebraic. Indeed first we observe that L is properly embedded in the affine part $\mathbb{C}^2$ since $\mathcal{F}_X$ possesses a meromorphic first integral on $\mathbb{C}^2$. Thus all the accumulation points of L are contained in Δ. Obviously if L accumulates a regular point of Δ then L intersects Σ infinitely many times as required. On the other hand, if L accumulates only points of Δ which are singularities of $\mathcal{F}_X$, then Remmert-Stein theorem shows that the closure of L is algebraic. Summarizing, using Jouanolou's or Borel-Nishino's theorem, we can suppose without loss of generality that all the leaves of $\widetilde{\mathcal{F}}_X$ intersects Σ an infinite number of times (and in fact these intersection points approximate the point $q = \Sigma \cap \Delta$).

To prove that $\mathcal{F}_X$ has infinitely many leaves cutting the exceptional divisor $\mathcal{E}$ more than once, we fix a neighborhood $\mathcal{U}$ of $\mathcal{E}$. Proposition (4.1) is now a consequence of the next proposition.

Proposition 4.3 *Under the above assumptions, there is an open neighborhood $V \subset \Sigma$ of q in Σ and an open subset $W \subset V$ of V with the following property: any leaf L passing through a point of W intersects the exceptional divisor $\mathcal{E}$ before leaving the neighborhood $\mathcal{U}$.*

In fact, since the set of leaves meeting $\mathcal{E}$ contains an open set and all of them (with possible exception of a finite number) cross Σ and accumulates Δ, Proposition (4.3) clearly shows the existence of infinitely many leaves (orbits of X) intersecting $\mathcal{E}$ more than one time thus providing the desired contradiction.

In order to prove Proposition (4.3) we keep the preceding setting and notations. We are naturally led to discuss the behavior of the leaves of $\mathcal{F}_X$ on a neighborhood of the point p_{ij} of intersection of the irreducible components D_i, D_j belonging to $\mathcal{E}$.

Now let us fix coordinates (x, y) around p_{ij} ($p_{ij} \simeq (0,0)$) such that $\{y = 0\} \subseteq D_i$ and $\{x = 0\} \subseteq D_j$. Without loss of generality we can suppose that the domain of definition of the (x, y)-coordinates contains the bidisc of radius 2. Next we fix a segment of vertical line Σ_x (resp. horizontal line Σ_y) passing through the point $(1, 0)$ (resp. $(0, 1)$). We assume that D_i is invariant under $\widetilde{\mathcal{F}}_X$ but D_j may or may not be invariant under $\widetilde{\mathcal{F}}_X$. Let us also make the following assumptions:

A) $p_{ij} \simeq (0, 0)$ is not a dicritical singularity of $\widetilde{\mathcal{F}}$.

B) $\widetilde{\mathcal{F}}_X$ has at least one eigenvalue different from *zero* at $p_{ij} \simeq (0, 0)$.

C) The vector field X whose associated foliation is $\widetilde{\mathcal{F}}$ is meromorphic semi-complete in the domain of the coordinates (x, y).

We are going to discuss a variant of the so-called *Dulac's transform*, namely if leaves intersecting Σ_x necessarily cut Σ_y. Precisely we fix a neighborhood $\mathbf{U}$ of $\{x = 0\} \cup \{y = 0\}$, we then have:

Lemma 4.4 *Under the preceding assumptions, there is an open neighborhood $V_x \subset \Sigma_x$ of $(1, 0)$ in Σ_x and an open set $W_x \subset V_x$ with the following property: any leaf L of $\mathcal{F}$ passing through a point of W_x meets Σ_y before leaving U. In particular, if D_j is not invariant by $\mathcal{F}$, then the leaves of $\mathcal{F}$ passing through points of W_x cross the axis $\{x = 0\}$ before leaving $\mathbf{U}$. In addition, by choosing V_x very small, the ratio between the area of W_x and the area of V_x becomes arbitrarily close to* 1.

Before proving this lemma, let us deduce the proof of Proposition (4.3).

Proof of Proposition (4.3): Recall that $\widetilde{\mathcal{F}}_X$, the proper transform of $\mathcal{F}_X$, has only simple singularities in $\mathcal{E}$. In particular, if $\mathbf{p} \in \mathcal{E}$ is a dicritical singularity of $\widetilde{\mathcal{F}}_X$, then $\widetilde{\mathcal{F}}_X$ has 2 eigenvalues different from *zero* at $\mathbf{p}$. Hence $\widetilde{\mathcal{F}}_X$ is linearizable around $\mathbf{p}$ and, as a consequence, there is a small neighborhood $U_{\mathbf{p}}$ of $\mathbf{p}$ such that any regular leaf L of $\widetilde{\mathcal{F}}_X$ entering $U_{\mathbf{p}}$ must cross $\mathcal{E}$ before leaving $U_{\mathbf{p}}$. This applies in particular if $\mathbf{p}$ coincides with the intersection of two irreducible components D_i, D_j of $\mathcal{E}$.

On the other hand Δ is invariant by $\widetilde{\mathcal{F}}_X$ and all but a finite number of leaves of $\widetilde{\mathcal{F}}_X$ intersect Σ in arbitrarily small neighborhoods of $q = \Sigma \cap \Delta$. In particular if $\widetilde{\mathcal{F}}_X$ has a dicritical singularity on Δ, then the statement follows from the argument above.

Suppose now that $\widetilde{\mathcal{F}}_X$ does not have a dicritical singularity on Δ. Let D_1 be another irreducible component of $\mathcal{E}$ which intersects Δ at p_{01}. Note that p_{01} is not a dicritical singularity of $\widetilde{\mathcal{F}}_X$ by the preceding discussion. If D_1 is not invariant by $\widetilde{\mathcal{F}}_X$, then Lemma (4.4) allows us to find infinitely many leaves of $\widetilde{\mathcal{F}}_X$ intersecting $\mathcal{E}$. Thus the proposition would follow. On the other hand, if D_1 is invariant by $\widetilde{\mathcal{F}}_X$, then Lemma (4.4) still allows us to find a local transverse section Σ_1 through a regular point q_1 of D_1 with the desired property namely: excepted for a set of leaves whose volume can be made arbitrarily small (modulo choosing V_x sufficiently small), all leaves of $\widetilde{\mathcal{F}}_X$ meet Σ_1 in arbitrarily small neighborhoods of $q_1 = \Sigma_1 \cap D_1$. We then continue the procedure replacing Δ by D_1. Since we eventually will find an irreducible component of $\mathcal{E}$ which is not invariant by $\widetilde{\mathcal{F}}_X$ or contains a dicritical singularity, the proposition is proved. This also concludes the proof of Proposition (4.1). $\square$

The rest of the section is devoted to the proof of Lemma (4.4). Let us begin with the easier case in which D_j is not invariant by $\widetilde{\mathcal{F}}_X$.

Lemma 4.5 *The vector field $\widetilde{X}$ vanishes with order* 1 *on D_j. It also has poles of order* 1 *on D_i and the origin* $(0,0)$ *is a LJ-singularity of $\widetilde{\mathcal{F}}_X$.*

Proof: By assumption D_j is contained in $\mathcal{E}$ and is not invariant by $\widetilde{\mathcal{F}}_X$. In particular $\widetilde{X}$ cannot have poles on D_j since there is no strictly meromorphic semi-complete vector field in dimension 1. Neither can $\widetilde{X}$ be regular on D_j otherwise certain points of $\mathbb{C}^2$ would reach the infinity in finite time, thus contradicting the fact that $\widetilde{X}$ is complete. Finally the order of $\widetilde{X}$ on D_j cannot be greater than 2, otherwise $\widetilde{X}$ would not be semi-complete. Besides, if this order is 2, then infinitely many orbits of X will be compactified into rational curves. This would imply that $\mathcal{F}_X$ has an algebraic first integral which is impossible. This shows that $\widetilde{X}$ vanishes with order 1 on D_j.

Because $\widetilde{X}$ vanishes on D_j and D_j is not invariant by the associated foliation $\widetilde{\mathcal{F}}_X$, it follows from Section 4 that either $(0,0)$ is a LJ-singularity of $\widetilde{\mathcal{F}}_X$ or $\widetilde{\mathcal{F}}_X$ is linearizable with eigenvalues $1,-1$ at $(0,0)$. In the latter case the conclusions of Lemma (4.4) are obvious. Thus we can suppose that $(0,0)$ is a LJ-singularity. It follows from Lemma (3.5) that $\widetilde{X}$ must have a pole divisor of order 1 on D_i. $\square$

Proof of Lemma (4.4) when D_j is not invariant by $\widetilde{\mathcal{F}}_X$: Modulo blowing-up $(0,0)$, the problem is immediately reduced to the discussion of the Dulac's transform between the strong and the weak invariant manifolds of the saddle-node determined by

$$x(1+y)\,dy \;+\; y^2\,dx\;.$$

Thinking of this foliation as a differential equation, we obtain the solution

$$x(T) = \frac{x_0 e^T}{1 - y_0 T} \quad \text{and} \quad y(T) = \frac{y_0}{1 - y_0 T}\,.$$

Let us fix $x_0 = 1$. Given y_0, we search for T_0 so that $y(T_0) = 1$. Furthermore we also require that the norm of $x(T)$ stays "small" during the procedure. This is clearly possible if y_0 is real negative (sufficiently close to *zero*). Actually we set $T_0 = (1-y_0)/y_0 \in \mathbb{R}_-$ so that T can be chosen real negative during the procedure. Thus the norm of $x(T)$ will remain controlled by that of y_0.

On the other hand, let y_0 be in the transverse section Σ_x and suppose that y_0 is not real positive. Then the orbit of y_0 under the local holonomy of the strong invariant manifold converges to *zero* and is asymptotic to $\mathbb{R}_-$. Indeed this local holonomy is represented by a local diffeomorphism of the form $z \mapsto z + z^2 +$ h.o.t. and the local topological dynamics of these diffeomorphisms is simple and well-understood (known as a "flower", in the present case this dynamics is also called the parabolic bifurcation). Hence for a sufficiently large iterate of y_0 (without leaving V_x), the above "Dulac's transform" is well-defined. Thus the statement of Lemma (4.4) is verified as long as we take $W_x = V_x \setminus \mathbb{R}_-$ in the above coordinates. $\square$

From now on we can suppose that D_j is invariant under $\mathcal{F}_X$. We have three cases to check:

1) $\mathcal{F}_X$ has two eigenvalues different from *zero* at $(0,0)$ and is locally linearizable.

2) $\mathcal{F}_X$ has two eigenvalues different from *zero* at $(0,0)$ but is not locally linearizable. In this case the quotient of the eigenvalues is real negative.

3) $\mathcal{F}_X$ defines a saddle-node at $(0,0)$.

In the Case 1 the verification is automatic and left to the reader. Case 2 follows from [M-M] (note that our convention of signs is opposite to the convention of [M-M]). So we just need to consider the case of saddle-nodes. Of course all the possible saddle-nodes necessarily have a convergent weak invariant manifold. Without loss of generality we can suppose that D_i is the strong invariant manifold so that D_j is the weak invariant manifold (the other possibility is analogous). All the background material about saddle-nodes used in what follows can be found in [M-R].

Thanks to Lemma (3.6), we can find coordinates (x,y) as above where the vector field $\widetilde{X}$ becomes

$$\widetilde{X} = y^{-k}[(x(1+\lambda y^p) + yR)\partial/\partial x \; + \; y^{p+1}\partial/\partial y]\,.$$

Since our problem depends only on the foliation associated to $\widetilde{X}$, we drop the factor y^{-k} in the sequel. We also notice that $\widetilde{X}$ is regular on D_j. The argument which is going to be employed here is a generalization of the one employed to deal with the saddle-node appearing after blowing-up the LJ-singularity in the previous case.

Following [M-R] we consider open sets $V_i \subset \mathbb{C}$, $i = 0, \ldots, 2p-1$, defined by $V_i = \{z \in \mathbb{C}\ ;\ (2i+1)\pi/2p - \pi/p < \arg z < (2i+1)\pi/2p + \pi/p\}$. The V_i's, $i = 1, \ldots, 2p-1$, define a covering of $\mathbb{C}^*$ and, besides, each V_i intersects only V_{i-1} and V_{i+1} (where $V_{-1} = V_{2p-1}$). We also let W_i^+ (resp. W_i^-) be defined by

$$W_i^+ = \left\{ \frac{(4i-1)\pi}{2p} < \arg z < \frac{(4i+1)\pi}{2p} \right\} \quad \text{and} \quad W_i^- = \left\{ \frac{(4i+1)\pi}{2p} < \arg z < \frac{(4i+3)\pi}{2p} \right\}.$$

for $i = 0, \ldots, 2p-1$. We point out that $\mathrm{Re}\,(y^p) < 0$ (resp. $\mathrm{Re}\,(y^p) > 0$) provided that $y \in W_i^-$ (resp. $y \in W_i^+$). In addition we have $W_i^+ = V_{2i} \cap V_{2i+1}$ and $W_i^- = V_{2i+1} \cap V_{2i+2}$

(unless $p = 1$ where $V_0 \cap V_1 = W_0^+ \cup W_0^-$). Given $\varepsilon > 0$, we set

$$U_{i,V} = \{(x,y) \in \mathbb{C}^2 \, ; \, \|x\| < \varepsilon \, , \, \|y\| < \varepsilon \text{ and } y \in V_i\} \, .$$

According to Hukuara-Kimura-Matuda, there is a bounded holomorphic mapping $\phi_{U_{i,V}}(x,y) = (\varphi_{U_{i,V}}(x,y), y)$ defined on $U_{i,V}$ which brings the vector field $\widetilde{X}$ to the form

$$x(1 + \lambda y^p)\partial/\partial x \; + \; y^{p+1}\partial/\partial y \, . \tag{4}$$

The vector field in (4) can be integrated to give

$$x(T) = \frac{x_0 e^T}{\sqrt[p]{(1 - p y_0^p T)^\lambda}} \text{ and } y(T) = \frac{y_0}{\sqrt[p]{1 - p y_0^p T}} \, . \tag{5}$$

Proof of Lemma (4.4): We keep the preceding notations. On $U_{i,V}$ we consider a normalizing mapping $\phi_{U_{i,V}}$ such that $\widetilde{X}$ is as in (4). In this coordinate we fix a vertical line Σ_x as before and let $\Sigma_{y,i}$ denote the intersection of the horizontal line through $(0,1)$, Σ_y, with the sector V_i. Again we want to know which leaves of $\widetilde{\mathcal{F}}$ passing through a point of Σ_x will intersect Σ_y as well. Thus starting with $x_0 = 1$, we search for $y(T_0) = 1$. Thanks to equations (5), it is enough to choose $T_0 = (1 - y_0^p)/p y_0^p$. In particular $T_0 \in \mathbb{R}_-$ provided that $y_0^p \in \mathbb{R}_-$. The formula for $x(T)$ in (5) shows that $x(T)$ remains in the fixed neighborhood $\mathbf{U}$ of $\{x = 0\} \cup \{y = 0\}$ provided that we keep $T \in \mathbb{R}_-$ during the procedure and choose y_0 sufficiently small. More generally if the real part $\mathrm{Re}\,(y_0^p)$ is negative and the quotient between imaginary and real parts is bounded, then the same argument applies. In other words, if y_0 belongs to a compact subsector of $W^-_{[i+1/2]-1}$, the set W_j^- contained in V_i, then the Dulac's transform in question is well-defined modulo choosing y_0 uniformly small.

The local holonomy associated to D_i (the strong invariant manifold) has the form $\mathsf{h}(z) = z + z^{p+1} + \text{h.o.t.}$ The dynamical picture corresponding to this diffeomorphims is still a "flower". However, in general, it is not true that the orbit of a "generic" point will intersect a fixed $W^-_{[i+1/2]-1}$ since h may have invariant sectors. However, the above argument can be applied separately for each i. Clearly to each i fixed we have a different $\Sigma_y \cap V_i$ associated. Nonetheless they are all equivalent since the weak invariant manifold of the foliation is convergent. It follows that apart from a finite number of curves whose union has empty interior, the leaf through a point of Σ_x meets Σ_y before leaving the neighborhood $\mathbf{U}$. As mentioned the case where D_i is the (convergent) weak invariant manifold and D_j is the strong invariant manifold is analogous. The statement follows. $\square$

5 Arrangements of simple singularities

Now we are going to study the possible arrangements of simple semi-complete singularities over a rational curve of self-intersection -1 which are obtained by blowing-up a semi-complete vector field on a neighborhood of $(0,0) \in \mathbb{C}^2$.

We shall make a number of assumptions which are always satisfied in our cases. We denote by $\widetilde{\mathbb{C}}^2$ the blow-up of $\mathbb{C}^2$ at the origin and by $\pi : \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ the corresponding blow-up map. Given a vector field Y (resp. foliation $\mathcal{F}$) defined on a neighborhood U of the origin, π naturally induces a vector field $\widetilde{Y}$ (resp. foliation $\widetilde{\mathcal{F}}$) defined on $\pi^{-1}(U)$. Furthermore Y is semi-complete on U if and only if $\widetilde{Y}$ is semi-complete on $\pi^{-1}(U)$.

Given a meromorphic vector field $Y = fZ/g$ with f, g, Z holomorphic, we call the vector field fZ the *holomorphic part* of Y. In this section it is discussed the nature of a meromorphic semi-complete vector field Y defined on a neighborhood of the origin $(0,0) \in \mathbb{C}^2$ which satisfies the following assumptions:

1. $Y = x^{k_1} y^{k_2} f_1 Z$ where Z is a holomorphic vector field having an isolated singularity at $(0,0) \in \mathbb{C}^2$, f_1 is holomorphic function and $k_1, k_2 \in \mathbb{Z}$.
2. The regular orbits of Y contain at most 1 singular point. Furthermore the order of Y at this singular point is one.
3. The regular orbits of Y contains at most 1 period (i.e. there is only one homology class containing loops on which the integral of the time-form is different from zero).
4. The foliation $\mathcal{F}$ associated to Y (or to Z) has both eigenvalues equal to *zero* at $(0,0) \in \mathbb{C}^2$.
5. The blow-up $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ is such that every singularity $\tilde{p} \in \pi^{-1}(0)$ of $\widetilde{\mathcal{F}}$ is simple.
6. $\mathcal{F}$ is not dicritical at $(0,0)$.

Before continuing let us introduce two basic definitions. Assume that $\mathcal{F}$ is a singular holomorphic foliation defined on a neighborhood of a singular point p. Let $\mathcal{S}$ be a smooth separatrix of $\mathcal{F}$ at p. We want to define the *order* of $\mathcal{F}$ with respect to $\mathcal{S}$ at p, $\mathrm{ord}_{\mathcal{S}}(\mathcal{F}, p)$ (also called the multiplicity of $\mathcal{F}$ along $\mathcal{S}$), and the *index* of $\mathcal{S}$ w.r.t. $\mathcal{F}$ at p, $\mathrm{Ind}_p(\mathcal{F}, \mathcal{S})$ (cf. [C-S]). In order to do that, we consider coordinates (x, y) where $\mathcal{S}$ is given by $\{y = 0\}$ and a holomorphic 1-form $\omega = F(x,y)\,dy - G(x,y)\,dx$ defining $\mathcal{F}$ and having an isolated singularity at p. Then we let

$$\mathrm{ord}_{\mathcal{S}}(\mathcal{F}, p) = \mathrm{ord}\,(F(x,0)) \quad \text{at} \quad 0 \in \mathbb{C} \quad \text{and} \tag{6}$$

$$\mathrm{Ind}_p(\mathcal{F}, \mathcal{S}) = \mathrm{Res}\, \frac{\partial}{\partial y} \left(\frac{G}{F} \right) (x, 0)\, dx \, . \tag{7}$$

In the above formulas $\mathrm{ord}\,(F(x,0))$ stands for the order of the function $x \mapsto F(x,0)$ at $0 \in \mathbb{C}$ and Res for the residue of the 1-form in question.

Let $p_1, \ldots, p_r$ denote the singularities of $\widetilde{\mathcal{F}}$ belonging to $\pi^{-1}(0)$. Since $\pi^{-1}(0)$ naturally defines a separatrix for every p_i, we can consider both $\mathrm{ord}_{\pi^1(0)}(\widetilde{\mathcal{F}}, p_i)$ and $\mathrm{Ind}_{p_i}(\widetilde{\mathcal{F}}, \pi^{-1}(0))$. Easy calculations and the Residue Theorem then provides (cf. [M-M] , [C-S]):

$$\mathrm{ord}_{(0,0)}(\mathcal{F}) + 1 = \sum_{i=1}^{r} \mathrm{ord}_{\pi^{-1}(0)}(\widetilde{\mathcal{F}}, p_i) \, , \tag{8}$$

$$\sum_{i=1}^{r} \mathrm{Ind}_{p_i}(\widetilde{\mathcal{F}}, \pi^{-1}(0)) = -1 \, . \tag{9}$$

On the other hand the order of $\pi^{-1}(0)$ as a divisor of zeros or poles of $\widetilde{Y}$ is

$$\mathrm{ord}_{\pi^{-1}(0)}\widetilde{Y} = \mathrm{ord}_{(0,0)}(f) + \mathrm{ord}_{(0,0)}(\mathcal{F}) - \mathrm{ord}_{(0,0)}(g) - 1\,. \tag{10}$$

In particular if this order is *zero*, then $\widetilde{Y}$ is regular on $\pi^{-1}(0)$.

To abridge notations, the local singular foliation induced by the linear vector field

$$(x+y)\partial/\partial x + y\partial/\partial y$$

will be called a LJ-singularity (where LJ stands for linear and in the Jordan form).

To simplify the statement of the main result of this section, namely Proposition (5.1), we first introduce 3 types, or models, of vector fields. Let us keep the preceding notation.

Model $Z_{1,11}$**:** Let $\mathcal{F}_{1,11}$ be the foliation associated to $Z_{1,11}$ and $\widetilde{\mathcal{F}}_{1,11}$ its blow-up. Then $\widetilde{\mathcal{F}}_{1,11}$ contains 3 singularities p_1, p_2, p_3 on $\pi^{-1}(0)$ whose eigenvalues are respectively $1, 1$, $-1, 1$ and $-1, 1$. The singularity p_1 is a LJ-singularity and the blow-up $\widetilde{Z}_{1,11}$ has a pole of order 1 on $\pi^{-1}(0)$. The separatrix of p_2 (resp. p_3) transverse $\pi^{-1}(0)$ is a pole divisor of $\widetilde{Z}_{1,11}$ of order 1 as well. Finally $\widetilde{Z}_{1,11}$ has a curve of zeros passing through p_1 which is not invariant under $\widetilde{\mathcal{F}}_{1,11}$.

Model $Z_{0,12}$**:** With similar notations, $\widetilde{\mathcal{F}}_{0,12}$ has 3 singularities p_1, p_2, p_3 on $\pi^{-1}(0)$ of eigenvalues equal to $1, 0$, $-1, 2$ and $-1, 2$. The singularity p_1 is a saddle-node with strong invariant manifold contained in $\pi^{-1}(0)$. The separatrix of p_2 (resp. p_3) transverse to $\pi^{-1}(0)$ is a pole of order $d \neq 0$ of $\widetilde{Z}_{0,12}$. The $\pi^{-1}(0)$ is a pole of order $2d-1$ of $\widetilde{Z}_{0,12}$. There is no other component of the divisor of zeros or poles of $\widetilde{Z}_{0,12}$.

Model $Z_{1,00}$**:** $\widetilde{\mathcal{F}}_{1,00}$ still has 3 singularities p_1, p_2, p_3 whose eigenvalues are $-1, 1$, $1, 0$ and $1, 0$. The singularities p_2, p_3 are saddle-nodes with strong invariant manifolds contained in $\pi^{-1}(0)$. The separatrix of p_1 transverse to $\pi^{-1}(0)$ is a pole of $\widetilde{Z}_{1,00}$ of order $d \neq 0$. The exceptional divisor $\pi^{-1}(0)$ is a pole of order $d-1$ and there is no other component of the divisor of zeros or poles of $\widetilde{Z}_{1,00}$,

Note in particular that Formula (8) implies that $\mathcal{F}_{1,11}$ (resp. $\mathcal{F}_{0,12}$, $\mathcal{F}_{1,00}$) has a singularity of order 2 at the origin.

Proposition 5.1 *Let Y, $\widetilde{Y}$ be as above. Assume that the order of $\widetilde{Y}$ on $\pi^{-1}(0)$ is different from zero. Then the structure of the singularities of $\widetilde{\mathcal{F}}$ on $\pi^{-1}(0)$ is equal to that of one of the models $Z_{1,11}$, $Z_{0,12}$ or $Z_{1,00}$.*

Lemma 5.2 *Denote by λ_1^i, λ_2^i the eigenvalues of $\widetilde{\mathcal{F}}$ at p_i ($i = 1, \ldots, r$). Then one of the following possibilities holds:*

($\imath$) $\lambda_1^i/\lambda_2^i = -n/m$ *where n, m belong to $\mathbb{N}^*$.*

($\imath\imath$) p_i *is a saddle-node (i.e. the eigenvalues are* 1 *and zero) whose strong invariant manifold coincides with* $\pi^{-1}(0)$.

($\imath\imath\imath$) p_i *is a LJ-singularity.*

Furthermore there may exist at most one LJ-singularity and, when such singularity does exist, all the remaining singularities are as in ($\imath$).

Proof: First let us suppose that one of the eigenvalues λ_1^i, λ_2^i vanishes. In this case $\widetilde{\mathcal{F}}$ defines a saddle-node at p_i. Moreover p_i belongs to the divisor of zeros or poles of $\widetilde{Y}$, so that Theorem (3.6) shows that the strong invariant manifold of $\widetilde{\mathcal{F}}$ at p_i coincides with $\pi^{-1}(0)$ as required.

On the other hand, if both λ_1^i, λ_2^i are different from *zero*, then they satisfy condition ($\imath$) as a consequence of Lemma (3.2). Finally it remains only to consider the case where p_1 is a LJ-singularity. Thus in local coordinates (x,t), $\{x=0\} \subset \pi^{-1}(0)$, around p_1, $\widetilde{Y}$ is given by

$$x^{-1}h[(t+x)\partial/\partial t + x\partial/\partial x]\,.$$

Hence the regular orbits of $\widetilde{Y}$ contain a *zero* of $\widetilde{Y}$ corresponding to their intersection with $h=0$. Because of condition 2, this implies that only one of the p_i's can be a LJ-singularity. Furthermore, by the same reason, the holonomy of $\pi^{-1}(0) \setminus \{p_1, \ldots, p_r\}$ has to be trivial. In particular none of the remaining singularities can be a saddle-node. $\square$

Combining the information contained in the preceding lemma with Formula (8) we obtain:

Corollary 5.3 *The order of* $\mathcal{F}$ *at* $(0,0) \in \mathbb{C}$ *equals* $r-1$*, i.e.* $\mathrm{ord}_{(0,0)}(\mathcal{F}) = r-1$. $\square$

The case where p_1 is a LJ-singularity is indeed easy to analyse. After the preceding lemma and the fact that the holonomy of $\pi^{-1}(0) \setminus \{p_1, \ldots, p_r\}$ is trivial, we conclude that all the remaining singularities $p_2, \ldots, p_r$ have eigenvalues 1 and -1. Now using formulas (8) and (9) we conclude that $\widetilde{Y}$ is as in the model $Z_{1,11}$.

Hereafter we suppose without loss of generality that none of the p_i's is a LJ-singularity. For $s \leq r$, we denote by $p_1, \ldots, p_r$ the singularities of $\widetilde{\mathcal{F}}$ where $\widetilde{\mathcal{F}}$ has two non-vanishing eigenvalues (whose quotient has the form $-n/m$, $m,n \in \mathbb{N}$). The remaining $p_{s+1}, \ldots, p_r$ singularities are therefore saddle-nodes. Recall that the strong invariant manifolds of these saddle-nodes coincide with $\pi^{-1}(0)$ thanks to Theorem (3.6). Next we have:

Lemma 5.4 *At least one of the* p_i*'s is a saddle-node (i.e.* s *is strictly less than* r*).*

Proof: The proof relies on Section 4 of [Re4]. Suppose for a contradiction that none of the p_i's is a saddle-node. Given that there is no LJ-singularity, it follows that the quotient λ_1^i/λ_2^i is negative rational for every $i = 1, \ldots, r$. Hence the hypotheses of Proposition 4.2 of [Re4] are verified. It results that X has one of the normal forms indicated in that proposition. As is easily seen, all those vector fields have orbits with 2 distinct periods which is impossible in our case. The lemma is proved. $\square$

To complete the proof of Proposition (5.1) we proceed as follows. For $i \in \{1, \ldots, s\}$, we consider local coordinates (x_i, t_i), $\{x_i = 0\} \subset \pi^{-1}(0)$, around p_i. In these coordinates $\widetilde{Y}$ has the form

$$\widetilde{Y} = x_i^{(\mathrm{ord}_{\pi^{-1}(0)}(\widetilde{Y}))} t_i^{d_i} h_i [m_i x_i (1 + \mathrm{h.o.t.}) \partial/\partial x_i - n_i t_i (1 + \mathrm{h.o.t}) \partial/\partial t_i] \tag{11}$$

where $d_i \in \mathbb{Z}$, $m_i, n_i \in \mathbb{N}$ and h_i is holomorphic but not divisible by either x_i, t_i. Similarly, around the saddle-nodes singularities $p_{s+1}, \ldots, p_r$, we have

$$\widetilde{Y} = x_i^{(\mathrm{ord}_{\pi^{-1}(0)}(\widetilde{Y}))} h_i [x_i^{p_i+1} \partial/\partial x_i + t_i (1 + \mathrm{h.o.t}) \partial/\partial t_i] \, . \tag{12}$$

We claim that $h_i(0, 0) \neq 0$. This is clear in equation (12) thanks to Theorem (3.6). As to equation (11), let us suppose for a contradiction that $h_i(0, 0) = 0$. Hence the regular leaves of $\widetilde{Y}$ have a zero corresponding to the intersection of these leaves with $\{h_i = 0\}$. Given condition 2, it results that only one of the h_i's may verify $h_i(0, 0) = 0$. Without loss of generality we suppose that $h_1(0, 0) = 0$. From Lemma (3.2) and Remark (3.3), it follows that (x_i, t_i) can be chosen so as to have $\widetilde{Y} = (xy)^a (x - y)(x_i \partial/\partial x_i - t_i \partial/\partial t_i)$. Formula (9) then shows that all the remaining singularities have to be saddle-nodes since the sum of the indices is -1. Nonetheless, again condition 2, implies that the holonomy of $\pi^{-1}(0) \setminus \{p_1, \ldots, p_r\}$ is trivial. Thus no saddle-node can appear on $\pi^{-1}(0)$. In other words r must be equal to 1 which is impossible.

Proof of Proposition(5.1): Considering the normal forms (11) and (12), we can suppose that $h_i(0, 0) \neq 0$. Set $\epsilon_i = (\mathrm{ord}_{\pi^{-1}(0)}(\widetilde{Y})) m_i - n_i d_i$ so that $\epsilon_i \in \{-1, 0, 1\}$ (cf. Lemma (3.2)). Alternatively we let $d_i = (\mathrm{ord}_{\pi^{-1}(0)}(\widetilde{Y})) m_i / n_i - \epsilon_i / n_i$.

On the other hand, Formula (9), in the present context, becomes

$$\sum_{i=1}^{r} m_i / n_i = 1 \, ,$$

where $m_i = 0$ if and only if p_i is a saddle-node and $n_i \neq 0$. Since all m_i, n_i are non-negative, only one of the n_i's can be equal to 1 provided that $m_i \neq 0$. In this case, we must have $m_i = 1$ as well and the remaining singularities are saddle-nodes. We claim that this implies that $h_i(0, 0)$ in (11) is always different from *zero*. Indeed if, say $h_1(0, 0) = 0$, then $m + 1 = n_1 = 1$ and the remaining singularities are saddle-nodes. The fact that the holonomy associated to the strong invariant manifold of a saddle-node is has order infinity, implies that this case cannot be produced. The resulting contradiction establishes the claim.

Now the fact that $h_i(0, 0) \neq 0$ show that $\sum_{i=1}^{r} d_i = \mathrm{ord}_{(0,0)}(f) - \mathrm{ord}_{(0,0)}(g)$. Therefore

$$\begin{aligned} \mathrm{ord}_{(0,0)}(f) - \mathrm{ord}_{(0,0)} &= \sum_{i=1}^{r} d_i = (\mathrm{ord}_{\pi^{-1}(0)}(\widetilde{Y}))(\sum_{i=1}^{r} m_i / n_i = 1) - \sum_{i=1}^{r} \epsilon_i / n_i \\ &= -\sum_{i=1}^{r} \epsilon_i / n_i + \mathrm{ord}_{(0,0)}(f) + r - 1 - \mathrm{ord}_{(0,0)}(g) - 1 \, . \end{aligned}$$

In other words, one has

$$\sum_{i=1}^{s}(1-\epsilon_i/n_i) = 2+s-r < 2\,. \tag{13}$$

As mentioned, only one of the n_i's may be equal to 1. In this case the remaining singularities are saddle-nodes and we obtain the model $Z_{1,00}$.

Next assume that all the n_i's are strictly greater than 1. In particular $1-\epsilon_i/n_i \geq 1/2$. The only new possibility is to have $n_1 = n_2 = 2$ and $r - s = 1$. Thus we obtain the model $Z_{0,12}$ completing the proof of our proposition. □

Remark 5.5 To complement the description of the Models $Z_{1,11}$, $Z_{0,12}$ and $Z_{1,00}$, we want to point out that excepted for the saddle-nodes, all the singularities appearing in the exceptional divisor after blowing-up are linearizable. Indeed this results from the finiteness of the local holonomies associated to their separatrizes. To check that these holonomies are finite we just have to use an argument analogous to the one employed in Remark (3.3).

As a consequence of the above fact, we conclude that the two saddle-nodes appearing as singularities of $\widetilde{\mathcal{F}}_{1,00}$ are identical. In particular either both have convergent weak invariant manifold or both have divergent weak invariant manifold.

6 The combinatorics of the reduction of singularities

In this last section we are going to prove our main results. Since we are going to work in local coordinates, we can consider a meromorphic semi-complete vector field Y defined around $(0,0) \in \mathbb{C}^2$. As usual let $\mathcal{F}$ be the foliation associated to Y. In view of Seidenberg's theorem [Se], there exists a sequence of punctual blow-ups π_j together with singular foliations $\widetilde{\mathcal{F}}^j$,

$$\mathcal{F} = \widetilde{\mathcal{F}}^0 \xleftarrow{\pi_1} \widetilde{\mathcal{F}}^1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} \widetilde{\mathcal{F}}^r\,, \tag{14}$$

where $\widetilde{\mathcal{F}}^j$ is the blow-up of $\widetilde{\mathcal{F}}^{j-1}$, such that all singularities of $\widetilde{\mathcal{F}}^r$ are simple. Furthermore each π_j is centered at a singular point where $\widetilde{\mathcal{F}}^{j-1}$ has vanishing eigenvalues. The sequence $(\widetilde{\mathcal{F}}^j, \pi_j)$ is said to be the *resolution tree* of $\mathcal{F}$. Fixed $j \in \{1,\ldots,r\}$, we denote by $\mathcal{E}^j$ the total exceptional divisor $(\pi_1\circ\cdots\circ\pi_j)^{-1}(0,0)$ and by D^j the irreducible component of $\mathcal{E}^j$ introduced by π_j. Note that D^j is a rational curve given as $\pi_j^{-1}(\tilde{p}^{j-1})$ where $\tilde{p}^{j-1}$ is a singularity of $\widetilde{\mathcal{F}}^{j-1}$. Finally we identify curves and their proper transforms in the obvious way. Also $\widetilde{Y}^j$ will stand for the corresponding blow-up of Y.

Throughout this section $Y, \mathcal{F}$ are supposed to verify the following assumptions:

A. $Y = y^{-k} f Z$ where $k \geq 2$, Z is a holomorphic vector field having an isolated singularity at $(0,0) \in \mathbb{C}^2$ and f is a holomorphic function.

B. Assumptions 2 and 3 of Section 4.

C. The origin is not a dicritical singularity of $\mathcal{F}$.

It immediately results from the above assumptions that the axis $\{y=0\}$ is a smooth separatrix of $\mathcal{F}$. Letting $Z = f\partial/\partial x + g\partial/\partial y$, recall that the multiplicity of $\mathcal{F}$ along $\{y=0\}$

(or the order of $\mathcal{F}$ w.r.t. $\{y=0\}$ and $(0,0)$) is by definition the order at $0 \in \mathbb{C}$ of the function $x \mapsto f(x,0)$.

The main result of this section is Theorem (6.1) below.

Theorem 6.1 *Let $Y, \mathcal{F}$ be as above. Suppose that the divisor of zeros/poles of $\widetilde{Y}^r$ contains $\mathcal{E}^r$ (i.e. there is no component D^j of $\mathcal{E}^r$ where $\widetilde{Y}^r$ is regular). Then the multiplicity of $\mathcal{F}$ along $\{y=0\}$ is at most 2 (in particular the order of $\mathcal{F}$ at $(0,0)$ is not greater than 2).*

As a by-product of our proof, the cases in which the multiplicity of $\mathcal{F}$ along $\{y=0\}$ is 2 are going to be characterized as well. Also note that assumption **C** ensures that all the components D^j are invariant by $\mathcal{F}^r$. Moreover none of the singularities of $\widetilde{\mathcal{F}}^r$ is dicritical. In what follows we shall obtain the proof of Theorem (6.1) through a systematic analyse of the structure of the resolution tree of $\mathcal{F}$.

Remark 6.2 Let $\mathcal{F}$ be a foliation defined on a neighborhood of $(0,0) \in \mathbb{C}^2$ and consider a separatrix $\mathcal{S}$ of $\mathcal{F}$. Denote by $\widetilde{\mathcal{F}}$ the blow-up of $\mathcal{F}$ and by $\widetilde{\mathcal{S}}$ the proper transform of $\mathcal{S}$. Naturally $\widetilde{\mathcal{S}}$ constitutes a separatrix for some singularity p of $\widetilde{\mathcal{F}}$. In the sequel the elementary relation

$$\mathrm{Ind}_p(\widetilde{\mathcal{F}}, \widetilde{\mathcal{S}}) = \mathrm{Ind}_{(0,0)}(\mathcal{F}, \mathcal{S}) - 1 \tag{15}$$

will often be used.

Consider the resolution tree (14) of $\mathcal{F}$. By assumption $\mathcal{E}^r$ contains a rational curve D^r of self-intersection -1. Blowing-down (collapsing) this curve yields a foliation $\widetilde{\mathcal{F}}^{r_1}$ together with the total exceptional divisor $\mathcal{E}^{r_1}$. If $\mathcal{E}^{r_1}$ contains a rational curve with self-intersection -1 where all the singularities of $\widetilde{\mathcal{F}}^{r_1}$ are simple, we then continue by blowing-down this curve. Proceeding recurrently in this way, we eventually arrive to a foliation $\widetilde{\mathcal{F}}^{r_1}$, $1 \leq r_1 < r$, together with an exceptional divisor $\mathcal{E}^{r_1}$ such that every irreducible component of $\mathcal{E}^{r_1}$ with self-intersection -1 contains a singularity of $\widetilde{\mathcal{F}}^{r_1}$ with vanishing eigenvalues. Let $\widetilde{Y}^{r_1}$ be the vector field corresponding to $\widetilde{\mathcal{F}}^{r_1}$, $\mathcal{E}^{r_1}$, using Proposition (5.1) we conclude the following:

Lemma 6.3 *$\mathcal{E}^{r_1}$ contains (at least) one rational curve D^{r_1} of self-intersection -1. Moreover if p is a singularity of $\widetilde{\mathcal{F}}^{r_1}$ belonging to D^{r_1} then either p is simple for $\widetilde{\mathcal{F}}^{r_1}$ or $\widetilde{Y}^{r_1}$ has one of the normal forms $Z_{1,11}$, $Z_{0,12}$, $Z_{1,00}$ around p. Finally there is at least one such singularity p_1 which is not simple for $\widetilde{\mathcal{F}}^{r_1}$.* □

The next step is to consider the following description of the models $Z_{1,11}$, $Z_{0,12}$, $Z_{1,00}$ (and their respective associated foliations $\mathcal{F}_{1,11}$, $\mathcal{F}_{0,12}$, $\mathcal{F}_{1,00}$) which results at once from the definition of these models given in Section 5. While this description is slightly less precise than the previous one, it emphasizes the properties more often used in the sequel.

• $Z_{1,11}$, $\mathcal{F}_{1,11}$: $\mathcal{F}_{1,11}$ has exactly 2 separatrizes $\mathcal{S}_1, \mathcal{S}_2$ which are smooth, transverse and of index *zero*. $\mathcal{F}_{1,11}$ has order 2 at the origin. The multiplicity of $\mathcal{F}_{1,11}$ along $\mathcal{S}_1$, $\mathcal{S}_2$ is 2. The vector field $Z_{1,11}$ has poles of order 1 on each of the separatrizes $\mathcal{S}_1$, $\mathcal{S}_2$.

• $Z_{0,12},\ \mathcal{F}_{0,12}$: $\mathcal{F}_{0,12}$ has order 2 at the origin and 2 smooth, transverse separatrizes coming from the separatrizes associated to the singularities of eigenvalues $-1, 2$ of $\widetilde{\mathcal{F}}_{0,12}$ (they are denoted $\mathcal{S}_1, \mathcal{S}_2$ and called strong separatrizes of $\mathcal{F}_{0,12}$). The multiplicity of $\mathcal{F}_{0,12}$ along $\mathcal{S}_1,\ \mathcal{S}_2$ is 2 and the corresponding indices are both -1. $\mathcal{F}_{0,12}$ has still a formal third separatrix $\mathcal{S}_3$, referred to as the weak separatrix of $\mathcal{F}_{0,12}$, which may or may not be convergent. The vector field $Z_{0,12}$ has poles of order $d \in \mathbb{N}^*$ on both $\mathcal{S}_1, \mathcal{S}_2$.

• $Z_{1,00},\ \mathcal{F}_{1,00}$: $\mathcal{F}_{1,00}$ has order 2 and 1 smooth separatrix $\mathcal{S}_1$ coming from the singularity of $\widetilde{\mathcal{F}}_{1,00}$ whose eigenvalues are $-1, 1$ which will be called the strong separatrix of $\mathcal{F}_{1,00}$. Note that the multiplicity of $\mathcal{F}_{1,00}$ along $\mathcal{S}_1$ is 2 and the index of $\mathcal{S}_1$ is *zero*. $\widetilde{\mathcal{F}}_{1,00}$ also has 2 additional formal separatrizes $\mathcal{S}_2, \mathcal{S}_3$ coming from the weak invariant manifold of the saddle-nodes singularities of $\widetilde{\mathcal{F}}_{1,00}$ and, accordingly, called the weak separatrizes of $\mathcal{F}_{1,00}$. Naturally the weak separatrizes of $\mathcal{F}_{1,00}$ may or may not converge. Finally the vector field $Z_{1,00}$ has poles of order $d \in \mathbb{N}^*$ on $\mathcal{S}_1$.

Remark 6.4 The content of this remark will not be proved in these notes and therefore will not be formally used either. Nonetheless it greatly clarifies the structure of the combinatorial discussion that follows. Consider a meromorphic vector field X having a smooth separatrix $\mathcal{S}$. Using appropriate coordinates (x, y) we can identify $\mathcal{S}$ with $\{y = 0\}$ and write X as $y^d(f\partial/\partial x + yg\partial/\partial y)$ where $d \in \mathbb{Z}$ and $f(x, 0)$ is a non-trivial meromorphic function. We define the *asymptotic order* of X at $\mathcal{S}$ (at $(0, 0)$), $\mathrm{ord\,asy}_{(0,0)}(X, \mathcal{S})$, by means of the formula

$$\mathrm{ord\,asy}_{(0,0)}(X, \mathcal{S}) = \mathrm{ord}_0 f(x, 0) + d.\mathrm{Ind}_{(0,0)}(\mathcal{F}, \mathcal{S}) \tag{16}$$

where $\mathcal{F}$ is the foliation associated to X. It can be proved that $0 \leq \mathrm{ord\,asy}_{(0,0)}(X, \mathcal{S}) \leq 2$ provided that X is semi-complete. Besides, if $\mathcal{S}$ is induced by a global rational curve (still denoted by $\mathcal{S}$) and $p_1, \ldots, p_s$ are the singularities of $\mathcal{F}$ on $\mathcal{S}$, then we have

$$\sum_{i=1}^{s} \mathrm{ord\,asy}_{p_i}(X, \mathcal{S}) = 2 \tag{17}$$

provided that X is semi-complete on a neighborhood of $\mathcal{S}$. Note that the above formula indeed generalizes formula (13).

Now we go back to the vector field $\widetilde{Y}^{r_1}$ on a neighborhood of D^{r_1}. Again we denote by $p_1, \ldots, p_s$ the singularities of $\widetilde{\mathcal{F}}^{r_1}$ on D^{r_1} and, without loss of generality, assume that $\widetilde{Y}^{r_1}$ admits one of the normal forms $Z_{1,11},\ Z_{0,12},\ Z_{1,00}$ around p_1.

Lemma 6.5 *All the singularities $p_2, \ldots, p_s$ are simple for $\widetilde{\mathcal{F}}^{r_1}$.*

Proof: We have to prove that it is not possible to exist two singularities with one of the normal forms $Z_{1,11},\ Z_{0,12},\ Z_{1,00}$ on D^{r_1}. Clearly we cannot have two singularities of type $Z_{1,11}$ otherwise a "generic" regular orbit of $\widetilde{Y}^{r_1}$ would contain two singular points which contradicts assumption **B**. Indeed the fact that a "generic" regular orbit of $\widetilde{Y}^{r_1}$ effectively intersects the divisor of zeros of both singularities results from the method employed in the preceding section. In the present case the discussion is simplified since the singularity appearing in the intersection of the two irreducible components of the exceptional divisor is linear with eigenvalues $-1, 1$ (cf. description of $Z_{1,11}$).

To prove that the other combinations are also impossible, it is enough to repeat the argument employed in Section 5, in particular using the fact that the order $d_1 \neq 0$ of $\widetilde{Y}^{r_1}$ on D^{r_1} does not depend of the singularity p_i. If the reader takes for grant Formula (17), this verification can easily be explained. In fact, the asymptotic order of $Z_{0,12}$ (resp. $Z_{1,00}$) with respect to its strong separatrizes is already 2. Furthermore the asymptotic order of $Z_{1,11}$ with respect to its separatrizes is 1. Since the asymptotic order of a semi-complete singularity cannot be negative, it becomes obvious that two such singularities cannot co-exist on D^{r_1} provided that Y^{r_1} is semi-complete. $\square$

Now we analyse each of the three possible cases.

• The normal form of $\widetilde{Y}^{r_1}$ around p_1 is $Z_{1,11}$: First recall that D^{r_1} is an irreducible component of order 1 of the pole divisor of $\widetilde{Y}^{r_1}$. Since the number of singularities of regular orbits of $\widetilde{Y}^{r_1}$ is at most 1, it follows that none of the remaining singularities $p_2, \ldots, p_s$ can be a LJ-singularity for $\widetilde{\mathcal{F}}^{r_1}$. Otherwise there would be another curve of *zeros* of $\widetilde{Y}^{r_1}$ which is not invariant by $\widetilde{\mathcal{F}}^{r_1}$ so that "generic" regular orbits of $\widetilde{Y}^{r_1}$ would have 2 singularities (cf. above). By the same reason the holonomy of $D^{r_1} \setminus \{p_2, \ldots, p_s\}$ with respect to $\widetilde{\mathcal{F}}^{r_1}$ must be trivial. This implies that none of the singularities $p_2, \ldots, p_s$ is a saddle-node for $\widetilde{\mathcal{F}}^{r_1}$. In fact, by virtue of Theorem (3.6), a saddle-node must have strong invariant manifold contained in D^{r_1} which ensures that the above mentioned holonomy is non-trivial. Then we conclude that all the remaining singularities $p_2, \ldots, p_s$ have eigenvalues $m_i, -n_i$ ($i = 2, \ldots, s$) with $m_i, n_i \in \mathbb{N}^*$. Once again the fact that the holonomy of $D^{r_1} \setminus \{p_2, \ldots, p_s\}$ is trivial shows that $m_i/n_i \in \mathbb{N}^*$. Finally Formula (9) implies that $s = 2$ and $m_2 = n_2 = 1$. An immediate application of Formulas (8) and (10) (or yet Formula (17)) shows that the separatrix of p_2 transverse to D^{r_1} is a component of order 1 of the pole divisor of $\widetilde{Y}^{r_1}$. Finally we denote by $Z_{1,11}^{(1)}$ (resp. $\mathcal{F}_{1,11}^{(1)}$) the local vector field (resp. holomorphic foliation) resulting from the collapsing of D^{r_1}. Summarizing one has:

• $Z_{1,11}^{(1)}, \ \mathcal{F}_{1,11}^{(1)}$: The foliation $\mathcal{F}_{1,11}^{(1)}$ has exactly two separatrizes $\mathcal{S}_1, \mathcal{S}_2$ which are smooth, transverse and of indices respectively equal to 1 and 0. The order of $\mathcal{F}_{1,11}^{(1)}$ at the origin is 2 as well as the multiplicity of $\mathcal{F}_{1,11}^{(1)}$ along $\mathcal{S}_1, \mathcal{S}_2$. The vector field $Z_{1,11}^{(1)}$ has poles of order 1 on $\mathcal{S}_1, \mathcal{S}_2$.

Now let us discuss the second case.

• The normal form of $\widetilde{Y}^{r_1}$ around p_1 is $Z_{0,12}$: Recall that D^{r_1} is an irreducible component of order $d \neq 0$ of the pole divisor of $\widetilde{Y}^{r_1}$. Repeating the argument of the previous section, we see that the singularities $p_2, \ldots, p_s$ can be neither saddle-nodes nor LJ-singularities. Again this can directly be seen from Formula (17): by virtue of Lemma (3.5) and Theorem (3.6), both types of singularities in question have asymptotic order equal to 1. Nonetheless the asymptotic order of $Z_{0,12}$ is already 2 which implies the claim. It follows that $\widetilde{\mathcal{F}}^{r_1}$ has eigenvalues $m_i, -n_i$ at each of the remaining singularities $p_2, \ldots, p_s$ ($m_i, n_i \in \mathbb{N}^*$). In particular the index of D^{r_1} w.r.t. $\widetilde{\mathcal{F}}^{r_1}$ around each p_i is strictly negative. Hence Formula (9) shows that p_1 is, in fact, the unique singularity of $\widetilde{\mathcal{F}}^{r_1}$ on D^{r_1}. We denote by $Z_{0,12}^{(1)}$ (resp. $\mathcal{F}_{0,12}^{(1)}$) the local vector field (resp. holomorphic foliation) arising from the collapsing of D^{r_1}. Thus:

- $Z_{0,12}^{(1)},\ \mathcal{F}_{0,12}^{(1)}$: The order of $\mathcal{F}_{0,12}^{(1)}$ at the origin is 1, besides the linear part of $\mathcal{F}_{0,12}^{(1)}$ is nilpotent. $\mathcal{F}_{0,12}^{(1)}$ has one strong separatrix $\mathcal{S}_1$ obtained through the strong separatrix of $\mathcal{F}_{0,12}$ which is transverse to D^{r_1}. This separatrix is smooth and has index *zero*, furthermore the multiplicity of $\mathcal{F}_{0,12}^{(1)}$ along $\mathcal{S}_1$ is 2. The foliation $\mathcal{F}_{0,12}^{(1)}$ has still another formal weak separatrix which may or may not converge. Finally the vector field $Z_{0,12}^{(1)}$ has poles of order $d \neq 0$ on $\mathcal{S}_1$.

Finally we have:

- The normal form of $\widetilde{Y}^{r_1}$ around p_1 is $Z_{1,00}$: Note that D^{r_1} is an irreducible component of order $d \neq 0$ of the pole divisor of $\widetilde{Y}^{r_1}$ (cf. description of the vector field $Z_{1,00}$). As before the remaining singularities cannot be saddle-nodes or LJ-singularities (the asymptotic order of $Z_{1,00}$ w.r.t. D^{r_1} is already 2). It follows that $\widetilde{\mathcal{F}}^{r_1}$ has eigenvalues $m_i, -n_i$ at the remaining singularities $p_2, \ldots, p_s$ ($m_i, n_i \in \mathbb{N}$). Around each singularity p_i ($i = 2, \ldots, s$), the vector field $\widetilde{Y}^{r_1}$ can be written as

$$x_i^{-d} t_i^{k_i} h_i[m_i x_i(1 + \text{h.o.t.})\partial/\partial x_i - n_i t_i(1 + \text{h.o.t.})\partial/\partial t_i]$$

where $h_i(0,0) \neq 0$. We just have to repeat the argument of Section 5, here we summarize the discussion by using the "fact" that the asymptotic order of $\widetilde{Y}^{r_1}$ w.r.t. D^{r_1} has to be *zero* at each p_i. Indeed, this gives us that $k_i = -1 + dm_i/n_i$. Comparing this with Lemma (3.2), it results that $n_i = 1$. Hence Formula (9) informs us that $s = 2$ and $m_2 = n_2 = 1$. It also follows that $k_2 = d - 1$. Let us denote by $Z_{1,00}^{(1)}$ (resp. $\mathcal{F}_{1,00}^{(1)}$) the local vector field (resp. holomorphic foliation) arising from the collapsing of D^{r_1}.

- $Z_{1,00}^{(1)},\ \mathcal{F}_{1,00}^{(1)}$: The order of $\mathcal{F}_{1,00}^{(1)}$ at the origin is 2 and it has one strong separatrix $\mathcal{S}_1$ obtained through the separatrix of p_2 which is transverse to D^{r_1}. This separatrix is smooth and has index *zero*, furthermore the multiplicity of $\mathcal{F}_{1,00}^{(1)}$ along $\mathcal{S}_1$ is 2. The foliation $\mathcal{F}_{1,00}^{(1)}$ has still two formal weak separatrizes which may or may not converge. Finally the vector field $Z_{1,00}^{(1)}$ has poles of order $d \neq 0$ on $\mathcal{S}_1$.

Summarizing what precedes, we easily obtain the following analogue of Lemma (6.3):

Lemma 6.6 *$\mathcal{E}^{r_2}$ contains (at least) one rational curve D^{r_2} of self-intersection -1. Moreover if p is a singularity of $\widetilde{\mathcal{F}}$ belonging to D^{r_2} then either p is simple for $\widetilde{\mathcal{F}}^{r_2}$ or $\widetilde{Y}^{r_2}$ has one of the normal forms $Z_{1,11},\ Z_{0,12},\ Z_{1,00},\ Z_{1,11}^{(1)},\ Z_{0,12}^{(1)},\ Z_{1,00}^{(1)}$ around p. Finally there is at least one such singularity p_1 which is not simple for $\widetilde{\mathcal{F}}^{r_2}$.* □

The argument is now by recurrence, we shall discuss only the next step in details. Again $p_1, \ldots, p_s$ are the singularities of $\widetilde{\mathcal{F}}^{r_2}$ on D^{r_2} and, without loss of generality, $\widetilde{Y}^{r_2}$ admits one of the normal forms $Z_{1,11},\ Z_{0,12},\ Z_{1,00},\ Z_{1,11}^{(1)},\ Z_{0,12}^{(1)},\ Z_{1,00}^{(1)}$ around p_1.

Lemma 6.7 *All the singularities $p_2, \ldots, p_s$ are simple for $\widetilde{\mathcal{F}}^{r_2}$.*

Proof: The proof is as in Lemma (6.5). Since no leaf of $\widetilde{\mathcal{F}}^{r_2}$ can meet the divisor of zeros of $\widetilde{Y}^{r_2}$ in more than one point, it results that we can have at most one singularity $Z_{1,11}$ or $Z_{1,11}^{(1)}$ on D^{r_2}. The fact that the models $Z_{0,12},\ Z_{1,00},\ Z_{0,12}^{(1)}$ and $Z_{1,00}^{(1)}$ cannot be combined among them or with $Z_{1,11},\ Z_{1,11}^{(1)}$ follows from the natural generalization of the method of Section 5 (which is again explained by the fact that these 3 vector fields have asymptotic order 2 w.r.t. D^{r_2}). The lemma is proved. $\square$

So we have obtained three new possibilities according to the normal form of $\widetilde{Y}^{r_2}$ around p_1 is $Z_{1,11}^{(1)},\ Z_{0,12}^{(1)}$ or $Z_{1,00}^{(1)}$. Let us analyse them separately.

• The normal form of $\widetilde{Y}^{r_2}$ around p_1 is $Z_{1,11}^{(1)}$: Note that $\mathcal{F}_{1,11}^{(1)}$ has two separatrizes which may coincide with D^{r_2}, one with index *zero* and other with index 1. In any case, D^{r_2} is an irreducible component of order 1 of the divisor of poles of $\widetilde{Y}^{r_2}$. Suppose first that the index of D^{r_2} w.r.t. $\mathcal{F}^{r_2}$ at p_1 is *zero*. The discussion then goes exactly as in the case of $Z_{1,11}$. We conclude that $s = 2$ and that $\mathcal{F}^{r_2}$ has eigenvalues $-1, 1$ at p_2. The fact that the holonomy of $D^{r_2} \setminus \{p_1, p_2\}$ w.r.t. $\mathcal{F}^{r_2}$ is trivial also implies that $\mathcal{F}^{r_2}$ is linearizable at p_2. Formulas (8) and (10) show that the separatrix of p_2 transverse to D^{r_2} is a component with order 1 of the divisor of poles of $\widetilde{Y}^{r_2}$. Finally we denote by $Z_{1,11}^{(2)}$ (resp. $\mathcal{F}_{1,11}^{(2)}$) the local vector field (resp. holomorphic foliation) arising from the collapsing of D^{r_2}. One has

• $Z_{1,11}^{(2)},\ \mathcal{F}_{1,11}^{(2)}$: The foliation $\mathcal{F}_{1,11}^{(2)}$ has exactly 2 separatrizes $\mathcal{S}_1, \mathcal{S}_2$ which are smooth, transverse and of indices respectively equal to *zero* and 2. $\mathcal{F}_{1,11}^{(2)}$ has order 2 at the origin and its multiplicity along $\mathcal{S}_1,\ \mathcal{S}_2$ is 2. The vector field $Z_{1,11}^{(2)}$ has poles of order 1 on each of the separatrizes $\mathcal{S}_1$, $\mathcal{S}_2$.

Now let us prove that the index of D^{r_2} w.r.t. $\mathcal{F}^{r_2}$ at p_1 cannot be 1. Suppose for a contradiction that this index is 1. Again the triviality of the holonomy of the regular leaf contained in D^{r_2} implies that all the singularities $p_2, \ldots, p_s$ are linearizable with eigenvalues $-1, 1$. Hence Formula (9) ensures that $s = 3$. In turn, Formula (10) shows that the separatrix of p_2 (resp. p_3) transverse to D^{r_2} is a component with order 1 of the pole divisor of $\widetilde{Y}^{r_2}$. This is however impossible in view of Formula (17). Alternate, we can observe that the divisor of poles of the vector field obtained by collapsing D^{r_2} consists of three smooth separatrizes. By virtue of assumptions **A, B, C** one of them must be the proper transform of $\{y = 0\}$. In particular its order as component of the pole divisor should be $k \geq 2$, thus providing the desired contradiction.

Next one has:

• The normal form of $\widetilde{Y}^{r_2}$ around p_1 is $Z_{0,12}^{(1)}$: The divisor D^{r_2} constitutes a separatrix of $\widetilde{\mathcal{F}}^{r_2}$ at p_1. Besides $\widetilde{Y}^{r_2}$ has poles of order $d \neq 0$ on D^{r_2} (cf. description of $Z_{0,12}^{(1)}$). Note also that the index of D^{r_2} w.r.t. $\widetilde{\mathcal{F}}^{r_2}$ at p_1 is *zero*. Our standard method shows that the remaining singularities cannot be saddle-nodes or LJ-singularities (the asymptotic order of $Z_{0,12}^{(1)}$ w.r.t. D^{r_2} is already 2). Thus $\widetilde{\mathcal{F}}^{r_2}$ has eigenvalues $m_i, -n_i$ at the singularity p_i, $i = 2, \ldots s$ $(m_i, n_i \in \mathbb{N})$. On a neighborhood of p_i, $\widetilde{Y}^{r_2}$ is given in appropriate coordinates by

$$x_i^{-d} t_i^{k_i} h_i [m_i x_i (1 + \text{h.o.t.}) \partial / \partial x_i - n_i t_i (1 + \text{h.o.t.}) \partial / \partial t_i]$$

where $h_i(0,0) \neq 0$. It is enough to repeat the discussion of Section 5. Using the "fact" that the asymptotic order of $\widetilde{Y}^{r_1}$ w.r.t. D^{r_1} has to be *zero* at each p_i, this can be summarized as follows. The asymptotic order is given by $k_i + 1 - dm_i/n_i$, since it must equal *zero*, one has $k_i = -1 + dm_i/n_i$. Comparing with Lemma (3.2), we conclude that $n_i = 1$. Hence Formula (9) implies that $s = 2$ and $m_2 = n_2 = 1$. In particular $k_2 = d - 1$. We then denote by $Z_{0,12}^{(2)}$ (resp. $\mathcal{F}_{0,12}^{(2)}$) the local vector field (resp. holomorphic foliation) arising from the collapsing of D^{r_2}.

• $Z_{0,12}^{(2)}$, $\mathcal{F}_{0,12}^{(2)}$: The order of $\mathcal{F}_{0,12}^{(2)}$ at the origin is 2. $\mathcal{F}_{0,12}^{(2)}$ has one strong separatrix $\mathcal{S}_1$ obtained through the strong separatrix of $\mathcal{F}_{0,12}^{(1)}$ which is transverse to D^{r_2}. This separatrix is smooth and has index *zero*, furthermore the multiplicity of $\mathcal{F}_{0,12}^{(2)}$ along $\mathcal{S}_1$ is 2. The foliation $\mathcal{F}_{0,12}^{(2)}$ has still another formal weak separatrix which may or may not converge. Finally the vector field $Z_{0,12}^{(2)}$ has poles of order $d \neq 0$ on $\mathcal{S}_1$.

Finally let us consider $Z_{1,00}^{(1)}$.

• The normal form of $\widetilde{Y}^{r_2}$ around p_1 is $Z_{1,00}^{(1)}$: The discussion is totally analoguous to the case $Z_{1,00}$. After collapsing D^{r_2}, we obtain a local vector field $Z_{1,00}^{(2)}$ (resp. holomorphic foliation $\mathcal{F}_{1,00}^{(2)}$) with the following characteristics:

• $Z_{1,00}^{(2)}$, $\mathcal{F}_{1,00}^{(2)}$: The order of $\mathcal{F}_{1,00}^{(2)}$ at the origin is 2 and it has one strong separatrix $\mathcal{S}_1$ obtained through the separatrix of p_2 which is transverse to D^{r_2}. This separatrix is smooth and has index *zero*, furthermore the multiplicity of $\mathcal{F}_{1,00}^{(2)}$ along $\mathcal{S}_1$ is 2. The foliation $\mathcal{F}_{1,00}^{(2)}$ has still two formal weak separatrizes which may or may not converge. Finally the vector field $Z_{1,00}^{(2)}$ has poles of order $d \neq 0$ on $\mathcal{S}_1$.

Let us inductively define a sequence of vector fields $Z_{1,11}^{(n)}$ by combining over a rational curve with self-intersection -1 a model $Z_{1,11}^{(n-1)}$ with a linear singularity p_2 having eigenvalues $-1, 1$. The rational curve in question induces a separatrix of index *zero* for $Z_{1,11}^{(n-1)}$ and both separatrizes of p_2 are components having order 1 of the pole divisor of the corresponding vector field. The model $Z_{1,11}^{(n)}$ is then obtained by collapsing the mentioned rational curve. Similarly we also define the sequences $Z_{1,00}^{(n)}$, $Z_{0,12}^{(n)}$.

Proof of Theorem (6.1): Let Y, $\mathcal{F}$ be as in the statement of this theorem. Suppose first that the order of $\mathcal{F}$ at $(0,0)$ is greater than one. We consider a resolution tree (14) for $\mathcal{F}$ and the recurrent procedure discussed above. Whenever we collapse a rational curve with self-intersection -1 contained in one of the exceptional divisors $\mathcal{E}^j$, it results a singularity which is either simple or belongs to the list $Z_{1,11}$, $Z_{1,11}^{(n)}$, $Z_{1,00}$, $Z_{1,00}^{(n)}$, $Z_{0,12}$, $Z_{0,12}^{(n)}$. After a finite number of steps of the above procedure, we arrive to the original vector field Y (resp. foliation $\mathcal{F}$). Therefore Y must admit one of the above indicated normal forms. Since the divisor of poles of Y is constituted by the axis $\{y = 0\}$, the cases $Z_{1,11}$, $Z_{1,11}^{(n)}$, $Z_{0,12}$ cannot be produced (their divisor of poles consist of two irreducible components). The case $Z_{0,12}^{(1)}$ cannot be produced either since the order of the associated foliation is supposed to be greater than or equal to 2. Thus we conclude that $Y = Z_{1,00}^{(n)}$ or, for $n \geq 2$, $Y = Z_{0,12}^{(n)}$ and the theorem follows in this case.

Next suppose that the order of $\mathcal{F}$ at $(0,0)$ is 1. Clearly if the linear part of $\mathcal{F}$ at $(0,0)$ has rank 2 (i.e. the corresponding holomorphic vector field with isolated singularities has 2 non-vanishing eigenvalues at $(0,0)$), then the conclusion is obvious. Next suppose that $\mathcal{F}$ is a saddle-node. If $\{y=0\}$ corresponds to the strong invariant manifold of $\mathcal{F}$ then the statement is obvious. On the other hand, $\{y=0\}$ cannot be the weak invariant manifold thanks to Theorem (3.6).

It only remains to check the case where the linear part of $\mathcal{F}$ at $(0,0)$ is nilpotent. The blow-up $\widetilde{Y}$ (resp. $\widetilde{\mathcal{F}}$) of Y (resp. $\mathcal{F}$) is such that $\widetilde{\mathcal{F}}$ has a single singularity $p \in \pi^{-1}(0)$. In addition the order of $\widetilde{\mathcal{F}}$ at p is necessarily 2. On a neighborhood of p, the vector field $\widetilde{Y}$ is given by $x^{-k}y^{-k}Z$ where Z is as in assumption **A**. The assumptions **B**, **C** are clearly verified as well. An inspection in the preceding discussion immediately shows that it applies equally well to this vector field $\widetilde{Y}$. We conclude that $\widetilde{Y}$ is given on a neighborhood of p by the model $Z_{0,12}$. Hence Y is the model $Z_{0,12}^{(1)}$ completing the proof of the theorem. □

• Conclusion:

Proof of Theorem A: Let X be a complete polynomial vector field in $\mathbb{C}^2$ with degree 2 or greater. We denote by $\mathcal{F}_X$ the singular holomorphic foliation induced by X on $\mathbb{CP}(2)$. We know from Lemma (2.3) that the line at infinity $\Delta \subset \mathbb{CP}(2)$ is invariant under $\mathcal{F}_X$. On the other hand there is a dicritical singularity p_1 of $\mathcal{F}_X$ belonging to Δ. Hence Proposition (4.1) ensures that $\mathcal{F}_X$ has a meromorphic first integral on $\mathbb{CP}(2)$. Besides the generic leaves of $\mathcal{F}_X$ in $\mathbb{CP}(2)$ are, up to normalization, rational curves (i.e. isomorphic to $\mathbb{CP}(1)$). According to Saito and Suzuki (cf. [Sz2]), up to polynomial automorphisms of $\mathbb{C}^2$, $\mathcal{F}_X$ is given by a first integral R having one of the following forms:

ı) $R(x,y) = x$;

ıı) $R(x,y) = x^n y^m$, with g.c.d$(m,n) = 1$ and $m,n \in \mathbb{Z}$;

ııı) $R(x,y) = x^n(x^l y + P(x))^m$, with g.c.d$(m,n) = 1$ and $m,n \in \mathbb{Z}$, $l \geq 1$. Moreover P is a polynomial of degree at most $l-1$ satisfying $P(0) \neq 0$.

To each of these first integrals there corresponds the foliations associated to the vector fields $X_1 = \partial/\partial x$, $X_2 = mx\partial/\partial x - ny\partial/\partial y$ and $X_3 = mx^{l+1}\partial/\partial x - [(n+lm)x^l y + nP(x) + mxP'(x)]\partial/\partial y$. Therefore the original vector field X has the form $X = Q.X_i$, where Q is a polynomial and $i = 1,2,3$. If $i=1$ then it follows at once that Q has to be as in the in order to produce a complete vector field X. Assume now that $i=2$. Using for instance Lemma (3.2), we see that P has again the form indicated in the statement unless $X_2 = x\partial/\partial x - y\partial/\partial y$ in which case we can also have $P = (xy)^a(x-y)$. Nonetheless we immediately check that the resulting vector field X is not complete in this case.

Finally let us assume that $i=3$. It is again easy to see that that the resulting vector field cannot be complete. This follows for example from the fact that $(0,0)$ is a singularity of X_3 with trivial eigenvalues (cf. [Re3]). The theorem is proved. □

Proof of Theorem B: Let us suppose for a contradiction that none of the singularities $p-1,\dots,p_k$ of $\mathcal{F}_X$ in Δ is dicritical. We write X as $F.Z$ where F is a polynomial of degree

$n \in \mathbb{N}$ and Z is a polynomial vector field of degree $d-n$ and having isolated zeros (where d is the degree of X).

We consider the restriction of X to a neighborhood of p_i. Clearly this restriction satisfies assumptions **A**, **B** and **C** of Section 5. In particular Theorem (6.1) applies to show that the multiplicity of $\mathcal{F}$ along Δ is at most 2. Moreover, if this multiplicity is 2, then X admits one of the normal forms $Z_{1,11},\ Z_{1,11}^{(n)},\ Z_{1,00},\ Z_{1,00}^{(n)},\ Z_{0,12},\ Z_{0,12}^{(n)}$ on a neighborhood of p_i.

From Lemma (2.6) we know that $k \leq 3$. Since the sum of the multiplicities of $\mathcal{F}$ along Δ at each p_i is equal to $d-n+1$, it follows that $d-n \leq 5$.

However, if $k=3$, Corollary (2.5) shows that the top-degree homogeneous component of X is as in the cases 5, 6, 7 of the corollary in question. Simple calculations guarantees that it is not possible to realize the models $Z_{1,11}, \ldots, Z_{0,12}^{(n)}$ in this way. The case $k=1$ being trivial, we just need to consider the case $k=2$. Now we have $d-n \leq 3$ and again it is very easy to conclude that none of these possibilities lead to a complete polynomial vector field. The resulting contradiction proves the theorem. $\square$

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